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DIFFERENTIAL TENSOR ALGEBRAS AND ENDOLENGTH VECTORS

BY

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Abstract. We introduce the notions of central endolength and central endolength vector, and we study their behavior under base field extension for finite-dimensional algebras over perfect fields and for almost admissible ditalgebras.

1. Introduction. In [5], for a given generically tame finite-dimensional algebra Λ over an infinite perfect field, parametrizations were provided for indecomposable Λ-modules with dimension less than or equal to d, for each natural number d. It is typical of this parametrization, for a base field not algebraically closed and not real closed, to have an infinite number of isomorphism classes of indecomposable modules with dimension greater than d, for an infinite number of integers d. To see an example, consider the Kronecker algebra

$$\Gamma = \; 1\cdot \rightrightarrows \cdot 2$$

over the rational field $\mathbb{Q}$ and the Γ-$\mathbb{Q}[x]$-bimodule

$$B = \; \mathbb{Q}[x] \overset{x}{\underset{1}{\rightrightarrows}} \mathbb{Q}[x] \; .$$

For any prime p and any natural number n we have the irreducible monic polynomial $x^n - p$ and the indecomposable Γ-module

$$B \otimes_{\mathbb{Q}[x]} ((\mathbb{Q}[x]/\langle x^n - p\rangle) = M_{n,p} \cong \; \mathbb{Q}^n \overset{A_{n,p}}{\underset{I}{\rightrightarrows}} \mathbb{Q}^n \; ,$$

where I is the identity matrix and

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$$A_{n,p} = \begin{pmatrix} 0 & 0 & \dots & 0 & p \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}.$$

Notice that $M_{n,p}$ is a quasi-simple Γ-module, but $\dim_{\mathbb{Q}}(M_{n,p}) = 2n$ and the dimension vector of $M_{n,p}$ is (n,n).

Also interesting is the infiniteness assumption on the base field: there is a strong feeling that parametrizations are also possible on generically tame finite-dimensional algebras over finite fields, and so we have to replace dimension by something else.

Already in [12] it was proposed to use the notion of endolength in order to extend the concepts of wildness and tameness. Following this idea, in [24] a parametrization was obtained for finite-dimensional modules of an almost admissible ditalgebra (Definition 2.1) for a fixed endolength vector (Definition 2.6) and simple group of self-extensions, and in [6] this notion of endolength vector was used to obtain parametrizations for generically tame finite-dimensional algebras over real closed fields.

Here we propose to use the central endolength and the central endolength vector (Definition 2.11) when the base field is perfect; in Theorem 2.13 we prove that the central endolength is kept, for indecomposable finite-dimensional modules, when we pass to the indecomposable direct summands obtained through extension of the base field by algebraic closure. Moreover, the behavior of the central endolength vector for elementary algebras and elementary ditalgebras (Theorem 2.14) suggests that this notion is a good generalization of the concept of dimension vector.

Then it is necessary to see if there are analogs for central endolength vectors and central endolength of some usual results for dimension vectors and dimension: in Section 3 we review classical equivalences for infinite representation type for algebras and constructible almost admissible ditalgebras.

In Section 4 we analyze a norm (we call it the endonorm) for almost admissible ditalgebras and its behavior for endolength vectors and central endolength vectors under reduction functors. We also prove a version of Brauer–Thrall I (BT-I) for almost admissible ditalgebras (Theorem 4.16).

With this tool we find an equivalent condition for an almost admissible ditalgebra $\mathcal{A}$ to be generically trivial (Proposition 4.14), and in Section 5 we see that the isomorphism classes of indecomposable finite-dimensional $\mathcal{A}$-modules with trivial group of self-extensions are determined by their central endolength vectors (Theorem 5.3).

In Section 6 we provide an example of a subcategory determined by a central endolength vector which is covered by a finite number of one-parameter families, and this covering is good in the sense that almost all the isomorphism classes of the one-parameter families have that central endolength vector (Theorem 6.15); also the generic modules associated are algebraically bounded and have the same central endolength vector.

We think it is possible to use a parametrization similar to the one of the above-mentioned example in other cases, and thus get a generalized version of tameness. Moreover, in [25] it is proved that Λ^K tame is equivalent to Λ being semigenerically tame; the results of Section 6 suggest that the notion of semigeneric tameness should be close to the usual notion of tameness.

2. Endolength vectors. Throughout the paper, k will denote a perfect field, perhaps finite, and Λ a finite-dimensional k-algebra; K will denote an algebraic closure of k.

Furthermore, $\mathcal{A} = (T, \delta)$ will denote a layered triangular ditalgebra (see [7]) with layer $(R, W = W_0 \oplus W_1)$, so R is a k-algebra, $W = W_0 \oplus W_1$ as R-R-bimodules and T is the tensor algebra $T_R(W) = R \oplus W \oplus W \otimes_R W \oplus \cdots \oplus W^{\otimes^n} \oplus \cdots$.

We say that the elements of W_1 are of *degree one* and those of R and W_0 are of *degree zero*. Then there is an induced structure of a graded k-algebra over T, i.e. $T = \bigoplus_{i \in \mathbb{N} \cup \{0\}} [T]_i$ as vector spaces and $[T]_i [T]_j \subset [T]_{i+j}$ for all i, j.

Moreover, the *differential* $\delta : T \to T$ is a linear transformation such that:

- $\delta([T]_i) \subset [T]_{i+1}$ for all i.
- $\delta(h_1 h_2) = \delta(h_1) h_2 + (-1)^{\deg(h_1)} h_1 \delta(h_2)$ for all homogeneous elements $h_1, h_2 \in T$.
- $\delta^2 = 0$.

The triangularity of the layer means that there are filtrations of R-R-bimodules of W_0 and W_1 with good properties (see [7, Definition 5.1]).

Following [7] we write $A = [T]_0$ and $V = [T]_1$.

The objects of the category $\mathcal{A}$-Mod are all the A-modules. Given $M, N \in \mathcal{A}$-Mod, a morphism $f : M \to N$ in $\mathcal{A}$-Mod is a pair $f = (f^0, f^1)$, with $f^0 \in \mathrm{Hom}_k(M, N)$ and $f^1 \in \mathrm{Hom}_{A\text{-}A}(V, \mathrm{Hom}_k(M, N))$, satisfying $af^0(m) - f^0(am) = f^1(\delta(a))(m)$ for any $a \in A$ and $m \in M$.

$\mathcal{A}$-mod is the full subcategory of $\mathcal{A}$-Mod of all finite-dimensional objects.

The objects of $\mathcal{A}$-Mod are called $\mathcal{A}$-*modules*.

Definition 2.1. Let $\mathcal{A} = (T, \delta)$ be a layered triangular ditalgebra. We say that:

(1) $\mathcal{A}$ is *almost admissible* if $R \cong M_{m_1}(D_1) \times \cdots \times M_{m_n}(D_n)$ for some finite-dimensional division k-algebras $D_1, \ldots, D_n$, and the R-R-bimodule W is finitely generated.
(2) $\mathcal{A}$ is *admissible* if it is almost admissible and $R \cong D_1 \times \cdots \times D_n$ for some finite-dimensional division k-algebras $D_1, \ldots, D_n$.
(3) $\mathcal{A}$ is *elementary* if it is admissible and $R \cong k \times \cdots \times k$.

Let us recall that Λ is *elementary* if $\Lambda/\mathrm{rad}(\Lambda) \cong \prod_{i=1}^{n} k$ (see [3, p. 65]). In particular, if Λ is elementary then it is basic.

Remark 2.2. An admissible ditalgebra $\mathcal{A}$ with $W_1 = 0$ is a k-species (see [26]).

We will apply the results of [7] on ditalgebras; for the benefit of the reader we recall that for an almost admissible ditalgebra $\mathcal{A}$ the following hold:

- $\mathcal{A}$-Mod is an additive k-category and idempotents split in $\mathcal{A}$-Mod.
- $\mathcal{A}$-mod is a Krull–Schmidt category.
- Let $f = (f^0, f^1) : M \to N$ be a morphism in $\mathcal{A}$-Mod. Then f is an isomorphism in $\mathcal{A}$-Mod if and only if f^0 is an isomorphism, and if $M = N$ then f is nilpotent if and only if f^0 is nilpotent.
- If A is finite-dimensional then $\mathcal{A}$-mod has almost split sequences.

Notation. For $M \in \mathcal{A}$-Mod (respectively $M \in \Lambda$-Mod) we write $E_M = \mathrm{End}_{\mathcal{A}}(M)^{\mathrm{op}}$ (resp. $E_M = \mathrm{End}_{\Lambda}(M)^{\mathrm{op}}$), $D_M = E_M/\mathrm{rad}(E_M)$ and denote by Z_M the center of D_M.

Definition 2.3. For $M \in \mathcal{A}$-Mod there is a canonical structure of right E_M-module given by $m \cdot (f^0, f^1) = f^0(m)$. Then, for $M \in \mathcal{A}$-Mod (resp. $M \in \Lambda$-Mod), the *endolength* of M, denoted by $\mathrm{endol}(M)$, is its length as an E_M-module. We say that M is *endofinite* if $\mathrm{endol}(M) < \infty$. We say that M is *generic* if it is endofinite, indecomposable and has infinite dimension over k.

Here, for simplicity, we do not use the term *pregeneric module* of [5] because its definition is the same one we gave for generic module, and there are several statements in this paper that are expressed for almost admissible ditalgebras and for finite-dimensional k-algebras.

Definition 2.4. An almost admissible ditalgebra $\mathcal{A}$ (resp. the f.d. k-algebra Λ) is *generically trivial* if there are no generic modules in $\mathcal{A}$-Mod (resp. Λ-Mod).

Definition 2.5. A nonzero idempotent is *primitive* if it cannot be written as a sum of two nonzero orthogonal idempotents, and *centrally primitive* if it is central and cannot be written as the sum of two nonzero orthogonal central idempotents.

