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ON THE INDEX OF AN ODD PERFECT NUMBER

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Abstract. Suppose that N is an odd perfect number and q^α is a prime power with $q^\alpha \parallel N$. Define the index $m = \sigma(N/q^\alpha)/q^\alpha$. We prove that m cannot take the form p^{2u}, where u is a positive integer and $2u+1$ is composite. We also prove that, if q is the Euler prime, then m cannot take any of the 30 forms q_1, q_1^2, q_1^3, q_1^4, q_1^5, q_1^6, q_1^7, q_1^8, q_1q_2, $q_1^2q_2$, $q_1^3q_2$, $q_1^4q_2$, $q_1^5q_2$, $q_1^2q_2^2$, $q_1^3q_2^2$, $q_1^4q_2^2$, $q_1q_2q_3$, $q_1^2q_2q_3$, $q_1^3q_2q_3$, $q_1^4q_2q_3$, $q_1^2q_2^2q_3$, $q_1^2q_2^2q_3^2$, $q_1q_2q_3q_4$, $q_1^2q_2q_3q_4$, $q_1^3q_2q_3q_4$, $q_1^2q_2^2q_3q_4$, $q_1q_2q_3q_4q_5$, $q_1^2q_2q_3q_4q_5$, $q_1q_2q_3q_4q_5q_6$, $q_1q_2q_3q_4q_5q_6q_7$, where q_1, q_2, q_3, q_4, q_5, q_6, q_7 are distinct odd primes. A similar result is proved if q is not the Euler prime. These extend recent results of Broughan, Delbourgo, and Zhou. We also pose a related problem.

1. Introduction. For a positive integer N, let $\sigma(N)$ be the sum of all positive divisors of N. We call N *perfect* if $\sigma(N) = 2N$. It is well known that an even integer N is perfect if and only if $N = 2^{p-1}(2^p - 1)$, where p and $2^p - 1$ are both primes. The existence of odd perfect numbers is one of the oldest open problems. If N is an odd perfect number, Euler gave the standard factorization of $N = \gamma_0^{\tau_0}\gamma_1^{2\tau_1}\cdots\gamma_s^{2\tau_s}$, where $\gamma_0, \gamma_1, \ldots, \gamma_s$ are distinct odd primes and $\gamma_0 \equiv \tau_0 \equiv 1 \pmod 4$. We call $\gamma_0^{\tau_0}$ the *Euler factor* of N, and γ_0 the *Euler prime*. In 2007, Nielsen [Ni2] proved that $s \geq 8$. This has been superseded recently by proving that $s \geq 9$ (see Nielsen [Ni1]). Ochem and Rao [OR] proved that there are no odd perfect numbers below 10^{1500}.

Let N be an odd perfect number with $q^\alpha \parallel N$, where q^α is a prime power and $q^\alpha \parallel N$ means that $q^\alpha \mid N$ and $q^{\alpha+1} \nmid N$. Since $\sigma(N) = 2N$, we have

$$\sigma(N/q^\alpha)\sigma(q^\alpha) = \frac{2N}{q^\alpha} \cdot q^\alpha.$$

By $(q^\alpha, \sigma(q^\alpha)) = 1$, we have $q^\alpha \mid \sigma(N/q^\alpha)$. Define the *index* $m = \sigma(N/q^\alpha)/q^\alpha$. Then m is a positive integer and

$$m\sigma(q^\alpha) = \frac{2N}{q^\alpha}. \tag{1.1}$$

Dris and Luca [DL] proved that $m \geq 6$. Chen and Chen [CC] improved

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the result of [DL] by showing that $m \neq q_1, q_1^2, q_1^3, q_1^4, q_1q_2, q_1^2q_2$, where q_1, q_2 are primes. By (1.1), $2 \nmid m$ if and only if q is the Euler prime. Recently, Broughan, Delbourgo and Zhou [BDZ] extended the list by proving the following theorem.

Theorem A. *Suppose that N is an odd perfect number and q^α is a prime power with $q^\alpha \,\|\, N$. Let $m = \sigma(N/q^\alpha)/q^\alpha$.*

(1) *If q is the Euler prime, then m cannot take any of the eleven forms*

$$q_1, q_1^2, q_1^3, q_1^4, q_1^5, q_1^6, q_1q_2, q_1^2q_2, q_1^3q_2, q_1^2q_2^2, q_1q_2q_3,$$

where q_1, q_2, q_3 are distinct odd primes.

