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RATNER'S PROPERTY FOR SPECIAL FLOWS OVER IRRATIONAL ROTATIONS UNDER FUNCTIONS OF BOUNDED VARIATION. II

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Abstract. We consider special flows over the rotation on the circle by an irrational α under roof functions of bounded variation. The roof functions, in the Lebesgue decomposition, are assumed to have a continuous singular part coming from a quasi-similar Cantor set (including the devil's staircase case). Moreover, a finite number of discontinuities is allowed. Assuming that α has bounded partial quotients, we prove that all such flows are weakly mixing and enjoy the weak Ratner property. Moreover, we provide a sufficient condition on the roof function for stability of Ratner's cocycle property of the resulting special flow.

1. Introduction. The paper is a continuation of [7], that is, we will continue the study of Ratner's property (originally, H-property [13]) in the class of special flows $\mathcal{T}^f := (T_t^f)_{t\in\mathbb{R}}$ determined by the rotation $Tx = x+\alpha$, on the additive circle $\mathbb{T}$, by an irrational number α with bounded partial quotients and with f a function of bounded variation. In [7], in order to study Ratner's property, we used the Lebesgue decomposition

$$f = f_{\mathrm{j}} + f_{\mathrm{a}} + f_{\mathrm{s}} + \tilde{S}\{\cdot\}, \tag{1}$$

where f_{j} is the jump function (with countably many jumps d_i), f_{a} is absolutely continuous on $\mathbb{T}$, f_{s} is singular and continuous on $[0,1]$, $\tilde{S} := \int_{\mathbb{T}} f'\,d\lambda$ (λ denotes Lebesgue measure on $\mathbb{T}$) and $\{t\}$ stands for the fractional part of $t\in\mathbb{R}$. In [7], we assumed that

$$f_{\mathrm{s}} = 0 \quad\text{and}\quad \tilde{S}\neq 0 \tag{2}$$

(in that case $\tilde{S} = \sum_{i=1}^{\infty} d_i$). We recall that some particular cases of (2) have already been studied in [2, 4]. In [7], under the assumptions (2), we proved the weak mixing property of $\mathcal{T}^f$ and, under an additional condition concerning the rate of convergence of the series of jumps, the so called weak Ratner property (WR-property from now on) [1]. We also proved that (2)

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[1] The WR-property has the same dynamical consequences as the original H-property of Ratner (see Section 3).

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together with a condition on the set of jumps yields a stability result for the WR-property in the class of roof functions studied in [7] (for sufficiently small bounded variation perturbations satisfying (2)).

The situation becomes more complicated when $f_{\mathrm{s}} \neq 0$. We have already noticed in [7] that, in view of [6, 14], it may happen that for an irrational rotation there exists $f = f_{\mathrm{s}}$ with $f_{\mathrm{s}}(1) - f_{\mathrm{s}}(0) \neq 0$ such that the corresponding special flow $\mathcal{T}^f$ is isomorphic to the suspension flow, i.e. a special flow with constant roof function (in particular such a special flow is not even weakly mixing).

In the present paper, we always assume $f_{\mathrm{s}} \neq 0$ and focus on the following two problems:

1. WR-property and weak mixing for functions of bounded variation with singular part of devil's staircase type.
2. The stability of the WR-property under arbitrary (sufficiently small) bounded variation perturbations, answering a question of M. Lemańczyk [11].

We now pass to the description of the results.

We deal with the case when f_{s} is naturally defined by a Cantor set. More precisely, we consider quasi-similar Cantor sets. Each such set is obtained in the following way. We are given two bounded sequences $(m_i)_{i\geq 1}$, $(k_i)_{i\geq 1}$ of natural numbers satisfying ([2])

$$2 \leq m_i \leq [k_i/2], \quad i \geq 1.$$

First, divide $[0,1)$ into k_1 clopen (here by clopen we mean left-closed, right-open) subintervals of equal length and choose m_1 of them (including the first one and the last one) so that the closures (in $[0,1]$) of any two of them are disjoint. Then set $A_1 \subset [0,1]$ to be the union of the closures of the selected m_1 intervals. In the next step, we divide each of the previously selected intervals into k_2 clopen subintervals of equal length and we select m_2 subintervals (including the first one and the last one in each previously chosen subinterval), with the same configuration choice in each selected subinterval, so that the closures of any two of them are disjoint. We then set A_2 to be the union of the closures of all selected subintervals of step two. Clearly $A_2 \subset A_1$. Proceeding in the same manner, we obtain a sequence $A_1 \supset A_2 \supset \cdots$ of closed subsets of $[0,1]$ and we define the corresponding *quasi-similar* Cantor set $\mathcal{C}$ as $\mathcal{C} := \bigcap_{i=1}^{\infty} A_i$. Of course, it has Lebesgue measure 0. As in the classical case, there is a canonically associated continuous non-decreasing function $f = f(\mathcal{C}) : [0,1] \to \mathbb{R}$ with $f(0) = 0$, $f(1) = 1$ and $f' = 0$ on $[0,1] \setminus \mathcal{C}$ (or alternatively a non-increasing function f with the above properties but with $f(0) = 1, f(1) = 0$).

([2]) By $[x]$ we denote the integer part of $x \in \mathbb{R}$.

In this paper, we consider roof functions f of bounded variation and with Lebesgue decomposition (1)

$$f = f_{\mathrm{j}} + f_{\mathrm{a}} + f_{\mathrm{s}} + \tilde{S}\{\cdot\},$$

where

- f_{j} has finitely many discontinuity points $\beta_1, \ldots, \beta_k$ (with the corresponding jumps $d_1, \ldots, d_k$),
- $f_{\mathrm{s}} = f(\mathcal{C})$ (we consider both possibilities $f_{\mathrm{s}}(0) = 0$ or $f_{\mathrm{s}}(0) = 1$), and either
- $\tilde{S} \neq 0$, or
- $\tilde{S} = 0$, $f_{\mathrm{j}} = 0$, that is, $f = f_{\mathrm{s}} + f_{\mathrm{a}}$ (notice that this includes the classical devil's staircase).

Note that $\tilde{S} = S - d_0$, where S is the sum of the jumps of f ($S := \sum_{i=0}^{k} d_i$, where $d_0 = f_{\mathrm{s}}(1) - f_{\mathrm{s}}(0)$). The main results of the paper are summarized in the following.

THEOREM 1.1. *Assume that $Tx = x + \alpha$ is the rotation by an irrational $\alpha \in [0, 1)$ having bounded partial quotients. Let f be a bounded variation roof function, $f = f_{\mathrm{a}} + f_{\mathrm{j}} + f_{\mathrm{s}} + \tilde{S}\{\cdot\}$, where $f_{\mathrm{s}} = f(\mathcal{C})$ and $\mathcal{C}$ is a quasi-similar Cantor set. If either*

(i) *f_{j} has finitely many discontinuities, $\tilde{S} > 0$ and f_{s} is non-decreasing* ([3]), *or*
(ii) *$f = f_{\mathrm{s}} + f_{\mathrm{a}}$,*

then the special flow $(T_t^f)_{t \in \mathbb{R}}$ is weakly mixing and has the WR-property.