Observe that in the case of an admissible ditalgebra $\mathcal{A}$ the notions of primitive and centrally primitive coincide on R.

Definition 2.6. For an admissible ditalgebra $\mathcal{A}$ (resp. Λ), let $M \in \mathcal{A}$-Mod (resp. $M \in \Lambda$-Mod) be endofinite and let $1_R = e_1 + \cdots + e_n$ be a decomposition into centrally primitive idempotents (resp. let $1_\Lambda = e_1 + \cdots + e_n$ be a decomposition into orthogonal idempotents such that $\pi(e_i)$ is centrally primitive for each i, where $\pi : \Lambda \to \Lambda/\mathrm{rad}(\Lambda)$ is the canonical ring epimorphism). Then we consider the *endolength vector*

$$\underline{\ell}(M) = (\ell_{E_M}(e_1 M), \ldots, \ell_{E_M}(e_n M)),$$

where $\ell_{E_M}(e_j M)$ is the length of $e_j M$ as a right E_M-module.

Remark 2.7. Let be $z_1, \ldots, z_n$ a complete set of orthogonal centrally primitive idempotents of $\Lambda/\mathrm{rad}(\Lambda)$, and let $\pi : \Lambda \to \Lambda/\mathrm{rad}(\Lambda)$ be the canonical ring epimorphism. It is known that there exist idempotents $e'_1, \ldots, e'_n \in \Lambda$ such that $\pi(e'_j) = z_j$ for $j \in \{1, \ldots, n\}$. From the previous set of idempotents it is possible to obtain orthogonal idempotents $e_1, \ldots, e_n \in \Lambda$ such that $\pi(e_j) = z_j$ for each j (see [29, Proposition 1.1.25]). The orthogonality of $e_1, \ldots, e_n$ implies that $\mathrm{endol}(M) = \ell_{E_M}(e_1 M) + \cdots + \ell_{E_M}(e_n M)$ for $M \in \Lambda$-Mod.

Now fix j and assume that $e_j, e \in \Lambda$ are idempotents such that $e - e_j = r \in \mathrm{rad}(\Lambda)$. Let $M \in \Lambda$-Mod and consider the homomorphisms of E_M-modules $\alpha : e_j M \to eM$ and $\beta : eM \to e_j M$ given by $\alpha(e_j m) = ee_j m$ and $\beta(em) = e_j em$ for $m \in M$. Notice that $\beta\alpha(e_j m) = (e_j + e_j r e_j)e_j m$. Since $e_j r e_j$ belongs to the radical of $e_j \Lambda e_j$, it is quasi-invertible and so there exists $t \in e_j \Lambda e_j$ such that $t(e_j + e_j r e_j) = e_j = (e_j + e_j r e_j)t$. It follows that $\beta\alpha$ is an isomorphism of E_M-modules. In a similar way we can verify that $\alpha\beta$ is an isomorphism, and so $\ell_{E_M}(e_j M) = \ell_{E_M}(eM)$.

Notation. For an object V with the structure of a k-vector space and F a field extension of k we denote by V^F the object $V \otimes_k F$ (see [13] and [21]).

Remark 2.8. Given Λ and F a field extension of k (recall that k is perfect), $\Lambda = S \oplus \mathrm{rad}(\Lambda)$, by Wedderburn's principal theorem (see [13, Theorem 72.19] and [29, Theorem 2.5.37]), and $\Lambda^F = S^F \oplus \mathrm{rad}(\Lambda)^F$, where $\mathrm{rad}(\Lambda)^F$ can be identified with $\mathrm{rad}(\Lambda^F)$, by [21, Lemma 3.3(b)] (see also [13, Theorem 29.21 and Corollary 29.22]). There are similar claims for the endomorphism ring E_M of $M \in \Lambda$-mod (see [13, Lemma 29.5] and [21, Lemma 2.2(a)]), and for the endomorphism ring E_M of $M \in \mathcal{A}$-mod for an almost admissible ditalgebra $\mathcal{A}$ (see [5, proof of Lemma 5.1]).

Remark 2.9. Let $M \in \mathcal{A}$-Mod for $\mathcal{A}$ an admissible ditalgebra and assume that $e = (e^0, e^1) \in E_M$ is an idempotent. It is known (see, for example, [7, Lemma 5.11]) that there is an isomorphism $h : M \to M_1 \oplus M_2$ such that

$heh^{-1} = \big(\big(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\big),0\big)$. Then, by conjugation by h, we can identify E_{M_2} with $e(E_M)e$ and $\operatorname{rad} E_{M_2}$ with $e(\operatorname{rad} E_M)e$, so there are canonical isomorphisms $D_{M_2} \cong \pi(e)D_M\pi(e)$ and $Z_{M_2} \cong \pi(e)Z_M\pi(e)$, where $\pi : E_M \to D_M$ is the canonical epimorphism. Similar claims are well known for $M \in \Lambda$-Mod and e an idempotent of E_M.

Remark 2.10. Let $\mathcal{A}$ be an admissible ditalgebra and let $M \in \mathcal{A}$-mod (resp. $M \in \Lambda$-mod) be indecomposable. It is known that E_M is a finite-dimensional k-algebra (see [3, Proposition §II.1.1] and [7, p. 29]) and a local ring (see [3, Theorem §II.2.2] and [7, Lemma 5.12]), so D_M is a finite-dimensional division k-algebra; it follows that Z_M is a field and D_M is finite-dimensional over Z_M. Moreover, $\dim_{Z_M}(D_M) = c_M^2$ for a natural number c_M (see [29, Corollary 2.3.25]).

Definition 2.11. Let $M \in \mathcal{A}$-Mod for $\mathcal{A}$ an almost admissible ditalgebra (resp. $M \in \Lambda$-Mod) be such that $D_M = E_M/\operatorname{rad}(E_M)$ is a division ring finite-dimensional over its center Z_M, and write $c_M = \sqrt{\dim_{Z_M}(D_M)}$. We define the *central endolength vector* of M as $c\text{-}\underline{\ell}(M) = c_M\underline{\ell}(M)$, and its *central endolength* as $c\text{-endol}(M) = c_M \operatorname{endol}(M)$.

Remark 2.12. If k is a finite field and $M \in \mathcal{A}$-mod, for $\mathcal{A}$ an almost admissible ditalgebra (resp. $M \in \Lambda$-mod), is indecomposable, then, by Wedderburn's theorem on finite division rings, $c\text{-}\underline{\ell}(M) = \underline{\ell}(M)$.

Theorem 2.13 (cf. [19, Lemma 5.5]). *Let $\mathcal{A}$ be an admissible ditalgebra and let $M \in \mathcal{A}$-mod (resp. $M \in \Lambda$-mod) be indecomposable. Then there exists a Galois field extension F of k such that:*

(1) *There is an isomorphism of $\mathcal{A}^F$-modules (resp. Λ^F-modules) $M^F \cong N_1 \oplus \cdots \oplus N_t$, where N_i is indecomposable and $D_{N_i} \cong F$ for $i \in \{1,\dots,t\}$. Moreover, $\operatorname{endol}(N_i) = c\text{-endol}(N_i) = c\text{-endol}(M)$ for each i.*
(2) *Also, N_i^K is an indecomposable $\mathcal{A}^K$-module (resp. Λ^K-module) and $D_{N_i^K} \cong K$ for $i \in \{1,\dots,t\}$, and so $\operatorname{endol}(N_i) = \dim_K(N_i^K)$ for each i.*

Proof. Let us write $r_M = \operatorname{rad}(E_M)$.

Let $\{0\} = M_0 \le M_1 \le \cdots \le M_u = M$ be a composition series of M as a right E_M-module, hence $\operatorname{endol}(M) = u$. Observe that $M_{j+1}/M_j \cong D_M$ for $j \in \{0,\dots,u-1\}$.

By [18, Theorem 4.2.1] there exists a finite field extension F of Z_M such that $D_M \otimes_{Z_M} F \cong M_{c_M}(F)$. Since k is perfect and Z_M is a finite field extension of k we can choose F to be a Galois field extension of Z_M. Observe that Z_M is a simple extension of k. Then $Z_M \otimes_k F \cong F \times \cdots \times F$, where the number of factors is $[Z_M : k]$, and so

$$D_M \otimes_k F \cong D_M \otimes_{Z_M} Z_M \otimes_k F \cong D_M \otimes_{Z_M} (F \times \cdots \times F) \cong \mathop{\times}_{i=1}^{[Z_M:k]} M_{c_M}(F).$$

(See, for example, [14, Lemma 2.7] or [25, Lemma 2.11].)

By Remark 2.8 we find that $(E_M)^F \cong E_{M^F}$ and $(r_M)^F \cong \operatorname{rad}(E_{M^F})$, so $(D_M)^F \cong E_{M^F}/(r_M)^F$ and from [29, Corollary 1.7.24] we deduce that $(Z_M)^F$ is isomorphic to the center of D_{M^F}.

Then (see [7, Lemma 20.2] and [5, proof of Lemma 5.1]) $M_0^F \le M_1^F \le \cdots \le M_u^F$ is a series of E_{M^F}-submodules of M^F such that $M_{j+1}^F/M_j^F \cong \times_{i=1}^{[Z_M:k]} M_{c_M}(F)$.

Consider an associated decomposition of the unit as a sum of primitive orthogonal idempotents

$$1_{D_M^F} = e_{1,1}+e_{1,2}+\cdots+e_{1,c_M}+e_{2,1}+\cdots+e_{2,c_M}+\cdots+e_{[Z_M:k],1}+\cdots+e_{[Z_M:k],c_M}.$$

Applying Remark 2.9 we have $M^F \cong \bigoplus_{i=1}^{[Z_M:k]} \bigoplus_{j=1}^{c_M} N_{i,j}$, where the summand $N_{i,j}$ is associated to the idempotent $e_{i,j}$. We see that $D_{N_{i,j}} \cong F$, so $N_{i,j}$ is indecomposable, and also $N_{i,j} \cong N_{i',j'}$ if and only if $i = i'$.

