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SCATTERING THEORY FOR A NONLINEAR SYSTEM OF WAVE EQUATIONS WITH CRITICAL GROWTH

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Abstract. We consider scattering properties of the critical nonlinear system of wave equations with Hamilton structure

$$\begin{cases} u_{tt} - \Delta u = -F_1(|u|^2, |v|^2)u, \\ v_{tt} - \Delta v = -F_2(|u|^2, |v|^2)v, \end{cases}$$

for which there exists a function $F(\lambda, \mu)$ such that

$$\frac{\partial F(\lambda,\mu)}{\partial \lambda} = F_1(\lambda,\mu), \qquad \frac{\partial F(\lambda,\mu)}{\partial \mu} = F_2(\lambda,\mu).$$

By using the energy-conservation law over the exterior of a truncated forward light cone and a dilation identity, we get a decay estimate for the potential energy. The resulting global-in-time estimates imply immediately the existence of the wave operators and the scattering operator.

1. Introduction. In this note, we continue our study from [3, 4] on the following nonlinear system of wave equations with Hamilton structure:

$$(1.1)\qquad \begin{cases} u_{tt} - \Delta u = -F_1(|u|^2, |v|^2)u, \\ v_{tt} - \Delta v = -F_2(|u|^2, |v|^2)v, \\ u(0) = \varphi_1(x), \quad u_t(0) = \psi_1(x), \\ v(0) = \varphi_2(x), \quad v_t(0) = \psi_2(x), \\ (\varphi_j, \psi_j) \in \dot{H}^1 \times L^2, \quad j = 1, 2, \end{cases}$$

where we assume the existence of a function $F(\lambda, \mu)$ such that

$$\frac{\partial F(\lambda,\mu)}{\partial \lambda} = F_1(\lambda,\mu), \qquad \frac{\partial F(\lambda,\mu)}{\partial \mu} = F_2(\lambda,\mu).$$

To ensure that the potential energy of problem (1.1) tends to zero as $t \to \infty$, which will play an important role in the proof of our result, we need to assume that F, F_1, F_2 satisfy the following assumptions similar to those

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in [3, 4]:

(H1) $$|F_1| + |u^2 F_{11}| + |uvF_{12}| + |F_2| + |uvF_{21}| + |v^2 F_{22}| \le C(|u|^{2^*-2} + |v|^{2^*-2}),$$

where $F_{11} = \partial F_1/\partial\lambda,\ F_{12} = \partial F_1/\partial\mu,\ F_{21} = \partial F_2/\partial\lambda,\ F_{22} = \partial F_2/\partial\mu$;

(H2) $$F(|u|^2, |v|^2) \ge 0, \qquad F(0,0) = 0;$$

(H3) $$|u|^{2^*} + |v|^{2^*} \le C_0 F(|u|^2, |v|^2);$$

(H4) $$\frac{n-1}{2}\,|u|^2 F_1(|u|^2, |v|^2) + \frac{n-1}{2}\,|v|^2 F_2(|u|^2, |v|^2) \ge \frac{n+1}{2}\,F(|u|^2, |v|^2)$$

for $|u|$ or $|v|$ larger than a fixed constant M;

(H5) $$\begin{aligned}&|F_1(|u_1|^2, |v_1|^2)u_1 - F_1(|u_2|^2, |v_2|^2)u_2| \\ &\quad + |F_2(|u_1|^2, |v_1|^2)v_1 - F_2(|u_2|^2, |v_2|^2)v_2| \\ &\le C(|u_1|^{2^*-2} + |v_1|^{2^*-2} + |u_2|^{2^*-2} + |v_2|^{2^*-2})(|u_1 - u_2| + |v_1 - v_2|).\end{aligned}$$

Note that (H1) and (H2) imply an inequality which is reverse to (H3):

$$F(|u|^2, |v|^2) \le C(|u|^{2^*} + |v|^{2^*}). \tag{1.2}$$

It is easy to verify that e.g. the function $F(|u|^2, |v|^2) = |u|^6 + |u|^4|v|^2 + |u|^2|v|^4 + |v|^6$ satisfies (H1)–(H5) in the space dimension $n = 3$. For the physical background and related research on the wave equation, we refer the reader to [1, 3–7] and the references therein.

Let us end this section by recalling what we have done in our previous papers. On the basis of a dilation identity derived through the Lagrangian associated with problem (1.1), we prove in [3] that the "potential energy" cannot concentrate at any given point. We combine this fact with the Strichartz estimate to improve the regularity of a solution with finite-energy initial data. That reasoning is completed by standard energy estimates.

In [4], we study the well-posedness of problem (1.1) in the energy space under assumptions on nonlinearities slightly more general than those in (H1)–(H5). By showing through an approximation argument that the energy and the dilation identities hold true for weak solutions, we prove that problem (1.1) has a unique solution (u, v) such that

$$\begin{aligned}(u, v, u_t, v_t) &\in C(\mathbb{R}; \dot H^1 \times \dot H^1 \times L^2 \times L^2) \\ &\quad \cap L^q_{\mathrm{loc}}(\mathbb{R}; \dot B^{1/2}_q \times \dot B^{1/2}_q \times \dot B^{-1/2}_q \times \dot B^{-1/2}_q).\end{aligned} \tag{1.3}$$

Here, the Besov space $\dot B^s_{p,q}$ is defined as the set of those functions for which the following norm is finite:

$$\|f\|_{\dot B^s_{p,q}} \equiv \left\{ \int_0^\infty \sup_{|y|\le t} [t^{-s}\|\tau_y f - f\|_{L^p}]^q \,\frac{dt}{t} \right\}^{1/q},$$

where τ_y denotes the space translation by $y \in \mathbb{R}^n$ (cf. [2, p. 493, eq. (3.15)]). We limit ourselves to the particular case of this space for $p = q$ and we write $\dot{B}^s_q(\mathbb{R}^n) = \dot{B}^s_{q,q}(\mathbb{R}^n) \cap L^{q^*}(\mathbb{R}^n)$ for $q = 2(n+1)/(n-1)$ and $q^* = 2n(n+1)/(n^2-2n-1)$.

