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ON RINGS OF CONSTANTS OF DERIVATIONS IN TWO VARIABLES IN POSITIVE CHARACTERISTIC

BY

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Abstract. Let k be a field of chracteristic $p > 0$. We describe all derivations of the polynomial algebra $k[x,y]$, homogeneous with respect to a given weight vector, in particular all monomial derivations, with the ring of constants of the form $k[x^p, y^p, f]$, where $f \in k[x,y] \setminus k[x^p, y^p]$.

Introduction. A. Nowicki and M. Nagata proved in [4] that if d is a nonzero k-derivation of $k[x,y]$, where k is a field of characteristic $p > 0$, then $k[x,y]^d$, the ring of constants of d, is a free $k[x^p, y^p]$-module. Morover they showed that if $p = 2$, then $k[x,y]^d = k[x^p, y^p, f]$ for some $f \in k[x,y]$. W. Li proved in [2] that the rank of $k[x,y]^d$ as a free $k[x^p, y^p]$-module equals 1 or p.

It is natural to ask, for arbitrary p, when a k-derivation of $k[x,y]$ has the ring of constants of the form $k[x^p, y^p, f]$, where $f \in k[x,y] \setminus k[x^p, y^p]$. In this paper we answer this question for derivations which are homogeneous with respect to a given weight vector (Theorem 11, Corollary 12). This is a generalization of the results of [1].

In the last section we obtain a description, for arbitrary p, of all monomial derivations of $k[x,y]$ with rings of constants of the form $k[x^p, y^p, f]$, where $f \in k[x,y] \setminus k[x^p, y^p]$ (Theorem 16, Corollary 17). Note that the rings of constants of all monomial derivations for $p = 2$ and $p = 3$ were computed by S.-I. Okuda in [5], using his adaptation of van den Essen's algorithm for the case of positive characteristic.

1. Preliminaries. Throughout this paper k is a field of characteristic $p > 0$. We denote by $k[X]$ the polynomial k-algebra $k[x_1, \dots, x_n]$ and by $k[X^p]$ the k-subalgebra $k[x_1^p, \dots, x_n^p]$. In the case of two variables we will just write $k[x,y]$.

A k-linear mapping $d\colon k[X] \to k[X]$ is called a *k-derivation* of $k[X]$ if $d(fg) = fd(g) + gd(f)$ for all $f, g \in k[X]$. Every k-derivation d of $k[X]$ is

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of the form $g_1 \cdot \partial/\partial x_1 + \cdots + g_n \cdot \partial/\partial x_n$ for some polynomials $g_1, \ldots, g_n \in k[X]$, that is, d is uniquely determined by the conditions $d(x_1) = g_1, \ldots, d(x_n) = g_n$. If d is a k-derivation of $k[X]$, then we denote by $k[X]^d$ the *ring of constants* of d:

$$k[X]^d = \{f \in k[X] : d(f) = 0\}.$$

Note that $k[X^p] \subseteq k[X]^d$, so $k[X]^d$ is a $k[X^p]$-algebra.

We introduce the notion of γ-homogeneity analogously to [3, 2.1]. Consider a vector $\gamma = (\gamma_1, \ldots, \gamma_n) \in k^n \setminus \{(0, \ldots, 0)\}$. For every $r \in k$ denote by $k[X]^{\gamma}_{(r)}$ the k-linear span of all monomials $x_1^{l_1} \cdots x_n^{l_n}$ such that

$$l_1\gamma_1 + \cdots + l_n\gamma_n = r.$$

If no monomial satisfies this equality, then $k[X]^{\gamma}_{(r)} = 0$. We obtain a grading of $k[X]$ by the additive group of the field k. Polynomials belonging to $k[X]^{\gamma}_{(r)}$ are called *γ-forms of degree r*. In particular, x_i is a γ-form of degree γ_i for $i = 1, \ldots, n$. If $\gamma_1 = \cdots = \gamma_n$, then the γ-forms are exactly the p-homogeneous polynomials in the sense of [1].