Moreover, $M_0^F e_{i,j} \le M_1^F e_{i,j} \le \cdots \le M_u^F e_{i,j}$ is a sequence of $e_{i,j}(E_M^F)e_{i,j}$-modules that can be identified with a sequence of $E_{N_{i,j}}$-submodules of $N_{i,j}$. It is immediate that $M_{s+1}^F e_{i,j}/M_s^F e_{i,j} \cong \bigoplus_{h=1}^{c_M} F$; consequently, the endolength of $Ne_{i,j}$ is $c_M \operatorname{endol}(M)$.

By Remark 2.8 we get $(D_M)^K \cong (\times_{i=1}^{[Z_M:k]} M_{c_M}(F))^K \cong \times_{i=1}^{[Z_M:k]} M_{c_M}(K)$, so the endomorphism ring of $N_{i,j}^K$ is isomorphic to K and its endolength coincides with its dimension as a K-vector space.

We apply a similar argument when we consider Λ instead of $\mathcal{A}$. ∎

THEOREM 2.14. *Let $\mathcal{A}$ be an elementary ditalgebra* (*resp. let Λ be elementary*) *and let $M \in \mathcal{A}$-mod* (*resp. $M \in \Lambda$-mod*) *be indecomposable. Then $M^K \cong N_1 \oplus \cdots \oplus N_t$, where N_i is an indecomposable $\mathcal{A}^K$-module* (*resp. Λ^K-module*) *and $c\text{-}\underline{\ell}(M) = c\text{-}\underline{\ell}(N_i) = \underline{\dim}(N_i)$ for each i, where $\underline{\dim}(N_i)$ denotes the dimension vector of N_i.*

Proof. It is not hard to verify that $\mathcal{A}^K$ is elementary and that there is a canonical bijection between the primitive orthogonal idempotents of R and those of R^K. Now we only need to apply the proof of Theorem 2.13 to eM, where $e \in R$ is a centrally primitive idempotent.

The argument for Λ elementary is similar (use Remarks 2.7 and 2.8). ∎

REMARK 2.15. Example 4.7 of [21] shows that Theorem 2.14 is not true for Λ not elementary. However, next we see that we can associate to Λ a closely related elementary finite-dimensional algebra.

Proposition 2.16. *Given Λ there exists a Galois field extension F of k, and a finite-dimensional elementary F-algebra Λ_0, such that Λ^F is Morita equivalent to Λ_0.*

Proof. It is known that $\Lambda/\mathrm{rad}(\Lambda) \cong \prod_{i=1}^{n} M_{m_i}(D_i)$, where D_i is a finite-dimensional division k-algebra for each i.

As in the proof of Theorem 2.13 there exist finite field extensions E_i of Z_i such that $D_i \otimes_{Z_i} E_i \cong M_{c_i}(E_i)$. So there exists a Galois extension F of k that contains E_i as an intermediate field for each i, and then

$$D_i \otimes_k F \cong D_i \otimes_{Z_i} Z_i \otimes_k F \cong \mathop{\times}_{j=1}^{[Z_i:k]} D_i \otimes_{Z_i} F \cong \mathop{\times}_{j=1}^{[Z_i:k]} M_{c_i}(F).$$

The functor of [3, Proposition §II.2.5] determines an elementary finite-dimensional F-algebra Λ_0 and a Morita equivalence $\mathrm{Hom}_{\Lambda^F}(P,-) : \Lambda^F\text{-Mod} \to \Lambda_0\text{-Mod}$, where $P_1, \dots, P_t$ is a complete set of representatives of indecomposable projective Λ^F-modules, $P = \bigoplus_{i=1}^{t} P_i$ and $\Lambda_0 = \mathrm{End}_{\Lambda}(P)^{\mathrm{op}}$. ∎

Now we want to calculate the effect of the Morita equivalence of the previous proposition on endolength vectors, and so on central endolength vectors, using a proof communicated to us by R. Bautista.

Proposition 2.17. *Let Γ be the reduced form of Λ (see* [3, p. 35]*) and $H : \Gamma\text{-Mod} \to \Lambda\text{-Mod}$ the corresponding Morita equivalence. Then $N \in \Gamma\text{-Mod}$ is endofinite if and only if $H(N)$ is endofinite. Moreover, there are fixed positive integers $m_1, \dots, m_n$ such that $\underline{\ell}(H(N)) = (m_1d_1, \dots, m_nd_n)$ (resp. $c\text{-}\underline{\ell}(H(N)) = (m_1d_1, \dots, m_nd_n)$ when N is indecomposable and D_N is finite-dimensional over Z_N) where $\underline{\ell}(N) = (d_1, \dots, d_n)$ ($c\text{-}\underline{\ell}(N) = (d_1, \dots, d_n)$).*

Proof. Let $\Lambda = Q_1 \oplus \dots \oplus Q_n$, where $Q_i = P_{i,1} \oplus \dots \oplus P_{i,m_i}$, each $P_{i,j}$ is indecomposable and $P_{i,j} \cong P_{i',j'}$ if and only if $i = i'$.

Let $\{e_{1,1}, \dots, e_{1,m_1}, \dots, e_{n,1}, \dots, e_{n,m_n}\}$ be the associated set of primitive orthogonal idempotents given by $\Lambda e_{i,j} = P_{i,j}$. Observe that $Q_i = \Lambda\hat{e}_i$, where $\hat{e}_i = e_{i,1} + \dots + e_{i,m_i}$.

We choose $P = P_{1,1} \oplus P_{2,1} \oplus \dots \oplus P_{n,1}$ and $\Gamma = \mathrm{End}_{\Lambda}(P)^{\mathrm{op}}$. Then there are equivalences $\mathrm{Hom}_{\Lambda}(P,-) : \Lambda\text{-Mod} \to \Gamma\text{-Mod}$ and $P \otimes_{\Gamma} - : \Gamma\text{-Mod} \to \Lambda\text{-Mod}$.

Given $M \in \Lambda\text{-Mod}$ it is easy to verify that $\mathrm{Hom}_{\Lambda}(Q_i, M)$ and $\hat{e}_iM$ are isomorphic as right E_M-modules.

Now, for each i, let $f_i : P \to P$ be the idempotent induced by the identity on $P_{i,1}$. As before, for $N \in \Gamma\text{-Mod}$ we have $\ell_{E_N}(\mathrm{Hom}_{\Gamma}(\Gamma f_i, N)) = \ell_{E_N}(f_iN)$.

The isomorphism of vector spaces

$$\alpha : \mathrm{Hom}_{\Gamma}(\Gamma f_i, N) \to \mathrm{Hom}_{\Lambda}(P \otimes_{\Gamma} \Gamma f_i, P \otimes_{\Gamma} N)$$

is also an isomorphism of right E_N-modules, because we can identify E_N with $E_{P\otimes_\Gamma N}$. Having in mind that $P\otimes_\Gamma \Gamma f_i \cong Pf_i = P_{i,1}$ and writing $M = P\otimes_\Gamma N$ we have the identities

$$\begin{aligned}\ell_{E_M}(\hat{e}_i M) &= m_i \ell_{E_M}(\mathrm{Hom}_\Lambda(P_{i,1}, M)) = m_i \ell_{E_M}(\mathrm{Hom}_\Gamma(\Gamma f_i, N))\\ &= m_i \ell_{E_N}(f_i N),\end{aligned}$$

and the claim for endolength vectors follows.

For central endolength vectors we only need to observe that $c_N = c_{H(N)}$, because H is an equivalence of categories. ■

3. Infinite representation type: equivalences

Definition 3.1. Let us recall that an almost admissible ditalgebra $\mathcal{A}$ (resp. Λ) is of *finite representation type* if there are a finite number of isomorphism classes of indecomposable modules in $\mathcal{A}$-mod (resp. Λ-mod), and of *infinite representation type* otherwise.

We make a slight adaptation of some notions in [11], and we say that $\mathcal{A}$ (resp. Λ) is *c-unbounded* if for any natural number d there exists an indecomposable $M\in\mathcal{A}$-mod (resp. $M\in\Lambda$-mod) such that c-endol$(M)\geq d$, and *c-strongly unbounded* if there is a sequence of natural numbers $d_1 < d_2 < \cdots$ such that for any j there are an infinite number of isomorphism classes of indecomposables $M\in\mathcal{A}$-mod (resp. $M\in\Lambda$-mod) such that c-endol$(M) = d_j$.

Now we prove a result very similar to one of [12].

Theorem 3.2 (cf. [12, Theorem of p. 156]). *Recall that k is a perfect field and Λ is a finite-dimensional k-algebra. The following are equivalent:*

(1) *Λ is of infinite representation type.*
(2) *Λ is c-unbounded.*
(3) *Λ is c-strongly unbounded.*
(4) *Λ is not generically trivial.*

Proof. By [1, Corollary 4.8], or [2, Theorem A], if Λ (or Λ^K) is of finite representation type then it is generically trivial. The converse follows by [12, Theorem, p. 156].

By [20, Theorem 3.3] (see also [21]), Λ is of infinite representation type if and only if Λ^K is of infinite representation type.

If $M\in\Lambda^K$-mod is indecomposable then $D_M\cong K$, and so for Λ^K the notions of c-unbounded and c-strongly unbounded are equivalent to the usual concepts of unbounded and strongly unbounded.

Now, the equivalence between (1), (2) and (3) is known for Λ^K: these are the Brauer–Thrall conjectures (see, for example, [3, pp. 221 and 222]).

