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ADDITION AND SUBSPACE THEOREMS FOR ASYMPTOTIC LARGE INDUCTIVE DIMENSION

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Abstract. We prove the addition and subspace theorems for asymptotic large inductive dimension. We investigate a transfinite extension of this dimension and show that it is trivial.

0. Asymptotic dimension asdim of a metric space was defined by Gromov for studying asymptotic invariants of discrete groups [4]. This dimension can be considered as an asymptotic analogue of the Lebesgue covering dimension dim. Dranishnikov has introduced the dimension asInd which is analogous to large inductive dimension Ind (see [1]). It is known that $\operatorname{asdim} X = \operatorname{asInd} X$ for each proper metric space with $\operatorname{asdim} X < \infty$. The problem of coincidence of asdim and asInd is still open in the general case [2].

Among basic theorems of classical dimension theory are the sum, addition and subspace theorems for different dimensions and classes of topological spaces. Here we mention some of them related to dimension Ind. (All the above mentioned facts from classical dimension theory can be found in [3].)

1. *Countable sum theorem*: If a strongly hereditarily normal space X can be represented as the union of a sequence $F_1, F_2, \ldots$ of closed subsets such that $\operatorname{Ind} F_i \le n$ for each $i \in \mathbb{N}$, then $\operatorname{Ind} X \le n$.

2. *Addition theorem*: If a hereditarily normal space X is represented as the union of two subspaces X_1 and X_2, then $\operatorname{Ind} X \le \operatorname{Ind} X_1 + \operatorname{Ind} X_2 + 1$.

Let us remark that the above theorems do not generalize to the class of all normal spaces. We have only a weaker result:

3. If a normal space X is represented as the union of two closed subsets X_1 and X_2, then $\operatorname{Ind} X \le \operatorname{Ind} X_1 + \operatorname{Ind} X_2$.

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And finally:

4. *Subspace theorem*: For each closed subset M of a normal space X we have $\operatorname{Ind} M \leq \operatorname{Ind} X$. Let us remark that if X is a strongly hereditarily normal space, then the condition that M is closed can be dropped.

There is no countable sum theorem for asInd. Indeed, the space of integers $\mathbb{Z}$ has asymptotic dimension 1 but it is a countable union of its points which have asymptotic dimension -1. However we prove the subspace and addition theorems for asInd in this paper.

Extending the codomain of Ind to ordinal numbers we obtain the transfinite extension trInd of the dimension Ind. It is known that there exists a space S_α such that $\operatorname{trInd} S_\alpha = \alpha$ for each countable ordinal number α. Zarichnyi has proposed to consider transfinite extension of asInd and conjectured that this extension is trivial. We prove this conjecture: if a space has a transfinite asymptotic dimension, then its dimension is finite.

The paper is organized as follows: in Section 1 we give some necessary definitions and introduce some notations, in Section 2 we prove a theorem which can be considered a weak version of the countable sum theorem, in Section 3 we obtain the main results and in Section 4 we show that the transfinite extension of asInd is trivial.

1. Let $A_1, A_2 \subset X$ be two disjoint closed subsets in a topological space. We recall that a *partition* between A_1 and A_2 is a subset $C \subset X$ such that there are open disjoint sets U_1, U_2 with $X \setminus C = U_1 \cup U_2$, $A_1 \subset U_1$ and $A_2 \subset U_2$. Clearly a partition C is a closed subset of X.

We recall the definition of the large inductive dimension Ind (see [3]): $\operatorname{Ind} X = -1$ iff $X = \emptyset$; $\operatorname{Ind} X \leq n$ if for any two disjoint closed subsets $A_1, A_2 \subset X$ there is a partition C with $\operatorname{Ind} C \leq n-1$.

We will define the dimension asInd for the class of proper metric spaces. We recall that a metric space is *proper* if every closed ball is compact. Assume that some base point $x_0 \in X$ is chosen in each proper metric space X. We denote by d the generic metric. If X is a metric space and $A \subset X$ we denote by $N_r(A)$ the closed r-neighborhood of A in X: $N_r(A) = \{x \in X \mid d(x, A) \leq r\}$, and by $B_r(A)$ the open r-neighborhood: $B_r(A) = \{x \in X \mid d(x, A) < r\}$.

A subset W of a metric space X is called an *asymptotic neighborhood* of a set $A \subset X$ if $\lim_{r\to\infty} d(A \setminus B_r(x_0), X \setminus W) = \infty$. We call two subsets $A_1, A_2 \subset X$ *asymptotically disjoint* if

$$\lim_{r\to\infty} d(A_1 \setminus B_r(x_0), A_2 \setminus B_r(x_0)) = \infty.$$

It is easy to see that A_1 and A_2 are asymptotically disjoint iff $X \setminus A_2$ is an asymptotic neighborhood of A_1 and $X \setminus A_1$ is an asymptotic neighborhood of A_2.

A map $\phi : X \to I = [0,1]$ is called *slowly oscillating* if for any $r, \varepsilon > 0$, there exists $D > 0$ such that $\operatorname{diam}\phi(B_r(x)) < \varepsilon$ for all x with $d(x, x_0) \geq D$. If $C_h(X)$ is the set of all continuous slowly oscillating functions $\phi : X \to I$, then the *Higson compactification* is the closure of the image of X under the embedding $\Phi : X \to I^{C_h(X)}$ defined as $\Phi(x) = (\phi(x) \mid \phi \in C_h(X)) \in I^{C_h(X)}$. We denote the Higson compactification of a proper metric space X by cX and the remainder $cX \setminus X$ by νX. The compactum νX is called the *Higson corona*. Note that νX need not be metrizable.