The main tool used to prove the WR-property for special flows over rotations, the so called *Ratner's cocycle property*, has been introduced in [3] (we also used it in [7]). We recall its definition now. Let X be a compact metric Abelian group (with metric denoted by d) with $\mathscr{B}$ the σ-algebra of Borel subsets of X and μ Haar measure on X. Let $Tx = x + x_0$ be an ergodic rotation on $(X, \mathscr{B}, \mu)$. Assume that $f \in L^\infty(X, \mathscr{B}, \mu)$ and $f \geq c$ for some constant $c > 0$. Assume $P \subset \mathbb{R} \setminus \{0\}$ is a compact set.

DEFINITION 1.2 ([3]). One says that the special flow (T_t^f) given by T and f has *Ratner's cocycle property* (relative to the set P) if for every $\epsilon > 0$ and $N \in \mathbb{N}$ there exist $\kappa = \kappa(\epsilon) > 0$, $\delta = \delta(\epsilon, N) > 0$ ([4]) and a subset $Z = Z(\epsilon, N) \in \mathscr{B}$ with $\mu(Z) > 1 - \epsilon$ such that if $x, x' \in Z$, $0 < d(x, x') < \delta$, then there are $M = M(x, x') = M(\epsilon, N, x, x') \geq N$ and $L = L(x, x') =$

([3]) The same result is obtained for $\tilde{S} < 0$ and f_{s} non-increasing.

([4]) If neccesary, we will write $\kappa_f(\epsilon), \delta_f(\epsilon, N)$, etc. to emphasize the dependence of these parameters on f.

$L(\epsilon,N,x,x')\geq N$ such that $L/M\geq\kappa$ and there exists $p=p(\epsilon,N,x,x')\in P$ such that

$$\frac{1}{L}\left|\{n\in\mathbb{Z}\cap[M,M+L]:\left|f^{(n)}(x)-f^{(n)}(x')-p\right|<\epsilon\}\right|>1-\epsilon \ (^5).$$

As proved in [1], if $(T_t^f)_{t\in\mathbb{R}}$ has Ratner's cocycle property and is weakly mixing, then it enjoys the WR-property. All special flows considered in [2, 3, 4, 7] are weakly mixing and have Ratner's cocycle property. We note in passing that assuming weak mixing one can ask whether the WR-property is in fact equivalent to Ratner's cocycle property; we will not consider this problem here, postponing it to the forthcoming paper [8].

In the notes [11], M. Lemańczyk formulated the natural question whether Ratner's cocycle property is stable under sufficiently small bounded variation perturbations. We will provide a positive answer to this question (see Section 5) assuming that:

- The sets $Z=Z(\epsilon,N)$ in Definition 1.2 are equal to X for all $\epsilon>0$ and $N\in\mathbb{N}$.
- For every $\epsilon>0, N\in\mathbb{N}$ and $x,x'\in X$ the (partial) functions $(x,x')\mapsto M(\epsilon,N,x,x')d(x,x')$ defined for $x\neq x'$, $d(x,x')<\delta$, $\epsilon>0$, $N\in\mathbb{N}$ are bounded away from 0 and ∞ (uniformly in ϵ,N,x,x').
- For every $\epsilon>0$, $N\in\mathbb{N}$ and x,x' as above, we have $|f^{(n)}(x)-f^{(n)}(x')-p|<\epsilon$ for each $n\in\mathbb{Z}\cap[M,M+L]$ (6).

In Theorem 5.3 below, we prove that under the above three assumptions, whenever $(T_t^f)_{t\in\mathbb{R}}$ has Ratner's cocycle property, so does $(T_t^{f+g})_{t\in\mathbb{R}}$ whenever g has sufficiently small variation.

Finally, we notice that by using the methods of [2], the assertion of Theorem 1.1 in case (i) can be strengthened to the mild mixing property of the corresponding special flows.

2. Basic notions. We will use the notation from [7]. We denote by $\mathbb{T}$ the circle group $\mathbb{R}/\mathbb{Z}$, which will be identified with the interval $[0,1)$, with addition modulo 1. For a real number t let $\|t\|$ be its distance to the nearest integer (note that $\|t\|=\|-t\|$ and $\|qt\|\leq q\|t\|$ for $q\in\mathbb{N}$). For an irrational $\alpha\in\mathbb{T}$ denote by $(q_n)_{n=0}^{\infty}$ its *sequence of denominators*, that is,

$$\frac{1}{2q_nq_{n+1}}<\left|\alpha-\frac{p_n}{q_n}\right|<\frac{1}{q_nq_{n+1}},$$

where $q_0=1$, $q_1=a_1$, $q_{n+1}=a_{n+1}q_n+q_{n-1}$, $p_0=0$, $p_1=1$, $p_{n+1}=a_{n+1}p_n+p_{n-1}$, and $[0;a_1,a_2,\dots]$ is the continued fraction expansion of α. One says that α has *bounded partial quotients* if the sequence $(a_n)_{n=1}^{\infty}$ is

(5) We denote $f^{(n)}(x)=f(x)+f(Tx)+\dots+f(T^{n-1}x)$ for $n\geq 1$.

(6) All examples from [2, 4, 7] enjoy the above three properties.

bounded. In this case, if we set $C := \sup\{a_n : n \in \mathbb{N}\} + 1$ then $q_{n+1} \le Cq_n$ and

$$\frac{1}{2Cq_n} \le \frac{1}{2q_{n+1}} < \|q_n\alpha\| < \frac{1}{q_{n+1}} < \frac{1}{q_n}$$

for each $n \in \mathbb{N}$. The following lemma is well-known.

LEMMA 2.1. *Let $\alpha \in \mathbb{T}$ be irrational with bounded partial quotients. Then there exist positive constants C_1, C_2 such that for every $k \in \mathbb{N}$ the lengths of the intervals $J_1, \dots, J_k$ of the partition of $\mathbb{T}$ by $0, \alpha, \dots, (k-1)\alpha$ satisfy*

$$C_2/k \le |J_j| < C_1/k \quad \text{for each } j = 1, \dots, k.$$

LEMMA 2.2. *Let α be irrational with bounded partial quotients. Consider points $x_1, \dots, x_l \in \mathbb{T}$ such that there exists $q \in \mathbb{N}$ so that for every $s, t \in 1, \dots, l$,*

$$x_s - x_t = r_{st}/q \quad \text{with } |r_{st}| \in \mathbb{N} \cap [0, q].$$

If C_2 is the constant given by Lemma 2.1, then for every $m \in \mathbb{N}$ the length of each interval of the partition of $\mathbb{T}$ by the points $x_s + j\alpha$, $s = 1, \dots, l$ and $j = 0, \dots, m-1$, is at least $C_2/(q^2 m)$.