A k-derivation d of $k[X]$ is called *γ-homogeneous of degree s*, where $s \in k$, if $d(k[X]^{\gamma}_{(r)}) \subseteq k[X]^{\gamma}_{(r+s)}$ for every $r \in k$, that is, $d(x_i) \in k[X]^{\gamma}_{(\gamma_i+s)}$ for $i = 1, \ldots, n$. Denote by E^{γ} the derivation of the form

$$\gamma_1 x_1 \cdot \frac{\partial}{\partial x_1} + \cdots + \gamma_n x_n \cdot \frac{\partial}{\partial x_n},$$

which is γ-homogeneous of degree 0. Observe that

$$E^{\gamma}(x_1^{l_1} \cdots x_n^{l_n}) = (l_1\gamma_1 + \cdots + l_n\gamma_n) \cdot x_1^{l_1} \cdots x_n^{l_n},$$

so a polynomial f is a γ-form of degree r if and only if $E^{\gamma}(f) = rf$. This is a weight analog of the Euler formula (compare [3, 2.1.1], [1, 1.4]). In particular, $k[x, y]^{\gamma}_{(0)}$ is the ring of constants of E^{γ}.

For every $f \in k[X]$ let

$$C_k(f) = k(x_1^p, \ldots, x_n^p)[f] \cap k[X],$$

where $k(x_1^p, \ldots, x_n^p)$ is the subfield of $k(x_1, \ldots, x_n)$ generated by $x_1^p, \ldots, x_n^p$. The following fact immediately follows from [4, Proposition 1.2].

Proposition 1. *If d is a nonzero k-derivation of $k[x, y]$ such that $k[x, y]^d \neq k[x^p, y^p]$, then $k[x, y]^d = C_k(f)$ for some (and then for any) $f \in k[x, y]^d \setminus k[x^p, y^p]$.* ∎

We denote by $\bar{f}$ the greatest common divisor of $\partial f/\partial x_1, \ldots, \partial f/\partial x_n$ (defined up to a nonzero scalar factor). We write $f \sim g$, where f, g are polynomials, if $f = ag$ for some $a \in k \setminus \{0\}$. We use the same convention for derivations, i.e. we write $d_1 \sim d_2$ if $d_1 = ad_2$ for some $a \in k \setminus \{0\}$.

It is easy to verify that Corollary 2.4, Proposition 2.6, Theorem 3.2 and Corollary 3.3 from [1] hold true for γ-forms, so we obtain the following result.

PROPOSITION 2. *If $f \in k[X] \setminus k[X^p]$ is a γ-form of a nonzero degree, then the following conditions are equivalent*:

(i) $C_k(f) = k[X^p][f]$,
(ii) *f has no multiple factors and no factors from $k[X^p] \setminus k$,*
(iii) $\overline{f} \sim 1$. ∎

2. γ-homogeneous derivations of $k[x,y]$. For a polynomial $f \in k[x,y]$ we denote by d_f the *jacobian derivation* with respect to f:

$$d_f = \frac{\partial f}{\partial x} \cdot \frac{\partial}{\partial y} - \frac{\partial f}{\partial y} \cdot \frac{\partial}{\partial x}.$$

If f is a γ-form of degree r, where $\gamma = (\lambda, \mu)$, then d_f is a γ-homogeneous derivation of degree $r - \lambda - \mu$. Note that $d_f = d_g$ if and only if $f - g \in k[x^p, y^p]$.

We can reformulate Proposition 4.1 and generalize Proposition 4.3 from [1] in the following way.

PROPOSITION 3. *Let d be a nonzero k-derivation of $k[x,y]$ such that $k[x,y]^d \neq k[x^p, y^p]$, and let $f \in k[x,y]^d \setminus k[x^p, y^p]$. Then*

$$\overline{f} \cdot d \sim \gcd(d(x), d(y)) \cdot d_f,$$

where $\overline{f} = \gcd(\partial f/\partial x, \partial f/\partial y)$. In particular, if $d(x)$, $d(y)$ are coprime and $\overline{f} \sim 1$, then $d \sim d_f$. ∎

COROLLARY 4. *Let d be a nonzero k-derivation of $k[x,y]$. If $d(f) = 0$ for some $f \in k[x,y]^{\gamma}_{(0)} \setminus k[x^p, y^p]$, then $k[x,y]^d = k[x,y]^{\gamma}_{(0)}$.*

