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GAGLIARDO–NIRENBERG INEQUALITIES IN LOGARITHMIC SPACES

BY

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Abstract. We obtain interpolation inequalities for derivatives:

$$\int M_{q,\alpha}(|\nabla f(x)|)\,dx \le C\Big[\int M_{p,\beta}(\Phi_1(x,|f|,|\nabla^{(2)}f|))\,dx + \int M_{r,\gamma}(\Phi_2(x,|f|,|\nabla^{(2)}f|))\,dx\Big],$$

and their counterparts expressed in Orlicz norms:

$$\|\nabla f\|^2_{(q,\alpha)} \le C\|\Phi_1(x,|f|,|\nabla^{(2)}f|)\|_{(p,\beta)}\,\|\Phi_2(x,|f|,|\nabla^{(2)}f|)\|_{(r,\gamma)},$$

where $\|\cdot\|_{(s,\kappa)}$ is the Orlicz norm relative to the function $M_{s,\kappa}(t)=t^s(\ln(2+t))^\kappa$. The parameters $p,q,r,\alpha,\beta,\gamma$ and the Carathéodory functions Φ_1,Φ_2 are supposed to satisfy certain consistency conditions. Some of the classical Gagliardo–Nirenberg inequalities follow as a special case. Gagliardo–Nirenberg inequalities in logarithmic spaces with higher order gradients are also considered.

1. Introduction and statement of results. The purpose of this paper is to obtain variants of interpolation inequalities for derivatives:

$$\|\nabla^{(k)}f\|_{L^q} \le C\|f\|_{L^p}^{1-k/m}\|\nabla^{(m)}f\|_{L^r}^{k/m} \tag{1.1}$$

(where $f\in W^{m,1}_{\mathrm{loc}}(\mathbb{R}^n)$, the symbol $\nabla^{(k)}f$ stands for the k-th gradient of $f:\mathbb{R}^n\to\mathbb{R}$, i.e. the vector $(D^\alpha f)_{|\alpha|=k}$, $p,q,r\in[1,\infty]$, $\frac{1}{q}=\big(1-\frac{k}{m}\big)\frac{1}{p}+\frac{k}{m}\frac{1}{r}$, $0<k<m$ and k,m are positive integers), expressed in logarithmic-type Orlicz spaces instead of L^p, L^q and L^r.

Inequalities of the form (1.1) have been extensively investigated and have evolved in many directions (see [5, 6, 8, 10, 11, 19, 21, 24, 27, 29, 30, 33, 36] and their references), but their generalizations to Orlicz spaces are nearly missing in the literature. In 1996 Bang [1] (see also [2–4]) proved variants of (1.1) for a one-variable function, within the same Orlicz space L^M.

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The authors have recently obtained inequalities of the form

$$\int M(|\nabla f|)\,dx \le C\Big(\int H(|f|)\,dx + \int J(|\nabla^{(2)} f|)\,dx\Big), \tag{1.2}$$

$$\|\nabla f\|_{(M)}^2 \le C\|f\|_{(H)}\|\nabla^{(2)} f\|_{(J)} \tag{1.3}$$

for functions of n variables in Orlicz spaces L^M, L^H and L^J defined by possibly distinct N-functions M, H, J which satisfy certain compatibility conditions (see [26]). In this work we adapt this abstract approach to the N-functions $M_{s,\kappa}(t) = t^s(\ln(2+t))^\kappa$, with related Orlicz norms denoted by $\|\cdot\|_{(s,\kappa)}$.

The parameters in Theorems 1.1–1.3 below will be subject to the following two consistency conditions:

(**A**) $\beta,\gamma \in \mathbb{R}$, $p, r > 1$, $(q > 2, \alpha \in \mathbb{R}$ or $q = 2, \alpha \ge 0)$ and $\frac{2}{q} = \frac{1}{p} + \frac{1}{r}, \frac{2\alpha}{q} \le \frac{\beta}{p} + \frac{\gamma}{r}$,

(**B**) $\beta,\gamma \in \mathbb{R}$, $\alpha < 0$, $1 < p, r$, $q = 2$, $\frac{1}{p} + \frac{1}{r} = 1$, $\beta(r-1) + \gamma \ge 0$.

We are mostly concerned with logarithmic variants of inequality (1.1) in the case when $k = 1$, $m = 2$. One of our results is the following logarithmic variant of the Gagliardo–Nirenberg inequality.

THEOREM 1.1. *Suppose that $p, q, r, \alpha, \beta, \gamma$ are real numbers such that Condition* (**A**) *or* (**B**) *is satisfied. Then for any smooth function $f : \mathbb{R}^n \to \mathbb{R}$ with bounded support one has*

$$\begin{aligned}\int |\nabla f|^q (\ln(2 + |\nabla f|))^\alpha\,dx \\ \le C\Big[\int |f|^p(\ln(2+|f|))^\beta\,dx + \int |\nabla^{(2)} f|^r (\ln(2+|\nabla^{(2)} f|))^\gamma\,dx\Big],\end{aligned} \tag{1.4}$$

and also

$$\|\nabla f\|_{(q,\alpha)}^2 \le C\|f\|_{(p,\beta)}\|\,\|\nabla^{(2)} f\|_{(r,\gamma)}, \tag{1.5}$$

with a constant C independent of f.

In the particular case $\alpha = \beta = \gamma = 0$, we obtain the classical Gagliardo–Nirenberg inequality (1.1) restricted here to $q \ge 2$, while for $p = q = r \ge 2$, $\alpha = \beta = \gamma$ (negative values of α permitted only for $q > 2$) and a scalar function f, we retrieve Bang's result from [1]. Observe that q is in this case the harmonic mean of p and r, and if $p = q = r$ and (**A**) holds then α does not exceed the arithmetic mean of β and γ.

The special cases of (1.4) when α, β or γ is zero follow from our previous work [25], where we dealt with variants of (1.4) in logarithmic spaces $L^s(\ln(\mu + L^a))^\alpha$ with $\mu \in \{1, 2\}$, under the restriction that one of the spaces considered: for f, $|\nabla f|$ or $|\nabla^{(2)} f|$, was the homogeneous space L^s (see Remark 4.3).

