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A QUANTITATIVE VERSION OF THE CONVERSE TAYLOR THEOREM: $C^{k,\omega}$-SMOOTHNESS

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Abstract. We prove a uniform version of the converse Taylor theorem in infinite-dimensional spaces with an explicit relation between the moduli of continuity for mappings on a general open domain. We show that if the domain is convex and bounded, then we can extend the estimate up to the boundary.

The converse to the Taylor theorem is a well-known result (see e.g. [LS] or [AD]). We could not find in the literature a version of this theorem for mappings with uniformly continuous derivatives that deals explicitly with the moduli of continuity, so we prove such a version below (Theorem 9). Usually when dealing with quantitative uniform estimates for derivatives of mappings on general open domains there are troubles when we approach the boundary. We show that if the domain is convex and bounded these problems can be avoided.

All vector spaces considered are real. We denote by $B(x,r)$, resp. $U(x,r)$ the closed, resp. open ball in a normed linear space centred at x with radius $r>0$. By B_X we denote the closed unit ball of a normed linear space X, i.e. $B_X = B(0,1)$. Let X, Y be normed linear spaces and $n\in\mathbb{N}$. We denote by $\mathcal{L}^s({}^nX;Y)$ the space of symmetric n-linear mappings from X to Y with the norm $\|M\| = \sup_{x_1,\dots,x_n\in B_X}\|M(x_1,\dots,x_n)\|$, and by $\mathcal{P}({}^nX;Y)$ the space of n-homogeneous polynomials from X to Y with the norm $\|P\| = \sup_{x\in B_X}\|P(x)\|$. Further, $\mathcal{P}^n(X;Y)$ denotes the space of polynomials of degree at most n from X to Y with the norm $\|P\| = \sup_{x\in B_X}\|P(x)\|$. We will use the following convention: for $k\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$ and $x\in X$ we denote

$$^k x = \underbrace{x,\dots,x}_{k\text{ times}}.$$

If $P\in\mathcal{P}({}^nX;Y)$, then we denote by $\check{P}$ the uniquely determined symmetric n-linear mapping that gives rise to the polynomial P, i.e. $P(x)=\check{P}({}^nx)$. We start by recalling a few well-known results on polynomials that will be needed later on. They can be found e.g. in [M].

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Lemma 1. *Let X, Y be normed linear spaces, $n \in \mathbb{N}$, $P \in \mathcal{P}(^nX;Y)$, and $x, y \in X$. Then*

$$P(x+y) = \sum_{j=0}^{n} \binom{n}{j} \check{P}(^jx, {}^{n-j}y).$$

Theorem 2. *Let $n \in \mathbb{N}_0$. There are numbers $a_{kj} \in \mathbb{R}$, $k, j = 0, \ldots, n$, such that whenever X, Y are normed linear spaces, $P \in \mathcal{P}^n(X;Y)$, and $P_k \in \mathcal{P}(^kX;Y)$ are such that $P = \sum_{k=0}^{n} P_k$, then $P_k(x) = \sum_{j=0}^{n} a_{kj} P\bigl(\frac{j}{n}x\bigr)$ for every $x \in X$.*

In particular, there are constants $K_{n,k} > 0$ such that

$$\|Q_k(x)\| \le K_{n,k} \max_{0 \le j \le n} \left\| Q\left(\frac{j}{n}x\right) \right\|$$

whenever X, Y are normed linear spaces, $Q \in \mathcal{P}^n(X;Y)$, $x \in X$, and $k \in \{0, \ldots, n\}$, where Q_k is the k-homogeneous summand of Q.

The next lemma shows that the polarisation formula applied to a non-homogeneous polynomial extracts its “leading term”.

Lemma 3 ([MO]). *Let X, Y be normed linear spaces, $n \in \mathbb{N}$, and let $P \in \mathcal{P}^n(X;Y)$ be such that $P = \sum_{k=0}^{n} P_k$ where $P_k \in \mathcal{P}(^kX;Y)$. Then*

$$\check{P}_n(x_1, \ldots, x_n) = \frac{1}{2^n n!} \sum_{\varepsilon_j = \pm 1} \varepsilon_1 \cdots \varepsilon_n P\Bigl(a + \sum_{j=1}^{n} \varepsilon_j x_j\Bigr)$$

for every $a, x_1, \ldots, x_n \in X$.

