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GENERALIZED POLY-CAUCHY POLYNOMIALS AND THEIR INTERPOLATING FUNCTIONS

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Abstract. We give a generalization of poly-Cauchy polynomials and investigate their arithmetical and combinatorial properties. We also study the zeta functions which interpolate the generalized poly-Cauchy polynomials.

1. Introduction. Let $n \geq 0$, $k \geq 1$ be integers. The *poly-Cauchy polynomials of the first kind* $c_n^{(k)}(z)$ are defined by

$$c_n^{(k)}(z) = \underbrace{\int_0^1 \ldots \int_0^1}_{k} (x_1 \cdots x_k + z)(x_1 \cdots x_k - 1 + z) \cdots (x_1 \cdots x_k - n + 1 + z)\, dx_1 \cdots dx_k$$

(see [13]). If $z = 0$, then $c_n^{(k)}(0) = c_n^{(k)}$ are the *poly-Cauchy numbers of the first kind* introduced in [15]. If $k = 1$, then $c_n^{(1)}(z) = c_n(z)$ are the classical *Cauchy polynomials* (see e.g. [6]). If $z = 0$ and $k = 1$, then $c_n^{(1)}(0) = c_n$ are the classical *Cauchy numbers* defined by

$$c_n = \int_0^1 x(x-1)\cdots(x-n+1)\, dx$$

(see e.g. [7, 20]). We remark that $b_n := c_n/n!$ are also called *Bernoulli numbers of the second kind* (see e.g. [1], [10]).

Before the terminology of Cauchy numbers appeared in Comtet's book [7], the concept was first introduced by Nörlund [21, pp. 146–147] in 1924.

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There, the *higher order Bernoulli numbers* $B_n^{(r)}$ were defined by

$$\left(\frac{x}{e^x-1}\right)^r = \sum_{n=0}^\infty B_n^{(r)}\frac{x^n}{n!} \qquad (|x|<2\pi),$$

or

$$\left(\frac{\ln(1+x)}{x}\right)^r = r\sum_{n=0}^\infty \frac{B_n^{(r+n)}}{r+n}\frac{x^n}{n!} \qquad (|x|<1).$$

See also [9, pp. 257, 259]. Then

$$B_n^{(n)} = \int_0^1 (x-1)\cdots(x-n)\,dx,$$

or

$$B_{n+1}^{(n)} = -n\int_0^1 x(x-1)\cdots(x-n)\,dx.$$

Hence, $c_n = -B_n^{(n-1)}/(n-1)$.

The concept of the Cauchy polynomials was first introduced by Ch. Jordan [12, p. 130] in 1928. There, the Bernoulli polynomials of the second kind were defined by the derivative of the binomial coefficient:

$$D\psi_n(x) = \binom{x}{n-1}.$$

Hence, $\psi_n(x) = c_n(-x)/n!$. The Bernoulli numbers of the second kind b_n (see [12, p. 131]) were also defined by

$$b_n = \psi_{n+1}(1) - \psi_{n+1}(0) = \int_0^1 \binom{x}{n}\,dx.$$

Hence, as stated above, $b_n = c_n/n!$.

A relation between the Bernoulli polynomials of the second kind and the higher order Bernoulli polynomials was pointed out by Carlitz [4] in 1961. Define β_n and $\beta_n^{(z)}$ by

$$\frac{x}{\ln(1+x)} = \sum_{n=0}^\infty \beta_n\frac{x^n}{n!} \quad \text{and} \quad \left(\frac{x}{\ln(1+x)}\right)^r (1+x)^z = \sum_{n=0}^\infty \beta_n^{(r)}(z)\frac{x^n}{n!}.$$

That is, $\beta_n = c_n$. Then Carlitz showed that

$$\beta_n^{(r)}(z) = B_n^{(n-r+1)}(z+1)$$

(see [4, (2.11)]). See also [24, 19]. We remark that $\beta_n^{(1)}(z)$ are also called Bernoulli polynomials of the second kind (see e.g. [10], [23, §4.3.2]).

The generating function of poly-Cauchy polynomials [13, Theorem 2] is given by

$$(1+x)^z \operatorname{Lif}_k(\ln(1+x)) = \sum_{n=0}^{\infty} c_n^{(k)}(z)\frac{x^n}{n!},$$

where

$$\operatorname{Lif}_k(z) = \sum_{m=0}^{\infty} \frac{z^m}{m!(m+1)^k}$$

is the kth *polylogarithm factorial* function [15] or simply the *polyfactorial* function. An explicit formula for $c_n^{(k)}(z)$ (see [13, Theorem 1]) is

$$c_n^{(k)}(z) = \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} (-1)^{n-m} \sum_{i=0}^{m} \binom{m}{i} \frac{z^i}{(m-i+1)^k}, \tag{1.1}$$

where $\left[{n \atop m}\right]$ are the (unsigned) *Stirling numbers of the first kind*, arising as coefficients of the rising factorial

$$x(x+1)\cdots(x+n-1) = \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} x^m$$

(see e.g. [11]).

The concept of poly-Cauchy numbers is an analogue of that of *poly-Bernoulli numbers* $B_n^{(k)}$ (see [14]) defined by

$$\frac{\operatorname{Li}_k(1-e^{-x})}{1-e^{-x}} = \sum_{n=0}^{\infty} B_n^{(k)} \frac{x^n}{n!},$$

where

$$\operatorname{Li}_k(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^k}$$

is the kth polylogarithm function. When $k = 1$, $B_n = B_n^{(1)}$ is the classical Bernoulli number with $B_1^{(1)} = 1/2$, whose generating function is

$$\frac{xe^x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}.$$

An explicit formula for $B_n^{(k)}$ (see [14, Theorem 1]) is

$$B_n^{(k)} = (-1)^n \sum_{m=0}^{n} \left\{ {n \atop m} \right\} \frac{(-1)^m m!}{(m+1)^k} \qquad (n \geq 0,\, k \geq 1), \tag{1.2}$$

where $\left\{ {n \atop m} \right\}$ are the *Stirling numbers of the second kind*, determined by

$$\left\{ {n \atop m} \right\} = \frac{1}{m!} \sum_{j=0}^{m} (-1)^j \binom{m}{j} (m-j)^n$$

(see e.g. [11]). A relation between the denominator of B_{2n} and the Stirling numbers of the second kind via the greatest common divisor is investigated in [18].