2. Global space-time estimate. Our first goal is to improve the result from [4] and to obtain global-in-time estimates of solutions to (1.1).

THEOREM 2.1. *Assume that* F, F_1, F_2 *satisfy* (H1)–(H5). *Then problem* (1.1) *has a unique solution satisfying*

$$(u, v, u_t, v_t) \in C(\mathbb{R}; \dot{H}^1 \times \dot{H}^1 \times L^2 \times L^2) \cap L^q(\mathbb{R}; \dot{B}^{1/2}_q \times \dot{B}^{1/2}_q \times \dot{B}^{-1/2}_q \times \dot{B}^{-1/2}_q), \tag{2.1}$$

where $\dot{B}^s_q(\mathbb{R}^n) = \dot{B}^s_{q,q}(\mathbb{R}^n) \cap L^{q^*}(\mathbb{R}^n)$ *with* $q = 2(n+1)/(n-1)$ *and* $q^* = 2n(n+1)/(n^2-2n-1)$.

Note that, by [4], problem (1.1) has a unique solution satisfying (1.3). Hence, to prove Theorem 2.1, we only need to verify that there exists $T_0 > 0$ such that for $I = [T_0, \infty)$, the following quantities are finite:

$$\begin{aligned} &\|u\|_{L^q(I;\dot{B}^{1/2}_q(\mathbb{R}^n))}, \qquad \|v\|_{L^q(I;\dot{B}^{1/2}_q(\mathbb{R}^n))}, \\ &\|u_t\|_{L^q(I;\dot{B}^{-1/2}_q(\mathbb{R}^n))}, \qquad \|v_t\|_{L^q(I;\dot{B}^{-1/2}_q(\mathbb{R}^n))}. \end{aligned} \tag{2.2}$$

As we shall see below, to this end, we should first prove that $\|u(t)\|_{L^{2^*}(\mathbb{R}^n)}$ and $\|v(t)\|_{L^{2^*}(\mathbb{R}^n)}$ tend to zero as $t \to \infty$. However, it follows from our assumptions (1.2) and (H3) that it suffices to show the following result.

PROPOSITION 2.2. *Let* (u, v) *be a solution of* (1.1), *and let* F, F_1, F_2 *satisfy* (H2)–(H4). *Then*

$$g(t) = \lim_{t\to\infty} \frac{1}{2} \int_{\mathbb{R}^n} F(|u(x,t)|^2, |v(x,t)|^2)\, dx = 0. \tag{2.3}$$

Proof. Since the initial data have finite energy, we obtain

$$\int_{|x|\ge R} e(u,v)(x,0)\, dx \to 0 \quad \text{as } R \to \infty, \tag{2.4}$$

where

$$e(u,v) = \frac{1}{2}\,(|u_t|^2 + |v_t|^2 + |\nabla u|^2 + |\nabla v|^2 + F). \tag{2.5}$$

Applying the energy conservation law on the exterior of a truncated forward light cone, for every $t \ge 0$ one gets

$$\int_{|x|>R+t} e(u,v)\, dx + \mathrm{Flux}(u, v; M^t_0) \to 0 \quad \text{as } R \to \infty, \tag{2.6}$$

where the Flux on the mantle is given by (cf. [7, p. 137])

$$
\begin{aligned}
(2.7)\quad &\mathrm{Flux}(u, v; M_a^b) \\
&= \frac{1}{\sqrt{2}} \int\limits_{M_a^b} \left\{ (-u_t \nabla u - v_t \nabla v) \cdot \frac{-x}{|x|} + e(u,v) \times 1 \right\} d\sigma \\
&= \frac{1}{\sqrt{2}} \int\limits_{M_a^b} \left\{ \frac{1}{2} \left| \frac{x}{|x|}\, u_t + \nabla u \right|^2 + \frac{1}{2} \left| \frac{x}{|x|}\, v_t + \nabla v \right|^2 + \frac{1}{2}\, F \right\} d\sigma
\end{aligned}
$$

with

$$M_a^b = \{(x,t) \in \mathbb{R}^n \times [a,b] : |x| = R + t\}. \tag{2.8}$$

By identity (2.7), the Flux is nonnegative. Since $e(u,v)$ contains the potential energy term $\frac{1}{2}F$, it follows from (2.5)–(2.7) that

$$
\begin{aligned}
\frac{1}{2} \int\limits_{|x|>R+t} F\, dx &\le \int\limits_{|x|>R+t} e(u,v)\, dx \\
&\le \int\limits_{|x|>R+t} e(u,v)\, dx + \mathrm{Flux}(u, v; M_0^t) \to 0 \quad \text{ as } R \to \infty.
\end{aligned}
$$

Therefore, to complete the proof of Proposition 2.2, it suffices to show that

$$\frac{1}{2} \int\limits_{|x|\le R+t} F dx \to 0 \quad \text{ as } t \to \infty. \tag{2.9}$$