The following lemma is useful for estimating the norm of a homogeneous polynomial using its values on an arbitrary ball.

Lemma 4. *Let X, Y be normed linear spaces, $n \in \mathbb{N}$, let $P \in \mathcal{P}^n(X;Y)$ be such that $P = \sum_{k=0}^{n} P_k$, $P_k \in \mathcal{P}(^kX;Y)$, and let $a \in X$. Then*

$$\|P_n(x)\| \le \frac{n^n}{n!} \sup_{t \in [-1,1]} \|P(a+tx)\|$$

for every $x \in X$. In particular, for any $r > 0$,

$$\sup_{x \in B(0,r)} \|P_n(x)\| \le \frac{n^n}{n!} \sup_{x \in B(a,r)} \|P(x)\|.$$

Proof. By Lemma 3,

$$\|P_n(x)\| = n^n \left\| P_n\left(\frac{x}{n}\right) \right\| \le \frac{n^n}{2^n n!} \sum_{\varepsilon_j = \pm 1} \left\| P\Bigl(a + \frac{x}{n} \sum_{j=1}^{n} \varepsilon_j\Bigr) \right\|$$
$$\le \frac{n^n}{n!} \sup_{t \in [-1,1]} \|P(a+tx)\|. \blacksquare$$

Let X, Y be normed linear spaces, $U \subset X$ open, $f\colon U \to Y$, and $x \in U$. We denote by $Df(x)$ the Fréchet derivative of f at x; and by $Df(x)[h]$ the evaluation of this derivative at the direction $h \in X$. Similarly we denote by $D^k f(x)$ the kth Fréchet derivative of f at x; by $d^k f(x)$ the k-homogeneous polynomial corresponding to the symmetric k-linear mapping $D^k f(x)$; and by $d^k f(x)[h]$ its evaluation at $h \in X$, i.e. $d^k f(x)[h] = D^k f(x)[^k h]$.

We say that f is *C^k-smooth* if $D^k f$ (i.e. the mapping $x \mapsto D^k f(x)$) is continuous in the domain. We denote by $C^k(U;Y)$ the vector space of all C^k-smooth mappings from U into Y. For convenience we also put $C^0(U;Y) = C(U;Y)$, the continuous mappings.

Lemma 5. *Let X, Y be normed linear spaces, $U \subset X$ open, $f\colon U \to Y$, $k \in \mathbb{N}$, and $a \in U$. Then $d^k f(a)$ exists if and only if $D(d^{k-1}f)(a)$ exists.*

Proof. Denote by $I\colon \mathcal{P}(^{k-1}X;Y) \to \mathcal{L}^s(^{k-1}X;Y)$ the canonical isomorphism $I(P) = \check{P}$ from the polarisation formula. Then $D^{k-1}f = I \circ d^{k-1}f$ and $d^{k-1}f = I^{-1} \circ D^{k-1}f$ and so the equivalence follows from the chain rule. ∎

Notice that by the polarisation formula, $f \in C^k(U;Y)$ if and only if the mappings $x \mapsto d^j f(x)$, $j = 1, \dots, k$, are continuous on U. Recall the following two versions of Taylor's theorem:

Theorem 6 (Peano's form of Taylor's theorem). *Let X, Y be normed linear spaces, $U \subset X$ an open set, $f\colon U \to Y$, $a \in U$, $k \in \mathbb{N}$, and suppose that $D^k f(a)$ exists. Then*

$$\Bigl\| f(x) - \sum_{j=0}^{k} \frac{1}{j!} d^j f(a)[x-a] \Bigr\| = o(\|x-a\|^k), \qquad x \to a.$$