(2) *If q is not the Euler prime and the Euler prime divides N to a power greater than 1, then m cannot take any of the seven forms*

$$2, 2q_1, 2q_1^2, 2q_1^3, 2q_1^4, 2q_1q_2, 2q_1^2q_2,$$

where q_1, q_2 are distinct odd primes.

(3) *If q is not the Euler prime and the Euler prime divides N to the power 1, then m cannot take any of the five forms*

$$2, 2q_1, 2q_1^2, 2q_1^3, 2q_1q_2,$$

where q_1, q_2 are distinct odd primes.

In this paper, we first prove two general theorems and then extend the above list as a corollary.

Theorem 1.1. *Suppose that N is an odd perfect number and q^α is a prime power with $q^\alpha \,\|\, N$. Let $m = \sigma(N/q^\alpha)/q^\alpha$. Then m cannot take the form p^{2u}, where u is a positive integer and $\sigma(p^{2u})$ is composite. In particular, m cannot take the form p^{2u}, where u is a positive integer and $2u + 1$ is composite.*

Motivated by Theorem 1.1, we pose the following problem.

Problem 1.2. *Is there any odd prime q such that*

$$\frac{p^q - 1}{p - 1}$$

is always composite for all primes p?

If q is such an odd prime, then m in Theorem 1.1 cannot take the form p^{q-1}.

Theorem 1.3. *Suppose that N is an odd perfect number and q^α is a prime power with $q^\alpha \,\|\, N$. Let $m = \sigma(N/q^\alpha)/q^\alpha = 2^\beta q_1^{\beta_1} \cdots q_u^{\beta_u}$, where $q_1, \dots, q_u$ are distinct odd primes and $\beta, \beta_1, \dots, \beta_u$ are integers with $\beta_1 \geq \cdots \geq \beta_v > \beta_{v+1} = \cdots = \beta_u = 1$ and $\beta \in \{0, 1\}$. If $2 \mid m$ and the Euler prime divides N to the power 1, let $w = 1$; otherwise, let $w = 0$. Then*

(i) $v + w + \beta_1 + \cdots + \beta_u > k_1(s)$, *where*

$$k_1(s) = \lfloor s - 1 - (\log(s+2) - \log 2)/\log 3 \rfloor;$$

(ii) $u + w + \beta_1 + \cdots + \beta_u > k_2(s)$, *where*

$$k_2(s) = \lfloor s - 1 - (\log(s+2) - \log 3)/\log 4 \rfloor;$$

(iii) $v + \beta_1 + \cdots + \beta_u > k_3(s)$ *if* $2 \nmid m$, *where*

$$k_3(s) = \lfloor s - 1 - (\log(s+2) - \log 4)/\log 3 \rfloor.$$

Here $\lfloor x \rfloor$ *denotes the largest integer not exceeding* x.

In the following corollary, we underline the terms excluded by the condition $s \geq 9$.

Corollary 1.4. *Suppose that* N *is an odd perfect number and* q^α *is a prime power with* $q^\alpha \parallel N$. *Let* $m = \sigma(N/q^\alpha)/q^\alpha$.

(1) *If* q *is the Euler prime, then* m *cannot take any of the* 19 *forms*

$$\underline{q_1^7}, q_1^8, q_1^4 q_2, \underline{q_1^5 q_2}, \underline{q_1^3 q_2^2}, q_1^4 q_2^2, q_1^2 q_2 q_3, q_1^3 q_2 q_3,$$
$$\underline{q_1^4 q_2 q_3}, \underline{q_1^2 q_2^2 q_3}, q_1^2 q_2^2 q_3^2, q_1 q_2 q_3 q_4, q_1^2 q_2 q_3 q_4, \underline{q_1^3 q_2 q_3 q_4}, \underline{q_1^2 q_2^2 q_3 q_4},$$
$$q_1 q_2 q_3 q_4 q_5, \underline{q_1^2 q_2 q_3 q_4 q_5}, q_1 q_2 q_3 q_4 q_5 q_6, \underline{q_1 q_2 q_3 q_4 q_5 q_6 q_7},$$

where $q_1, q_2, q_3, q_4, q_5, q_6, q_7$ *are distinct odd primes.*

(2) *If* q *is not the Euler prime and the Euler prime divides* N *to a power greater than* 1, *then* m *cannot take any of the* 14 *forms*

