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COMPACT OPERATORS AND APPROXIMATION SPACES

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Abstract. We investigate compact operators between approximation spaces, paying special attention to the limit case. Applications are given to embeddings between Besov spaces.

1. Introduction. The theory of approximation spaces is a useful and flexible tool which allows one to study not only problems in function spaces, but also in spaces of operators and sequence spaces. For the classical theory see, for example, the papers by Butzer and Scherer [5] and Pietsch [21], and the books by Peetre [18], Triebel [24], Petrushev and Popov [20] and DeVore and Lorentz [12]; while for the limiting theory see the papers by Cobos and Resina [11], Cobos and Milman [9], Fehér and Grässler [14] and the references cited there. Given a quasi-Banach space X and scalar parameters $0 < \alpha < \infty$, $0 < p, q \le \infty$ and $\gamma \in \mathbb{R}$, the classical theory deals with spaces X_p^α, and the limiting theory with $X_q^{(0,\gamma)}$ (see Section 2 for definitions of approximation spaces).

Outstanding examples of spaces X_p^α are Besov spaces $B_{p,q}^s$, Lorentz sequence spaces $\ell_{p,q}$ and the spaces of operators $\mathfrak{L}_{p,q}^{(a)}(E,F)$ consisting of all bounded linear operators between the Banach spaces E and F whose approximation numbers belong to $\ell_{p,q}$. Examples of $X_q^{(0,\gamma)}$ spaces are the Besov spaces $B_{p,q}^{0,\gamma}$ with smoothness close to zero and the Lorentz–Zygmund operator spaces $\mathfrak{L}_{\infty,q,\gamma}^{(a)}(E,F)$.

The study of compact operators is a natural question in this setting, which was considered by Fugarolas [15] and Almira and Luther [1]. Results of [15] characterize compact subsets of X_p^α for $p < \infty$, while results of [1] refer to compact operators but just in the setting of Banach spaces, i.e. when X is a Banach space and $1 \le p \le \infty$.

Here we continue these investigations, focusing our attention mainly on limiting approximation spaces. This is done in Section 3. Then, in Section 4, we give applications of our results to embeddings between Besov spaces.

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2. Preliminaries. In what follows, $(X, \|\cdot\|_X)$ stands for a quasi-Banach space. By an *approximation family* in X we mean a sequence $(G_n)_{n\in\mathbb{N}_0}$ of subsets of X satisfying the following conditions:

$$G_0 = \{0\} \quad \text{and} \quad \lambda G_n \subseteq G_n \qquad \text{for any scalar } \lambda \text{ and } n \in \mathbb{N}, \tag{2.1}$$

$$G_n \subseteq G_{n+1} \qquad \text{for any } n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \tag{2.2}$$

$$G_n + G_m \subseteq G_{n+m} \qquad \text{for any } n, m \in \mathbb{N}. \tag{2.3}$$

For $f \in X$, we set

$$E_n(f) = E_n(f; X) = \inf\{\|f - g\|_X : g \in G_{n-1}\}, \qquad n \in \mathbb{N}.$$

Let $\alpha > 0$ and $0 < p \le \infty$. The (*classical*) *approximation space* X_p^α consists of all $f \in X$ which have a finite quasi-norm

$$\|f\|_{X_p^\alpha} = \Big(\sum_{n=1}^{\infty} (n^\alpha E_n(f))^p \, n^{-1}\Big)^{1/p}$$

(the sum should be replaced by a supremum if $p = \infty$). We refer to [21], [20] and [12] for a general theory of these spaces.

Let $0 < q \le \infty$ and $\gamma \in \mathbb{R}$. The *limiting approximation space* $X_q^{(0,\gamma)}$ is formed by all those $f \in X$ having a finite quasi-norm

$$\|f\|_{X_q^{(0,\gamma)}} = \Big(\sum_{n=1}^{\infty} \big((1 + \log n)^\gamma E_n(f)\big)^q n^{-1}\Big)^{1/q}$$

(see [9] and [14]).