Theorem 7. *Let X, Y be normed linear spaces, $U \subset X$ an open convex set, $k \in \mathbb{N}$, and $f \in C^k(U;Y)$. Then for any $x \in U$ and $h \in X$ satisfying $x + h \in U$ we have*

$$\Bigl\| f(x+h) - \sum_{j=0}^{k} \frac{1}{j!} d^j f(x)[h] \Bigr\| \le \frac{1}{k!} \Bigl(\sup_{t \in [0,1]} \|d^k f(x+th) - d^k f(x)\| \Bigr) \cdot \|h\|^k.$$

Let X, Y be normed linear spaces, $U \subset X$ an open set, $f\colon U \to Y$, and $k \in \mathbb{N}_0$. We say that f is *T^k-smooth at $x \in U$* if there exists a polynomial $P \in \mathcal{P}^k(X;Y)$ satisfying $P(0) = f(x)$ and

$$f(x+h) - P(h) = o(\|h\|^k), \qquad h \to 0. \tag{1}$$

We say that f is *T^k-smooth on U* if it is T^k-smooth at every point $x \in U$.

We remark that the polynomial in (1) is uniquely determined. It is easy to see that T^{k+1}-smoothness implies T^k-smoothness. If f is T^1-smooth at x, then f is Fréchet differentiable at x with $Df(x) = P_1$, the 1-homogeneous term of P.

Theorem 6 implies that a C^k-smooth mapping is also T^k-smooth and the approximating polynomial is given by $\sum_{j=0}^{k} \frac{1}{j!} d^j f(x)$. The converse is not true in general: consider $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = x^{k+1} \sin \frac{1}{x^k}$, $f(0) = 0$. Then f is T^k-smooth but not even C^1-smooth. Nevertheless, under certain uniformity assumptions the converse does hold.

Theorem 8 (Converse Taylor theorem). *Let X, Y be normed linear spaces, $U \subset X$ an open set, $f\colon U \to Y$, and $k \in \mathbb{N}_0$. Then $f \in C^k(U;Y)$ if and only if f is a T^k-smooth mapping satisfying*

$$\lim_{\substack{(y,h)\to(x,0)\\ h\neq 0}} \frac{\|R(y,h)\|}{\|h\|^k} = 0 \tag{2}$$

for every $x \in U$, where $R(x,h) = f(x+h) - P^x(h)$ and the polynomials $P^x \in \mathcal{P}^k(X;Y)$ come from the definition of T^k-smoothness at x. In this case $P^x = \sum_{j=0}^{k} \frac{1}{j!} d^j f(x)$.

The proof of the uniform version below reuses significant parts of the proof of this theorem. For this reason (and also for the reader's convenience) we give our version of the proof. The main ideas are the same as in [AD].

Proof. $\Rightarrow$ If $f \in C^k(U;Y)$, then f is T^k-smooth, and $P^x = \sum_{j=0}^{k} \frac{1}{j!} d^j f(x)$ by the uniqueness of the Taylor polynomial. Fix $x \in U$ and choose any $\varepsilon > 0$. Let $\delta > 0$ be such that $B(x, 2\delta) \subset U$ and $\|d^k f(z) - d^k f(x)\| < \varepsilon$ for $z \in B(x, 2\delta)$. By Theorem 7,

$$\|R(y,h)\| \leq \frac{1}{k!}\Big(\sup_{z\in B(y,\delta)} \|d^k f(z) - d^k f(y)\|\Big) \cdot \|h\|^k \leq \frac{2\varepsilon}{k!}\|h\|^k$$

whenever $y \in B(x,\delta)$ and $h \in B(0,\delta)$, from which (2) follows.

$\Leftarrow$ We use induction on k. For $k = 0$ the assertion is obvious, since both T^0-smoothness and C^0-smoothness mean just the continuity of f. So assume that $k \in \mathbb{N}$ and the theorem holds for $k-1$.

