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ON THE CONVERGENCE OF MOMENTS IN THE CLT FOR TRIANGULAR ARRAYS WITH AN APPLICATION TO RANDOM POLYNOMIALS

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Abstract. We give a proof of convergence of moments in the Central Limit Theorem (under the Lyapunov–Lindeberg condition) for triangular arrays, yielding a new estimate of the speed of convergence expressed in terms of νth moments. We also give an application to the convergence in the mean of the pth moments of certain random trigonometric polynomials built from triangular arrays of independent random variables, thereby extending some recent work of Borwein and Lockhart.

1. Introduction and results. This paper concerns the convergence of moments of order ν in the CLT for triangular arrays of independent random variables, and more precisely the speed of convergence. We obtain a general estimate of the speed of convergence and a new explicit simple form when $\nu > 5$ (with a good control for large ν based on the use of an optimal Rosenthal inequality). This proof is based on a classical approach (replacing step by step the variables under consideration with normal variables, see e.g. Billingsley [3]). Next, we give an application to the convergence in the mean of the pth moments of certain random trigonometric polynomials built from triangular arrays of independent random variables.

Before entering into the matter, we will discuss known facts related to the convergence of moments in the CLT. Let $\{X_{n,k},\ 1 \le k \le k_n,\ n \ge 1\}$ be a triangular array of real centered independent, square integrable random variables and set, for every $n \ge 1$ and $1 \le j, k \le k_n$,

$$\sigma_{n,j}^2 = \mathbb{E}\, X_{n,j}^2, \qquad s_{n,k}^2 = \sum_{j=1}^{k} \sigma_{n,j}^2, \qquad s_n = s_{n,k_n},$$

$$S_{n,k} = \sum_{j=1}^{k} X_{n,j}, \qquad S_n = S_{n,k_n}.$$

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Following Brown [5], we introduce the (generalized) *Lindeberg condition* of order $\nu \geq 2$:

$$(\mathcal{L}_\nu) \qquad \sum_{j=1}^{k_n} \mathbb{E}\left\{|X_{n,j}|^\nu \mathbf{1}_{\{|X_{n,j}|>\varepsilon s_n\}}\right\} = o(s_n^\nu), \qquad (\forall \varepsilon > 0)\ n \to \infty.$$

This is also (see [2]) called *Lyapunov's condition.* As already noticed by Brown, this condition is equivalent, for $\nu > 2$, to

$$(\mathcal{L}'_\nu) \qquad \sum_{j=1}^{k_n} \mathbb{E}\,|X_{n,j}|^\nu = o(s_n^\nu), \qquad n \to \infty.$$

It is a well known fact (*Lindeberg theorem*, see [3] or [11]) that under $(\mathcal{L}_2)$, $\{S_n/s_n\}$ converges in law to the standard normal law. Now, since $\mathbb{E}\,(S_n^2/s_n^2)$ $= 1$, we have

$$\lim_{n\to\infty} \mathbb{E}\,(S_{n,k}^2/s_{n,k}^2) = 1 = m_2,$$

where $m_2 = \mathbb{E}\,W^2$ and W is a variable with standard normal law. More generally, for $\nu > 0$, write $m_\nu := \mathbb{E}\,|W|^\nu$. If $0 < \nu \leq 2$, we have the following proposition.

Proposition 1.1. *Let $\{X_{n,k},\ 1 \leq k \leq k_n,\ n \geq 1\}$ be a triangular array of real centered independent, square integrable random variables. Let $0 < \nu \leq 2$. Assume that $(\mathcal{L}_2)$ holds. Then*

$$\lim_{n\to\infty} \frac{\mathbb{E}\,|S_n|^\nu}{s_n^\nu} = m_\nu.$$

Now, consider the case $\nu > 2$. We first give a general estimate for the νth moment of the sum of n independent random variables. Then we extend the above moment convergence result to the case $\nu > 2$. Moreover, an estimate of the rate of convergence is provided. In the case $\nu > 5$, this estimate turns out to be very simple.

Theorem 1.2. *Let $\nu > 2$. Let $\{Y_k, 1 \leq k \leq n\}$ be real centered independent random variables with finite moment of order ν. Write $S_n = \sum_{k=1}^n Y_k$ and $s_n = (\sum_{k=1}^n \mathbb{E}Y_n^2)^{1/2}$. Then there exists a universal constant C such that*

$$\left|\mathbb{E}\left(\frac{|S_n|}{s_n}\right)^\nu - m_\nu\right| \leq C\,\frac{\sum_{k=1}^n \mathbb{E}\,|Y_k|^\nu}{s_n^\nu} \qquad \textit{for } 2 < \nu \leq 3,$$

$$\left|\mathbb{E}\left(\frac{|S_n|}{s_n}\right)^\nu - m_\nu\right| \leq C\left(\frac{\sum_{k=1}^n \mathbb{E}\,|Y_k|^\nu}{s_n^\nu} + \frac{\sum_{k=1}^n \mathbb{E}\,|Y_k|^3}{s_n^3}\right) \qquad \textit{for } 3 < \nu \leq 5,$$

and, for $\nu > 5$,

$$\left|\mathbb{E}\left(\frac{|S_n|}{s_n}\right)^\nu - m_\nu\right| \le \left(C\,\frac{\nu}{\log\nu}\right)^\nu \left\{\frac{\sum_{k=1}^n \mathbb{E}\,|Y_k|^\nu}{s_n^\nu} + \frac{\sum_{k=1}^n \mathbb{E}\,|Y_k|^3}{s_n^3} + \frac{\sum_{k=1}^n \mathbb{E}\,|Y_k|^{\nu-3}}{s_n^{\nu-3}}\,\frac{\sum_{k=1}^n \mathbb{E}\,|Y_k|^3}{s_n^3}\right\}.$$