Let C be a subset of a proper metric space X. We denote by C' the intersection $\operatorname{cl} C \cap \nu X$ where cl is the closure in the Higson compactification cX. Clearly, two sets A_1 and A_2 are asymptotically disjoint iff their traces A_1' and A_2' in the Higson corona are disjoint. Note that for each $r > 0$ we have $N_r(C)' = C'$.

Let $A_1, A_2 \subset X$ be two asymptotically disjoint subsets of a proper metric space X. A closed subset $C \subset X$ is called an *asymptotic separator* for A_1 and A_2 if its trace C' is a partition for A_1' and A_2' in νX.

We define $\operatorname{asInd} X = -1$ if and only if X is bounded; $\operatorname{asInd} X \leq n$ if for any two asymptotically disjoint sets $A, B \subset X$ there is an asymptotic separator C with $\operatorname{asInd} C \leq n-1$. Naturally we say $\operatorname{asInd} X = n$ if $\operatorname{asInd} X \leq n$ and it is not true that $\operatorname{asInd} X \leq n-1$. We set $\operatorname{asInd} X = \infty$ if $\operatorname{asInd} X > n$ for each $n \in \mathbb{N}$ (see [1]).

For each $A \subset Y \subset X$ we define $\operatorname{ex}_Y A = Y' \setminus (Y \setminus A)'$. Clearly, $\operatorname{ex}_Y A$ is an open set in Y'. We denote by ω^ω the set of all functions $\tau : \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$ such that $\tau(0) = 0$. For each $\tau \in \omega^\omega$ we set $V_\tau^Y(A) = \{y \in Y \mid d(y, x_0) \geq \tau([d(y, A)])\}$ where $[\cdot]$ is the integer part. Clearly, $A \subset V_\tau^Y(A)$ for each $\tau \in \omega^\omega$. If $X = Y$ we use the simpler notations $\operatorname{ex} A$ and $V_\tau(A)$.

Lemma 1. *The family $\{\operatorname{ex}(V_\tau(A)) \mid \tau \in \omega^\omega\}$ forms a base of neighborhoods of the set A' in the space νX.*

Proof. Let us show first that $A' \subset \operatorname{ex}(V_\tau(A))$ for each $\tau \in \omega^\omega$. It is enough to show that A and $X \setminus V_\tau(A)$ are asymptotically disjoint. Fix any $D > 0$. Put $R = \max\{\tau(l) \mid l \in \{0, \dots, [D]\}\}$. Then for each $x \in (X \setminus V_\tau(A)) \setminus B_R(x_0)$ we have $R \leq d(x, x_0) < \tau([d(x, A)])$. Thus $d(x, A) \geq [D] + 1 > D$.

Consider now any closed subset B of νX such that $B \cap A' = \emptyset$. Then there exists a continuous function $f : \nu X \to [0,1]$ such that $f(B) \subset \{0\}$ and $f(A') \subset \{1\}$. We can extend f to a continuous function $g : cX \to [0,1]$ such that $A \subset g^{-1}(1)$. Put $C = g^{-1}[0, 1/2] \cap X$. The sets C and A are asymptotically disjoint. For each $n \in \mathbb{N}$ there exists $R(n) > 0$ such that $d(C \setminus B_{R(n)}(x_0), A) \geq n$. Put $\tau(0) = 0$ and $\tau(n) = [R(n+1)]$. Choose any $c \in C$. Then $d(c, x_0) < R([d(c, A)] + 1) = \tau([d(c, A)])$. So, $C \subset X \setminus V_\tau(A)$

and $B \subset C' \subset (X \setminus V_\tau(A))' \subset \nu X \setminus \operatorname{ex} V_\tau(A)$. Hence $\{\operatorname{ex}(V_\tau(A)) \mid \tau \in \omega^\omega\}$ forms the base of neighborhoods of A' in νX and the lemma is proved.

2. Let X be a proper metric space and let X_0 be an unbounded subset of X. We say that X_0 is a *kernel* of X if there exists a sequence $(k_i)_{i=0}^\infty$ of natural numbers such that $k_i \to \infty$ and $B_{k_i}(x) = \{x\}$ for each $i \in \mathbb{N} \cup \{0\}$ and $x \in X \setminus N_i(X_0)$. We suppose that $x_0 \in X_0$ where x_0 is the base point of X.

LEMMA 2. *If X_0 is a kernel of a proper metric space X then the family $\{V_\tau(X_0)) \mid \tau \in \omega^\omega\}$ forms a base of clopen neighborhoods of the set X_0' in the space νX.*

Proof. It follows from Lemma 1 that it is enough to prove that $V_\tau(X_0)' \cap (X \setminus V_\tau(X_0))' = \emptyset$. Suppose the contrary: there exists $x \in V_\tau(X_0)' \cap (X \setminus V_\tau(X_0))'$. Let U be a neighborhood of x in cX. Then there exist two sequences (a_i) and (b_i) in $V_\tau(X_0) \cap U$ and $(X \setminus V_\tau(X_0)) \cap U$ respectively such that $0 < d(a_i, b_i) \le r$ for some $r > 0$ and $a_i, b_i \in X \setminus B_i(x_0)$. Choose any $n_0 \in \mathbb{N}$ such that $k_n > r$ for each $n \ge n_0$ where (k_n) is a sequence from the definition of kernel. Then $a_i, b_i \in N_{n_0}(X_0)$ for each $i \in \mathbb{N}$. Hence $\emptyset \ne \operatorname{cl} U \cap N_{n_0}(X_0)' = \operatorname{cl} U \cap X_0'$ and $x \in X_0$, which is a contradiction.