We will prove the following more general variant of Theorem 1.1:

THEOREM 1.2. *Suppose that* $p, q, r, \alpha, \beta, \gamma$ *are real numbers satisfying* $(\mathbf{A})$ *or* $(\mathbf{B})$ *and let* $\Phi_1, \Phi_2 : \mathbb{R}^n \times \mathbb{R}^2 \to \mathbb{R}$ *be Carathéodory functions* (*i.e. measurable with respect to* $x \in \mathbb{R}^n$ *and continuous with respect to* $(\lambda_1, \lambda_2) \in \mathbb{R}^2$) *such that* $\Phi_1(x, \lambda_1, \lambda_2)\Phi_2(x, \lambda_1, \lambda_2) = \lambda_1\lambda_2$ *a.e. Take any smooth function* $f : \mathbb{R}^n \to \mathbb{R}$ *with bounded support. Then*, *setting*

$$w_1(x) = \Phi_1(x, |f|, |\nabla^{(2)} f|), \qquad w_2(x) = \Phi_2(x, |f|, |\nabla^{(2)} f|),$$

we have

$$\int M_{q,\alpha}(|\nabla f(x)|)\, dx \le C\Big[\int M_{p,\beta}(w_1(x))\, dx + \int M_{r,\gamma}(w_2(x))\, dx\Big], \tag{1.6}$$

and also

$$\|\nabla f\|^2_{(q,\alpha)} \le C\|w_1\|_{(p,\beta)}\, \|w_2\|_{(r,\gamma)}, \tag{1.7}$$

both inequalities holding with a constant C *independent of* f.

For $\Phi_1(x, \lambda_1, \lambda_2) = \omega(x)\lambda_1^{\theta_1}\lambda_2^{\theta_2}$, $\Phi_2(x, \lambda_1, \lambda_2) = \frac{1}{\omega(x)}\lambda_1^{1-\theta_1}\lambda_2^{1-\theta_2}$, where $\omega : \mathbb{R}^n \to (0, \infty)$ is a measurable, a.e. positive function, we obtain the following theorem.

THEOREM 1.3. *Suppose that* $p, q, r, \alpha, \beta, \gamma$ *are given real numbers such that Condition* $(\mathbf{A})$ *or* $(\mathbf{B})$ *is satisfied, let* $(\theta_1, \theta_2) \in [0, 1]^2 \setminus \{(0, 0), (1, 1)\}$ *and let* ω *be an arbitrary a.e. positive measurable function. Then for any smooth function* $f : \mathbb{R}^n \to \mathbb{R}$ *with bounded support one has*

$$\begin{aligned} &\int M_{q,\alpha}(|\nabla f|)\, dx \\ &\quad \le C\Big[\int M_{p,\beta}(|f|^{\theta_1}|\nabla^{(2)} f|^{\theta_2}\omega)\, dx + \int M_{r,\gamma}(|f|^{1-\theta_1}|\nabla^{(2)} f|^{1-\theta_2}\omega^{-1})\, dx\Big], \end{aligned} \tag{1.8}$$

and also

$$\|\nabla f\|^2_{(q,\alpha)} \le C\|\, |f|^{\theta_1}|\nabla^{(2)} f|^{\theta_2}\omega\|_{(p,\beta)}\|\, \|\, |f|^{1-\theta_1}|\nabla^{(2)} f|^{1-\theta_2}\omega^{-1}\|_{(r,\gamma)}, \tag{1.9}$$

both inequalities holding with a constant C *independent of* f, (θ_1, θ_2) *and* ω.

Observe that Theorem 1.1 is a particular case of Theorem 1.3 (it corresponds to $\theta_1 = 1$, $\theta_2 = 0$ and $\omega \equiv 1$), but Theorem 1.3 (and so also Theorem 1.2) is more general.

Yet another choice of parameters: $\theta_1 = \theta_2 = 1/2$, $p = q = r$, $\alpha = \beta = \gamma$ and $\omega \equiv 1$ in Theorem 1.3 yields the following result.

THEOREM 1.4. *Suppose that either* $q > 2, \alpha \in \mathbb{R}$ *or* $q = 2, \alpha \ge 0$. *Then for every smooth function* $f : \mathbb{R}^n \to \mathbb{R}$ *with bounded support we have*

$$\int M_{q,\alpha}(|\nabla f|)\, dx \le C\int M_{q,\alpha}(\sqrt{|f|\, |\nabla^{(2)} f|})\, dx, \tag{1.10}$$

and also

$$\|\nabla f\|_{(q,\alpha)} \le C\|\sqrt{|f|\, |\nabla^{(2)} f|}\,\|_{(q,\alpha)}, \tag{1.11}$$

both inequalities holding with a constant C *independent of* f.

For completeness we write down the statement of Theorem 1.3 in homogeneous spaces ($\alpha = \beta = \gamma = 0$).

COROLLARY 1.1. *If p, q, r are real numbers such that $q \geq 2$, $p, r > 1$ and $2/q = 1/p + 1/r$, then for any $(\theta_1, \theta_2) \in [0,1]^2 \setminus \{(0,0),(1,1)\}$, $f \in C_0^\infty(\mathbb{R}^n)$ and any a.e. positive measurable function ω we have*

$$(1.12) \quad \Big(\int |\nabla f|^q \, dx\Big)^{2/q} \\ \leq C\Big(\int (|f|^{\theta_1} |\nabla^{(2)} f|^{\theta_2} \omega)^p \, dx\Big)^{1/p} \Big(\int (|f|^{1-\theta_1} |\nabla^{(2)} f|^{1-\theta_2} \omega^{-1})^r \, dx\Big)^{1/r},$$

with a constant C independent of f, (θ_1, θ_2) and ω.

We also point out two special cases of Corollary 1.1.

COROLLARY 1.2 ($\theta_1 = \theta_2 = 1/2$, $\omega \equiv 1$, $p = q = r$). *If $q \geq 2$ and $f \in C_0^\infty(\mathbb{R}^n)$, then*

$$\int |\nabla f|^q \, dx \leq C \int (|f| \, |\nabla^{(2)} f|)^{q/2} \, dx,$$

with a constant C independent of f.

COROLLARY 1.3 ($\theta_2 = 0$). *If p, q, r are real numbers such that $q \geq 2$, $p, r > 1$ and $2/q = 1/p + 1/r$, then for any $\theta \in [0,1]$, $f \in C_0^\infty(\mathbb{R}^n)$ and an arbitrary measure $\mu(dx) = \omega(x)\, dx$ with a positive weight ω, we have*

$$(1.13) \quad \Big(\int |\nabla f|^q \, dx\Big)^{2/q} \leq C\Big(\int |f|^{\theta p} \, d\mu\Big)^{1/p} \Big(\int (|f|^{1-\theta} |\nabla^{(2)} f|)^r \omega^{-r/p} \, dx\Big)^{1/r},$$

with a constant C independent of f, θ and ω.

