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ON INVOLUTIONS OF ITERATED BUNDLE FUNCTORS

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Abstract. We introduce the concept of an involution of iterated bundle functors. Then we study the problem of the existence of an involution for bundle functors defined on the category of fibered manifolds with m-dimensional bases and of fibered manifold morphisms covering local diffeomorphisms. We also apply our results to prolongation of connections.

Introduction Let $\mathcal{M}f$ be the category of smooth manifolds and all smooth maps, and $\mathcal{FM}_m$ be the category of fibered manifolds with m-dimensional bases and of fibered manifold morphisms covering local diffeomorphisms. Our starting point are the facts that every product preserving bundle functor on $\mathcal{M}f$ is a Weil functor T^A (cf. [7], [11]), and every fiber product preserving bundle functor on $\mathcal{FM}_m$ can be identified with a triple (A, H, t) (cf. [12]), where A is a Weil algebra and H and t are described in Section 1 below. From such a point of view, the Weil algebras can be regarded as a general technique for investigating the geometric properties of all these functors. In particular, the iteration $T^A T^B$ of Weil functors is determined by the tensor product $A \otimes B$ of Weil algebras, and for every pair $F = T^A$ and $G = T^B$ of Weil functors there is a natural equivalence

$$FG \to GF \tag{1}$$

induced by the exchange isomorphism $A \otimes B \to B \otimes A$. On the other hand, the iteration of fiber product preserving bundle functors on $\mathcal{FM}_m$ is more sophisticated; see [2]. Moreover, there is the following open problem:

Problem 1. *For which pairs F and G of bundle functors on $\mathcal{FM}_m$ does there exist a natural exchange equivalence* (1)?

The aim of this paper is to study this problem in a systematic way. In Section 1 we recall the Weil description of fiber product preserving bundle functors on $\mathcal{FM}_m$. We also present some important examples of such functors, which we need in what follows. In Section 2 we introduce the general

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concept of an involution of iterated bundle functors. Roughly speaking, it is a natural equivalence (1) which interchanges the related projections. Further, in Section 3 we present a nontrivial example of an involution of iterated bundle functors on $\mathcal{FM}_m$. The main result of this paper is proved in Section 4. We classify all bundle functors F on $\mathcal{FM}_m$ with the point property, which admit a natural exchange equivalence (1), where G is a higher order jet functor. As a direct consequence we prove nonexistence of involutions of iterated higher order jet functors. The rest of the paper is devoted to applications of our results to prolongation of connections. In particular, in Section 5 we introduce prolongation of higher order connections to vertical Weil bundles and in Section 6 we classify all fiber product preserving bundle functors F on $\mathcal{FM}_m$ which admit a natural operator transforming higher order connections on a fibered manifold $Y \to M$ into connections on $FY \to M$.

In what follows we use the terminology and notation from the book [11]. We denote by $\mathcal{M}f_m \subset \mathcal{M}f$ the subcategory of m-dimensional manifolds and their local diffeomorphisms, by $\mathcal{FM} \supset \mathcal{FM}_m$ the category of fibered manifolds and fiber respecting mappings, and by $\mathcal{FM}_{m,n}$ the subcategory of fibered manifolds with m-dimensional bases and n-dimensional fibres and their local fibered diffeomorphisms. All manifolds and maps are assumed to be infinitely differentiable.

