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THE AR-PROPERTY OF THE SPACES OF CLOSED CONVEX SETS

BY

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Abstract. Let $\mathrm{Conv}_{\mathrm{H}}(X)$, $\mathrm{Conv}_{\mathrm{AW}}(X)$ and $\mathrm{Conv}_{\mathrm{W}}(X)$ be the spaces of all non-empty closed convex sets in a normed linear space X admitting the Hausdorff metric topology, the Attouch–Wets topology and the Wijsman topology, respectively. We show that every component of $\mathrm{Conv}_{\mathrm{H}}(X)$ and the space $\mathrm{Conv}_{\mathrm{AW}}(X)$ are AR. In case X is separable, $\mathrm{Conv}_{\mathrm{W}}(X)$ is locally path-connected.

1. Introduction. Throughout the paper, $X = (X, \|\cdot\|)$ *is a normed linear space*. There are various topologies on the set $\mathrm{Cld}(X)$ of all non-empty closed sets in X (cf. [2]). Let $C(X)$ be the set of all continuous real-valued functionals of X. Each $A \in \mathrm{Cld}(X)$ can be identified with the continuous functional $X \ni x \mapsto d(x, A) = \inf_{a\in A} \|x - a\|$. Thus, we can regard $\mathrm{Cld}(X) \subset C(X)$. The *Hausdorff metric topology* τ_{H}, the *Attouch–Wets topology* τ_{AW} and the *Wijsman topology* τ_{W} are respectively defined by restricting the topologies on $C(X)$ of uniform convergence, of uniform convergence on bounded sets and of pointwise convergence ([1]). Obviously, $\tau_{\mathrm{H}} \supset \tau_{\mathrm{AW}} \supset \tau_{\mathrm{W}}$. The spaces $\mathrm{Cld}(X)$ with these topologies are denoted by $\mathrm{Cld}_{\mathrm{H}}(X)$, $\mathrm{Cld}_{\mathrm{AW}}(X)$ and $\mathrm{Cld}_{\mathrm{W}}(X)$, respectively. The first two spaces are always metrizable, but the last is metrizable if and only if X is separable ([2, Theorem 2.1.5]). In [4], [1] and [3], we have studied when these spaces (or their components) are AR's.

Given $\mathcal{S}(X) \subset \mathrm{Cld}(X)$, the set $\mathcal{S}(X)$ with the topologies τ_{H}, τ_{AW} and τ_{W} is denoted by $\mathcal{S}_{\mathrm{H}}(X)$, $\mathcal{S}_{\mathrm{AW}}(X)$ and $\mathcal{S}_{W}(X)$, respectively. In this paper, we consider the subset $\mathrm{Conv}(X) \subset \mathrm{Cld}(X)$ consisting of all non-empty closed convex sets in X. Note that $\mathrm{Conv}_{\mathrm{H}}(X)$ is not connected. In fact, $\mathrm{Conv}_{\mathrm{H}}(\mathbb{R})$ has four components and $\mathrm{Conv}_{\mathrm{H}}(\mathbb{R}^n)$ has uncountably many components if $n > 1$ (see Remarks 1 and 2). In this paper, we show that every component

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([1]) These definitions are valid for an arbitrary metric space X and they depend on a metric on X.

of $\mathrm{Conv_H}(X)$ and the space $\mathrm{Conv_{AW}}(X)$ are AR (Theorems 2.2 and 3.4). In case X is separable, it is proved that $\mathrm{Conv_W}(X)$ is locally path-connected (Theorem 4.5). As a related subject, the space of compact convex sets with the Hausdorff metric is studied in [6].

2. The Hausdorff metric topology. Recall that the *Hausdorff metric* d_{H} is defined on $\mathrm{Cld}(X)$ as follows:

$$\begin{aligned} d_{\mathrm{H}}(A,B) &= \sup_{x\in X} |d(x,A) - d(x,B)| \\ &= \max\Big\{ \sup_{x\in B} d(x,A),\ \sup_{x\in A} d(x,B)\Big\}, \end{aligned}$$

where we allow $d_{\mathrm{H}}(A,B)=\infty$, but d_{H} induces a topology on $\mathrm{Cld}(X)$ like a metric does. It should be noted that d_{H} is a metric on each component of $\mathrm{Cld_H}(X)$ (cf. [4, Introduction]).

The *convex hull* of $A \subset X$ is denoted by $\langle A\rangle$, so the closure $\mathrm{cl}\langle A\rangle$ is the closed convex hull of A.

LEMMA 2.1. *For each $A, B \in \mathrm{Cld_H}(X)$, $d_{\mathrm{H}}(\mathrm{cl}\langle A\rangle, \mathrm{cl}\langle B\rangle) \le d_{\mathrm{H}}(A,B)$.*