Proof. If $f \in [x,y]^{\gamma}_{(0)}$, then $E^{\gamma}(f) = 0$, so $k[x,y]^d = k[x,y]^{E^{\gamma}} = k[x,y]^{\gamma}_{(0)}$, by Proposition 1. ∎

COROLLARY 5. *Let $\gamma = (\lambda, \mu)$ and let $f \in k[x,y] \setminus k[x^p, y^p]$ be a γ-form of degree 0 such that $\overline{f} \sim 1$.*

(a) *If $\lambda, \mu \neq 0$, then $d_f \sim E^{\gamma}$.*
(b) *If $\lambda = 0$, $\mu \neq 0$, then $y d_f \sim E^{\gamma}$.*
(c) *If $\lambda \neq 0$, $\mu = 0$, then $x d_f \sim E^{\gamma}$.*

Proof. Applying Proposition 3 to $d = E^{\gamma}$, we obtain the following formula:

$$\gcd(\lambda x, \mu y) \cdot d_f \sim \overline{f} \cdot E^{\gamma}. \blacksquare$$

Recall Proposition 2.7 from [1] in the case of two variables.

PROPOSITION 6. *Let $f, g \in k[x,y]$. Then $k[x^p, y^p, f] = k[x^p, y^p, g]$ if and only if $f - ag \in k[x^p, y^p]$ for some $a \in k \setminus \{0\}$.* ∎

The following proposition is a generalization of Proposition 4.4 from [1]. This proof is new; the proof in [1] was partially specific to homogeneity without weights.

Proposition 7. *Let* $f \in k[x,y]^{\gamma}_{(0)} \setminus k[x^p,y^p]$, *where* $\gamma = (\lambda,\mu)$. *Then the following conditions are equivalent*:

(i) $k[x,y]^{\gamma}_{(0)} = k[x^p,y^p,f]$,

(ii) $\left.\begin{array}{l}\lambda+\mu=0,\ f=axy+g\\ \text{or } \lambda=0,\ f=ax+g\\ \text{or } \mu=0,\ f=ay+g\end{array}\right\}$ *for some* $a \in k\setminus\{0\}$ *and* $g \in k[x^p,y^p]$,

(iii) $\overline{f} \sim 1$.

Proof. (i)⇒(ii). Assume that $k[x,y]^{\gamma}_{(0)} = k[x^p,y^p,f]$. If $\lambda+\mu=0$, then all monomials of degree 0 are of the form $x^{mp+l}y^{np+l}$, where $m,n,l \geq 0$, so $k[x,y]^{\gamma}_{(0)} = k[x^p,y^p,xy]$, and, by Proposition 6, $f - axy \in k[x^p,y^p]$ for some $a \in k\setminus\{0\}$. If $\lambda = 0$, $\mu \neq 0$, then all monomials of degree 0 are of the form $x^l y^{np}$, where $l,n \geq 0$, and we have $k[x,y]^{\gamma}_{(0)} = k[x^p,y^p,x]$, so (Proposition 6) $f - ax \in k[x^p,y^p]$ for some $a \in k\setminus\{0\}$. Analogously, if $\lambda \neq 0$, $\mu = 0$, then $k[x,y]^{\gamma}_{(0)} = k[x^p,y^p,y]$, so $f - ay \in k[x^p,y^p]$ for some $a \in k\setminus\{0\}$.

Now, let $\lambda,\mu \neq 0$ and $\lambda+\mu \neq 0$. Note that λ,μ are linearly dependent over the prime subfield $\mathbb{F}_p$ of k, because $k[x,y]^{\gamma}_{(0)} \neq k[x^p,y^p]$. Consider integers $j,l \in \{2,\dots,p-1\}$ such that $j\lambda+\mu=0$ and $\lambda+l\mu=0$. In this case the monomials x^jy and xy^l are γ-homogeneous of degree 0, so $x^jy, xy^l \in k[x^p,y^p,f]$. Following the method from Example 4.3 in [4], we consider polynomials $u(T), v(T) \in k[x^p,y^p][T]$ such that $x^jy = u(f)$, $xy^l = v(f)$. We obtain the following equalities:

$$jx^{j-1}y = u'(f)\cdot\frac{\partial f}{\partial x},\qquad x^j = u'(f)\cdot\frac{\partial f}{\partial y},\qquad y^l = v'(f)\cdot\frac{\partial f}{\partial x},$$

from which we deduce that $u'(f) = cx^{j-1}$ for some $c \in k\setminus\{0\}$, so $x^{j-1} \in k[x^p,y^p,f]$. This is a contradiction, because $E^{\gamma}(x^{j-1}) \neq 0$.