As a corollary we obtain

THEOREM 1.3. *Let $\nu > 2$. Let $\{X_{n,k},\ 1 \le k \le k_n,\ n \ge 1\}$ be a triangular array of real centered independent random variables having moments of order ν. Assume that $(\mathcal{L}'_\nu)$ holds. Then $(\mathbb{E}\,|S_n|^\nu)/s_n^\nu$ converges to m_ν as $n \to \infty$, with the speed given by Theorem* 1.2 (*see Lemma* 2.1 *below*). *Further if $\nu \ge 3$, the estimate of the rate of convergence can be simplified*:

$$\left|\mathbb{E}\left(\frac{|S_n|}{s_n}\right)^\nu - m_\nu\right| \le \left(C\,\frac{\nu}{\log\nu}\right)^\nu \max_{h\in\{1,1/(\nu-2)\}} \left(\frac{\sum_{k=1}^{k_n} \mathbb{E}\,|X_{n,k}|^\nu}{s_n^\nu}\right)^h,$$

where C is a universal constant.

For this result (*without* speed of convergence), one usually refers to [2], [6] or [7]. See also [13] and the references therein (notably the works of Kruglov) for an extension to non-homogeneous Markov chains. In [4], the proof relied on a version of this result, i.e., on the study of an array of variables, while the references cited there ([2] and [6]) deal only with a simple sequence of independent variables. In a series of papers ([5]–[8]) Brown was interested in related problems and stated a version of Theorem 1.2 in [7], but without an explicit proof.

There are, however, also convergence results with speed of convergence, and we shall now compare them with Theorems 1.2 and 1.3. In an earlier paper by von Bahr [1] concerning sequences of independent random variables, the rate of convergence is specified (see Theorem 4, p. 816), but instead of $|\mathbb{E}\,(|S_n|/s_n)^\nu - m_\nu|$, a more complicated difference is estimated: $|\mathbb{E}\,(|S_n|/s_n)^\nu - P - m_\nu|$, where P is an expression built up from the Fourier inversion formula allowing one to express $\mathbb{E}\,(|S_n|/s_n)^\nu$ by means of the first ν terms of the Fourier expansion of the characteristic function of $|S_n|/s_n$, with a suitable control of the error. In [10], an estimate in terms of truncated moments is given in the iid case when $\nu < 4$, $\nu \ne 2$. Generalizations of this result to higher moments, using Chebyshev–Cramér expansion, are also given. In [9], the case of independent random variables is considered and results similar to ours are obtained when $2 < \nu < 4$.

Before passing to applications to random polynomials, we should also mention that we could not find in the existing literature a complete detailed proof of the convergence of moments in the CLT for triangular arrays of independent random variables.

Recall now a convergence result recently obtained by Borwein and Lockhart [4], on the L^p-norms of some random polynomials built from sequences of iid random variables. Given a sequence $\{X, X_n,\, n \geq 1\}$ of iid centered real random variables, with unit variance, if $\mathbb{E}\,|X|^\nu < \infty$ for some $\nu > 2$, then

$$\lim_{n\to\infty} \frac{1}{2\pi n^{\nu/2}} \int_0^{2\pi} \mathbb{E}\,|q_n(\theta)|^\nu \, d\theta = \Gamma(1+\nu/2),$$

where $q_n(\theta) = \sum_{k=1}^n X_k e^{ik\theta}$ and $\Gamma(s) = \int_0^\infty u^{s-1} e^{-u}\, du$ is the usual Gamma function.

Using Theorem 1.3, we will also prove the following extension of this result to triangular arrays of independent random variables.

THEOREM 1.4. *Let $\{X_{n,k},\, 1 \leq k \leq k_n,\, n \geq 1\}$ be a triangular array of real centered independent random variables with $\mathbb{E}\, X_{n,k}^2 = 1$, satisfying the Lindeberg condition $(\mathcal{L}_\nu)$ of order $\nu \geq 2$. Then*

$$\lim_{n\to\infty} \frac{1}{2\pi k_n^{\nu/2}} \int_0^{2\pi} \mathbb{E}\,|q_n(\theta)|^\nu \, d\theta = \Gamma(1+\nu/2),$$

where $q_n(\theta) = \sum_{k=1}^{k_n} X_{n,k} e^{ik\theta}$.

2. Proofs of Proposition 1.1 and Theorem 1.2. Throughout the rest of the paper, C denotes a universal constant, which may vary at each occurrence. Before going into the proof itself we will discuss the relations between conditions $(\mathcal{L}_\nu)$ and $(\mathcal{L}'_\nu)$ and give certain useful (although simple) estimates.

LEMMA 2.1. *Let $\{X_{n,k},\, 1 \leq k \leq k_n,\, n \geq 1\}$ be a triangular array of real random variables. Then*:

(i) *For any $0 < \mu \leq \nu$, condition $(\mathcal{L}_\nu)$ implies $(\mathcal{L}_\mu)$.*
(ii) *For any $\mu > 2$, conditions $(\mathcal{L}_\mu)$ and $(\mathcal{L}'_\mu)$ are equivalent.*
(iii) *For any $2 < \mu < \nu$, we have*

$$\frac{\sum_{k=1}^{k_n} \mathbb{E}\,|X_{n,k}|^\mu}{s_n^\mu} \leq 2\bigg(\frac{\sum_{k=1}^{k_n} \mathbb{E}\,|X_{n,k}|^\nu}{s_n^\nu}\bigg)^{(\mu-2)/(\nu-2)}.$$

Proof. (i) Let $\varepsilon > 0$. As

$$\frac{\sum_{k=1}^{k_n} \mathbb{E}\,\{|X_{n,k}|^\mu \mathbf{1}_{\{|X_{n,k}|>\varepsilon s_n\}}\}}{s_n^\mu} \leq \varepsilon^{\mu-\nu} \frac{\sum_{k=1}^{k_n} \mathbb{E}\,\{|X_{n,k}|^\nu \mathbf{1}_{\{|X_{n,k}|>\varepsilon s_n\}}\}}{s_n^\nu},$$

the claimed implication is immediate.