Note that on the right hand side of (1.13) we can have the terms $\int |f|^s \, d\mu$ with $s = \theta p$ smaller than 1 and an arbitrary weighted measure $\mu(dx) = \omega(x)dx$, with a positive weight ω. In that case $\|f\|_{L^s_\mu} = (\int |f|^s \, d\mu)^{1/s}$ is no longer a norm.

In this paper we deal mostly with derivatives of order $0, 1$ and 2, but some generalizations to higher order derivatives are also possible. In Theorem 4.3 we generalize some cases of Theorem 1.1 to higher order derivatives. We also obtain stronger variants of inequalities (1.4), (1.6) and (1.8) (Theorem 4.1). Moreover, we get nonlinear variants of inequalities (1.6), namely inequalities between Young functionals $I_1 = \int M_{q,\alpha}(|\nabla f|)\, dx$, $I_0 = \int M_{p,\beta}(w_1)\, dx$ and $I_2 = \int M_{r,\gamma}(w_2)\, dx$, with w_1 and w_2 introduced in Theorem 1.2, where I_1 is estimated from above by a nonlinear expression involving I_0 and I_2. The precise statement is given in Theorem 4.2.

In the proof of Theorem 1.3 we adapt abstract techniques described in [26]. These techniques specialized to logarithmic Orlicz spaces require an additional and independent analysis (see also Remark 4.3). The results obtained (Theorems 1.1–1.4) are in general new, while the results in homogeneous spaces (Corollaries 1.1–1.3) are covered by the abstract approach of [26]. On the other hand, the additional results in Section 4 (Theorems 4.1–4.3) are based on the special structure of logarithmic Orlicz spaces and have no abstract counterparts in [26]. In our opinion, the importance of logarithmic Orlicz spaces in various disciplines of analysis and PDE's (e.g. [7], [9], [12, Section 4.3], [13]–[18], [20], [22], [23], [31], [37, Theorems 11.7 and Corollary 15.4], and references therein) justifies separate investigation of the logarithmic-type Gagliardo–Nirenberg inequalities.

Notation. If A is a vector or a matrix, we denote by $|A|$ its Euclidean norm induced by the standard scalar product $\langle\cdot,\cdot\rangle$, while A^t stands for its transposition.

By q^* we will denote the Hölder conjugate of $1<q<\infty$, and by C a general constant whose value can change even within the same line. When the domain of integration is not specified, it is meant to be the whole of $\mathbb{R}^n$. If F is an N-function, we denote by F^* its Legendre transform, defined by $F^*(t)=\sup_{s\geq 0}[st-F(s)]$.

Let $M,N:[0,\infty)\to[0,\infty)$ be two given functions. If $N(\lambda)\leq CM(k\lambda)$ for $\lambda\geq\lambda_0$ (resp. for $0\leq\lambda\leq\lambda_0$; for $\lambda\geq 0$) with constants C,k independent of x, then we say that M *dominates* N *at infinity* (resp. *near zero*; *globally*). This relation is denoted by $M\succ N$. We say that M is *equivalent* to N (written $M\sim N$) when $M\succ N$ and $N\succ M$. It is not hard to check (see e.g. Theorems 2.1 and 3.1 of [28]) that this domination is reversed by taking the Legendre transform of N-functions: $M\succ N$ (at infinity, near zero, globally) implies $N^*\succ M^*$ (at infinity, near zero, globally). Note that if M satisfies the Δ_2-condition then $M\succ N$ if and only if $N(\lambda)\leq CM(\lambda)$ with some constant C independent of λ.

2. Preliminaries. We will be dealing with the functions

$$M_{q,\alpha}(t):=t^q(\ln(2+t))^\alpha \quad \text{where } q>1,\ \alpha\in\mathbb{R}. \tag{2.1}$$

Within this range of parameters q,α they are all N-functions (i.e. convex, $M_{q,\alpha}(0)=0$, $\lim_{t\to 0+}M_{q,\alpha}(t)/t=0$, $\lim_{t\to\infty}M_{q,\alpha}(t)/t=\infty$). Therefore the set

$$L_{(q,\alpha)}=\Big\{f:\mathbb{R}^n\to\mathbb{R}\text{ measurable:}\\ \text{for some } K>0,\ \int M_{q,\alpha}(|f(x)|/K)\,dx<\infty\Big\}$$

becomes a Banach space when equipped with the Luxemburg norm

$$\|f\|_{(q,\alpha)}:=\inf\Big\{K>0:\int M_{q,\alpha}(|f(x)|/K)\,dx\leq 1\Big\}.$$

This is an Orlicz space defined by $M_{q,\alpha}$. Note that for $\alpha=0$ it coincides with the usual L^q space. The functions $M_{q,\alpha}$ satisfy the Δ_2-*condition*, i.e. $M_{q,\alpha}(2t)\leq CM_{q,\alpha}(t)$ with a constant $C=C(q,\alpha)$ independent of $t\geq 0$. It is known that

$$\int M_{q,\alpha}\left(\frac{|f(x)|}{\|f\|_{(q,\alpha)}}\right) dx = 1, \qquad \|f\|_{(q,\alpha)} \le \int M_{q,\alpha}(|f(x)|)\, dx + 1. \tag{2.2}$$

For details we refer the reader to [28, Chapter 1].

For later use observe that

$$M_{q,\alpha} \circ M_{\mu,\kappa} \sim M_{q\mu,q\kappa+\alpha}. \tag{2.3}$$

Finally, let us prove a lemma.

