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WIENER AMALGAM SPACES WITH RESPECT TO QUASI-BANACH SPACES

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Abstract. We generalize the theory of Wiener amalgam spaces on locally compact groups to quasi-Banach spaces. As a main result we provide convolution relations for such spaces. Also we weaken the technical assumption that the global component is invariant under right translations, which is new even for the classical Banach space case. To illustrate our theory we discuss in detail an example on the $ax + b$ group.

1. Introduction. Wiener amalgam spaces consist of functions on a locally compact group defined by a (quasi-)norm that mixes, or amalgamates, a local criterion with a global criterion. The most general definition of Wiener amalgams so far was provided by Feichtinger in the early 1980's in a series of papers [4–6]. We refer to [12] for some historical notes and for an introduction to Wiener amalgams on the real line.

Wiener amalgams have proven to be a very useful tool for instance in time-frequency analysis [11] (e.g. the Balian–Low theorem [12]) and sampling theory. Our interest in those spaces arose from coorbit space theory [7–9, 14] which provides a group-theoretical approach to function spaces like Besov and Triebel–Lizorkin spaces as well as modulation spaces.

It seems that Wiener amalgams with respect to quasi-Banach spaces have not yet been considered in full generality, except for a few results for Wiener amalgams on $\mathbb{R}^d$ in [10]. So this paper deals with basic properties of Wiener amalgams $W(B,Y)$ with a quasi-Banach space Y as global component and one of the spaces $B = L^1, L^\infty$ or M (the space of complex Radon measures) as local component. Moreover, we also remove the technical assumption imposed by Feichtinger [4] that the global component Y has to be invariant under right translation. Thus, some of our results are even new for the classical case of Banach spaces Y.

One of our main achievements is a convolution relation for Wiener amalgams. As a special case it turns out that $W(L^\infty, L^p)$ is a convolution algebra for $0 < p \leq 1$ if the underlying group is an IN group, e.g. $\mathbb{R}^d$. This result is

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interesting since for non-discrete groups there are no convolution relations available for L^p if $p < 1$. The problem comes from possible p-integrable singularities which are not integrable. So the integral defining the convolution $F * G$ does not even exist for all $F \in L^p$ even if G is very nice, e.g. continuous with compact support. Of course, the local component L^∞ of $W(L^\infty, L^p)$ prohibits such singularities. So our results indicate that whenever treating quasi-Banach spaces in connection with convolution, one is almost forced to use Wiener amalgam spaces.

To illustrate our results we also treat a class of spaces Y on the $ax + b$ group such that $W(L^\infty, Y)$ is right translation invariant (and thus admits convolution relations) although Y is not.

For a quasi-Banach space $(B, \|\cdot \,|B\|)$, we denote the quasi-norm of a bounded operator $T : B \to B$ by $|\!|\!|T \,|\, B|\!|\!|$. The symbol $A \asymp B$ indicates throughout the paper that there are constants $C_1, C_2 > 0$ such that $C_1 A \leq B \leq C_2 A$ (independent of other quantities on which A, B might depend). We normally use the symbol C for a generic constant whose precise value might be different at each occurrence.

2. Basic properties. Let $\mathcal{G}$ be a locally compact group. Integration on $\mathcal{G}$ will always be with respect to a left Haar measure. We denote by $L_x F(y) = F(x^{-1}y)$ and $R_x F(y) = F(yx)$, $x, y \in \mathcal{G}$, the left and right translation operators. Furthermore, let Δ be the Haar module on $\mathcal{G}$. For a Radon measure μ we define $(A_x \mu)(k) = \mu(R_x k)$, $x \in \mathcal{G}$, for a continuous function k with compact support. We may identify a function $F \in L^1$ with a measure $\mu_F \in M$ by $\mu_F(k) = \int F(x)k(x)\,dx$. Then clearly $A_x F = \Delta(x^{-1}) R_{x^{-1}} F$. Further, we define the involutions $F^\vee(x) = F(x^{-1})$, $F^\nabla(x) = \overline{F(x^{-1})}$, $F^*(x) = \Delta(x^{-1})\overline{F(x^{-1})}$.

A quasi-norm $\|\cdot\|$ on some linear space Y is defined in the same way as a norm, with the only difference that the triangle inequality is replaced by $\|f + g\| \leq C(\|f\| + \|g\|)$ with some constant $C \geq 1$. It is well-known (see e.g. [1, p. 20] or [13]) that there exists an equivalent quasi-norm $\|\cdot \,|\, Y\|$ on Y and an exponent p with $0 < p \leq 1$ such that $\|\cdot \,|\, Y\|$ satisfies the p-triangle inequality, i.e., $\|f + g \,|\, Y\|^p \leq \|f \,|\, Y\|^p + \|g \,|\, Y\|^p$. ($C$ and p are related by $C = 2^{1/p} - 1$.) We can choose $p = 1$ if and only if Y is a Banach space. We always assume that such a p-norm on Y is chosen and denote it by $\|\cdot \,|\, Y\|$. If Y is complete with respect to the topology defined by the metric $d(f, g) = \|f - g \,|\, Y\|^p$ then it is called a *quasi-Banach space*.

Let Y be a quasi-Banach space of measurable functions on $\mathcal{G}$, which contains the characteristic function of any compact subset of $\mathcal{G}$. We assume Y to be *solid*, i.e., if $F \in Y$ and G is measurable and satisfies $|G(x)| \leq |F(x)|$ a.e. then also $G \in Y$ and $\|G \,|\, Y\| \leq \|F \,|\, Y\|$.

The Lebesgue spaces $L^p(\mathcal{G})$, $0 < p \le \infty$, provide natural examples of such spaces Y, and the usual quasi-norm in $L^p(\mathcal{G})$ is a p-norm if $0 < p \le 1$. If w is some positive measurable weight function on $\mathcal{G}$ then we further define $L^p_w = \{F \text{ measurable} : Fw \in L^p\}$ with $\|F \,|\, L^p_w\| := \|Fw \,|\, L^p\|$. A continuous weight w is called *submultiplicative* if $w(xy) \le w(x)w(y)$ for all $x, y \in \mathcal{G}$.

Now let B be one of the spaces $L^\infty(\mathcal{G}), L^1(\mathcal{G})$ or $M(\mathcal{G})$, the space of complex Radon measures. Choose some relatively compact neighborhood Q of $e \in \mathcal{G}$. We define the control function by

$$K(F, Q, B)(x) := \|(L_x\chi_Q)F \,|\, B\|, \quad x \in \mathcal{G}, \tag{2.1}$$

if F is locally contained in B, in symbols $F \in B_{\mathrm{loc}}$. The *Wiener amalgam space* $W(B, Y)$ is then defined as

$$W(B, Y) := W(B, Y, Q) := \{F \in B_{\mathrm{loc}} : K(F, Q, B) \in Y\}$$

with quasi-norm

$$\|F \,|\, W(B, Y, Q)\| := \|K(F, Q, B) \,|\, Y\|. \tag{2.2}$$

Here B is called the *local component* and Y the *global component*. It follows from the solidity of Y and from the quasi-norm properties of $\|\cdot \,|\, B\|$ and $\|\cdot \,|\, Y\|$ that (2.2) is indeed a quasi-norm. Since B is a Banach space it is easy to see that (2.2) is also a p-norm (with p being the exponent of the quasi-norm of Y). We emphasize that in general we do not require here that Y is right translation invariant in contrast to the classical papers of Feichtinger [4, 5].