The spaces X_p^α and $X_q^{(0,\gamma)}$ are complete. Moreover, it is not hard to check that

$$X_p^\alpha \hookrightarrow X_q^{(0,\gamma)} \qquad \text{for } 0 < \alpha < \infty,\ 0 < p, q \le \infty,\ \gamma \in \mathbb{R}, \tag{2.4}$$

where $\hookrightarrow$ means continuous embedding. Note that if $\gamma < -1/q$, then $X_q^{(0,\gamma)} = X$. So the only case of interest for the limiting spaces is $\gamma \ge -1/q$.

It is shown in [11] that, even when $\gamma = 0$, the theory of spaces $X_q^{(0,0)}$ does not follow by taking $\alpha = 0$ in the theory of classical approximation spaces.

Next we recall some important examples. Let $X = L_p([0, 2\pi])$ be the Lebesgue space of periodic measurable functions, and let G_n be the set of all trigonometric polynomials of degree less than or equal to n. Then X_q^α coincides with the Besov space $B_{p,q}^\alpha$ (see [23]) and $X_q^{(0,\gamma)}$ is the Besov space of logarithmic smoothness $B_{p,q}^{0,\gamma}$ (see [13]).

Let $X = \ell_\infty$ be the space of bounded sequences, and let G_n be the subset of sequences having at most n coordinates different from 0. It turns out that $(E_n(x; \ell_\infty))$ is the non-increasing rearrangement of the sequence x. Hence X_q^α coincides with the Lorentz sequence space $\ell_{1/\alpha,q}$ (see [21]).

If $X = \mathfrak{L}(E,F)$ is the Banach space of all bounded linear operators between the Banach spaces E and F, and we take G_n as the subset of operators $R \in \mathfrak{L}(E,F)$ such that $\operatorname{rank}(R) \le n$, then the sequence $(E_n(T;\mathfrak{L}(E,F)))$ coincides with the sequence $(a_n(T))$ of the approximation numbers of T. Thus X_q^α is the space $\mathfrak{L}^{(a)}_{1/\alpha,q}(E,F)$ of all operators $T \in \mathfrak{L}(E,F)$ such that $(a_n(T)) \in \ell_{1/\alpha,q}$ (see [21]). The space $X_q^{(0,\gamma)}$ coincides with the Lorentz–Zygmund operator space

$$\mathfrak{L}^{(a)}_{\infty,q,\gamma}(E,F) = \Big\{T \in \mathfrak{L}(E,F) : \Big(\sum_{n=1}^{\infty}\big((1+\log n)^\gamma a_n(T)\big)^q n^{-1}\Big)^{1/q} < \infty\Big\}$$

(see [7] and [11]).

3. Compact operators. We start with a consequence of the interpolation properties of approximation spaces.

THEOREM 3.1. *Let X, Y be quasi-Banach spaces and let $(G_n)_{n\in\mathbb{N}_0}$, $(F_n)_{n\in\mathbb{N}_0}$ be approximation families in X and Y, respectively. Suppose that $0<\alpha<\infty$, $0<q\le\infty$ and $\gamma > -1/q$. Let $T \in \mathfrak{L}(X,Y)$ be such that for some $c>0$ we have*

$$T(G_n) \subseteq F_m \quad \textit{whenever } m \ge cn. \tag{3.1}$$

If $T : X \to Y$ is compact, then the restrictions

$$T : X_q^{(0,\gamma)} \to Y_q^{(0,\gamma)} \quad \textit{and} \quad T : X_q^\alpha \to Y_q^\alpha$$

are also compact.