Proof. Let $a = \sum_{i=1}^n t_i a_i \in \langle A\rangle$, where $a_i \in A$, $t_i > 0$ and $\sum_{i=1}^n t_i = 1$. For each $\varepsilon > 0$ and $i = 1,\dots,n$, we can choose $b_i \in B$ so that $\|a_i - b_i\| < d_{\mathrm{H}}(A,B)+\varepsilon$. Let $b = \sum_{i=1}^n t_i b_i \in \langle B\rangle$. Then

$$\|a-b\| \le \sum_{i=1}^n t_i\|a_i - b_i\| < d_{\mathrm{H}}(A,B)+\varepsilon,$$

hence $d(a,\mathrm{cl}\langle B\rangle) = d(a,\langle B\rangle) < d_{\mathrm{H}}(A,B)+\varepsilon$. Thus, $d(a,\mathrm{cl}\langle B\rangle) \le d_{\mathrm{H}}(A,B)$ for every $a \in \langle A\rangle$. Similarly, $d(b,\mathrm{cl}\langle A\rangle) \le d_{\mathrm{H}}(A,B)$ for every $b \in \langle B\rangle$. Consequently, $d_{\mathrm{H}}(\mathrm{cl}\langle A\rangle, \mathrm{cl}\langle B\rangle) \le d_{\mathrm{H}}(A,B)$. ∎

By Lemma 2.1 above, the map

$$\mathrm{Cld_H}(X) \ni A \mapsto \mathrm{cl}\langle A\rangle \in \mathrm{Conv_H}(X)$$

is a uniformly continuous retraction. In [4], it is proved that $\mathrm{Cld_H}(X)$ is an ANR and each component of $\mathrm{Cld_H}(X)$ is a uniform AR (in the sense of Michael [5]) with respect to the Hausdorff metric d_{H}, hence so is the space $\mathrm{Conv_H}(X)$, that is,

THEOREM 2.2. *The space $\mathrm{Conv_H}(X)$ is an ANR and each component of $\mathrm{Conv_H}(X)$ is a uniform AR with respect to the Hausdorff metric d_{H}.* ∎

Now, let $\mathrm{Conv^B}(X) \subset \mathrm{Conv}(X)$ be the subset consisting of all bounded closed convex sets. As is easily observed, $\mathrm{Conv_H^B}(X)$ is closed and open in $\mathrm{Conv_H}(X)$. Moreover, the space $\mathrm{Conv_H^B}(X)$ is path-connected. Indeed, for each $A, B \in \mathrm{Conv_H^B}(X)$,

$$\mathbf{I} \ni t \mapsto \mathrm{cl}((1-t)A + tB) \in \mathrm{Conv_H^B}(X)$$

is continuous. Thus, $\mathrm{Conv}_{\mathrm{H}}^{\mathrm{B}}(X)$ is a component of $\mathrm{Conv}_{\mathrm{H}}(X)$. Hence, we have the following:

THEOREM 2.3. *The space* $\mathrm{Conv}_{\mathrm{H}}^{\mathrm{B}}(X)$ *is a uniform AR.* ∎

REMARK 1. As is easily observed,

$$\begin{aligned}\mathrm{Conv}_{\mathrm{H}}(\mathbb{R}) = \{\mathbb{R}\} &\cup \{(-\infty, a] \mid a \in \mathbb{R}\} \\ &\cup \{[a, \infty) \mid a \in \mathbb{R}\} \cup \{[a, b] \mid a \leq b \in \mathbb{R}\},\end{aligned}$$

where $\mathbb{R}$ is an isolated point of $\mathrm{Conv}_{\mathrm{H}}(\mathbb{R})$, the second and third summands are isometric to $\mathbb{R}$, and the last one ($= \mathrm{Conv}_{\mathrm{H}}^{\mathrm{B}}(\mathbb{R})$) is isometric to the space $\{(a, b) \in \mathbb{R}^2 \mid a \leq b\}$ with the metric defined as follows:

$$d((a, b), (a', b')) = \max\{|a - a'|, |b - b'|\}.$$

Every 1-dimensional normed linear space X is linearly isometric to $\mathbb{R}$, hence $\mathrm{Conv}_{\mathrm{H}}(X)$ can be identified with $\mathrm{Conv}_{\mathrm{H}}(\mathbb{R})$. Thus, $\mathrm{Conv}_{\mathrm{H}}(X)$ when $\dim X = 1$ is of no interest.

REMARK 2. For Euclidean space $\mathbb{R}^n$ with $n > 1$, $\mathrm{Conv}_{\mathrm{H}}^{\mathrm{B}}(\mathbb{R}^n)$ is the space of compact convex sets. It is proved in [6] that $\mathrm{Conv}_{\mathrm{H}}^{\mathrm{B}}(\mathbb{R}^n)$ is homeomorphic to the Hilbert cube with one point removed. Every n-dimensional normed linear space X is Lipschitz homeomorphic to $\mathbb{R}^n$ by a linear isomorphism, which implies that $\mathrm{Conv}_{\mathrm{H}}^{\mathrm{B}}(X)$ is homeomorphic to $\mathrm{Conv}_{\mathrm{H}}^{\mathrm{B}}(\mathbb{R}^n)$. Thus, the space $\mathrm{Conv}_{\mathrm{H}}^{\mathrm{B}}(X)$ is known for $\dim X < \infty$.

Moreover, in case $\dim X > 1$, $\mathrm{Conv}_{\mathrm{H}}(X)$ has uncountably many components. Indeed, let $\mathbf{S}_X$ be the unit sphere of X. Then $\operatorname{card} \mathbf{S}_X > \aleph_0$. For each pair $v \neq v' \in \mathbf{S}_X$, $\mathbb{R}_+ v$ and $\mathbb{R}_+ v'$ do not belong to the same component, hence $\mathrm{Conv}_{\mathrm{H}}(X)$ has at least $\operatorname{card} \mathbf{S}_X$ many components.

