COLLOQUIUM MATHEMATICUM

VOL. 136 2014 NO. 1

UNIQUENESS OF DECOMPOSITION OF PSEUDO-RIEMANNIAN SUPERALGEBRAS

BY

KELI ZHENG (Changchun and Harbin), LIANGYUN CHEN (Changchun) and YONGZHENG ZHANG (Changchun)

Abstract. This paper is primarily concerned with pseudo-Riemannian superalgebras, which are superalgebras endowed with pseudo-Riemannian non-degenerate supersymmetric consistent bilinear forms. Decompositions of pseudo-Riemannian superalgebras whose left centers are isotropic and whose left centers are not isotropic are investigated.

1. Introduction. In the last few decades one of the most active and fertile subjects in algebra is the recently developed theory of graded algebras and so called superalgebras. The prefix *super*- comes from the theory of supersymmetry in theoretical physics [BPZ, F]. Superalgebras and their representations (supermodules) provide an algebraic framework for supersymmetry [DJ, K, V]. Sometimes the study of such objects is called *superlinear algebra* [VS]. In the related field of supergeometry, superalgebras also play an important role. For example, they enter the definitions of graded manifolds, supermanifolds and superschemes [Ma].

In mathematics and theoretical physics, a *superalgebra* S is a $\mathbb{Z}_2$-graded algebra [KMZ]. This means that there exists a direct sum decomposition $S = S_{\bar{0}} \oplus S_{\bar{1}}$ together with a bilinear multiplication $S \times S \to S$ such that $S_\alpha S_\beta \subseteq S_{\alpha+\beta}$, where the subscripts belong to $\mathbb{Z}_2$ and $\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$ is the residue class ring modulo 2. Elements of S_α are said to be *homogeneous*, and the *degree* of a homogeneous element x is $\bar{0}$ or $\bar{1}$ according to whether it is in $S_{\bar{0}}$ or $S_{\bar{1}}$. Elements of degree $\bar{0}$ are said to be *even*, and those of degree $\bar{1}$ are *odd*. If x and y are two homogeneous elements of S, then so is their product xy.

The motivation for studying pseudo-Riemannian algebras comes from the study of Lie groups with left-invariant pseudo-metrics [AM, Mi]. In some sense pseudo-Riemannian algebras are related to pseudo-Riemannian connections, which are pseudo-metric connections such that the torsion is

2010 *Mathematics Subject Classification*: Primary 17A30; Secondary 17A70, 16W55, 17D99.

Key words and phrases: pseudo-Riemannian superalgebra, indecomposable ideal, orthogonal decomposition, isometry.

DOI: 10.4064/cm136-1-10

zero and parallel translation preserves the bilinear form on the tangent spaces [S]. Recently, certain classes of pseudo-Riemannian algebras have been investigated. In [CZ1], Chen and Zhu found that there is a remarkable geometry on pseudo-Riemannian Novikov algebras, and studied a special class of pseudo-Riemannian Novikov algebras. They also proved in [CZ2] that the underlying Lie algebras of pseudo-Riemannian associative fermionic Novikov algebras are 2-step nilpotent and that pseudo-Riemannian associative fermionic Novikov algebras are 3-step nilpotent. Pseudo-Riemannian bilinear forms and pseudo-Riemannian Leibniz algebras, i.e., Leibniz algebras with pseudo-Riemannian non-degenerate symmetric bilinear forms, were considered by Lin and Chen [LC]. More generally, algebras with pseudo-Riemannian non-degenerate symmetric bilinear forms were investigated in [CLZ].

The purpose of this paper is to study pairs (S, λ), where $\lambda : S \times S \to \mathbb{C}$ is a non-degenerate supersymmetric bilinear form on the superalgebra S such that

$$\lambda(xy, z) + (-1)^{\alpha\beta}\lambda(y, xz) = 0 \tag{1.1}$$

for all $x \in S_\alpha$, $y \in S_\beta$ and $z \in S$ and

$$\lambda(x, y) = 0 \quad \text{for all } x \in S_{\bar{0}},\ y \in S_{\bar{1}},$$

where $\alpha, \beta \in \mathbb{Z}_2$. Such pairs will be called *pseudo-Riemannian superalgebras*. In particular, if S is a Lie superalgebra, then the pseudo-Riemannian superalgebra (S, λ) is in fact a quadratic Lie superalgebra (see e.g. [ABB, BB, ZM] and the references therein). Therefore, pseudo-Riemannian superalgebras are natural generalizations of quadratic Lie superalgebras.

The aim of this paper is to generalize some of the beautiful results proved in [CLZ, ZM]. A brief summary of the relevant concepts and propositions on superalgebras and bilinear forms is presented in Section 2. Section 3 shows some properties of pseudo-Riemannian superalgebras whose left centers are not isotropic. In Section 4, we consider pseudo-Riemannian superalgebras whose left centers are isotropic and determine that the orthogonal decomposition of any pseudo-Riemannian superalgebra into indecomposable non-degenerate ideals is unique up to an isometry if the left centers equal the centers. In particular, we establish conditions allowing us to construct pseudo-Riemannian superalgebras from certain other pseudo-Riemannian superalgebras.

Throughout this paper, all superalgebras are assumed to be finite-dimensional over the complex number field $\mathbb{C}$ and all subspaces of superalgebras are assumed to be homogeneous. Without explicit mention, α and β will always be assumed to be elements of $\mathbb{Z}_2$.

2. Preliminaries. In this section, some necessary definitions and propositions will be given. Let S be a superalgebra with a bilinear multiplication $S \times S \to S$ denoted by $(x, y) \mapsto xy$.

The following notation will be used. Let $I^\perp$ denote the subspace of S orthogonal to I with respect to a bilinear form $\lambda\colon S \times S \to \mathbb{C}$, i.e.,

$$I^\perp = \{x \in S \mid \lambda(x, y) = 0 \text{ for any } y \in S\}.$$

Denote by $LC(S)$ the *left center* of S, i.e.,

$$LC(S) = \{x \in S \mid yx = 0 \text{ for any } y \in S\},$$

and by $Z(S)$ the *center* of S, i.e.,

$$Z(S) = \{x \in S \mid xy = yx = 0 \text{ for any } y \in S\}.$$

Definition 2.1. A subspace $I = I_{\bar{0}} \oplus I_{\bar{1}}$ of S is called a *left* (resp. *right*) *ideal* of S if $SI \subseteq I$ (resp. $IS \subseteq I$), where $I_\alpha = I \cap S_\alpha$. If I is both a left and a right ideal, then I is an *ideal*. The superalgebra S is called *abelian* if $S \neq 0$ and $xy = 0$ for all $x, y \in S$.

Definition 2.2. Let S be a superalgebra over $\mathbb{C}$.