If we replace t by $t+R$, (2.6), (2.8) and (2.9) can be rewritten as

$$\int\limits_{|x|>t} e(u,v)\, dx + \mathrm{Flux}(u, v; M_R^t) \to 0 \quad \text{ as } R \to \infty, \tag{2.6$'$}$$

$$M_a^b = \{(x,t) \in \mathbb{R}^n \times [a,b] : |x| = t\}, \tag{2.8$'$}$$

$$\frac{1}{2} \int\limits_{|x|\le t} F\, dx \to 0, \qquad t \to \infty. \tag{2.9$'$}$$

To prove (2.9′), we use the following dilation identity obtained in [3, 4]:

$$\mathrm{div}_{x,t} \left(-tP_0, tQ_0 + \frac{n-1}{2}\, u_t u + \frac{n-1}{2}\, v_t v \right) - R_0 = 0, \tag{2.10}$$

where

$$
\begin{aligned}
Q_0 &= \frac{1}{2}\, |u'|^2 + \frac{1}{2}\, |v'|^2 + \frac{1}{2}\, F + u_t\, \frac{x \cdot \nabla u}{t} + v_t\, \frac{x \cdot \nabla v}{t}, \\
P_0 &= \left(\frac{1}{2}\, |u_t|^2 + \frac{1}{2}\, |v_t|^2 - \frac{1}{2}\, |\nabla u|^2 - \frac{1}{2}\, |\nabla v|^2 - \frac{1}{2}\, F \right) \frac{x}{t} \\
&\quad + \left(\frac{n-1}{2}\, \frac{u}{t} + u_t + \frac{x \cdot \nabla u}{t} \right) \nabla u + \left(\frac{n-1}{2}\, \frac{v}{t} + v_t + \frac{x \cdot \nabla v}{t} \right) \nabla v \\
R_0 &= \frac{n-1}{2}\, F_1 |u|^2 + \frac{n-1}{2}\, F_2 |v|^2 - \frac{n+1}{2}\, F.
\end{aligned}
$$

Integrating identity (2.10) over $K(T,S)=\{(x,t)\in\mathbb{R}^n\times[T,S]:T\le t\le S,$ $|x|<t\}$, we obtain

$$(2.11)\qquad 0=\int\limits_{D_S}\Big(SQ_0+\frac{n-1}{2}\,u_tu+\frac{n-1}{2}\,v_tv\Big)\,dx$$
$$-\int\limits_{D_T}\Big(TQ_0+\frac{n-1}{2}\,u_tu+\frac{n-1}{2}\,v_tv\Big)\,dx$$
$$-\frac{1}{\sqrt{2}}\int\limits_{M_T^S}\Big(tQ_0+\frac{n-1}{2}\,u_tu+\frac{n-1}{2}\,v_tv+x\cdot P_0\Big)\,d\sigma$$
$$+\int\int\limits_{K(T,S)}R_0\,dx\,dt\equiv \mathrm{I}_1+\mathrm{I}_2+\mathrm{I}_3+\mathrm{I}_4,$$

where

$$D_T=\{(x,t):|x|\le T\},\qquad M_T^S=\{(x,t):T\le t\le S,\ |x|=t\}.$$

Note that $t=|x|$ on M_T^S, hence we rewrite the term I_3 in (2.11) as

$$(2.12)\qquad \mathrm{I}_3=-\frac{1}{\sqrt{2}}\int\limits_{M_T^S}\Big(tQ_0+\frac{n-1}{2}\,u_tu+\frac{n-1}{2}\,v_tv+x\cdot P_0\Big)\,d\sigma$$
$$=-\frac{1}{\sqrt{2}}\int\limits_{M_T^S}\Big(\frac{|x|}{2}\,|u'|^2+\frac{|x|}{2}\,|v'|^2+\frac{|x|}{2}\,F+u_tx\cdot\nabla u+v_tx\cdot\nabla v+\frac{n-1}{2}\,u_tu$$
$$+\frac{n-1}{2}\,v_tv+\frac{n-1}{2}\,\frac{u}{|x|}\,x\cdot\nabla u+u_tx\cdot\nabla u+\frac{1}{|x|}\,(x\cdot\nabla u)^2$$
$$+\frac{n-1}{2}\,\frac{v}{|x|}\,x\cdot\nabla v+v_tx\cdot\nabla v+\frac{1}{|x|}\,(x\cdot\nabla v)^2$$
$$-\frac{|x|}{2}\,|\nabla u|^2-\frac{|x|}{2}\,|\nabla v|^2+\frac{|x|}{2}\,|u_t|^2+\frac{|x|}{2}\,|v_t|^2-\frac{|x|}{2}\,F\Big)\,d\sigma$$
$$=-\frac{1}{\sqrt{2}}\int\limits_{M_T^S}\Big(|x|\,|u_t|^2+|x|\,|v_t|^2+2u_tx\cdot\nabla u+2u_tx\cdot\nabla v+\frac{n-1}{2}\,u_tu$$
$$+\frac{n-1}{2}\,v_tv+\frac{1}{|x|}\,(x\cdot\nabla u)^2+\frac{1}{|x|}\,(x\cdot\nabla v)^2$$
$$+\frac{n-1}{2}\,\frac{u}{|x|}\,x\cdot\nabla u+\frac{n-1}{2}\,\frac{v}{|x|}\,x\cdot\nabla v\Big)\,d\sigma$$
$$=-\frac{1}{\sqrt{2}}\int\limits_{M_T^S}\Big[|x|\Big(\frac{x\cdot\nabla u}{|x|}+u_t\Big)^2+\frac{n-1}{2}\,u\Big(\frac{x\cdot\nabla u}{|x|}+u_t\Big)$$
$$+|x|\Big(\frac{x\cdot\nabla v}{|x|}+v_t\Big)^2+\frac{n-1}{2}\,v\Big(\frac{x\cdot\nabla v}{|x|}+v_t\Big)\Big]\,d\sigma,$$