(ii) We only need to prove one implication, since the other is trivial. Let $\varepsilon > 0$ and assume that $(\mathcal{L}_\mu)$ holds for some $\mu > 2$. We have

$$\text{(2.1)}\quad \sum_{k=1}^{k_n} \mathbb{E}\,|X_{n,k}|^\mu \le (\varepsilon s_n)^{\mu-2} \sum_{k=1}^{k_n} \mathbb{E}\,\{|X_{n,k}|^2 \mathbf{1}_{\{|X_{n,k}| \le \varepsilon s_n\}}\} + \sum_{k=1}^{k_n} \mathbb{E}\,\{|X_{n,k}|^\mu \mathbf{1}_{\{|X_{n,k}| > \varepsilon s_n\}}\}$$
$$\le (\varepsilon s_n)^{\mu-2} \sum_{k=1}^{k_n} \mathbb{E}\,|X_{n,k}|^2 + \sum_{k=1}^{k_n} \mathbb{E}\,\{|X_{n,k}|^\mu \mathbf{1}_{\{|X_{n,k}| > \varepsilon s_n\}}\}$$
$$\le \varepsilon^{\mu-2} s_n^\mu + \sum_{k=1}^{k_n} \mathbb{E}\,\{|X_{n,k}|^\mu \mathbf{1}_{\{|X_{n,k}| > \varepsilon s_n\}}\},$$

which proves $(\mathcal{L}'_\mu)$.

(iii) Let $2 < \mu < \nu$. By (2.1), for any $\varepsilon > 0$ we have

$$\sum_{k=1}^{k_n} \mathbb{E}\,|X_{n,k}|^\mu \le \varepsilon^{\mu-2} s_n^\mu + \sum_{k=1}^{k_n} \mathbb{E}\,\{|X_{n,k}|^\mu \mathbf{1}_{\{|X_{n,k}| > \varepsilon s_n\}}\}$$
$$\le \varepsilon^{\mu-2} s_n^\mu + (\varepsilon s_n)^{\mu-\nu} \sum_{k=1}^{k_n} \mathbb{E}\,|X_{n,k}|^\nu.$$

So

$$\text{(2.3)}\qquad \frac{\sum_{k=1}^{k_n} \mathbb{E}\,|X_{n,k}|^\mu}{s_n^\mu} \le \varepsilon^{\mu-2} + \varepsilon^{\mu-\nu}\, \frac{\sum_{k=1}^{k_n} \mathbb{E}\,|X_{n,k}|^\nu}{s_n^\nu}.$$

Taking now $\varepsilon = (\sum_{k=1}^{k_n} \mathbb{E}\,|X_{n,k}|^\nu / s_n^\nu)^{1/(\nu-2)}$ leads to the conclusion. ∎

Remark. One can prove (iii) without the coefficient 2, using Hölder's inequality in some appropriate spaces.

Proof of Proposition 1.1. Assume first that $0 < \nu \le 2$. So $(\mathcal{L}_2)$ is satisfied by assumption and (S_n/s_n) converges in law to W (Lindeberg's theorem). Now write, for $C > 0$,

$$\mathbb{E}\,\frac{|S_n|^\nu}{s_n^\nu} = \mathbb{E}\left[\frac{|S_n|^\nu}{s_n^\nu}\,\mathbf{1}_{\{|S_n| \le C s_n\}}\right] + \mathbb{E}\left[\frac{|S_n|^\nu}{s_n^\nu}\,\mathbf{1}_{\{|S_n| < C s_n\}}\right].$$

The second term is less than $1/C^{2-\nu}$, and for fixed C, the first term tends to $\mathbb{E}\,[|W|^\nu \mathbf{1}_{\{|W| \le C\}}]$ by Lindeberg's theorem, which proves the result. ∎

Proof of Theorem 1.2. Let $\nu > 2$ and $n \ge 1$. We proceed by combining Billingsley's approach in [3] with the Taylor formula. Let $\{\eta_k,\ 1 \le k \le n\}$ be independent centered normal variables with $\mathbb{E}\,\eta_k^2 = \sigma_k^2 = \mathbb{E}\,Y_k^2$ that we take independent of $\{Y_k\}_{1 \le k \le n}$. For $1 \le k \le n$ and t real, put

$$\xi_k = \sum_{1 \le j < k} Y_j + \sum_{k < j \le n} \eta_{n,j}, \qquad u_{n,k}(t) = |\xi_{n,k} + tY_k|^\nu, \qquad v_{n,k}(t) = |\xi_{n,k} + t\eta_k|^\nu.$$

Let $\psi_\nu(x) := \mathrm{sgn}(x)|x|^\nu$, and write the Taylor formula of order 2 for $u_{n,k}$ and $v_{n,k}$:

$$\begin{aligned}
u_{n,k}(1) &= u_{n,k}(0) + \nu Y_k \psi_{\nu-1}(\xi_{n,k}) \\
&\qquad + \nu(\nu-1)\int_0^1 (1-s) Y_k^2 u_{n,k}(s)^{(\nu-2)/\nu} ds, \\
v_{n,k}(1) &= v_{n,k}(0) + \nu \eta_k \psi_{\nu-1}(\xi_{n,k}) \\
&\qquad + \nu(\nu-1)\int_0^1 (1-s) \eta_k^2 v_{n,k}(s)^{(\nu-2)/\nu} ds.
\end{aligned} \tag{2.4}$$

Notice that $u_{n,k}(0) = v_{n,k}(0)$ and $u_{n,k-1}(1) = v_{n,k}(1)$. Hence, by summing the differences of the previous equalities over k, next taking expectations and using independence, we obtain

$$\begin{aligned}
&\mathbb{E}\,|S_n|^\nu - \mathbb{E}\,|W_n|^\nu \\
&= \sum_{k=1}^n \Big\{ \nu(\nu-1)\int_0^1 (1-s)\mathbb{E}\,[Y_k^2(u_{n,k}(s)^{(\nu-2)/\nu} - u_{n,k}(0)^{(\nu-2)/\nu})]\, ds \\
&\qquad + \nu(\nu-1)\int_0^1 (1-s)\mathbb{E}\,[\eta_k^2(v_{n,k}(0)^{(\nu-2)/\nu} - v_{n,k}(s)^{(\nu-2)/\nu})]\, ds \Big\},
\end{aligned} \tag{2.5}$$

where W_n is a normal variable with variance s_n^2.