Proof. Consider any two points $x_s + j\alpha, x_t + i\alpha$ as above. Then

$$q\|(x_s + j\alpha) - (x_t + i\alpha)\| = q\left\|\frac{r_{st}}{q} + (j-i)\alpha\right\| \ge \|q(j-i)\alpha\| \ge \frac{C_2}{qm},$$

by Lemma 2.1 applied to $k = qm$. ∎

REMARK 2.3. Take any $z_1, z_2 \in \mathbb{T}$ with $z_1 - z_2 = r/q$ for some $|r| < q$ ($r \in \mathbb{Z}$). Fix an interval $I \subset \mathbb{T}$ and assume that for some integers t_0, t_1 with $t_0 < t_1$, we have $z_1 + t_0\alpha, z_2 + t_1\alpha \in I$. Then

$$t_1 - t_0 > \frac{C_2}{q^2|I|}. \tag{3}$$

Indeed, by Lemma 2.2 with $l = 2$, $x_1 = z_1 + t_0\alpha$, $x_2 = z_2 + t_0\alpha$ and $m = t_1 - t_0$, we have

$$|I| \ge \left\|\frac{r}{q} + (t_1 - t_0)\alpha\right\| \ge \frac{C_2}{q^2(t_1 - t_0)},$$

whence (3) holds.

Assume that T is an ergodic automorphism on $(X, \mathscr{B}, \mu)$. We will always assume that T is also aperiodic. A measurable function $f : X \to \mathbb{R}$ determines a cocycle $f^{(\cdot)}(\cdot) : \mathbb{Z} \times X \to \mathbb{R}$ given by $f^{(m)}(x) = f(x) + f(Tx) + \dots + f(T^{m-1}x)$, $m \ge 1$, and

$$f^{(m+n)}(x) = f^{(m)}(x) + f^{(n)}(T^m x)$$

for all $m, n \in \mathbb{Z}$. If $f : X \to \mathbb{R}$ is a strictly positive L^1 function, then $\mathcal{T}^f = (T_t^f)_{t \in \mathbb{R}}$ will denote the corresponding special flow under f acting on $(X^f, \mathscr{B}^f, \mu^f)$, where $X^f := \{(x, s) \in X \times \mathbb{R} : x \in X,\ 0 \le s < f(x)\}$, and

$\mathscr{B}^f$ (resp. μ^f) is the restriction of $\mathscr{B} \otimes \mathscr{B}(\mathbb{R})$ (resp. $\mu \times \lambda$) to X^f. Under the action of the flow $\mathcal{T}^f$, each point in X^f moves vertically with unit speed and we identify the point $(x, f(x))$ with $(Tx, 0)$. More precisely, if $(x,s) \in X^f$ then

$$T_t^f(x,s) = (T^n x, s + t - f^{(n)}(x)),$$

where $n \in \mathbb{Z}$ is unique such that $f^{(n)}(x) \leq s + t < f^{(n+1)}(x)$. Assume additionally that X is a metric space with metric d. Then X^f is a metrizable space with the metric

$$d_1((x,t),(y,s)) = d(x,y) + |t - s|.$$

It is not hard to see that $\mathcal{T}^f$ satisfies the following "almost continuity" condition ([3]): for every $\epsilon > 0$ there exists $X(\epsilon) \in \mathscr{B}^f$ with $\mu^f(X(\epsilon)) > 1 - \epsilon$ such that for every $\epsilon' > 0$ there exists $\epsilon_1 > 0$ such that

$$d_1(T_u^f(x,t), T_{u'}^f(x,t)) < \epsilon'$$

for all $(x,t) \in X(\epsilon)$ and $u, u' \in [-\epsilon_1, \epsilon_1]$.

PROPOSITION 2.4 (Denjoy–Koksma inequality; see e.g. [10]). *If $f : \mathbb{T} \to \mathbb{R}$ is a function of bounded variation then*

$$\Big|\sum_{k=0}^{q_n - 1} f(x + k\alpha) - q_n \int_{\mathbb{T}} f \, d\lambda\Big| \leq \operatorname{Var} f$$

for every $x \in \mathbb{T}$ and $n \in \mathbb{N}$.

3. Weak Ratner property. In this section we recall the notion of weak Ratner property (WR-property) introduced in [3] and we list results from [3] needed in what follows. Let X be a σ-compact metric space with metric d. Let $\mathscr{B}$ denote the σ-algebra of Borel sets in and μ a probability measure on $(X, \mathscr{B})$. Assume that $\mathcal{S} = (S_t)_{t \in \mathbb{R}}$ is a flow on $(X, \mathscr{B}, \mu)$. Let $P \subset \mathbb{R} \setminus \{0\}$ be compact and $t_0 \in \mathbb{R} \setminus \{0\}$.

DEFINITION 3.1 ([3]). The flow $(S_t)_{t \in \mathbb{R}}$ is said to have *property $R(t_0, P)$* if for every $\epsilon > 0$ and $N \in \mathbb{N}$ there exist $\kappa = \kappa(\epsilon)$, $\delta = \delta(\epsilon, N) > 0$ and a subset $Z = Z(\epsilon, N) \in \mathscr{B}$ with $\mu(Z) > 1 - \epsilon$ such that if $x, x' \in Z$, x' is not in the orbit of x, and $d(x, x') < \delta$, then there are $M = M(x, x') \geq N$ and $L = L(x, x') \geq N$ with $L/M \geq \kappa$ and there exists $p = p(x, x') \in P$ such that

$$\frac{1}{L}|\{n \in \mathbb{Z} \cap [M, M + L] : d(S_{nt_0}(x), S_{nt_0 + p}(x')) < \epsilon\}| > 1 - \epsilon.$$

Moreover, we say that $(S_t)_{t \in \mathbb{R}}$ has the *weak Ratner property* or *WR-property* (relative to P if the set of $s \in \mathbb{R}$ such that $(S_t)_{t \in \mathbb{R}}$ has the $R(s, P)$ property is uncountable.