$$2q_1^5, \underline{2q_1^6}, 2q_1^3 q_2, \underline{2q_1^4 q_2}, 2q_1^2 q_2^2, \underline{2q_1^3 q_2^2}, 2q_1 q_2 q_3, 2q_1^2 q_2 q_3,$$
$$\underline{2q_1^3 q_2 q_3}, \underline{2q_1^2 q_2^2 q_3}, 2q_1 q_2 q_3 q_4, \underline{2q_1^2 q_2 q_3 q_4}, 2q_1 q_2 q_3 q_4 q_5, \underline{2q_1 q_2 q_3 q_4 q_5 q_6},$$

where $q_1, q_2, q_3, q_4, q_5, q_6$ *are distinct odd primes.*

(3) *If* q *is not the Euler prime and the Euler prime divides* N *to the power* 1, *then* m *cannot take any of the nine forms*

$$2q_1^4, \underline{2q_1^5}, 2q_1^2 q_2, \underline{2q_1^3 q_2}, \underline{2q_1^2 q_2^2}, 2q_1 q_2 q_3, \underline{2q_1^2 q_2 q_3}, 2q_1 q_2 q_3 q_4, \underline{2q_1 q_2 q_3 q_4 q_5},$$

where q_1, q_2, q_3, q_4, q_5 *are distinct odd primes.*

With more arguments, we can exclude $m = q_1^7, q_1^3 q_2^2, q_1^2 q_2^2 q_3$ by assuming only $s \geq 8$.

2. Lemmas. For any positive integer n, denote by $d(n)$ the number of positive divisors of n. Suppose that N is an odd perfect number with $q^\alpha \parallel N$, where q^α is a prime power. In this paper, we always write the standard factorization of N as

$$N = p_1^{\lambda_1} \cdots p_s^{\lambda_s} q^\alpha,$$

such that

$$\sigma(p_i^{\lambda_i}) = m_i q^{\mu_i}, \quad i = 1, \ldots, k, \qquad \sigma(p_i^{\lambda_i}) = q^{\mu_i}, \quad i = k+1, \ldots, s, \tag{2.1}$$

where $m_i \geq 2$ and $q \nmid m_i$ for $i = 1, \ldots, k$. Then (1.1) becomes

$$m\frac{q^{\alpha+1}-1}{q-1} = 2p_1^{\lambda_1}\cdots p_k^{\lambda_k}p_{k+1}^{\lambda_{k+1}}\cdots p_s^{\lambda_s}. \tag{2.2}$$

By the definition of m and (2.1), we have

$$mq^{\alpha} = \sigma(p_1^{\lambda_1}\cdots p_s^{\lambda_s}) = m_1\cdots m_k q^{\mu_1+\cdots+\mu_s}. \tag{2.3}$$

It follows from (2.2) that $m \mid 2p_1^{\lambda_1}\cdots p_s^{\lambda_s}$. So $q \nmid m$. Noting that $q \nmid m_i$ for $i = 1, \ldots, k$, by (2.3) we have

$$m = m_1\cdots m_k, \qquad \alpha = \mu_1+\cdots+\mu_s. \tag{2.4}$$

Write $m = p_{k+1}^{\alpha_{k+1}}\cdots p_s^{\alpha_s}m'$ with $(m', p_{k+1}\cdots p_s) = 1$ and $\alpha_{k+1} \geq \cdots \geq \alpha_s$. For convenience, let $\alpha_i = 0$ for all $i > s$. By (2.2) we have $\lambda_i \geq \alpha_i$ for $k+1 \leq i \leq s$. Now (2.2) becomes

$$m'\frac{q^{\alpha+1}-1}{q-1} = 2p_1^{\lambda_1}\cdots p_k^{\lambda_k}p_{k+1}^{\lambda_{k+1}-\alpha_{k+1}}\cdots p_s^{\lambda_s-\alpha_s}. \tag{2.5}$$

Noting that p_j and q are odd primes, by (2.1) we know that all λ_j $(k+1 \leq j \leq s)$ are positive even integers.

Now we present some lemmas which will be used later.