By [21, Lemma 2.5] we see, for $L, M \in \Lambda$-mod indecomposables, that L^K and M^K have a common direct summand if and only if $L \cong M$; by Theorem 2.13 we conclude that Λ is c-unbounded (resp. c-strongly unbounded) if and only if Λ^K is unbounded (resp. strongly unbounded). ■

Observe, by [22, Lemmas 3.3, 4.1 and 4.2] and Theorem 2.13 and Propositions 2.16 and 2.17, that it was enough to prove Theorem 3.2 for Λ elementary.

Definition 3.3. Let $\mathcal{A}$ be an almost admissible ditalgebra. We say that $\mathcal{A}$ (resp. Λ) is *limited* if there exist a finite list $\{D_1, \dots, D_t\}$ of finite-dimensional k-division rings such that for any $M \in \mathcal{A}$-mod (resp. $M \in \Lambda$-mod) indecomposable there exists $j_M \in \{1, \dots, t\}$ with $D_M \cong D_{j_M}$.

The next result follows straightforwardly from Theorem 3.2.

Corollary 3.4. *Let k be a finite field. Then Λ is of finite representation type if and only if Λ is limited.*

4. Reduction functors, endolength vectors and a norm. We are assuming the notation and results of [4] and [7] for reduction functors (see also [24]).

Recall that, for an almost admissible ditalgebra $\mathcal{A} = (T, \delta)$ with layer $(R, W_0 \oplus W_1)$, δ is a homomorphism of R-R-bimodules (see e.g. [7, Definition 4.5]). Then, for $1_R = e_1 + \cdots + e_n$ a decomposition into centrally primitive idempotents and $e = e_{i_1} + \cdots + e_{i_t}$, where $\{i_1, \dots, i_t\}$ is a non-empty and proper subset of $\{1, \dots, n\}$, there is an associated almost admissible ditalgebra $\mathcal{A}_e$ with layer $(eR, eW_0e \oplus eW_1e)$ and a reduction functor $F_e : \mathcal{A}_e$-Mod $\to \mathcal{A}$-Mod, called *idempotent deletion,* which is full and faithful, and is dense in the subcategory of $\mathcal{A}$-Mod of all objects M such that $(1-e)M = 0$.

Recall k is a perfect field, so $R \otimes_k R$ is a semisimple finite-dimensional k-algebra. Therefore $W_0 = \mathrm{Ker}(\delta_{|W_0}) \oplus U_0$ and $W_1 = \delta(U_0) \oplus U_1$ as R-R-bimodules.

If $U_0 \neq 0$, then there is an almost admissible ditalgebra $\mathcal{A}_r$ with layer $(R, \mathrm{Ker}(\delta_{|W_0}) \oplus U_1)$ and a reduction functor $F_r : \mathcal{A}_r$-Mod $\to \mathcal{A}$-Mod, which we are going to call *regularization*, which is an equivalence of categories. The usual notion of regularization functor is more general (see [7]), but the one above is good enough for our purposes.

When $U_0 = 0$ and $\mathrm{Ker}(\delta_{|W_0}) \neq 0$ we can use a reduction functor, just denoted by $F^X : \mathcal{A}^X$-Mod $\to \mathcal{A}$-Mod, which in the notation of [4] is a combination of absorption and reduction by an admissible bimodule. This functor includes both Edge Reduction and Unraveling a Loop of [11]. In the terminology of [7] the layer (R, W_0'), where W_0' is a direct R-R-summand of $\mathrm{Ker}(\delta_{|W_0})$, determines an initial subalgebra, and in our context we can identify it with the (tensor) k-algebra $\varGamma = T_R(W_0') = R \oplus W_0' \oplus W_0' \otimes_R W_0' \oplus \cdots$.

Depending on the objects of $\mathcal{A}$-Mod that we want to study, we choose a $\varGamma$-module X with appropriate properties (see e.g. [7, Sections 12–14]) which is in general finite-dimensional and such that $X = \bigoplus_{i=1}^{t} X_i$, where X_i is indecomposable for each i, and $i \neq j$ implies $X_i \not\cong X_j$. Then $\mathcal{A}^X$ has a layer $(S, X^* \otimes_R W_0'' \otimes X \oplus W_1^X)$, where $S \cong \mathrm{End}_{\varGamma}(X)^{\mathrm{op}}/\mathrm{rad}(\mathrm{End}_{\varGamma}(X)^{\mathrm{op}})$, $W_0 = W_0' \oplus W_0''$ as R-R-bimodules and $X^* = \mathrm{Hom}_{k\text{-}S}(X, S)$.

Let $\mathrm{res} : \mathcal{A}\text{-Mod} \to \varGamma\text{-Mod}$ be the restriction functor.

The functor F^X is full and faithful, and it is dense in the subcategory of $\mathcal{A}$-Mod of all objects M such that $\mathrm{res}(M) \in \mathrm{add}\, X$.

Let f_j be the idempotent in S induced by the identity on X_j, for $j \in \{1, \ldots, t\}$. Then for $N \in \mathcal{A}^X$-mod we observe that $\mathrm{res}(F^X(N)) \cong a_1X_1 \oplus \cdots \oplus a_tX_t$ as $\varGamma$-modules, where $N \cong a_1Sf_1 \oplus \cdots \oplus a_tSf_t$ as S-modules.

Definition 4.1. Recall (see [5, Definition 4.2]) that an almost admissible ditalgebra $\mathcal{A}$ is *constructible* if there is a finite sequence of reduction functors, restricted to those called idempotent deletion, regularization or type F^X,

$$\mathcal{D}_t\text{-Mod} \xrightarrow{F_t} \mathcal{D}_{t-1}\text{-Mod} \xrightarrow{F_{t-1}} \cdots \xrightarrow{F_2} \mathcal{D}_1\text{-Mod} \xrightarrow{F_1} \mathcal{D}_0\text{-Mod} = \mathcal{D}^{\varLambda}\text{-Mod},$$

where $\mathcal{D}^{\varLambda}$ is the Drozd ditalgebra of $\varLambda$, and there is an isomorphism of layered ditalgebras $\mathcal{A} \cong \mathcal{D}_t$.

Corollary 4.2. *Let $\mathcal{A}$ be a constructible ditalgebra. Then the following are equivalent:*

(1) *$\mathcal{A}$ is of infinite representation type.*
(2) *$\mathcal{A}$ is c-unbounded.*
(3) *$\mathcal{A}$ is c-strongly unbounded.*
(4) *$\mathcal{A}$ is not generically trivial.*

Proof. Let us assume the notation of Definition 4.1. By [5, Lemma 4.4, items 3 and 4], [5, Corollary 4.5] and Theorem 3.2, the statement is true for $\mathcal{D}^{\varLambda}$-Mod. By [7, Lemmas 25.2, 25.3, 25.4 and 25.7], the statement is true for $\mathcal{D}_j$-Mod for each j. ■

The next proposition was proved in [24] for admissible ditalgebras and in [7, main part of Lemma 25.7] for seminested ditalgebras.

Proposition 4.3. *Let $\mathcal{A}$ be an almost admissible ditalgebra, let $F : \mathcal{A}^z\text{-Mod} \to \mathcal{A}\text{-Mod}$ be a reduction functor and let $M \in \mathcal{A}^z\text{-Mod}$ be indecomposable and endofinite. In the following cases there is a fixed matrix t_F such that $\underline{\ell}(F(M))^t = t_F(\underline{\ell}(M)^t)$:*

(1) *If* $F = F_r$ *is the regularization functor, then* t_F *is the identity matrix.*
(2) *If* $F = F_e$ *is the idempotent deletion functor, then after a suitable numbering of the idempotents,* t_F *is a matrix with* $(t_F)_{i,j} = 1$ *if* $i = j$*, and* $(t_F)_{i,j} = 0$ *if* $i \neq j$*.*
(3) *If* $F = F^X$*, and* $1_R = e_1 + \cdots + e_n$ *is the canonical decomposition into centrally primitive orthogonal idempotents, and* $1_S = f_1 + \cdots + f_s$ *is the canonical decomposition into central primitive orthogonal idempotents, then* $(t_{F^X})_{i,j}$ *is the rank of* $e_i X f_j$ *over* Sf_j*.*

Proof. Claims (1) and (2) are immediate from the definitions. For the third claim we can adapt the argument of [7, Lemma 25.7]. ∎

Now we see an analogue of Proposition 2.17.

COROLLARY 4.4. *Let* $\mathcal{A}$ *be an almost admissible ditalgebra, and let* $\mathcal{A}^b$ *be its basification* (*see* [5, Proposition 3.3])*, which is obtained through the initial subalgebra given by the layer* $(R, 0)$ *and* $X = L_1 \oplus \cdots \oplus L_n$*, where* $1_R = e_1 + \cdots + e_n$ *is a decomposition into centrally primitive idempotents and* L_i *is a simple* Re_i*-module, and* $F^X : \mathcal{A}^b\text{-Mod} \to \mathcal{A}\text{-Mod}$ *is the associated reduction functor. Then there are fixed integers* $m_1, \ldots, m_n$ *such that, for* $M \in \mathcal{A}^b\text{-Mod}$ *endofinite and indecomposable with* $\underline{\ell}(M) = (d_1, \ldots, d_n)$ (*resp.* c-$\underline{\ell}(M) = (d_1, \ldots, d_n)$) *we have* $\underline{\ell}(F^X(M)) = (m_1 d_1, \ldots, m_n d_n)$ (*resp.* c-$\underline{\ell}(F^X(M)) = (m_1 d_1, \ldots, m_n d_n)$)*.*

Proof. We use the notation of Proposition 4.3. It is clear that $e_i X f_j = 0$ for $i \neq j$ and that $e_i X f_i = L_i$. Moreover, the rank of L_i over $S_i = \mathrm{End}_R(L_i)^{\mathrm{op}}$ is m_i, where $Re_i \cong M_{m_i}(D_i)$. ∎

The functor $F^X : \mathcal{A}^b\text{-Mod} \to \mathcal{A}\text{-Mod}$ of the previous corollary is an equivalence of categories, so for most results, we only need to develop proofs for admissible ditalgebras; however, we would rather have definitions for almost admissible ditalgebras.