Proof. Let $0<p\le\infty$ and $\delta > -1/p$, such that $\delta + 1/p > \gamma + 1/q$. Set $\theta = (\gamma+1/q)/(\delta+1/p)$. It follows from [14, Theorem 4] that

$$(X, X_p^{(0,\delta)})_{\theta,q} = X_q^{(0,\gamma)}. \tag{3.2}$$

On the other hand,

$$(X, X_q^{2\alpha})_{1/2\,,q} = X_q^\alpha \tag{3.3}$$

(see [19]).

Assumption (3.1) implies that for any $f \in X$,

$$E_m(Tf;Y) \le \|T\|_{X,Y}\, E_n(f;X) \qquad \text{whenever } c(n-1)+1 \le m < cn+1,\ n=1,2,\ldots$$

(see [21, Theorem 3.3]). Hence the restrictions

$$T : X_p^{(0,\delta)} \to Y_p^{(0,\delta)} \quad \text{and} \quad T : X_q^{2\alpha} \to Y_q^{2\alpha}$$

are bounded. Now, using the compactness theorem for the real method in the quasi-Banach case (see [10, Theorem 3.1]), we derive that

$$T : X_q^{(0,\gamma)} \to Y_q^{(0,\gamma)} \quad \text{and} \quad T : X_q^{\alpha} \to Y_q^{\alpha}$$

are compact. ∎

REMARK 3.2. Theorem 3.1 was established by Almira and Luther [1, Corollary 7.5] in the particular case where $q \geq 1$, and X, Y are Banach spaces and the approximation families $(G_n)_{n\in\mathbb{N}_0}, (F_n)_{n\in\mathbb{N}_0}$ are formed by finite-dimensional subspaces.

The following result requires another type of assumptions. We set

$$\mu_n = 2^{2^n}, \qquad n = 0, 1, 2, \ldots,$$

and, as usual, U_X stands for the closed unit ball of X.

THEOREM 3.3. *Let X, Y be quasi-Banach spaces and let $(G_n)_{n\in\mathbb{N}_0}$ be an approximation family in X. Let $0 < q \leq \infty$, $\gamma > -1/q$ and let $T \in \mathfrak{L}(X, Y)$. Then a necessary and sufficient condition for $T : X_q^{(0,\gamma)} \to Y$ to be compact is that $T(G_n \cap U_X)$ is precompact in Y for any $n \in \mathbb{N}$.*

Proof. Set

$$r_n = \Big(\sum_{k=1}^{n} (1 + \log k)^{\gamma q} k^{-1}\Big)^{1/q}, \qquad n \in \mathbb{N}.$$

If $f \in G_n \cap U_X$, we have

$$\begin{aligned}\|f\|_{X_q^{(0,\gamma)}} &= \Big(\sum_{k=1}^{n} \big((1 + \log k)^{\gamma} E_k(f)\big)^q k^{-1}\Big)^{1/q} \\ &\leq \Big(\sum_{k=1}^{n} (1 + \log k)^{\gamma q} k^{-1}\Big)^{1/q} = r_n.\end{aligned}$$

So $G_n \cap U_X$ is bounded in $X_q^{(0,\gamma)}$. If $T : X_q^{(0,\gamma)} \to Y$ is compact, it follows that $T(G_n \cap U_X)$ is precompact in Y for any $n \in \mathbb{N}$.

In order to show that the condition is sufficient, we recall that without loss of generality we may assume that X and Y are ρ-normed for some $0 < \rho < q$ (see [4, Lemma 3.10.1]). Let $1/r = 1/\rho - 1/q$. We shall also use the following representation (see [11] and [14]): The space $X_q^{(0,\gamma)}$ is formed by all $f \in X$ such that there is a representation $f = \sum_{n=0}^{\infty} g_n$ (convergence in X) with $g_n \in G_{\mu_n}$ and

$$\Big(\sum_{n=0}^{\infty} \big(2^{n(\gamma+1/q)} \|g_n\|_X\big)^q\Big)^{1/q} < \infty.$$

Furthermore,

$$\|f\|_{(0,\gamma),q} = \inf\Big\{ \Big(\sum_{n=0}^{\infty} \big(2^{n(\gamma+1/q)} \|g_n\|_X\big)^q\Big)^{1/q} : f = \sum_{n=0}^{\infty} g_n,\ g_n \in G_{\mu_n}\Big\} \tag{3.4}$$

defines an equivalent quasi-norm to $\|\cdot\|_{X_q^{(0,\gamma)}}$. We shall work with the quasi-norm $\|\cdot\|_{(0,\gamma),q}$.