(a) A bilinear form $\lambda\colon S \times S \to \mathbb{C}$ is *pseudo-Riemannian* if
$$\lambda(xy, z) + (-1)^{\alpha\beta}\lambda(y, xz) = 0$$
for all $x \in S_\alpha$, $y \in S_\beta$ and $z \in S$.

(b) A bilinear form $\lambda\colon S \times S \to \mathbb{C}$ is *supersymmetric* if $\lambda(x, y) = (-1)^{\alpha\beta}\lambda(y, x)$ for all $x \in S_\alpha$ and $y \in S_\beta$, and *consistent* if $\lambda(x, y) = 0$ for all $x \in S_{\bar{0}}$ and $y \in S_{\bar{1}}$.

(c) The pair (S, λ) is a *pseudo-Riemannian superalgebra* if $\dim_{\mathbb{C}} S$ is finite and λ is a pseudo-Riemannian non-degenerate supersymmetric consistent bilinear form on S.

(d) The left center $LC(S)$ of a pseudo-Riemannian superalgebra (S, λ) is *isotropic* if $LC(S) \subseteq LC(S)^\perp$.

From now on we shall consider only consistent bilinear forms.

Proposition 2.3. *The pair (S, λ) is a pseudo-Riemannian superalgebra if and only if $(S_{\bar{0}}, \lambda_0)$ is a pseudo-Riemannian superalgebra and there exists a skew-symmetric non-degenerate bilinear form λ_1 on the $S_{\bar{0}}$-module $S_{\bar{1}}$ such that*

$$\lambda_0(yx, z) = \lambda_1(x, yz) \qquad \text{for any } x, y \in S_{\bar{1}},\ z \in S_{\bar{0}}, \tag{2.1}$$

$$\lambda_1(yx, z) = -\lambda_1(x, yz) \qquad \text{for any } x, z \in S_{\bar{1}},\ y \in S_{\bar{0}}. \tag{2.2}$$

Proof. Since λ is consistent, it is easy to see that the restriction of λ to $S_{\bar{0}}$ is a pseudo-Riemannian non-degenerate symmetric bilinear form, as also is the restriction of λ to $S_{\bar{1}}$. Clearly, (2.1) and (2.2) follow from λ being pseudo-Riemannian.

Conversely, let $(S_{\bar{0}}, \lambda_0)$ be a pseudo-Riemannian superalgebra and λ_1 a non-degenerate skew-symmetric bilinear form on $S_{\bar{1}}$ such that (2.1) and (2.2) are satisfied. Consider the form λ defined on S by

$$\lambda(S_{\bar{0}}, S_{\bar{1}}) = 0 = \lambda(S_{\bar{1}}, S_{\bar{0}}),$$
$$\lambda(x, y) = \lambda_{\bar{0}}(x, y) \text{ for } x, y \in S_{\bar{0}}, \qquad \lambda(x, y) = \lambda_{\bar{1}}(x, y) \text{ for } x, y \in S_{\bar{1}}.$$

Then it is easy to see that λ is a pseudo-Riemannian non-degenerate supersymmetric consistent bilinear form on S. Therefore, (S, λ) is a pseudo-Riemannian superalgebra. ∎

DEFINITION 2.4. Let (S, λ) be a pseudo-Riemannian superalgebra. If there exist non-trivial and non-degenerate ideals I_1 and I_2 such that $S = I_1 \oplus I_2$, then (S, λ) is called *decomposable*; otherwise it is *indecomposable*. Furthermore, if $\lambda(I_1, I_2) = 0$, then the decomposition $S = I_1 \oplus I_2$ is said to be *orthogonal*.

DEFINITION 2.5. Let (S, λ) be a pseudo-Riemannian superalgebra. An automorphism π of S is called an *isometry* if π preserves the bilinear form, i.e., $\lambda(\pi(x), \pi(y)) = \lambda(x, y)$ for any $x, y \in S$.

PROPOSITION 2.6. *If (S, λ) is a pseudo-Riemannian superalgebra, then $LC(S) = (SS)^\perp$. As a consequence,* $\dim_{\mathbb{C}} LC(S) + \dim_{\mathbb{C}} SS = \dim_{\mathbb{C}} S$.

Proof. Assume that $x \in LC(S) \cap S_\alpha$, i.e., $yx = 0$ for any $y \in S_\beta$. Then $\lambda(yx, z) = 0$ for any $z \in S$. It follows that $-(-1)^{\alpha\beta}\lambda(x, yz) = 0$, so $LC(S) \subseteq (SS)^\perp$.

Conversely, assume that $x \in (SS)^\perp$ and $d(x) = \alpha \in \mathbb{Z}_2$, i.e., $\lambda(x, yz) = 0$ for any $y \in S_\beta$ and $z \in S$. It follows that $-(-1)^{\alpha\beta}\lambda(yx, z) = 0$ since λ is a pseudo-Riemannian bilinear form. By the non-degeneracy of λ, we have $yx = 0$, so $x \in LC(S)$. Therefore, $(SS)^\perp \subseteq LC(S)$. ∎

PROPOSITION 2.7. *Suppose that (S, λ) is a pseudo-Riemannian superalgebra and I is an ideal of S. Then $I^\perp$ is a left ideal and $II^\perp = 0$.*

Proof. Assume that x is an arbitrary homogeneous non-zero element of the ideal $I \cap S_\alpha$. Then $\lambda(x, yz) = -(-1)^{\alpha\beta}\lambda(yx, z) = 0$ for any $y \in S_\beta$ and $z \in I^\perp$. It follows that $yz = 0$, i.e., $I^\perp$ is a left ideal of S and $II^\perp = 0$. ∎

PROPOSITION 2.8. *Suppose that (S, λ) is a pseudo-Riemannian superalgebra. Then there exists a decomposition $S = \bigoplus_{i=1}^{l} S_i$ into indecomposable non-degenerate ideals.*

Proof. We shall use induction on $\dim_{\mathbb{C}} S$. If S is one-dimensional, then the result is clear. The inductive hypothesis gives a decomposition of S into indecomposable non-degenerate ideals for $\dim_{\mathbb{C}} S < n$. If S is n-dimensional, then let I be a non-degenerate proper super-subalgebra of S, i.e., $\dim_{\mathbb{C}} I < \dim_{\mathbb{C}} S$. It is clear that $I \cap I^\perp = 0$. Then $S = I \oplus I^\perp$. From the inductive hypothesis and since $\dim_{\mathbb{C}} I^{\perp} < \dim_{\mathbb{C}} S$, we conclude that I and $I^{\perp}$ can be decomposed into indecomposable non-degenerate ideals. This completes the proof. ■

PROPOSITION 2.9. *Let S be an abelian superalgebra. If λ is a non-degenerate supersymmetric bilinear form on S, then (S, λ) is a pseudo-Riemannian superalgebra. Furthermore, there exists an orthogonal decomposition $S = S_1 \oplus \cdots \oplus S_n$ into indecomposable non-degenerate ideals such that* $\dim_{\mathbb{C}} S_i = 1$, $1 \leq i \leq n$.