DEFINITION 4.5. We will work with the partial order on $\mathbb{Z}^n$ given by $(d_1, \ldots, d_n) \geq (d'_1, \ldots, d'_n)$ iff $d_i \geq d'_i$ for $i \in \{1, \ldots, n\}$.

REMARK 4.6. The matrices t_F of Proposition 4.3 preserve order strictly, i.e. if $\underline{d}$ and $\underline{d'}$ are integer vectors of appropriate size and $\underline{d} > \underline{d'}$, then $t_F(\underline{d})^t > t_F(\underline{d'})^t$.

The next definition is closely related to the norm used in [24]; notice that the endonorm is different from the norm of [7, 25.1 and 28.1].

DEFINITION 4.7. Let $\mathcal{A}$ be an almost admissible ditalgebra, where $1_R = e_1 + \cdots + e_n$ is a decomposition into centrally primitive orthogonal idempotents. For $M \in \mathcal{A}\text{-Mod}$ endofinite with endolength vector $\underline{\ell}(M)$ we define the *endonorm* of M, denoted by $\|M\|$, as the number $\underline{\ell}(M) W (\underline{\ell}(M))^t$, where

$W = (w_{i,j})$ is an $n \times n$ matrix with

$$w_{i,j} = \frac{\dim_k(e_i W_0 e_j)}{\dim_k(Re_i)\dim_k(Re_j)}.$$

REMARK 4.8. For $\mathcal{A}$ an admissible ditalgebra and $M \in \mathcal{A}$-mod indecomposable, $\|M\|(\dim_k(D_M))^2$ is equal to the norm defined in [5, 7.1].

REMARK 4.9. For $\mathcal{A}$ an admissible ditalgebra and $M \in \mathcal{A}$-Mod indecomposable, if $\|M\| = 0$ then M has to be indecomposable as an R-module. Also $E_M \cong \mathrm{End}_R(M) \oplus \mathrm{Hom}_R(W_1 \otimes_R M, M)$ as k-vector spaces and $\mathrm{Hom}_R(W_1 \otimes_R M, M) \subset \mathrm{rad}(E_M)$.

PROPOSITION 4.10 (cf. [5, Lemma 7.2]). *Let $\mathcal{A}$ be an admissible ditalgebra and $1_R = e_1 + \cdots + e_n$ a decomposition into centrally primitive orthogonal idempotents. Let $M \in \mathcal{A}$-*Mod *be indecomposable and endofinite, and assume that $\underline{\ell}(M)$ has $m > 1$ nonzero coordinates. Then*

$$\mathrm{endol}(M) \leq \frac{mb}{m-1}\|M\|,$$

where b is the least common multiple of $\dim_k(Re_1), \ldots, \dim_k(Re_n)$.

Proof. Without loss of generality we can assume $(d_1, \ldots, d_m, 0, \ldots, 0) = \underline{\ell}(M)$, where $d_1, \ldots, d_m$ are nonzero. Since M is indecomposable, there are pairs $\{d_{h_{i,1}}, d_{h_{i,2}}\}$, for $i \in \{1, \ldots, m-1\}$, such that $h_{i,j} \in \{1, \ldots, m\}$, the sets $\{d_1, \ldots, d_m\}$ and $\{\{d_{h_{1,1}}, d_{h_{1,2}}\}, \ldots, \{d_{h_{m-1,1}}, d_{h_{m-1,2}}\}\}$ form a connected graph, and $w_{h_{i,1},h_{i,2}} \neq 0$.

Using induction we can prove that

$$\frac{m}{m-1}(d_{h_{1,1}}d_{h_{1,2}} + \cdots + d_{h_{m-1,1}}d_{h_{m-1,2}}) \geq d_1 + \cdots + d_m.$$

Finally, notice that $bw_{i,j}$ is an integer for all i and j, so $\frac{mb}{m-1}\|M\| \geq \frac{m}{m-1}(d_{h_{1,1}}d_{h_{1,2}} + \cdots + d_{h_{m-1,1}}d_{h_{m-1,2}})$. ■

REMARK 4.11. Suppose that $\mathcal{A}$ is an admissible ditalgebra with layer $(R, W_0 \oplus W_1)$, $1_R = e_1 + \cdots + e_n$ a decomposition into centrally primitive orthogonal idempotents, $F^X : \mathcal{A}^X \to \mathcal{A}$ a reduction functor, $W_{0,0}$ a direct R-R-summand of W_0, L the matrix with (i,j) entry

$$\frac{\dim_k(e_i W_{0,0} e_j)}{\dim_k(Re_i)\dim_k(Re_j)},$$

and L_X the matrix with (a,b) entry

$$\frac{\dim_k(f_a X^* \otimes_R W_{0,0} \otimes_R X f_b)}{\dim_k(Sf_a)\dim_k(Sf_b)},$$

where (S, W^X) is the layer of $\mathcal{A}^X$ and $1_S = f_1 + \cdots + f_t$ a decomposition into centrally primitive orthogonal idempotents. It is easy to show that $L_X =$

$(t_{FX})^t L t_{FX}$, where t_{FX} is the matrix of Proposition 4.3(3), and therefore it is easy to prove the next result.

LEMMA 4.12. *Let $\mathcal{A}$ be an admissible ditalgebra, let $F : \mathcal{A}^z\text{-Mod} \to \mathcal{A}\text{-Mod}$ be any of the reduction functors analyzed in Proposition 4.3, and let $N \in \mathcal{A}^z\text{-Mod}$. Then N is endofinite if and only if $F(N) = M$ is endofinite, and in that case:*

(1) *if F is idempotent deletion, then $\|N\| = \|M\|$,*
(2) *if F is regularization and $\underline{\ell}(M)$ has nonzero coordinates, then $\|N\| < \|M\|$,*
(3) *if $F = F^X$ and $\underline{\ell}(M)$ has nonzero coordinates, then $\|N\| < \|M\|$.*

Proof. Computations similar to those of [7, Section 25] or [5, Lemma 7.3]. ∎

The endonorm of an endofinite object may be a rational number, but we still have situations where we can apply the usual induction arguments (see [6]).

DEFINITION 4.13. Let $\mathcal{A}$ be an almost admissible ditalgebra. If $\underline{d}$ is an endolength vector we denote by $\operatorname{ind}\mathcal{A}(\underline{d})$ the full subcategory of $\mathcal{A}$-mod of indecomposable modules M with $\underline{\ell}(M) = \underline{d}$. We say that $\operatorname{ind}\mathcal{A}(\underline{d})$ is *finite* if it contains a finite number of isomorphism classes, otherwise it is *infinite.* In a similar way we denote by $c\text{-}\operatorname{ind}\mathcal{A}(\underline{d})$ the full subcategory of $\mathcal{A}$-mod of indecomposable modules M with $c\text{-}\underline{\ell}(M) = \underline{d}$, where now $\underline{d}$ is a central endolength vector, and we call this subcategory finite or infinite depending on the number of isomorphism classes contained in $c\text{-}\operatorname{ind}\mathcal{A}(\underline{d})$.

Observe that $\|M_1\| = \|M_2\|$ for any $M_1, M_2 \in \operatorname{ind}\mathcal{A}(\underline{d})$, so we can associate to $\operatorname{ind}\mathcal{A}(\underline{d})$ the number $\|\underline{d}\| = \|M\|$ for any $M \in \operatorname{ind}\mathcal{A}(\underline{d})$.

PROPOSITION 4.14. *Let $\mathcal{A}$ be an almost admissible ditalgebra. The following are equivalent:*

(1) *$\operatorname{ind}\mathcal{A}(\underline{d})$ is finite for each $\underline{d}$.*
(2) *$c\text{-}\operatorname{ind}\mathcal{A}(\underline{d})$ is finite for each $\underline{d}$.*
(3) *$\mathcal{A}$ is generically trivial.*

Proof. It is clear that (1) implies (2).

Now assume (2); we will prove (3) by induction on the endonorm. By Corollary 4.4 we can assume that $\mathcal{A}$ is admissible with $R \cong D_1 \times \cdots \times D_n$.

Let b be the least common multiple of $\dim_k(D_1), \ldots, \dim_k(D_n)$. Then for any $M \in \mathcal{A}$-Mod endofinite and indecomposable the number $\|M\|$ is in $b^{-1}\mathbb{N} \cup \{0\}$.

By Remark 4.9 there is no generic module $G \in \mathcal{A}$-Mod such that $\|G\| = 0$. Assume that for an admissible ditalgebra $\mathcal{A}'$ fulfilling (2) there is no generic module $G' \in \mathcal{A}'$-Mod such that $\|G'\| \in \{0, 1/b, \ldots, m/b\}$.

Let $\underline{d}$ be an endolength vector such that $\|\underline{d}\| = (m+1)/b$.

If $F : \mathcal{A}^z \to \mathcal{A}$ is idempotent deletion or regularization then, by Proposition 4.3, there is a unique endolength vector $\underline{d}'$ such that F induces an equivalence between $\operatorname{ind} \mathcal{A}^z(\underline{d}')$ and $\operatorname{ind} \mathcal{A}(\underline{d})$. Also we see for $N \in \operatorname{ind} \mathcal{A}^z(\underline{d}')$ that $\|N\| \in b^{-1}\mathbb{N} \cup \{0\}$.

Then, by Lemma 4.12, we can assume that $\underline{d}$ has nonzero coordinates.

If we use regularization then we can apply the induction hypothesis.

So it only remains to analyze the effect of a suitable functor $F^X : \mathcal{A}^X \to \mathcal{A}$. Recall that the associated initial subalgebra is a hereditary finite-dimensional k-algebra, which we denote by Γ.