LEMMA 3. *If X_0 is a kernel of X then* $\operatorname{Ind}(\nu X \setminus V_\tau(X_0)') \le 0$ *for each $\tau \in \omega^\omega$.*

Proof. Since $\nu X \setminus V_\tau(X_0)'$ is compact, it is enough to prove that the space $\nu X \setminus V_\tau(X_0)'$ has a base of clopen sets [3, Theorem 1.6.5].

Choose any $x \in \nu X \setminus V_\tau(X_0)'$ and its open neighborhood $U \subset \nu X \setminus V_\tau(X_0)'$. Take a continuous function $f : \nu X \to [0,1]$ such that $f(x) = 0$ and $f(\nu X \setminus U) \subset \{1\}$. Extend f to a continuous function $g : cX \to [0,1]$. Put $A = g^{-1}[0,1/3] \cap X$ and $C = g^{-1}[2/3,1] \cap X$. Then C and A are asymptotically disjoint. Moreover, $x \in A'$ and $\nu X \setminus U \subset C'$. For each $n \in \mathbb{N}$ there exists $R(n) > 0$ such that $d(C \setminus B_R(x_0), A \setminus B_R(x_0)) \ge n$. Put $\tau(0) = 0$ and $\tau(n) = [R(n+1)]$. We can show that $V_\tau(A)'$ is a clopen neighborhood of x such that $V_\tau(A)' \subset U$ using the same reasoning as in Lemmas 1 and 2. The lemma is proved.

Let us define a preorder $\le^*$ in ω^ω as follows: $\tau \le^* \sigma$ iff there exists $n \in \mathbb{N}$ such that $\tau(i) \le \sigma(i)$ for each $i \ge n$. It is easy to check that $V_\tau(A)' \subset V_\sigma(A)'$ if $\sigma \le^* \tau$.

THEOREM 1. *Let X_0 be a kernel of a proper metric space X such that* $\operatorname{asInd} X_0 \le k \ge 0$. *Then* $\operatorname{asInd} X \le k$.

Proof. We use induction on k. Let $\operatorname{asInd} X_0 \le 0$. Then $\operatorname{Ind} X_0' \le 0$ (see [1]). Consider any two asymptotically disjoint sets A and B in X. We can represent X_0' as $K \cup L$ where K and L are disjoint closed subsets of X_0' such that $A' \cap X_0' \subset K$ and $B' \cap X_0' \subset L$. Then $A' \cup K$ and $B' \cup L$ are disjoint closed subsets of νX. Choose two disjoint open subsets U_1 and U_2 of νX such that $A' \cup K \subset U_1$ and $B' \cup L \subset U_2$. Then $U_1 \cup U_2$ is a neighborhood of X_0 in νX and there exists $\tau \in \omega^\omega$ such that $V_\tau(X_0)' \subset U_1 \cup U_2$. The sets $A' \cap (\nu X \setminus V_\tau(X_0)')$ and $B' \cap (\nu X \setminus V_\tau(X_0)')$ are disjoint closed subsets of the 0-dimensional space $\nu X \setminus V_\tau(X_0)'$, so there exist open disjoint subsets O_1, O_2 of $\nu X \setminus V_\tau(X_0)'$ such that $A' \cap (\nu X \setminus V_\tau(X_0)') \subset O_1$, $B' \cap (\nu X \setminus V_\tau(X_0)') \subset O_2$ and $O_1 \cup O_2 = \nu X \setminus V_\tau(X_0)'$. Put $V_1 = (U_1 \cap V_\tau(X_0)') \cup O_1$ and $V_2 = (U_2 \cap V_\tau(X_0)') \cup O_2$. Since $V_\tau(X_0)'$ is a clopen subset of νX, the sets V_1 and V_2 are open. Moreover, they are disjoint, $V_1 \cup V_2 = \nu X$ and $A' \subset V_1$ and $B' \subset V_2$. Thus, the empty space is a partition between A' and B' in νX. Hence, the empty space is an asymptotic separator between A and B in X and $\operatorname{asInd} X \le 0$.

Suppose that the theorem is proved for each $i < n \ge 1$. Consider the case when $\operatorname{asInd} X_0 \le n$. Let A and B be any asymptotically disjoint subsets of X. Then $A' \cap X_0'$ and $B' \cap X_0'$ are disjoint closed subsets of X_0' and we can choose a continuous function $f' : X_0' \to [0,1]$ such that $f'(A' \cap X_0') \subset \{0\}$ and $f'(B' \cap X_0') \subset \{1\}$. We extend it to a continuous function $f : \operatorname{cl} X_0 \to [0,1]$. The sets $A_1 = (f^{-1}[0,1/3]) \cap X_0$ and $B_1 = (f^{-1}[2/3,1]) \cap X_0$ are asymptotically disjoint and we can choose an asymptotic separator $L \subset X_0$ between them such that $\operatorname{asInd} L < n$.

Write $X = \bigcup_{i=0}^\infty X_i$ where $X_i = N_i(X_0) \setminus N_{i-1}(X_0)$ for $i \in \mathbb{N}$. It follows from the definition of kernel that for each $R > 0$ there exists $i(R) \in \mathbb{N}$ such that $\bigcup_{k=i(R)}^\infty X_k$ is R-discrete and $d(\bigcup_{k=i(R)}^\infty X_k, \bigcup_{k=0}^{i(R)-1} X_k) \ge R$.