THEOREM 3.2 ([3]). *Let (X, d) be a σ-compact metric space, $\mathscr{B}$ the σ-algebra of Borel subsets of X, and μ a probability Borel measure on $(X, \mathscr{B})$.*

Let $(S_t)_{t\in\mathbb{R}}$ be a weakly mixing flow on $(X,\mathscr{B},\mu)$ that has the WR-property relative to a compact subset $P\subset\mathbb{R}\setminus\{0\}$. Assume that $(S_t)_{t\in\mathbb{R}}$ satisfies the "almost continuity" condition (⁷). *Let $(T_t)_{t\in\mathbb{R}}$ be an ergodic flow on $(Y,\mathscr{C},\nu)$ and let ρ be an ergodic joining of $(S_t)_{t\in\mathbb{R}}$ and $(T_t)_{t\in\mathbb{R}}$. Then either $\rho=\mu\times\nu$, or ρ is a finite extension of ν.*

We recall that each special flow $\mathcal{T}^f=(T_t^f)_{t\in\mathbb{R}}$ over an ergodic base T is ergodic. In particular, the set $\{t\in\mathbb{R}: T_t^f \text{ is not ergodic}\}$ is countable.

PROPOSITION 3.3 ([3]). *Let X be a compact, metric (with a metric d) Abelian group with Haar measure μ. Assume that $Tx=x+x_0$ is an ergodic rotation of $(X,\mathscr{B},\mu)$. Let $f\in L^\infty(X,\mathscr{B},\mu)$ be positive and bounded away from zero. Assume that the special flow $\mathcal{T}^f=(T_t^f)_{t\in\mathbb{R}}$ has Ratner's cocycle property relative to a compact subset $P\subset\mathbb{R}\setminus\{0\}$. Suppose that $\gamma>0$ is such that the γ-time automorphism $T_\gamma^f: X^f\to X^f$ is ergodic. Then $\mathcal{T}^f$ has the $R(\gamma,P)$-property. In particular, $\mathcal{T}^f$ enjoys the WR-property.*

3.1. Properties of bounded variation functions. Let $T:\mathbb{T}\to\mathbb{T}$ be an irrational rotation by α with the sequence of denominators $(q_n)_{n=1}^\infty$, and $f:\mathbb{T}\to\mathbb{R}$ a function of bounded variation. It follows from the Lebesgue decomposition (see e.g. [5] and [7]) that f can be written as

$$f=f_{\mathrm{a}}+f_{\mathrm{j}}+f_{\mathrm{s}}+\tilde{S}\{\cdot\}, \tag{4}$$

where f_{j} is the jump function on $\mathbb{T}$ with jumps $\{d_i\}_{i=1}^\infty$ at $\{\beta_i\}_{i=1}^\infty$ respectively ($\beta_1=0$ and d_1 can be equal to zero), f_{s} is singular, continuous on $[0,1]$ ($d_0:=f_{\mathrm{s}}(1)-f_{\mathrm{s}}(0)$), and f_{a} is absolutely continuous on $\mathbb{T}$. Moreover, $\tilde{S}=\int_{\mathbb{T}}f'\,d\lambda$. Let $S:=\sum_{i=0}^\infty d_i$ (then $\tilde{S}+d_0=S$) be the sum of (all) jumps of f. Recall that the space $\mathrm{BV}(\mathbb{T})$ of functions of bounded variation is a Banach space with the norm $\|f\|:=\operatorname{Var}f+\|f\|_{L^1}$.

We will now focus on singular continuous functions which come from quasi-similar Cantor sets (see Introduction). By construction, each such set is of the form $\mathcal{C}=\bigcap_{i=1}^\infty A_i$ and for each $i\in\mathbb{N}$ the endpoints of all chosen intervals in A_i are multiples of $1/(k_1\dots k_i)$, and there are $2m_1\dots m_i$ of them. For each $n\in\mathbb{N}$ we define an absolutely continuous function f_n such that $f_n'=0$ on $[0,1]\setminus\bigcup_{i=1}^n A_i$ and f_n is linear on each of the intervals of A_n; moreover $f_n(0)=0$, $f_n(1)=1$ and

$$f_n'(x)=\frac{k_1\dots k_n}{m_1\dots m_n}\ (^8)$$

(⁷) The "almost continuity" condition which we gave for $\mathcal{T}^f$ can be naturally carried over to a more general situation.

(⁸) On each interval in A_n, hence of length $\frac{1}{k_1\dots k_n}$, f_n increases by $\frac{1}{m_1\dots m_n}$, so indeed $f_n(1)=1$.

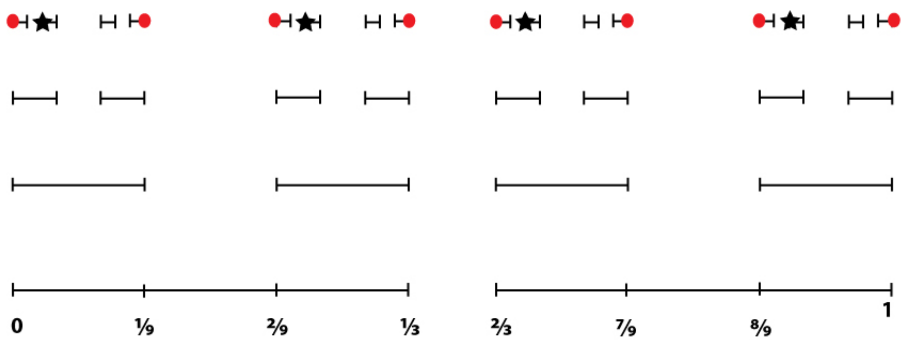
(we can also choose $f_n(0)=1$, $f_n(1)=0$ and $f_n'(x)=-\frac{k_1\dots k_n}{m_1\dots m_n}$). Then the sequence $(f_n)_{n=1}^{\infty}$ converges uniformly to some continuous, singular non-decreasing function $f_{\mathrm{s}}:[0,1]\to\mathbb{R}$. Of course, $f_{\mathrm{s}}'=0$ everywhere except on $\mathcal{C}$ (similarly, by considering $f_n(0)=0$, $f_n(1)=1$ and $f_n'(x)=-\frac{k_1\dots k_n}{m_1\dots m_n}$, we get a non-increasing f_{s}).

Remark 3.4. Fix $i_0\in\mathbb{N}$. For each $n>i_0$ we will define an equivalence relation $\backsim$ on the boundary ∂A_n of A_n. Namely for $x,y\in\partial A_n$ we set

$$x\backsim y \iff |x-y|=\frac{r_{xy}}{k_1\dots k_{i_0}}$$

for some (unique) $r_{xy}\in\mathbb{Z}$ with $0\le r_{xy}\le k_1\dots k_{i_0}$.