Remark 2.1. The restriction of the local component B to the spaces L^1, L^∞ and M is done for the sake of simplicity. One can certainly extend our considerations to more general spaces B, e.g. L^p-spaces with $0 < p \le \infty$ (cf. [4, 12]). However, convolution relations as in Section 5 will not hold any more when taking $B = L^p$ for $p < 1$.

Let us first make some easy observations.

Lemma 2.1. *We have the following continuous embeddings*:

(a) $W(L^\infty, Y) \hookrightarrow Y$.
(b) $W(L^\infty, Y) \hookrightarrow W(L^1, Y) \hookrightarrow W(M, Y)$.

Proof. (a) Since $|F(x)| \le \sup_{u \in U} |F(u^{-1}x)|$ for a compact neighborhood U of $e \in \mathcal{G}$ the assertion follows from the solidity of Y.

The statement (b) follows immediately from $L^\infty(Q) \hookrightarrow L^1(Q) \hookrightarrow M(Q)$ for any compact set $Q \subset \mathcal{G}$. ∎

Let us now investigate whether $W(B, Y, Q)$ is independent of Q and whether it is complete. It will turn out that both properties are connected to the right translation invariance of $W(B, Y)$. In order to clarify this we need certain discrete sets in $\mathcal{G}$ and associated sequence spaces.

DEFINITION 2.1. Let $X = (x_i)_{i \in I}$ be some discrete set of points in $\mathcal{G}$ and V a relatively compact neighborhood of e in $\mathcal{G}$.

(a) X is called *V-dense* if $\mathcal{G} = \bigcup_{i \in I} x_i V$.
(b) X is called *relatively separated* if for all compact sets $K \subset \mathcal{G}$ there exists a constant C_K such that $\sup_{j \in I} \#\{i \in I : x_i K \cap x_j K \neq \emptyset\} \leq C_K$.
(c) X is called *V-well-spread* (or *simply well-spread*) if it is both relatively separated and V-dense for some V.

The existence of V-well-spread sets for arbitrarily small V is proven in [6].

Given the function space Y, a well-spread family $X = (x_i)_{i \in I}$ and a relatively compact neighborhood Q of $e \in \mathcal{G}$ we define the sequence space

$$Y_d := Y_d(X) := Y_d(X, Q) := \Big\{ (\lambda_i)_{i \in I} : \sum_{i \in I} |\lambda_i| \chi_{x_i Q} \in Y \Big\}, \tag{2.3}$$

with natural norm $\|(\lambda_i)_{i \in I} \,|\, Y_d\| := \| \sum_{i \in I} |\lambda_i| \chi_{x_i Q} \,|\, Y\|$. Here, $\chi_{x_i Q}$ denotes the characteristic function of the set $x_i Q$. If the quasi-norm of Y is a p-norm, $0 < p \leq 1$, then also Y_d has a p-norm. Suppose for instance $Y = L^p_m$, $0 < p \leq \infty$, with some positive continuous weight function m. If in addition m is *moderate*, i.e., $m(xy) \leq m(x) w(y)$ for all $x, y \in \mathcal{G}$ and some function w, then it is easily seen that $Y_d = \ell^p_{\widetilde{m}}$ with $\widetilde{m}(i) = m(x_i)$.

Although we will not require the right translation invariance of Y in general, we state the following easy observation in case it holds.

LEMMA 2.2. *If Y is right translation invariant then the definition of $Y_d = Y_d(X, U)$ does not depend on U.*

Proof. Let V, U be relatively compact sets with non-void interior. Then there exist a finite number of points y_j, $j = 1, \ldots, n$, such that $V = \bigcup_{j=1}^n U y_j$. This implies

$$\sum_{i \in I} |\lambda_i| \chi_{x_i V} \leq \sum_{j=1}^n \sum_{i \in I} |\lambda_i| \chi_{x_i U y_j} = \sum_{j=1}^n R_{y_j^{-1}} \Big(\sum_{i \in I} |\lambda_i| \chi_{x_i U} \Big).$$

By solidity and the p-triangle inequality we obtain

$$\begin{aligned} \Big\| \sum_{i \in I} |\lambda_i| \chi_{x_i V} \,\Big|\, Y \Big\| &\leq \Big(\sum_{j=1}^n |\!|\!| R_{y_j^{-1}} \,|\, Y |\!|\!|^p \Big\| \sum_{i \in I} |\lambda_i| \chi_{x_i U} \,\Big|\, Y \Big\|^p \Big)^{1/p} \\ &= C \Big\| \sum_{i \in I} |\lambda_i| \chi_{x_i U} \,\Big|\, Y \Big\|. \end{aligned}$$

Exchanging the roles of V and U shows the reverse inequality. ∎

The following concept will also be very useful.

DEFINITION 2.2. Suppose U is a relatively compact neighborhood of $e \in \mathcal{G}$. A collection of functions $\Psi = (\psi_i)_{i\in I}$, $\psi_i \in C_0(\mathcal{G})$, is called a *bounded uniform partition of unity of size* U (for short U-BUPU) if the following conditions are satisfied:

(1) $0 \le \psi_i(x) \le 1$ for all $i \in I$, $x \in \mathcal{G}$,
(2) $\sum_{i\in I} \psi_i(x) \equiv 1$,
(3) there exists a well-spread family $(x_i)_{i\in I}$ such that $\operatorname{supp} \psi_i \subset x_i U$.

The construction of BUPU's with respect to arbitrary well-spread sets is standard.

We call $W(B, Y)$ *right translation invariant* if for any relatively compact neighborhood Q of e the space $W(B, Y, Q)$ is right translation invariant and the right translations $R_x : W(B, Y, Q) \to W(B, Y, Q)$ are bounded operators. (In case $B = M$ we replace R_x by A_x in this definition.)

Now we are prepared to state the basic properties of Wiener amalgams.

THEOREM 2.3. *The following statements are equivalent*:

(i) $W(L^\infty, Y) = W(L^\infty, Y, Q)$ *is independent of the choice of the neighborhood* Q *of* e (*with equivalent norms for different choices*).
(ii) *For all relatively separated sets* X *the space* $Y_d = Y_d(X, Q)$ *is independent of the choice of the neighborhood* Q *of* e (*with equivalent norms for different choices*).
(iii) $W(L^\infty, Y) = W(L^\infty, Y, Q)$ *is right translation invariant* (*for all choices of* Q).

If one (*and hence all*) *of these conditions are satisfied then also* $W(B, Y) = W(B, Y, Q)$ *is independent of the choice of* Q. *Moreover*, *the expression*

$$\|F \,|\, W(B, Y_d)\| := \|(\|F\psi_i \,|\, B\|)_{i\in I} \,|\, Y_d(X)\| \tag{2.4}$$

defines an equivalent quasi-norm on $W(B, Y)$, *where* $(\psi_i)_{i\in I}$ *is a BUPU corresponding to the well-spread set* X.