In this paper, we give a generalization of the poly-Cauchy polynomials and investigate several of their arithmetical and combinatorial properties. We also study the zeta functions which interpolate the generalized poly-Cauchy polynomials.

2. Definitions and basic properties. Let $n \ge 0$, $k \ge 1$ be integers, and q and $l_1, \dots, l_k$ be non-zero real numbers. Define

$$c_{n,q,(l_1,\dots,l_k)}^{(k)}(z) = \underbrace{\int_0^{l_1} \dots \int_0^{l_k}}_{k} (x_1 \cdots x_k + z)(x_1 \cdots x_k - q + z) \cdots \big(x_1 \cdots x_k - (n-1)q + z\big)\, dx_1 \cdots dx_k\,.$$

If $l_1 = \dots = l_k = 1$, then $c_{n,q,(1,\dots,1)}^{(k)}(-z) = c_{n,q}(z)$ are the *poly-Cauchy polynomials* (of the first kind) *with parameter* q. Note that z is replaced by $-z$ in [16]. By the definition, we can see that

$$c_{n,q,L}^{(k)}(z+q) - c_{n,q,L}^{(k)}(z) = nqc_{n-1,q,L}^{(k)}(z). \tag{2.1}$$

The polynomials $c_{n,q,(l_1,\dots,l_k)}^{(k)}(z)$ can be expressed in terms of Stirling numbers of the first kind $\left[{n \atop m} \right]$. For simplicity, from now on, we write $c_{n,q,L}^{(k)}(z) = c_{n,q,(l_1,\dots,l_k)}^{(k)}(z)$ with $L = (l_1, \dots, l_k)$ and $l = l_1 \cdots l_k$.

THEOREM 2.1. *For integers $n \ge 0$ and $k \ge 1$, we have*

$$c_{n,q,L}^{(k)}(z) = \sum_{m=0}^{n} \left[{n \atop m} \right] (-q)^{n-m} \sum_{i=0}^{m} \binom{m}{i} \frac{l^{m-i+1} z^i}{(m-i+1)^k}.$$

REMARK. The integer k must be positive in the definition of $c_{n,q,L}^{(k)}(z)$, but k can be 0 or a negative integer in the above expression. If $l = l_1 \cdots l_k = 1$, then Theorem 2.1 reduces to [16, Theorem 5(1)].

Proof of Theorem 2.1. Since

$$x(x-1)\cdots(x-n+1) = \sum_{m=0}^{n} \left[{n \atop m} \right] (-1)^{n-m} x^m,$$

we have

$$\begin{aligned}c_{n,q,L}^{(k)}(z) &= \int_0^{l_1}\dots\int_0^{l_k}\sum_{m=0}^{n}\begin{bmatrix}n\\m\end{bmatrix}(-1)^{n-m}(x_1\cdots x_k+z)^m q^{n-m}\,dx_1\cdots dx_k\\ &= \sum_{m=0}^{n}\begin{bmatrix}n\\m\end{bmatrix}(-q)^{n-m}\sum_{i=0}^{m}\binom{m}{i}z^i\int_0^{l_1}\dots\int_0^{l_k}(x_1\cdots x_k)^{m-i}\,dx_1\cdots dx_k\\ &= \sum_{m=0}^{n}\begin{bmatrix}n\\m\end{bmatrix}(-q)^{n-m}\sum_{i=0}^{m}\binom{m}{i}\frac{l^{m-i+1}z^i}{(m-i+1)^k}. \ \blacksquare\end{aligned}$$

The generating function of the polynomial $c_{n,q,L}^{(k)}(z)$ $(q\neq 0)$ is given by using the polyfactorial function.

THEOREM 2.2. *For integers n and k with $n\ge 0$, we have*

$$(1+qx)^{z/q}l\,\mathrm{Lif}_k\left(\frac{l\ln(1+qx)}{q}\right)=\sum_{n=0}^{\infty}c_{n,q,L}^{(k)}(z)\frac{x^n}{n!}.$$

REMARK. If $l=1$, then Theorem 2.2 reduces to [16, Theorem 6(1)]. Note that z is changed to $-z$ in [16].

Proof of Theorem 2.2. Since

$$\frac{\big(\ln(1+x)\big)^m}{m!}=(-1)^m\sum_{n=m}^{\infty}\begin{bmatrix}n\\m\end{bmatrix}\frac{(-x)^n}{n!},$$

by Theorem 2.1 we have

$$\begin{aligned}&\sum_{n=0}^{\infty}c_{n,q,L}^{(k)}(z)\frac{x^n}{n!}\\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\begin{bmatrix}n\\m\end{bmatrix}(-q)^{n-m}\sum_{i=0}^{m}\binom{m}{i}\frac{l^{m-i+1}z^i}{(m-i+1)^k}\,\frac{x^n}{n!}\\ &= \sum_{m=0}^{\infty}(-q)^{-m}\sum_{i=0}^{m}\binom{m}{i}\frac{l^{m-i+1}z^i}{(m-i+1)^k}\sum_{n=m}^{\infty}\begin{bmatrix}n\\m\end{bmatrix}\frac{(-qx)^n}{n!}\\ &= \sum_{m=0}^{\infty}\frac{1}{m!}\left(\frac{\ln(1+qx)}{q}\right)^m\sum_{i=0}^{m}\binom{m}{i}\frac{l^{m-i+1}z^i}{(m-i+1)^k}\\ &= \sum_{i=0}^{\infty}\frac{1}{i!}\left(\frac{z\ln(1+qx)}{q}\right)^i\sum_{m=i}^{\infty}\frac{l^{m-i+1}}{(m-i)!(m-i+1)^k}\left(\frac{\ln(1+qx)}{q}\right)^{m-i}\\ &= \sum_{i=0}^{\infty}\frac{1}{i!}\left(\frac{z\ln(1+qx)}{q}\right)^i\sum_{\nu=0}^{\infty}\frac{l^{\nu+1}}{\nu!(\nu+1)^k}\left(\frac{\ln(1+qx)}{q}\right)^{\nu}\\ &= (1+qx)^{z/q}l\,\mathrm{Lif}_k\left(\frac{l\ln(1+qx)}{q}\right). \ \blacksquare\end{aligned}$$

The generating function of the polynomial $c_{n,q,L}^{(k)}$ can be written in the form of iterated integrals.