where $|u'|^2 = |\nabla u|^2 + |u_t|^2$. If we parameterize M_T^S by $y \mapsto (y, |y|)$ and set $\overline{u}(y) = u(y, |y|), \overline{v}(y) = v(y, |y|)$, then

$$\begin{aligned} d\sigma &= \sqrt{2}\, dy, \\ \overline{u}_r &\equiv y \cdot \frac{\nabla \overline{u}}{|y|} = \frac{x \cdot \nabla u}{|x|} + u_t = u_r + u_t, \\ \overline{v}_r &\equiv y \cdot \frac{\nabla \overline{v}}{|y|} = \frac{x \cdot \nabla v}{|x|} + v_t = v_r + v_t \end{aligned}$$

where $\nabla \overline{u} = \sum_{j=0}^n \partial_j \overline{u}$ and $\nabla u = \sum_{j=1}^n \partial_j u$. Therefore

$$\begin{aligned} \mathrm{I}_3 = &- \int_T^S \int_{\Sigma^{n-1}} \left(r\overline{u}_r^2 + \frac{n-1}{2}\, \overline{u}\, \overline{u}_r + r\overline{v}_r^2 + \frac{n-1}{2}\, \overline{v}\, \overline{v}_r \right) r^{n-1}\, dr\, d\sigma(\omega) \\ = &- \int_T^S \int_{\Sigma^{n-1}} r \left(\left| \overline{u}_r + \frac{n-1}{2r}\, \overline{u} \right|^2 + \left| \overline{v}_r + \frac{n-1}{2r}\, \overline{v} \right|^2 \right) r^{n-1}\, dr\, d\sigma(\omega) \\ &+ \int_T^S \int_{\Sigma^{n-1}} \frac{n-1}{2} (\overline{u}\, \overline{u}_r + \overline{v}\, \overline{v}_r) r^{n-1}\, dr\, d\sigma(\omega) \\ &+ \int_T^S \int_{\Sigma^{n-1}} \frac{(n-1)^2}{4} (\overline{u}^2 + \overline{v}^2) r^{n-2}\, dr\, d\sigma(\omega). \end{aligned} \tag{2.13}$$

Note that

$$\begin{aligned} &\int_T^S \int_{\Sigma^{n-1}} \frac{n-1}{2}\, \overline{u}\, \overline{u}_r r^{n-1}\, dr\, d\sigma(\omega) \\ &= \frac{1}{2} \int_{\Sigma^{n-1}} \int_T^S \frac{n-1}{2}\, \partial_r (\overline{u}^2(r\omega)) r^{n-1}\, dr\, d\sigma(\omega) \\ &= \frac{1}{2} \int_{\Sigma^{n-1}} \frac{n-1}{2}\, \overline{u}^2(S\omega) S^{n-1}\, d\sigma(\omega) - \frac{1}{2} \int_{\Sigma^{n-1}} \frac{n-1}{2}\, \overline{u}^2(T\omega) T^{n-1}\, d\sigma(\omega) \\ &\quad - \left(\frac{n-1}{2} \right)^2 \int_{\Sigma^{n-1}} \int_T^S \overline{u}^2(r\omega) r^{n-2}\, dr\, d\sigma(\omega) \\ &= \frac{n-1}{4} \int_{\partial D_S} u^2\, d\sigma - \frac{n-1}{4} \int_{\partial D_T} u^2\, d\sigma \\ &\quad - \frac{(n-1)^2}{4} \int_{\Sigma^{n-1}} \int_T^S \overline{u}^2(r\omega) r^{n-2}\, dr\, d\sigma(\omega), \end{aligned}$$

and

$$\int_T^S \int_{\Sigma^{n-1}} \frac{n-1}{2}\,\overline{v}\overline{v}_r r^{n-1}\,dr\,d\sigma(\omega) = \frac{n-1}{4}\int_{\partial D_S} v^2\,d\sigma - \frac{n-1}{4}\int_{\partial D_T} v^2\,d\sigma$$
$$- \frac{(n-1)^2}{4}\int_{\Sigma^{n-1}}\int_T^S \overline{v}^2(r\omega) r^{n-2}\,dr\,d\sigma(\omega).$$

Hence, the expression in (2.13) reduces to

$$\text{(2.14)}\quad \mathrm{I}_3 = -\int_T^S \int_{\Sigma^{n-1}} r\left(\left|\overline{u}_r + \frac{n-1}{2r}\overline{u}\right|^2 + \left|\overline{v}_r + \frac{n-1}{2r}\overline{v}\right|^2\right) r^{n-1}\,dr\,d\sigma(\omega)$$
$$+ \frac{n-1}{4}\int_{\partial D_S}(u^2+v^2)\,d\sigma - \frac{n-1}{4}\int_{\partial D_T}(u^2+v^2)\,d\sigma.$$