Now we estimate

$$\begin{aligned}
|u_{n,k}(s)^{(\nu-2)/\nu} - u_{n,k}(0)^{(\nu-2)/\nu}| &= |\,|\xi_{n,k} + sY_k|^{\nu-2} - |\xi_{n,k}|^{\nu-2}|, \\
|v_{n,k}(s)^{(\nu-2)/\nu} - v_{n,k}(0)^{(\nu-2)/\nu}| &= |\,|\xi_{n,k} + s\eta_k|^{\nu-2} - |\xi_{n,k}|^{\nu-2}|.
\end{aligned}$$

We will use the following well-known estimates:

$$\begin{aligned}
&\forall x, y \ge 0,\ \forall \alpha \in (0,1], && |x^\alpha - y^\alpha| \le |x-y|^\alpha, \\
&\forall x, y \ge 0,\ \forall \alpha > 0, && (x+y)^\alpha \le 2^{(\alpha-1)^+}(x^\alpha + y^\alpha) \le 2^\alpha (x^\alpha + y^\alpha),
\end{aligned}$$

where $(\alpha-1)^+ = \max(0, \alpha-1)$.

Case 1. Assume first that $0 < \nu - 2 < 1$. Let $0 < s \le 1$. We have

$$\begin{aligned}
|u_{n,k}(s)^{(\nu-2)/\nu} - u_{n,k}(0)^{(\nu-2)/\nu}| &\le |sY_k|^{\nu-2}, \\
|v_{n,k}(s)^{(\nu-2)/\nu} - v_{n,k}(0)^{(\nu-2)/\nu}| &\le |s\eta_k|^{\nu-2}.
\end{aligned}$$

Consequently,

$$|\mathbb{E}\,|S_n|^\nu - \mathbb{E}\,|W_n|^\nu| \le \nu \sum_{k=1}^n (\mathbb{E}\,|Y_k|^\nu + \mathbb{E}\,|\eta_k|^\nu). \tag{2.7}$$

Now $\mathbb{E}\,|\eta_k|^\nu = \sigma_k^\nu \mathbb{E}\,|W|^\nu$ and $\sigma_k^\nu = (\mathbb{E}\,Y_k^2)^{\nu/2} \le \mathbb{E}\,|Y_k|^\nu$. Hence the theorem is proved in this case.

CASE 2. Assume that $\nu \geq 3$. Let $0 < s \leq 1$. We have

$$(2.8)\quad \begin{aligned} &|u_{n,k}(s)^{(\nu-2)/\nu} - u_{n,k}(0)^{(\nu-2)/\nu}| \\ &= \int_{\min\{|\xi_{n,k}|,|\xi_{n,k}+sY_k|\}}^{\max\{|\xi_{n,k}|,|\xi_{n,k}+sY_k|\}} (\nu-2)x^{\nu-3}\,dx \\ &\leq s(\nu-2)|Y_k|(|\xi_{n,k}| + s|Y_k|)^{\nu-3} \\ &\leq (\nu-2)2^{\nu-3}|Y_k|(|\xi_{n,k}|^{\nu-3} + s|Y_k|^{\nu-3}). \end{aligned}$$

A similar computation can be made for $v_{n,k}$. Hence, by using the independence of Y_k and $\xi_{n,k}$, we get

$$(2.9)\quad \begin{aligned} |\mathbb{E}\,|S_n|^\nu - \mathbb{E}\,|W_n|^\nu| \leq \nu(\nu-1)(\nu-2)2^{\nu-3}\Big\{ &\sum_{k=1}^n (\mathbb{E}\,|Y_k|^\nu + \mathbb{E}\,|\eta_k|^\nu) \\ &+ \sum_{k=1}^n \mathbb{E}\,|\xi_{n,k}|^{\nu-3}(\mathbb{E}\,|Y_k|^3 + \mathbb{E}\,|\eta_{n,k}|^3)\Big\}. \end{aligned}$$

First we examine the term $\mathbb{E}\,|\xi_{n,k}|^{\nu-3}$.

If $0 \leq \nu - 3 \leq 2$, then $\mathbb{E}\,|\xi_{n,k}|^{\nu-3} \leq s_n^{\nu-3}$.

If $\nu > 5$, using the Jensen inequality and the fact that the variables $\{Y_k, 1 \leq k \leq n\}$ are centered, we have

$$(2.10)\quad \begin{aligned} \mathbb{E}\,|\xi_{n,k}|^{\nu-3} &\leq 2^{\nu-3}\Big(\mathbb{E}\Big(\Big|\sum_{1\leq j<k} Y_k\Big|^{\nu-3}\Big) + \mathbb{E}\Big(\Big|\sum_{k<j\leq k_n} \eta_k\Big|^{\nu-3}\Big)\Big) \\ &\leq 2^{\nu-2}(\mathbb{E}\,|S_n|^{\nu-3} + s_n^{\nu-3}\mathbb{E}\,|W|^{\nu-3}). \end{aligned}$$

By using Rosenthal's inequality (for the best constant, see [12]), we have

$$(2.11)\quad \mathbb{E}\,|S_n|^{\nu-3} \leq \Big(C\,\frac{\nu}{\log\nu}\Big)^{\nu-3}\Big(s_n^{\nu-3} + \sum_{k=1}^n \mathbb{E}\,|Y_k|^{\nu-3}\Big).$$