COROLLARY 2.3. *Let q be a real number with $q \neq 0$. For $k = 1$, we have*

$$(1+qx)^{z/q}\,\frac{q((1+qx)^{l_1/q}-1)}{\ln(1+qx)} = \sum_{n=0}^{\infty} c_{n,q,l_1}^{(1)}(z)\frac{x^n}{n!}.$$

For $k > 1$, we have (with $k-1$ integrals)

$$(1+qx)^{z/q}\frac{q}{\ln(1+qx)}\int_0^x \frac{q}{(1+qx)\ln(1+qx)}\int_0^x \cdots \frac{q}{(1+qx)\ln(1+qx)}$$
$$\times \int_0^x \frac{q\big((1+qx)^{l/q}-1\big)}{\ln(1+qx)}\,\underbrace{dx\cdots dx}_{k-1} = \sum_{n=0}^{\infty} c_{n,q,L}^{(k)}(z)\frac{x^n}{n!}.$$

REMARK. If $l = 1$, then Corollary 2.3 reduces to [16, Corollary 3(1)]. Note that z is changed to $-z$ in [16].

Proof of Corollary 2.3. For $k = 1$,

$$\mathrm{Lif}_1(z) = \frac{e^z-1}{z}.$$

For $k > 1$, we have

$$\mathrm{Lif}_k(z) = \frac{1}{z}\sum_{m=0}^{\infty}\frac{z^{m+1}}{m!(m+1)^k} = \frac{1}{z}\int_0^z \sum_{m=0}^{\infty}\frac{z^m}{m!(m+1)^{k-1}}\,dz = \frac{1}{z}\int_0^z \mathrm{Lif}_{k-1}(z)$$
$$= \underbrace{\frac{1}{z}\int_0^z \frac{1}{z}\int_0^z \cdots \frac{1}{z}\int_0^z}_{k-1} \frac{e^z-1}{z}\,\underbrace{dz\cdots dz}_{k-1}.$$

Setting $z = l\ln(1+qx)/q$ and multiplying by $(1+qx)^{z/q}l$, we get the result. ∎

3. Poly-Cauchy polynomials of the second kind. In [15], the concept of *poly-Cauchy numbers of the second kind* $\widehat{c}_n^{(k)}$ is also introduced. They are defined by

$$\widehat{c}_n^{(k)} = \underbrace{\int_0^1 \cdots \int_0^1}_{k} (-x_1\cdots x_k)(-x_1\cdots x_k-1)$$
$$\cdots(-x_1\cdots x_k-n+1)\,dx_1\cdots dx_k$$

and their generating function is given by

$$\mathrm{Lif}_k(-\ln(1+x)) = \sum_{n=0}^{\infty}\widehat{c}_n^{(k)}\frac{x^n}{n!}.$$

The poly-Cauchy numbers of the second kind $\widehat{c}_n^{(k)}$ can also be expressed in terms of Stirling numbers of the first kind (see [15, Theorem 4]).

Proposition 3.1. *We have*

$$\widehat{c}_n^{(k)} = (-1)^n \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} \frac{1}{(m+1)^k} \,.$$

Similarly to generalized poly-Cauchy polynomials of the first kind $c_{n,q,L}^{(k)}(z)$, we define the *poly-Cauchy polynomials of the second kind* $\widehat{c}_{n,q,L}^{(k)}(z)$ $(n \ge 0, k \ge 1)$ by

$$\widehat{c}_{n,q,L}^{(k)}(z) = \int_0^{l_1} \cdots \int_0^{l_k} (-x_1 \cdots x_k - z)(-x_1 \cdots x_k - q - z) \cdots (-x_1 \cdots x_k - (n-1)q - z)\, dx_1 \cdots dx_k .$$

When $z = 0$ and $q = l_1 = \cdots = l_k = 1$, then $\widehat{c}_{n,1,L}^{(k)}(0) = \widehat{c}_n^{(k)}$ are the poly-Cauchy numbers of the second kind. By the definition, we can see that

$$\widehat{c}_{n,q,L}^{(k)}(z) - \widehat{c}_{n,q,L}^{(k)}(z-q) = -nq\widehat{c}_{n-1,q,L}^{(k)}(z). \tag{3.1}$$

Similarly to Theorem 2.1, $\widehat{c}_{n,q,L}^{(k)}$ can also be expressed in terms of Stirling numbers of the first kind $\left[\begin{smallmatrix} n \\ m \end{smallmatrix}\right]$:

Theorem 3.2. *For $n \ge 0$ and $k \ge 1$, we have*

$$\widehat{c}_{n,q,L}^{(k)}(z) = (-1)^n \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} q^{n-m} \sum_{i=0}^{m} \binom{m}{i} \frac{l^{m-i+1} z^i}{(m-i+1)^k} .$$

Remark. The integer k must be positive in the definition of $\widehat{c}_{n,q,L}^{(k)}(z)$, but k can be 0 or a negative integer in the above expression. If $l = l_1 \cdots l_k = 1$, then Theorem 3.2 reduces to [16, Theorem 5(2)].

Theorem 3.3. *The generating function of the polynomial $\widehat{c}_{n,q,L}^{(k)}(z)$ is given by*

$$\frac{l}{(1+qx)^{z/q}} \operatorname{Lif}_k\left(-\frac{l\ln(1+qx)}{q}\right) = \sum_{n=0}^{\infty} \widehat{c}_{n,q,L}^{(k)}(z) \frac{x^n}{n!} .$$

Remark. If $l = 1$, then Theorem 3.3 reduces to [16, Theorem 6(2)]. Note that z is changed to $-z$ in [16].

The generating function of the polynomial $\widehat{c}_{n,q,L}^{(k)}(z)$ can be written in the form of iterated integrals.