Proof. Since S is abelian, we know that any super-subspace is an ideal. Note that $\lambda(xy, z) = 0 = (-1)^{\alpha\beta}\lambda(y, xz)$ for any $x \in S_\alpha$, $y \in S_\beta$ and $z \in S$. Then $\lambda(xy, z) - (-1)^{\alpha\beta}\lambda(y, xz) = 0$, i.e., (S, λ) is a pseudo-Riemannian superalgebra. Every element of the standard orthogonal basis of S generates an indecomposable non-degenerate one-dimensional ideal S_i, $1 \leq i \leq n$, such that the decomposition $S = S_1 \oplus \cdots \oplus S_n$ is orthogonal. ■

3. Pseudo-Riemannian superalgebras whose left centers are not isotropic. First of all, we give an example of a pseudo-Riemannian superalgebra whose left center is not isotropic.

EXAMPLE 3.1. Let (H, λ_H) be an abelian pseudo-Riemannian superalgebra and (J, λ_J) be a pseudo-Riemannian superalgebra with product $\circ$. Set

$$so(J) = \{A \in \mathrm{End}_\theta(J) \mid \lambda_J(A(x), y) + (-1)^{\alpha\theta}\lambda_J(x, A(y)) = 0\},$$

where $x \in J_\alpha$, $y \in J$ and $\mathrm{End}_\theta(J) = \{f \in \mathrm{End}(J) \mid f(J_\alpha) \subseteq J_{\theta+\alpha}$ for $\theta \in \mathbb{Z}_2\}$. Given an even linear mapping $L\colon H \to so(J)$ denoted by $x \mapsto L_x$, define a product $*$ on a vector superspace $S = H + J$ (direct sum as super-subspace) by

$$\begin{aligned} x * y &= 0 && \text{for any } x, y \in H, \\ x * y &= 0 && \text{for any } x \in J,\ y \in H, \\ x * y &= x \circ y && \text{for any } x, y \in J, \\ x * y &= L_x(y) && \text{for any } x \in H,\ y \in J. \end{aligned}$$

and define a supersymmetric bilinear form λ on S by

$$\begin{aligned} \lambda(x, y) &= \lambda_H(x, y) && \text{for any } x, y \in H, \\ \lambda(x, y) &= \lambda_I(x, y) && \text{for any } x, y \in J, \\ \lambda(x, y) &= 0 && \text{for any } x \in H,\ y \in J. \end{aligned}$$

An easy verification shows that (S, λ) is a pseudo-Riemannian superalgebra whose left center H is not isotropic.

Next, we give some theorems on pseudo-Riemannian superalgebras whose left centers are not isotropic.

THEOREM 3.2. *Let (S,λ) be a pseudo-Riemannian superalgebra whose left center is not isotropic. Then there exists a sequence $S = S_0 \supset S_1 \supset \cdots \supset S_n$ of non-degenerate supersubalgebras of S such that S_i is an ideal of S_{i-1}, the quotient superalgebra S_{i-1}/S_i is abelian for each $i \in \{1,\dots,n\}$, and the left center of S_n is isotropic.*

Proof. Since the left center $LC(S)$ is not isotropic, there exists a maximal subspace H_1 of $LC(S)$ such that $\lambda|_{H_1\times H_1}$ is non-degenerate. Then $H_1 \cap H_1^{\perp} = 0$. Hence $S = H_1 \oplus H_1^{\perp}$ and the restrictions of λ to H_1 and $H_1^{\perp}$ are non-degenerate. Let $S_1 = H_1^{\perp}$. For $a \in S_\alpha$, $h \in (H_1)_\beta$ and $h' \in (S_1)$, we have $\lambda(h, ah') = -(-1)^{\alpha\beta}\lambda(ah, h') = 0$. Hence $ah' \in H_1^{\perp} = S_1$ and so S_1 is a left ideal of S. For $a \in S_\alpha$, $h \in (H_1)_\beta$ and $h'' \in (S_1)$, we have

$$\lambda(h, h''a) = -(-1)^{\alpha\beta}\lambda(h''h, a) = -(-1)^{\alpha\beta}\lambda(0, a) = 0.$$

Thus $h''a \in H_1^{\perp} = S_1$ and so S_1 is a right ideal of S. Therefore, S_1 is an ideal of S. Using the above methods, we can show that S_i is an ideal of S_{i-1} for every $i \in \{1,\dots,n\}$ and the left center of S_n is isotropic.

Next we prove that the quotient superalgebra S_{i-1}/S_i is abelian for each $i \in \{1,\dots,n\}$. The equality $\lambda(xy, h) = -(-1)^{\alpha\beta}\lambda(y, xh) = 0$ for any $x \in (S_{i-1})_\alpha$, $y \in (S_{i-1})_\beta$ and $h \in H_i$ shows that $S_{i-1}S_{i-1} \subseteq H_i^{\perp} = S_i$. Hence $(S_{i-1}/S_i)(S_{i-1}/S_i) = S_{i-1}S_{i-1}/S_i = 0$. ∎

THEOREM 3.3. *Let (S,λ) be a pseudo-Riemannian superalgebra whose left center equals its center. If the left center is not isotropic, then there exist non-degenerate ideals S_1 and S_2 of S such that $S = S_1 \oplus S_2$, where $\lambda(S_1, S_2) = 0$, $S_1S_1 = 0$ and the left center of S_2 is isotropic.*

Proof. According to Theorem 3.2, we may assume that $S_2 = S_n$. Then the left center of S_2 is isotropic and S/S_2 is abelian. Hence $S_1S_1 = 0$. Since $S_1 = S_2^{\perp}$ and $S_2 = S_1^{\perp}$, we have $\lambda(S_1, S_2) = 0$. ∎

Hereafter, we write $L(e_1,\dots,e_i)$ for the subspace spanned by $e_1,\dots,e_i$.