Lemma 2.1. *Let α, μ and γ be positive integers, and p and q be odd primes such that*

$$\frac{p^{\lambda+1}-1}{p-1} = q^{\mu}, \quad p^{\gamma} \,\Big|\, \frac{q^{\alpha+1}-1}{q-1}.$$

Then $p^{\gamma-1} \mid \alpha+1$ if $\mu > 1$, and $p^{\gamma} \mid \alpha+1$ if $\mu = 1$.

Lemma 2.1 follows from the proof of [BDZ, Lemma 2].

Lemma 2.2 ([CC, Lemma 4] or [Ni2, Lemma 4]). *If N is an odd perfect number with $q^{\alpha} \,\|\, N$, then $d(\alpha+1) \leq s+1$.*

Lemma 2.3 (Ljunggren [Lj], see also [EGSS, p. 359]). *The only integer solutions (x, n, y) with $|x| > 1$, $n > 2, y > 0$ to the equation $(x^n-1)/(x-1) = y^2$ are $(7, 4, 20)$ and $(3, 5, 11)$, i.e. $(7^4-1)/(7-1) = 20^2$ and $(3^5-1)/(3-1) = 11^2$.*

Lemma 2.4 ([EGSS, p. 363]). *The only solutions in non-zero integers with $n > 1$ to the equation $y^n = x^2+x+1$ are $n = 3$, $y = 7$ and $x = 18$ or $x = -19$.*

Lemma 2.5. *At most one of the λ_j $(k+1 \leq j \leq s)$ is 2.*

Proof. If λ_j is 2, then $p_j^2 + p_j + 1 = q^{\mu_j}$. Noting that p_j is a positive prime, by Lemma 2.4, we have $\mu_j = 1$. Since q is fixed, there is at most one prime p with $p^2+p+1 = q$. Now Lemma 2.5 follows. ∎

LEMMA 2.6. *Let $\delta = 1$ if $2 \nmid m$, otherwise $\delta = 0$, and let $\delta_i = 1$ if $\lambda_i > 2$ and $\delta_i = 0$ if $\lambda_i = 2$. Then*

$$2^\delta \prod_{j=k+1}^{s} p_j^{\max\{\lambda_j - \alpha_j - \delta_j, 0\}} \mid \alpha + 1 \tag{2.6}$$

and

$$(\delta + 1) \prod_{j=k+1}^{s} \max\{\lambda_j - \alpha_j - \delta_j + 1, 1\} \le d(\alpha + 1) \le s + 1. \tag{2.7}$$

Proof. It is clear that $2 \mid \alpha + 1$ if and only if q is the Euler prime. So $2^\delta \mid \alpha + 1$. From (2.1) and (2.5) we have

$$\frac{p_j^{\lambda_j+1} - 1}{p_j - 1} = q^{\mu_j}, \qquad j = k+1, \dots, s,$$
$$p_j^{\lambda_j - \alpha_j} \,\Big|\, \frac{q^{\alpha+1} - 1}{q - 1}, \qquad j = k+1, \dots, s.$$

If $\lambda_i = 2$, then, by Lemma 2.4 and p_i being a prime, we have $p_i^2 + p_i + 1 = q$. Noting that all λ_j $(k+1 \le j \le s)$ are positive even integers, by Lemma 2.1, we have

$$p_j^{\max\{\lambda_j - \alpha_j - \delta_j, 0\}} \mid \alpha + 1, \qquad j = k+1, \dots, s.$$

Thus (2.6) follows immediately and (2.7) follows from (2.6) and Lemma 2.2. ∎

REMARK. By Lemma 2.5, at most one of the δ_i is zero.

LEMMA 2.7 ([BDZ, Lemma 8]). *If the index m is a square, then $\alpha = 1$.*

LEMMA 2.8. *If the index m is a square, then $k = s - 1$ or s.*

Proof. By Lemma 2.7, we have $\alpha = 1$. By (2.4), exactly one of the μ_i $(1 \le i \le s)$ is 1 and the others are 0. Since $\mu_i > 0$ $(k+1 \le i \le s)$, we have $k = s - 1$ or s. ∎

LEMMA 2.9. *Let the notations be as in Theorem* 1.3 *and Lemma* 2.6. *Then none of the following three statements can happen:*

(i) *$k \le k_1(s)$ and $\alpha_{k_1(s)+1} \le 1$;*
(ii) *$k \le k_2(s)$ and $\alpha_{k_2(s)+1} = 0$;*
(iii) *$2 \nmid m$, $k \le k_3(s)$ and $\alpha_{k_3(s)+1} \le 1$.*

Proof. By Lemma 2.5, at most one of the λ_j $(k+1 \le j \le s)$ is 2.