1. The foundations. We first recall that the natural transformations $T^A \to T^B$ of two Weil functors are in bijection with the algebra homomorphisms $A \to B$ (cf. [11]). I. Kolář and the second author [12] have recently characterized all fiber product preserving functors on $\mathcal{FM}_m$ and their natural transformations in terms of Weil algebras. We recall that a bundle functor F on $\mathcal{FM}_m$ is said to be *of order* r if from $j^r_y f = j^r_y g$ it follows that $F_y f = F_y g$ for any two fibered manifolds $p : Y \to M$, $q : Z \to N$, two $\mathcal{FM}_m$-morphisms $f, g : Y \to Z$ and every point $y \in Y$. Write $\mathbb{D}^r_m = J^r_0(\mathbb{R}^m, \mathbb{R})$ for the algebra of all r-jets of $\mathbb{R}^m$ to $\mathbb{R}$ with source $0 \in \mathbb{R}^m$ and denote by G^r_m the rth jet group in dimension m. According to [12], all fiber product preserving bundle functors on $\mathcal{FM}_m$ of order r are in bijection with the triples (A, H, t), where A is a Weil algebra, H is the homomorphism of G^r_m into the group $\mathrm{Aut}(A)$ of all automorphisms of A and $t : \mathbb{D}^r_m \to A$ is an equivariant algebra homomorphism. In such a case we write $F = T^{(A,H,t)}$.

Let F be a natural bundle on $\mathcal{M}f_n$. Given a fibered manifold $Y \to M$, the *F-vertical functor* V^F on $\mathcal{FM}_{m,n}$ is defined fiberwise by

$$V^F Y = \bigcup_{x \in M} F(Y_x)$$

and analogously for morphisms. If $F = T^A$ is a Weil functor determined by the Weil algebra A, then V^{T^A} is the *vertical Weil functor*, which will be denoted by V^A. Clearly, this is a fiber product preserving functor on $\mathcal{FM}_m \supset \mathcal{FM}_{m,n}$. Denoting by r its order, the corresponding triple of V^A is of the form (A, H, t), where $H : G^r_m \to \{\mathrm{id}_A\} \subset \mathrm{Aut}(A)$ is the trivial group homomorphism and $t : \mathbb{D}^r_m \to \mathbb{R}\cdot 1 \subset A$ is the trivial algebra homomorphism.

It is well known that in the theory of higher order jets it is useful to distinguish between holonomic, nonholonomic and semiholonomic jets; see e.g. [5] and [13]. For every fibered manifold $Y \to M$ we have its classical r-jet prolongation $J^rY \to M$, which is called *holonomic*. Further, the functor $\widetilde{J}^r$ of *nonholonomic jets* is defined by iteration

$$\widetilde{J}^r = J^1 \circ \cdots \circ J^1 \qquad (r \text{ times}).$$

Moreover, the rth *semiholonomic prolongation* $\overline{J}^rY \subset \widetilde{J}^rY$ is defined by the following induction. Write $\overline{J}^0Y = Y$, $\overline{J}^1Y = J^1Y$ and assume we have defined $\overline{J}^{r-1}Y \subset \widetilde{J}^{r-1}Y$ such that the restriction of the projection $\beta_{r-1} : \widetilde{J}^{r-1}Y \to \widetilde{J}^{r-2}Y$ maps $\overline{J}^{r-1}Y$ into $\overline{J}^{r-2}Y$. Then we have an induced map $J^1\beta_{r-1} : J^1\overline{J}^{r-1}Y \to J^1\overline{J}^{r-2}Y$ and we can define

$$\overline{J}^rY = \{U \in J^1\overline{J}^{r-1}Y \mid \beta_r(U) = J^1\beta_{r-1}(U) \in \overline{J}^{r-1}Y\}.$$

One can also define other kinds of subspaces in $\widetilde{J}^rY$. For example, the rth *sesquiholonomic prolongation* $\widehat{J}^rY$ of Y is defined by

$$\widehat{J}^rY = J^1(J^{r-1}Y) \cap \overline{J}^rY;$$

see e.g. [13]. Of course, $\widehat{J}^2Y$ coincides with $\overline{J}^2Y$ and for $r > 2$ we have

$$J^rY \subset \widehat{J}^rY \subset \overline{J}^rY \subset \widetilde{J}^rY.$$

Clearly, all the functors J^r, $\widehat{J}^r$, $\overline{J}^r$ and $\widetilde{J}^r$ preserve fiber products. I. Kolář [10] has recently introduced the general concept of an rth order jet functor on $\mathcal{FM}_m$. This is a fiber product preserving bundle subfunctor of the rth nonholonomic prolongation that contains the rth holonomic one. It is interesting to point out that all second order jet functors are J^2, $\overline{J}^2$ and $\widetilde{J}^2$ (see [10]).