Further, as $\mathbb{E}\,|W|^m \leq C(m/\log m)^m$, $m = 2, 3, \ldots$, for every $\nu \geq 3$ we arrive at

$$(2.12)\quad \mathbb{E}\,|\xi_{n,k}|^{\nu-3} \leq \Big(C\,\frac{\nu}{\log\nu}\Big)^{\nu-3}\Big(s_n^{\nu-3} + \sum_{k=1}^n \mathbb{E}\,|Y_k|^{\nu-3}\Big).$$

Now, by incorporating (2.12) into (2.9) and in view of the assumption that $\eta_k \stackrel{\mathcal{D}}{=} \mathcal{N}(0, \sigma_k)$, we obtain

$$(2.13)\quad \begin{aligned} |\mathbb{E}\,|S_n|^\nu - \mathbb{E}\,|W_n|^\nu| \leq \Big(C\,\frac{\nu}{\log\nu}\Big)^{\nu}\Big\{ &\sum_{k=1}^n (\mathbb{E}\,|Y_k|^\nu + \sigma_k^\nu) \\ &+ \Big(s_n^{\nu-3} + \sum_{k=1}^n \mathbb{E}\,|Y_k|^{\nu-3}\Big)\sum_{k=1}^n (\mathbb{E}\,|Y_k|^3 + \sigma_k^3)\Big\}. \end{aligned}$$

But $\sigma_k^\nu = (\mathbb{E}\,|Y_k|^2)^{\nu/2} \le \mathbb{E}\,|Y_k|^\nu$, and also $\sigma_k^3 = (\mathbb{E}\,|Y_k|^2)^{3/2} \le \mathbb{E}\,|Y_k|^3$, so that

$$|\mathbb{E}\,|S_n|^\nu - \mathbb{E}\,|W_n|^\nu| \le \Big(C\,\frac{\nu}{\log\nu}\Big)^\nu \Big\{ \sum_{k=1}^n \mathbb{E}\,|Y_k|^\nu + \Big(s_n^{\nu-3} + \sum_{k=1}^n \mathbb{E}\,|Y_k|^{\nu-3}\Big) \sum_{k=1}^n \mathbb{E}\,|Y_k|^3 \Big\}.$$

Dividing both sides of (2.13) by s_n^ν, and developing the right hand side, leads to

$$\Big|\mathbb{E}\Big(\frac{|S_n|}{s_n}\Big)^\nu - m_\nu\Big| \le \Big(C\,\frac{\nu}{\log\nu}\Big)^\nu \Big\{ \frac{\sum_{k=1}^n \mathbb{E}\,|Y_k|^\nu}{s_n^\nu} + \frac{\sum_{k=1}^n \mathbb{E}\,|Y_k|^3}{s_n^3} + \frac{\sum_{k=1}^n \mathbb{E}\,|Y_k|^{\nu-3}}{s_n^{\nu-3}} \cdot \frac{\sum_{k=1}^n \mathbb{E}\,|Y_k|^3}{s_n^3} \Big\}. \tag{2.14}$$

Theorem 1.2 now follows from estimates (2.7) and (2.14). ∎

Now, we turn to the proof of Theorem 1.3. Let $\{X_{n,k},\ 1 \le k \le k_n,\ n \ge 1\}$ be a triangular array of independent variables. For every $n \ge 1$, apply Theorem 1.2 to the k_n variables $\{X_{n,k},\ 1 \le k \le k_n\}$. The first statement of the theorem follows from assumption $(\mathcal{L}'_\nu)$ and the first two assertions of Lemma 2.1.

Let us prove the second statement. We only prove the case $\nu > 5$, the case $3 \le \nu \le 5$ follows similarly. By applying Lemma 2.1(iii) successively with $\mu = 3$ and $\mu = \nu - 3$, we have

$$\begin{aligned} \frac{\sum_{k=1}^{k_n} \mathbb{E}\,[|X_{n,k}|^3]}{s_n^3} &\le 2\Big(\frac{\sum_{k=1}^{k_n} \mathbb{E}\,[|X_{n,k}|^\nu]}{s_n^\nu}\Big)^{1/(\nu-2)}, \\ \frac{\sum_{k=1}^{k_n} \mathbb{E}\,|X_{n,k}|^{\nu-3}}{s_n^{\nu-3}} &\le 2\Big(\frac{\sum_{k=1}^{k_n} \mathbb{E}\,|X_{n,k}|^\nu}{s_n^\nu}\Big)^{(\nu-5)/(\nu-2)}. \end{aligned} \tag{2.15}$$

Thus

$$\frac{\sum_{k=1}^{k_n} \mathbb{E}\,|X_{n,k}|^{\nu-3}}{s_n^{\nu-3}} \cdot \frac{\sum_{k=1}^{k_n} \mathbb{E}\,|X_{n,k}|^3}{s_n^3} \le 4\Big(\frac{\sum_{k=1}^{k_n} \mathbb{E}\,|X_{n,k}|^\nu}{s_n^\nu}\Big)^{(\nu-4)/(\nu-2)}. \tag{2.16}$$

By inserting estimates (2.15), (2.16) into (2.14) we get

$$\begin{aligned} &\Big|\mathbb{E}\Big(\frac{|S_n|}{s_n}\Big)^\nu - m_\nu\Big| \\ &\quad\le \Big(C\,\frac{\nu}{\log\nu}\Big)^\nu \Big\{ \frac{\sum_{k=1}^{k_n} \mathbb{E}\,|X_{n,k}|^\nu}{s_n^\nu} + \Big(\frac{\sum_{k=1}^{k_n} \mathbb{E}\,|X_{n,k}|^\nu}{s_n^\nu}\Big)^{1/(\nu-2)} \\ &\qquad + \Big(\frac{\sum_{k=1}^{k_n} \mathbb{E}\,|X_{n,k}|^\nu}{s_n^\nu}\Big)^{(\nu-4)/(\nu-2)} \Big\} \\ &\quad\le \Big(C\,\frac{\nu}{\log\nu}\Big)^\nu \max_{h \in \{1, (\nu-4)/(\nu-2), 1/(\nu-2)\}} \Big(\frac{\sum_{k=1}^{k_n} \mathbb{E}\,|X_{n,k}|^\nu}{s_n^\nu}\Big)^h. \end{aligned} \tag{2.17}$$

The second statement of Theorem 1.3 follows from the fact that for $\nu > 5$, $1/(\nu - 2) \le (\nu - 4)/(\nu - 2) \le 1$. The proof is now complete. ■

The following result concerning triangular arrays of weighted independent random variables is now a rather straightforward consequence of Theorem 1.3.