In any case, $\mathrm{Conv}_{\mathrm{H}}(X)$ has the unique singular point X, that is,

PROPOSITION 2.4. *As a point*, X *is isolated in* $\mathrm{Conv}_{\mathrm{H}}(X)$.

Proof. It suffices to show that $d_{\mathrm{H}}(A, X) = \infty$ if $A \in \mathrm{Conv}_{\mathrm{H}}(X) \setminus \{X\}$. Let $a_0 \in A$. There exists some $x_0 \in X \setminus \{0\}$ such that $a_0 + [1, \infty)x_0 \subset X \setminus A$. Otherwise, for every $x \in X$, $a_0 + [1, \infty)x$ and $a_0 + [1, \infty)(-x)$ meet A, which implies $a_0 + x \in A$ by the convexity of A. Hence, $a_0 + X \subset A$, which contradicts $A \neq X$.

For each $t \geq 1$, since $d(a_0 + tx_0, A) > 0$, there exists $a \in A$ such that

$$\|a_0 + tx_0 - a\| < 2d(a_0 + tx_0, A) \leq 2d_{\mathrm{H}}(A, X).$$

Since $(1 - t^{-1})a_0 + t^{-1}a \in A$ by the convexity of A, it follows that

$$\begin{aligned}\|a_0 + tx_0 - a\| &= t\|a_0 + x_0 - ((1 - t^{-1})a_0 + t^{-1}a)\| \\ &\geq td(a_0 + x_0, A).\end{aligned}$$

This means that $d_{\mathrm{H}}(A, X) = \infty$. ∎

3. The Attouch–Wets Topology. The space $\mathrm{Cld}_{\mathrm{AW}}(X)$ has the following admissible metric:

$$d_{\mathrm{AW}}(A, A') = \sup_{k\in\mathbb{N}} \min\Big\{k^{-1}, \sup_{x\in k\mathbf{B}_X} |d(x,A) - d(x,A')|\Big\},$$

where $\mathbf{B}_X$ is the closed unit ball in X. It should be noted that the operator

$$\mathrm{Cld}_{\mathrm{AW}}(X) \ni A \mapsto \mathrm{cl}\langle A\rangle \in \mathrm{Conv}_{\mathrm{AW}}(X)$$

is not continuous even if $X = (\mathbb{R}, |\cdot|)$. Indeed, $[0,n] = \mathrm{cl}\langle\{0,n\}\rangle$ for each $n\in\mathbb{N}$. As is easily observed, $\lim_{n\to\infty}\{0,n\} = \{0\}$ but $\lim_{n\to\infty}[0,n] = [0,\infty)$ in $\mathrm{Cld}_{\mathrm{AW}}(\mathbb{R})$. Therefore, we need a different approach than in the case of $\mathrm{Conv}_{\mathrm{H}}(X)$.

We observe the following relation between the metrics d_{AW} and d_{H}:

LEMMA 3.1. *Let* $A\in\mathrm{Conv}(X)$, $r \geq d(0,A)+1$ *and* $0<\delta\leq(3r+2)^{-1} < 1/4$. *Then*

$$A'\in\mathrm{Conv}(X),\ d_{\mathrm{AW}}(A,A')<\delta,\ |r-r'|<\delta \\ \Rightarrow\ d_{\mathrm{H}}(A\cap 3r\mathbf{B}_X, A'\cap 3r'\mathbf{B}_X) < 9\delta.$$

Proof. Choose $m\in\mathbb{N}$ so that $3r\leq m < 3r+1$, whence $\delta<(m+1)^{-1}$. On the other hand, there exists $a\in A$ with $\|a\| < d(0,A)+1/4 \leq r-3/4$, whence $a\in r\mathbf{B}_X\subset(m+1)\mathbf{B}_X$. Since $d_{\mathrm{AW}}(A,A')<\delta<(m+1)^{-1}$, it follows that

$$\sup_{x\in(m+1)\mathbf{B}_X} |d(x,A)-d(x,A')| < \delta.$$

Thus, $\|a-a'\|<\delta$ for some $a'\in A'$, whence

$$\|a'\| \leq \|a\|+\delta < r-3/4+\delta < r-1/2 < r'.$$

These a, a' are fixed in the following argument.

For each $x\in A\cap 3r\mathbf{B}_X\subset m\mathbf{B}_X$, there is $x'\in A'$ such that $\|x-x'\|<\delta$. If $\|x'\|\leq 3r'$ then $x'\in A'\cap 3r'\mathbf{B}_X$, hence $d(x, A'\cap 3r'\mathbf{B}_X)<\delta$. In case $\|x'\|>3r'$, we want to find $y\in A'\cap 3r'\mathbf{B}_X$ replacing x'. Note that $\|x'\| > 3r' > 3(r-\delta)\geq 3(1-\delta) > 9\delta$. Since A' is convex and $a'\in A'\cap r'\mathbf{B}_X$, we have

$$y = \left(1-\frac{6\delta}{\|x'\|}\right)x' + \frac{6\delta}{\|x'\|}\,a' \in A',$$

whence

$$\|y\| \leq \|x'\| - 6\delta + \frac{6\delta\|a'\|}{\|x'\|} \leq \|x'\| - 6\delta + 6\delta\,\frac{r'}{3r'} \\ = \|x'\| - 4\delta \leq \|x\| - 3\delta \leq 3(r-\delta) < 3r',$$

hence $y\in A'\cap 3r'\mathbf{B}_X$. Moreover, observe that

$$\|x'-y\| \leq \frac{6\delta}{\|x'\|}\,\|x'-a'\| \leq 6\delta + \frac{6\delta\|a'\|}{\|x'\|} < 6\delta + 6\delta\frac{r'}{3r'} = 8\delta.$$