For any $\mathcal{FM}_m$-object $Y \to M$ we can define its *vertical r-jet prolongation*

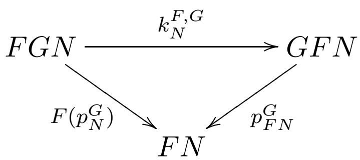
$$J^r_vY = \{j^r_x\sigma \mid \sigma : M \to Y_x,\, x \in M\}$$

over Y. Any $\mathcal{FM}_m$-map $f : Y_1 \to Y_2$ over $\underline{f} : M_1 \to M_2$ induces a fibered map $J^r_vf : J^r_vY_1 \to J^r_vY_2$ covering f such that $J^r_vf(j^r_x\sigma) = j^r_{\underline{f}(x)}(f \circ \sigma \circ \underline{f}^{-1})$, $j^r_x\sigma \in J^r_vY_1$. Then $J^r_v : \mathcal{FM}_m \to \mathcal{FM}$ is a fiber product preserving bundle functor. Quite analogously to the functor J^r, one can also define higher order vertical jet functors $\widetilde{J}^r_v$, $\overline{J}^r_v$ and $\widehat{J}^r_v$.

2. Involution of iterated bundle functors. In general, an involution is a nonidentical mapping f that is its own inverse, so that $f \circ f$ is the identity. Moreover, it is well known that the canonical involution of the iterated tangent bundle interchanges both the projections of TTN into TN. Now let F and G be arbitrary product preserving bundle functors on $\mathcal{M}f$ and denote by $p_N^F : FN \to N$ and $p_N^G : GN \to N$ the bundle projections. By [11], there is a natural equivalence

$$k^{F,G} : FG \to GF \tag{2}$$

such that for every manifold N the following diagram commutes:



So $k^{F,G}$ interchanges the related projections and generalizes the classical involution of TTN to an arbitrary couple F and G of product preserving functors on $\mathcal{M}f$.

However, if F and G are arbitrary fiber product preserving bundle functors on $\mathcal{FM}_m$, then we have the open Problem 1. The first author and I. Kolář [2] have recently constructed the natural isomorphism

$$\kappa_Y^{A,G} : V^A(GY \to M) \to G(V^A Y \to M) \tag{3}$$

for every fibered manifold $Y \to M$. As a particular case we obtain the classical exchange map $V(J^1Y \to M) \to J^1(VY \to M)$ by H. Goldschmidt and S. Sternberg [6] and also the isomorphism $\kappa_Y^{A,\widetilde{J}^r}$ from [1]. Further, the isomorphism $\kappa_Y^{A,J^r} : V^A(J^rY \to M) \to J^r(V^AY \to M)$ was constructed in a direct way by I. Kolář [9]. One finds directly that $\kappa^{A,G}$ is an involution in the following sense.

Definition 1. Let F and G be two bundle functors on $\mathcal{FM}_m$ and denote by $p_Y^F : FY \to Y$ and $p_Y^G : GY \to Y$ the bundle projections. A natural equivalence $A : FG \to GF$ is called an *involution* if

$$F(p_Y^G) = p_{FY}^G \circ A_Y$$

for every fibered manifold $Y \to M$.

So the involution A interchanges the projections $F(p_Y^G) : FGY \to FY$ and $p_{FY}^G : GFY \to FY$. By [11], there is no jet involution $J^1J^1 \to J^1J^1$. On the other hand, M. Modugno [15] has introduced an involution $\mathrm{ex}_\Lambda : J^1J^1 \to J^1J^1$ depending on a classical linear connection Λ on the base manifold M.