Proof. We first prove that (ii) implies that (2.4) defines an equivalent quasi-norm on $W(B, Y)$. Let Q be a relatively compact neighborhood of $e \in \mathcal{G}$. Then there exists an open set $U = U^{-1}$ with $U^2 \subset Q$. Choose a BUPU $(\phi_i)_{i\in I}$ of size U. If $x_i U \subset zQ$ then for $F \in B_{\mathrm{loc}}$ we have

$$\|F\phi_i \,|\, B\| \le \|F\chi_{x_iU} \,|\, B\| \le \|F\chi_{zQ} \,|\, B\| = K(F, Q, B)(z).$$

This yields

$$\sum_{i\in I} \|F\phi_i \,|\, B\|\chi_{x_iU}(z) = \sum_{i,\, x_i\in zU^{-1}} \|F\phi_i \,|\, B\| \le CK(F, Q, B)(z) \tag{2.5}$$

since $(x_i)_{i\in I}$ is relatively separated. By solidity we obtain

$$\|(\|F\phi_i \,|\, B\|)_{i\in I} \,|\, Y_d(X,U)\| \le C\|F \,|\, W(B,Y,Q)\|.$$

Moreover, we have

$$K(F,Q,B)(z) = \|\chi_{zQ} F \,|\, B\| = \Big\|\chi_{zQ} \sum_{i\in I} F\phi_i \,\Big|\, B\Big\| \le \sum_{i,\, zQ\cap x_i U\neq\emptyset} \|F\phi_i \,|\, B\| \le \sum_{i\in I} \|F\phi_i \,|\, B\| \chi_{x_i U Q^{-1}}(z). \tag{2.6}$$

By solidity this yields

$$\|F \,|\, W(B,Y,Q)\| \le \|(\|F\phi_i \,|\, B\|)_{i\in I} \,|\, Y_d(X,UQ^{-1})\|.$$

Thus, the independence of $Y_d(X,U)$ from U implies that the norm in (2.4) is equivalent to the norm in $W(B,Y)$. Moreover, since Q was arbitrary this also shows that $W(B,Y) = W(B,Y,Q)$ is independent of the choice of Q. Specializing to $B = L^\infty$ we have thus also shown (ii)⇒(i).

As the next step we prove that (iii) implies (ii). Let U,V be relatively compact neighborhoods of e. Choose a neighborhood $Q = Q^{-1}$ of $e\in G$ such that $Q^2 \subset V$. Observe that

$$K\Big(\sum_{i\in I} |\lambda_i|\chi_{x_i Q}, Q\Big)(y) = \sup_{z\in yQ} \sum_{i\in I} |\lambda_i| \chi_{x_i Q}(z) \le \sum_{i\in I} |\lambda_i| \chi_{x_i Q^2}(y) \le \sum_{i\in I} |\lambda_i| \chi_{x_i V}(y).$$

The right translation invariance of $W(L^\infty,Y,Q)$ together with Lemma 2.2 applied to $W(L^\infty,Y)$ and the trivial inequality $|F(x)| \le \sup_{z\in xQ}|F(z)|$ thus imply

$$\Big\|\sum_{i\in I} |\lambda_i| \chi_{x_i U} \,\Big|\, Y\Big\| \le \Big\|K\Big(\sum_{i\in I} |\lambda_i|\chi_{x_i U}, Q, L^\infty\Big) \,\Big|\, Y\Big\| \le \Big\|K\Big(\sum_{i\in I} |\lambda_i|\chi_{x_i Q}, Q, L^\infty\Big) \,\Big|\, Y\Big\| \le \Big\|\sum_{i\in I} |\lambda_i| \chi_{x_i V} \,\Big|\, Y\Big\|. \tag{2.7}$$

Exchanging the roles of U and V shows the reverse inequality.

Finally, we prove (i)⇒(iii). Let $F \in W(L^\infty,Y)$ and $y \in \mathcal{G}$. We can find a compact neighborhood $V^{(y)}$ of e such that $Qy \subset V^{(y)}$. We obtain

$$K(R_y F, Q, L^\infty)(x) = \|(L_x\chi_Q)(R_y F)\|_\infty = \|(R_{y^{-1}} L_x \chi_Q) F\|_\infty = \|(L_x\chi_{Qy})F\|_\infty \le \|(L_x \chi_{V^{(y)}})F\|_\infty.$$

By assumption and solidity, this yields

$$\|R_y F \,|\, W(L^\infty,Y)\| \le C\|K(R_y F,Q,L^\infty) \,|\, Y\| \le C\|K(F,V^{(y)},L^\infty) \,|\, Y\| \le C'(y)\|F \,|\, W(L^\infty,Y)\|.$$

This concludes the proof. ∎

REMARK 2.2.

(a) The proof of the equivalence of the quasi-norm in (2.4) still works (with slight changes) upon replacing the BUPU $(\psi_i)_{i\in I}$ by the characteristic functions χ_{x_iU}. Thus, if $Y_d = Y_d(X,Q)$ is independent of the choice of Q then also the expression

$$\|(\|F\chi_{x_iQ} \mid B\|)_{i\in I} \mid Y_d\|$$

defines an equivalent quasi-norm on $W(B,Y)$.

(b) Analyzing the proof that (ii) implies (i) one recognizes that it is actually enough to require that for all neighborhoods Q of e there exists some relatively separated Q-dense set X such that $Y_d(X,U)$ is independent of the choice of U. The theorem then shows that $Y_d(X,U)$ is automatically independent of U for all relatively separated sets X.

COROLLARY 2.4. *If $W(L^\infty,Y)$ is right translation invariant then $(W(L^\infty,Y))_d = Y_d$.*

Proof. This follows immediately from inequality (2.7). ∎

Let us now investigate the completeness of the spaces $W(B,Y)$ and Y_d.

LEMMA 2.5. *Y_d is complete, and convergence in Y_d implies coordinatewise convergence.*

Proof. Let $\Lambda^n = (\lambda_i^{(n)})_{i\in I}$, $n\in\mathbb{N}$, be a Cauchy sequence in Y_d. This means that the functions $F_n = \sum_{i\in I}\lambda_i^{(n)}\chi_{x_iU}$ form a Cauchy sequence in Y. Since Y is complete the limit $F = \lim_{n\in\mathbb{N}} F_n$ exists. It follows from the solidity that F has the form $F = \sum_{i\in I}\lambda_i\chi_{x_iU}$ with $\lambda_i = \lim_{n\to\infty}\lambda_i^{(n)}$. Clearly, $(\lambda_i)_{i\in I}\in Y_d$ is the limit of Λ^n. ∎

THEOREM 2.6. *If $W(L^\infty,Y)$ is right translation invariant then $W(B,Y)$ is complete.*