By Theorem 3.2 we see that Γ is of finite representation type, and so we choose $X = X_1 \oplus \cdots \oplus X_r$, where $\{X_1, \ldots, X_r\}$ is a complete set of representatives of isomorphism classes of indecomposable Γ-modules ($i \neq i'$ implies $X_i \ncong X_{i'}$).

Then we have, thanks to Proposition 4.3, a finite number of vectors $\underline{d}'_1, \ldots, \underline{d}'_t$ such that F induces an equivalence between $\bigcup_{j=1}^{t} \operatorname{ind} \mathcal{A}^X(\underline{d}'_j)$ and $\operatorname{ind} \mathcal{A}(\underline{d})$.

As Γ is of finite representation type, it is known that there is a function $\sigma : \{1, \ldots, r\} \to \{1, \ldots, n\}$ such that $D_{X_i} \cong D_{\sigma(i)}$ (see, for example, [3]). Then, by Remark 4.11, we get $\|\underline{d}'_j\| \in b^{-1}\mathbb{N} \cup \{0\}$ for $j \in \{1, \ldots, t\}$.

By Lemma 4.12 and the induction hypothesis there is no generic module $G' \in \mathcal{A}^X$ such that $\underline{\ell}(G') \in \{\underline{d}'_1, \ldots, \underline{d}'_t\}$, so there is no generic module $G \in \mathcal{A}$-Mod such that $\underline{\ell}(G) = \underline{d}$.

The proof that (3) implies (1) is very similar to that for the previous implication. ∎

With a similar argument to the one in the proof of the proposition above, and applying Corollary 3.4, we get the next result.

Proposition 4.15. *Let $\mathcal{A}$ be an almost admissible ditalagebra over the finite field k. Then $\mathcal{A}$ is generically trivial if and only if it is limited.*

The next theorem is a version of BT-I.

Theorem 4.16. *Let $\mathcal{A}$ be an almost admissible ditalgebra. Then $\mathcal{A}$ is of infinite representation type if and only if it is c-unbounded.*

Proof. It is clear that if $\mathcal{A}$ is of finite representation type then it is not c-unbounded.

Now assume that $\mathcal{A}$ is of infinite representation type. If $c\text{-}\operatorname{ind} \mathcal{A}(\underline{d})$ is finite for each $\underline{d}$, then $\mathcal{A}$ is c-unbounded.

On the other hand, if there is a $\underline{d}$ such that $c\text{-}\operatorname{ind} \mathcal{A}(\underline{d})$ is infinite, we can use an inductive argument similar to the proof of Proposition 4.14, changing endolength vectors to central endolength vectors; in some step we have to apply a reduction functor F^X associated to a k-algebra of infinite representation type, and by Theorem 3.2, this algebra is c-unbounded, thus $\mathcal{A}$ is c-unbounded. ■

5. Trivial group of self-extensions. Now we see that the endonorm defined in the previous section also deals very well with finite-dimensional modules with trivial group of self-extensions.

DEFINITION 5.1. We say that Γ is a *k-triangular matrix algebra* if

$$\Gamma = \begin{pmatrix} D_1 & 0 \\ B & D_2 \end{pmatrix},$$

where D_1 and D_2 are finite-dimensional k-algebras and division rings over k, and B is a D_2-D_1-bimodule of finite dimension as a k-vector space with k acting centrally over B. We associate to $M = (V_1, V_2, \phi : B \otimes_{D_1} V_1 \to V_2) \in \Gamma\text{-mod}$ the quotient $\dim_{D_1}(V_1)/\dim_{D_2}(V_2)$. We denote this map as $q : \Gamma\text{-mod} \to [0, \infty]$.

LEMMA 5.2. *Let Γ be a k-triangular matrix algebra.*

(1) *Let Γ be of finite representation type, $A \in \Gamma\text{-mod}$ indecomposable and τ the Auslander–Reiten translation. Then $q(A) < q(\tau^{-1}(A))$ when A is not injective, and $q(\tau(A)) < q(A)$ when A is not projective.*
(2) *Let Λ be of infinite representation type, L a regular Γ-module, A a preprojective Γ-module, C a preinjective Γ-module and τ the Auslander–Reiten translation. Then $q(A) < q(L) < q(C)$, $q(A) < q(\tau^{-1}(A))$ and $q(\tau(C)) < q(C)$.*
(3) *Let $M \in \Gamma\text{-mod}$ be such that $\mathrm{Ext}_\Gamma(M, M) = 0$ and $M \cong X_1 \oplus \cdots \oplus X_n$, where X_i is indecomposable for each i. Then all elements of $\{X_1, \ldots, X_n\}$ are preprojective or preinjective. Moreover the set $\{X_1, \ldots, X_n\}$ has one or two isomorphism classes. In the latter case, these isomorphism classes are connected by one arrow in the Auslander–Reiten quiver.*
(4) *Assume $M, N \in \Gamma\text{-mod}$ such that $\mathrm{Ext}_\Gamma(M, M) = 0$ and $\mathrm{Ext}_\Gamma(N, N) = 0$. If $q(M) = q(N)$ then there are positive integers m, n such that $\bigoplus_{i=1}^m M \cong \bigoplus_{j=1}^n N$. Moreover, $M \cong N$ as Γ-modules if and only if $M \cong N$ as $D_1 \times D_2$-modules.*

Proof. This lemma is well known: it follows for example from [28, pp. 362 and 363]. For a detailed proof see [17]. ■

THEOREM 5.3. *Let $\mathcal{A}$ be an almost admissible ditalgebra, and let $M, N \in \mathcal{A}\text{-mod}$ be such that $\mathrm{Ext}_\mathcal{A}(M, M) = 0$ and $\mathrm{Ext}_\mathcal{A}(N, N) = 0$.*

(1) *$M \cong N$ as $\mathcal{A}$-modules if and only if $M \cong N$ as R-modules.*
(2) *If M and N are indecomposable and $\underline{\ell}(M) = \underline{\ell}(N)$ then $M \cong N$.*
(3) *If M and N are indecomposable and $c\text{-}\underline{\ell}(M) = c\text{-}\underline{\ell}(N)$ then $M \cong N$.*

Proof. By Proposition 4.4 we can assume that $\mathcal{A}$ is admissible.

For (1) we can assume, without loss of generality, that $\|M\| \geq \|N\|$, and we will prove the claim by induction on $\|M\|$.

One implication is well known, so assume that $M \cong N$ as R-modules; then there exists a rational number r such that $r\underline{\ell}(M) = \underline{\ell}(N)$, thus $\|M\| = 0$ implies $\|N\| = 0$. Also, for $\|M\| = 0$ it is easy to see that the structure of A-module of M is just its structure as an R-module, thus $\|M\| = 0$ and $M \cong N$ as R-modules implies $M \cong N$ as $\mathcal{A}$-modules.

Let b be as in Proposition 4.14 and so $b\|M\|$ is an integer.

Now assume that (1) is true for any admissible ditalgebra $\mathcal{B}$ with layer $(S, W_{\mathcal{B}})$, and $L_1, L_2 \in \mathcal{B}$-mod satisfy $\mathrm{Ext}_{\mathcal{B}}(L_1, L_1) = 0$, $\mathrm{Ext}_{\mathcal{B}}(L_2, L_2) = 0$, $L_1 \cong L_2$ as S-modules, $\|L_2\| \leq \|L_1\| < \|M\|$, and $b\|L_1\|$ and $b\|L_2\|$ are integers.

Applying idempotent deletion we can assume that the vector $\underline{\ell}(M)$ has only positive coordinates.

Now we have two cases:

Case 1. We use the regularization functor $F_r : \mathcal{A}_r\text{-Mod} \to \mathcal{A}\text{-Mod}$ of Proposition 4.3. If $L_1, L_2 \in \mathcal{A}_r$-mod are such that $F_r(L_1) \cong M$ and $F_r(L_2) \cong N$ then, by [4, Proposition 11.5], we can apply the induction hypothesis to L_1 and L_2, so that $M \cong N$ in $\mathcal{A}$-Mod.

Case 2. We can choose a direct R-R-summand W_0' of W_0 such that $\delta(W_0') = 0$, and central orthogonal primitive idempotents of R, e_{i_0}, e_{j_0}, such that W_0' is a simple $e_{j_0}R$-$e_{i_0}R$-bimodule.

Let $\Gamma = T_R(W_0') = R_0 \times T_{Re_{i_0} \times Re_{j_0}}(W_0') = R_0 \times \Gamma_0$. By [4, Proposition 9.5] there is an epimorphism $\pi_X : \mathrm{Ext}_{\mathcal{A}}(X, X) \to \mathrm{Ext}_{\Gamma}(X, X)$ for each $X \in \mathcal{A}$-Mod, so $\mathrm{Ext}_{\Gamma_0}((e_{i_0} + e_{j_0})M, (e_{i_0} + e_{j_0})M) = 0$ and $\mathrm{Ext}_{\Gamma_0}((e_{i_0} + e_{j_0})N, (e_{i_0} + e_{j_0})N) = 0$. It follows that $i_0 \neq j_0$, i.e., W_0' is not a loop (see, for example, [8, Lemma 6.3]). By Lemma 5.2 we have $(e_{i_0} + e_{j_0})M \cong (e_{i_0} + e_{j_0})N$ as Γ_0-modules, and so $M \cong N$ as Γ-modules. It follows, for $L_1, L_2 \in \mathcal{A}^X$-Mod such that $F^X(L_1) \cong M$ and $F^X(L_2) \cong N$, that $L_1 \cong L_2$ as S-modules, where (S, W^X) is the layer of $\mathcal{A}^X$. By [4, Lemma 10.5] we have $\mathrm{Ext}_{\mathcal{A}^X}(L_1, L_1) = 0$ and $\mathrm{Ext}_{\mathcal{A}^X}(L_2, L_2) = 0$. By Lemma 4.12 we get $\|L_1\|, \|L_2\| < \|M\|$. By the identity $\mathrm{Ext}_{\Gamma}(M, M) = 0$ we see that any indecomposable direct summand H of M has trivial group of self-extensions, i.e. H is a preprojective or a preinjective Γ-module, thus D_H is isomorphic to some D_i, where $R \cong D_1 \times \cdots \times D_n$; then, by Remark 4.11, we deduce that $b\|L_1\|$ and $b\|L_2\|$ are integers. The induction hypothesis implies $L_1 \cong L_2$ in $\mathcal{A}^X$-Mod and so $M \cong N$ in $\mathcal{A}$-Mod.