Using the fact that ex_Λ interchanges the projections $p^{J^1}_{J^1Y}$ and $J^1(p^{J^1}_Y)$ of J^1J^1Y to J^1Y, he has constructed a connection $\widetilde{J}^1\Gamma$ on $J^1(J^1Y \to M)$ from a connection $\Gamma : Y \to J^1Y$ by applying ex_Λ to the jet extension of Γ. Clearly, this geometric construction has a general character and one easily shows

PROPOSITION 1. *Let F and G be two bundle functors on $\mathcal{FM}_m$ and consider a section $\Gamma : Y \to GY$. If $A : FG \to GF$ is an involution, then the composition*

$$\mathcal{F}\Gamma := A_Y \circ F\Gamma$$

is a section $FY \to GFY$.

REMARK 1. Of course, there are nonidentical natural equivalences which are not involutions. Indeed, let $\mathrm{id}_A \neq \varphi \in \mathrm{Aut}(A)$ be an automorphism of a Weil algebra A. Then $\varphi \otimes \varphi : A \otimes A \to A \otimes A$ induces the required natural equivalence $V^AV^A \to V^AV^A$.

3. Involution of vertical jet functors. Until now, the map (3) was the only known example of an involution. In this section we introduce an involution

$$C^{r,s} : J^r_v J^s_v \to J^s_v J^r_v$$

of iterated higher order vertical jet functors. Write $\mathbb{R}^{m,n}$ for the product fibered manifold $\mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$. In what follows we identify sections of $\mathbb{R}^{m,n}$ with maps $\mathbb{R}^m \to \mathbb{R}^n$. Further, we use the notation

$$j^r_0 j^s_0(f(x, \underline{x})) = j^r_0(x \to j^s_0(\underline{x} \to f(x, \underline{x}))) \in (J^r_v)_0 J^s_v(\mathbb{R}^{m,n}).$$

DEFINITION 2. Define a linear isomorphism

$$C^{r,s}_{m,n} : (J^r_v)_0 J^s_v(\mathbb{R}^{m,n}) \to (J^s_v)_0 J^r_v(\mathbb{R}^{m,n}), \qquad C^{r,s}_{m,n}(j^r_0 j^s_0(f(x, \underline{x}))) = j^s_0 j^r_0(f(\underline{x}, x)). \tag{4}$$

Using standard arguments one can easily show that the definition of $C^{r,s}_{m,n}$ is correct. For, if $|\alpha| > r$ and $|\beta| > p$, then we have

$$C^{r,s}_{m,n}(j^r_0 j^s_0(x^\alpha \underline{x}^\beta g(x, \underline{x}))) = j^s_0 j^r_0((\underline{x})^\alpha x^\beta g(\underline{x}, x)) = 0.$$

Now we prove the following invariance condition:

LEMMA 1. *Let $\Phi : \mathbb{R}^{m,n} \to \mathbb{R}^{m,k}$ be an $\mathcal{FM}_m$-map of the form*

$$\Phi(x, y) = (\varphi(x), \phi(x, y))$$

such that $\varphi(0) = 0$. Then

$$C^{r,s}_{m,k}(J^r_v J^s_v \Phi(v)) = J^s_v J^r_v \Phi(C^{r,s}_{m,n}(v))$$

for any $v \in (J^r_v)_0 J^s_v(\mathbb{R}^{m,n})$.

Proof. We have

$$\begin{aligned}
&C^{r,s}_{m,k}(J^r_v J^s_v \Phi(j^r_0 j^s_0(f(x,\underline{x}))))\\
&\qquad = C^{r,s}_{m,k}(j^r_0 j^s_0(\phi(\varphi^{-1}(\underline{x}), f(\varphi^{-1}(x),\varphi^{-1}(\underline{x})))))\\
&\qquad = j^s_0 j^r_0(\phi(\varphi^{-1}(x), f(\varphi^{-1}(\underline{x}),\varphi^{-1}(x))))\\
&\qquad = J^s_v J^r_v \Phi(j^s_0 j^r(f(\underline{x},x))) = J^s_v J^r_v \Phi(C^{r,s}_{m,n}(j^r_0 j^s_0(f(x,\underline{x})))). \quad \blacksquare
\end{aligned}$$