Next using the fact that $|\nabla\mu|^2 - \mu_r^2 = |\nabla_\omega \mu|^2/r^2$, we obtain

$$\text{(2.15)}\quad \mathrm{I}_1 = \int_{D_S}\left(SQ_0 + \frac{n-1}{2}\,u_t u + \frac{n-1}{2}\,v_t v\right)dx$$
$$= \int_{D_S}\left\{S\left[\frac{1}{2}\,|u_t|^2 + \frac{1}{2}\left(u_r + \frac{n-1}{2r}\,u\right)^2 + \frac{1}{2r^2}\,|\nabla_\omega u|^2\right.\right.$$
$$\left. + \frac{1}{2}\,|v_t|^2 + \frac{1}{2}\left(v_r + \frac{n-1}{2r} v\right)^2 + \frac{1}{2r^2}\,|\nabla_\omega v|^2 + \frac{1}{2}\,F\right]$$
$$\left. + r\left(u_r + \frac{n-1}{2r}\,u\right)u_t + r\left(v_r + \frac{n-1}{2r}\,v\right)v_t\right\}dx$$
$$- \frac{n-1}{4}\int_{\partial D_S}(u^2+v^2)\,d\sigma + \frac{(n-1)(n-3)}{8}\int_{D_S} S\,\frac{|u|^2+|v|^2}{r^2}\,dx.$$

Similarly, we have

$$\text{(2.16)}\quad \mathrm{I}_2 = -\int_{D_T}\left(TQ_0 + \frac{n-1}{2}\,u_t u + \frac{n-1}{2}\,v_t v\right)dx$$
$$= -\int_{D_T}\left\{T\left[\frac{1}{2}\,|u_t|^2 + \frac{1}{2}\left(u_r + \frac{n-1}{2r}\,u\right)^2 + \frac{1}{2r^2}\,|\nabla_\omega u|^2\right.\right.$$
$$\left. + \frac{1}{2}\,|v_t|^2 + \frac{1}{2}\left(v_r + \frac{n-1}{2r}\,v\right)^2 + \frac{1}{2r^2}\,|\nabla_\omega v|^2 + \frac{1}{2}F\right]$$
$$\left. + r\left(u_r + \frac{n-1}{2r}\,u\right)u_t + r\left(v_r + \frac{n-1}{2r}\,v\right)v_t\right\}dx$$
$$+ \frac{n-1}{4}\int_{\partial D_T}(u^2+v^2)\,d\sigma - \frac{(n-1)(n-3)}{8}\int_{D_T} T\,\frac{|u|^2+|v|^2}{r^2}\,dx.$$

Finally, assumption (H4) means $\mathrm{I}_4 \geq 0$.

Now, let $T = \varepsilon S$ for some $0 < \varepsilon < 1$. Substituting (2.14)–(2.16) into (2.11) and using Hardy's inequality

$$\int \frac{|\mu|^2}{|x|^2}\,dx \le C\int |\nabla \mu|^2\,dx$$

we deduce that

$$\begin{aligned}(2.17)\qquad S\int_{D_S}\frac{1}{2}F\,dx &\le C\varepsilon S E_0\\ &+\int_{\varepsilon S}^{S}\int_{\Sigma^{n-1}} r\left(\left|\overline{u}_r+\frac{n-1}{2r}\,\overline{u}\right|^2+\left|\overline{v}_r+\frac{n-1}{2r}\,\overline{v}\right|^2\right)r^{n-1}\,dr\,d\sigma(\omega).\end{aligned}$$

Observe that by direct computation, we have

$$\begin{aligned}&\int_{\varepsilon S}^{S}\int_{\Sigma^{n-1}} r\left(\left|\overline{u}_r+\frac{n-1}{2r}\,\overline{u}\right|^2+\left|\overline{v}_r+\frac{n-1}{2r}\,\overline{v}\right|^2\right)r^{n-1}\,dr\,d\sigma(\omega)\\
&\qquad=\frac{1}{\sqrt{2}}\int_{M^S_{\varepsilon S}} r\left(\left|u_r+u_t+\frac{n-1}{2r}\,u\right|^2+\left|v_r+v_t+\frac{n-1}{2r}\,v\right|^2\right)d\sigma\\
&\qquad\le\sqrt{2}\int_{M^S_{\varepsilon S}} r(|u_r+u_t|^2+|v_r+v_t|^2)\,d\sigma\\
&\qquad\quad+\frac{2}{\sqrt{2}}\left(\frac{n-1}{2}\right)^2\int_{M^S_{\varepsilon S}} r\left(\left|\frac{u}{r}\right|^2+\left|\frac{v}{r}\right|^2\right)d\sigma\\
&\qquad\le\sqrt{2}\,S\int_{M^S_{\varepsilon S}}\left(\left|\frac{x}{|x|}\,u_t+\nabla u\right|^2+\left|\frac{x}{|x|}\,v_t+\nabla v\right|^2\right)d\sigma\\
&\qquad\quad+\frac{(n-1)^2}{2\sqrt{2}}\int_{M^S_{\varepsilon S}}\left(\frac{u^2}{|x|}+\frac{v^2}{|x|}\right)d\sigma\\
&\qquad\equiv \mathrm{I}+\mathrm{II}.\end{aligned}$$