Proof. Let $(\psi_i)_{i\in I}$ be a BUPU of size U. By Theorem 2.3, $\|\cdot \mid W(B,Y_d)\|$ defined in (2.4) is an equivalent quasi-norm on $W(B,Y)$. Assume that F_n, $n\in\mathbb{N}$, is a Cauchy sequence in $W(B,Y)$. This implies that $(\|F_n\psi_i \mid B\|)_{i\in I}$ is a Cauchy sequence in Y_d and by Lemma 2.5 the sequence $(F_n\psi_i)_{n\in\mathbb{N}}$ is a Cauchy sequence in B for each $i\in I$. Since B is complete the limit $\lim_{n\to\infty}F_n\psi_i = F^{(i)}$ exists for each $i\in I$. Set $F := \sum_{i\in I}F^{(i)}$. Clearly, $\operatorname{supp}F^{(i)}\subset x_iU$. Furthermore,

$$\begin{aligned}\|F\psi_i \mid B\| &= \Big\|\sum_{j\in I}F^{(j)}\psi_i\,\Big|\,B\Big\| = \Big\|\sum_{j\,:\,x_iU\cap x_iU}F^{(j)}\psi_i\,\Big|\,B\Big\|\\ &\le \sum_{j\,:\,x_jU\cap x_iU}\|\lim_{n\to\infty}F_n\psi_j\psi_i \mid B\| \le C\|F^{(i)} \mid B\|.\end{aligned}$$

By completeness of Y_d, the sequence $(\|F^{(i)}|B\|)_{i\in I}$ is contained in Y_d, and hence $F\in W(B,Y)$. Furthermore, we have

$$F=\sum_{i\in I}F^{(i)}=\sum_{i\in I}\lim_{n\to\infty}F_n\psi_i=\lim_{n\to\infty}F_n\sum_{i\in I}\psi_i=\lim_{n\to\infty}F_n.$$

Thus, F is the limit of F_n in $W(B,Y)$, and hence $W(B,Y)$ is complete. ■

3. Left translation invariance. Also the left translation invariance is an important property. In this section we assume that $W(L^\infty,Y)$ is right translation invariant, so that $W(B,Y)$ is complete and independent of the choice of the neighborhood Q according to Theorems 2.6 and 2.3.

LEMMA 3.1. *If $W(L^\infty,Y)$ is left translation invariant then Y_d is continuously embedded into $\ell^\infty_{1/r}$ with $r(i):=\|\!|L_{x_i^{-1}}\,|\,W(L^\infty,Y)|\!\|$.*

Proof. Let U be some compact neighborhood of e and $(\lambda_i)_{i\in I}\in Y_d$. With $C:=\|\chi_U\,|\,W(L^\infty,Y)\|$ we obtain by Corollary 2.4 and solidity

$$\begin{aligned}C|\lambda_i|&=|\lambda_i|\,\|\chi_U\,|\,W(L^\infty,Y)\|=|\lambda_i|\,\|L_{x_i^{-1}}\chi_{x_iU}\,|\,W(L^\infty,Y)\|\\&\le\|\!|L_{x_i^{-1}}\,|\,W(L^\infty,Y)|\!\|\,\|\,|\lambda_i|\chi_{x_iU}\,|\,W(L^\infty,Y)\|\\&\le r(i)\Big\|\sum_{j\in I}|\lambda_j|\chi_{x_jU}\,\Big|\,W(L^\infty,Y)\Big\|\le r(i)\|(\lambda_i)_{i\in I}\,|\,Y_d\|.\end{aligned}$$

This completes the proof. ■

LEMMA 3.2. *If $W(L^\infty,Y)$ is left translation invariant then $W(L^\infty,Y)$ is continuously embedded into $L^\infty_{1/r}$, where $r(x):=\|\!|L_{x^{-1}}\,|\,W(L^\infty,Y)|\!\|$.*

Proof. By Theorem 2.3, $Y_d=Y_d(X,Q)$ is independent of the choice of Q and the quasi-norm $\|\cdot\,|\,W(L^\infty,Y_d)\|$ defined in (2.4) is equivalent to the quasi-norm of $W(L^\infty,Y)$. Since Y_d is continuously embedded into $\ell^\infty_{1/r}$ by Lemma 3.1 and $(L^\infty_{1/r})_d=\ell^\infty_{1/r}$ we obtain

$$\begin{aligned}C_1\|F\,|\,W(L^\infty,L^\infty_{1/r})\|&\le\|F\,|\,W(L^\infty,\ell^\infty_r)\|\le\|F\,|\,W(L^\infty,Y_d)\|\\&\le C_2\|F\,|\,W(L^\infty,Y)\|\end{aligned}\tag{3.1}$$

for all $F\in W(L^\infty,Y)$. Further, it is easy to see that $W(L^\infty,L^\infty_{1/r})=L^\infty_{1/r}$. ■

In some cases one has translation invariant spaces Y. Then we have the following estimates of the norm of the left translation operators in $W(L^\infty,Y)$.

LEMMA 3.3. *If Y is left translation invariant then $W(B,Y)$ is left translation invariant and $\|\!|L_y\,|\,W(B,Y)|\!\|\le\|\!|L_y\,|\,Y|\!\|$.*

Proof. We have

$$K(L_yF,Q,B)(x) = \|(L_x\chi_Q)(L_yF)\,|\,B\| = \|(L_{y^{-1}x}\chi_Q)F\,|\,B\| \\ = (L_yK(F,Q,B))(x).$$

This yields

$$\|L_yF\,|\,W(B,Y)\| = \|L_yK(F,Q,B)\,|\,Y\| \le |||L_y\,|\,Y|||\,\|F\,|\,W(B,Y)\|,$$

and the proof is complete. ∎

4. Conditions ensuring translation invariance. Given a concrete space Y, according to the previous results, there is a need to check whether $W(L^\infty,Y)$ is right translation invariant. Moreover, we will see later that also the right translation invariance of $W(M,Y)$ is important in order to have convolution relations.

LEMMA 4.1. *If $W(L^\infty,Y)$ is right translation invariant then $W(M,Y)$ is also right translation invariant.*

Proof. Let $\mu\in W(M,Y)$, $y\in\mathcal{G}$ and Q be a compact neighborhood of e. Then there exist a finite number of points y_k, $k=1,\dots,n$, such that $Qy^{-1}\subset\bigcup_{k=1}^n y_kQ$. For the control function we obtain

$$K(A_y\mu,Q,M)(x) = \|(L_x\chi_Q)A_y\mu\,|\,M\| = |\mu|(R_yL_x\chi_Q) = |\mu|(L_x\chi_{Qy^{-1}}) \\ \le \sum_{k=1}^n |\mu|(L_x\chi_{y_kQ}) = \sum_{k=1}^n R_{y_k}K(\mu,Q,M)(x).$$