LEMMA 2.1. *Suppose that* $\mu > 1$, $\kappa \in \mathbb{R}$ *and* $\widetilde{\kappa} \ge \kappa_1 = -\kappa(\mu^* - 1)$. *Then there exists a constant* $C > 0$ *such that for all* $u, v \ge 0$,

$$uv \le M_{\mu,\kappa}(u) + CM_{\mu^*,\widetilde{\kappa}}(v). \tag{2.4}$$

Proof. This is immediate: as $M_{\mu,\kappa}(u) \sim u^\mu(\ln u)^\kappa$ for u large, we have $M^*_{\mu,\kappa}(v) \sim M_{\mu^*,\kappa_1}(v)$ for v large (see [28, Theorem 7.1]). On the other hand, for u small we have $M_{\mu,\kappa}(u) \sim u^\mu$, thus $M^*_{\mu,\kappa}(v) \sim v^{\mu^*} \sim M_{\mu^*,\kappa_1}$ for v small. Therefore $M^*_{\mu,\kappa} \sim M_{\mu^*,\kappa_1}$ globally.

If $\widetilde{\kappa} \ge \kappa_1$, then $M_{\mu^*,\widetilde{\kappa}}$ dominates M_{μ^*,κ_1} globally, and so, for $u, v \ge 0$,

$$uv \le M_{\mu,\kappa}(u) + M^*_{\mu,\kappa}(v) \le M_{\mu,\kappa}(u) + CM_{\mu^*,\kappa_1}(v) \le M_{\mu,\kappa}(u) + CM_{\mu^*,\widetilde{\kappa}}(v)$$

with a constant $C > 0$. ■

3. Proofs of Theorems 1.1–1.4. As indicated in Section 1, we only need to show Theorem 1.2. The remaining results: Theorems 1.1, 1.3, 1.4 (together with Corollaries 1.1–1.3) follow as consequences.

Proof of Theorem 1.2. The proof is carried out in several steps.

STEP 1. We show that

$$I := \int M_{q,\alpha}(|\nabla f|)\, dx \le C \int M_{q-2,\alpha}(|\nabla f|)|f|\,|\nabla^{(2)} f|\, dx \tag{3.1}$$

with a constant C not depending on f (with a slight abuse of notation: the number $q-2$ can be smaller than 1 here, but the formula (2.1) defining $M_{q-2,\alpha}$ remains valid).

The proof of this inequality is basically taken from [25]; we sketch it here to make the paper self-contained.

As $M_{q,\alpha}(|\lambda|) = M_{q-2,\alpha}(\lambda)\langle\lambda,\lambda\rangle$, where $\lambda = (\lambda_1, \ldots, \lambda_n)$, after integrating by parts we obtain

$$I = -\int \operatorname{div}(S(\nabla f(x)))f(x)\, dx, \tag{3.2}$$

where $S = (S_1, \ldots, S_n)$ and $S_i(\lambda) = M_{q-2,\alpha}(|\lambda|)\lambda_i$ (since $q \ge 2$ this integration by parts is allowed according to the Nikodym ACL Characterization Theorem, see [32, Th. 2, Sec. 1.1.3]).

In particular $S_i(\nabla f) = M_{q-2,\alpha}(|\nabla f|)\frac{\partial f}{\partial x_i}$, and so

$$\operatorname{div} S(\nabla f) = \frac{M'_{q-2,\alpha}(|\nabla f|)}{|\nabla f|}[\nabla f]^t\,[\nabla^{(2)} f][\nabla f] + M_{q-2,\alpha}(|\nabla f|)\Delta f.$$

Elementarily we check that $M'_{q-2,\alpha}(t) \sim M_{q-2,\alpha}(t)t^{-1}$ on the positive half-line. Moreover $|v^t A v| \le |A|\,|v|^2$ and $|\operatorname{tr} A| \le \sqrt{n}\,|A|$ (so that $|\Delta f| \le \sqrt{n}\,|\nabla^{(2)} f|$). This gives

$$|\operatorname{div} S(\nabla f)| \le C M_{q-2,\alpha}(|\nabla f|)|\nabla^{(2)} f|,$$

and together with (3.2) completes the proof of (3.1).

Step 2. Now assume that (**A**) holds. We show that in this case, for all $u, v, w \ge 0$,

$$M_{q-2,\alpha}(u)vw \le M_{q,\alpha}(u) + C\left[M_{p,\beta}(v) + M_{r,\gamma}(w)\right]. \tag{3.3}$$

To see this, first observe that

$$M_{q-2,\alpha}(s)t^2 \le M_{q,\alpha}(s) + M_{q,\alpha}(t). \tag{3.4}$$

This is immediate: if $t \le s$, then $M_{q-2,\alpha}(s)t^2 = M_{q,\alpha}(s)(t/s)^2 \le M_{q,\alpha}(s)$. Since $M_{q-2,\alpha}$ is increasing, for $s \le t$ one has $M_{q-2,\alpha}(s)t^2 \le M_{q-2,\alpha}(t)t^2 = M_{q,\alpha}(t)$.

Next, take $\mu = 2p/q$, $\kappa = (\beta - \alpha)/q$, $\widetilde{\kappa} = (\gamma - \alpha)/q$. Under current restriction on the parameters, it is not hard to check that $\widetilde{\kappa} \ge \kappa_1 = -\kappa(\mu^* - 1)$. Therefore the assumptions of Lemma 2.1 are satisfied and (2.4) can be applied, resulting in the following series of inequalities:

$$\begin{aligned} M_{q-2,\alpha}(u)vw &\le M_{q,\alpha}(u) + CM_{q,\alpha}(\sqrt{vw}) \\ &\le M_{q,\alpha}(u) + CM_{q,\alpha}\left(M_{\mu,\kappa}(\sqrt{v}) + M_{\mu^*,\widetilde{\kappa}}(\sqrt{w})\right) \\ &\le M_{q,\alpha}(u) + C[M_{q,\alpha} \circ M_{\mu,\kappa}(\sqrt{v}) + M_{q,\alpha} \circ M_{\mu^*,\widetilde{\kappa}}(\sqrt{w})] \end{aligned}$$

(the last inequality follows from the fact that for every nondecreasing function F satisfying the Δ_2-condition one has $F(a+b) \le F(2\max(a,b)) \le F(2a) + F(2b) \le C(F(a) + F(b))$). Using now the property (2.3) we see that $M_{q,\alpha} \circ M_{\mu,\kappa}(\sqrt{v}) \sim M_{p,\beta}(v)$ and $M_{q,\alpha} \circ M_{\mu^*,\widetilde{\kappa}}(\sqrt{w}) \sim M_{r,\gamma}(w)$, so that (3.3) follows.