COROLLARY 3.4. *For $k=1$, we have*

$$\frac{1}{(1+qx)^{z/q}}\frac{q\big(1-(1+qx)^{-l_1/q}\big)}{\ln(1+qx)}=\sum_{n=0}^\infty \widehat{c}_{n,q,l_1}^{(1)}\frac{x^n}{n!}.$$

For $k>1$, we have (with $k-1$ integrals)

$$\frac{1}{(1+qx)^{z/q}}\frac{q}{\ln(1+qx)}\int_0^x\frac{q}{(1+qx)\ln(1+qx)}\int_0^x\cdots\frac{q}{(1+qx)\ln(1+qx)}$$

$$\times\int_0^x\frac{q(1-(1+qx)^{-l/q})}{\ln(1+qx)}\underbrace{dx\cdots dx}_{k-1}=\sum_{n=0}^\infty \widehat{c}_{n,q,L}^{(k)}\frac{x^n}{n!}.$$

REMARK. If $l=1$, then Corollary 3.4 reduces to [16, Corollary 3(2)]. Note that z is changed to $-z$ in [16].

4. Properties of poly-Cauchy numbers. There are relations between both kinds of poly-Cauchy polynomials if $q=1$.

THEOREM 4.1. *Let k be an integer. Then for $n\ge 1$, we have*

$$(-1)^n\frac{c_{n,1,L}^{(k)}(z)}{n!}=\sum_{m=1}^n\binom{n-1}{m-1}\frac{\widehat{c}_{m,1,L}^{(k)}(z)}{m!},$$

$$(-1)^n\frac{\widehat{c}_{n,1,L}^{(k)}(z)}{n!}=\sum_{m=1}^n\binom{n-1}{m-1}\frac{c_{m,1,L}^{(k)}(z)}{m!}.$$

Proof. We prove the second identity. The first one can be proved similarly and its proof is omitted. By using the identity (see e.g. [11, Chapter 6])

$$\frac{(-1)^i}{n!}\begin{bmatrix}n\\i\end{bmatrix}=\sum_{m=l}^n\frac{(-1)^m}{m!}\binom{n-1}{m-1}\begin{bmatrix}m\\i\end{bmatrix}$$

and Theorems 2.1 and 3.2, we have

$$\begin{aligned}\text{RHS}&=\sum_{m=1}^n\binom{n-1}{m-1}\frac{1}{m!}\sum_{\lambda=1}^m\begin{bmatrix}m\\\lambda\end{bmatrix}(-1)^{m-\lambda}\sum_{i=0}^\lambda\frac{l^{\lambda-i+1}z^i}{(\lambda-i+1)^k}\\&=\sum_{\lambda=1}^n\sum_{m=\lambda}^n\frac{(-1)^{m-\lambda}}{m!}\binom{n-1}{m-1}\begin{bmatrix}m\\\lambda\end{bmatrix}\sum_{i=0}^\lambda\binom{\lambda}{i}\frac{l^{\lambda-i+1}z^i}{(\lambda-i+1)^k}\\&=\frac{1}{n!}\sum_{\lambda=1}^n\begin{bmatrix}n\\\lambda\end{bmatrix}\sum_{i=0}^\lambda\binom{\lambda}{i}\frac{l^{\lambda-i+1}z^i}{(\lambda-i+1)^k}=\text{LHS}.\ \blacksquare\end{aligned}$$

By differentiating $c_{n,q,L}^{(k)}(z)$ or $\widehat{c}_{n,q,L}^{(k)}(z)$, we have the following:

Proposition 4.2. *For integers n, k with $n \ge 0$ and a real number $q \neq 0$, we have*

$$\frac{d}{dz} c_{n,q,L}^{(k)}(z) = -n! \sum_{\lambda=0}^{n-1} \frac{(-q)^{n-\lambda-1}}{(n-\lambda)\lambda!} c_{\lambda,q,L}^{(k)}(z),$$

$$\frac{d}{dz} \widehat{c}_{n,q,L}^{(k)} = n! \sum_{\lambda=0}^{n-1} \frac{(-q)^{n-\lambda-1}}{(n-\lambda)\lambda!} \widehat{c}_{\lambda,q,L}^{(k)}(z).$$

Proof. We prove the first identity. Differentiating both sides of the formula in Theorem 2.2 with respect to z, we have

$$-\frac{\ln(1+qx)}{q}(1+qx)^{z/q} l \operatorname{Lif}_k\left(\frac{l\ln(1+qx)}{q}\right) = \sum_{n=0}^{\infty} \frac{d}{dz} c_{n,q,L}^{(k)}(z) \frac{x^n}{n!}.$$

Then

$$\begin{aligned}
\text{LHS} &= \left(\sum_{m=1}^{\infty} \frac{(-1)^m q^{m-1} x^m}{m}\right)\left(\sum_{\lambda=0}^{\infty} c_{\lambda,q,L}^{(k)}(z) \frac{x^\lambda}{\lambda!}\right) \\
&= \sum_{n=1}^{\infty} \sum_{\lambda=0}^{n-1} \frac{(-1)^{n-\lambda} q^{n-\lambda-1} c_{\lambda,q,L}^{(k)}(z)}{(n-\lambda)\lambda!} x^n \\
&= \sum_{n=1}^{\infty} (-n!) \sum_{\lambda=0}^{n-1} \frac{(-q)^{n-\lambda-1} c_{\lambda,q,L}^{(k)}(z)}{(n-\lambda)\lambda!} \frac{x^n}{n!}
\end{aligned}$$

and

$$\text{RHS} = \sum_{n=1}^{\infty} \frac{d}{dz} c_{n,q,L}^{(k)}(z) \frac{x^n}{n!}.$$

The second identity is proven similarly. ∎

In some special cases we have simpler results. If $q = l = k = 1$, then by Theorems 2.1 and 3.2 we get simplified products.