Take any $\epsilon > 0$. Since $\gamma + 1/q > 0$, there is $N \in \mathbb{N}$ such that

$$\Big(\sum_{n>N} 2^{-n(\gamma+1/q)r}\Big)^{1/r} \leq \frac{\epsilon}{2^{2+1/\rho}\|T\|_{X,Y}}. \tag{3.5}$$

Let $\epsilon_0, \ldots, \epsilon_N$ be positive numbers such that $(\sum_{n=0}^N \epsilon_n^\rho)^{1/\rho} = \epsilon/2^{1+1/\rho}$. By the assumption on T, for any $n = 0, 1, \ldots, N$, there is a finite set $V_n \subseteq Y$ such that

$$T(G_{\mu_n} \cap 2^{1-n(\gamma+1/q)}U_X) \subseteq \bigcup_{v \in V_n} \{v + \epsilon_n U_Y\}. \tag{3.6}$$

Put

$$W = \Big\{\sum_{n=0}^N v_n : v_n \in V_n,\, 0 \leq n \leq N\Big\}.$$

It is clear that W is finite. Let us check that W is an ϵ-net of $T(U_{X_q^{(0,\gamma)}})$ in Y.

Given any $f \in U_{X_q^{(0,\gamma)}}$, we can find a representation $f = \sum_{n=0}^\infty g_n$ with $g_n \in G_{\mu_n}$ and

$$\Big(\sum_{n=0}^\infty \big(2^{n(\gamma+1/q)}\|g_n\|_X\big)^q\Big)^{1/q} \leq 2.$$

Thus $\|g_n\|_X \leq 2^{1-n(\gamma+1/q)}$, and so $g_n \in G_{\mu_n} \cap 2^{1-n(\gamma+1/q)}U_X$. Using (3.6), for $n = 0, 1, \ldots, N$, we can find $v_n \in V_n$ such that $\|Tg_n - v_n\|_Y \leq \epsilon_n$. Let $w = \sum_{n=0}^N v_n \in W$. Applying the Hölder inequality and using (3.5), we get

$$\begin{aligned}
\|Tf - w\|_Y &= \Big\|\sum_{n=0}^\infty Tg_n - \sum_{n=0}^N v_n\Big\|_Y \\
&\leq 2^{1/\rho}\Big[\Big(\sum_{n=0}^N \|Tg_n - v_n\|_Y^\rho\Big)^{1/\rho} + \Big(\sum_{n>N} \|Tg_n\|_Y^\rho\Big)^{1/\rho}\Big] \\
&\leq 2^{1/\rho}\Big[\Big(\sum_{n=0}^N \epsilon_n^\rho\Big)^{1/\rho} + \|T\|_{X,Y}\Big(\sum_{n>N} \|g_n\|_X^\rho\Big)^{1/\rho}\Big] \\
&\leq \frac{\epsilon}{2} + 2^{1/\rho}\|T\|_{X,Y}\Big(\sum_{n>N} 2^{-n(\gamma+1/q)r}\Big)^{1/r}\Big(\sum_{n>N} (2^{n(\gamma+1/q)}\|g_n\|_X)^q\Big)^{1/q} \\
&\leq \frac{\epsilon}{2} + 2^{1/\rho}\|T\|_{X,Y}\frac{\epsilon}{2^{2+1/\rho}\|T\|_{X,Y}}2 = \epsilon.
\end{aligned}$$