Since L is an asymptotic separator in X_0 between A_1 and B_1, the set L' is a partition in X_0' between A_1' and B_1'. Thus we can choose open disjoint sets O_A and O_B in X_0' such that $A_1' \subset O_A$, $B_1' \subset O_B$ and $X_0' \setminus L' = O_A \cup O_B$.

For each $\tau \in \omega^\omega$ we can represent $X_0 \setminus V_\tau(L)$ as a union of two disjoint sets A_τ and B_τ such that $A_\tau' \subset O_A$ and $B_\tau' \subset O_B$. Moreover, we can suppose that for each $\tau \le^* \sigma$ there exists $R > 0$ such that $A_\sigma \supset A_\tau \setminus B_R(x_0)$ and $B_\sigma \supset B_\tau \setminus B_R(x_0)$. Define for each $\tau \in \omega^\omega$ two subsets C_τ, D_τ of X as follows:

$$C_\tau = \bigcup_{i=0}^\infty \{x \in X_i \mid N_i(x) \cap X_0 \subset A_\tau\},$$

$$D_\tau = \bigcup_{i=0}^\infty \{x \in X_i \mid N_i(x) \cap B_\tau \ne \emptyset\}.$$

We have $C_\tau \cap D_\tau = \emptyset$ and

$$X \setminus (C_\tau \cup D_\tau) \subset \bigcup_{i=0}^\infty \{x \in X_i \mid N_i(x) \cap V_\tau^{X_0}(L) \ne \emptyset\}.$$

Let us show that $\operatorname{ex} C_\tau \supset \operatorname{ex}_{X_0} A_\tau$. Choose any point $x \in \operatorname{ex}_{X_0} A_\tau$. Then there exists $Z \subset X_0$ such that Z and $X_0 \setminus A_\tau$ are asymptotically disjoint and $x \in Z'$. Choose any $a > 0$. Since $k_i \to \infty$, there exists $n_0 \in \mathbb{N}$ such that $k_{n_0} \geq a$. Choose any $R > 0$ such that

$$d(Z \setminus B_R(x_0), (X_0 \setminus A_\tau) \setminus B_R(x_0)) \geq a + n_0.$$

Consider any $z \in Z \setminus B_{R+n_0}(x_0)$ and $y \in \bigcup_{i=0}^\infty \{x \in X_i \mid N_i(x) \cap (X_0 \setminus A_\tau) \neq \emptyset\}$. We have $d(z, y) \geq a$. Hence the sets Z and $\bigcup_{i=0}^\infty \{x \in X_i \mid N_i(x) \cap (X_0 \setminus A_\tau) \neq \emptyset\}$ are asymptotically disjoint and $x \in \operatorname{ex} C_\tau$. Analogously we can show that $\operatorname{ex} D_\tau \supset \operatorname{ex}_{X_0} B_\tau$.

Now consider any $x \in C'_\tau \cap D'_\tau$. Then for every neighborhood V of x in the Higson compactification cX there exist two sequences (c_i) in $V \cap C_\tau$ and (d_i) in $V \cap D_\tau$ such that $0 < d(c_i, d_i) \leq r$ for some $r > 0$ and $c_i, d_i \in X \setminus B_i(x_0)$. Choose $n_0 \in \mathbb{N}$ such that $k_{n_0+1} > r$. So, $c_i, d_i \in \bigcup_{k=0}^{n_0} X_k$ and $\emptyset \neq \operatorname{cl} V \cap (\bigcup_{k=0}^{n_0} X_k)' = \operatorname{cl} V \cap X'_0$. Hence $x \in X'_0$. Moreover, $x \in X'_0 \setminus (\operatorname{ex}_{X_0} A_\tau \cup \operatorname{ex}_{X_0} B_\tau) = V_\tau^{X_0}(L)'$.

Define

$$S_k(M) = \bigcup_{i=0}^\infty \{x \in X_i \mid N_{ki}(x) \cap M \neq \emptyset\}$$

for any $M \subset X_0$ and $k \in \mathbb{N}$. The set $S_1(M)$ is denoted simply by $S(M)$. Then

$$\begin{aligned}\nu X \setminus (\operatorname{ex} D_\tau \cup \operatorname{ex} C_\tau) = (X \setminus D_\tau)' \cap (X \setminus C_\tau)' &\subset (X \setminus (D_\tau \cup C_\tau))' \\ &\cup (D'_\tau \cap C'_\tau) \subset (S(V_\tau^{X_0}(L)))'.\end{aligned}$$

Put $K_j = S(B_j(x_0))$ for each $j \in \mathbb{N}$. Then K_j and X_0 are asymptotically disjoint for each $j \in \mathbb{N}$. There exists $\sigma_j \in \omega^\omega$ such that $V_{\sigma_j}(X_0)' \cap K'_j = \emptyset$. Define $\sigma_0 \in \omega^\omega$ as follows: $\sigma_0(0) = 0$ and $\sigma_0(i) = \max\{\sigma_j(i) \mid j \leq i\}$. Then $\sigma_j \leq^* \sigma_0$ for each $j \in \mathbb{N}$. Hence $\operatorname{cl} \bigcup_{j=1}^\infty K'_j \subset \nu X \setminus V_{\sigma_0}(X_0)'$.