(i) Suppose that $k \le k_1(s)$ and $\alpha_{k_1(s)+1} \le 1$. Then $0 \le \alpha_i \le 1$ for all $k_1(s) + 1 \le i \le s$. Thus, since all λ_j $(k+1 \le j \le s)$ are positive even integers, the left side of (2.7) is

$$(\delta+1)\prod_{j=k+1}^{s}\max\{\lambda_j-\alpha_j-\delta_j+1,1\} \geq \prod_{j=k_1(s)+1}^{s}(\lambda_j-\delta_j)\geq 2\cdot 3^{s-k_1(s)-1}\geq s+2,$$

a contradiction with (2.7).

(ii) Suppose that $k\leq k_2(s)$ and $\alpha_{k_2(s)+1}=0$. Then $\alpha_i=0$ for all $k_2(s)+1\leq i\leq s$. Thus, noting that all λ_j $(k+1\leq j\leq s)$ are positive even integers, the left side of (2.7) is

$$(\delta+1)\prod_{j=k+1}^{s}\max\{\lambda_j-\alpha_j-\delta_j+1,1\} \geq \prod_{j=k_2(s)+1}^{s}(\lambda_j-\delta_j+1)\geq 3\cdot 4^{s-k_2(s)-1}\geq s+2,$$

a contradiction with (2.7).

(iii) Suppose that $2\nmid m$, $k\leq k_3(s)$ and $\alpha_{k_3(s)+1}\leq 1$. Then $0\leq\alpha_i\leq 1$ for all $k_3(s)+1\leq i\leq s$. Thus, noting that all λ_j $(k+1\leq j\leq s)$ are positive even integers, the left side of (2.7) is

$$(\delta+1)\prod_{j=k+1}^{s}\max\{\lambda_j-\alpha_j-\delta_j+1,1\} \geq 2\prod_{j=k_3(s)+1}^{s}(\lambda_j-\delta_j)\geq 2\cdot 2\cdot 3^{s-k_3(s)-1}\geq s+2,$$

a contradiction with (2.7). ■

3. Proof of Theorem 1.1. Suppose that $m=p^{2u}$, where u is a positive integer and $\sigma(p^{2u})$ is composite. By (2.4) we have $p\mid m_i$ $(1\leq i\leq k)$. By (2.1) we have $p_i\neq p$ $(1\leq i\leq k)$. So $k\leq s-1$, $p_{k+1}=p$ and $\alpha_{k+1}=2u$. It follows from Lemmas 2.7 and 2.8 that $\alpha=1$ and $k=s-1$. Thus $\mu_s=1$ (by (2.4)), $p_s=p$ and $\alpha_s=2u$. By (2.1), we see that $\sigma(p^{\lambda_s})=q$ is a prime. Noting $\lambda_s\geq\alpha_s=2u$ and $\sigma(p^{2u})$ is composite, we have $\lambda_s>\alpha_s=2u$. It follows from (2.5) and $\alpha=1$ that $p\mid q+1$. By $\sigma(p^{\lambda_s})=q$ we have $p\mid q-1$. Thus $p\mid 2$, a contradiction.

This completes the proof of Theorem 1.1.

4. Proofs of Theorem 1.3 and Corollary 1.4

Proof of Theorem 1.3. If q^α is the Euler factor of N, then $q\equiv 1 \pmod 4$, $2\mid\lambda_i$ $(1\leq i\leq s)$, $2\nmid m$ and $2\mid\alpha+1$. If q^α is not the Euler factor of N, then $2\mid m$, $4\nmid m$ and $2\nmid\alpha+1$. We always assume that $2\mid m_1$ if q^α is not the Euler factor of N. It is known that $m_1 \neq 2$ if the Euler prime divides N to a power greater than 1 (see [BDZ, p. 6]). Recall that $m = 2^\beta q_1^{\beta_1} \cdots q_u^{\beta_u}$, where $\beta, \beta_1, \ldots, \beta_u$ are non-negative integers with $\beta_1 \geq \cdots \geq \beta_v > \beta_{v+1} = \cdots = \beta_u = 1$ and $\beta \in \{0, 1\}$, and $w = 1$ if $2 \mid m$ and the Euler prime divides N to the power 1, otherwise $w = 0$. For convenience, let $\beta_i = 0$ for all $i > u$. By (2.4), we have

$$k \leq w + \beta_1 + \cdots + \beta_u, \qquad \alpha_{k+i} \leq \beta_i \quad (i \geq 1).$$