Step 3: *Conclusion under condition* (**A**). Applying (3.1) we get

$$I \le \frac{1}{2}\int M_{q-2,\alpha}(|\nabla f|) \cdot (2C|f|\,|\nabla^{(2)} f|)\,dx. \tag{3.5}$$

Since, by definition of w_1 and w_2, $|f(x)|\,|\nabla^{(2)} f(x)| = w_1(x)w_2(x)$, applying (3.3) and using the Δ_2-condition we find that I is not greater than

$$\frac{1}{2}\,I + C\int M_{p,\beta}(w_1(x))\,dx + C\int M_{r,\gamma}(w_2(x))\,dx,$$

(with C possibly different than in (3.5)), which after rearranging yields (1.6).

In order to prove (1.7), fix $t_1, t_2 > 0$ and write the inequality (3.5) for $\widetilde{f} = f/t_1t_2$. We get

$$I \le \frac{1}{2}\int M_{q-2,\alpha}(|\nabla \widetilde{f}|)(2C\widetilde{w}_1\widetilde{w}_2)\,dx,$$

where $\widetilde{w}_i = w_i/t_i^2$ (because $|\widetilde{f}|\,|\nabla^{(2)}\widetilde{f}| = \widetilde{w}_1\widetilde{w}_2$).

Using (3.3) and repeating the subsequent steps with f, w_1 and w_2 replaced by $\widetilde{f}, \widetilde{w}_1$ and $\widetilde{w}_2$ we obtain

$$\int M_{q,\alpha}(|\nabla \widetilde{f}|)\,dx \le C\Big(\int M_{p,\beta}(\widetilde{w}_1)\,dx + \int M_{r,\gamma}(\widetilde{w}_2)\,dx\Big),$$

with a constant C independent of f and t_1, t_2. Now choose $t_1^2 = \|w_1\|_{(q,\beta)}$, $t_2^2 = \|w_2\|_{(r,\gamma)}$. As $t_i = 0$ implies $w_1w_2 = 0$, which by (3.1) forces $f \equiv 0$ (as f is compactly supported and smooth), we can assume that $t_1, t_2 > 0$. Moreover, we have $\int M_{p,\beta}(\widetilde{w}_1)\,dx = \int M_{p,\beta}(w_1/\|w_1\|_{(p,\beta)})\,dx = 1$, and similarly $\int M_{r,\gamma}(\widetilde{w}_2)\,dx = \int M_{r,\gamma}(w_2/\|w_1\|_{(r,\gamma)})\,dx = 1$. We end up with $\int M_{q,\alpha}(|\nabla \widetilde{f}|)\,dx \le C$. This together with (2.2) gives $\|\nabla \widetilde{f}\|_{(q,\alpha)} \le C+1$, so that

$$\|\nabla f\|^2_{(q,\alpha)} \le (C+1)\|w_1\|_{(p,\beta)}\,\|w_2\|_{(r,\gamma)}.$$

STEP 4: *Conclusion under condition* (**B**). First, apply (3.1), but instead of using (3.3) observe that for $q = 2$ and $\alpha < 0$ the function $M_{q-2,\alpha}$ is bounded. Therefore (using the same notation as above)

$$I \le C\int |f|\,|\nabla^{(2)} f|\,dx = C\int w_1w_2\,dx.$$

The conditions imposed on the parameters β and γ imply that (see Lemma 2.1) $w_1w_2 \le M_{p,\beta}(w_1) + CM_{q,\gamma}(w_2)$, and consequently

$$I \le C\Big(\int M_{p,\beta}(w_1)\ dx + \int M_{r,\gamma}(w_2)\,dx\Big),$$

which proves (1.6) in this case. The proof of (1.7) goes now along the same lines as in Step 3 and so we skip it. ∎

4. Extensions and remarks. We start with the following result which shows that inequality (1.6) in Theorem 1.2 and its special variants: inequalities (1.4) and (1.8), can be transformed into a stronger form, where one of the summands can be made arbitrarily small. We obtain:

THEOREM 4.1. *Suppose that $p, q, r, \alpha, \beta, \gamma$ are real numbers satisfying* (**A**) *or* (**B**) *and let $\Phi_1, \Phi_2 : \mathbb{R}^n \times \mathbb{R}^2 \to \mathbb{R}$ be Carathéodory functions such that $\Phi_1(x, \lambda_1, \lambda_2)\Phi_2(x, \lambda_1, \lambda_2) = \lambda_1\lambda_2$ a.e. Take any smooth function $f : \mathbb{R}^n \to \mathbb{R}$*

with bounded support and define

$$w_1(x) = \Phi_1(x, |f|, |\nabla^{(2)} f|), \qquad w_2(x) = \Phi_2(x, |f|, |\nabla^{(2)} f|),$$

$$h_{s,\kappa}(\delta) = \begin{cases} M_{s,\kappa}(\delta) & \text{for } \kappa \geq 0, \\ \delta^s \ln(2 + 1/\delta)^{-\kappa} & \text{for } \kappa < 0. \end{cases} \tag{4.1}$$

Then there exists a constant $C = C(\beta, \gamma)$ such that for any $\delta > 0$,

$$\begin{aligned} &\int M_{q,\alpha}(|\nabla f(x)|)\,dx \\ &\quad \leq C\Big(h_{p,\beta}(\delta)\int M_{p,\beta}(w_1(x))\,dx + h_{r,\gamma}(\delta^{-1})\int M_{r,\gamma}(w_2(x))\,dx\Big). \end{aligned} \tag{4.2}$$

In particular, for every $\varepsilon > 0$ there exists a constant C_ε, depending on ε, p, r, β and γ, such that

$$\int M_{q,\alpha}(|\nabla f(x)|)\,dx \leq \varepsilon \int M_{p,\beta}(w_1(x))\,dx + C_\varepsilon \int M_{r,\gamma}(w_2(x))\,dx, \tag{4.3}$$

$$\int M_{q,\alpha}(|\nabla f(x)|)\,dx \leq C_\varepsilon \int M_{p,\beta}(w_1(x))\,dx + \varepsilon \int M_{r,\gamma}(w_2(x))\,dx. \tag{4.4}$$

Proof. Take any $\delta > 0$ and apply (1.6) with $\widetilde{w}_1 = \delta w_1$ and $\widetilde{w}_2 = w_2/\delta$ replacing w_1 and w_2. Then it suffices to prove that for $s > 1$ and $\kappa \in \mathbb{R}$ we have

$$M_{s,\kappa}(\delta\lambda) \leq C h_{s,\kappa}(\delta) M_{s,\kappa}(\lambda) \quad \text{for } \delta, \lambda \geq 0, \tag{4.5}$$

with C depending on κ only. To obtain (4.5), first note that

$$\ln(2 + \delta\lambda) \leq C \ln(2 + \delta) \ln(2 + \lambda), \tag{4.6}$$

with C independent of δ and λ. Indeed, if $\delta \leq \lambda$, then the left hand side is not greater than $\ln(2 + \lambda^2) \sim \ln(2 + \lambda)$. Also, $\ln(2 + \delta) \geq \ln 2 > 0$, which completes the proof of (4.6).