It follows that

$$\|x-y\| \le \|x-x'\| + \|x'-y\| < \delta + 8\delta = 9\delta.$$

Thus, we have $d(x, A' \cap 3r'\mathbf{B}_X) < 9\delta$ for each $x \in A \cap 3r\mathbf{B}_X$.

In the above argument, replace A, a, x, r with A', a', x', r', respectively. The inclusion $A' \cap 3r'\mathbf{B}_X \subset m\mathbf{B}_X$ might be false, but we have $A' \cap 3r'\mathbf{B}_X \subset 3(r+\delta)\mathbf{B}_X \subset (m+1)\mathbf{B}_X$. The rest of the argument is valid under this replacement. Thus, we can show that $d(x', A \cap 3r\mathbf{B}_X) < 9\delta$ for each $x' \in A' \cap 3r'\mathbf{B}_X$. Consequently, $d_{\mathrm{H}}(A \cap 3r\mathbf{B}_X, A' \cap 3r'\mathbf{B}_X) < 9\delta$. ∎

THEOREM 3.2. *As topological spaces*, $\mathrm{Conv}^{\mathrm{B}}_{\mathrm{AW}}(X) = \mathrm{Conv}^{\mathrm{B}}_{\mathrm{H}}(X)$, *hence the space* $\mathrm{Conv}^{\mathrm{B}}_{\mathrm{AW}}(X)$ *is an AR.*

Proof. Since $\tau_{\mathrm{H}} \supset \tau_{\mathrm{AW}}$, it is enough to see that $\mathrm{id} : \mathrm{Conv}^{\mathrm{B}}_{\mathrm{AW}}(X) \to \mathrm{Conv}^{\mathrm{B}}_{\mathrm{H}}(X)$ is continuous at each $A \in \mathrm{Conv}^{\mathrm{B}}_{\mathrm{AW}}(X)$. For each $\varepsilon > 0$, choose $n \in \mathbb{N}$ so that

$$n > d(0, A) + 1, \quad A \subset (3n-1)\mathbf{B}_X, \quad 9(3n+2)^{-1} < \varepsilon.$$

Let $A' \in \mathrm{Conv}^{\mathrm{B}}_{\mathrm{AW}}(X)$ with $d_{\mathrm{AW}}(A, A') < (3n+2)^{-1}$. Then $A' \subset 3n\mathbf{B}_X$. Indeed, since $A \subset (3n-1)\mathbf{B}_X$ and $d_{\mathrm{AW}}(A, A') < 1$, we have $a' \in A'$ with $\|a'\| < 3n$. If $\|a''\| > 3n$ for some $a'' \in A'$, then we can find $a''' \in A'$ with $\|a'''\| = 3n$ because of the convexity of A'. Since $d_{\mathrm{AW}}(A, A') < (3n+2)^{-1}$ and $a''' \in A' \cap 3n\mathbf{B}_X$, there exists $a \in A$ such that $\|a - a'''\| < (3n+2)^{-1} \le 1/5$, whence $\|a\| > \|a'''\| - 1/5 = 3n - 1/5$, which contradicts $A \subset (3n-1)\mathbf{B}_X$. Now, it follows from Lemma 3.1 that

$$d_{\mathrm{H}}(A, A') = d_{\mathrm{H}}(A \cap 3n\mathbf{B}_X, A' \cap 3n\mathbf{B}_X) < 9(3n+2)^{-1} < \varepsilon.$$

Thus, we have the result. ∎

The following fact was observed in the proof of [1, Fact 2]:

FACT. *For every* $A \in \mathrm{Cld}(X)$ *with* $A \cap k\mathbf{B}_X \ne \emptyset$,

$$d(x, A) = d(x, A \cap 3k\mathbf{B}_X) \quad \text{for each } x \in k\mathbf{B}_X. \ ∎$$

THEOREM 3.3. $\mathrm{Conv}^{\mathrm{B}}_{\mathrm{AW}}(X)$ *is homotopy dense in* $\mathrm{Conv}_{\mathrm{AW}}(X)$, *that is, there is a homotopy* $\varphi : \mathrm{Conv}_{\mathrm{AW}}(X) \times \mathbf{I} \to \mathrm{Conv}_{\mathrm{AW}}(X)$ *such that*

$$\varphi_0 = \mathrm{id} \quad \textit{and} \quad \varphi(\mathrm{Conv}_{\mathrm{AW}}(X) \times (0,1]) \subset \mathrm{Conv}^{\mathrm{B}}_{\mathrm{AW}}(X).$$

Proof. Define $\varphi : \mathrm{Conv}_{\mathrm{AW}}(X) \times \mathbf{I} \to \mathrm{Conv}_{\mathrm{AW}}(X)$ as follows:

$$\varphi(A, t) = \begin{cases} A & \text{if } t = 0, \\ A \cap 3 \cdot \dfrac{d(0, A) + 1}{t} \mathbf{B}_X & \text{if } t > 0. \end{cases}$$

Then $\varphi(\mathrm{Conv}_{\mathrm{AW}}(X) \times (0,1]) \subset \mathrm{Conv}^{\mathrm{B}}_{\mathrm{AW}}(X)$. It remains to show the continuity of φ.