THEOREM 3.4. *Let (S,λ) be a pseudo-Riemannian superalgebra whose left center equals its center and is not isotropic. If the decomposition $S = S_1 \oplus S_2$ is orthogonal such that S_1 and S_2 are non-degenerate, $LC(S_1)$ is isotropic and $S_2 \in LC(S)$, then the decomposition is unique up to an isometry.*

Proof. Let $S = S_1' \oplus S_2'$ be another such decomposition. Then

$$S_1S_1 = S_1S_1' = S_1'S_1 = S_1'S_1'.$$

It follows that $LC(S_1) \subseteq LC(S_1)^{\perp} = S_1S_1 = S_1'S_1'$ since the left center of S_1 is isotropic and by Proposition 2.6. Since $LC(S) = Z(S)$, we have

$LC(S_1) \subseteq LC(S) \cap S_1'S_1' = LC(S_1')$. Similarly, $LC(S_1') \subseteq LC(S_1)$. That is, $LC(S_1) = LC(S_1')$. By Proposition 2.6, we have $\dim_{\mathbb{C}} S_1 = \dim_{\mathbb{C}} S_1'$. Hence also $\dim_{\mathbb{C}} S_2 = \dim_{\mathbb{C}} S_2'$. Let $\{e_1, \dots, e_{m+k}, u_1, \dots, u_{n+l}\}$ be a basis of S_1 such that $(S_1)_{\bar{0}} = L(e_1, \dots, e_{m+k})$, $(S_1)_{\bar{1}} = L(u_1, \dots, u_{n+l})$, $(LC(S_1))_{\bar{0}} = L(e_1, \dots, e_k)$, $(LC(S_1))_{\bar{1}} = L(u_1, \dots, u_l)$, $(S_1S_1)_{\bar{0}} = L(e_1, \dots, e_m)$, $(S_1S_1)_{\bar{1}} = L(u_1, \dots, u_n)$ and

$$\begin{aligned}
\lambda(e_i, e_j) &= \delta_{ij}, && k+1 \le i, j \le m,\\
\lambda(u_g, u_h) &= \delta_{gh}, && l+1 \le g, h \le n,\\
\lambda(e_i, e_{m+j}) &= \delta_{ij}, && 1 \le i, j \le k,\\
\lambda(u_g, u_{n+h}) &= \delta_{gh}, && 1 \le g, h \le l,\\
\lambda(e_i, e_j) &= 0, && 1 \le i, j \le k,\\
\lambda(u_g, u_h) &= 0, && 1 \le g, h \le l,\\
\lambda(e_i, e_j) &= 0, && m+1 \le i, j \le m+k,\\
\lambda(u_g, u_h) &= 0, && n+1 \le g, h \le n+l,
\end{aligned}$$

where δ_{st} is Kronecker's delta.

Now we consider the projections

$$\pi_1 \colon S_1 \to S_1', \quad \pi_2 \colon S_2 \to S_2'$$

which are isomorphisms. It is clear that $\pi_1|_{S_1S_1} = \mathrm{id}$. Then $\lambda(\pi_1(e_i), \pi_1(e_j)) = \lambda(e_i, e_j)$ and $\lambda(\pi_1(u_g), \pi_1(u_h)) = \lambda(u_g, u_h)$ for $1 \le i \le m+k$, $1 \le j \le m$, $1 \le g \le n+l$, $1 \le h \le n$.

Assume that $e_p = e_{p_3} + e_{p_4}$ for $m+1 \le p \le m+k$ and $u_a = u_{a_3} + u_{a_4}$ for $n+1 \le a \le n+l$, where $e_{p_3} \in (S_1')_{\bar{0}}$, $e_{p_4} \in (S_2')_{\bar{0}}$, $u_{a_3} \in (S_1')_{\bar{1}}$, $u_{a_4} \in (S_2')_{\bar{1}}$. For $m+1 \le q \le m+k$ and $n+1 \le b \le n+l$, we have

$$\begin{aligned}
0 = \lambda(e_p, e_q) = \lambda(e_{p_3}, e_{q_3}) + \lambda(e_{p_4}, e_{q_4}),\\
0 = \lambda(u_a, u_b) = \lambda(u_{a_3}, u_{b_3}) + \lambda(u_{a_4}, u_{b_4}).
\end{aligned}$$

Let $c_{pq} = \lambda(e_{p_4}, e_{q_4})$ for $p \ne q$, $2c_{pp} = \lambda(e_{p_4}, e_{p_4})$, $d_{ab} = \lambda(u_{a_4}, u_{b_4})$ for $a \ne b$, $2d_{aa} = \lambda(u_{a_4}, u_{a_4})$ and

$$e'_{p_3} = e_{p_3} + \sum_{g=p}^{m+k} c_{pg} e_{g-m}, \qquad u'_{a_3} = e_{a_3} + \sum_{h=p}^{n+l} c_{ah} e_{h-n}.$$

It is easy to see that

$$\begin{aligned}
\lambda(e'_{p_3}, e'_{p_3}) &= \lambda\Big(e_{p_3} + \sum_{g=p}^{m+k} c_{pg} e_{g-m}, e_{p_3} + \sum_{g=p}^{m+k} c_{pg} e_{g-m}\Big)\\
&= \lambda(e_{p_3}, e_{p_3}) + 2\lambda\Big(e_{p_3}, \sum_{g=p}^{m+k} c_{pg} e_{g-m}\Big)\\
&= \lambda(e_{p_3}, e_{p_3}) + 2c_{pp} = 0, \quad m+1 \le p \le m+k,
\end{aligned}$$

$$\begin{aligned}
\lambda(u'_{a_3}, u'_{a_3}) &= \lambda\Big(u_{a_3} + \sum_{h=a}^{n+l} c_{ah} u_{h-n}, u_{a_3} + \sum_{h=a}^{n+l} c_{ah} u_{h-n}\Big) \\
&= \lambda(u_{a_3}, u_{a_3}) + 2\lambda\Big(u_{a_3}, \sum_{h=a}^{n+l} c_{ah} u_{h-n}\Big) \\
&= \lambda(u_{a_3}, u_{a_3}) + 2d_{aa} = 0, \quad n+1 \le a \le n+l, \\
\lambda(e'_{p_3}, e'_{q_3}) &= \lambda(e_{p_3}, e_{q_3}) + \lambda\Big(e_{p_3}, \sum_{s=q}^{m+k} c_{qs} e_{s-m}\Big) + \lambda\Big(e_{q_3}, \sum_{t=p}^{m+k} c_{pt} e_{t-m}\Big) \\
&= \lambda(e_{p_3}, e_{q_3}) + c_{pq} = 0, \quad m+1 \le p < q \le m+k, \\
\lambda(u'_{p_3}, u'_{q_3}) &= \lambda(u_{a_3}, u_{b_3}) + \lambda\Big(u_{a_3}, \sum_{s=b}^{n+l} c_{bs} u_{g-n}\Big) + \lambda\Big(u_{b_3}, \sum_{t=a}^{n+l} c_{at} u_{t-n}\Big) \\
&= \lambda(u_{a_3}, u_{b_3}) + d_{ab} = 0, \quad n+1 \le a < b \le n+l.
\end{aligned}$$

Define $\pi'_1 \colon S_1 \to S'_1$ by

$$\begin{aligned}
&\pi'_1(e_j) = e_j, \quad 1 \le j \le m, && \pi'_1(u_k) = u_k, \quad 1 \le k \le n, \\
&\pi'_1(e_j) = e'_{j_3}, \quad m+1 \le j \le m+k, && \pi'_1(u_k) = u'_{k_3}, \quad n+1 \le k \le n+l.
\end{aligned}$$

An easy verification shows that π'_1 is also an isomorphism from S_1 to S'_1 and preserves the bilinear form. Then $\pi = (\pi'_1, \pi_2)$ is an isometry of S. ∎

4. Pseudo-Riemannian superalgebras whose left centers are isotropic. Theorem 3.2 shows that left centers of pseudo-Riemannian superalgebras play an important role. In this section we obtain some results on pseudo-Riemannian superalgebras whose left centers are isotropic.