(ii)⇒(i). Consider arbitrary $a \in k\setminus\{0\}$ and $g \in k[x^p,y^p]$. If $\lambda+\mu=0$, then $k[x,y]^{\gamma}_{(0)} = k[x^p,y^p,xy] = k[x^p,y^p,f]$ for $f = axy+g$. If $\lambda = 0$ and $f = ax+g$, then $k[x,y]^{\gamma}_{(0)} = k[x^p,y^p,x] = k[x^p,y^p,f]$. Analogously, if $\mu = 0$ and $f = ay+g$, then $k[x,y]^{\gamma}_{(0)} = k[x^p,y^p,y] = k[x^p,y^p,f]$.

(ii)⇒(iii). Obviously, in each case f belongs to $k[x,y]^{\gamma}_{(0)} \setminus k[x^p,y^p]$ and $\partial f/\partial x$, $\partial f/\partial y$ are coprime.

(iii)⇒(ii). If $\lambda,\mu \neq 0$, then, by Corollary 5, $d_f = cE^{\gamma}$ for some $c \in k\setminus\{0\}$, so we obtain a system of partial differential equations $\partial f/\partial x = c\mu y$ and $\partial f/\partial y = -c\lambda x$. Note that

$$c\mu = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = -c\lambda,$$

so $\lambda+\mu = 0$. In this case the general solution is of the form $f = c\mu xy + g$, where $g \in k[x^p,y^p]$.

If $\lambda = 0$, then (Corollary 5) $yd_f = cE^\gamma$ for some $c \in k \setminus \{0\}$, we have a system $\partial f/\partial x = c\mu$, $\partial f/\partial y = 0$, and the solution is $f = c\mu x + g$, where $g \in k[x^p, y^p]$. Analogously, if $\mu = 0$, then $\partial f/\partial x = 0$ and $\partial f/\partial y = -c\lambda$, so $f = -c\lambda y + g$, where $g \in k[x^p, y^p]$. ■

COROLLARY 8. *Let d be a nonzero k-derivation of $k[x,y]$ such that $k[x,y]^d \neq k[x^p, y^p]$, and let $f \in k[x,y]^d \setminus k[x^p, y^p]$ be a γ-form. Then $k[x,y]^d = k[x^p, y^p, f]$ if and only if $\overline{f} \sim 1$.*

Proof. This follows from Propositions 1 and 2 if f is a γ-form of a nonzero degree, and from Proposition 7 and Corollary 4 if f is a γ-form of degree 0. ■

The next two propositions explain some relations between γ-homogeneity of derivations and γ-homogeneity of polynomials.

LEMMA 9. *Let $f \in k[x,y] \setminus k[x^p, y^p]$. If d_f is a γ-homogeneous k-derivation of $k[x,y]$, then there exists a γ-form $h \in k[x,y] \setminus k[x^p, y^p]$ such that $f - h \in k[x^p, y^p]$.*

Proof. Assume that d_f is γ-homogeneous of degree s. This means that $\partial f/\partial x$ and $\partial f/\partial y$ are γ-forms of degrees $s + \mu$ and $s + \lambda$, respectively.