Proposition 4.3. *For $n \ge 1$, we have*

$$\frac{d}{dz} c_{n,1,1}^{(1)}(z) = \frac{d}{dz} c_n^{(1)}(z) = n \prod_{i=0}^{n-2} (z-i),$$

$$\frac{d}{dz} \widehat{c}_{n,1,1}^{(1)}(z) = \frac{d}{dz} \widehat{c}_n^{(1)}(z) = (-1)^n n \prod_{i=1}^{n-1} (z+i).$$

Proof. By Theorem 2.1, we have

$$\begin{aligned}\frac{d}{dz}c_n^{(1)}(z) &= \sum_{m=1}^{n}\left[{n \atop m}\right](-1)^{n-m}\sum_{i=1}^{m}\binom{m}{i}\frac{iz^{i-1}}{m-i+1}\\ &= \sum_{m=1}^{n}\left[{n \atop m}\right](-1)^{n-m}\sum_{i=0}^{m-1}\binom{m}{i}z^{i}\\ &= \sum_{m=1}^{n}\left[{n \atop m}\right](-1)^{n-m}\big((z+1)^m - z^m\big).\end{aligned}$$

Since

$$\sum_{k=0}^{m}\left\{{m \atop k}\right\}(z)_k = z^k$$

with $(z)_k = z(z-1)\cdots(z-k+1)$ and

$$\sum_{m=k}^{n}(-1)^{n-m}\left[{n \atop m}\right]\left\{{m \atop k}\right\} = \begin{cases}1 & \text{if } k=n,\\ 0 & \text{if } k\neq n\end{cases}$$

(see e.g. [11, Ch. 6]), we get

$$\begin{aligned}\frac{d}{dz}c_n^{(1)}(z) &= \sum_{m=1}^{n}(-1)^{n-m}\left[{n \atop m}\right]\sum_{k=0}^{m}\left\{{m \atop k}\right\}\big((z+1)_k - (z)_k\big)\\ &= \sum_{m=1}^{n}(-1)^{n-m}\left[{n \atop m}\right]\sum_{k=0}^{m}\left\{{m \atop k}\right\}k\prod_{i=0}^{k-2}(z-i)\\ &= \sum_{k=0}^{n}\left(\sum_{m=k}^{n}(-1)^{n-m}\left[{n \atop m}\right]\left\{{m \atop k}\right\}\right)k\prod_{i=0}^{k-2}(z-i) = n\prod_{i=0}^{n-2}(z-i).\end{aligned}$$

The second identity can be proven similarly because

$$(-1)^m\big((z+1)^m - z^m\big) = \sum_{k=0}^{m}\left\{{m \atop k}\right\}(-1)^k k\prod_{i=1}^{k-1}(z+i). \quad \blacksquare$$

The derivative of $c_{n,q,L}^{(k)}(z)$ or $\hat{c}_{n,q,L}^{(k)}(z)$ with respect to the parameter l_j is the following.

Proposition 4.4. *For each $j = 1,\dots,k$, we have*

$$\frac{\partial}{\partial l_j}c_{n,q,L}^{(k)}(z) = \frac{c_{n,q,L}^{(k-1)}(z)}{l_j}, \qquad \frac{\partial}{\partial l_j}\widehat{c}_{n,q,L}^{(k)}(z) = \frac{\widehat{c}_{n,q,L}^{(k-1)}(z)}{l_j}.$$

Proof. By Theorem 2.1,

$$\frac{\partial}{\partial l_j} c_{n,q,L}^{(k)}(z) = \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} (-q)^{n-m} \sum_{i=0}^{m} \binom{m}{i} \frac{(l_1 \cdots l_k)^{m-i} \cdot (l_1 \cdots l_k)/l_j \cdot z^i}{(m-i+1)^{k-1}}$$
$$= c_{n,q,L}^{(k-1)}(z)/l_j.$$

The second identity is proven similarly. ∎

5. Functions interpolating generalized poly-Cauchy polynomials. Let k be a positive integer. For $s \in \mathbb{C}$ with $\Re(s) > 0$ and $z > -1$ define

$$Z_{k,q,L}(s,z) := \frac{1}{\Gamma(s)} \int_0^1 t^{s-1}(1-qt)^{z/q} \operatorname{Lif}_k\left(\frac{l \ln(1-qt)}{q}\right) dt. \tag{5.1}$$

By the change of the variables $t = (1-e^{-qu/l})/q$, this can be written as

$$Z_{k,q,L}(s,z) = \frac{1}{\Gamma(s)} \int_0^\infty \left(\frac{1-e^{-qu/l}}{q}\right)^{s-1} e^{-(z+q)u/l} \operatorname{Lif}_k(-u)\, du. \tag{5.2}$$

THEOREM 5.1. *The function $Z_{k,q,L}(-n,z)$ can be extended to an entire function, and its values at non-positive integers are given by*

$$Z_{k,q,L}(-n,z) = c_{n,q,L}^{(k)}(z) \quad (n = 0,1,2,\ldots).$$

REMARK. If $q = 1$ and $l_1 \cdots l_k = 1$, then Theorem 5.1 reduces to [13, Proposition 6.2].

Proof of Theorem 5.1. The proof of the analytic continuation is similar to that of [13, Proposition 6.2]. By (5.1) and Theorem 2.2, for $n = 0,1,2,\ldots$ we have

$$Z_{k,q,L}(-n,z) = \lim_{s\to -n} \frac{1}{\Gamma(s)} \int_0^1 t^{s-1}(1-qt)^{z/q} l \operatorname{Lif}_k\left(\frac{l \ln(1-qt)}{q}\right) dt$$
$$= \lim_{s\to -n} \frac{1}{\Gamma(s)} \sum_{m=0}^\infty \frac{(-1)^m c_{m,q,L}^{(k)}(z)}{m!} \int_0^1 t^{m-n-1}\, dt$$
$$= \lim_{s\to -n} \frac{1}{\Gamma(s)} \sum_{m=0}^\infty \frac{(-1)^m c_{m,q,L}^{(k)}(z)}{m!} \frac{1}{m+s}$$
$$= \frac{(-1)^n c_{n,q,L}^{(k)}(z)}{n!} \lim_{s\to -n} \frac{1}{\Gamma(s)\cdot(n+s)} = c_{n,q,L}^{(k)}(z). \quad ∎$$

Theorem 5.1 gives the values of $Z_{k,q,L}(s,z)$ at negative integers. The values at positive integers are expressed by using values of the polylogarithm functions $\mathrm{Li}_k(z)$.

THEOREM 5.2. *Let n and k be positive integers. For $z \geq 0$, we have*

$$Z_{k,q,L}(n,z) = \frac{1}{(n-1)!q^{n-1}} \sum_{i=0}^{n-1} \binom{n-1}{i} (-1)^{i+1} \mathrm{Li}_k\left(-\frac{l}{(i+1)q+z}\right).$$

REMARK. If $q = l = 1$, Theorem 5.2 reduces to [13, Proposition 6.3].