PROPOSITION 4.1. *Suppose that (S, λ) is a pseudo-Riemannian superalgebra whose left center is isotropic. Then (S, λ) is decomposable if and only if there exist non-trivial ideals I_1 and I_2 of S such that $S = I_1 \oplus I_2$.*

Proof. The necessity is obvious. Conversely, assume that there exist non-trivial homogeneous ideals I_1 and I_2 of S such that $S = I_1 \oplus I_2$. It is enough to show that $\lambda|_{I_1 \times I_1}$ and $\lambda|_{I_2 \times I_2}$ are non-degenerate. Assume that $\lambda|_{I_1 \times I_1}$ is degenerate. Then there exists a non-zero element x of I_1 such that $\lambda(x, I_1) = 0$. If $x \in I_1 I_1$, then $\lambda(x, I_2) \subseteq \lambda(I_1 I_1, I_2) = \lambda(I_1, I_1 I_2)$ (or $-\lambda(I_1, I_1 I_2)) = 0$. It follows that $x = 0$ as λ is non-degenerate. This contradicts the assumption $x \neq 0$, and therefore $x \notin I_1 I_1$. By Proposition 2.6 and since $LC(S)$ is isotropic, we have $LC(S) \subseteq LC(S)^\perp = SS$. Thus $x \notin LC(I_1)$, i.e., there exists $y \in I$ such that $yx \neq 0$. Since λ is non-degenerate, there exists a homogeneous element z of S such that $\lambda(yx, z) \neq 0$. Thus $\lambda(x, yz) = -(-1)^{\alpha\beta} \lambda(yx, z) \neq 0$ for any $x \in (I_1)_\alpha$, $y \in (I_1)_\beta$. It follows that

$yz \in I_1$ because I_1 is an ideal of S and $y \in I_1$. This contradicts the choice of x, i.e., $\lambda|_{I_1 \times I_1}$ is non-degenerate. Similarly, $\lambda|_{I_2 \times I_2}$ is non-degenerate. ∎

In the following, we show that the decomposition of a pseudo-Riemannian superalgebra whose left center is isotropic into non-degenerate indecomposable ideals is unique up to an automorphism.

Let (S, λ) be a pseudo-Riemannian superalgebra whose left center is isotropic. Suppose that

$$S = S_1 \oplus \cdots \oplus S_n \quad \text{and} \quad S = S_1' \oplus \cdots \oplus S_m',$$

where S_i, S_j, $1 \le i \le n$, $1 \le j \le m$, are indecomposable non-degenerate ideals of S.

Note that $S_1 S_1 \neq 0$. Indeed, assume that $S_1 S_1 = 0$. Thus $S_1 \subseteq LC(S)$, which contradicts that $LC(S)$ is isotropic. Since $S_1 S_1 = S_1 S = \bigoplus_{j=1}^{m} S_1 S_j'$, we have $S_1 S_j' \neq 0$ for some j. Without loss of generality, we assume that $S_1 S_1' \neq 0$. Let $H_1 = \bigoplus_{i=2}^{n} S_i$ and $H_1' = \bigoplus_{j=2}^{m} S_j'$, which are non-degenerate ideals of S by Proposition 4.1.

Lemma 4.2. *$S_1 \cap H_1' = 0$ and $S_1' \cap H_1 = 0$.*

Proof. Let $B_1 = S_1 \cap S_1'$ and $B_2 = S_1 \cap H_1'$. Clearly, $S_1 S_1 = S_1 S = S_1 S_1' \oplus S_1 H_1' \subseteq S_1 \cap S_1' \oplus S_1 \cap H_1' = B_1 \oplus B_2$.

(1) If $S_1 = B_1 \oplus B_2$, then both B_1 and B_2 are non-degenerate ideals of S_1, hence also of S. Since S_1 is indecomposable and $B_1 \neq 0$, we have $B_2 = 0$, i.e., $S_1 \cap H_1' = 0$.

(2) If $S_1 \neq B_1 \oplus B_2$, then there exists $x \in S_1$ such that $x \notin B_1 \oplus B_2$. Therefore, $x = x_1 + x_2$, where $x_1 \in S_1'$ and $x_2 \in H_1'$. Using the other decomposition, we have

$$x_1 = x_1^1 + x_1^2 \quad \text{and} \quad x_2 = x_2^1 + x_2^2,$$

where $x_1^1, x_1^2 \in S_1$ and $x_2^1, x_2^2 \in H_1$. So $x = x_1^1 + x_1^2 + x_2^1 + x_2^2$. Then $x = x_1^1 + x_2^1$ and $x_1^2 + x_2^2 = 0$. An easy verification shows that

$$S_1 x_1^1 \subseteq S_1 S_1', \quad x_1^1 S_1 \subseteq S_1' S_1,$$
$$S_1 x_2^1 \subseteq S_1 H_1', \quad x_2^1 S_1 \subseteq H_1' S_1.$$

If $x_1^1 \notin B_1 \oplus B_2$, then we can let $B_1^{(1)} = B_1 + \mathbb{C} x_1^1$ and $B_2^{(1)} = B_2$.

If $x_1^1 \in B_1 \oplus B_2$, then $x_2^1 \notin B_1 \oplus B_2$. Let $B_1^{(1)} = B_1$ and $B_2^{(1)} = B_2 + \mathbb{C} x_2^1$. One may easily verify that both $B_1^{(1)}$ and $B_2^{(1)}$ are ideals of S_1 satisfying $B_1^{(1)} \cap B_2^{(1)} = 0$. If $S_1 = B_1^{(1)} \oplus B_2^{(1)}$, then $B_2^{(1)} = 0$ by a similar argument to that in (1). Accordingly, $S_1 \cap H_1' = 0$.