By solidity, the p-triangle inequality and independence of $W(M,Y,Q)$ from the choice of Q we get

$$\begin{aligned}\|A_y\mu\,|\,W(M,Y)\|^p &\le \Big\|\sum_{k=1}^n R_{y_k}K(\mu,Q,M)\,\Big|\,Y\Big\|^p\\ &\le \sum_{k=1}^n \|R_{y_k}K(\mu,Q,M)\,|\,W(L^\infty,Y)\|^p\\ &\le \sum_{k=1}^n |||R_{y_k}\,|\,W(L^\infty,Y)|||^p\|K(\mu,Q,M)\,|\,W(L^\infty,Y)\|^p\\ &\le \sum_{k=1}^n |||R_{y_k}\,|\,W(L^\infty,Y)|||^p\|K(\mu,Q^2,M)\,|\,Y\|^p\\ &\le C\sum_{k=1}^n |||R_{y_k}\,|\,W(L^\infty,Y)|||^p\|\mu\,|\,W(M,Y)\|^p.\end{aligned}$$

This concludes the proof. ∎

Another criterion for the right translation invariance of $W(B,Y)$ is:

COROLLARY 4.2. *If Y is right translation invariant then also $W(B,Y) = W(B,Y,Q)$ is right translation invariant and independent of Q.*

Proof. By Lemma 2.2, $Y_d = Y_d(X,U)$ is independent of U. Thus, Theorem 2.3 implies that $W(B,Y) = W(B,Y,Q)$ is independent of Q and $W(L^\infty, Y)$ is right translation invariant. Lemma 4.1 implies that $W(M,Y)$ is also right translation invariant. Clearly, $W(L^1,Y)$ is a subspace of $W(M,Y)$ that is right translation invariant if $W(M,Y)$ is right translation invariant. Thus, we proved the assertion for all admissible choices $B = L^\infty, L^1, M$. ∎

Recall that $\mathcal{G}$ is called an *IN group* if there exists a compact neighborhood of e such that $xQ = Qx$ for all $x \in \mathcal{G}$.

LEMMA 4.3. *Let $\mathcal{G}$ be an IN group and assume Y to be right translation invariant. Then $|||R_y \,|\, W(L^\infty, Y)||| \le |||R_y \,|\, Y|||$ and $|||A_y \,|\, W(M,Y)||| \le |||R_y \,|\, Y|||$.*

Proof. Choose Q to be a compact invariant neighborhood of e, i.e., $yQ = Qy$ for all $y \in \mathcal{G}$. This yields

$$\begin{aligned} K(R_yF, Q, L^\infty)(x) &= \|(L_x\chi_Q)R_yF\|_\infty = \|(L_x\chi_{Qy})F\|_\infty = \|(L_x\chi_{yQ})F\|_\infty \\ &= \|(L_{xy}Q)F\|_\infty = K(F,Q,L^\infty)(xy) \end{aligned}$$

and thus,

$$\|R_yF \,|\, W(L^\infty,Y)\| = \|R_yK(F,Q,L^\infty) \,|\, Y\| \le |||R_y \,|\, Y|||\, \|F \,|\, W(L^\infty,Y)\|.$$

The proof for $B = M$ is similar. ∎

We remark that Y does not necessarily need to be translation invariant for $W(L^\infty, Y)$ to be translation invariant (see Section 6). The following criterions allow us to check left or right translation invariance of $W(L^\infty, Y)$ without using translation invariance of Y.

LEMMA 4.4. *Let U be some compact neighborhood of $e \in \mathcal{G}$. Let $X = (x_i)_{i\in I}$ be some well-spread set in $\mathcal{G}$. Denote by $x^{-1}X$, $x \in \mathcal{G}$, the well-spread set $(x^{-1}x_i)_{i\in I}$. If there is a function $k(x)$ such that*

$$\|(\lambda_i)_{i\in I} \,|\, Y_d(x^{-1}X,U)\| \le k(x)\|(\lambda_i)_{i\in I} \,|\, Y_d(X,U)\|$$

for all $(\lambda_i)_{i\in I} \in Y_d(X)$ then $W(B,Y)$ is left translation invariant with

$$|||L_x \,|\, W(B,Y)||| \le Ck(x).$$

Proof. Let $(\psi)_{i\in I}$ be some BUPU corresponding to X. Since (2.4) defines an equivalent norm on $W(B,Y)$ we obtain

$$\begin{aligned} \|L_xF \,|\, W(B,Y)\| &\le C\|(\|(L_xF)\psi_i \,|\, B\|_{i\in I} \,|\, Y_d(X,U)\| \\ &\le C\|(\|F(L_{x^{-1}}\psi_i) \,|\, B\|)_{i\in I} \,|\, Y_d(X,U)\| \\ &\le Ck(x)\|(\|F(L_{x^{-1}}\psi_i)\|)_{i\in I} \,|\, Y_d(x^{-1}X,U)\|. \end{aligned}$$

The system $(L_{x^{-1}}\psi_i)_{i\in I}$ is a BUPU corresponding to the well-spread set $x^{-1}X$. Thus, using once more the equivalence of the norm (2.4) with the norm in $W(B,Y)$ we obtain $\|L_xF\,|\,W(B,Y)\| \le C'k(x)\|F\,|\,W(B,Y)\|$. ∎

REMARK 4.1. If $Y_d(X,U)$ is independent of the choice of the neighborhood U then we already know from Theorem 2.3 that $W(L^\infty,Y)$ is right translation invariant. If $h(x)$ is a function such that

$$\|(\lambda_i)_{i\in I}\,|\,Y_d(X,Ux)\| \le h(x)\|(\lambda_i)_{i\in I}\,|\,Y_d(X,U)\|$$

for all $(\lambda_i)_{i\in I} \in Y_d(X)$ then a similar argument to the previous proof shows that

$$|||R_x\,|\,W(L^\infty,Y)||| \le Ch(x).$$

5. Convolution relations. Let us now prove the main results of this article concerning convolution relations of Wiener amalgams with quasi-Banach spaces as global components (cf. [7, 8] for the classical case of Banach spaces).

THEOREM 5.1. *Let $0 < p \le 1$ be such that the quasi-norm of Y satisfies the p-triangle inequality and assume that $W(L^\infty,Y)$ is right translation invariant.*

(a) *Set $w(x) := |||A_x\,|\,W(M,Y)|||$. Then*

$$W(M,Y) * W(L^\infty,L^p_w) \hookrightarrow W(L^\infty,Y)$$

with a corresponding estimate for the quasi-norms.

(b) *Set $v(x) := \Delta(x^{-1})|||R_{x^{-1}}\,|\,W(L^\infty,Y)|||$. Then*

$$W(L^\infty,Y) * W(L^\infty,L^p_v) \hookrightarrow W(L^\infty,Y)$$

with a corresponding estimate for the quasi-norms.

Proof. (a) It follows from Theorem 2.3 that any $G \in W(L^\infty,L^p_w)$ has a decomposition $G = \sum_{i\in I} L_{x_i}G_i$ with $G_i \in L^\infty$, $\operatorname{supp} G_i \subset Q = Q^{-1}$ for some compact Q and $\sum_{i\in I}\|G_i\|_\infty^p w(x_i)^p \le C\|G\,|\,W(L^\infty,L^p_w)\|^p < \infty$.