Claims (2) and (3) can be proved in a similar way: first we observe that the identity $\underline{\ell}(M) = \underline{\ell}(N)$ (resp. c-$\underline{\ell}(M) = c$-$\underline{\ell}(N)$) implies that $\|M\| = 0$ if and only if $\|N\| = 0$, and so $M \cong N$ in $\mathcal{A}$-mod in the first stage of induction.

In the induction step everything works in the same way when using the regularization functor.

When we deal with the F^X functor and consider M and N as Γ-modules, we observe that $\underline{\ell}(M) = \underline{\ell}(N)$ (resp. c-$\underline{\ell}(M) = c$-$\underline{\ell}(N)$) implies that $q(e_{i_0} + e_{j_0}M) = q(e_{i_0} + e_{j_0}M)$ and then, by Lemma 5.2(4), there are integers a and a' such that $\bigoplus_{u=1}^{a}(1 - e_{i_0} - e_{j_0})M \cong \bigoplus_{v=1}^{a'}(1 - e_{i_0} - e_{j_0})N$.

Notice that the vector $\underline{\ell}(M)$ is obtained by dividing the k-dimension of e_iM by the k-dimension of D_M for each i.

By the previous two paragraphs we get $\bigoplus_{u=1}^{a} M \cong \bigoplus_{v=1}^{a'} N$ as Γ-modules.

Furthermore, it follows that $\dim_k(D_N)/\dim_k(D_M) = a/a'$ (resp. $(\dim_k(D_N)c_M)/(\dim_k(D_M)c_N) = a/a'$): since F^X is a full faithful functor, we can deduce that $\dim_k(D_{L_2})/\dim_k(D_{L_1}) = a/a'$ (resp. $(\dim_k(D_{L_2})c_{L_1})/(\dim_k(D_{L_1})c_{L_2}) = a/a'$).

The isomorphism $\bigoplus_{u=1}^{a} M \cong \bigoplus_{v=1}^{a'} N$ as Γ-modules implies $\bigoplus_{u=1}^{a} L_1 \cong \bigoplus_{v=1}^{a'} L_2$ as S-modules; then, using the previous identities, we conclude that $\underline{\ell}(L_1) = \underline{\ell}(L_2)$ (resp. c-$\underline{\ell}(L_1) = c$-$\underline{\ell}(L_2)$). ∎

6. One-parameter families

DEFINITION 6.1. Let $\mathcal{A}$ be an almost admissible ditalgebra. A generic module $G \in \mathcal{A}$-Mod (resp. $G \in \Lambda$-Mod) is *algebraically rigid* if for any algebraic field extension L/k the Λ^L-module G^L is generic. We say that the generic module G is *algebraically bounded* if there exists a finite field extension L/k and a natural number t such that $G^L \cong G_1 \oplus \cdots \oplus G_t$, where G_i is algebraically rigid for $i \in \{1, \ldots, t\}$.

For the next result we recall that for a connected f.d. hereditary algebra Γ there is a bilinear form defined on the Grothendieck group which induces a quadratic form q_Γ (see [26, pp. 269 and 270], and also [12, 8.3]).

PROPOSITION 6.2. *Let Γ be a k-triangular matrix algebra. If Γ has a positive semidefinite quadratic form then the only generic Γ-module, up to isomorphism, is algebraically bounded.*

Proof. The existence and uniqueness, up to isomorphism, of the generic Γ-module G, is known: see [27] and [28].

Also it it is known that the generic module associated to some of the Euclidean diagrams, namely $\widetilde{A}_n$, $\widetilde{D}_n$, $\widetilde{E}_6$, $\widetilde{E}_7$ and $\widetilde{E}_8$, has a ring of endomorphisms isomorphic to $k(x)$.

Then, applying [22, Lemma 3.2(a), Lemma 4.1 and Theorem 4.3] we deduce that there exists a finite field extension F of k such that G^F is a direct sum of a finite number of algebraically rigid Λ^F-modules, all of them with endolength invariant under base field extension, and so G is algebraically bounded. ∎

Definition 6.3. Let $\mathcal{A}$ be an almost admissible ditalgebra and let $\mathcal{H}_1$ and $\mathcal{H}_2$ be full subcategories of $\mathcal{A}$-mod. We say that $\mathcal{H}_1$ *covers* $\mathcal{H}_2$ if all but a finite number of iso-classes of objects of $\mathcal{H}_2$ intersect $\mathcal{H}_1$.

Remark 6.4. Let us consider the tensor k-algebra $A = T_D(B)$, where D is a finite-dimensional k-algebra and a division ring, and B is a D-D-bimodule such that it is simple as a left D-module. If we identify B with D as left D-modules then there exists an isomorphism of (unitary) rings and a k-linear transformation $\tau : D \to D$ determined by the identity $1_D \cdot d = \tau(d)$, where $d \in D$ and $- \cdot d$ denotes the right action of D over B. Then A is isomorphic to the twisted polynomial ring $D[x;\tau]$. We will call the k-algebra A a *twisted tensor algebra.* Also there is a functor H : $D[x;\tau]$-Mod $\to \Gamma$-Mod which is full and faithful, where $\Gamma = \left(\begin{smallmatrix} D & 0 \\ D\oplus B & D \end{smallmatrix}\right)$ (see [26] and [27]; in [14] there are detailed arguments and computations related to the previous facts). Then H sends indecomposable f.g. A-modules to regular Γ-modules, and the image of the set of simple $D[x;\tau]$-modules covers the subcategory of quasi-simple Γ-modules.

By the previous remark, Proposition 6.2 and Theorem 3.2 we have the next claim.

Corollary 6.5. *A twisted tensor algebra is of infinite representation type, c-unbounded and c-strongly unbounded. Moreover, it has one generic module up to isomorphism, and it is algebraically bounded.*

Remark 6.6. The proof of Proposition 6.2 shows that k-triangular matrix algebras of tame representation type and twisted tensor algebras are semigenerically tame in the sense of [25].

For a ring A we denote by A_s the localization of A at the element s. We are going to localize at central elements.

Remark 6.7. Let Γ be a k-triangular algebra of tame representation type (and infinite representation type) or a twisted tensor algebra. It is known that up to isomorphism there is a unique generic Γ-module G, and also an associated Γ-O-bimodule B, such that B is finitely generated as an O-module and O is a bounded principal ideal domain ([15] and [10]). Also $B \otimes_O -$ is full and faithful. Moreover, for each height h, the subcategory of regular modules with height $\leq h$ is covered by the Γ-modules of the form $B \otimes_O N$, where N is an indecomposable in O-mod of length bounded from above by dh, with d fixed (use [10, Proposition 5.2(3)]). Also, for Q the skew ring of fractions of O, we have $B \otimes_O Q \cong G$ (see [27], [28] and [10]).

Definition 6.8. Consider the context of Remark 6.7. We will call Q a *tame division ring*. Let w be an element in the center of O; we say that O_w is a *tame PID*.

Proposition 6.9. *Let B be an associated Γ-O-bimodule as in Remark 6.7. Then there exists an element w in the center of O such that c-$\underline{\ell}(G) = c$-$\underline{\ell}(B \otimes_O L)$, where G is a generic Γ-module and L is any simple O_w-module.*

Proof. Let $1_R = e_1 + \cdots + e_n$ be a decomposition into centrally primitive orthogonal idempotents.

Since O is bounded, for Z the center of O and K the classical field of fractions of Z, we get $Q \cong O \otimes_Z K$.

Since $e_i B$ is finitely generated over O for $i \in \{1, \ldots, n\}$, and O is a bounded PID, there exists an element $w_0 \in Z$ such that $e_i B \otimes_O O_{w_0}$ is free and finitely generated over O_{w_0} for each i.

Since G is a generic module of a k-triangular matrix algebra or a twisted tensor algebra, there are integers m_i such that $e_i B \otimes_O Q \cong Q^{m_i}$ as right Q-modules for each i, so the rank of $e_i B \otimes_O O_{w_0}$ over O_{w_0} is m_i.

By [27, p. 560] there exists an element w_1 in the Formanek center of O_{w_0} such that $O_{w_0 w_1}$ is an Azumaya algebra over $Z_{w_0 w_1}$. Then $w_1 \in Z_{w_0}$ (see [29, Definition 6.1.14 and p. 446]) and $O_{w_0 w_1}$ is free over $Z_{w_0 w_1}$.

It is known (see for example [29, Proposition 1.10.12]) that the center of $O_{w_0 w_1}$ is $Z_{w_0 w_1}$, and that the center of Q is K; hence the rank of $O_{w_0 w_1}$ over $Z_{w_0 w_1}$ is equal to $\dim_K(Q) = c^2$, where c is a positive integer.

Observe that c-$\underline{\ell}(G) = c(m_1, \ldots, m_n)$.

Now $O_{w_0 w_1}$ is an Azumaya algebra and so there is a bijective correspondence between the ideals of $Z_{w_0 w_1}$ and the ideals of $O_{w_0 w_1}$, given by sending I to $I O_{w_0 w_1}$, with inverse sending J to $J \cap Z_{w_0 w_1}$ (see [29, Corollary 5.3.25]). It follows that for each maximal ideal $\mathfrak{m}$ of $O_{w_0 w_1}$ we have a canonical isomorphism $O_{w_0 w_1}/\mathfrak{m} \cong O_{w_0 w_1} \otimes_{Z_{w_0 w_1}} (Z_{w_0 w_1}/\mathfrak{m} \cap Z_{w_0 w_1})$, and so $\dim_{Z_{w_0 w_1}/\mathfrak{m} \cap Z_{w_0 w_1}}(O_{w_0 w_1}/\mathfrak{m}) = c^2$.