Now (4.5) follows immediately from (4.6) when $\kappa \geq 0$, while for negative κ we have

$$\begin{aligned} M_{s,\kappa}(\delta\lambda) &= \delta^s \left(\frac{\ln(2 + \delta\lambda)}{\ln(2 + \lambda)}\right)^{\kappa} M_{s,\kappa}(\lambda) \leq \delta^s \left(\sup_{\lambda > 0} \frac{\ln(2 + \lambda)}{\ln(2 + \delta\lambda)}\right)^{-\kappa} M_{s,\kappa}(\lambda) \\ &\leq \delta^s \left(\sup_{\lambda > 0} \frac{\ln(2 + \lambda\delta^{-1})}{\ln(2 + \lambda)}\right)^{-\kappa} M_{s,\kappa}(\lambda) \\ &\leq C_\kappa \delta^s (\ln(2 + \delta^{-1}))^{-\kappa} M_{s,\kappa}(\lambda), \end{aligned}$$

where for the last inequality we have used (4.6).

This gives (4.2). To derive (4.3) and (4.4) we observe that $\lim_{\delta\to 0} h_{s,\kappa}(\delta) = 0$, so we can find δ such that $Ch_{p,\beta}(\delta) = \varepsilon$ (for (4.3)) or $Ch_{r,\gamma}(\delta^{-1}) = \varepsilon$ (for (4.4)). ∎

Now we will derive multiplicative variants of inequality (1.6) in Theorem 1.2, involving not Orlicz norms, but Orlicz functionals. Consequently, inequalities (1.4) and (1.8) will also have multiplicative counterparts involving Orlicz functionals.

The result presented below is restricted to the case $\beta,\gamma \geq 0$. If $\beta < 0$ or $\gamma < 0$, then a similar statement holds, but with the third and fourth factors in (4.7) different.

THEOREM 4.2. *Suppose that* $p,q,r,\alpha,\beta,\gamma$ *are real numbers satisfying* (**A**) *or* (**B**), $\beta,\gamma \geq 0$, *and let* Φ_1, Φ_2, w_1, w_2 *and* f *be as in Theorem* 4.1. *Then there exists a constant* $C = C(p,r,\beta,\gamma) > 0$ *such that*

$$\begin{aligned}(4.7)\qquad &\Big(\int M_{q,\alpha}(|\nabla f(x)|)\,dx\Big)^{2/q}\\ &\leq C\Big(\int M_{p,\beta}(w_1(x))\,dx\Big)^{1/p}\Big(\int M_{r,\gamma}(w_2(x))\,dx\Big)^{1/r}\\ &\quad\times\bigg(\ln\bigg(2+\frac{\int M_{p,\beta}(w_1(x))\,dx}{\int M_{r,\gamma}(w_2(x))\,dx}\bigg)\bigg)^{\gamma/r}\bigg(\ln\bigg(2+\frac{\int M_{r,\gamma}(w_2(x))\,dx}{\int M_{p,\beta}(w_1(x))\,dx}\bigg)\bigg)^{\beta/p}.\end{aligned}$$

Proof. Set

$$a := \int M_{q,\alpha}(|\nabla f(x)|)\,dx,\quad b := C\int M_{p,\beta}(w_1(x))\,dx,\quad c := C\int M_{r,\gamma}(w_2(x))\,dx,$$

where C is the constant from (4.2). Then (4.2) reads

$$a \leq M_{p,\beta}(\delta)b + M_{r,\gamma}(\delta^{-1})c, \tag{4.8}$$

where $\delta > 0$ can be taken arbitrary.

Now observe that $M'_{s,\kappa}(\lambda) \sim M_{s,\kappa}(\lambda)/\lambda$, and so the minimum of the right hand side of (4.8) with respect to $\delta > 0$ is achieved at a point δ_0 for which

$$C_1\,\frac{c}{b} \leq R(\delta_0) \leq C_2\,\frac{c}{b},\quad \text{where}\quad R(\lambda) := \frac{M_{p,\beta}(\lambda)}{M_{r,\gamma}(\lambda^{-1})}, \tag{4.9}$$

with constants C_1, C_2 independent of c and b. As

$$R(\lambda) \sim \frac{\lambda^{p+r}}{|\ln\lambda|^{\gamma}} = \frac{1}{(1/\lambda)^{p+r}(\ln(1/\lambda))^{\gamma}} \qquad \text{for } \lambda \text{ close to } 0,$$

and $R(\lambda) \sim \lambda^{p+r}(\ln\lambda)^{\beta}$ for λ large, and $(\lambda|\ln\lambda|)^{-1} \sim \lambda/|\ln\lambda|$ for both small and large values of λ (here ϕ^{-1} denotes the inverse function to ϕ), we verify that the inverse function to R satisfies

$$R^{-1}(\lambda) \sim \bigg(\lambda\,\frac{(\ln(2+\lambda^{-1}))^{\gamma}}{(\ln(2+\lambda))^{\beta}}\bigg)^{1/(p+r)}. \tag{4.10}$$

Using (4.6) and (4.9) we establish that

$$\widetilde{C}_1 R^{-1}(c/b) \leq \delta_0 \leq \widetilde{C}_2 R^{-1}(c/b), \tag{4.11}$$

with $\widetilde{C}_1, \widetilde{C}_2$ independent of b and c. Moreover, we have

$$M_{p,\beta}\circ R^{-1}(\lambda) \sim \lambda^{\frac{p}{p+r}}(\ln(2+\lambda^{-1}))^{\frac{\gamma p}{p+r}}(\ln(2+\lambda))^{\frac{\beta r}{p+r}}. \tag{4.12}$$