For $\mu \in W(M,Y)$ we estimate the control function of $\mu * (L_{x_i}G_i)$ by

$$\begin{aligned}
K(\mu * (L_{x_i}G_i), Q, L^\infty)(x) &= \sup_{z\in xQ} |\mu * (L_{x_i}G_i)(z)| \\
&= \sup_{z\in xQ}\Big|\int (L_yL_{x_i}G_i)(z)\,d\mu(y)\Big| \le \|G_i\|_\infty \sup_{q\in Q}\int L_{yx_i}\chi_Q(xq)\,d|\mu|(y) \\
&\le \|G_i\|_\infty \int \chi_{Q^2}((yx_i)^{-1}x)\,d|\mu|(y) = \|G_i\|_\infty\int \chi_{Q^2}(x^{-1}yx_i)\,d|\mu|(y) \\
&= \|G_i\|_\infty \int R_{x_i}L_x\chi_{Q^2}(y)\,d|\mu|(y) = \|G_i\|_\infty \|(L_x\chi_{Q^2})(A_{x_i}\mu)\,|\,M\| \\
&= \|G_i\|_\infty K(A_{x_i}\mu, Q^2, M)(x).
\end{aligned}$$

Thus, we have

$$\|\mu * L_{x_i}G_i \,|\, W(L^\infty, Y)\| \le \|G_i\|_\infty \|K(A_{x_i}\mu, Q^2, M) \,|\, Y\| \\ \le C\|G_i\|_\infty \|A_{x_i}\mu \,|\, W(M,Y)\|.$$

Pasting the pieces together yields

$$\begin{aligned} \|\mu * G \,|\, W(L^\infty, Y)\|^p &= \Big\| \sum_{i\in I} \mu * L_{x_i}G_i \,\Big|\, W(L^\infty, Y)\Big\|^p \\ &\le \sum_{i\in I} \|\mu * L_{x_i}G_i \,|\, W(L^\infty, Y)\|^p \le C\sum_{i\in I} \|G_i\|_\infty^p \|A_{x_i}\mu \,|\, W(M,Y)\|^p \\ &\le C\sum_{i\in I} \|G_i\|_\infty^p |\!|\!| A_{x_i} \,|\, W(M,Y)|\!|\!|^p \|\mu \,|\, W(M,Y)\|^p \\ &\le C\|\mu \,|\, W(M,Y)\|^p \|G \,|\, W(L^\infty, L^p_w)\|^p. \end{aligned} \tag{5.1}$$

(b) Since $W(L^\infty, Y) \subset W(M,Y)$ all the computations done in (a) are still valid. We only have to replace $\|A_{x_i}\mu \,|\, W(M,Y)\|$ by $\|A_{x_i}\mu \,|\, W(L^\infty, Y)\| = \Delta(x_i^{-1})\|R_{x^{-1}}\mu \,|\, W(L^\infty,Y)\|$ in (5.1) to deduce (b). ∎

THEOREM 5.2. *Assume Y is such that $W(L^\infty, Y)$ is left and right translation invariant. Set $v(x) := |\!|\!| L_{x^{-1}} \,|\, W(L^\infty, Y)|\!|\!|$. Then*

$$W(L^\infty, L^p_v) * W(L^\infty, Y^\vee)^\vee \hookrightarrow W(L^\infty, Y).$$

Proof. Let $F \in W(L^\infty, L^p_v)$ and $G \in W(L^\infty, Y)$. Similarly to the proof of Theorem 5.1 we may write $F = \sum_{i\in I} L_{x_i}F_i$ with $\operatorname{supp} F_i \subset Q = Q^{-1}$ (compact) and $\sum_{i\in I} \|F_i\|_\infty^p v(x_i)^p \le C\|F \,|\, W(L^\infty, L^p_v)\|$. We obtain

$$\begin{aligned} &K(F_i * G, Q, L^\infty)(x) \\ &\quad = \sup_{z\in xQ} |F_i * G(z)| \le \sup_{z\in xQ} \Big| \int_{x_iQ} F_i(y) L_y G(z)\, dy\Big| \\ &\quad \le \|F_i\|_\infty \sup_{q\in Q} \int \chi_Q(y)|(R_qG)(y^{-1}x)|\, dy \le C\|F_i\|_\infty \int \chi_{Q^2}(y)|G^\vee(x^{-1}y)|\, dy \\ &\quad \le C\|F_i\|_\infty \int L_{x^{-1}}\chi_{Q^2}(y)|G^\vee(y)|\, dy \le C'\|F_i\|_\infty K(G^\vee, Q^2, L^\infty)(x^{-1}). \end{aligned}$$

This yields

$$\|F_i * G \,|\, W(L^\infty, Y)\| \le C\|F_i\|_\infty \|K(G^\vee, Q^2, L^\infty)^\vee \,|\, Y\| \\ \le C\|F_i\|_\infty \|G \,|\, W(L^\infty, Y^\vee)^\vee\|.$$

Pasting the pieces together we get

$$\begin{aligned}\|F*G\,|\,W(L^\infty,Y)\|^p &= \Big\|\sum_{i\in I}(L_{x_i}F_i)*G\,\Big|\,W(L^\infty,Y)\Big\|^p\\ &\le \sum_{i\in I}\|L_{x_i}(F_i*G)\,|\,W(L^\infty,Y)\|^p\\ &\le C\sum_{i\in I}|\!|\!|L_{x_i}\,|\,W(L^\infty,Y)|\!|\!|^p\|F_i\|_\infty^p\|G\,|\,W(L^\infty,Y^\vee)^\vee\|^p\\ &\le C'\|F\,|\,W(L^\infty,L^p_v)\|^p\|G\,|\,W(L^\infty,Y^\vee)^\vee\|^p.\end{aligned}$$

This concludes the proof. ∎

From the previous theorem we see that the involution $^\vee$ has some relevance. In the case of IN groups we have the following result.

LEMMA 5.3. *If $\mathcal{G}$ is an IN group then $W(L^\infty,Y^\vee)^\vee = W(L^\infty,Y)$ with equivalent norms.*

Proof. Let Q be an invariant compact neighborhood of e. Then also Q^{-1} is invariant. For the control function we obtain

$$\begin{aligned}K(F^\vee,Q,L^\infty)(x) &= \|(L_x\chi_Q)F^\vee\|_\infty = \|(L_x\chi_Q)^\vee F\|_\infty = \|(R_x\chi_{Q^{-1}})F\|_\infty\\ &= \|\chi_{Q^{-1}x^{-1}}F\|_\infty = \|\chi_{x^{-1}Q^{-1}}F\|_\infty = K(F,Q^{-1},L^\infty)(x^{-1}).\end{aligned}$$

This shows the claim. ∎

Theorem 5.2 implies a convolution relation for Wiener amalgam spaces with respect to weighted L^p-spaces.

COROLLARY 5.4. *Let w be a submultiplicative weight and $0 < p \le 1$. Then*

$$W(L^\infty,L^p_w)*W(L^\infty,L^p_{w^*})^\vee \hookrightarrow W(L^\infty,L^p_w).$$

In particular, if $\mathcal{G}$ is an IN group then $W(L^\infty,L^p_w)*W(L^\infty,L^p_w) \hookrightarrow W(L^\infty,L^p_w)$ with a corresponding quasi-norm estimate.