Let us show that

$$(*) \qquad \bigcap_{\tau \in \omega^\omega} (S(V_\tau^{X_0}(L)))' \cap V_{\sigma_0}(X_0)' \subset (S_2(L))'.$$

Choose any $x \notin (S_2(L))' \cup (\nu X \setminus V_{\sigma_0}(X_0)')$. Then there exists $Z \subset X$ such that $x \in Z'$ and the sets Z, $S_2(L) \cup (X \setminus V_{\sigma_0}(X_0))$ are asymptotically disjoint. Set $O(Z) = \{y \in X_0 \mid$ there exists $i \in \mathbb{N} \cup \{0\}$ and $x \in X_i \cap Z$ such that $d(y, x) \leq i\}$.

Suppose that there exist $r \in \mathbb{N}$ and two sequences (y_i) in $O(Z)$ and (l_i) in L such that $d(y_i, l_i) \leq r$ and $l_i, y_i \in X_0 \setminus B_i(x_0)$. For each i choose $x_i^{j_i} \in X_{j_i} \cap Z$ such that $d(x_i^{j_i}, y_i) \leq j_i$. Consider two cases:

1. There exists $n_0 \in \mathbb{N}$ such that $j_i \leq n_0$ for each $i \in \mathbb{N}$. Then $d(x_i^{j_i}, l_i) \leq n_0 + r$, contrary to the asymptotic disjointness of Z and $S_2(L)$.

2. In the contrary case we can suppose that $j_i \geq r$ for each $i \in \mathbb{N}$ and $j_i \to \infty$. Then $d(x_i^{j_i}, l_i) \leq j_i + r \leq 2j_i$. Since $x_i^{j_i} \in X_{j_i}$, we have $x_i^{j_i} \in S_2(L) \cap Z$ and we obtain a contradiction again. So, the sets $O(Z)$ and L are asymptotically disjoint.

We can choose $\tau \in \omega^\omega$ such that $O(Z)$ and $V_\tau^{X_0}(L)$ are asymptotically disjoint. Let us show that Z and $S(V_\tau^{X_0}(L))$ are asymptotically disjoint. Suppose the contrary. Then there exist $r \in \mathbb{N}$ and two sequences (z_i) in Z and (s_i) in $S(V_\tau^{X_0}(L))$ such that $d(z_i, s_i) \leq r$ and $z_i, s_i \in X_0 \setminus B_i(x_0)$. Consider two cases:

1. There exists $n_0 \in \mathbb{N}$ such that $z_i, s_i \in \bigcup_{i=0}^{n_0} X_i$. We can choose $y_i \in O(Z)$ and $l_i \in V_\tau^{X_0}(L)$ for each $i \in \mathbb{N}$ such that $d(y_i, z_i) \leq n_0$ and $d(l_i, s_i) \leq n_0$. Then $d(y_i, l_i) \leq 2n_0 + r$, contrary to the asymptotic disjointness of $O(Z)$ and $V_\tau^{X_0}(L)$.

2. We can suppose that $z_i, s_i \notin \bigcup_{j=0}^{n_0} X_j$ where $k_n > r$ for each $n \geq n_0$. Then $z_i = s_i$ for each $i \in \mathbb{N}$ and we can choose $y_i \in O(Z) \cap V_\tau^{X_0}(L)$. Moreover, since Z and $X \setminus V_{\sigma_0}(X_0) \supset \bigcup_{j=1}^{\infty} K_j$ are asymptotically disjoint, we can assume that $d(y_i, x_0) \to \infty$. We obtain a contradiction again and Z and $S(V_\tau^{X_0}(L))$ are asymptotically disjoint. Hence $x \notin (S(V_\tau^{X_0}(L)))'$ and we have proved $(*)$.

Put
$$V_A = \bigcup_{\tau \in \omega^\omega} \operatorname{ex} C_\tau \quad \text{and} \quad V_B = \bigcup_{\tau \in \omega^\omega} \operatorname{ex} D_\tau.$$

Choose $x \in V_A \cap V_B$. Then there exist $\tau_1, \tau_2 \in \omega^\omega$ such that $x \in \operatorname{ex} C_{\tau_1} \cap \operatorname{ex} D_{\tau_2}$. Put $\tau = \max\{\tau_1, \tau_2\}$. Since $\operatorname{ex} C_\tau \cap \operatorname{ex} D_\tau = \emptyset$, we have $x \in (\operatorname{ex} C_{\tau_1} \setminus \operatorname{ex} C_\tau) \cup (\operatorname{ex} D_{\tau_2} \setminus \operatorname{ex} D_\tau)$. Consider the case when $x \in \operatorname{ex} C_{\tau_1} \setminus \operatorname{ex} C_\tau$. Choose $n \in \mathbb{N}$ such that $A_{\tau_1} \setminus A_\tau \subset B_n(x_0)$. Then $x \in S(B_n(x_0))'$. The same holds in the case when $x \in \operatorname{ex} D_{\tau_2} \setminus \operatorname{ex} D_\tau$. Thus $V_A \cap V_B \subset \bigcup_{j=1}^{\infty} K_j' \subset \nu X \setminus V_{\sigma_0}(X_0)'$.