For each $A \in \mathrm{Conv}_{\mathrm{AW}}(X)$, $t \in (0,1]$ and $\varepsilon > 0$, let

$$\delta = \min\left\{\frac{\varepsilon}{9}, \left(3 \cdot \frac{d(0,A)+1}{t} + 2\right)^{-1}\right\} > 0.$$

Choose $\gamma > 0$ so that $\gamma < \delta$ and

$$|d(0,A) - s| < \gamma,\ |t - t'| < \gamma \ \Rightarrow\ t' > 0,\ \left|\frac{d(0,A)+1}{t} - \frac{s+1}{t'}\right| < \delta.$$

Let $A' \in \mathrm{Conv}_{\mathrm{AW}}(X)$ and $t' \in \mathbf{I}$ with $d_{\mathrm{AW}}(A, A') < \gamma$ and $|t - t'| < \gamma$. Then $|d(0,A) - d(0,A')| < \gamma$ and $t' > 0$, hence

$$\left|\frac{d(0,A)+1}{t} - \frac{d(0,A')+1}{t'}\right| < \delta.$$

By Lemma 3.1, we have $d_{\mathrm{H}}(\varphi(A,t), \varphi(A',t')) < 9\delta \le \varepsilon$. This means that φ is continuous at (A,t) because $\mathrm{id} : \mathrm{Conv}_{\mathrm{H}}(X) \to \mathrm{Conv}_{\mathrm{AW}}(X)$ is continuous.

To see the continuity of φ at $(A, 0)$, for each $\varepsilon > 0$, choose $k \in \mathbb{N}$ so that $k^{-1} < \varepsilon$ and $A \cap (k-1)\mathbf{B}_X \neq \emptyset$. Let $A' \in \mathrm{Conv}_{\mathrm{AW}}(X)$ with $d_{\mathrm{AW}}(A, A') < k^{-1}$ and $0 < t' < k^{-1}$. Then

$$A' \cap k\mathbf{B}_X \neq \emptyset \quad \text{and} \quad 3k\mathbf{B}_X \subset 3 \cdot \frac{d(0,A')+1}{t'}\,\mathbf{B}_X.$$

Using the Fact, for every $x \in k\mathbf{B}_X$, we have

$$\begin{aligned} |d(x, \varphi(A,0)) - d(x, \varphi(A',t'))| &= |d(x,A) - d(x, A' \cap 3k\mathbf{B}_X)| \\ &= |d(x,A) - d(x,A')| \le d_{\mathrm{AW}}(A,A') < k^{-1}, \end{aligned}$$

hence $d_{\mathrm{AW}}(\varphi(A,0), \varphi(A',t')) < k^{-1} < \varepsilon$. This completes the proof. ■

Recall that a metrizable space is an AR if it contains an AR as a homotopy dense subset. Then, combining Theorems 3.3 and 3.2, we have the following result:

THEOREM 3.4. *The space* $\mathrm{Conv}_{\mathrm{AW}}(X)$ *is an AR.* ■

As is easily observed, $\mathrm{Cld}^{\mathrm{B}}_{\mathrm{AW}}(X)$ is not open in the space $\mathrm{Cld}_{\mathrm{AW}}(X)$. Nevertheless, we have the following:

PROPOSITION 3.5. *The subspace* $\mathrm{Conv}^{\mathrm{B}}_{\mathrm{AW}}(X) \subset \mathrm{Conv}_{\mathrm{AW}}(X)$ *is open.*

Proof. For each $A \in \mathrm{Conv}^{\mathrm{B}}_{\mathrm{AW}}(X)$, choose $k \in \mathbb{N}$ so that $A \subset k\mathbf{B}_X$. If $A' \in \mathrm{Conv}_{\mathrm{AW}}(X)$ and $d_{\mathrm{AW}}(A,A') < (k+1)^{-1}$ then $A' \subset (k+1)\mathbf{B}_X$. Indeed, take $a \in A$. Since $\|a\| \le k < k+1$, it follows that $d(a, A') < (k+1)^{-1}$, that is, $\|a - a'\| < (k+1)^{-1}$ for some $a' \in A'$. Then $\|a'\| < k+1$. Now, assume that $A' \not\subset (k+1)\mathbf{B}_X$, that is, $\|a''\| > k+1$ for some $a'' \in A'$. Choose $0 < s < 1$ so that $\|(1-s)a' + sa''\| = k+1$. Then $(1-s)a' + sa'' \in A'$ because A' is convex. However,

$$d((1-s)a' + sa'', A) \ge d((1-s)a' + sa'', k\mathbf{B}_X) = 1 > (k+1)^{-1},$$

which contradicts $d_{\mathrm{AW}}(A,A') < (k+1)^{-1}$. Thus, $A' \in \mathrm{Conv}^{\mathrm{B}}_{\mathrm{AW}}(X)$. ■