Proof. The first assertion is a direct consequence of Theorem 5.2, and the second then follows from Lemma 5.3. ∎

In particular, if $\mathcal{G}$ is an IN group then $W(L^\infty,L^p_w)$, $0\le p\le 1$, is a quasi-Banach algebra under convolution. Since commutative groups are clearly IN groups this result applies in particular to Wiener amalgams on $\mathcal{G}=\mathbb{R}^d$. Moreover, if $\mathcal{G}$ is discrete then we recover the well-known relation $\ell^p_w(\mathcal{G})*\ell^p_w(\mathcal{G})\hookrightarrow \ell^p_w(\mathcal{G})$, $0<p\le 1$.

6. An example on the $ax+b$ group. In this section we provide an example of a non-translation invariant space Y such that $W(L^\infty,Y)$ is right translation invariant. We consider the n-dimensional $ax+b$ group $\mathcal{G}=\mathbb{R}^n\rtimes\mathbb{R}^*_+$ where $\mathbb{R}^*_+$ denotes the multiplicative group of positive real numbers. The group law in $\mathcal{G}$ reads $(x,a)\cdot(y,b) = (x+ay, ab)$. The $ax+b$ group has left Haar measure

$$\int_{\mathcal{G}} f(x)\,dx = \int_{\mathbb{R}^n}\int_0^\infty f(x,a)\,\frac{da}{a^{n+1}}\,dx$$

and modular function $\Delta(x,a) = a^{-n}$. The $ax+b$ group plays an important role in wavelet analysis and the theory of Besov and Triebel–Lizorkin spaces.

Let $0 < p, q \le \infty$. With some positive measurable weight function v on $\mathcal{G}$ we define the mixed norm space $L^{p,q}(v)$ on $\mathcal{G}$ as the collection of measurable functions whose quasi-norm

$$\|F \mid L^{p,q}(v)\| := \left(\int_0^\infty \left(\int_{\mathbb{R}^n} |F(x,a)|^p v(x,a)\,dx\right)^{q/p} \frac{da}{a^{n+1}}\right)^{1/q}$$

is finite (with obvious modification in the cases $p=\infty$ or $q=\infty$). This quasi-norm is actually an r-norm where $r := \min\{1,p,q\}$. If $v \equiv 1$ we write $L^{p,q}$. If $p = q$ then clearly $L^{p,p} = L^p(\mathcal{G})$. It is easy to see by an integral transformation that $L^{p,q}$ is invariant under left and right translations. We remark that for reasons to become clear later v is treated as a measure here, so if v does not vanish on a set of positive measure then $L^{\infty,\infty}(v) = L^\infty(\mathcal{G})$.

With a similar argument to [12, Proposition 2.4] (see also [3]), one shows (using the right translation invariance of the unweighted $L^{p,q}$ space) that $L^{p,q}(v)$, $0 < p, q < \infty$, is right translation invariant if and only if

$$v((x,a)\cdot(y,b)) \le v(x,a)w(y,b) \tag{6.1}$$

for some submultiplicative function w (possibly depending on p,q). Now assume that $v(x,a)$ is a function of x only. Then condition (6.1) means that the quotient

$$\frac{v((x,a)(y,b))}{v(x,a)} = \frac{v(x+ay)}{v(x)} \tag{6.2}$$

is bounded by a submultiplicative function w of y only. However, since the right hand side also depends on $a \in (0,\infty)$ this can be satisfied only in special cases (e.g. if v is bounded from above and below). In particular, the typical choice $v_s(x,a) = v_s(x) = (1+|x|)^s$, $s \in \mathbb{R}$, does not satisfy (6.1) for any submultiplicative weight w on $\mathcal{G}$ if $s \neq 0$ (although it is even submultiplicative as a function on $\mathbb{R}^n$ if $s \ge 0$). In particular, $L^{p,q}(v)$ is not right translation invariant for many non-trivial choices of v.

In the following we introduce a class of weight functions v for which $W(L^\infty, L^{p,q}(v))$ is right translation invariant. This class, however, contains weights v that do not satisfy (6.1), i.e., $L^{p,q}(v)$ is not right translation invariant, in general.

Let $B(x,r)$ denote the ball in $\mathbb{R}^n$ of radius r centered at $x \in \mathbb{R}^n$. A positive measurable weight function v on $\mathbb{R}^n$ is said to satisfy the *doubling condition* if there exists a constant C such that

$$\int_{B(x,2r)} v(y)\,dy \leq C \int_{B(x,r)} v(y)\,dy \tag{6.3}$$

for all $x \in \mathbb{R}^n$ and $r \in (0,\infty)$. This condition is equivalent to the existence of constants c, α such that

$$\int_{B(x,tr)} v(y)\,dy \leq ct^\alpha \int_{B(x,r)} v(y)\,dy \quad \text{for all } x \in \mathbb{R}^n,\ r \in (0,\infty),\ t \geq 1. \tag{6.4}$$

For instance the weights in the Muckenhoupt classes A_p, $p > 1$, satisfy the doubling condition [2]. A typical example of a weight in $A_\infty = \bigcup_{p>1} A_p$ is $v^{(s)}(x) = |x|^s$, $s > -n$. So doubling weights may have zeros or poles. A further example of a doubling weight is $v_s(x) = (1+|x|)^s$, $s \in \mathbb{R}$. For a construction of a doubling weight which is not contained in A_∞ we refer to [2].

We extend a doubling weight v on $\mathbb{R}^n$ to $\mathcal{G} = \mathbb{R}^n \rtimes \mathbb{R}_+^*$ by setting $v(x,t) = v(x)$ for $(x,t) \in \mathcal{G}$. Let $L^{p,q}(v)$ be the associated mixed norm space as defined above. We will use Theorem 2.3 to prove that $W(L^\infty, L^{p,q}(v))$ is right translation invariant. In particular, let us study the associated sequence space $(L^{p,q}(v))_d$.

Lemma 6.1. *Let $0<p<\infty$, $0<q\leq\infty$ and v be a weight function on $\mathbb{R}^n$. Let $X = (x_{k,j}, a_j)_{(k,j)\in I:=\mathbb{Z}^n\times\mathbb{Z}}$ be some well-spread set in $\mathcal{G} = \mathbb{R}^n \rtimes \mathbb{R}_+^*$. If v satisfies the doubling condition* (6.3) *then $(L^{p,q}(v))_d = (L^{p,q}(v))_d(X,U)$ is independent of the choice of the neighborhood U of e in $\mathcal{G}$, and an equivalent norm on $(L^{p,q}(v))_d(X)$ is given by*

$$\|(\lambda_i)_{i\in I} \mid \ell^{p,q}(\widetilde{v})\| = \Big(\sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}^n} |\lambda_{k,j}|^p \widetilde{v}_{k,j}\Big)^{q/p} a_j^{-n}\Big)^{1/q}$$

where $\widetilde{v}_{k,j} = \int_{B(x_{k,j},a_j)} v(y)\,dy$ (with the usual modification for $q=\infty$).

Moreover, $W(L^\infty, L^{p,q}(v))$ is right translation invariant if and only if v satisfies the doubling condition.