We have $V_A \supset O_A \supset A' \cap X_0'$ and $V_B \supset O_B \supset B' \cap X_0'$. Hence there exists $\tau \in \omega^\omega$ such that $V_A \supset V_\tau(X_0)' \cap A'$ and $V_B \supset V_\tau(X_0)' \cap B'$. Moreover, we can suppose that $(S_2(L) \cap V_\tau(X_0))' \cap (A' \cup B') = \emptyset$ and $\sigma_0 \leq^* \tau$. Then $U_A^1 \cap U_B^2 = \emptyset$ where $U_A^1 = V_A \cap V_\tau(X_0)'$, $U_B^1 = V_B \cap V_\tau(X_0)'$. Thus U_A^1 and U_B^1 are disjoint open subsets of νX such that $U_A^1 \supset V_\tau(X_0)' \cap A'$ and $U_B^1 \supset V_\tau(X_0)' \cap B'$. Since $\operatorname{Ind}(\nu X \setminus V_\tau(X_0)') \leq 0$, there exist open (in $\nu X \setminus V_\tau(X_0)'$) disjoint sets U_A^2 and U_B^2 such that $U_A^2 \supset (\nu X \setminus V_\tau(X_0)') \cap A'$, $U_B^2 \supset (\nu X \setminus V_\tau(X_0)') \cap B'$ and $\nu X \setminus V_\tau(X_0)' = U_A^2 \cup U_B^2$. Since $V_\tau(X_0)'$ is clopen, U_A^2 and U_B^2 are open in νX.

The sets $U_A = U_A^1 \cup U_A^2$ and $U_B = U_B^1 \cup U_B^2$ are open disjoint subsets of νX such that $A' \subset U_A$, $B' \subset U_B$ and $\nu X \setminus (U_A \cup U_B) \subset (S_2(L) \cap V_\tau(X_0))'$.

Thus $S_2(L) \cap V_\tau(X_0)$ is an asymptotic separator in X between A and B. Since L is a kernel of $S_2(L) \cap V_\tau(X_0)$, we have $\operatorname{asInd}(S_2(L) \cap V_\tau(X_0)) < n$ by the inductive assumption and the theorem is proved.

3. In this section we prove the subspace and addition theorems.

THEOREM 2. *Let X be a proper metric space and $Y \subset X$. Then* $\operatorname{asInd} Y \leq \operatorname{asInd} X$.

Proof. We apply induction on $\operatorname{asInd} X$. The case when $\operatorname{asInd} X = -1$ is trivial. The case when $\operatorname{asInd} X = 0$ follows from the equivalence of $\operatorname{asInd} X = 0$ and $\operatorname{Ind} \nu X = 0$ for each proper metric space X (see [1]).

Assume that we have proved the theorem for each X with $\operatorname{asInd} X < n \geq 1$. Consider a proper metric space X with $\operatorname{asInd} X = n$. Suppose that A and B are asymptotically disjoint subsets of Y and C is an asymptotic separator in X between V_A and V_B with $\operatorname{asInd} C < n$ where V_A and V_B are asymptotic neighborhoods of A and B respectively such that V_A and V_B are asymptotically disjoint. We will build an asymptotic separator L in Y between A and B such that $\operatorname{asInd} L < n$ using a construction from [2, Lemma 5.4].

Set $D_0 = C$. Put $Z = Y \setminus (V_A \cup V_B)$. Let

$$D_k = (N_k(D_0) \cap Z) \setminus B_k\Big(\bigcup_{i=0}^{k-1} D_i\Big)$$

for $k \in \mathbb{N}$. There exists a subset L_k of D_k which is k-discrete and for each $x \in D_k$ there exists $y \in L_k$ such that $d(x, y) \leq k$. Put $L = \bigcup_{i=1}^{\infty} L_i$. It was shown in [2] that L is an asymptotic separator in Y between A and B. It is easy to see that C is a kernel in $C \cup L$. Hence $\operatorname{asInd}(C \cup L) < n$ and we have $\operatorname{asInd} L < n$ by the inductive assumption. The theorem is proved.

LEMMA 4. *Let X be a proper metric space and $f : cX \to [0, 1]$ be a continuous function. Then for any $a, b \in [0, 1]$ with $a < b$,*

$$(f^{-1}([a, b]) \cap X)' = (f^{-1}([0, b]) \cap X)' \cap (f^{-1}([a, 1]) \cap X)'.$$

Proof. Choose any $x \in (f^{-1}([0, b]) \cap X)' \cap (f^{-1}([a, 1]) \cap X)'$ and any neighborhood V of x in cX. Then there exist two sequences (a_i), (b_i) in $V \cap X$ such that $f(a_i) \geq a$, $f(b_i) \leq b$, $d(a_i, b_i) \leq r$ for some $r > 0$ and $a_i, b_i \in X \setminus B_i(x_0)$ for each $i \in \mathbb{N}$. We can suppose that $f(a_i) \to c_1$ and $f(b_i) \to c_2$. Since $f|X$ is slowly oscillating, we have $c_1 = c_2 = c \in [a, b]$. Thus $a < c \leq b$ or $a \leq c < b$. Consider the case $a \leq c < b$. There exists $n_0 \in \mathbb{N}$ such that $f(b_n) \in [a, b]$ for each $n \geq n_0$. Hence $\operatorname{cl} V \cap (f^{-1}([a, b]) \cap X)' \neq \emptyset$ and $x \in (f^{-1}([a, b]) \cap X)'$. The proof is analogous in the case $a < c \leq b$. The inclusion $(f^{-1}([a, b]) \cap X)' \subset (f^{-1}([0, b]) \cap X)' \cap (f^{-1}([a, 1]) \cap X)'$ is trivial and the lemma is proved. ■

THEOREM 3. *Let X be a proper metric space and $X = Y \cup Z$ where Y and Z are unbounded sets. Then* $\operatorname{asInd} X \leq \operatorname{asInd} Y + \operatorname{asInd} Z$.