THEOREM 2.2. *Let $\{\xi_{n,k},\ 1 \le k \le k_n,\ n \ge 1\}$ be a triangular array of real centered independent random variables such that $\mathbb{E}\,\xi_{n,k}^2 = 1$ and let $\{a_{n,k},\ 1 \le k \le k_n,\ n \ge 1\}$ be a triangular array of real numbers with $\sum_{k=1}^{k_n} a_{n,k}^2 = 1$. Assume that for some $\nu > 2$,*

$$\sup_{n\ge 1} \sup_{1\le k\le k_n} \mathbb{E}\,|\xi_{n,k}|^\nu < \infty, \qquad \sum_{k=1}^{k_n} |a_{n,k}|^\nu = o(1).$$

Put $S_n = \sum_{k=1}^{k_n} a_{n,k}\xi_{n,k}$. Then

$$\lim_{n\to\infty} \mathbb{E}\,|S_n|^\nu = m_\nu.$$

Furthermore, if $\nu \ge 3$, then $\sum_{k=1}^{k_n} |a_{n,k}|^3 = o(1)$ and the estimate of the speed of convergence in the above limit takes the form

$$|\mathbb{E}\,|S_n|^\nu - m_\nu| \le \left(C\,\frac{\nu}{\log \nu}\right)^\nu \sum_{k=1}^{k_n} |a_{n,k}|^3.$$

Proof. The convergence results from Theorem 1.3. The fact that $\sum_{k=1}^{k_n} |a_{n,k}|^3 = o(1)$ follows easily from our assumptions on $\{a_{n,k},\ 1 \le k \le k_n,\ n \ge 1\}$ (or may be deduced from Lemma 2.1). To obtain the speed, we apply Theorem 1.2, noticing that (since $\sum_{k=1}^{k_n} |a_{n,k}|^2 = 1$) we have, for $\nu \ge 3$, $\sum_{k=1}^{k_n} |a_{n,k}|^\nu \le \sum_{k=1}^{k_n} |a_{n,k}|^3$, and for $\nu \ge 5$, $\sum_{k=1}^{k_n} |a_{n,k}|^{\nu-3} \le 1$.

3. L^p-norms of random trigonometric polynomials. The following theorem extends Borwein–Lockhart's result of [4] to triangular arrays of independent random variables.

THEOREM 3.1. *Let $\{X_{n,k},\ 1 \le k \le k_n,\ n \ge 1\}$ be a triangular array of real centered independent random variables with $\mathbb{E}\,X_{n,k}^2 = 1$, satisfying the Lindeberg condition $(\mathcal{L}_\nu)$ of order $\nu \ge 2$. Then*

$$\lim_{n\to\infty} \frac{1}{2\pi k_n^{\nu/2}} \int_0^{2\pi} \mathbb{E}\,|q_n(\theta)|^\nu\, d\theta = \Gamma(1 + \nu/2),$$

where $q_n(\theta) = \sum_{k=1}^{k_n} X_{n,k} e^{ik\theta}$.

It will follow from the proof that the result remains true under the slightly weaker assumption that there exists $0 < m < M$ such that $m \le$

$\mathbb{E}\, X_{n,k}^2 \le M$ for any $n \ge 1$ and any $1 \le k \le k_n$. For any positive integer n and $\theta \in \mathbb{R} - \pi\mathbb{Z}$, put

$$\begin{aligned} C_n(\theta) &= \sum_{k=1}^{k_n} X_k \cos(k\theta)\Big(\sum_{k=1}^{k_n} \cos^2(k\theta)\Big)^{-1/2}, \\ S_n(\theta) &= \sum_{k=1}^{k_n} X_k \sin(k\theta)\Big(\sum_{k=1}^{k_n} \sin^2(k\theta)\Big)^{-1/2}. \end{aligned} \tag{3.1}$$

The proof of Theorem 3.1 makes use of the following lemma.

LEMMA 3.2. *Under the assumption of the previous theorem, for any sequence* $\{\theta_n\}$ *of real numbers which are not multiples of* π *we have*

$$\lim_{n\to\infty} \mathbb{E}\,|C_n(\theta_n)|^\nu = \mathbb{E}\,|W|^\nu, \qquad \lim_{n\to\infty} \mathbb{E}\,|S_n(\theta_n)|^\nu = \mathbb{E}\,|W|^\nu.$$

REMARK. The result is in fact true without any restriction on $\{\theta_n\}$. This is clear for C_n. The only problem is that S_n is not well defined, but $|S_n|$ may be extended by continuity to multiples of π.