It is easy to see (cf. equation (2.7)) that

$$\begin{aligned}(2.18)\qquad \mathrm{I}&=\sqrt{2}\,S\int_{M^S_{\varepsilon S}}\left(\left|\frac{x}{|x|}\,u_t+\nabla u\right|^2+\left|\frac{x}{|x|}\,v_t+\nabla v\right|^2\right)d\sigma\\ &\le CS[\mathrm{Flux}(u,v;M^S_{\varepsilon S})],\end{aligned}$$

and

$$\text{(2.19)}\qquad \mathrm{II} = \frac{(n-1)^2}{2\sqrt{2}} \int\limits_{M^S_{\varepsilon S}} \left(\frac{u^2}{t} + \frac{v^2}{t}\right) d\sigma = \frac{(n-1)^2}{2\sqrt{2}} \Big(\int\limits_{M^S_{\varepsilon S}} t^{-n/2}\, d\sigma\Big)^{2/n}$$
$$\times \Big\{ \Big(\int\limits_{M^S_{\varepsilon S}} u^{2^*}\, d\sigma\Big)^{(n-2)/n} + \Big(\int\limits_{M^S_{\varepsilon S}} v^{2^*}\, d\sigma\Big)^{(n-2)/n} \Big\}$$
$$\le C\Big(\int\limits_0^S \int\limits_{\Sigma^{n-1}} t^{-n/2} t^{n-1}\, dt\, d\sigma(\omega)\Big)^{2/n} \Big(\int\limits_{M^S_{\varepsilon S}} (u^{2^*} + v^{2^*})\, d\sigma\Big)^{(n-2)/n}$$
$$\le CS\Big\{\int\limits_{M^S_{\varepsilon S}} \frac{F}{2}\, d\sigma\Big\}^{(n-2)/n} \le CS[\mathrm{Flux}(u,v;M^S_{\varepsilon S})]^{(n-2)/n}.$$

Substituting estimates (2.18) and (2.19) into (2.17) and dividing by S, we obtain

$$\text{(2.20)}\qquad \int\limits_{D_S} \frac{1}{2} F\, dx \le C\varepsilon E_0 + C[\mathrm{Flux}(u,v;M^S_{\varepsilon S})] + C[\mathrm{Flux}(u,v;M^S_{\varepsilon S})]^{(n-2)/n}.$$

From (2.6′), letting $S \to \infty$ and then $\varepsilon \to 0$, we get (2.9′). ∎

Assumption (H3) and Proposition 2.2 immediately imply the following result.

PROPOSITION 2.3. *Let* (u,v) *be a solution of* (1.1), *and let* F, F_1, F_2 *satisfy* (H2)–(H4). *Then*

$$\lim_{|t|\to\infty} \int\limits_{\mathbb{R}^n} (|u(x,t)|^{2^*} + |v(x,t)|^{2^*})\, dx = 0. \ ∎$$

Proof of Theorem 2.1. We ought to show that $u, v \in L^q([T_0,\infty); \dot{B}^{1/2}_q)$ for some T_0. By Proposition 2.3, for any fixed $\varepsilon_0 > 0$ one can choose T_0 such that

$$\int\limits_{\mathbb{R}^n} (|u(x,t)|^{2^*} + |v(x,t)|^{2^*})\, dx \le \varepsilon_0, \qquad \forall t > T_0.$$

As in [3, proof of Proposition 3.1], for every $T > T_0$, we can derive the inequalities

$$\|u\|_{q,T_0,T} + \|v\|_{q,T_0,T} \le CE_0^{1/2} + C \sup_{T_0\le t\le T} \|u\|^\beta_{L^{2^*}(\mathbb{R}^n)} \|u\|^\gamma_{q,T_0,T}$$
$$+ C \sup_{T_0\le t\le T} \|v\|^\beta_{L^{2^*}(\mathbb{R}^n)} \|v\|^\gamma_{q,T_0,T}$$
$$\le CE_0^{1/2} + C\varepsilon_0^{\beta/2^*} (\|u\|^\gamma_{q,T_0,T} + \|v\|^\gamma_{q,T_0,T})$$
$$\le CE_0^{1/2} + C\varepsilon_0^{\beta/2^*} (\|u\|_{q,T_0,T} + |v\|_{q,T_0,T})^\gamma,$$

where $\|u\|_{q,T_0,T} = (\int_{T_0}^T \|u(t)\|^q_{\dot{B}^{1/2}_q}\,dt)^{1/q}$ and

$$\beta = (1-\alpha)(2^*-2) > 0, \qquad \gamma = \alpha(2^*-2)+1 > 1, \qquad \alpha = (n-2)/(n-1).$$

For ε_0 sufficiently small, the above inequality implies

$$\|u\|_{q,T_0,T} + \|v\|_{q,T_0,T} \le 2CE_0$$

for all $T > T_0$. Letting $T \to \infty$ we complete the proof of Theorem 2.1. ∎

3. Scattering theory. As we have proved the global-in-time existence of solutions to problem (1.1), the following questions arise. What is the asymptotic behavior of the solution (u,v) as $t \to \pm\infty$? Does it converge to a solution of the corresponding free system

$$\begin{cases} u_{tt} - \Delta u = 0, \\ v_{tt} - \Delta v = 0, \end{cases} \tag{3.1}$$

in the sense of $\dot{H}^1 \times \dot{H}^1$ norm? These questions will be discussed in this section; in other words, we will construct the scattering operator for problem (1.1) and we shall study its properties.

For simplicity of exposition, let $(u^\pm, v^\pm)$ be the solutions of system (3.1) with the initial data $(\varphi_1^\pm, \varphi_2^\pm, \psi_1^\pm, \psi_2^\pm)$, respectively. We also denote by (u,v) the solution to problem (1.1) with the initial data $(\varphi_1, \varphi_2, \psi_1, \psi_2)$.