Notice that if $x\backsim y$ and $y\backsim z$ then since $|x-z|\le 1$ we have $x\backsim z$. Hence $\backsim$ is an equivalence relation. The set A_{i_0} consists of $m_1\dots m_{i_0}$ closed intervals, say $I_1,\dots,I_{m_1\dots m_{i_0}}$, of length $1/(k_1\dots k_{i_0})$ each, separated by gaps whose lengths are multiples of $1/(k_1\dots k_{i_0})$. Clearly $\partial A_n\subset A_{i_0}$. It follows that for each $i=1,\dots,k_1\dots k_{i_0}$ and each equivalence class E of $\backsim$, we have $1\le|E\cap I_i|\le 2$, and $|E\cap I_i|=2$ iff $0\in E$. Hence, the number of classes is $|\partial A_n\cap I_1|-1=2m_{i_0+1}\dots m_n-1$. Moreover, the class E containing 0 has $2m_1\dots m_{i_0}$ elements, while all remaining classes have $m_1\dots m_{i_0}$ elements. In the picture below, $i_0=2$, $n=4$. Colors denote different equivalence classes.



Assume now that x_0,y_0 are in the same equivalence class of $\backsim$ and let $I\subset\mathbb{T}$ be an interval. Given $t_1,t_2\in\mathbb{Z}$ we then have

$$\text{(5)}\qquad \text{if}\quad x_0+t_1\alpha,\,y_0+t_2\alpha\in I\quad\text{then}\quad |t_1-t_2|\ge\frac{C_2}{(k_1\dots k_{i_0})^2|I|}.$$

Indeed, (5) follows directly from Remark 2.3 (with $z_1=x_0$, $z_2=y_0$) because $|x_0-y_0|=r/(k_1\dots k_{i_0})$ for some $r\le k_1\dots k_{i_0}$.

4. Weak mixing. We will state a criterion which implies weak mixing of special flows over ergodic rotations and under bounded roof functions. First we recall a lemma.

LEMMA 4.1 ([3, Lemma 5.2]). *Let $T : (X, \mathscr{B}, \mu) \to (X, \mathscr{B}, \mu)$ be an ergodic automorphism and let $A \in \mathscr{B}$. For every $\theta, \gamma, \tau > 0$ there exist $N = N(\theta, \gamma, \tau) \in \mathbb{N}$ and $Z = Z(\theta, \gamma, \tau) \in \mathscr{B}$ with $\mu(Z) > 1 - \gamma$ such that for every $M, L \in \mathbb{N}$ with $M, L > N$ and $L/M > \tau$ we have*

$$\left| \frac{1}{L} \sum_{j=M}^{M+L} \chi_A(T^j x) - \mu(A) \right| < \theta$$

for all $x \in Z$. ∎

We will now prove a criterion for a special flow over an ergodic rotation to be weakly mixing.

PROPOSITION 4.2. *Assume that $T : (X, \mathscr{B}, \mu) \to (X, \mathscr{B}, \mu)$ is an ergodic rotation of a compact metric Abelian group X (with a translation invariant metric d) and $f \in L^\infty(X, \mathscr{B}, \mu)$ a positive function which is bounded away from zero. Let $P' \subset \mathbb{R}$ be a compact set. Assume that there exists $\eta_0 > 0$ such that for every $0 < \eta < \eta_0$, every $\epsilon > 0$ and $N \in \mathbb{N}$ there exist $\kappa = \kappa(\epsilon, \eta) > 0$, $\delta = \delta(\epsilon, N) > 0$ and a subset $Z' = Z'(\epsilon, N) \in \mathscr{B}$ with $\mu(Z') > 1 - \epsilon$ such that if $x, x' \in Z'$ and $0 < d(x, x') < \delta$, then there are $M_1 = M_1(x, x'), M_2 = M_2(x, x') \geq N$ and $L_1 = L(x, x'), L_2 = L_2(x, x') \geq N$ such that $L_1/M_1, L_2/M_2 \geq \kappa$ and there exists $p = p(x, x') \in P'$ such that*

$$\frac{1}{L_1} \left| \{ n \in \mathbb{Z} \cap [M_1, M_1 + L_1] : \left| f^{(n)}(x) - f^{(n)}(x') - p \right| < \epsilon \} \right| > 1 - \epsilon$$

and

$$\frac{1}{L_2} \left| \{ n \in \mathbb{Z} \cap [M_2, M_2 + L_2] : \left| f^{(n)}(x) - f^{(n)}(x') - p - \eta \right| < \epsilon \} \right| > 1 - \epsilon.$$

Then the special flow $\mathcal{T}^f = (T_t^f)_{t \in \mathbb{R}}$ is weakly mixing (9). *Moreover, it has Ratner's cocycle property relative to the set $P := (P' + [-\eta_0, \eta_0]) \setminus (-\eta_0/4, \eta_0/4)$.*

Proof. In view of [12], to prove weak mixing we need to show that the equation

$$e^{2\pi i s f(x)} = \frac{\psi(Tx)}{\psi(x)} \tag{6}$$

has no measurable solution $\psi : X \to \mathbb{S}^1$ for any $s \in \mathbb{R} \setminus \{0\}$. Assume that such s, ψ exist. Without loss of generality, we can assume that

$$s > \frac{1}{2\eta_0}. \tag{7}$$

Fix $1/3 > \zeta > 0$ (one more restriction on ζ will appear later).

(9) Notice that here we must assume that T is aperiodic since the assumptions of Proposition 4.2 are satisfied for X being a finite set, while $\mathcal{T}^f$ in this case is not weakly mixing (it has discrete spectrum).

By Egorov's theorem, there exists a set $A_\zeta \subset X$ with $\mu(A_\zeta) > 1-\zeta$ such that $\psi|_{A_\zeta}$ is uniformly continuous. Therefore there exists $\delta_0 > 0$ such that for all $x, x', y, y' \in A_\zeta$, if $d(x,y), d(x',y') < \delta_0$ then

$$\left|\frac{\psi(x')}{\psi(y')}\frac{\psi(y)}{\psi(x)} - 1\right| < \frac{1}{3}. \tag{8}$$

In view of (6), for any $n \in \mathbb{N}$ and $x, y \in A_\zeta$ we get

$$e^{2\pi i s f^{(n)}x} = \frac{\psi(T^n x)}{\psi(x)} \quad \text{and} \quad e^{2\pi i s f^{(n)}y} = \frac{\psi(T^n y)}{\psi(y)},$$

whence

$$e^{2\pi i s (f^{(n)}x - f^{(n)}y)} = \frac{\psi(T^n x)}{\psi(x)}\frac{\psi(y)}{\psi(T^n y)}. \tag{9}$$