Using Lemma 1, we can extend $C^{r,s}_{m,n}$ to an $\mathcal{FM}_m$-natural fibered map

$$C^{r,s}_Y : J^r_v J^p_v Y \to J^s_v J^r_v Y.$$

One can see that $C^{s,r}_Y \circ C^{r,s}_Y = \mathrm{id}_{J^r_v J^s_v Y}$ and $\beta^s \circ C^{r,s}_Y = J^r_v(\beta^s)$, where β^s is the bundle projection of $J^s_v Y \to Y$. Hence $C^{r,s}$ is an involution.

We remark that this involution can be used for prolongation of connections. By a *vertical s-connection* on $Y \to M$ we understand a section $\Gamma : Y \to J^s_v Y$ of $J^s_v Y \to Y$. Proposition 1 now becomes

PROPOSITION 2. *Let Γ be a vertical s-connection on $Y \to M$. Then*

$$C^r(\Gamma) = C^{r,s}_Y \circ J^r_v \Gamma : J^r_v Y \to J^s_v J^r_v Y$$

is a vertical s-connection on $J^r_v Y \to M$.

REMARK 2. The rth order nonholonomic vertical bundle $\widetilde{J}^r_v Y$ is defined by the iteration $\widetilde{J}^1_v Y = J^1_v Y$ and $\widetilde{J}^r_v Y = J^1_v(\widetilde{J}^{r-1}_v Y)$. Then we have an involution $\widetilde{C}^{1,1}_Y := C^{1,1}_Y : \widetilde{J}^1_v \widetilde{J}^1_v Y \to \widetilde{J}^1_v \widetilde{J}^1_v Y$. Using induction, one can extend this map to an involution of nonholonomic vertical jet functors

$$\widetilde{C}^{r,s}_Y : \widetilde{J}^r_v \widetilde{J}^s_v Y \to \widetilde{J}^s_v \widetilde{J}^r_v Y.$$

Indeed, we first define

$$\widetilde{C}^{r,1}_Y := \widetilde{C}^{1,1}_{\widetilde{J}^{r-1}_v Y} \circ J^1(\widetilde{C}^{r-1,1}_Y) : \widetilde{J}^r_v \widetilde{J}^1_v Y \to \widetilde{J}^1_v \widetilde{J}^r_v Y$$

and then we set $\widetilde{C}^{r,s}_Y := J^1(\widetilde{C}^{r,s-1}_Y) \circ \widetilde{C}^{r,1}_{\widetilde{J}^{s-1} Y}$.

4. Jet involutions

LEMMA 2. *Let r be some natural number and let $F : \mathcal{FM}_m \to \mathcal{FM}$ be a bundle functor. Suppose that for some natural number n the bundle functor $\widetilde{F}^n : \mathcal{M}f_m \to \mathcal{FM}$ defined by $\widetilde{F}^n M = F(M \times \mathbb{R}^n)$, $\widetilde{F}^n f = F(f \times \mathrm{id}_{\mathbb{R}^n})$ is of minimal order $s \geq 1$. Then there is no involution $A : FJ^r \to J^r F$.*

Proof. If there is an A in question, then we have the natural operator

$$A(\Gamma) = A \circ F\Gamma : FY \to J^r FY$$

transforming rth order holonomic connections $\Gamma : Y \to J^r Y$ on $Y \to M$ into rth order holonomic connections $A(\Gamma)$ on $FY \to M$. In particular, given the rth order trivial connection Γ_M on $M \times \mathbb{R}^n \to M$ (defined by