If $S_1 \neq B_1^{(1)} \oplus B_2^{(1)}$, then we may repeat the discussion in (2). Since $\dim_{\mathbb{C}} S_1 < \infty$, we may choose $B_1^{(k)}$ and $B_2^{(k)}$ such that $S_1 = B_1^{(k)} \oplus B_2^{(k)}$. Similarly, we can show that both $B_1^{(k)}$ and $B_2^{(k)}$ are non-degenerate ideals of S_1. Then we can obtain $B_2^{(k)} = 0$ by a similar method to (1). As a consequence, $S_1 \cap H_1' = 0$. Similarly, $S_1' \cap H_1 = 0$. ∎

LEMMA 4.3. *Let $\pi_1\colon S_1 \to S_1'$ be the restriction of the projection $\pi\colon S \to S_1'$ to the ideal S_1 of S. Then π_1 is an isomorphism and preserves the bilinear form.*

Proof. Since $\ker \pi_1 \subseteq S_1 \cap H_1'$, Lemma 4.2 shows that π_1 is injective. Then $\dim_{\mathbb{C}} S_1 \leq \dim_{\mathbb{C}} S_1'$. Similarly, $\dim_{\mathbb{C}} S_1' \leq \dim_{\mathbb{C}} S_1$. Therefore, $\dim_{\mathbb{C}} S_1 = \dim_{\mathbb{C}} S_1'$. It is clear that $\pi_1(xy) = \pi_1(x)\pi_1(y)$ for any $x, y \in S_1$. Then π_1 is an isomorphism from S_1 to S_1'. Assume that $x = x_1 + x_2$, where $x \in S_1$, $x_1 \in S_1'$ and $x_2 \in H_1'$. Clearly, we have $S_1' x_2 = 0$ and $H_1' x_2 = H_1'(x - x_1) = H_1' x \subseteq H_1' \cap S_1 = 0$. Thus $x_2 \in LC(S)$. Therefore, $\lambda(x, x) = \lambda(x_1, x_1)$ or $\lambda(x, x) = \lambda(x_1, x_1) + 2\lambda(x_1, x_2)$. Let $x_1 = h_1 + h_2$, where $h_1 \in H_1$ and $h_2 \in H_1^{\perp}$. It follows that $h_1 \in LC(H_1) \subseteq LC(S)$, because $H_1 h_1 = H_1(x_1 - h_2) = 0$. Then we have

$$\lambda(x_1, x_2) = \lambda(h_1 + h_2, x_2) = \lambda(h_1, x_2) + \lambda(h_2, x_2) = 0.$$

Therefore, $\lambda(x, x) = \lambda(x_1, x_1) = \lambda(\pi_1(x), \pi_1(x))$, i.e., π_1 preserves the bilinear form. ∎

Furthermore, we have

$$S_1 S_1 = S_1 S_1' = S_1' S_1 = S_1' S_1',$$
$$S_1 H_1' = H_1' S_1 = S_1' H_1 = H_1 S_1' = 0.$$

Applying the above method for $j = 2, \ldots, n$, we can obtain the following theorem.

THEOREM 4.4. *Let (S, λ) be a pseudo-Riemannian superalgebra whose left center is isotropic. Suppose that $S = S_1 \oplus \cdots \oplus S_n$ and $S = S_1' \oplus \cdots \oplus S_m'$, where S_i, S_j', $1 \leq i \leq n$, $1 \leq j \leq m$, are indecomposable non-degenerate ideals of S. Then:*

(1) *$n = m$.*
(2) *Changing the subscripts if necessary, we can obtain*

$$\dim_{\mathbb{C}} S_j = \dim_{\mathbb{C}} S_j',$$
$$S_j S_j = S_j S_j' = S_j' S_j = S_j' S_j',$$
$$S_j S_k' = S_j' S_k = 0, \quad j \neq k.$$

(3) *The projections $\pi_i\colon S_i \to S_i'$, $1 \leq i \leq n$, are isomorphisms and preserve the bilinear form. Furthermore, $\pi = (\pi_1, \ldots, \pi_n)$ is an isometry of S.*

PROPOSITION 4.5. *Let (S, λ) be a decomposable pseudo-Riemannian superalgebra whose left center equals its center. If the left center is isotropic,* *then there exist non-degenerate ideals S_1 and S_2 of S such that the decomposition $S = S_1 \oplus S_2$ is orthogonal.*

Proof. Since (S, λ) is decomposable, we have $S = S_1 \oplus S_2$, where $\lambda|_{S_1 \times S_1}$ and $\lambda|_{S_2 \times S_2}$ are non-degenerate. Therefore, $S = S_1 \oplus S_1^\perp$ and $S_1 S_1^\perp = 0$. If x is an arbitrary element of $S_1^\perp$, then $x = x_{\bar{0}} + x_{\bar{1}} = (x_1)_{\bar{0}} + (x_1)_{\bar{1}} + (x_2)_{\bar{0}} + (x_2)_{\bar{1}}$, where $(x_1)_{\bar{0}} \in (S_1)_{\bar{0}}$, $(x_1)_{\bar{1}} \in (S_1)_{\bar{1}}$, $(x_2)_{\bar{0}} \in (S_2)_{\bar{0}}$, $(x_2)_{\bar{1}} \in (S_2)_{\bar{1}}$. Since both $S_1 = (S_1)_{\bar{0}} \oplus (S_1)_{\bar{1}}$ and $S_2 = (S_2)_{\bar{0}} \oplus (S_2)_{\bar{1}}$ are ideals, we have, for any $y \in (S_1)_\alpha$, $z \in S_1$,

$$\begin{aligned}
\lambda(yx_1, z) &= \lambda(y((x_1)_{\bar{0}} + (x_1)_{\bar{1}}), z) = \lambda(y(x_1)_{\bar{0}}, z) + \lambda(y(x_1)_{\bar{1}}, z) \\
&= -\lambda((x_1)_{\bar{0}}, yz) - (-1)^\alpha \lambda((x_1)_{\bar{1}}, yz) \\
&= \lambda(-x_{\bar{0}} + (x_2)_{\bar{0}}, yz) + (-1)^\alpha \lambda(-x_{\bar{1}} + (x_2)_{\bar{1}}, yz) \\
&= \lambda((x_2)_{\bar{0}}, yz) + (-1)^\alpha \lambda((x_2)_{\bar{1}}, yz) \\
&= -\lambda(y(x_2)_{\bar{0}}, z) - \lambda(y(x_2)_{\bar{1}}, z) \\
&= -\lambda(y((x_2)_{\bar{0}} + (x_2)_{\bar{1}}), z) = 0.
\end{aligned}$$

As $\lambda|_{S_1 \times S_1}$ is non-degenerate, we obtain $S_1 x_1 = 0$. Then $x_1 = (x_1)_{\bar{0}} + (x_1)_{\bar{1}} \in LC(S) = Z(S)$. It follows that $xy = ((x_1)_{\bar{0}} + (x_1)_{\bar{1}} + (x_2)_{\bar{0}} + (x_2)_{\bar{1}})y = 0$ for any $y \in S_1$. Therefore, $S_1^\perp S_1 = 0$, i.e., $S_1^\perp$ is an ideal of S. Since S is uniquely decomposable, we have $S_2 = S_1^\perp$. Similarly, $S_1 = S_2^\perp$. ■

Similarly to Theorem 4.4 and Proposition 4.5, we have the following results.