4. The Wijsman topology. For each $x \in X$ and $r > 0$, we define

$$U^-(x,r) = \{A \in \mathrm{Cld}(X) \mid d(x,A) < r\},$$
$$U^+(x,r) = \{A \in \mathrm{Cld}(X) \mid d(x,A) > r\}.$$

These sets form an open subbasis for $\mathrm{Cld}_{\mathrm{W}}(X)$. As mentioned in the introduction, $\mathrm{Cld}_{\mathrm{W}}(X)$ is (separable) metrizable if and only if X is separable. This is true even if $\mathrm{Cld}_{\mathrm{W}}(X)$ is replaced with $\mathrm{Conv}_{\mathrm{W}}(X)$. In fact, the following holds:

PROPOSITION 4.1. *If* $\mathrm{Conv}_{\mathrm{W}}(X)$ *is first countable then* X *is separable.*

Proof. Assume that $\mathrm{Conv}_{\mathrm{W}}(X)$ is first countable and X is non-separable. Then there is a δ-discrete uncountable subset $D \subset X$ for some $\delta > 0$, i.e., $\|x - y\| \geq \delta$ for each $x \neq y \in D$. By the first countability of $\mathrm{Conv}_{\mathrm{W}}(X)$, we have a countable neighborhood basis $\{W_i \mid i \in \mathbb{N}\}$ of $X \in \mathrm{Conv}_{\mathrm{W}}(X)$. For each $i \in \mathbb{N}$, we can choose a finite set $F_i \subset X$ and $\varepsilon_i > 0$ so that $\bigcap_{p \in F_i} U^-(p, \varepsilon_i) \subset W_i$ and $\lim_{i\to\infty} \varepsilon_i = 0$. Observe that $P = \bigcup_{i\in\mathbb{N}} \langle F_i \rangle$ is separable, that is, it contains a countable dense subset Q. Then $d(x_0, Q) > \delta/3$ for some $x_0 \in D$. Otherwise, there would be a function $q : D \to Q$ such that $\|x - q(x)\| < \delta/2$, which implies that q is injective by the δ-discreteness of D. This is a contradiction because D is uncountable and Q is countable. Note that $U^-(x_0, \delta/3)$ is a neighborhood of X in $\mathrm{Conv}_{\mathrm{W}}(X)$. Now, we can choose $i \in \mathbb{N}$ so that $\varepsilon_i < \delta/3$ and

$$\bigcap_{p\in F_i} U^-(p, \varepsilon_i) \subset W_i \subset U^-(x_0, \delta/3),$$

whence $\langle F_i \rangle \in U^-(x_0, \delta/3)$. It follows that

$$d(x_0, Q) = d(x_0, P) \leq d(x_0, \langle F_i \rangle) < \delta/3,$$

which is a contradiction. ■

REMARK 3. The space $\mathrm{Conv}^{\mathrm{B}}_{\mathrm{W}}(\ell_2)$ is separable. However, $\mathrm{Conv}^{\mathrm{B}}_{\mathrm{AW}}(\ell_2)$ is not separable. Indeed, let $V = \{e_n \mid n \in \mathbb{N}\}$ be the canonical orthonormal basis. Let $A \neq A' \subset V$. We may assume $A \setminus A' \neq \emptyset$. Let $e_n \in A \setminus A'$. For each $x = (x_i)_{i\in\mathbb{N}} \in \langle A' \rangle$, we have $\|e_n - x\| \geq 1$ because $x_n = 0$. Therefore, $d(e_n, \mathrm{cl}\langle A' \rangle) \geq 1$. It follows that $d_{\mathrm{AW}}(\mathrm{cl}\langle A \rangle, \mathrm{cl}\langle A' \rangle) \geq 1/2$. Thus, $D = \{\mathrm{cl}\langle A \rangle \mid A \subset V\}$ is discrete in $\mathrm{Conv}^{\mathrm{B}}_{\mathrm{AW}}(\ell_2)$ and $\operatorname{card} D = 2^{\aleph_0}$.

It should be noticed that if $\dim X < \infty$ then $\mathrm{Conv}_{\mathrm{W}}(X) = \mathrm{Conv}_{\mathrm{AW}}(X)$ as spaces [2, Theorem 3.1.4].

Let $\mathrm{Conv}^{\mathrm{P}}(X)$ be the subset of $\mathrm{Conv}(X)$ consisting of all convex polyhedra in X, that is,

$$\mathrm{Conv}^{\mathrm{P}}(X) = \{\langle F \rangle \in \mathrm{Conv}(X) \mid F \in \mathrm{Fin}(X)\},$$

where $\mathrm{Fin}(X)$ is the set of all non-empty finite sets in X. We denote by $\mathrm{Conv}^{\mathrm{s}}(X)$ the subset of $\mathrm{Conv}(X)$ consisting of all separable closed convex sets.