Set $\eta = 1/(2s)$ and fix $\epsilon > 0$. Then $\kappa = \kappa(\epsilon)$ is determined. We will now use Lemma 4.1 with the following parameters: $A = A_\zeta$, $0 < \theta < 1/3 - \zeta$, $\gamma = \zeta$ and $\tau = \kappa$. It follows that there exist N_0 and $Z \in \mathscr{B}$ with $\mu(Z) > 1-\zeta$ such that for every $M, L > N_0$ with $L/M \geq \kappa$ we have

$$\left|\frac{1}{L}\sum_{j=M}^{M+L} \chi_{A_\zeta}(T^j x) - \mu(A_\zeta)\right| < \theta \tag{10}$$

for all $x \in Z$. By (10), since $\zeta < 1/3$, for $x \in Z$ the number of $j \in [M, M+L]$ for which $T^j x \in A_\zeta$ is at least $\frac{2}{3}L$. Hence, for any $x, y \in Z$ the number of $j \in [M, M+L]$ for which simultaneously $T^j x, T^j y \in A_\zeta$ is at least $\frac{1}{3}L$. By assumption, there exist $\delta = \delta(\epsilon, N)$ and $Z' = Z'(\epsilon, N)$ with $\mu(Z') > 1-\epsilon$ such that for any $x, y \in Z'$ with $0 < d(x,y) < \delta$ there are $M_1 = M_1(x,x'), M_2 = M_2(x,x') \geq N_0$ and $L_1 = L(x,y), L_2 = L_2(x,y) \geq N_0$ such that $L_1/M_1, L_2/M_2 \geq \kappa$ and there exists $p = p(x,y) \in P$ such that

$$\frac{1}{L_1}\left|\{n \in \mathbb{Z} \cap [M_1, M_1+L_1] : \left|f^{(n)}(x) - f^{(n)}(y) - p\right| < \epsilon\}\right| > 1-\epsilon$$

and

$$\frac{1}{L_2}\left|\left\{n \in \mathbb{Z} \cap [M_2, M_2+L_2] : \left|f^{(n)}(x) - f^{(n)}(y) - p - \frac{1}{2s}\right| < \epsilon\right\}\right| > 1-\epsilon.$$

Notice that the set $A_\zeta \cap Z \cap Z'$ has positive measure (if ζ is small enough) and therefore, since μ is continuous, there are different points in this set arbitrarily close to each other. So fix $x, y \in A_\zeta \cap Z \cap Z'$ such that $0 < d(x,y) < \min(\delta, \delta_0)$. Then the number of $j \in [M_1, M_1+L_1]$ such that $T^j x, T^j y \in A_\zeta$ is at least $\frac{1}{3}L_1$. Hence there exists $j_1 \in [M_1, M_1+L_1]$ such that $T^{j_1}x, T^{j_1}y \in A_\zeta$ and $|f^{(j_1)}(x) - f^{(j_1)}(y) - p| < \epsilon$. By (9),

$$e^{2\pi i s (f^{(j_1)}x - f^{(j_1)}y)} = \frac{\psi(T^{j_1} x)}{\psi(x)}\frac{\psi(y)}{\psi(T^{j_1} y)}. \tag{11}$$

In view of (8) and (11),

$$|e^{2\pi is(f^{(j_1)}x-f^{(j_1)}y)}-1|<1/3.$$

On the other hand, $p-\epsilon<|f^{(j_1)}x-f^{(j_1)}y|<p+\epsilon$. Hence, by letting $0<\epsilon\to 0$, we obtain $|e^{2\pi isp}-1|<1/3$. Doing the same for M_2, L_2, we get $|e^{2\pi is(p+\frac{1}{2s})}-1|<1/3$, or $|e^{2\pi isp}(-1)-1|<1/3$, which is an obvious contradiction.

Notice that for every $p\in\mathbb{R}$, $\max(|p|,|p+\eta_0/2|)\ge\eta_0/4$. So Ratner's cocycle property is satisfied with respect to P as defined in the statement of the proposition (for every $x,x'\in Z$ we choose either p or $p+\eta_0/2$, both being in $P'+[-\eta,\eta]$). ∎

5. Stability of Ratner's property. In this section we show that Ratner's cocycle property in the class of special flows over an irrational rotation by α having bounded partial quotients and the roof function in $\mathrm{BV}(\mathbb{T})$ is stable under small perturbations in the variation norm. We assume throughout that α has bounded partial qoutients and $C:=\sup_{n\ge1}a_n+1$.

LEMMA 5.1. *Let $f\in\mathrm{BV}(\mathbb{T})$ and $D\in\mathbb{N}$. Then for every $s\in\mathbb{N}$,*

$$\max_{0\le k<Dq_s}\sup_{\|x-y\|<1/q_s}|f^{(k)}(x)-f^{(k)}(y)|\le 2D\operatorname{Var}f.$$

Proof. Fix $s\in\mathbb{N}$ and $k=cq_s+d$ with $d<q_s$ and $c\le D-1$. We will prove that for any $x,y\in\mathbb{T}$,

$$\max_{j\le q_s}\sup_{\|x-y\|<1/q_s}|f^{(j)}(x)-f^{(j)}(y)|<2\operatorname{Var}f. \tag{12}$$

Then (12) and Proposition 2.4 will give the required result because

$$\begin{aligned}|f^{(k)}(x)-f^{(k)}(y)|&\le\sum_{i=1}^{c}|f^{(q_s)}(T^{(i-1)q_s}x)-f^{(q_s)}(T^{(i-1)q_s}y)|\\&\quad+|f^{(d)}(T^{cq_s}x)-f^{(d)}(T^{cq_s}y)|\\&\le 2c\operatorname{Var}f+2\operatorname{Var}f\le 2D\operatorname{Var}f.\end{aligned}$$

To prove (12), consider the points $x+i\alpha$, $i=0,\dots,j$, and reorder them to get $0\le x+i_1\alpha<x+i_2\alpha<\dots<x+i_j\alpha$. It follows that the distance between any two such points is at least $1/(2q_s)$. Indeed, for every $u,v\in\{0,\dots,j\}$ we have

$$\|(x+u\alpha)-(x+v\alpha)\|\ge\|(u-v)\alpha\|\ge\frac{1}{2q_s}.$$

Set $A_1:=\{x+i_k\alpha : k \text{ even}\}$ and $A_2:=\{x+i_k\alpha : k\text{ odd}\}$. Then the distance between any two points in A_1 and in A_2 is at least $1/q_s$. It follows that the points $x+i_k\alpha, y+i_k\alpha$ with k even yield a partition of $\mathbb{T}$ into some intervals, and all intervals $[x+i_k\alpha, y+i_k\alpha)$ (or with endpoints interchanged) are members of this partition. Hence

$$\operatorname{Var} f \geq \sum_{r=1}^{[j/2]} |f(x+i_{2r}\alpha) - f(y+i_{2r}\alpha)| \geq \Big|\sum_{r=1}^{[j/2]} f(x+i_{2r}\alpha) - f(y+i_{2r}\alpha)\Big|.$$

The same holds for A_2. Finally, we get $2\operatorname{Var} f \geq |f^{(j)}(x) - f^{(j)}(y)|$. ■

REMARK 5.2. Notice that by the cocycle identity, for each $N \geq 1$,

$$\max_{k<Dq_s} \sup_{\|x-y\|<1/q_s} |(f^{(N+k)}(x) - f^{(N+k)}(y)) - (f^{(N)}(x) - f^{(N)}(y))| \leq 2D\operatorname{Var} f.$$

Moreover, (by the proof of Lemma 5.1) the same inequality holds if $k = Dq_s$.