If f_r is the γ-homogeneous component of f of degree $r \in k$, then $\partial f_r/\partial x$ is the γ-homogeneous component of $\partial f/\partial x$ of degree $r - \lambda$, so $\partial f_r/\partial x = 0$ for $r \neq s + \lambda + \mu$. Analogously, $\partial f_r/\partial y$ is the γ-homogeneous component of $\partial f/\partial y$ of degree $r - \mu$, so $\partial f_r/\partial y = 0$ for $r \neq s + \lambda + \mu$. This implies that $f_r \in k[x^p, y^p]$ for $r \neq s + \lambda + \mu$, and we may put $h = f_{s+\lambda+\mu}$. ■

LEMMA 10. *If d is a nonzero γ-homogeneous k-derivation of $k[x,y]$ such that $k[x,y]^d = k[x^p, y^p, f]$, where $f \in k[x,y] \setminus k[x^p, y^p]$, then there exists a γ-form $h \in k[x,y]^d \setminus k[x^p, y^p]$ such that $f - h \in k[x^p, y^p]$.*

Proof. By γ-homogeneity of d, all γ-homogeneous components of f belong to $k[x,y]^d$. If the γ-homogeneous component of f of degree 0 does not belong to $k[x^p, y^p]$, then $k[x,y]^d = k[x,y]^\gamma_{(0)}$ by Corollary 4, so $f \in k[x,y]^\gamma_{(0)}$, and we may apply the implication (i)⇒(ii) from Proposition 7.

Now assume that the γ-homogeneous component of f of degree 0 belongs to $k[x^p, y^p]$. Let f_r be the γ-homogeneous component of f of degree $r \neq 0$, so $f_r \in k[x,y]^d$, and, by the assumption, $f_r = u(f)$ for some polynomial $u(T) \in k[x^p, y^p][T]$. Then $rf_r = E^\gamma(f_r) = E^\gamma(f) \cdot u'(f)$.

Assume that $f_r \neq 0$. Then $\deg f_r \leq \deg E^\gamma(f)$, where deg denotes the ordinary degree of a polynomial, so the above equality implies that $rf_r = cE^\gamma(f)$ for some $c \in k \setminus \{0\}$. Hence $E^\gamma(f)$ is a γ-form of degree r and f_r is the only nonzero γ-homogeneous component of f of a nonzero degree, so we may put $h = f_r$. ■

Now we are ready to prove the following theorem.

Theorem 11. *Let k be a field of characteristic $p > 0$, let d be a nonzero γ-homogeneous k-derivation of $k[x,y]$ such that $d(x)$ and $d(y)$ are coprime, and let $f \in k[x,y] \setminus k[x^p,y^p]$. Then*

$$k[x,y]^d = k[x^p,y^p,f]$$

if and only if $d \sim d_f$.

Proof. ($\Rightarrow$) If $k[x,y]^d = k[x^p,y^p,f]$ for some $f \in k[x,y] \setminus k[x^p,y^p]$, then (Lemma 10) there exists a γ-form $h \in k[x,y]$ such that $f - h \in k[x^p,y^p]$, that is, $k[x,y]^d = k[x^p,y^p,h]$. Then $\overline{h} \sim 1$ by Corollary 8, so, by Proposition 3, $d \sim d_h \sim d_f$.

($\Leftarrow$) If $d \sim d_f$, then (Lemma 9) $d \sim d_h$ for some γ-form $h \in k[x,y] \setminus k[x^p,y^p]$ such that $f - h \in k[x^p,y^p]$. Since $d(x)$ and $d(y)$ are coprime, that is, $\overline{h} \sim 1$, we deduce by Corollary 8 that $k[x,y]^d = k[x^p,y^p,h] = k[x^p,y^p,f]$. ∎

Corollary 12. *Let d be a nonzero γ-homogeneous k-derivation of $k[x,y]$ such that $d(x)$ and $d(y)$ are coprime. Then $k[x,y]^d = k[x^p,y^p,f]$ for some $f \in k[x,y] \setminus k[x^p,y^p]$ if and only if d is a jacobian derivation.* ∎

3. Monomial derivations of $k[x,y]$. A k-derivation $d\colon k[x,y] \to k[x,y]$ is called *monomial* if $d(x) = x^t y^u$ and $d(y) = x^v y^w$ for some integers $t,u,v,w \geq 0$. We will consider a slightly more general case:

$$(*) \qquad \begin{cases} d(x) = \alpha x^t y^u, \\ d(y) = \beta x^v y^w, \end{cases}$$

where $\alpha, \beta \in k$.