PROPOSITION 4.2. *For each* $A \in \mathrm{Cld}^{\mathrm{s}}_{\mathrm{W}}(X)$ *and* $a_1, \dots, a_n \in A$, *there exists a path* $f : \mathbf{I} \to \mathrm{Conv}^{\mathrm{s}}_{\mathrm{W}}(X)$ *such that* $f(0) = A$, $f(1) = \langle \{a_1, \dots, a_n\} \rangle$, $f((0,1]) \subset \mathrm{Conv}^{\mathrm{P}}_{\mathrm{W}}(X)$ *and* $f(t) \supset f(t')$ *for* $t < t'$.

Proof. Let $\{x_i \mid i \in \mathbb{N}\}$ be a dense set in A. For each $k \in \mathbb{N}$, let $A_k = A_0 \cup \{x_1, \dots, x_k\}$, where $A_0 = \{a_1, \dots, a_n\}$. The desired path $f : \mathbf{I} \to \mathrm{Conv}_{\mathrm{W}}(X)$ can be defined as follows:

$$f(t) = \begin{cases} A & \text{if } t = 0, \\ \langle A_{k-1} \cup \{(2 - 2^k t)x_k + (2^k t - 1)a_1\} \rangle & \text{if } 2^{-k} \leq t \leq 2^{-k+1}. \end{cases}$$

We have to verify the continuity of f. By Lemma 2.1, $f|(0,1] : (0,1] \to \mathrm{Conv}_{\mathrm{H}}(X)$ is continuous. Since $\tau_{\mathrm{H}} \supset \tau_{\mathrm{W}}$, $f|(0,1] : (0,1] \to \mathrm{Conv}_{\mathrm{W}}(X)$ is also continuous, hence f is continuous at $t > 0$. To see the continuity of f at $t = 0$, let

$$f(0) = A \in \bigcap_{i=1}^{n} U^-(p_i, r_i) \cap \bigcap_{j=1}^{m} U^+(q_j, s_j), \qquad p_i, q_j \in X,\ r_i, s_j > 0.$$

Since $\{x_i \mid i \in \mathbb{N}\}$ is dense in A, we can choose $\nu(1), \dots, \nu(n) \in \mathbb{N}$ so that $\|p_i - x_{\nu(i)}\| < r_i$. Let $k = \max\{\nu(1), \dots, \nu(n)\}$. Then, as is easily observed,

$$0 < t \leq 2^{-k} \ \Rightarrow\ f(t) \in \bigcap_{i=1}^{n} U^-(p_i, r_i) \cap \bigcap_{j=1}^{m} U^+(q_j, s_j),$$

hence f is continuous at 0. ■

COROLLARY 4.3. *If* X *is separable, then for each* $A \in \mathrm{Conv}_{\mathrm{W}}(X)$, *there is a path* $f : \mathbf{I} \to \mathrm{Conv}_{\mathrm{W}}(X)$ *such that* $f(0) = A$ *and* $f((0,1]) \subset \mathrm{Conv}^{\mathrm{P}}_{\mathrm{W}}(X)$. ■

When X is separable, the assertion below follows from the above corollary, but it can be easily proved without separability.

PROPOSITION 4.4. *The subspace* $\mathrm{Conv}^{\mathrm{P}}_{\mathrm{W}}(X) \subset \mathrm{Conv}_{\mathrm{W}}(X)$ *is dense.*

Proof. For each $A \in \mathrm{Conv}_{\mathrm{W}}(X)$ and each neighborhood $\mathcal{U}$ of A in $\mathrm{Cld}_{\mathrm{W}}(X)$, there are $p_i, q_j \in X$ and $r_i, s_j > 0$ such that

$$A \in \bigcap_{i=1}^{n} U^-(p_i, r_i) \cap \bigcap_{j=1}^{m} U^+(q_j, s_j) \subset \mathcal{U}.$$

Choose $a_1, \dots, a_n \in A$ so that $\|p_i - a_i\| < r_i$ and define $A_0 = \{a_1, \dots, a_n\} \in$

$\mathrm{Fin}(X)$. Then, as is easily observed,

$$\langle A_0\rangle \in \bigcap_{i=1}^{n} U^-(p_i, r_i) \cap \bigcap_{j=1}^{m} U^+(q_j, s_j) \subset \mathcal{U},$$

that is, $\mathcal{U}$ meets $\mathrm{Conv}^{\mathrm{P}}_{\mathrm{W}}(X)$. Hence, $\mathrm{Conv}^{\mathrm{P}}_{\mathrm{W}}(X)$ is dense in $\mathrm{Conv}_{\mathrm{W}}(X)$. ∎

Now, we show the following:

THEOREM 4.5. *The space* $\mathrm{Conv}^{\mathrm{s}}_{\mathrm{W}}(X)$ *is locally path-connected. Thus, if* X *is separable then* $\mathrm{Conv}_{\mathrm{W}}(X)$ *is locally path-connected.*

Proof. For each $A \in \mathrm{Conv}^{\mathrm{s}}_{\mathrm{W}}(X)$ and each neighborhood $\mathcal{U}$ of A in $\mathrm{Cld}_{\mathrm{W}}(X)$, take $p_i, q_j \in X$, $r_i, s_j > 0$ and $A_0 = \{a_1, \dots, a_n\} \subset A$ as in the proof of Proposition 4.4. Since $\tau_{\mathrm{H}} \supset \tau_{\mathrm{W}}$, we can choose $\delta > 0$ so that

$$d_{\mathrm{H}}(\langle A_0\rangle, B) < \delta \;\Rightarrow\; B \in \bigcap_{i=1}^{n} U^-(p_i, r_i) \cap \bigcap_{j=1}^{m} U^+(q_j, s_j) \subset \mathcal{U}$$

and $\delta < r_i - \|p_i - a_i\|$ for each $i = 1, \dots, n$. Then A has the following neighborhood $\mathcal{V}$ in $\mathrm{Cld}_{\mathrm{W}}(X)$:

$$\mathcal{V} = \bigcap_{i=1}^{n} U^-(a_i, \delta) \cap \bigcap_{j=1}^{m} U^+(q_j, s_j) \subset \mathcal{U}.$$