This shows that $T(U_{X_q^{(0,\gamma)}})$ is precompact in Y and completes the proof. ∎

Since the closed unit ball of any finite-dimensional topological vector space is compact (see [16, §15.5(1)]), as a direct consequence of Theorem 3.3 we obtain the following

Corollary 3.4. *Let X be a quasi-Banach space and let $(G_n)_{n\in\mathbb{N}_0}$ be an approximation family in X. Let $0 < q \leq \infty$ and $\gamma > -1/q$.*

If for each $n \in \mathbb{N}$, the set G_n is a finite-dimensional linear subspace of X, then the embedding $X_q^{(0,\gamma)} \hookrightarrow X$ is compact.

Remark 3.5. Let $0 < \alpha < \infty$ and $0 < p \leq \infty$. In the assumptions of Corollary 3.4, it follows from (2.4) that $X_p^\alpha \hookrightarrow X$ is also compact. This result and also the corresponding ones for spaces $X_q^{(0,\gamma)}$ were proved by Almira and Luther [1, Theorem 2.1(ii)] in the special case where X is a Banach space and $p, q \geq 1$.

Corollary 3.6. *Let X, Y be quasi-Banach spaces and let $(G_n)_{n\in\mathbb{N}_0}$ be an approximation family in X, formed by finite-dimensional subspaces. Let $0 < q \leq \infty$, $\gamma > -1/q$ and $\epsilon > 0$. If $T \in \mathfrak{L}(X_q^{(0,\gamma)}, Y)$, then $T : X_q^{(0,\gamma+\epsilon)} \to Y$ is compact.*

Proof. Let $\delta = \epsilon - 1/q$. By the reiteration theorem (see [14, Theorem 2]), we have $X_q^{(0,\gamma+\epsilon)} = (X_q^{(0,\gamma)})_q^{(0,\delta)}$. Since Corollary 3.4 shows that the embedding $(X_q^{(0,\gamma)})_q^{(0,\delta)} \hookrightarrow X_q^{(0,\gamma)}$ is compact, we conclude that $T : X_q^{(0,\gamma+\epsilon)} \to Y$ is also compact. ■

A similar result to Corollary 3.6 is valid for spaces X_p^α. It is a consequence of Remark 3.5 and [21, Theorem 3.2].

Corollary 3.7. *Let X be a quasi-Banach space and let $(G_n)_{n\in\mathbb{N}_0}$ be an approximation family in X, formed by finite-dimensional subspaces. If $0 < q_1, q_2 \leq \infty$ and $\gamma_1 + 1/q_1 > \gamma_2 + 1/q_2$, then $X_{q_1}^{(0,\gamma_1)} \hookrightarrow X_{q_2}^{(0,\gamma_2)}$ is compact.*

Proof. Let $\epsilon > 0$ be such that $\gamma_1 - \epsilon + 1/q_1 > \gamma_2 + 1/q_2$. By [14, Lemma 2], we have $X_{q_1}^{(0,\gamma_1-\epsilon)} \hookrightarrow X_{q_2}^{(0,\gamma_2)}$. Then the result follows from Corollary 3.6. ■

The next result concerns reflexivity of approximation spaces.

Corollary 3.8. *Let X be a Banach space and let $(G_n)_{n\in\mathbb{N}_0}$ be an approximation family in X, formed by finite-dimensional subspaces. Let $1 < q < \infty$, $0 < \alpha < \infty$ and $\gamma > -1/q$. Then the spaces $X_q^{(0,\gamma)}$ and X_q^α are reflexive.*

Proof. Take $1 < p < \infty$ and $\delta + 1/p > \gamma + 1/q$. By Corollary 3.4, the embedding $X_p^{(0,\delta)} \hookrightarrow X$ is compact and therefore weakly compact. Consequently, reflexivity of $X_q^{(0,\gamma)}$ follows from (3.2) and the interpolation properties of weakly compact operators (see [3, Proposition II.3.1]).