Proof of Lemma 3.2. We first recall a result of [4]: there exists a constant $C > 0$, independent of m, such that for every $1 \le k \le m$,

$$\inf_\theta \sum_{l=1}^{m} \frac{\cos^2(l\theta)}{\cos^2(k\theta)} \ge Cm, \qquad \inf_\theta \sum_{l=1}^{m} \frac{\sin^2(l\theta)}{\sin^2(k\theta)} \ge Cm. \tag{3.2}$$

Notice that the sine sum is well defined, by continuity, for multiples of π. Consider the array $\{Y_{n,k},\ 1 \le k \le k_n,\ n \ge 1\}$ of random variables defined for $n, k \ge 1$ by

$$Y_{n,k} = X_{n,k} \cos(k\theta_n)\Big[\sum_{l=1}^{k_n} \cos^2(l\theta_n)\Big]^{-1/2}.$$

To prove the lemma it suffices to show that this array satisfies condition $(\mathcal{L}_\nu)$. Observe that $s_n = 1$ in this case. Fix $\varepsilon > 0$. Then

$$\sum_{k=1}^{k_n} \mathbb{E}\,[|Y_{n,k}|^\nu \mathbf{1}_{\{|Y_{n,k}|>\varepsilon\}}] \le \sum_{k=1}^{k_n} \Big(\frac{1}{Ck_n}\Big)^{\nu/2} \mathbb{E}\,[|X_{n,k}|^\nu \mathbf{1}_{\{|X_{n,k}|>\varepsilon\sqrt{Ck_n}\}}] = o(1),$$

since $\{X_{n,k},\ 1 \le k \le k_n,\ n \ge 1\}$ satisfies $(\mathcal{L}_\nu)$. We conclude thanks to Theorem 1.2. The case of S_n follows from the same arguments using the second part of the lemma. ∎

The proof of Theorem 1.4 can now be finished as in [4]. We include it for the sake of completeness. We first prove that, for any θ not a multiple of π, the sequence of pairs $\{(C_n(\theta), S_n(\theta)),\ n \ge 1\}$ converges in law to (W_1, W_2), where W_1, W_2 are independent with standard normal law.

We show first that for any $\alpha, \beta \in \mathbb{R}$, $\{\alpha C_n(\theta) + \beta S_n(\theta)\}$ converges in law to $\sqrt{\alpha^2+\beta^2}\, W_1$.

Let $\alpha, \beta \in \mathbb{R}$. Let $\{Y_{n,k},\, 1 \le k \le k_n,\, n \ge 1\}$ be defined by $Y_{n,k} := X_{n,k}(\alpha \cos(k\theta) + \beta \sin(k\theta))$. We want to apply the Lindeberg theorem, so we need to check whether condition $(\mathcal{L}_2)$ is satisfied. In this case, $s_n = (\sum_{k=1}^{k_n} (\alpha \cos(k\theta) + \beta \sin(k\theta))^2)^{1/2}$.

Now, for any $\theta \notin \pi\mathbb{Z}$, both sequences

$$\left\{ \frac{1}{k_n} \sum_{k=1}^{k_n} \cos^2(k\theta),\, n \ge 1 \right\} \quad \text{and} \quad \left\{ \frac{1}{k_n} \sum_{k=1}^{k_n} \sin^2(k\theta),\, n \ge 1 \right\}$$

converge to $1/2$, and the sequence

$$\left\{ \frac{1}{k_n} \sum_{k=1}^{k_n} \cos(k\theta) \sin(k\theta),\, n \ge 1 \right\}$$

converges to 0. Hence, $\{s_n/\sqrt{k_n}\}$ converges to $\sqrt{\alpha^2+\beta^2}$, and for n large enough, we have, for any $\varepsilon > 0$,

$$\begin{aligned} (3.3) \qquad & \frac{1}{s_n^2} \sum_{k=1}^{k_n} \mathbb{E} \left[|Y_{n,k}|^2 \mathbf{1}_{\{|Y_{n,k}| > \varepsilon s_n\}} \right] \\ & \le \left(\frac{2}{\sqrt{k_n(\alpha^2+\beta^2)}} \right)^2 \sum_{k=1}^{k_n} \mathbb{E}\, |X_{n,k}|^2 \mathbf{1}_{\{|X_{n,k}| > \frac{\varepsilon}{2} \sqrt{k_n(\alpha^2+\beta^2)}\}} = o(1), \end{aligned}$$

since $\{X_{n,k},\, 1 \le k \le k_n,\, n \ge 1\}$ satisfies $(\mathcal{L}'_\nu)$, so $(\mathcal{L}_2)$. Hence, by the Lindeberg theorem, $\{\alpha C_n(\theta) + \beta S_n(\theta)\}$ converges in law to $\sqrt{\alpha^2+\beta^2}\, W_1$.

Since this is true for any $\alpha, \beta \in \mathbb{R}$, we deduce that the sequence of pairs $\{(C_n(\theta), S_n(\theta)),\, n \ge 1\}$ converges in law to (W_1, W_2), where (W_1, W_2) is a Gaussian vector. Moreover we already saw that the covariances

$$\mathbb{E}\left[C_n(\theta) S_n(\theta) \right] = \frac{\sum_{k=1}^{k_n} \cos(k\theta) \sin(k\theta)}{(\sum_{k=1}^{k_n} \cos^2(k\theta))^{1/2} (\sum_{k=1}^{k_n} \sin^2(k\theta))^{1/2}}$$

tend to 0 as $n \to \infty$.

So W_1 and W_2 are uncorrelated, hence independent. We deduce that $\{k_n^{-1/2} |q_n(\theta)|,\, n \ge 1\}$ converges in law to $((W_1^2 + W_2^2)/2)^{1/2}$ for any $\theta \in \mathbb{R} - \pi\mathbb{Z}$. Observe now that

$$(3.4) \qquad \frac{|q_n(\theta)|^\nu}{k_n^{\nu/2}} \le (C_n^2(\theta) + S_n^2(\theta))^{\nu/2} \le 2^{\nu/2-1} (|C_n(\theta)|^\nu + |S_n(\theta)|^\nu).$$