$\mathit{\Gamma}_M(x,y)=j^r_x(y)$, where $y: M\to M\times\mathbb{R}^n$ is the constant section), we have the rth order connection $A(\mathit{\Gamma}_M)$ on $\widetilde{F}^nM\to M$. Denote by $\widetilde{A}(\mathit{\Gamma}_M)$ the first order underlying connection of $A(\mathit{\Gamma}_M)$. Since $\mathit{\Gamma}_M$ is invariant with respect to $\mathcal{FM}_{m,n}$-maps of the form $f\times\mathrm{id}_{\mathbb{R}^n}$, the connection $\widetilde{A}(\mathit{\Gamma}_M)$ is $\mathcal{M}f_m$-natural. Denoting by $X^{\widetilde{A}(\mathit{\Gamma}_M)}$ the horizontal lift of a vector field X on M with respect to $\widetilde{A}(\mathit{\Gamma}_M)$, we have the $\mathcal{M}f_m$-natural operator $B: T\rightsquigarrow T\widetilde{F}^n$ given by

$$B(X)=X^{\widetilde{A}(\mathit{\Gamma}_M)}.$$

Clearly, B is a zero order operator. Further, as $B(X)$ is projectable over X, we have

$$B=\widetilde{\mathcal{F}}^n+\mathcal{V}$$

for some vertical type $\mathcal{M}f_m$-natural operator $\mathcal{V}$, where $\widetilde{\mathcal{F}}^n$ is the flow operator. As $\widetilde{F}^n$ is of minimal order s, so is the flow operator $\widetilde{\mathcal{F}}^n$. By Lemma 1 from [14], $\mathcal{V}$ is of order $\le s-1$. Then B is of minimal order $s\ge 1$, which is a contradiction. ∎

Remark 3. Under the assumptions of Lemma 2 there is no involution $A: FJ\to JF$, where J is an arbitrary rth order jet functor. Indeed, in the proof of Lemma 2 it suffices to replace J^r by J.

We recall that a bundle functor F on $\mathcal{FM}_m$ is said to have the *point property* if $FQ=Q$ for every m-dimensional manifold Q, which is viewed as a fibered manifold $\mathrm{id}_Q: Q\to Q$.

Lemma 3. *Let $F:\mathcal{FM}_m\to\mathcal{FM}$ be a bundle functor with the point property. Suppose that there is an $\mathcal{FM}_m$-natural equivalence (not necessarily involution) $FJ^r\cong J^rF$ for some natural number r. Then F is fiber product preserving.*

Proof. Write

$$g_l=\dim F_0\mathbb{R}^{m,l}.$$

Similarly to the case of bundle functors on $\mathcal{M}f$ with the point property (see 38.18 in [11]) we have

$$g_{k+l}\ge g_k+g_l. \tag{5}$$

By induction we obtain

$$g_k\ge kg_1\quad \text{for } k=1,2,\ldots$$

Further, let

$$h=\dim J^r_0\mathbb{R}^{m,1}.$$

Since $FJ^r\cong J^rF$, we have

$$g_{hq}=hg_q\quad \text{for } q=1,2,\ldots. \tag{6}$$

Using induction, from (6) we get

(7) $$g_{h^p} = h^p g_1 \quad \text{for } p = 1, 2, \ldots$$

Suppose now

(8) $$g_{k_0} > k_0 g_1 \quad \text{for some } k_0 \geq 1.$$

From (5) and (8) we get by induction

$$g_k > k g_1 \quad \text{for } k = k_0, k_0 + 1, \ldots.$$

But this contradicts (7). Thus we have proved

(9) $$g_k = k g_1 \quad \text{for all } k = 1, 2, \ldots.$$

Just as for bundle functors on $\mathcal{M}f$ with the point property (see 38.18 in [11]), from (9) and the point property of F we deduce that F is fiber product preserving. ∎

REMARK 4. Analyzing the proof we see that Lemma 3 is true for any fiber product preserving bundle functor $G : \mathcal{FM}_m \to \mathcal{FM}$ instead of J^r.