The proof for X_q^α is similar but using (3.3) and Remark 3.5. ■

The remaining part of this section is devoted to operators with image in an approximation space.

THEOREM 3.9. *Let X, Y be quasi-Banach spaces and let $(F_n)_{n\in\mathbb{N}_0}$ be an approximation family in Y. Let $0<\alpha<\infty$, $0<p<\infty$ and $T\in\mathfrak{L}(X,Y)$. Then $T:X\to Y_p^\alpha$ is compact if and only if the following conditions hold:*

(a) *$T:X\to Y$ is compact,*

(b) $\sup\big\{\big(\sum_{m=n}^\infty(2^{m\alpha}E_{2^m}(Tf))^p\big)^{1/p}:\|f\|_X\le 1\big\}\to 0$ *as $n\to\infty$.*

Proof. First we show that (b) implies $T\in\mathfrak{L}(X,Y_p^\alpha)$. There is an $N\in\mathbb{N}$ such that

$$\Big(\sum_{m=N}^\infty(2^{m\alpha}E_{2^m}(Tf))^p\Big)^{1/p}\le 1\qquad\text{for any } f\in U_X.$$

Hence, for any $f\in X$, we get

$$\begin{aligned}\|Tf\|_{Y_p^\alpha}&\le c_1\Big(\sum_{m=0}^\infty(2^{m\alpha}E_{2^m}(Tf))^p\Big)^{1/p}\\&\le c_2\Big[\Big(\sum_{m=0}^{N-1}(2^{m\alpha}E_{2^m}(Tf))^p\Big)^{1/p}+\|f\|_X\Big]\\&\le c_2\Big[\Big(\sum_{m=0}^{N-1}2^{m\alpha p}\Big)^{1/p}\|T\|_{X,Y}+1\Big]\|f\|_X=c_3\|f\|_X.\end{aligned}$$

Consequently, $T\in\mathfrak{L}(X,Y_p^\alpha)$.

Now the result follows by using [15, Theorem 1]. ■

Recall that the approximation family $(F_n)_{n\in\mathbb{N}_0}$ of Y is called *linear* if there exists a uniformly bounded sequence of linear projections P_n mapping Y onto F_n. If this is the case, we have

$$\|h-P_{n-1}h\|_Y\le cE_n(h),\qquad h\in Y,\ n\in\mathbb{N}. \tag{3.7}$$

Next we show that condition (a) is not needed if the family $(F_n)_{n\in\mathbb{N}_0}$ is linear and finite-dimensional.

THEOREM 3.10. *Let X, Y be quasi-Banach spaces and let $(F_n)_{n\in\mathbb{N}_0}$ be a linear approximation family in Y, formed by finite-dimensional subspaces. Let $0<\alpha<\infty$, $0<p<\infty$ and $T\in\mathfrak{L}(X,Y)$. Then a necessary and sufficient condition for $T:X\to Y_p^\alpha$ to be compact is*

$$\sup\Big\{\Big(\sum_{m=n}^\infty(2^{m\alpha}E_{2^m}(Tf))^p\Big)^{1/p}:\|f\|_X\le 1\Big\}\to 0\qquad \text{as } n\to\infty.$$

Proof. The argument is similar to [15, Theorem 2]. If the condition holds, then $T \in \mathfrak{L}(X, Y_p^\alpha)$ as we showed in the proof of Theorem 3.9. We now take any $\epsilon > 0$ and construct an ϵ-net for $T(U_X)$ in Y.