Proof of Theorem 5.2. By (5.2), we have

$$Z_{k,q,L}(n,z) = \frac{1}{(n-1)!} \int_0^\infty \frac{1}{q^{n-1}} \sum_{i=0}^{n-1} \binom{n-1}{i} (-1)^i e^{-(qi+q+z)u/l} \sum_{m=0}^\infty \frac{(-u)^m}{m!(m+1)^k}\, du.$$

The change of variables $u = lv/(qi+q+z)$ shows that

$$Z_{k,q,L}(n,z) = \frac{1}{(n-1)!q^{n-1}} \sum_{i=0}^{n-1} \binom{n-1}{i} (-1)^i \int_0^\infty e^{-v} \sum_{m=0}^\infty \frac{(-1)^m v^m l^{m+1}}{m!(m+1)^k((i+1)q+z)^{m+1}}\, dv.$$

Since $m! = \int_0^\infty e^{-v} v^m\, dv$, we obtain

$$\begin{aligned} Z_{k,q,L}(n,z) &= \frac{1}{(n-1)!q^{n-1}} \sum_{i=0}^{n-1} \binom{n-1}{i} (-1)^i \sum_{m=0}^\infty \frac{(-1)^m l^{m+1}}{(m+1)^k((i+1)q+z)^{m+1}} \\ &= \frac{1}{(n-1)!q^{n-1}} \sum_{i=0}^{n-1} \binom{n-1}{i} (-1)^{i+1} \mathrm{Li}_k\left(-\frac{l}{(i+1)q+z}\right). \quad \blacksquare \end{aligned}$$

The values at positive integers are also expressed by using multiple zeta star values defined by

$$\zeta_n^*(k_1,\dots,k_r) = \sum_{n \geq m_1 \geq \cdots \geq m_r \geq 1} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}}.$$

THEOREM 5.3. *Let $|l/q| \leq 1$. Then, for $k \geq 1$, $n \geq 1$ and $0 \leq z < |q|$, we have*

$$Z_{k,q,L}(n,z) = \frac{1}{n!} \sum_{m=1}^\infty \frac{(-1)^{m+1} l^m}{m^k} \sum_{j=0}^\infty \binom{m+j-1}{m-1} \frac{\zeta_n^*(\{1\}_{m+j-1})(-z)^j}{q^{n+m+j-1}},$$

where

$$\zeta_n^*(\{1\}_r) = \begin{cases} \displaystyle\sum_{n \geq m_1 \geq \cdots \geq m_r \geq 1} \frac{1}{m_1 \cdots m_r} & (r \geq 1), \\ 0 & (r = 0). \end{cases}$$

In particular, when $z=0$, we have

$$Z_{k,q,L}(n,0)=\frac{1}{n!}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}l^m}{m^k}\frac{\zeta_n^*(\{1\}_{m-1})}{q^{n+m-1}}.$$

REMARK. If $q=l=1$, then Theorem 5.3 reduces to [13, Theorem 6.5].

Proof of Theorem 5.3. By Theorem 5.2, we have

$$Z_{k,q,L}(n,z)=\frac{1}{(n-1)!q^{n-1}}\sum_{i=0}^{n-1}\binom{n-1}{i}\sum_{m=1}^{\infty}\frac{(-1)^{i+m+1}}{m^k}\left(\frac{l}{(i+1)q+z}\right)^m.$$

Since for $0\le z<|q|$,

$$\frac{1}{((i+1)q+z)^m}=\frac{1}{(i+1)^m q^m}\sum_{j=0}^{\infty}\binom{m+j-1}{m-1}\left(-\frac{z}{(i+1)q}\right)^j,$$

we get

$$Z_{k,q,L}(n,z)=\frac{1}{(n-1)!}\sum_{m=1}^{\infty}\frac{l^m}{m^k}\sum_{j=0}^{\infty}\binom{m+j-1}{m-1}\frac{1}{q^{n+m+j-1}}\times\sum_{i=1}^{n}\binom{n}{i}\frac{(-1)^{i+m}(-z)^j}{i^{m+j-1}}.$$

Since for $r\ge 0$,

$$\sum_{i=1}^{n}\binom{n}{i}\frac{(-1)^{i-1}}{i^r}=\zeta_n^*(\{1\}_r)$$

(see [22, (2)]) and so $\lim_{r\to\infty}\zeta_n^*(\{1\}_r)=n$ $(n=1,2,\dots)$, we have

$$Z_{k,q,L}(n,z)=\frac{1}{n!}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}l^m}{m^k}\sum_{j=0}^{\infty}\binom{m+j-1}{m-1}\frac{\zeta_n^*(\{1\}_{m+j-1})(-z)^j}{q^{n+m+j-1}}. \quad\blacksquare$$

6. The second case. For $s\in\mathbb{C}$ with $\Re(s)>0$ and $z>-1$ define

$$\widehat{Z}_{k,q,L}(s,z):=\frac{1}{\Gamma(s)}\int_0^1\frac{t^{s-1}}{(1-qt)^{z/q}}l\,\mathrm{Lif}_k\left(-\frac{l\ln(1-qt)}{q}\right)dt, \tag{6.1}$$

or equivalently,

$$\widehat{Z}_{k,q,L}(s,z)=\frac{1}{\Gamma(s)}\int_0^{\infty}\left(\frac{1-e^{-qu/l}}{q}\right)^{s-1}e^{(z-q)u/l}\,\mathrm{Lif}_k(u)\,du. \tag{6.2}$$

THEOREM 6.1. *The function $\widehat{Z}_{k,q,L}(-n,z)$ can be extended to an entire function, and its values at non-positive integers are given by*

$$\widehat{Z}_{k,q,L}(-n,z)=\widehat{c}_{n,q,L}^{k}(z)\qquad(n=0,1,2,\dots).$$

Proof. By (6.1) and Theorem 3.3, for $n = 0, 1, 2, \ldots$, we have

$$\begin{aligned}\widehat{Z}_{k,q,L}(-n,z) &= \frac{1}{\Gamma(-n)}\int_0^1 \frac{t^{-n-1}}{(1-qt)^{z/q}} l\,\mathrm{Lif}_k\left(-\frac{l\ln(1-qt)}{q}\right)dt \\ &= \sum_{m=0}^{\infty}\frac{\widehat{c}^{(k)}_{m,q,L}(z)(-1)^m}{m!}\frac{1}{\Gamma(-n)}\int_0^1 t^{m-n-1}\,dt \\ &= \frac{\widehat{c}^{(k)}_{n,q,L}(z)(-1)^n}{n!}\frac{n!(-1)^n}{2\pi i}\cdot 2\pi i = \widehat{c}^{k}_{n,q,L}(z). \ \blacksquare\end{aligned}$$

REMARK. If $q = l = 1$, then Theorem 6.1 reduces to [13, Proposition 7.2].