Proof. We apply induction on asInd $= 0$. We have $\operatorname{Ind} Y' = \operatorname{Ind} Z' = 0$ (see [1]). Since $\nu X = Y' \cup Z'$, we have $\operatorname{Ind} \nu X = 0$ [3, Theorem 2.2.7]. Thus $\operatorname{asInd} X = 0$ (see [1]).

Assume that the theorem is proved for each Y and Z with $\operatorname{asInd} Y \leq m \geq 0$, $\operatorname{asInd} Z \leq l \geq 0$ and $m+l < n \geq 1$. Consider the case when $m+l = n$. Let A and B be asymptotically disjoint subsets of X. Then $A' \cap B' = \emptyset$. Choose a continuous function $f : \nu X \to [0,1]$ such that $f(A') \subset \{0\}$ and $f(B') \subset \{1\}$ and extend it to a continuous function $g : cX \to [0,1]$. Then the sets $g^{-1}[0,1/3] \cap X$ and $g^{-1}[2/3,1] \cap X$ are asymptotically disjoint. We can choose an asymptotic separator L_1 in Y between $g^{-1}[0,1/3] \cap Y$ and $g^{-1}[2/3,1] \cap Y$ such that $\operatorname{asInd} L_1 < m$. Put $L_2 = g^{-1}[1/3,2/3] \cap Z$. Then $\operatorname{asInd} L_2 \leq \operatorname{asInd} Z \leq l$. We know that $\operatorname{asInd} L_1 \cup L_2 \leq m - 1 + l < n$ by the inductive assumption.

Let us show that $L_1 \cup L_2$ is an asymptotic separator between A and B. It is easy to see that $K = g^{-1}[1/3,2/3] \cap X$ is an asymptotic separator between A and B in X. Put $V_1 = \operatorname{ex}(g^{-1}[0,1/3) \cap X)$ and $V_2 = \operatorname{ex}(g^{-1}(2/3,1] \cap X)$. Then V_1 and V_2 are disjoint open sets in νX such that $A' \subset V_1$ and $B' \subset U_1$. It follows from Lemma 4 that $\nu X \setminus K' \subset V_1 \cup U_1$. Since L_1 is an asymptotic separator in Y between $g^{-1}[0,1/3] \cap Y$ and $g^{-1}[2/3,1] \cap Y$, there exist two disjoint open sets O_1 and O_2 in Y' such that $O_1 \supset (g^{-1}[0,1/3] \cap Y)'$, $O_2 \supset (g^{-1}[2/3,1] \cap Y)'$ and $Y' \setminus L_1' = O_1 \cup O_2$. Put $V_2 = O_1 \setminus Z'$ and $U_2 = O_2 \setminus Z'$. Then $V = V_1 \cup V_2$ and $U = U_1 \cup U_2$ are open disjoint subsets of νX such that $A' \subset V$, $B' \subset U$ and $\nu X \setminus (L_1 \cup L_2)' \subset V \cup U$. The theorem is proved.

4. In this section we investigate the transfinite extension of asInd. Recall the definition of the transfinite large inductive dimension trInd (see [3]): $\operatorname{trInd} X = -1$ iff $X = \emptyset$; $\operatorname{trInd} X \leq \alpha$ for an ordinal number if for any disjoint closed subsets $A_1, A_2 \subset X$ there is a partition C with $\operatorname{Ind} C \leq \alpha$.

Define the transfinite extension $\operatorname{trasInd} X$ analogously: $\operatorname{trasInd} X = -1$ if and only if X is bounded; $\operatorname{trasInd} X \leq \alpha$ where α is an ordinal number if for any asymptotically disjoint sets $A, B \subset X$ there is an asymptotic separator C with $\operatorname{trasInd} C \leq \beta$ for some $\beta < \alpha$. Naturally we say $\operatorname{trasInd} X = \alpha$ if $\operatorname{trasInd} X \leq \alpha$ and it is not true that $\operatorname{trasInd} X \leq \beta$ for some $\beta < \alpha$. We set $\operatorname{trasInd} X = \infty$ if $\operatorname{trasInd} X \leq \alpha$ for no ordinal number α.

The proof of the following theorem is the same as that of Theorem 2.

THEOREM 4. *Let X be a proper metric space and $Y \subset X$. Then* $\operatorname{trasInd} Y \leq \operatorname{trasInd} X$.

Let X be a proper metric space and $\{A_i \mid i \in \mathbb{N}\}$ a countable family of subsets. We say that this family is *asymptotically discrete* if for each $i \in \mathbb{N}$ the sets A_i and $\bigcup_{j \neq i} A_j$ are asymptotically disjoint. We say that X

is *asymptotically S-like* if it can be represented as the union of a sequence $X_1, X_2, \dots$ of subsets such that $\operatorname{asInd} X_i \geq i$ and the family $\{X_i \mid i \in \mathbb{N}\}$ is asymptotically discrete. The class of all asymptotically S-like spaces is denoted by $\mathcal{S}$.

LEMMA 5. *If a proper metric space X is asymptotically S-like, then* $\operatorname{trasInd} X = \infty$.

Proof. Suppose the contrary. Then there exists $X \in \mathcal{S}$ such that $\operatorname{trasInd} X < \infty$. Put $\xi = \min\{\alpha \mid$ there exists $X_\alpha \in \mathcal{S}$ such that $\operatorname{trasInd} X_\alpha = \alpha\}$. Clearly, $\xi \geq \omega$ where ω is the first infinite ordinal number.