Let P_n be the projection associated to F_n and let c_Y be the constant in the triangle inequality of Y. By (3.7) and the assumption, we can find an $N \in \mathbb{N}$ such that

$$\Big(\sum_{m=N}^{\infty}\big(2^{m\alpha}\|Tf - P_{2^m-1}(Tf)\|_Y\big)^p\Big)^{1/p} \le \frac{\epsilon}{2c_Y}, \qquad f \in U_X.$$

In particular, we have

$$\|Tf - P_{2^N-1}(Tf)\|_Y \le \frac{\epsilon}{2c_Y}, \qquad f \in U_X.$$

Moreover, by compactness of $P_{2^N-1}T : X \to Y$ there is a finite subset $V = \{h_1, \ldots, h_k\} \subseteq Y$ such that

$$P_{2^N-1}T(U_X) \subseteq \bigcup_{j=1}^{k}\Big\{h_j + \frac{\epsilon}{2c_Y}U_Y\Big\}.$$

Therefore, for any $f \in U_X$, if we choose $h_j \in V$ such that $\|P_{2^N-1}Tf - h_j\|_Y \le \epsilon/2c_Y$, we obtain

$$\|Tf - h_j\|_Y \le c_Y\big(\|Tf - P_{2^N-1}Tf\|_Y + \|P_{2^N-1}Tf - h_j\|_Y\big) \le \epsilon.$$

This shows that the condition is sufficient. Necessity follows from Theorem 3.9. ■

REMARK 3.11. It is not hard to check that the techniques used in Theorems 3.9 and 3.10 also work to characterize compactness of operators with image in $Y_q^{(0,\gamma)}$ for $0 < q < \infty$ and $\gamma > -1/q$. The condition corresponding to (b) of Theorem 3.9 reads now

(b′) $\sup\big\{\big(\sum_{m=n}^{\infty}(2^{m(\gamma+1/q)}E_{\mu_m}(Tf))^q\big)^{1/q} : \|f\|_X \le 1\big\} \to 0$ as $n \to \infty$.

4. Besov spaces. In this final section we give applications of the previous results to Besov spaces. We start by writing down Corollary 3.4 for Besov spaces.

COROLLARY 4.1. *Let $0 < p, q \le \infty$ and $\gamma > -1/q$. Then the embedding $B_{p,q}^{0,\gamma} \hookrightarrow L_p$ is compact.*

The following result is a consequence of Corollary 3.7.

COROLLARY 4.2. *Let $0 < p, q \le \infty$, $\gamma > -1/q$ and $\epsilon > 0$. Then the embedding*

$$B_{p,q}^{0,\gamma+\epsilon} \hookrightarrow B_{p,q}^{0,\gamma} \tag{4.1}$$

is compact.

Let $1 \le p \le r \le \infty$, $1 \le q \le \infty$ and $\gamma > -1/q$. It is shown in [13, Corollary 5.3(ii)] that

$$B_{p,q}^{1/p-1/r,\gamma+1} \hookrightarrow B_{r,q}^{0,\gamma}.$$

Here

$$\begin{aligned} B_{p,q}^{s,\eta} &= (L_p)_q^{s,\eta} \\ &= \Big\{ f \in L_p : \|f\|_{B_{p,q}^{s,\eta}} = \Big(\sum_{n=1}^{\infty} \big(n^s (1+\log n)^{\eta} E_n(f; L_p) \big)^q n^{-1} \Big)^{1/q} < \infty \Big\}. \end{aligned}$$

Next we extend this result to the full range of parameters.

Subsequently, if W and Z are non-negative quantities depending on certain parameters, we write $W \lesssim Z$ if there is a constant $c > 0$ independent of the parameters in W and Z such that $W \le cZ$.