Fix $x \in U$ and let $\delta > 0$ be such that $U(x, 2\delta) \subset U$. We have

$$\begin{aligned} f(x+h+y) &= P^x(h+y) + R(x, h+y), \\ f(x+h+y) &= P^{x+h}(y) + R(x+h, y), \end{aligned} \tag{3}$$

for all $h, y \in U(0,\delta)$. Set $q(h,y) = P^{x+h}(y) - P^x(h+y)$. Denote by P_j^z the j-homogeneous summands of P^z, $j = 0, \dots, k$. By Lemma 1 we can write $q(h,y) = \sum_{j=0}^{k} q_j(h,y)$, where

$$q_j(h,y) = P_j^{x+h}(y) - \sum_{l=j}^{k} \binom{l}{j} \widecheck{P_l^x}(^{l-j}h, {}^j y).$$

Note that $q(h,\cdot) \in \mathcal{P}^k(X;Y)$ and $q_j(h,\cdot) \in \mathcal{P}({}^jX;Y)$, $j = 0,\dots,k$, for every $h \in U(0,\delta)$. By subtracting the equalities (3) we obtain $q(h,y) = R(x,h+y) - R(x+h,y)$. Thus for any $h,y \in U(0,\delta)$ such that $\|y\| \le \|h\|$, $y \neq 0$, and $y \neq -h$ we obtain

$$\begin{aligned}\|q(h,y)\| &\le \|R(x,h+y)\| + \|R(x+h,y)\| \\ &\le \left(2^k \frac{\|R(x,h+y)\|}{\|h+y\|^k} + \frac{\|R(x+h,y)\|}{\|y\|^k}\right)\|h\|^k.\end{aligned}$$

It follows (using also simpler versions of the above estimate if $y = 0$ or $y = -h$) that

$$\|q(h,y)\| = o(\|h\|^k), \qquad (h,y) \to (0,0),\ \|y\| \le \|h\|.$$

Applying Theorem 2 we get $\|q_j(h,y)\| \le K_{k,j} \max_{0\le l\le k} \left\| q\big(h, \tfrac{l}{k}y\big)\right\|$ for all $h \in U(0,\delta)$, $y \in Y$, and $j \in \{0,\dots,k\}$. Therefore

$$\|q_j(h,y)\| = o(\|h\|^k), \qquad (h,y) \to (0,0), \|y\| \le \|h\|.$$

So finally by taking the supremum over $y \in B(0,\|h\|)$ and using the j-homogeneity of $q_j(h,\cdot)$ we obtain

$$\begin{aligned}\|q_j(h,\cdot)\| &= \frac{1}{\|h\|^j} \sup_{\|y\|\le\|h\|} \|q_j(h,y)\| \\ &= \sup_{\|y\|\le\|h\|} \frac{\|q_j(h,y)\|}{\|h\|^j} = o(\|h\|^{k-j}), \qquad h \to 0,\end{aligned} \tag{4}$$

for each $j \in \{0,\dots,k\}$.

Since $q_k(h,\cdot) = P_k^{x+h} - P_k^x$, it follows that the mapping $x \mapsto P_k^x$ is continuous on U. Further, since for $h \neq 0$,

$$\begin{aligned}\frac{\|f(y+h) - \sum_{j=0}^{k-1} P_j^y(h)\|}{\|h\|^{k-1}} &\le \frac{\|R(y,h)\| + \|P_k^y(h)\|}{\|h\|^{k-1}} \\ &\le \left(\frac{\|R(y,h)\|}{\|h\|^k} + \|P_k^y\|\right)\|h\|,\end{aligned}$$

the continuity of $x \mapsto P_k^x$ implies

$$\lim_{\substack{(y,h)\to(x,0)\\ h\neq 0}} \frac{\|f(y+h) - \sum_{j=0}^{k-1} P_j^y(h)\|}{\|h\|^{k-1}} = 0$$

and so by the inductive hypothesis f is C^{k-1}-smooth and $P_j^x = \frac{1}{j!}d^jf(x)$, $j = 0,\dots,k-1$. Thus

$$q_{k-1}(h,y) = \frac{1}{(k-1)!}\big(d^{k-1}f(x+h)[y] - d^{k-1}f(x)[y]\big) - k\widecheck{P_k^x}(h, {}^{k-1}y)$$

and from (4) we get

$$\|d^{k-1}f(x+h) - d^{k-1}f(x) - k!\widetilde{P^x_k}(h,\cdot,\dots,\cdot)\| = o(\|h\|), \qquad h \to 0.$$