The function $\widehat{Z}_{k,q,L}(s,z)$ has similar properties to those of $Z_{k,q,L}(n,z)$. They are proven in the same manner, so we only state the results and omit their proofs.

THEOREM 6.2. *Let n and k be positive integers. For $z \ge 0$, we have*

$$\widehat{Z}_{k,q,L}(n,z) = \frac{1}{(n-1)!q^{n-1}}\sum_{i=0}^{n-1}\binom{n-1}{i}(-1)^i\,\mathrm{Li}_k\left(\frac{l}{(i+1)q-z}\right).$$

REMARK. If $q = l = 1$, then Theorem 6.2 reduces to [13, Proposition 7.3].

THEOREM 6.3. *Let $|l/q| \le 1$. Then for $k \ge 1$, $n \ge 1$ and $0 \le z < |q|$, we have*

$$\widehat{Z}_{k,q,L}(n,z) = \frac{1}{n!}\sum_{m=1}^{\infty}\frac{l^m}{m^k}\sum_{j=0}^{\infty}\frac{1}{q^{n+m+j-1}}\binom{m+j-1}{m-1}\zeta_n^*(\{1\}_{m+j-1})z^j.$$

In particular, when $z = 0$, we have

$$\widehat{Z}_{k,q,L}(n,0) = \frac{1}{n!}\sum_{m=1}^{\infty}\frac{l^m}{m^k}\frac{\zeta_n^*(\{1\}_{m-1})}{q^{n+m-1}} \qquad (k \ge 2).$$

REMARK. If $q = l = 1$, then Theorem 6.3 reduces to [13, Theorem 7.4].

7. Poly-Bernoulli polynomials with parameter q. Throughout this section, $l = l_1 \cdots l_k = 1$.

In [17], the first author and Cencki defined the Bernoulli numbers corresponding to the poly-Cauchy numbers with parameter q (see [16]) by

$$\frac{q\mathrm{Li}_k((1-e^{-qt})/q)}{1-e^{-qt}} = \sum_{n=0}^{\infty} B^{(k)}_{n,q}\frac{t^n}{n!}.$$

Hence, if $q = 1$, then $B^{(k)}_{n,1} = B^{(k)}_n$ are the poly-Bernoulli numbers [14].

As a general case of poly-Bernoulli numbers, the poly-Bernoulli numbers $B_{n,q}^{(k)}$ with parameter q can be expressed in terms of Stirling numbers of the second kind $\left\{ {n \atop m} \right\}$.

Lemma 7.1. *We have*

$$B_{n,q}^{(k)} = \sum_{m=0}^{n} \left\{ {n \atop m} \right\} \frac{(-q)^{n-m} m!}{(m+1)^k}.$$

We define the *poly-Bernoulli polynomials with parameter* q by

$$\frac{q\mathrm{Li}_k((1-e^{-qt})/q)}{1-e^{-qt}} e^{-tx} = \sum_{n=0}^{\infty} B_{n,q}^{(k)}(x) \frac{t^n}{n!}.$$

If $q = 1$, then $B_{n,1}^{(k)}(x) = B_n^{(k)}(x)$ are the *poly-Bernoulli polynomials* [8]. Note that we also have a different definition, where x and $-x$ are interchanged (see [3]). If $x = 0$, then $B_{n,q}^{(k)}(0) = B_{n,q}^{(k)}$ are the *poly-Bernoulli numbers with parameter* q.

Weighted Stirling numbers of the first kind and *of the second kind* (cf. [5] in slightly different meanings) are defined by the generating functions

$$\frac{(1-t)^z \bigl(-\ln(1-t)\bigr)^m}{m!} = \sum_{n=0}^{\infty} S_1(n,m,z) \frac{t^n}{n!},$$

$$\frac{e^{zt}(e^t-1)^m}{m!} = \sum_{n=0}^{\infty} S_2(n,m,z) \frac{t^n}{n!},$$

respectively, so that $S_1(n,m,0) = \left[{n \atop m} \right]$ and $S_2(n,m,0) = \left\{ {n \atop m} \right\}$. Poly-Bernoulli polynomials with parameter q are expressed explicitly by weighted Stirling numbers of the second kind $S_2(n,m,x)$.

Lemma 7.2. *For poly-Bernoulli polynomials with parameter* q, *we have*

$$B_{n,q}^{(k)}(x) = \sum_{m=0}^{n} S_2\left(n, m, \frac{x}{q}\right) \frac{(-q)^{n-m} m!}{(m+1)^k}.$$

There are relations between $B_{n,q}^{(k)}(x)$ and $c_{n,q}^{(k)}(x)$ (or $\widehat{c}_{n,q}^{(k)}(x)$). Note that x in $c_{n,q}^{(k)}(x)$ and $\widehat{c}_{n,q}^{(k)}(x)$ is changed to $-x$ in [16].