THEOREM 5.3. *Let $P \subset \mathbb{R} \setminus \{0\}$ be a nonempty compact set. Let $1/2 > \eta > 0$ be such that $P \subset \mathbb{R} \setminus (\eta, \eta)$ and let $(T_t^f)_{t\in\mathbb{R}}$ be the special flow over an irrational rotation by α with $f \in L^\infty(X, \mathscr{B}, \mu)$. Assume that $(T_t^f)_{t\in\mathbb{R}}$ has Ratner's cocycle property (see Definition 1.2) and there exist $R_1, R_2 > 0$ such that*

$$R_1 \leq M(x,x')\|x - x'\| \leq R_2.$$

Moreover, assume that for every $n \in \mathbb{Z} \cap [M, M+L]$,

$$|f^{(n)}(x) - f^{(n)}(x') - p| < \epsilon \quad \textit{for every } x, x' \in \mathbb{T}$$

(this does not depend on ϵ, N, see Introduction). Then Ratner's cocycle property is stable under bounded variation perturbations; more precisely, if $g \in \mathrm{BV}(\mathbb{T})$ with $\operatorname{Var} g < \frac{1}{8CR_2}\eta$ then $(T_t^{f+g})_{\in\mathbb{R}}$ has Ratner's cocycle property relative to some compact set $P' \subset \mathbb{R} \setminus \{0\}$.

Proof. Fix $\epsilon > 0$ and $N \in \mathbb{N}$. Define

$$\kappa_{f+g} = \kappa_{f+g}(\epsilon) := \frac{\epsilon}{4}\min\left(\kappa_f(\epsilon/2), \frac{1}{CR_2}\right)$$

and $\delta_{f+g} = \delta_{f+g}(\epsilon, N) = \delta_f(\epsilon/2, N)$ (cf. footnote 4). Take $0 < \|x - x'\| < \delta_{f+g}$. By assumption, there exist $M_f(x,x'), L_f(x,x')$ and $p_f(x,x')$ such that $M_f > N$, $1 > L_f/M_f > \kappa_f$ and for every $k \in [M_f, M_f + L_f]$,

$$|f^{(k)}(x) - f^{(k)}(x') - p_f| < \epsilon/2.$$

Let $s \in \mathbb{N}$ be unique such that

$$\frac{1}{q_{s+1}} \leq \|x - x'\| < \min\left(\frac{1}{q_s}, \delta_{f+g}\right).$$

Now denote $L'_f = \min(q_s, L_f)$ and consider the interval $[M_f, M_f + L'_f]$. It follows from the proof of (12), since we have $L'_f \leq q_s$ and $\|T^{M_f}x - T^{M_f}x'\| =$

$\|x-x'\| < 1/q_s$, that

$$\frac{1}{4} > 2\operatorname{Var} g > |g(T^{M_f}x) - g(T^{M_f}x')| + |g(T^{M_f+1}x) - g(T^{M_f+1}x')| + \cdots + |g(T^{M_f+L'_f}x) - g(T^{M_f+L'_f}x')|.$$

Therefore, there exists an interval I_g with $[M_f, M_f + L'_f] \supset I_g = [a_g, b_g]$ of length $[\epsilon L'_f]$ such that

$$\frac{\epsilon}{4} > |g(T^{a_g}x) - g(T^{a_g}x')| + |g(T^{a_g+1}x) - g(T^{a_g+1}x')| + \cdots + |g(T^{b_g}x) - g(T^{b_g}x')|;$$

in particular, for every $k \in I_g$,

$$|g^{(k)}(x) - g^{(k)}(x') - g^{(a_g)}(x) - g^{(a_g)}(x')| < \epsilon/2. \tag{13}$$

Let

$$M_{f+g} = a_g, \quad L_{f+g} = b_g - a_g \geq [\epsilon L'_f] \geq \frac{\epsilon}{2} L'_f = \frac{\epsilon}{2}\min(L_f, q_s).$$

Then, by definition, $N \leq M_f \leq M_{f+g} \leq M_f + L_f < 2M_f$ and

$$\frac{L_{f+g}}{M_{f+g}} \geq \frac{\epsilon}{2}\frac{\min(L_f, q_s)}{2M_f} \geq \frac{\epsilon}{4}\min\left(\frac{L_f}{M_f}, \frac{q_s}{M_f}\right) = \kappa_{f+g}.$$

Moreover, let $P' := P + [-\eta/2, \eta/2] \subset \mathbb{R} \setminus (-\eta/2, \eta/2)$ and

$$p_0(x, x') := g^{(M_{f+g})}(x) - g^{(M_{f+g})}(x').$$

It follows from Lemma 5.1 and the inequality $\operatorname{Var} g < \eta/(8CR_2)$ (because $M_{f+g} \leq 2M_f \leq 2CR_2 q_s$) that

$$|p_0| \leq 4CR_2 \frac{\eta}{8CR_2} = \eta/2.$$

Let $p'(x, x') := p_f(x, x') + p_0(x, x') \in P'$. In view of (13), for every $k \in [M_{f+g}, M_{f+g} + L_{f+g}] \subset [M_f, M_f + L_f]$,

$$\begin{aligned}
&|(f+g)^{(k)}(x) - (f+g)^{(k)}(x') - p'| \\
&\qquad \leq |f^{(k)}(x) - f^{(k)}(x') - p_f| + |g^{(k)}(x) - g^{(k)}(x') - p_0| \\
&\qquad < \epsilon/2 + \epsilon/2 = \epsilon.
\end{aligned}$$

This completes the proof. ■

6. Weak mixing and Ratner's property for some roof functions in $\mathrm{BV}(\mathbb{T})$**.** In this section we prove weak mixing and Ratner's property of a special flow $\mathcal{T}^f$, where $f \in \mathrm{BV}(\mathbb{T})$ is of the form $f = f_{\mathrm{a}} + f_{\mathrm{j}} + f_{\mathrm{s}} + S'\{\cdot\}$, where f_{a} is absolutely continuous on $\mathbb{T}$, f_{j} is piecewise constant (finitely many jumps) and f_{s} is a singular, continuous function on $[0,1]$ which comes from a quasi-similar Cantor set ($f_{\mathrm{s}} = f(\mathcal{C})$). Consider $\mathcal{C} = \bigcap_{i=1}^{\infty} A_i$ (with bounded sequences $(m_i)_{i\geq 1}$, $(k_i)_{i\geq 1}$, see Introduction). Take $x < y \in \mathbb{T}$ and let $n = n(x, y) \in \mathbb{N}$ be unique such that

$$\frac{1}{k_1 \dots k_n} < \|x - y\| \le \frac{1}{k_1 \dots k_{n-1}}. \tag{14}$$