We shall show that each $B \in \mathcal{V} \cap \mathrm{Conv}^{\mathrm{s}}_{\mathrm{W}}(X)$ can be connected with A by a path in $\mathcal{U} \cap \mathrm{Conv}^{\mathrm{s}}_{\mathrm{W}}(X)$, which means that $\mathrm{Conv}^{\mathrm{s}}_{\mathrm{W}}(X)$ is locally path-connected. Choose $x_1, \dots, x_n \in B$ so that $\|x_i - a_i\| < \delta$ and let $B_0 = \{x_1, \dots, x_n\}$. By Lemma 2.1, we can define a path $h : \mathbf{I} \to \mathrm{Conv}^{\mathrm{P}}_{\mathrm{H}}(X)$ as follows:

$$h(t) = \langle (1-t)a_1 + tx_1, \dots, (1-t)a_n + tx_n\rangle.$$

Then $h(0) = \langle A_0\rangle$ and $h(1) = \langle B_0\rangle$. Since $\mathrm{diam}_{\mathrm{H}}\, h(\mathbf{I}) < \delta$, we have

$$h(\mathbf{I}) \subset \bigcap_{i=1}^{n} U^-(p_i, r_i) \cap \bigcap_{j=1}^{m} U^+(q_j, s_j) \subset \mathcal{U}.$$

Since $\tau_{\mathrm{H}} \supset \tau_{\mathrm{W}}$, $h : \mathbf{I} \to \mathrm{Conv}^{\mathrm{P}}_{\mathrm{W}}(X)$ is also continuous. On the other hand, by Proposition 4.2, we have paths $f, g : \mathbf{I} \to \mathrm{Conv}^{\mathrm{s}}_{\mathrm{W}}(X)$ such that

$$f(0) = A \supset f(t) \supset \langle A_0\rangle = f(1) \;\text{ and }\; g(0) = B \supset g(t) \supset \langle B_0\rangle = g(1),$$

whence

$$f(t), g(t) \in \bigcap_{i=1}^{n} U^-(p_i, r_i) \cap \bigcap_{j=1}^{m} U^+(q_j, s_j) \subset \mathcal{U}.$$

By connecting the paths f, g and h, we obtain a path from A to B contained in $\mathcal{U} \cap \mathrm{Conv}^{\mathrm{s}}_{\mathrm{W}}(X)$. ∎

Finally, we show the following:

PROPOSITION 4.6. *The subset* $\mathrm{Conv}(X) \subset \mathrm{Cld}(X)$ *is closed with respect to any one of the topologies* τ_{H}, τ_{AW} *and* τ_{W}.

Proof. Since $\tau_{\mathrm{H}} \supset \tau_{\mathrm{AW}} \supset \tau_{\mathrm{W}}$, it suffices to prove that $\mathrm{Conv}(X)$ is closed in $\mathrm{Cld}_{\mathrm{W}}(X)$, equivalently $\mathrm{Cld}(X) \setminus \mathrm{Conv}(X)$ is open in $\mathrm{Cld}_{\mathrm{W}}(X)$.

For each $A \in \mathrm{Cld}(X) \setminus \mathrm{Conv}(X)$, there are $x, y \in A$ and $t \in \mathbf{I}$ such that $z = (1-t)x + ty \notin A$. Let $\delta = \frac{1}{2}d(z, A) > 0$ and

$$\mathcal{U} = U^-(x,\delta) \cap U^-(y,\delta) \cap U^+(z,\delta).$$

Then $\mathcal{U}$ is a neighborhood of A in $\mathrm{Cld}_{\mathrm{W}}(X)$. For each $A' \in \mathcal{U}$, there are $x', y' \in A'$ such that $\|x - x'\| < \delta$ and $\|y - y'\| < \delta$. Since $d(z, A') > \delta$ and

$$\|(1-t)x' + ty' - z\| \le (1-t)\|x' - x\| + t\|y' - y\| < \delta,$$

it follows that $(1-t)x' + ty' \notin A'$, hence A' is not convex, that is, $A' \in \mathrm{Cld}(X) \setminus \mathrm{Conv}(X)$. Thus, $\mathrm{Cld}(X) \setminus \mathrm{Conv}(X)$ is open in $\mathrm{Cld}_{\mathrm{W}}(X)$. ∎

COROLLARY 4.7. *For every Banach space* X, *the spaces* $\mathrm{Conv}_{\mathrm{H}}(X)$ *and* $\mathrm{Conv}_{\mathrm{AW}}(X)$ *are completely metrizable. If* X *is a separable Banach space then* $\mathrm{Conv}_{\mathrm{W}}(X)$ *is also completely metrizable.* ∎

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