For L a simple $O_{w_0 w_1}$-module with annihilator $\mathfrak{m}$, we have $O_{w_0 w_1}/\mathfrak{m} \cong M_a(D)$. Let $\dim_{Z(D)}(D) = b^2$ where $Z(D)$ is the center of D; by the previous paragraph we get $c^2 = a^2 b^2$.

We observe that $\operatorname{endol}(L) = a$; then, by [7, Lemma 31.4] and the fact that $B \otimes_O -$ is full and faithful, we find for $L' = B \otimes_O O_{w_0 w_1} \otimes_{O_{w_0 w_1}} L$ that $\underline{\ell}(L') = a(m_1, \ldots, m_n)$ and so c-$\underline{\ell}(L') = ba(m_1, \ldots, m_n) = c$-$\underline{\ell}(G)$. ∎

Notation. We recall that there is a canonical embedding $L_{\mathcal{A}}$: A-Mod $\to$ $\mathcal{A}$-Mod (see the beginning of Section 2 and [7, Remark 2.5]). We say that B is an *$\mathcal{A}$-O-bimodule* if B is an A-O-bimodule, and for any O-module N we will denote the $\mathcal{A}$-module $L_{\mathcal{A}}(B \otimes_O N)$ just by $B \otimes_O N$.

DEFINITION 6.10. Let $\mathcal{C}$ and $\mathcal{D}$ be additive k-categories and $H : \mathcal{C} \to \mathcal{D}$ a k-functor (see [3, p. 28]). We say that H is *sharp* if it preserves indecomposability and reflects isomorphism classes, and if for any indecomposable $M \in \mathcal{C}$ we have $H(\operatorname{rad} E_M) \subset \operatorname{rad} E_{H(M)}$ and the induced morphism of k-algebras $E_M/\operatorname{rad} E_M \to E_{H(M)}/\operatorname{rad} E_{H(M)}$ is a bijection.

DEFINITION 6.11. Consider an almost admissible ditalgebra $\mathcal{A}$ (resp. Λ). Let O be a tame PID and B a $\mathcal{A}$-O-bimodule (resp. Λ-O-bimodule) such that it is free and finitely generated as O-module and $B \otimes_O -$ is a sharp functor. Then we say that B is a *parametric module*. Let $\{S_i\}_{i\in I}$ be a complete list of representatives of isomorphism classes of simple O-modules. We say that $\{B \otimes_O S_i\}_{i\in I}$ is a *one-parameter family*.

LEMMA 6.12. *Let $\mathcal{A}$ be an almost admissible ditalgebra and $F^z : \mathcal{A}^z$-Mod $\to \mathcal{A}$-Mod a reduction functor as in Proposition* 4.3. *If B is a parametric module in $\mathcal{A}^z$ then $F^z(B)$ is a parametric module in $\mathcal{A}$.*

Proof. The claim is immediate for idempotent deletion and regularization. For F^X we can use [4, Corollary 5.4] to prove that $F^X(B)$ is an $\mathcal{A}$-O-bimodule; the rest follows by the properties of this functor. ■

THEOREM 6.13. *Let*

$$\mathcal{A}_t\text{-Mod} \xrightarrow{F_t} \mathcal{A}_{t-1}\text{-Mod} \xrightarrow{F_{t-1}} \cdots \xrightarrow{F_2} \mathcal{A}_1\text{-Mod} \xrightarrow{F_1} \mathcal{A}\text{-Mod}$$

be a sequence of reductions like those of Proposition 4.3, *where $\mathcal{A}$ is an almost admissible ditalgebra, and $\mathcal{A}_t = (T, \delta)$ is an admisible ditalgebra with layer $(S, W_0 \oplus W_1)$, such that $T_S(W_0) = \Gamma$ is a k-triangular matrix algebra of tame representation type or a twisted tensor algebra, and $\delta(W_0) = 0$. Let $H = F_1 \ldots F_t$. Let B be an associated Γ-O-bimodule with O a tame PID and Q the corresponding tame division ring. There exists an element w in the center of O such that c-$\underline{\ell}(H(B \otimes_O Q)) = c$-$\underline{\ell}(H(B \otimes_O S))$ for any simple O_w-module S. Also, for the generic $\mathcal{A}$-module $G = H(B \otimes_O Q)$, we have $E_G = D \oplus \operatorname{rad} E_G$, where $D \cong Q$.*

Proof. B is a parametric module by [25, Lemma 4.4].

By Lemma 6.12 we see that $H(B)$ is a parametric module.

By Propositions 4.3 and 6.9 we get the identity for central endolength vectors.

It is known, for $G' = B \otimes_O Q$, that $E_{G'} = D' \oplus \operatorname{rad} E_{G'}$, where $D \cong Q$; the reduction functors under consideration are full and faithful and so the final part of the claim follows. ■

DEFINITION 6.14. Let $\mathcal{A}$ be an almost admissible ditalgebra and $\underline{d}$ a central endolength vector. We say that $\underline{d}$ is *minimal of infinite representation type* (m.i.r.t.) if c-ind $\mathcal{A}(\underline{d})$ is of infinite representation type and if $\underline{d}' < \underline{d}$ then c-ind $\mathcal{A}(\underline{d}')$ is of finite representation type.

THEOREM 6.15. *Let $\mathcal{A}$ be an almost admissible ditalgebra such that $\mathcal{A}^K$ is tame, and let $\underline{d}$ be a central endolength vector m.i.r.t. Then c-$\operatorname{ind}\mathcal{A}(\underline{d})$ is covered by the union of a finite number of one-parameter families, and this union is covered by c-$\operatorname{ind}\mathcal{A}(\underline{d})$. With each of those one-parameter families there is associated a generic module G with endomorphism ring E_G, where $E_G = D_G \oplus \operatorname{rad} E_G$ and D_G is a tame division ring.*

Proof. We can repeat part of the argument for Proposition 4.14 in order to see that in some step of the reduction process we have to use a functor F^X, where the initial subalgebra associated with Γ is of infinite representation type: otherwise c-$\operatorname{ind}\mathcal{A}(\underline{d})$ would be finite.

So, using Remark 4.6, let $F^X : \mathcal{A}_1^X$-Mod $\to \mathcal{A}_1$-Mod where c-$\operatorname{ind}\mathcal{A}(\underline{d}_1)$ is m.i.r.t., and $F : \mathcal{A}_1$-Mod $\to \mathcal{A}$-Mod a composition of reduction functors such that $F(c\text{-}\operatorname{ind}\mathcal{A}(\underline{d}_1)) \subset c\text{-}\operatorname{ind}\mathcal{A}(\underline{d})$.

Thanks to idempotent deletion we can assume that $\underline{d}_1$ has no zero coordinates.

Let $\Gamma = T_R(W_0') = R_0 \times \Gamma_0$ be as in Case 2 of the proof of Theorem 5.3. We see that Γ_0 is not wild because $\mathcal{A}^K$ is not wild, so $\operatorname{ind}\mathcal{A}_1$ is generically tame, and the last admissible ditalgebra is not generically trivial because c-$\operatorname{ind}\mathcal{A}(\underline{d}_1)$ is infinite. Then Γ_0 is a skew tensor algebra ($i_0 = j_0$) or it is a k-triangular matrix algebra of tame representation type and infinite representation type.

We can assume that there is at least one $M \in c$-$\operatorname{ind}\mathcal{A}(\underline{d}_1)$ such that $(e_{i_0} + e_{j_0})M$ in the case $i_0 \neq j_0$, or $e_{i_0}M$ if $i_0 = j_0$, is not annihilated by the bimodule B: otherwise we just reduce the norm and keep going forward with the reduction functors.

Then $(e_{i_0} + e_{j_0})M$ is a regular module of height one (resp. $e_{i_0}M$ is a simple module) and so, for an infinite number of isomorphism classes N of quasi-simple modules in Γ_0 (simple modules if Γ is a skew tensor algebra) we have c-$\underline{\ell}(N) \leq c$-$\underline{\ell}(M)$; consequently, $\Gamma = \Gamma_0$ and c-$\operatorname{ind}\mathcal{A}(\underline{d}_1)$ is covered by the full subcategory of quasi-simple (or simple) Γ-modules. Thus c-$\operatorname{ind}\mathcal{A}(\underline{d}_1)$ is a one-parameter family.

Then, by Lemma 6.12 and Theorem 6.13, c-$\operatorname{ind}\mathcal{A}(\underline{d})$ is covered by the union of a finite number of one-parameter families, and to each of these families there is associated a generic module as described in the statement.

By Theorem 6.13 the union of the one-parameter families is covered by c-$\operatorname{ind}\mathcal{A}(\underline{d})$. ■

We think that the previous result should be true for a more general case, as suggested by the following corollary, based on [21] and [22].

COROLLARY 6.16. *If Λ^K is tame then, for any natural number d, all but a finite number of isomorphism classes of indecomposable finite-dimensional Λ-modules of central endolength equal to d lie in homogeneous tubes.*

Proof. By [9, Corollary E], it is known that almost all the isomorphism classes of indecomposable finite-dimensional Λ^K-modules of dimension d lie in homogeneous tubes.

By the proof of [21, Proposition 4.13], for $L \in \Lambda^K$-mod indecomposable with $\dim_K(L) = d$, there exists $M \in \Lambda$-mod such that L is a direct summand of M^K. By Theorem 2.13 we have c-endol$(M) = d$.

Now we can repeat the remaining part of the proof of [22, Corollary 5.3] and, applying [21, Theorem 3.8 and Proposition 4.2], verify the statement. ∎

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