Lemma 7.3. *For any* x *and* y, *we have*

$$B_{n,q}^{(k)}(x) = \sum_{l=0}^{n} \sum_{m=0}^{n} (-1)^{n-m} m! q^{n-l} S_2\left(n, m, \frac{x}{q}\right) S_2\left(m, l, -\frac{y}{q}\right) c_{l,q}^{(k)}(y),$$

$$B_{n,q}^{(k)}(x) = \sum_{l=0}^{n} \sum_{m=0}^{n} (-1)^{n} m! q^{n-l} S_2\left(n, m, \frac{x}{q}\right) S_2\left(m, l, \frac{y}{q}\right) \widehat{c}_{l,q}^{(k)}(y),$$

$$c_{n,q}^{(k)}(x) = \sum_{l=0}^{n}\sum_{m=0}^{n} \frac{(-1)^{n-m}}{m!} q^{n-l} S_1\left(n, m, -\frac{x}{q}\right) S_1\left(m, l, \frac{y}{q}\right) B_{l,q}^{(k)}(y),$$
$$\widehat{c}_{n,q}^{(k)}(x) = \sum_{l=0}^{n}\sum_{m=0}^{n} \frac{(-1)^{n}}{m!} q^{n-l} S_1\left(n, m, \frac{x}{q}\right) S_1\left(m, l, \frac{y}{q}\right) B_{l,q}^{(k)}(y).$$

As a general case of the Arakawa–Kaneko zeta function [2, 8], define

$$\xi_{k,q}(s,z) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{\mathrm{Li}_r((1-e^{-qt})/q)}{(1-e^{-qt})/q} e^{-zt} t^{s-1}\, dt.$$

THEOREM 7.4. *The function $\xi_{k,q}(s,z)$ can be extended to an entire function, and its values at non-positive integers are given by*

$$\xi_{k,q}(-n,z) = (-1)^n B_{n,q}^{(k)}(z) \qquad (n = 0, 1, 2, \dots).$$

Proof. We split $\xi_{k,q}(s,z)$ into two integrals:

$$\xi_{-n,q}(s,z) = \frac{1}{\Gamma(s)} \int_0^1 \frac{\mathrm{Li}_r((1-e^{-qt})/q)}{(1-e^{-qt})/q} e^{-zt} t^{s-1}\, dt + \frac{1}{\Gamma(s)} \int_1^\infty \frac{\mathrm{Li}_r((1-e^{-qt})/q)}{(1-e^{-qt})/q} e^{-zt} t^{s-1}\, dt.$$

The second integral converges absolutely for an arbitrary $s \in \mathbb{C}$ and vanishes at non-positive integers. For $\mathcal{R}(s) > 0$, the first integral can be written as

$$\frac{1}{\Gamma(s)} \int_0^1 \sum_{m=0}^\infty B_{m,q}^{(k)}(z) \frac{t^m}{m!} t^{s-1}\, dt = \frac{1}{\Gamma(s)} \sum_{m=0}^\infty \frac{B_{m,q}^{(k)}(z)}{m!} \cdot \frac{1}{m+s}.$$

Therefore,

$$\xi_{k,q}(-n,z) = \lim_{s\to -n} \xi_{k,q}(s,z) = \lim_{s\to -n} \frac{1}{\Gamma(s)\cdot(n+s)} \frac{B_{n,q}^{(k)}(z)}{n!} = \frac{n!}{(-1)^n} \frac{B_{n,q}^{(k)}(z)}{n!} = (-1)^n B_{n,q}^{(k)}(z). \blacksquare$$

For simplicity, we write $Z_{k,q}(n,z) = Z_{k,q,L}(n,z)$ when $l = l_1 \cdots l_k = 1$. We can show a duality formula between $Z_{k,q}(n,z)$ and $\xi_{k,q}(s,z)$.

THEOREM 7.5. *For integers $k \ge 2$ and $r \ge 2$ and a real number z with $1-q \le z < 2-q$, we have*

$$\sum_{n=1}^\infty \frac{\Gamma(n)}{n^r} Z_{k,q}(n,z) = \sum_{m=1}^\infty (-1)^{m+1} \frac{\xi_{r,q}(m, q+z)}{m^k}.$$

REMARK. If $q = 1$, then Theorem 7.5 reduces to [13, Corollary 6.6].

Proof of Theorem 7.5. We shall calculate

$$\int_0^\infty \frac{e^{-uz}\operatorname{Li}_r((1-e^{-qu})/q)\operatorname*{Lif}_k(-u)}{e^{qu}-1}\,du \tag{7.1}$$

in two ways. Firstly, (7.1) is equal to

$$\begin{aligned}
&\int_0^\infty \frac{e^{-uz}\operatorname{Li}_r((1-e^{-qu})/q)}{e^{qu}-1}\sum_{m=1}^\infty \frac{(-u)^{m-1}}{(m-1)!m^k}\,du \\
&\qquad= \sum_{m=1}^\infty \frac{(-1)^{m+1}}{m^k}\frac{1}{\Gamma(m)}\int_0^\infty \frac{e^{-uz}u^{m-1}\operatorname{Li}_r((1-e^{-qu})/q)}{e^{qu}-1}\,du \\
&\qquad= \sum_{m=1}^\infty (-1)^{m+1}\frac{\xi_{r,q,L}(m,q+z)}{m^k q}.
\end{aligned}$$

On the other hand, (7.1) is equal to

$$\begin{aligned}
&\int_0^\infty e^{-u(z+q)}\operatorname{Lif}_k(-u)\sum_{n=1}^\infty \frac{(1-e^{-qu})^{n-1}}{q^n n^r}\,du \\
&\quad= \sum_{n=1}^\infty \frac{1}{n^r q}\int_0^\infty e^{-u(z+q)}\left(\frac{1-e^{-qu}}{q}\right)^{n-1}\operatorname{Lif}_k(-u)\,du = \sum_{n=1}^\infty \frac{\Gamma(n)}{n^r q}Z_{k,q,L}(n,z).
\end{aligned}$$

Combining the two expressions, we get the result. ∎

Similarly, when $l = l_1\cdots l_k = 1$, we have a duality formula between $\widehat{Z}_{k,q}(n,z) := \widehat{Z}_{k,q,L}(n,z)$ and $\xi_{k,q}(s,z)$.

THEOREM 7.6. *For integers $k \ge 2$ and $r \ge 2$ and a real number z with $q-2 < z \le q-1$, we have*

$$\sum_{n=1}^\infty \frac{\Gamma(n)}{n^r}\widehat{Z}_{k,q}(n,z) = \sum_{m=1}^\infty \frac{\xi_{r,q}(m,q-z)}{m^k}.$$

REMARK. If $q = 1$, then Theorem 7.6 reduces to [13, Corollary 7.5].

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