THEOREM 1. *Let $F : \mathcal{FM}_m \to \mathcal{FM}$ be a bundle functor with the point property and let r be a natural number. Then there is an involution $B : FJ^r \to J^r F$ if and only if $F \cong V^A$ for some Weil algebra A.*

Proof. By (3), for $F = V^A$ such an involution exists. So it suffices to prove the "only if" part. By Lemma 3, F is fiber product preserving, so that $F = T^{(A,H,t)}$, where A is a Weil algebra, $H : G^q_m \to \mathrm{Aut}(A)$ is a group homomorphism and $t : \mathbb{D}^q_m \to A$ is a G^q_m-equivariant algebra homomorphism. By Lemma 2, $\widetilde{F}^1$ is of order zero, so that H is trivial. Further, using the equivariance of t with respect to the homotheties $\tau\,\mathrm{id}_{\mathbb{R}^m}$, $\tau \neq 0$, we get

$$t(j^q_0(\gamma(\tau x))) = H(j^q_0(\tau^{-1}\,\mathrm{id}))(t(j^q_0\gamma)) = t(j^q_0\gamma).$$

Letting $\tau \to 0$, we obtain $t(j^q_0\gamma) \in \mathbb{R}$. Then t is also trivial. Consequently, $F \cong V^A$. ∎

Quite similarly one can prove (using Remarks 3 and 4)

THEOREM 2. *Let $F : \mathcal{FM}_m \to \mathcal{FM}$ be a bundle functor with the point property and let J be any rth order jet functor. Then there is an involution $B : FJ \to JF$ if and only if $F \cong V^A$ for some Weil algebra A. In particular, there is an involution $B_1 : F\widetilde{J}^r \to \widetilde{J}^r F$ (resp. $B_2 : F\overline{J}^r \to \overline{J}^r F$, resp. $B_3 : F\widehat{J}^r \to \widehat{J}^r F$) if and only if $F \cong V^A$ for some Weil algebra A.*

Theorem 2 yields

PROPOSITION 3. *Let F be an rth order jet functor and G be an sth order jet functor. Then there is no involution $FG \to GF$.*

COROLLARY 1. *Let F be any of the functors J^r, $\widetilde{J}^r$, $\overline{J}^r$, $\widehat{J}^r$ and G be any of J^s, $\widetilde{J}^s$, $\overline{J}^s$, $\widehat{J}^s$. Then there is no involution $FG \to GF$.*

PROPOSITION 4. *Let F be any of the functors J^r, $\widetilde{J}^r$, $\overline{J}^r$, $\widehat{J}^r$ and G be any of J^s_v, $\widetilde{J}^s_v$, $\overline{J}^s_v$, $\widehat{J}^s_v$. Then there is no involution $GF \to FG$.*

5. Prolongation of connections to vertical Weil bundles. An rth *order nonholonomic connection* in the sense of C. Ehresmann is a section $\Gamma : Y \to \widetilde{J}^r Y$ (see [4] and [8]). This can be generalized in the following way.

DEFINITION 3. Let J be an rth order jet functor on $\mathcal{FM}_m$. An *rth order connection* on a fibered manifold $Y \to M$ is a section $\Gamma : Y \to JY$.

DEFINITION 4. An rth order connection $\Gamma : Y \to JY$ is called *holonomic*, *nonholonomic* or *semiholonomic* if it has values in $J^r Y$, $\widetilde{J}^r Y$ or $\overline{J}^r Y$, respectively.

Clearly, for $r = 1$ all such connections coincide. Proposition 1 immediately gives

PROPOSITION 5. *Let $\Gamma : Y \to JY$ be an rth order connection on $Y \to M$ and let F be a bundle functor on $\mathcal{FM}_m$ such that there is an involution $A : FJ \to JF$. Then $\mathcal{F}\Gamma := A_Y \circ F\Gamma$ is an rth order connection on $FY \to M$.*

Consider now the involution $\kappa^{A,G}$ given by (3), where $G = J$ is an arbitrary rth order jet functor.