THEOREM 4.6. *Let (S, λ) be a pseudo-Riemannian superalgebra whose left center equals its center and is isotropic. Suppose that $S = S_1 \oplus \cdots \oplus S_n$ and $S = S'_1 \oplus \cdots \oplus S'_m$ are orthogonal decompositions of S, where S_i, S'_j, $1 \le i \le n$, $1 \le j \le m$, are indecomposable non-degenerate ideals of S. Then:*

(1) *$n = m$.*
(2) *Changing the subscripts if necessary, we can obtain*

$$\begin{gathered}
\dim_{\mathbb{C}} S_j = \dim_{\mathbb{C}} S'_j, \\
S_j S_j = S_j S'_j = S'_j S_j = S'_j S'_j, \\
S_j S'_k = S'_j S_k = 0, \quad j \neq k.
\end{gathered}$$

(3) *The projections $\pi_i \colon S_i \to S'_i$, $1 \le i \le n$, are isomorphisms and preserve the bilinear form; moreover, $\pi = (\pi_1, \ldots, \pi_n)$ is an automorphism of S, i.e., the decomposition is unique up to an isometry.*

By Theorems 3.4 and 4.6, we can also obtain the following theorem.

THEOREM 4.7. *Let (S, λ) be a pseudo-Riemannian superalgebra whose left center equals its center. Then the orthogonal decomposition of S into indecomposable non-degenerate ideals is unique up to an isometry.*

Next we show how to construct a new one from two pseudo-Riemannian superalgebras whose left centers are isotropic.

THEOREM 4.8. *Let (S_1, λ_1) and (S_2, λ_2) be pseudo-Riemannian superalgebras whose left centers are isotropic, $S = S_1 \oplus S_2$ and λ a supersymmetric bilinear form on S such that $\lambda|_{S_1\times S_1} = \lambda_1$ and $\lambda|_{S_2\times S_2} = \lambda_2$. If*

$$\lambda(S_1S_1, S_2) = \lambda(S_2S_2, S_1) = 0,$$

then (S, λ) is a pseudo-Riemannian superalgebra whose left center is isotropic.

Proof. By Proposition 2.6 and since the left center of S is isotropic, we have $LC(S_1) \subseteq LC(S_1)^\perp = S_1S_1$. Notice that $\lambda_1(x, y) = 0$ for any $x \in S_{\bar{0}}$, $y \in S_{\bar{1}}$. Thus there exist a basis

$$\{e_1, \ldots, e_{i_1}, e_{i_1+1}, \ldots, e_{m_1}, e_{m_1+1}, \ldots, e_{m_1+i_1}\}$$

of $(S_1)_{\bar{0}}$ and a basis

$$\{u_1, \ldots, u_{i_2}, u_{i_2+1}, \ldots, u_{m_2}, u_{m_2+1}, \ldots, u_{m_2+i_2}\}$$

of $(S_1)_{\bar{1}}$ such that

$$\begin{aligned}
\lambda_1(e_i, e_j) &= \delta_{ij}, & & i_1+1 \le i, j \le m_1,\\
\lambda_1(u_l, u_k) &= \delta_{lk}, & & i_2+1 \le l, k \le m_2,\\
\lambda_1(e_i, e_{m_1+j}) &= \delta_{ij}, & & 1 \le i, j \le i_1,\\
\lambda_1(u_l, u_{m_2+k}) &= \delta_{lk}, & & 1 \le l, k \le i_2,\\
\lambda_1(e_i, e_j) &= 0, & & 1 \le i, j \le i_1,\\
\lambda_1(u_l, u_k) &= 0, & & 1 \le l, k \le i_2,\\
\lambda_1(e_i, e_j) &= 0, & & m_1+1 \le i, j \le m_1+i_1,\\
\lambda_1(u_l, u_k) &= 0, & & m_2+1 \le l, k \le m_2+i_2,
\end{aligned}$$

where

$$\begin{aligned}
LC(S_1)_{\bar{0}} &= L(e_1, \ldots, e_{i_1}), & LC(S_1)_{\bar{1}} &= L(u_1, \ldots, u_{i_2}),\\
(S_1S_1)_{\bar{0}} &= L(e_1, \ldots, e_{m_1}), & (S_1S_1)_{\bar{1}} &= L(u_1, \ldots, u_{m_2}).
\end{aligned}$$

Similarly, there exist a basis

$$\{h_1, \ldots, h_{i_3}, h_{i_3+1}, \ldots, h_{n_1}, h_{n_1+1}, \ldots, h_{n_1+i_3}\}$$

of $(S_2)_{\bar{0}}$ and a basis

$$\{v_1, \ldots, v_{i_4}, v_{i_4+1}, \ldots, v_{n_2}, v_{n_2+1}, \ldots, v_{n_2+i_4}\}$$

of $(S_2)_{\bar{1}}$ such that

$$\begin{aligned}
\lambda_2(h_i, h_j) &= \delta_{ij}, && i_3+1 \le i, j \le n_1,\\
\lambda_2(v_l, v_k) &= \delta_{lk}, && i_4+1 \le l, k \le n_2,\\
\lambda_2(h_i, h_{m_1+j}) &= \delta_{ij}, && 1 \le i, j \le i_3,\\
\lambda_2(v_l, v_{m_2+k}) &= \delta_{lk}, && 1 \le l, k \le i_4,\\
\lambda_2(h_i, h_j) &= 0, && 1 \le i, j \le i_3,\\
\lambda_2(v_l, v_k) &= 0, && 1 \le l, k \le i_4,\\
\lambda_2(h_i, h_j) &= 0, && n_1+1 \le i, j \le n_1+i_3,\\
\lambda_2(v_l, v_k) &= 0, && n_2+1 \le l, k \le n_2+i_4,
\end{aligned}$$

where

$$\begin{aligned}
LC(S_2)_{\bar 0} &= L(h_{n_1+1}, \dots, h_{n_1+i_3}), & LC(S_2)_{\bar 1} &= L(v_{n_2+1}, \dots, v_{n_2+i_4}),\\
(S_2S_2)_{\bar 0} &= L(h_{i_3+1}, \dots, h_{n_1+i_3}), & (S_2S_2)_{\bar 1} &= L(v_{i_4+1}, \dots, v_{n_2+i_4}).
\end{aligned}$$