Proof. It suffices to show the assertion for neighborhoods of the form $U(r,\beta) = B(0,r) \times (\beta^{-1},\beta) \subset \mathcal{G}$ with $r \in (0,\infty)$ and $\beta \in (1,\infty)$ since for an arbitrary compact neighborhood U of $e = (0,1) \in \mathcal{G}$ we can find $r_1, r_2, \beta_1, \beta_2$ such that $U(r_1,\beta_1) \subset U \subset U(r_2,\beta_2)$. Observe that

$$(x,a)U(r,\beta) = B(x,ar) \times (a\beta^{-1}, a\beta).$$

Using the relative separatedness of X we obtain, for $0<q<\infty$,

$$\|(\lambda_i)_{i\in I}\,|\,(L^{p,q}(v))_d(X,U(r,\beta))\|$$
$$= \Big(\int_0^\infty\Big(\int_{\mathbb{R}^n}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^n}|\lambda_{k,j}|^p\chi_{B(x_{k,j},a_jr)}(y)\chi_{(a_j\beta^{-1},a_j\beta)}(a)v(y)\,dy\Big)^{q/p}\frac{da}{a^{n+1}}\Big)^{1/q}$$
$$\asymp \Big(\sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}^n}|\lambda_{k,j}|^p\int_{B(x_{k,j},a_jr)}v(y)\,dy\Big)^{q/p}\int_{a_j\beta^{-1}}^{a_j\beta}\frac{da}{a^{n+1}}\Big)^{1/q}$$
$$\asymp \Big(\sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}^n}|\lambda_{k,j}|^p\int_{B(x_{k,j},a_jr)}v(y)\,dy\Big)^{q/p}a_j^{-n}\Big)^{1/q}.$$

The computation for $q=\infty$ is similar. Thus, $(L^{p,q}(v))_d(X,U(r,\beta))$ is independent of r and β if and only if for all $r,s\in(0,\infty)$ there exist constants $C_1(r,s),C_2(r,s)>0$ such that

$$C_1(r,s)\int_{B(x_{k,j},a_jr)}v(y)\,dy \le \int_{B(x_{k,j},a_js)}v(y)\,dy \le C_2(r,s)\int_{B(x_{k,j},a_jr)}v(y)\,dy \tag{6.5}$$

for all $(k,j)\in\mathbb{Z}^n\times\mathbb{Z}$. Let us assume without loss of generality that $r\le s$. Then the first inequality is clear. Moreover, by the doubling condition, in its equivalent form (6.4), we have

$$\int_{B(x_{k,j},a_js)}v(y)\,dy \le c(s/r)^\alpha\int_{B(x_{k,j},a_jr)}v(y)\,dy.$$

So (6.5) is satisfied with $C_1(r,s)=1$ and $C_2(r,s)=c(s/r)^\alpha$.

Since we may choose relatively separated sets of the form $(x_{j,k},a_j)$ of arbitrarily small density (e.g. $(ab^{-j}k,b^{-j})_{k\in\mathbb{Z}^n,\,j\in\mathbb{Z}}$ with small $a>0$, $b>1$), $W(L^\infty,L^{p,q}(v))$ is right translation invariant by Theorem 2.3 and Remark 2.2(b) if v is doubling. Conversely, if $W(L^\infty,L^{p,q}(v))$ is right translation invariant then (6.5) must hold for any choice of the relatively separated set $X=(x_{j,k},a_j)$ by Theorem 2.3. In particular, choosing $s=2$, $r=1$ in (6.5) we obtain

$$\int_{B(x,2a)}v(y)\,dy \le C_2\int_{B(x,a)}v(y)\,dy$$

for all $x\in\mathbb{R}^n$, $a\in(0,\infty)$, which clearly is the doubling condition. ∎

Since $L^{\infty,q}(v)=L^{\infty,q}$ the analogue of Lemma 6.1 for $p=\infty$ is trivial. It seems that in general $W(L^\infty,L^{p,q}(v))$ is not left invariant.

In order to state the convolution relation in Theorem 5.1 for our case we estimate the norm of the right translation operators on $W(L^\infty,L^{p,q}(v))$

using Remark 4.1. Let $U = U(r,\beta)$, $r > 0$, $\beta > 1$, be a neighborhood of $e = (0,1)$ as in the previous proof. For $(x,a),(y,b) \in \mathcal{G}$ we obtain

$$\begin{aligned}(x,a)\cdot U(r,\beta)\cdot(y,b) &= (B(x,ar)\times a(\beta^{-1},\beta))\cdot(y,b)\\ &= \{(z+sy,sb) : z\in B(x,ar), s\in ab(\beta^{-1},\beta)\}\\ &\subset \bigcup_{s\in a(\beta^{-1},\beta)} B(x+sy,ar)\times ab(\beta^{-1},\beta) \subset B(x,a(\beta|y|+r))\times ab(\beta^{-1},\beta).\end{aligned}$$

Let $X = (x_{k,j}, a_j)$ be a relatively separated set in $\mathcal{G}$. Proceeding as in the previous proof we deduce

$$\begin{aligned}&\|(\lambda_i)_{i\in I} \mid (L^{p,q}(v))_d(X, U(r,\beta)\cdot(y,b))\|\\ &\quad\le C\bigg(\sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}^n}|\lambda_{k,j}|^p \int\limits_{B(x_{k,j},a_j r(\frac{\beta}{r}|y|+1))} v(y)\,dy\Big)^{q/p}\int\limits_{a_jb\beta^{-1}}^{a_jb\beta}\frac{da}{a^{n+1}}\bigg)^{1/q}\\ &\quad\le C\bigg(\sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}^n}|\lambda_{k,j}|^p\Big(\frac{\beta}{r}|y|+1\Big)^{\alpha}\int\limits_{B(x_{k,j},a_j r)} v(y)\,dy\Big)^{q/p} b^{-n}a_j^{-n}\bigg)^{1/q}\\ &\quad\le C(1+|y|)^{\alpha/p} b^{-n/q}\|(\lambda_i)_{i\in I} \mid (L^{p,q}(v))_d(X,U(r,\beta))\|,\end{aligned}$$

where α is the exponent from (6.4). By Remark 4.1 we conclude that

$$|||R_{(y,b)} \mid W(L^\infty, L^{p,q}(v))||| \le C(1+|y|)^{\alpha/p} b^{-n/q},$$

and since $(y,b)^{-1} = (-b^{-1}y, b^{-1})$ we have

$$\Delta((y,b)^{-1})|||R_{(y,b)^{-1}} \mid W(L^\infty, L^{p,q}(v))||| \le Cb^{n(1+1/q)}(1+b^{-1}|y|)^{\alpha/p}.$$

Set $w(y,b) := b^{n(1+1/q)}(1+b^{-1}|y|)^{\alpha/p}$ and $r := \min\{1,p,q\}$. Then Theorem 5.1 tells us that

$$W(L^\infty, L^{p,q}(v)) * W(L^\infty, L^r_w) \hookrightarrow W(L^\infty, L^{p,q}(v)).$$

To the author's knowledge this is a new convolution relation on the $ax+b$-group even for $p,q \ge 1$.

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