Now consider an arbitrary nonzero k-derivation d of $k[x,y]$ and a polynomial $f \in k[x,y] \setminus k[x^p,y^p]$. By Corollary 8, if $\partial f/\partial x$ and $\partial f/\partial y$ are coprime, $d(f) = 0$ and f is a γ-form for some γ, then $k[x,y]^d = k[x^p,y^p,f]$. This is the way one can easily verify the following fact.

Example 13. Let m,n,r,s be nonnegative integers, $m,n \not\equiv -1 \pmod{p}$, let $\alpha, \beta \in k \setminus \{0\}$. The following k-derivations of $k[x,y]$ have the rings of contants of the form $k[x^p,y^p,f]$, where $f \in k[x,y] \setminus k[x^p,y^p]$:

$$\begin{cases} d_1(x) = \alpha x^{rp}, \\ d_1(y) = \beta y^{sp}, \end{cases} \qquad k[x,y]^{d_1} = k[x^p,y^p,\beta x y^{sp} - \alpha x^{rp} y],$$

$$\begin{cases} d_2(x) = \alpha x, \\ d_2(y) = -\alpha y, \end{cases} \qquad k[x,y]^{d_2} = k[x^p,y^p,xy],$$

$$\begin{cases} d_3(x) = \alpha y^n, \\ d_3(y) = \beta x^m, \end{cases} \qquad k[x,y]^{d_3} = k[x^p,y^p,(n+1)\beta x^{m+1} - (m+1)\alpha y^{n+1}],$$

$$\begin{cases} d_4(x) = \alpha x^{rp} y^n, \\ d_4(y) = \beta, \end{cases} \qquad k[x,y]^{d_4} = k[x^p,y^p,(n+1)\beta x - \alpha x^{rp} y^{n+1}],$$

$$\begin{cases} d_5(x) = 0, \\ d_5(y) = \beta, \end{cases} \qquad k[x,y]^{d_5} = k[x^p, y^p, x],$$

$$\begin{cases} d_6(x) = \alpha, \\ d_6(y) = \beta x^m y^{sp}, \end{cases} \qquad k[x,y]^{d_6} = k[x^p, y^p, \beta x^{m+1} y^{sp} - (m+1)\alpha y],$$

$$\begin{cases} d_7(x) = \alpha, \\ d_7(y) = 0, \end{cases} \qquad k[x,y]^{d_7} = k[x^p, y^p, y]. \ \blacksquare$$

We will show in Theorem 16 that the above derivations are, up to multiplication by a monomial, all derivations of the form $(*)$ such that $k[x,y]^d = k[x^p, y^p, f]$, where $f \in k[x,y] \setminus k[x^p, y^p]$. Note the following adaptation of Proposition 2.1.6 from [3]. The original proof remains valid in our situation.

LEMMA 14. *Let d be a k-derivation of $k[x,y]$ of the form $(*)$. Then there exists a vector $\gamma \in k^2 \setminus \{(0,0)\}$ such that d is a γ-homogeneous derivation.* ■

Recall that if d is a k-derivation of $k[x,y]$, then the polynomial

$$d^* = \frac{\partial(d(x))}{\partial x} + \frac{\partial(d(y))}{\partial y}$$

is called the *divergence* of d, and recall Lemma 5.1 from [1].

LEMMA 15. *Let d be a k-derivation of $k[x,y]$ and let*

$$d(x) = \sum_{0 \le j,l < p} a_{jl} x^j y^l, \qquad d(y) = \sum_{0 \le j,l < p} b_{jl} x^j y^l,$$

where $a_{jl}, b_{jl} \in k[x^p, y^p]$. Then d is a jacobian derivation if and only if

$$(**) \qquad d^* = 0, \qquad a_{0,p-1} = 0, \qquad b_{p-1,0} = 0. \ \blacksquare$$

Finally, we can prove the following theorem.

THEOREM 16. *Let k be a field of characteristic $p > 0$, and let d be a k-derivation of $k[x,y]$ of the form $(*)$. Then*

$$k[x,y]^d = k[x^p, y^p, f]$$

for some $f \in k[x,y] \setminus k[x^p, y^p]$ if and only if $d = x^j y^l \cdot d_i$, where $j, l \ge 0$, $i \in \{1, \ldots, 7\}$ and d_i is a derivation from Example 13 with $m, n, r, s \ge 0$, $m, n \not\equiv -1 \pmod p$, $\alpha, \beta \in k \setminus \{0\}$.