On the other hand, according to (4.11), and using the fact that $M_{p,\beta}$ satisfies the Δ_2-condition, we get

$$M_{p,\beta}(\delta_0)b \le M_{p,\beta}(\widetilde{C}_2 R^{-1}(c/b))b \le C_3(M_{p,\beta} \circ R^{-1}(c/b))b := \mathcal{A}, \tag{4.13}$$

and by (4.9),

$$M_{r,\gamma}(\delta_0^{-1})c \le C_1^{-1} M_{p,\beta}(\delta_0)b \le C_1^{-1}\mathcal{A}. \tag{4.14}$$

Now we apply (4.8) with $\delta = \delta_0$, and also (4.13), (4.14) and (4.12), to get

$$\begin{aligned} a &\le M_{p,\beta}(\delta_0)b + M_{r,\gamma}(\delta_0^{-1})c \le C_4\mathcal{A} \\ &\le C_5\, b^{\frac{r}{p+r}} c^{\frac{p}{p+r}} (\ln(2+b/c))^{\frac{\gamma r}{p+r}} (\ln(2+c/b))^{\frac{\beta r}{p+r}}, \end{aligned}$$

with C_5 independent of b and c, which completes the proof of (4.7). ∎

REMARK 4.1. Note that for $\beta = \gamma = 0$, (4.7) is exactly the Gagliardo–Nirenberg inequality restricted to $q \ge 2$.

The results of Theorem 1.1 can be iterated to higher derivatives. In particular we obtain the following theorem:

THEOREM 4.3. *Suppose that $k, m \in \mathbb{Z}_+$, $0 < k < m$ and $p, q, r, \alpha, \beta, \gamma$ are real numbers such that*

$$\frac{1}{q} = \left(1 - \frac{k}{m}\right)\frac{1}{p} + \frac{k}{m}\,\frac{1}{r}, \qquad p, r > 2, \qquad \frac{\alpha}{q} \le \left(1 - \frac{k}{m}\right)\frac{\beta}{p} + \frac{k}{m}\,\frac{\gamma}{r}. \tag{4.15}$$

Then for any smooth function $f : \mathbb{R}^n \to \mathbb{R}$ with bounded support,

$$\begin{aligned} &\int M_{q,\alpha}(|\nabla^{(k)} f(x)|)\,dx \\ &\qquad\le C\Big(\int M_{p,\beta}(|f(x)|)\,dx + \int M_{r,\gamma}(|\nabla^{(m)} f(x)|)\,dx\Big), \end{aligned} \tag{4.16}$$

$$\|\nabla^{(k)} f(x)\|_{(q,\alpha)} \le C\|f\|_{(p,\beta)}^{1-k/m} \|\nabla^{(m)} f\|_{(r,\gamma)}^{k/m}, \tag{4.17}$$

with a constant C independent of f.

Proof. We give the proof of (4.3) only, leaving (4.17) to the reader. As $M_{q,\widetilde{\alpha}} \le M_{q,\alpha}$ whenever $\widetilde{\alpha} \le \alpha$, it suffices to prove the theorem under the condition

$$\frac{1}{q} = \left(1 - \frac{k}{m}\right)\frac{1}{p} + \frac{k}{m}\,\frac{1}{r}, \qquad p, r > 2, \qquad \frac{\alpha}{q} = \left(1 - \frac{k}{m}\right)\frac{\beta}{p} + \frac{k}{m}\,\frac{\gamma}{r}. \tag{4.18}$$

For simplicity we will use the following notation. Let $D = \big\{\binom{x}{y} : x \in \mathbb{R} \setminus \{0\},\ y \in \mathbb{R}\big\}$ and define $h : D \to D$ and $G_s : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ for $s \in [0,1]$ by

$$h\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1/x \\ y/x \end{pmatrix}, \qquad G_s(\overline{\lambda}_1, \overline{\lambda}_2) = s\overline{\lambda}_1 + (1-s)\overline{\lambda}_2, \tag{4.19}$$

where $\overline{\lambda}_1, \overline{\lambda}_2 \in \mathbb{R}^2$. Then conditions (4.18) read

$$h\begin{pmatrix} q \\ \alpha \end{pmatrix} = G_{k/m}\left(h\begin{pmatrix} p \\ \beta \end{pmatrix}, h\begin{pmatrix} r \\ \gamma \end{pmatrix}\right), \qquad p, r > 2,\ \gamma, \beta \in \mathbb{R}. \tag{4.20}$$

We proceed by induction on $m \geq 2$ and prove that for $k \in \{1, \dots, m-1\}$, all $k, m, q, p, r, \alpha, \beta, \gamma$ satisfying (4.20) and arbitrary $\varepsilon > 0$ there exists a constant $C_\varepsilon = C(\varepsilon, k, m, p, r, \gamma, \beta) > 0$ such that for all $f \in C_0^\infty(\mathbb{R}^n)$,

$$I_{q,\alpha}(|\nabla^{(k)} f|) \leq \varepsilon I_{p,\beta}(|f|) + C_\varepsilon I_{r,\gamma}(|\nabla^{(m)} f|), \tag{4.21}$$

where $I_{s,\kappa}(g) = \int M_{s,\kappa}(|g|)\, dx$.

If $m = 2$ and $k = 1$, then (4.21) is just (4.3) and there is nothing to prove. Suppose then that (4.21) holds for all $m \in \{2, \dots, M\}$ and all $0 < k < m$, provided that the parameters $k, m, q, p, r, \alpha, \beta, \gamma$ satisfy (4.20). Now we take $m = M + 1$, $0 \leq k \leq M + 1$ and set

$$\overline{\lambda}_k := \begin{pmatrix} q_k \\ \alpha_k \end{pmatrix} = h^{-1} \circ G_{k/(M+1)}\left(h\begin{pmatrix} p \\ \beta \end{pmatrix}, h\begin{pmatrix} r \\ \gamma \end{pmatrix}\right). \tag{4.22}$$

In particular

$$\overline{\lambda}_0 = \begin{pmatrix} q_0 \\ \alpha_0 \end{pmatrix} = \begin{pmatrix} p \\ \beta \end{pmatrix}, \qquad \overline{\lambda}_{M+1} = \begin{pmatrix} q_{M+1} \\ \alpha_{M+1} \end{pmatrix} = \begin{pmatrix} r \\ \gamma \end{pmatrix}.$$

To abbreviate, we write $I_k = I_{q_k,\alpha_k}(|\nabla^{(k)} f|)$. In this notation, the induction step reduces to the proof of

$$I_k \leq \varepsilon I_0 + C_\varepsilon I_{M+1} \tag{4.23}$$

with $C_\varepsilon = C(\varepsilon, k, M, p, r, \beta, \gamma)$ and for all $k \in \{1, \dots, M\}$.