(i) Suppose that $v + w + \beta_1 + \cdots + \beta_u \leq k_1(s)$. Then

$$k + v \leq v + w + \beta_1 + \cdots + \beta_u \leq k_1(s).$$

Thus $k \leq k_1(s)$ and $\alpha_{k_1(s)+1} \leq \alpha_{k+v+1} \leq \beta_{v+1} \leq 1$, a contradiction to Lemma 2.9(i).

(ii) Suppose that $u + w + \beta_1 + \cdots + \beta_u \leq k_2(s)$. Then

$$k + u \leq u + w + \beta_1 + \cdots + \beta_u \leq k_2(s).$$

Thus $k \leq k_2(s)$ and $\alpha_{k_2(s)+1} \leq \alpha_{k+u+1} \leq \beta_{u+1} = 0$, a contradiction to Lemma 2.9(ii).

Part (iii) can be proved similarly.

This completes the proof of Theorem 1.3. ∎

Proof of Corollary 1.4. Nielsen [Ni2] proved that $s \geq 8$. This has been superseded by proving that $s \geq 9$ (see Nielsen [Ni1]). We have $k_1(8) = 5$, $k_2(8) = 6$, $k_3(8) = 6$, $k_1(9) = 6$, $k_2(9) = 7$ and $k_3(9) = 7$.

By Theorem 1.3(i), we have $v+w+\beta_1+\cdots+\beta_u > k_1(s)$. Thus, m cannot be any one of $2q_1^3q_2$, $\underline{2q_1^4q_2}$, $2q_1q_2q_3$, $2q_1^2q_2q_3$, $\underline{2q_1^3q_2q_3}$, $2q_1q_2q_3q_4$, $\underline{2q_1^2q_2q_3q_4}$, $2q_1q_2q_3q_4q_5$, $\underline{2q_1q_2q_3q_4q_5q_6}$ in Corollary 1.4(2) ($w = 0$) or any one of $2q_1^2q_2$, $\underline{2q_1^3q_2}$, $2q_1q_2q_3$, $\underline{2q_1^2q_2q_3}$, $2q_1q_2q_3q_4$, $\underline{2q_1q_2q_3q_4q_5}$ in Corollary 1.4(3) ($w = 1$).

By Theorem 1.3(ii), we have $u+w+\beta_1+\cdots+\beta_u > k_2(s)$. Thus, m cannot be any one of $2q_1^5$, $\underline{2q_1^6}$, $2q_1^2q_2^2$, $\underline{2q_1^3q_2^2}$ in Corollary 1.4(2) ($w = 0$) and $2q_1^4$, $\underline{2q_1^5}$, $\underline{2q_1^2q_2^2}$ in Corollary 1.4(3) ($w = 1$).

If $2 \nmid m$, then, by Theorem 1.3(iii), $v+\beta_1+\cdots+\beta_u > k_3(s)$. Thus, m cannot be any one of $q_1^4q_2$, $\underline{q_1^5q_2}$, $\underline{q_1^3q_2^2}$, $q_1^2q_2q_3$, $q_1^3q_2q_3$, $\underline{q_1^4q_2q_3}$, $\underline{q_1^2q_2^2q_3}$, $q_1q_2q_3q_4$, $q_1^2q_2q_3q_4$, $\underline{q_1^3q_2q_3q_4}$, $q_1q_2q_3q_4q_5$, $\underline{q_1^2q_2q_3q_4q_5}$, $q_1q_2q_3q_4q_5q_6$, $\underline{q_1q_2q_3q_4q_5q_6q_7}$ in Corollary 1.4(1).

Suppose that m is a square. By $s \geq 8$ and Lemma 2.7, we have $k \geq s - 1 \geq 7$. Thus, m cannot be any one of $q_1^4q_2^2$, $q_1^2q_2^2q_3^2$ in Corollary 1.4(1). By Theorem 1.1, we have $m \neq q_1^8$.

Finally, the remaining cases to exclude are $m = \underline{q_1^7}, \underline{q_1^2q_2^2q_3q_4}$ in Corollary 1.4(1) and $m = \underline{2q_1^2q_2^2q_3}$ in Corollary 1.4(2). Suppose that m has one of these forms. We will derive a contradiction.