By Lemma 3.2, the sequences $\{\mathbb{E}\, |C_n(\theta)|^\nu,\, n \ge 1\}$ and $\{\mathbb{E}\, |S_n(\theta)|^\nu,\, n \ge 1\}$ are uniformly bounded in $\theta \in \mathbb{R} - \pi\mathbb{Z}$ (actually, uniformly on $\mathbb{R}$ by the remark made after that lemma). Hence, in order to prove the theorem, it suffices to prove that $\{(\mathbb{E}\,|q_n(\theta)|^\nu)/k_n^{\nu/2},\, n \geq 1\}$ converges for almost all θ to $\Gamma(1+\nu/2)$. Let $K > 0$. Put $Z_n(\theta) = 2^{\nu/2-1}(|C_n(\theta)|^\nu + |S_n(\theta)|^\nu)$, $T = 2^{\nu/2-1}(|W_1|^\nu + |W_2|^\nu)$ and $U = ((W_1^2+W_2^2)/2)^{1/2}$. We may write

$$\frac{\mathbb{E}\,|q_n(\theta)|^\nu}{k_n^{\nu/2}} = \frac{\mathbb{E}\,[|q_n(\theta)|^\nu \mathbf{1}_{\{|q_n(\theta)|\leq K\sqrt{k_n}\}}]}{k_n^{\nu/2}} + \frac{\mathbb{E}\,[|q_n(\theta)|^\nu \mathbf{1}_{\{|q_n(\theta)|> K\sqrt{k_n}\}}]}{k_n^{\nu/2}}.$$

But

$$\begin{aligned}(3.5)\qquad & \frac{\mathbb{E}\,[|q_n(\theta)|^\nu \mathbf{1}_{\{|q_n(\theta)|> K\sqrt{k_n}\}}]}{k_n^{\nu/2}} \\ &\leq \mathbb{E}\,[Z_n(\theta)\mathbf{1}_{\{Z_n(\theta)>K\}}] \leq \mathbb{E}\,Z_n(\theta) - \mathbb{E}\,[Z_n(\theta)\mathbf{1}_{\{Z_n(\theta)\leq K\}}] \\ &\leq (\mathbb{E}\,Z_n(\theta) - \mathbb{E}\,T) + \mathbb{E}\,[T\mathbf{1}_{\{T>K\}}] \\ &\quad + (\mathbb{E}\,[T\mathbf{1}_{\{T\leq K\}}] - \mathbb{E}\,[Z_n(\theta)\mathbf{1}_{\{Z_n(\theta)\leq K\}}]).\end{aligned}$$

Since $\{|q_n(\theta)|/\sqrt{k_n},\, n \geq 1\}$ converges in law to U, it follows that $\{Z_n(\theta),\, n \geq 1\}$ converges in law to T. Further, by using Lemma 3.2 we see that $\mathbb{E}\,Z_n(\theta) \to \mathbb{E}\,T$ as $n \to \infty$. We therefore obtain

$$(3.6)\qquad \limsup_{n\to\infty} \left| \frac{\mathbb{E}\,|q_n(\theta)|^\nu}{k_n^{\nu/2}} - \mathbb{E}\,[U^\nu \mathbf{1}_{\{U\leq K\}}] \right| \leq \mathbb{E}\,[T\mathbf{1}_{\{T>K\}}].$$

Letting K go to infinity yields the claimed result. ■

Remark. In the last step of the proof, we proved that

$$(3.7)\qquad \lim_{n\to\infty} \frac{\mathbb{E}\,|q_n(\theta)|^\nu}{k_n^{\nu/2}} = \Gamma(1+\nu/2)$$

for any $\theta \in \mathbb{R} - \pi\mathbb{Z}$. This has, in view of an inequality due to Petrov, consequences on the probability that θ is not a root of q_n, for n large. Indeed, we have the following corollary.

Corollary 3.3. *Let $\nu_2 > \nu_1 \geq 2$. Let $\{X_{n,k},\, 1 \leq k \leq k_n,\, n \geq 1\}$ be a triangular array of real centered independent random variables with $\mathbb{E}\,X_{n,k}^2 = 1$, satisfying the Lindeberg condition $(\mathcal{L}_{\nu_2})$. For any $\theta \in \mathbb{R} - \pi\mathbb{Z}$, we have*

$$\liminf_{n\to\infty} \mathbb{P}\{q_n(\theta) \neq 0\} \geq \left(\frac{\Gamma(1+\nu_1/2)^{\nu_2}}{\Gamma(1+\nu_2/2)^{\nu_1}}\right)^{1/(\nu_2-\nu_1)},$$

where $q_n(\theta) = \sum_{k=1}^{k_n} X_{n,k}e^{ik\theta}$.

Proof. From Petrov's inequality [14, inequality (2), p. 392], if X is any random variable and $s > r > 0$, then $X \in L^s(\mathbb{P})$ implies

$$\mathbb{P}\{X \neq 0\}^{1/r-1/s} \geq \frac{\|X\|_r}{\|X\|_s}.$$

Applying now this inequality for $X = |q_n(\theta)|/k_n^{1/2}$, and using the above remark, we can easily conclude the proof. ■

We also have the following result.

THEOREM 3.4. *Let* $\{X_{n,k},\, 1 \le k \le k_n,\, n \ge 1\}$ *be a triangular array of real centered independent random variables satisfying the Lindeberg condition* $(\mathcal{L}_\nu)$ *of order* $\nu \ge 2$. *Let* $\{Y_{n,k},\, 1 \le k \le k_n,\, n \ge 1\}$ *be an independent copy of* $\{X_{n,k},\, 1 \le k \le k_n,\, n \ge 1\}$. *Then for any increasing sequence* $\{p_n,\, n \ge 1\}$ *of integers, the sequence of random trigonometric polynomials*

$$Z_n(\theta) = \sum_{k=1}^{k_n} (X_{n,k}\cos(p_k\theta) + Y_{n,k}\sin(p_k\theta))$$

satisfies

$$\lim_{n\to\infty} \frac{1}{2\pi s_n^{\nu/2}} \int_0^{2\pi} \mathbb{E}\,|Z_n(\theta)|^\nu\, d\theta = m_\nu.$$

Proof. Apply Theorem 1.2 to the array $\{X_{n,k}\cos(p_k\theta) + Y_{n,k}\sin(p_k\theta),$ $1 \le k \le k_n,\, n \ge 1\}$. ■

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