To get it, we first check that $q_i > 2$ for $i \in \{0, \dots, M+1\}$, and moreover, for all s, l, t such that $0 \leq s < l < t \leq M + 1$ we have

$$h(\overline{\lambda}_l) = G_{\frac{l-s}{t-s}}(h(\overline{\lambda}_s), h(\overline{\lambda}_t)).$$

By the inductive assumption, this implies that (4.21) holds true with parameters: $q = q_l$, $\alpha = \alpha_l$, $p = p_s$, $\beta = \alpha_s$, $r = q_t$, $\gamma = \alpha_t$, $k = l - s$, $m = t - s$, provided $0 < t - s \leq M$. An application of (4.21) to all $g = D^\alpha f$ with $|\alpha| = s$, with this range of parameters, together with the inequality

$$M_{q_l,\alpha_l}(|\nabla^{(l)} f|) \leq C \sum_{\alpha,\, |\alpha|=s} M_{q_l,\alpha_l}(|\nabla^{(l-s)} D^\alpha f|),$$

with C independent of f, implies that once $0 \leq s < l < t \leq M + 1$ and $t - s \leq M$, then we have

$$I_l \leq \varepsilon I_s + C_\varepsilon I_t \tag{4.24}$$

with $C_\varepsilon = C(\varepsilon, s, t, l, p, r, \alpha, \beta)$. This gives, for all $0 < k < M$,

$$I_k \leq \delta I_0 + C_\delta I_M \leq \delta I_0 + C_\delta(\varepsilon I_k + C_\varepsilon I_{M+1})$$

for every $\varepsilon, \delta > 0$. Choosing $\varepsilon = \varepsilon_\delta$ such that $C_\delta \varepsilon = 1/2$ and rearranging we obtain (4.23) for all $0 < k < M$. To get (4.23) with $k = M$ we use the inequalities

$$I_M \le \varepsilon I_{M-1} + C_\varepsilon I_{M+1} \quad \text{and} \quad I_{M-1} \le \delta I_0 + C_\delta I_M.$$

They imply

$$I_M \le \varepsilon\delta I_0 + \varepsilon C_\delta I_M + C_\varepsilon I_{M+1}$$

for every $\varepsilon, \delta > 0$. Take $\varepsilon \le \varepsilon_\delta$, where ε_δ satisfies $\varepsilon_\delta C_\delta = 1/2$. After rearranging we obtain

$$I_M \le 2\varepsilon\delta I_0 + 2C_\varepsilon I_{M+1},$$

which completes the induction argument and concludes the proof of the theorem. ∎

Remark 4.2. In [26] we have shown that if M is an N-function satisfying the Δ_2-condition with $M'(t)/t$ bounded near zero and F is an arbitrary N-function, then for every $f \in C_0^\infty(\mathbb{R}^n)$ we have

$$\int M(|\nabla f|)\,dx \le C\Big(\int M(F(\sqrt{|f|}))\,dx + \int M(F^*(\sqrt{|\nabla^{(2)} f|}))\,dx\Big),$$
$$\|\nabla f\|_{(M)}^2 \le C\|f\|_{(H)}\|\nabla^{(2)} f\|_{(J)},$$

where $H(t) = M(F(\sqrt{t}))$, $J(t) = M(F^*(\sqrt{t}))$, and the constant C is independent of f. Analogous results remain true for arbitrary Carathéodory functions $\Phi_1, \Phi_2 : \mathbb{R}^n \times \mathbb{R}^2 \to \mathbb{R}$ such that $\Phi_1(x, \lambda_1, \lambda_2)\Phi_2(x, \lambda_1, \lambda_2) = \lambda_1\lambda_2$ and $w_1(x) = \Phi_1(x, |f|, |\nabla^{(2)} f|)$ and $w_2(x) = \Phi_2(x, |f|, |\nabla^{(2)} f|)$ replacing $|f(x)|$ and $|\nabla^{(2)} f(x)|$. In the present paper we have shown that in the particular case of logarithmic-type functions $M(t) = M_{q,\alpha}(t)$ and $F(t) = M_{\mu,\kappa}(t)$, with parameters μ and κ suitably chosen, we end up with (1.4)–(1.13), illustrating the abstract approach of [26].

Remark 4.3. In our previous work [25] we have dealt with the following logarithmic inequalities:

$$\int |\nabla f|^q (\ln(\mu + |\nabla f|^a))^\alpha\,dx \le C\Big(\Big(\int |f|^p(\ln(\mu + |f|^b))^\beta\,dx\Big)^{1/p^*}\|\nabla^{(2)} f\|_r + \|\nabla^{(2)} f\|_r^r\Big), \tag{4.25}$$

$$\int |\nabla f|^q (\ln(\mu + |\nabla f|^a))^\alpha\,dx \le C\Big(\Big(\int |\nabla^{(2)} f|^r(\ln(\mu + |\nabla^{(2)} f|^b))^\gamma\,dx\Big)^{1/p^*}\|f\|_p + \|f\|_p^p\Big), \tag{4.26}$$

$$\int |\nabla f|^q\,dx \le C\Big(\int |f|^p(\ln(\mu + |f|^a))^\beta\,dx + \int |\nabla^{(2)} f|^r(\ln(\mu + |\nabla^{(2)} f|^b))^\gamma\,dx\Big)$$

where $\mu \in \{1, 2\}$. In the particular case when $\mu = 2$, $a = b = 1$, by the classical Young inequality ($xy \le x^p/p + y^{p^*}/p^*$, $p > 1$) applied to (4.25) and (4.26) we see that they both imply (1.4) for β or γ equal to 0. The last inequality in this series with $a = b = 1$ and $\mu = 2$ is just (1.4) for $\alpha = 0$. Note that (4.25) and (4.26) for $a = b = 1$ and $\mu = 2$ are in general stronger than the special case of (1.4) when β or γ equals zero. It turns out that the ranges of parameters in inequalities (4.25) and (4.26) under the restrictions $a = b = 1$ and $\mu = 2$ obtained in [25] and that in (1.4) of this paper are consistent.

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