Since $\lambda(S_1S_1, S_2) = \lambda(S_2S_2, S_1) = 0$ and λ is consistent, the matrix of λ with respect to the basis

$$\{e_1, \dots, e_{m_1+i_1}, h_1, \dots, h_{n_1+i_3}, u_1, \dots, u_{m_2+i_2}, v_1, \dots, v_{n_2+i_4}\}$$

of S is

$$G = \begin{pmatrix}
0 & 0 & \mathrm{I}_{i_1} & & & & & & & & & \\
0 & \mathrm{I}_{m_1-i_1} & 0 & & & & & & & & & \\
\mathrm{I}_{i_1} & 0 & 0 & F & & & & & & & & \\
 & & F' & 0 & 0 & \mathrm{I}_{i_3} & & & & & & \\
 & & & 0 & \mathrm{I}_{m_2-i_2} & 0 & & & & & & \\
 & & & \mathrm{I}_{i_3} & 0 & 0 & & & & & & \\
 & & & & & & 0 & 0 & \mathrm{I}_{i_2} & & & \\
 & & & & & & 0 & \mathrm{I}_{n_1-i_3} & 0 & & & \\
 & & & & & & \mathrm{I}_{i_2} & 0 & 0 & H & & \\
 & & & & & & & & H' & 0 & 0 & \mathrm{I}_{i_4}\\
 & & & & & & & & & 0 & \mathrm{I}_{n_2-i_4} & 0\\
 & & & & & & & & & \mathrm{I}_{i_4} & 0 & 0
\end{pmatrix},$$

where F and H are arbitrary matrices over $\mathbb{C}$, and F' and H' denote the transposes of F and H, respectively. An easy verification shows that $\det G \neq 0$. Thus λ is a non-degenerate supersymmetric bilinear form such that the identity (1.1) is satisfied. Hence (S, λ) is a pseudo-Riemannian superalgebra whose left center is isotropic. ∎

Remark 4.9. Let (S, λ) be a pseudo-Riemannian superalgebra. If $S = S_1 \oplus S_2$, then it is easy to see that $\lambda(S_1S_1, S_2) = \lambda(S_2S_2, S_1) = 0$.

Remark 4.10. Let the notations be as above. If $LC(S_1) = 0$ or $LC(S_2) = 0$, then the decomposition $S = S_1 \oplus S_2$ is orthogonal.

Proof. If $LC(S_1) = 0$, then $S_1 = S_1S_1$. It follows that

$$\lambda(S_1, S_2) = \lambda(S_1S_1, S_2) = \lambda(S_1, S_1S_2) \text{ (or } -\lambda(S_1, S_1S_2)) = 0.$$

Similarly, the result follows for $LC(S_2) = 0$. ∎

Acknowledgements. This research was partly supported by the National Natural Science Foundation of China (Grant No. 11171055), Natural Science Foundation of Jilin province (No. 201115006, 20130101068), Scientific Research Foundation for Returning Scholars of Ministry of Education of China and the Fundamental Research Funds for the Central Universities (No. 12SSXT139).

The authors thank Professors Yang Jiang and Ruimei Gao for their helpful comments and suggestions. We also give our special thanks to the referees for many helpful suggestions.

REFERENCES

[ABB] H. Albuquerque, E. Barreiro and S. Benayadi, *Quadratic Lie superalgebras with a reductive even part*, J. Pure Appl. Algebra 213 (2009), 724–731.

[AM] A. Aubert et A. Medina, *Groupes de Lie pseudo-riemanniens plats*, Tohoku Math. J. (2) 55 (2003), 487–506.

[BB] H. Benamer and S. Benayadi, *Double extension of quadratic Lie superalgebras*, Comm. Algebra 27 (1999), 67–88.

[BPZ] A. Belavin, A. Polyakov and A. Zamolodchikov, *Infinite conformal symmetry in two-dimensional quantum field theory*, Nuclear Phys. B 241 (1984), 333–380.

[CLZ] Z. Chen, K. Liang and F. Zhu, *Algebras with pseudo-Riemannian bilinear forms*, J. Korean Math. Soc. 48 (2011), 1–12.

[CZ1] Z. Chen and F. Zhu, *Pseudo-Riemannian Novikov algebras*, J. Phys. A 41 (2008), 1–9.

[CZ2] Z. Chen and F. Zhu, *A note on pseudo-Riemannian associative fermionic Novikov algebras*, Bull. Korean Math. Soc. 49 (2012), 353–358.

[DJ] P. Deligne and J. Morgan, *Notes on supersymmetry (following Joseph Bernstein)*, in: Quantum Fields and Strings: A Course for Mathematicians, Vol. 1, Amer. Math. Soc., New York, 1999, 41–97.

[F] P. Freund, *Introduction to Supersymmetry*, Cambridge Monogr. Math. Phys., Cambridge Univ. Press, Cambridge, 1986.

[K] V. Kac, *Lie superalgebras*, Adv. Math. 26 (1977), 8–96.

[KMZ] V. Kac, C. Martinez and E. Zelmanov, *Graded simple Jordan superalgebras of growth one*, Mem. Amer. Math. Soc. 150 (2001), no. 711, x+140 pp.

[LC] J. Lin and Z. Chen, *Leibniz algebras with pseudo-Riemannian bilinear forms*, Front. Math. China 5 (2010), 103–115.

[Ma] Yu. Manin, *Gauge Field Theory and Complex Geometry*, 2nd ed., Grundlehren Math. Wiss. 289, Springer, Berlin, 1997.

[Mi] J. Milnor, *Curvatures of left invariant metrics on Lie groups*, Adv. Math. 21 (1976), 293–329.

[S] A. Sagle, *Nonassociative algebras and Lagrangian mechanics on homogeneous spaces*, Algebras Groups Geom. 2 (1985), 478–494.

[V] V. Varadarajan, *Supersymmetry for Mathematicians: An Introduction*, Courant Lecture Notes in Math. 11, Amer. Math. Soc., New York, 2004.

[VS] W. Vasantha Kandasamy and F. Smarandache, *Super Linear Algebra*, InfoLearnQuest, Ann Arbor, MI, 2008.

[ZM] L. Zhu and D. Meng, *Uniqueness of the decomposition of Lie superalgebras and quadratic Lie superalgebras*, Sci. China Ser. A 45 (2002), 273–279.

Keli Zheng
School of Mathematics and Statistics
Northeast Normal University
130024 Changchun, P.R. China
and
Department of Mathematics
Northeast Forestry University
150040 Harbin, P.R. China
E-mail: zhengkl561@nenu.edu.cn

Liangyun Chen (corresponding author),
Yongzheng Zhang
School of Mathematics and Statistics
Northeast Normal University
130024 Changchun, P.R. China
E-mail: chenly640@nenu.edu.cn
zhyz@nenu.edu.cn

Received 17 March 2014;
revised 6 May 2014 (6200)