THEOREM 4.3. *Let $0 < p \le r \le \infty$, $0 < q \le \infty$ and $\gamma > -1/q$. Then*

$$B_{p,q}^{1/p-1/r,\gamma+1/\min(1,r,q)} \hookrightarrow B_{r,q}^{0,\gamma}.$$

Proof. If $p = r$, the embedding is clear. Suppose then that $p < r$ and let $\sigma = \min(1, r, q)$. According to [22, Theorem 3.3], there is a constant $c > 0$ such that given any $f \in B_{p,q}^{1/p-1/r,\gamma+1/\sigma}$ there is a representation $f = \sum_{k=0}^{\infty} g_k$ (convergence in L_p) with $g_k \in G_{2^k}$ and

$$\Big(\sum_{k=0}^{\infty} \big(2^{k(1/p-1/r)} (1+k)^{\gamma+1/\sigma} \|g_k\|_{L_p} \big)^q \Big)^{1/q} \le c \|f\|_{B_{p,q}^{1/p-1/r,\gamma+1/\sigma}}. \tag{4.2}$$

Set

$$h_0 = h_1 = 0 \quad \text{and} \quad h_j = \sum_{k=2^{j-2}-1}^{2^{j-1}-2} g_k \quad \text{for } j = 2, 3, \ldots.$$

Clearly $h_j \in G_{\mu_j}$. Using Nikol'skiĭ's inequality (see [17, 3.4.3] and [2]) and the Hölder inequality, we obtain

$$\begin{aligned} \|h_j\|_{L_r} &\lesssim \Big(\sum_{k=2^{j-2}-1}^{2^{j-1}-2} \|g_k\|_{L_r}^{\sigma} \Big)^{1/\sigma} \\ &\lesssim \Big(\sum_{k=2^{j-2}-1}^{2^{j-1}-2} \big(2^{k(1/p-1/r)} \|g_k\|_{L_p} \big)^{\sigma} \Big)^{1/\sigma} \\ &\lesssim 2^{j(1/\sigma-1/q)} \Big(\sum_{k=2^{j-2}-1}^{2^{j-1}-2} \big(2^{k(1/p-1/r)} \|g_k\|_{L_p} \big)^q \Big)^{1/q}. \end{aligned}$$

Consequently, by (3.4) and (4.2),

$$
\begin{aligned}
\|f\|_{B^{0,\gamma}_{r,q}} &\lesssim \Big(\sum_{j=0}^{\infty}\big(2^{j(\gamma+1/q)}\|h_j\|_{L_r}\big)^q\Big)^{1/q} \\
&\lesssim \Big(\sum_{j=2}^{\infty} 2^{j(\gamma+1/\sigma)q} \sum_{k=2^{j-2}-1}^{2^{j-1}-2} \big(2^{k(1/p-1/r)}\|g_k\|_{L_p}\big)^q\Big)^{1/q} \\
&\lesssim \Big(\sum_{j=2}^{\infty} \sum_{k=2^{j-2}-1}^{2^{j-1}-2} \big(2^{k(1/p-1/r)}(1+k)^{\gamma+1/\sigma}\|g_k\|_{L_p}\big)^q\Big)^{1/q} \\
&\lesssim \|f\|_{B^{1/p-1/r,\gamma+1/\sigma}_{p,q}}. \ \blacksquare
\end{aligned}
$$

Combining Theorem 4.3 and Corollary 4.2, we get the following result.

COROLLARY 4.4. *Let $0 < p \le r \le \infty$, $0 < q \le \infty$ and $\gamma > \eta > -1/q$. Then the embedding*

$$B^{1/p-1/r,\gamma+1/\min(1,r,q)}_{p,q} \hookrightarrow B^{0,\eta}_{r,q}$$

is compact.

REMARK 4.5. Another kind of compact embeddings involving Besov spaces of logarithmic smoothness can be found in [6] and [8]. There, the authors work with spaces on $\mathbb{R}^n$, defined by means of the modulus of continuity, and study their compact embeddings into Lorentz–Zygmund function spaces by using different tools than those considered here.

We finish the paper with the following consequence of Corollary 3.8.

COROLLARY 4.6. *Let $1 \le p \le \infty$, $1 < q < \infty$ and $\gamma > -1/q$. Then the space $B^{0,\gamma}_{p,q}$ is reflexive.*

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