DEFINITION 5. The *vertical A-prolongation* of an rth order connection $\Gamma : Y \to JY$ is an rth order connection $\mathcal{V}^A\Gamma$ on $V^A Y \to M$ defined by

$$\mathcal{V}^A\Gamma := \kappa^{A,J}_Y \circ V^A\Gamma : V^A Y \to JV^A Y. \tag{10}$$

In particular, if Γ is a holonomic, semiholonomic or nonholonomic connection on $Y \to M$, then its vertical prolongation $\mathcal{V}^A\Gamma$ is a connection of the same type on $V^A Y \to M$.

6. Existence of prolongation of higher order connections. Using the proof of Lemma 2 we easily obtain

LEMMA 4. *Let r, r' be natural numbers and let $F : \mathcal{FM}_m \to \mathcal{FM}$ be a bundle functor. Suppose that for some natural number n the natural bundle $\widetilde{F}^n : \mathcal{FM}_m \to \mathcal{FM}$ from Lemma 2 is of minimal order $s \geq 1$. Then there is no $\mathcal{FM}_m$-natural operator A transforming rth order holonomic connections Γ on $Y \to M$ into r'th order holonomic connections $A(\Gamma)$ on $FY \to M$.*

THEOREM 3. *Let $F : \mathcal{FM}_m \to \mathcal{FM}$ be a fiber product preserving bundle functor and let r, r' be natural numbers such that $r' \leq r$. Then there is an $\mathcal{FM}_m$-natural operator B transforming rth order holonomic connections Γ on $Y \to M$ into r'th order holonomic connections $B(\Gamma)$ on $FY \to M$ if and only if $F \cong V^A$ for some Weil algebra A.*

Proof. The proof of the "only if" part is quite similar to that of Theorem 1 (we use Lemma 4 instead of Lemma 2). Conversely, if $F = V^A$, then we have the involution $\kappa^{A,J^r} : V^A J^r \to J^r V^A$ and we can construct the rth order holonomic connection

$$B(\Gamma) = \kappa_Y^{A,J^r} \circ V^A(\Gamma) : V^A Y \to J^r V^A Y$$

on $V^A Y \to M$ from an rth order connection $\Gamma : Y \to J^r Y$ on $Y \to M$. For $r' \leq r$ we have the underlying r'th order connection on $V^A Y \to M$. ∎

REMARK 5. Clearly, Theorem 3 is also true for higher order connections in the sense of Definition 3. In particular, we can replace holonomic connections by nonholonomic or semiholonomic ones.

REMARK 6. In Theorem 3 it is essential that the bundle functor F in question preserves fiber products. However, in the case $r = 1$, Theorem 3 is not true for all bundle functors $F : \mathcal{FM}_m \to \mathcal{FM}$ with the point property. Indeed, in [3] we have constructed the connection $\mathcal{V}^H \Gamma$ on $V^H Y \to M$ from a connection Γ on $Y \to M$ for every (not necessarily Weil) bundle functor $H : \mathcal{M}f \to \mathcal{FM}$.

AN UNSOLVED PROBLEM. Modifying the proofs of Propositions 5 and 9 in our paper [3] one can obtain from Lemma 4 the following fact: *If a bundle functor $G : \mathcal{FM}_m \to \mathcal{FM}$ admits an $\mathcal{FM}_m$-natural operator A transforming rth order holonomic connections Γ on $Y \to M$ into r'th order holonomic connections $A(\Gamma)$ on $GY \to M$, then $G \cong V^F$ for some bundle functor $F : \mathcal{M}f \to \mathcal{FM}$.* Therefore it is interesting to answer the following question: do there exist $\mathcal{FM}_m$-natural operators A transforming rth order connections Γ on $Y \to M$ into r'th order connections $A(\Gamma)$ on $V^F Y \to M$ for non-Weil bundle functors $F : \mathcal{M}f \to \mathcal{FM}$ and natural numbers r and r' satisfying $r \geq r' \geq 2$?

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