DEFINITION.

(a) If for any $(\varphi_1^\pm, \varphi_2^\pm, \psi_1^\pm, \psi_2^\pm) \in X = \dot{H}^1 \times \dot{H}^1 \times L^2 \times L^2(\mathbb{R}^n)$ there exists $(\varphi_1, \varphi_2, \psi_1, \psi_2) \in X$ such that

$$\|(u, v, u_t, v_t) - (u^\pm, v^\pm, u_t^\pm, v_t^\pm)\|_X \to 0 \quad \text{as } t \to \pm\infty, \tag{3.2}$$

then problem (1.1) is said to *have the wave operator*. The functions $(\varphi_1^\pm, \varphi_2^\pm, \psi_1^\pm, \psi_2^\pm)$ are called the *asymptotic states* of (u, v, u_t, v_t) at $t = \pm\infty$.

(b) If for any $(\varphi_1, \varphi_2, \psi_1, \psi_2) \in X$, there exist $(\varphi_1^\pm, \varphi_2^\pm, \psi_1^\pm, \psi_2^\pm) \in X$ such that (3.2) holds true, then problem (1.1) is said to be *asymptotically complete*.

If the conditions in both (a) and (b) hold true, then the *wave operators* $W_\pm$ are

$$W_+(\varphi_1^+, \varphi_2^+, \psi_1^+, \psi_2^+) = W_-(\varphi_1^-, \varphi_2^-, \psi_1^-, \psi_2^-) = (\varphi_1, \varphi_2, \psi_1, \psi_2).$$

The main result of this section reads as follows.

THEOREM 3.1. *The wave operators $W_\pm$ and the scattering operator $S \equiv W_+^{-1} \circ W_-$ for problem* (1.1) *exist and are isomorphisms of $X = \dot{H}^1 \times \dot{H}^1 \times L^2 \times L^2(\mathbb{R}^n)$.*

Proof. We set $A = (-\Delta)^{1/2}$ and define

$$\begin{aligned}(3.3)\quad & U_0(t)(\varphi_1, \varphi_2, \psi_1, \psi_2) \\ &\equiv (\cos(At)\varphi_1 + A^{-1}\sin(At)\psi_1, \cos(At)\varphi_2 + A^{-1}\sin(At)\psi_2, \\ &\quad - A\sin(At)\varphi_1 + \cos(At)\psi_1, -A\sin(At)\varphi_2 + \cos(At)\psi_2).\end{aligned}$$

It is well known that the solution to the free system associated with (1.1),

$$\begin{cases} \mu_{tt} - \Delta\mu = 0, \\ \nu_{tt} - \Delta\nu = 0, \\ \mu(0) = \varphi_1(x), \quad \mu_t(0) = \psi_1(x), \\ \nu(0) = \varphi_2(x), \quad \nu_t(0) = \psi_2(x), \end{cases} \tag{3.4}$$

is given by

$$(\mu, \nu) = (\cos(At)\varphi_1 + A^{-1}\sin(At)\psi_1, \cos(At)\varphi_2 + A^{-1}\sin(At)\psi_2). \tag{3.5}$$

Hence,

$$(\mu, \nu, \mu_t, \nu_t) = U_0(t)(\varphi_1, \varphi_2, \psi_1, \psi_2). \tag{3.6}$$

STEP 1: *Asymptotic completeness.* For any $(\varphi_1, \varphi_2, \psi_1, \psi_2) \in X$, let

$$\begin{aligned}(3.7)\quad & (u^\pm(t), v^\pm(t), u_t^\pm(t), v_t^\pm(t)) = U_0(t)(\varphi_1, \varphi_2, \psi_1, \psi_2) \\ &\quad - \int_0^{\pm\infty} U_0(t-\tau)(0, 0, F_1(|u|^2, |v|^2)u, F_2(|u|^2, |v|^2)v)\, d\tau.\end{aligned}$$

Combining the Strichartz estimates, the nonlinear estimates from [3, Proposition 3.1], and Proposition 2.3, we obtain

$$\begin{aligned}(3.8)\quad & \|(u, v, u_t, v_t) - (u^\pm, v^\pm, u_t^\pm, v_t^\pm)\|_X \\ &\le \Big\| \int_t^{\pm\infty} \big(A^{-1}\sin A(t-\tau)F_1(|u|^2, |v|^2)u, A^{-1}\sin A(t-\tau)F_2(|u|^2, |v|^2)v, \\ &\qquad \cos A(t-\tau)F_1(|u|^2, |v|^2)u, \cos A(t-\tau)F_2(|u|^2, |v|^2)v\big)\, d\tau \Big\|_X \\ &\le C \sup_{\tau \in [t, \pm\infty)} \|u\|_{L^{2^*}}^\beta \|u\|_{L^q([t,\pm\infty); \dot{B}_q^{1/2})}^\gamma \\ &\quad + C \sup_{\tau \in [t, \pm\infty)} \|v\|_{L^{2^*}}^\beta \|v\|_{L^q([t,\pm\infty); \dot{B}_q^{1/2})}^\gamma \to 0 \quad \text{as } t \to \pm\infty\end{aligned}$$

where

$$\frac{1}{q} = \frac{n-1}{2(n+1)}, \quad \alpha = \frac{n-2}{n-1}, \quad \beta = (1-\alpha)(2^* - 2) > 0,$$
$$\gamma = \alpha(2^* - 2) + 1 > 1.$$