Therefore $d^kf(x)$ exists by Lemma 5, and consequently $d^kf(x) = k!P^x_k$ by Theorem 6, which finishes the proof. ■

A *modulus* is a non-decreasing function $\omega\colon [0,\infty) \to [0,\infty]$ continuous at 0 with $\omega(0) = 0$. The set of all moduli will be denoted by $\mathcal{M}$. An important subset of all moduli consists of the subadditive moduli. A nice feature of a subadditive modulus ω is that it is real-valued and uniformly continuous with modulus of continuity ω. It is easy to check that the minimal modulus of continuity of a uniformly continuous mapping defined on a convex subset of a normed linear space is subadditive.

Let X, Y be normed linear spaces, $U \subset X$ open, $k \in \mathbb{N}$, $f \in C^k(U;Y)$, and let $\omega \in \mathcal{M}$. We say that f is $C^{k,\omega}$*-smooth* on U if d^kf is uniformly continuous on U with modulus ω.

Let X, Y be normed linear spaces, $V \subset X$ an open set, $f\colon V \to Y$, and $k \in \mathbb{N}_0$. We say that f is UT^k*-smooth* on V if there exists $\omega \in \mathcal{M}$ such that for each $x \in V$ there is a polynomial $P \in \mathcal{P}^k(X;Y)$ satisfying

$$\|f(x+h) - P(h)\| \le \omega(\|h\|)\|h\|^k \quad \text{for } x+h \in V.$$

We note that UT^{k+1}-smoothness in general does not imply UT^k-smoothness—the function $f(x) = x^3$ is UT^2-smooth on $\mathbb{R}$ but it is not UT^1-smooth on $\mathbb{R}$.

Theorem 7 implies that a $C^{k,\omega}$-smooth mapping on a convex set U is UT^k-smooth on U with modulus $(1/k!)\omega$. The converse is contained in the next theorem. For a convex subset U of a normed linear space X we define its "ellipticity" $e_U = \inf\{R/r : B(x,r) \subset U \subset B(x,R),\ x \in U\}$ and we set $e_X = 1$.

THEOREM 9. *Let X, Y be normed linear spaces, $U \subset X$ an open set, $f\colon U \to Y$, and $k \in \mathbb{N}$. Suppose that f is UT^k-smooth on U and the modulus ω from the definition of UT^k-smoothness is subadditive. If $V \subset U$ is an open bounded subset satisfying* $\operatorname{dist}(V, X \setminus U) > 0$, *then f is $C^{k,m\omega}$-smooth on V, where $m > 0$ is a constant depending on k,* $\operatorname{diam} V$, *and* $\operatorname{dist}(V, X\setminus U)$.

Moreover, if $U = X$ or U is a convex bounded set, then f is $C^{k,m\omega}$-smooth on the whole of U, we do not need to assume that ω is subadditive, and $m = c_k e_U^k$, where $c_k > 0$ is a constant depending only on k.

Proof. First notice that $f \in C^k(U;Y)$ by Theorem 8. Let $V \subset U$ be a non-empty bounded open set for which $\operatorname{dist}(V, X \setminus U) > 0$. Put $\rho = \operatorname{diam} V$ and $\varepsilon = \min\{\rho, \operatorname{dist}(V, X \setminus U)\}$. Fix any $x \in V$ and $h \in X$ such that $x + h \in V$. We use the notation from the proof of Theorem 8. For any $y \in X$

satisfying $\|y\| \le \|h\|$ and $x+h+y \in U$ we have

$$\begin{aligned}(5)\qquad \|q(h,y)\| &\le \|R(x,h+y)\| + \|R(x+h,y)\| \\ &\le \omega(\|h+y\|)\|h+y\|^k + \omega(\|y\|)\|y\|^k \\ &\le \omega(2\|h\|)2^k\|h\|^k + \omega(\|h\|)\|h\|^k \le (2^{k+1}+1)\omega(\|h\|)\|h\|^k.\end{aligned}$$