Choose any $X \in \mathcal{S}$ such that $\operatorname{trasInd} X = \xi$. Let us represent X as the union $\bigcup_{i=1}^\infty X_i$ of subsets X_i such that $\operatorname{asInd} X_i \geq i$ and the family $\{X_i \mid i \in \mathbb{N}\}$ is asymptotically discrete. For each $i \in \mathbb{N}$ choose two asymptotically disjoint subsets A_i and B_i of X_i such that for each asymptotic separator L_i in X_i between A_i and B_i we have $\operatorname{trasInd} L_i \geq i - 1$.

We build by induction sets $C_i \subset A_i$ and $D_i \subset B_i$ such that

$$d\Big(C_i, \bigcup_{j=1}^{i} D_j\Big) \geq i \quad \text{and} \quad d\Big(D_i, \bigcup_{j=1}^{i} C_j\Big) \geq i \qquad \text{for each } i \in \mathbb{N}.$$

Since A_1 and B_1 are asymptotically disjoint, there exists $r > 0$ such that $d(C_1, D_1) \geq 1$ where $C_1 = A_1 \setminus B_r(x_0)$ and $D_1 = B_1 \setminus B_r(x_0)$. Suppose we have built C_k and D_k for each $k \leq n \geq 1$. Since $\bigcup_{j=1}^n X_j$ and X_{n+1} are asymptotically disjoint, there exists $r > 0$ such that

$$d\Big(\Big(\bigcup_{j=1}^{n} X_j\Big) \setminus B_r(x_0), X_{n+1} \setminus B_r(x_0)\Big) \geq n + 1.$$

Since A_{n+1} and B_{n+1} are asymptotically disjoint, there exists $t > 0$ such that

$$d(A_{n+1} \setminus B_r(x_0), B_{n+1} \setminus B_r(x_0)) \geq n + 1.$$

Put $s = \max\{r, t\}$ and $C_{n+1} = A_{n+1} \setminus B_s(x_0)$, $D_{n+1} = B_{n+1} \setminus B_s(x_0)$. It is easy to see that the sets $C = \bigcup_{i=1}^\infty C_i$ and $D = \bigcup_{i=1}^\infty D_i$ are asymptotically disjoint and for each asymptotic separator L_i in X_i between C_i and D_i we have $\operatorname{trasInd} L_i \geq i - 1$.

Choose any asymptotic separator L in X between C and D such that $\operatorname{trasInd} L < \xi$. Since the family $\{X_i\}$ is asymptotically discrete, $L_i = L \cap X_i$ is an asymptotic separator in X_i between C_i and D_i for each $i \in \mathbb{N}$. Hence $\operatorname{asInd} L_i \geq i - 1$. Thus, $L \in \mathcal{S}$ and we have a contradiction. The lemma is proved.

LEMMA 6. *Let X be a proper metric space such that* $\operatorname{trasInd} X < \infty$. *Then for each $x \in \nu X$ there exists a neighborhood V of x in cX such that* $\operatorname{asInd} V \cap X < \infty$.

Proof. Suppose the contrary. Then there exists $x \in \nu X$ such that $\operatorname{asInd} V \cap X = \infty$ for each neighborhood V of x in cX. Let us build by induction a sequence (L_i) of subsets of X and a sequence (V_i) of neighborhoods of x in cX such that L_k and $V_k \cap X$ are asymptotically disjoint for each $k \in \mathbb{N}$, $L_{n+1} \subset V_n$, $\operatorname{asInd} L_n = \infty$ and $V_1 \supset V_2 \supset \dots$.

We have, in particular, $\operatorname{asInd} X = \infty$. There exist two asymptotically disjoint subsets $A, B \subset X$ such that $\operatorname{asInd} L = \infty$ for each asymptotic separator L between A and B. We can assume that $x \notin B'$. Choose a continuous function $f : \nu X \to [0, 1]$ such that $f(x \cup A') \subset \{0\}$ and $f(B') \subset \{1\}$. Choose an asymptotic separator L_1 between $f^{-1}[0, 1/3] \cap X$ and $f^{-1}[2/3, 1] \cap X$ and put $V_1 = f^{-1}[0, 1/3] \cap X$.

Assume we have built L_i and V_i for each $i \le n \ge 1$. Then $\operatorname{trasInd} V_n = \infty$. So, there exist two asymptotically disjoint subsets C and D of V_n such that $\operatorname{asInd} L = \infty$ for each asymptotic separator L between C and D. We can choose L_{n+1} and V_{n+1} as before. The sequences (L_i) and (V_i) are built. It is easy to check that the family $\{L_i\}$ is asymptotically discrete. Put $L = \bigcup_{i=1}^{\infty} L_i$. So, $\operatorname{trasInd} L = \infty$ by Lemma 5 and $\operatorname{trasInd} X = \infty$ by Theorem 4.

THEOREM 5. *Let X be a proper metric space such that* $\operatorname{trasInd} X < \infty$. *Then* $\operatorname{asInd} X < \infty$.

Proof. For each $x \in \nu X$ we can choose a neighborhood V_x in cX such that $\operatorname{asInd} V_x \cap X < \infty$. Since νX is compact, we have $\nu X \subset \bigcup_{i=1}^{k} V_i = V$ where each V_i is an open set in cX with $\operatorname{asInd} V_i \cap X < \infty$. Then $\operatorname{asInd} V \cap X < \infty$ by Theorem 3. Moreover, there exists $r > 0$ such that $X \subset V \cup B_r(x_0)$. Hence $\operatorname{asInd} X < \infty$.

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