If we introduce the notation

$$(3.9)\qquad \begin{aligned}&(\Phi_1^\pm,\Phi_2^\pm,\Psi_1^\pm,\Psi_2^\pm)\\ &= \int_0^{\pm\infty}\big(-A^{-1}\sin(A\tau)F_1(|u|^2,|v|^2)u, -A^{-1}\sin(A\tau)F_2(|u|^2,|v|^2)v,\\ &\qquad\qquad \cos(A\tau)F_1(|u|^2,|v|^2)u,\cos(A\tau)F_2(|u|^2,|v|^2)v\big)\,d\tau,\end{aligned}$$

then (3.7) reduces to

$$(u^\pm(t),v^\pm(t),u_t^\pm(t),v_t^\pm(t)) = U_0(t)(\varphi_1-\Phi_1^\pm,\varphi_2-\Phi_2^\pm,\psi_1-\Psi_1^\pm,\psi_2-\Psi_2^\pm).$$

Therefore, we can define the operator $\widetilde{W}_\pm^{-1}$ on X by the formula

$$(3.10)\qquad \begin{aligned}(\varphi_1^\pm,\varphi_2^\pm,\psi_1^\pm,\psi_2^\pm) &= \widetilde{W}_\pm^{-1}(\varphi_1,\varphi_2,\psi_1,\psi_2)\\ &\equiv(\varphi_1-\Phi_1^\pm,\varphi_2-\Phi_2^\pm,\psi_1-\Psi_1^\pm,\psi_2-\Psi_2^\pm).\end{aligned}$$

Step 2: *Wave operator.* For any $(\varphi_1^\pm,\varphi_2^\pm,\psi_1^\pm,\psi_2^\pm)\in X$, the existence of the wave operators is equivalent to the existence of solutions to the integral equation

$$(3.11)\qquad \begin{aligned}(u,v,u_t,v_t) &= U_0(t)(\varphi_1^\pm,\varphi_2^\pm,\psi_1^\pm,\psi_2^\pm)\\ &\quad+\int_t^{\pm\infty}U_0(t-\tau)(0,0,F_1(|u|^2,|v|^2)u,F_2(|u|^2,|v|^2)v)\,d\tau\end{aligned}$$

which satisfy

$$\lim_{t\to\pm\infty}\Big\|\int_t^{\pm\infty}U_0(t-\tau)(0,0,F_1(|u|^2,|v|^2)u,F_2(|u|^2,|v|^2)v)\,d\tau\Big\|_X=0.$$

To deal with (3.11), consider the space

$$\begin{aligned}\mathcal{Y}(I)=\{&(u,v,u_t,v_t)\in C(I;X):\\ &(u,v,u_t,v_t)\in L^q(I;\dot B_q^{1/2}\times\dot B_q^{1/2}\times\dot B_q^{-1/2}\times\dot B_q^{-1/2})\}\end{aligned}$$

as well as its closed subset

$$B=\{(u,v,u_t,v_t)\in\mathcal{Y}(I):\|(u,v,u_t,v_t)\|_{\mathcal{Y}(I)}\le C_{t_0}\},$$

where either $I=[t_0,\infty)$ or $I=(-\infty,-t_0]$ and $\lim_{|t_0|\to\infty}C_{t_0}=0$ for

$$C_{t_0}=\|U_0(t_0)(\varphi_1^\pm,\varphi_2^\pm,\psi_1^\pm,\psi_2^\pm)\|_{L^q(I;\dot B_q^{1/2}\times\dot B_q^{1/2}\times\dot B_q^{-1/2}\times\dot B_q^{-1/2})}.$$

By a standard argument, we can get the local well-posedness of (3.11) in $B\subset\mathcal{Y}(I)$. Therefore, if we define the wave operators by

$$W_\pm:(\varphi_1^\pm,\varphi_2^\pm,\psi_1^\pm,\psi_2^\pm)\mapsto(\varphi_1,\varphi_2,\psi_1,\psi_2)=(u(0),v(0),u_t(0),v_t(0)),$$

then $W_{\pm}^{-1}$ exists and is equal to $\widetilde{W}_{\pm}^{-1}$ in (3.10). In fact, the initial data of equation (3.11) are given by

$$(\widetilde{\varphi}_1, \widetilde{\varphi}_2, \widetilde{\psi}_1, \widetilde{\psi}_2) = (\varphi_1^{\pm}, \varphi_2^{\pm}, \psi_1^{\pm}, \psi_2^{\pm}) + \int_0^{\pm\infty} U_0(-\tau)(0, 0, F_1(|u|^2, |v|^2)u, F_2(|u|^2, |v|^2)v)\, d\tau.$$

Hence, we obtain from (3.9) the identity

$$(\varphi_1^{\pm}, \varphi_2^{\pm}, \psi_1^{\pm}, \psi_2^{\pm}) = (\widetilde{\varphi}_1 - \Phi_1^{\pm}, \widetilde{\varphi}_2 - \Phi_2^{\pm}, \widetilde{\psi}_1 - \Psi_1^{\pm}, \widetilde{\psi}_2 - \Psi_2^{\pm}),$$

which means $W_{\pm}$ is invertible. Thus $W_{\pm}^{-1} = \widetilde{W}_{\pm}^{-1}$ are isomorphisms on $X = \dot{H}^1 \times \dot{H}^1 \times L^2 \times L^2$. Consequently, the scattering operator $S = W_+^{-1} \circ W_-$ is also an isomorphism on X.

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