CASE 1: $m = \underline{q_1^7}$. Then $k \le 7$ and $\delta = 1$. By (2.1) and (2.4), we have $q_1 \mid m_i$ $(1 \le i \le k)$ and $p_i \ne q_1$ $(1 \le i \le k)$. So $\alpha_{k+1} = 7$ and $\alpha_i = 0$ $(k+2 \le i \le s)$. Since $\lambda_{k+1} \ge \alpha_{k+1}$ and λ_{k+1} is even, we have $\lambda_{k+1} \ge 8$ and $\delta_{k+1} = 1$. If $\lambda_{k+1} = 8$, then

$$q^{\mu_{k+1}} = \frac{p_{k+1}^9 - 1}{p_{k+1} - 1} = \frac{p_{k+1}^9 - 1}{p_{k+1}^3 - 1}\,\frac{p_{k+1}^3 - 1}{p_{k+1} - 1}.$$

This implies that at least one of

$$\frac{p_{k+1}^9 - 1}{p_{k+1} - 1}, \quad \frac{p_{k+1}^9 - 1}{p_{k+1}^3 - 1}, \quad \frac{p_{k+1}^3 - 1}{p_{k+1} - 1}$$

is a square (q to an even power), a contradiction with Lemma 2.3. So $\lambda_{k+1} \ge 10$ and then $\lambda_{k+1} - \alpha_{k+1} - \delta_{k+1} + 1 \ge 3$. Since $s \ge 9$ and $k \le 7$, the left side of (2.7) is

$$\begin{aligned}(\delta+1) \prod_{j=k+1}^{s} &\max\{\lambda_j - \alpha_j - \delta_j + 1, 1\} \\ &\ge 2 \cdot 3 \cdot \prod_{j=k+2}^{s} (\lambda_j - \delta_j + 1) \ge 2 \cdot 3^{s-k} > s+1,\end{aligned}$$

a contradiction with (2.7). Now, we have proved that $m \ne q_1^7$.

CASE 2: $m = \underline{q_1^2 q_2^2 q_3 q_4}$. Then $k \le 6$ and $\delta = 1$. By Lemma 2.9(iii) and $k_3(9) = 7$, we have $\alpha_8 \ge 2$. So $k = 6$, $\alpha_7 = 2$, $\alpha_8 = 2$ and $\alpha_i \le 1$ $(9 \le i \le s)$. By $s \ge 9$, as all λ_j $(k+1 \le j \le s)$ are positive even integers and at most one of λ_j $(k+1 \le j \le s)$ is 2, the left side of (2.7) is

$$\begin{aligned}(\delta+1) \prod_{j=k+1}^{s} &\max\{\lambda_j - \alpha_j - \delta_j + 1, 1\} \\ &\ge 2(\lambda_7 - \delta_7 - 1)(\lambda_8 - \delta_8 - 1) \prod_{j=9}^{s} (\lambda_j - \delta_j) \\ &\ge \min\{2 \cdot 2 \cdot 3^{s-8}, 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3^{s-9}\} > s+1,\end{aligned}$$

a contradiction with (2.7).

CASE 3: $m = \underline{2q_1^2 q_2^2 q_3}$, and q is not the Euler prime and the Euler prime divides N to a power greater than 1. Then $k \le 5$. By Lemma 2.9(ii) and $k_2(9) = 7$, we may assume that $\alpha_8 \ge 1$. So $k = 5$, $\alpha_6 = 2$, $\alpha_7 = 2$, $\alpha_8 = 1$ and $\alpha_i = 0$ $(9 \le i \le s)$. By $s \ge 9$, since all λ_j $(k+1 \le j \le s)$ are positive even integers and at most one of λ_j $(k+1 \le j \le s)$ is 2, the left side of (2.7) is

$$(\delta+1)\prod_{j=k+1}^{s}\max\{\lambda_j-\alpha_j-\delta_j+1,1\}$$
$$\geq(\lambda_6-\delta_6-1)(\lambda_7-\delta_7-1)\prod_{j=8}^{s}(\lambda_j-\delta_j)\geq 2\cdot 2\cdot 3^{s-8}>s+1,$$

a contradiction with (2.7).

This completes the proof of Corollary 1.4. ∎

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