Therefore by Theorem 2,

$$\begin{aligned}\|d^kf(x+h)[y]-d^kf(x)[y]\| = k!\|q_k(h,y)\| &\le k!K_{k,k}\max_{0\le l\le k}\left\|q\left(h,\tfrac{l}{k}y\right)\right\| \\ &\le k!K_{k,k}(2^{k+1}+1)\omega(\|h\|)\|h\|^k\end{aligned}$$

for all $y \in X$ satisfying $\|y\| \le \frac{\varepsilon}{\rho}\|h\|$. Consequently, if $h \ne 0$,

$$\begin{aligned}\|d^kf(x+h)-d^kf(x)\| &= \frac{1}{\left(\frac{\varepsilon}{\rho}\|h\|\right)^k}\sup_{\|y\|\le\frac{\varepsilon}{\rho}\|h\|}\|d^kf(x+h)[y]-d^kf(x)[y]\| \\ &\le \frac{\rho^k}{\varepsilon^k}k!K_{k,k}(2^{k+1}+1)\omega(\|h\|).\end{aligned}$$

Note that if $U = X$ we can take the supremum over $\|y\| \le \|h\|$ (since $x+h+y$ always lies in the domain of f) and thus we do not need to restrict to bounded sets. Also, if ω is not subadditive, then using (5) we obtain $\|q(h,y)\| \le (2^k+1)\omega(2\|h\|)\|h\|^k$, and thus $\omega_1(t) \le k!K_{k,k}(2^k+1)\omega(2t)$, where ω_1 is the minimal modulus of continuity of d^kf on X. But since the modulus ω_1 is subadditive in this case, we get $\omega_1(t) \le 2\omega_1\left(\frac{1}{2}t\right) \le 2k!K_{k,k}(2^k+1)\omega(t)$.

Finally, suppose that U is convex and bounded. Without loss of generality we may assume that $B(0,r) \subset U \subset B(0,2e_Ur)$ for some $r>0$. Fix any $x \in U$ and $h \in X\setminus\{0\}$ such that $x+h \in U$. By switching the roles of x and $x+h$ if necessary, we may assume that $\|x\| \le \|x+h\|$. It follows that $\|h\| \le 2\|x+h\|$. Put

$$a = -\|h\|\frac{x+h}{2\|x+h\|} \quad \text{and} \quad s = \frac{\|h\|}{4e_U}$$

and note that by convexity, $B(x+h+a,s) \subset U$. Indeed, any $z \in B(x+h+a,s)$ can be expressed as a convex combination

$$z = \left(1-\frac{\|h\|}{2\|x+h\|}\right)(x+h) + \frac{\|h\|}{2\|x+h\|}w,$$

where $w = \frac{2\|x+h\|}{\|h\|}(z-x-h-a) \in B(0,r)$. Now if $y \in B(a,s)$, then $\|y\| \le \|a\|+s \le \|h\|$ and hence

$$\|q(h,y)\| \le (2^{k+1}+1)\omega(\|h\|)\|h\|^k$$

by (5). Thus using Lemma 4 we obtain

$$\begin{aligned}\|d^kf(x+h)-d^kf(x)\| &= \frac{1}{s^k}\sup_{y\in B(0,s)}\|d^kf(x+h)[y]-d^kf(x)[y]\| \\ &= \frac{k!}{s^k}\sup_{y\in B(0,s)}\|q_k(h,y)\| \leq \frac{k^k}{s^k}\sup_{y\in B(a,s)}\|q(h,y)\| \\ &\leq \frac{k^k}{s^k}(2^{k+1}+1)\omega(\|h\|)\|h\|^k \\ &= (4k)^k(2^{k+1}+1)e_U^k\omega(\|h\|).\end{aligned}$$

If ω is not subadditive, then analogously to the case $U = X$ we obtain $\omega_1(t) \leq 2(4k)^k(2^{k+1}+1)e_U^k\omega(t)$, where ω_1 is the minimal modulus of continuity of d^kf on U. ∎

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