Proof. We may assume that d is a nonzero derivation. If $\alpha, \beta \ne 0$, we put $j = \min(t, v)$ and $l = \min(u, w)$, if $\alpha = 0$, we put $j = v$, $l = w$, and if $\beta = 0$, we put $j = t$, $l = u$. Then $d = x^j y^l \cdot d_0$, where d_0 is a k-derivation of $k[x,y]$ such that $d_0(x)$ and $d_0(y)$ are coprime. By Lemma 14 the derivation d_0 is γ-homogeneous for some $\gamma \in k^2 \setminus \{(0,0)\}$, so, by Corollary 12, the ring of constants of d is of the form $k[x^p, y^p, f]$, where $f \in k[x,y] \setminus k[x^p, y^p]$, if and only if d_0 is a jacobian derivation. We verify the conditions $(**)$ from

Lemma 15 for all possible forms of d_0:

(a)
$$\begin{cases} d_0(x) = \alpha x^m, \\ d_0(y) = \beta y^n, \end{cases}$$

where $m, n \geq 0$, $\alpha, \beta \neq 0$. We have $d_0^* = m\alpha x^{m-1} + n\beta y^{n-1}$, $a_{0,p-1} = 0$ and $b_{p-1,0} = 0$. The conditions $(**)$ hold in two cases:

- $m \equiv 0 \pmod p$ and $n \equiv 0 \pmod p$, that is, $d_0 = d_1$,
- $m = 1$, $n = 1$ and $\alpha + \beta = 0$, that is, $d = d_2$.

(b)
$$\begin{cases} d_0(x) = \alpha y^n, \\ d_0(y) = \beta x^m, \end{cases}$$

where $m, n \geq 0$, $\alpha, \beta \neq 0$. In this case $d_0^* = 0$. The conditions $(**)$ are equivalent to $m, n \not\equiv -1 \pmod p$, that is, $d_0 = d_3$.

(c)
$$\begin{cases} d_0(x) = \alpha x^m y^n, \\ d_0(y) = \beta, \end{cases}$$

where $m, n \geq 0$, $\beta \neq 0$. We have $d_0^* = m\alpha x^{m-1} y^n$, $b_{p-1,0} = 0$. The conditions $(**)$ hold in two cases:

- $m \equiv 0 \pmod p$ and $n \not\equiv -1 \pmod p$, when $d_0 = d_4$,
- $\alpha = 0$, when $d_0 = d_5$.

(d)
$$\begin{cases} d_0(x) = \alpha, \\ d_0(y) = \beta x^m y^n, \end{cases}$$

where $m, n \geq 0$, $\alpha \neq 0$. We have $d_0^* = n\beta x^m y^{n-1}$, $a_{0,p-1} = 0$. The conditions $(**)$ hold in two cases:

- $m \not\equiv -1 \pmod p$ and $n \equiv 0 \pmod p$, when $d_0 = d_6$,
- $\beta = 0$, when $d_0 = d_7$.

Note that in each case a polynomial f such that $k[x,y]^d = k[x^p, y^p, f]$ can be easily obtained from the condition $d_0 = d_f$, that is, $\partial f/\partial x = d_0(y)$ and $\partial f/\partial y = -d_0(x)$. ∎

Corollary 17. *All monomial k-derivations of $k[x,y]$ such that $k[x,y]^d = k[x^p, y^p, f]$ for some $f \in k[x,y] \setminus k[x^p, y^p]$ are of the form $x^j y^l \cdot d_i$, where*:

(1) $j, l \geq 0$,
(2) $i \in \{1, \ldots, 7\}$, *but $i = 2$ only in the case of $p = 2$*,
(3) $d_1, \ldots, d_7$ *are derivations from Example* 13 *with* $m, n, r, s \geq 0$, $m, n \not\equiv -1 \pmod p$ *and* $\alpha = \beta = 1$. ∎

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