Then $f_{\mathrm{s}}(x) \neq f_{\mathrm{s}}(y)$ iff $[x, y] \cap \partial A_n \neq \emptyset$. Indeed, at step $n \geq 1$ of the construction of $\mathcal{C}$, $[0, 1]$ is divided into $k_1 \dots k_n$ intervals of length $1/(k_1 \dots k_n)$ each, $m_1 \dots m_n$ of which, say $I_1, \dots, I_{m_1 \dots m_n}$, form A_n. Now, x and y cannot belong to the same I_j, hence either

- $[x, y] \cap \partial A_n = \emptyset$, i.e. $[x, y] \cap I_j = \emptyset$ for each j and then

$$f_{\mathrm{s}}(x) = f_{\mathrm{s}}(y); \tag{15}$$

or

- $[x, y] \cap \partial A_n \neq \emptyset$. In this case

$$f_{\mathrm{s}}(x) \neq f_{\mathrm{s}}(y). \tag{16}$$

Indeed, there is some interval I_{i_0} such that $[x, y] \cap I_{i_0} \neq \emptyset$.

Moreover, it follows from (14) that

$$|f_{\mathrm{s}}(y) - f_{\mathrm{s}}(x)| \le \frac{1}{m_1 \dots m_{n-1}} \quad \text{(unless } 0 \in [x, y)). \tag{17}$$

Indeed, by definition, for every $n \in \mathbb{N}$, f_{s} is constant on each connected component of $\mathcal{C} \setminus A_{n-1}$, and for every interval $I' = [a', b')$ chosen at step $n - 1$, we have $f_{\mathrm{s}}(b') - f_{\mathrm{s}}(a') = 1/(m_1 \dots m_{n-1})$.

To prove both weak mixing and the WR-property for (T_t^f) we need a lemma.

Lemma 6.1 ([2, Lemma 6.1]). *Let $T : \mathbb{T} \to \mathbb{T}$ be the rotation by an irrational α with bounded partial quotients and let $f : \mathbb{T} \to \mathbb{R}$ be absolutely continuous. Then*

$$\sup_{0 \le n < q_{s+1}} \sup_{\|y - x\| < 1/q_s} |f^{(n)}(y) - f^{(n)}(x)| \to 0 \quad \textit{as } s \to \infty.$$

Remark 6.2. Following (step by step) the proof of Lemma 6.1 in [2] one can prove that for every $b \in \mathbb{N}$ we have

$$\sup_{0 \le n < q_{s+b}} \sup_{\|y - x\| < 1/q_s} |f^{(n)}(y) - f^{(n)}(x)| \to 0 \quad \text{as } s \to \infty.$$

Denote by $\{\beta_i\}_{i=0}^k$ the (finite) set of discontinuities of f with the corresponding jumps $\{d_i\}_{i=0}^k$ (we recall that $\beta_0 = 0$ and d_0 may be equal to 0).

Proof of Theorem 1.1. We will use Proposition 4.2 with $P' := [-2C \operatorname{Var} f, 2C \operatorname{Var} f]$. It follows that $f = f_{\mathrm{ac}} + f_{\mathrm{j}} + f_{\mathrm{s}} + \tilde{S}\{\cdot\} + c$, where f_{ac} is absolutely continuous with zero mean ($c := \int_{\mathbb{T}} f_{\mathrm{a}}(x)\, d\mu$). Let $f_{\mathrm{pl}}(x) := f_{\mathrm{j}}(x) + \tilde{S}\{x\} + c$, so

$$f = f_{\mathrm{ac}} + f_s + f_{\mathrm{pl}}. \tag{18}$$

Assume that $\tilde{S} > 0$. Let

$$0 < \eta := \min\left(\frac{\tilde{S}}{5C(2C+1)(k+1)}, \frac{1}{8}\right). \tag{19}$$

Fix $\epsilon > 0$ and $N \geq 1$. By Lemma 6.1, there exists s_0 such that for every $s \geq s_0$, we have

$$\sup_{0\leq n<q_{s+1}} \sup_{\|y-x\|<1/q_s} |f_{\mathrm{ac}}^{(n)}(y) - f_{\mathrm{ac}}^{(n)}(x)| < \frac{\epsilon}{8}. \tag{20}$$

Let $n_0 = n_0(\epsilon)$ be the unique natural number such that

$$\frac{2K}{m_1 \dots m_{n_0}} \leq \epsilon/8 < \frac{2K}{m_1 \dots m_{n_0-1}} \ ({}^{10}). \tag{21}$$

(i) First we have to define $\kappa = \kappa(\epsilon)$ ([11]). Set

$$\kappa(\epsilon) := \min\left(\frac{C_2}{C(k_1 \dots k_{n_0})^2}, \frac{\epsilon}{4\tilde{S}C}, \frac{1}{2(2C+1)(k+1)}\right) \tag{22}$$

(see Lemma 2.1). Let $s_1 \in \mathbb{N}$ be smallest with

(23)

$$q_{s_1} \geq \max\left(\tilde{S}q_{s_0}, \frac{16\tilde{S}}{\epsilon}, \frac{N}{\kappa}\right) \quad \text{and let} \quad \delta(\epsilon, N) := \min\left(\frac{1}{q_{s_1+1}}, \frac{1}{k_1 \dots k_{n_0}}\right).$$

Take any $x, y \in \mathbb{T}$ with $\|x - y\| < \delta$. Let $s \in \mathbb{N}$ be unique such that

$$\frac{1}{q_{s+1}} \leq \|x - y\| < \frac{1}{q_s} \ ({}^{12}). \tag{24}$$

Consider now the points $\mathbb{T} \ni \beta_i - j\alpha$, $i = 0, \dots, k$, $j = 0, \dots, q_{s+1} - 1$. Note that for every $j, t \in \{0, \dots, q_{s+1} - 1\}$,

$$\|(\beta_i - j\alpha) - (\beta_i - t\alpha)\| = \|(t-j)\alpha\| \geq \min_{|v|\in\{0,\dots,q_{s+1}-1\}} \|v\alpha\| \geq \frac{1}{2q_{s+1}}.$$

Therefore, for every i, the number of $j = 0, \dots, q_{s+1} - 1$ such that $\beta_i - j\alpha \in [x, y)$ is at most

$$[2q_{s+1}\|x - y\|] + 1 \overset{(24)}{\leq} \left[\frac{2q_{s+1}}{q_s}\right] + 1 \leq 2C + 1.$$

So we can divide the time interval $[q_s, q_{s+1}]$ into $(2C + 1)(k + 1)$ clopen ([13]) intervals $I_1, \dots, I_{(2C+1)(k+1)}$ such that $\beta_i - v\alpha \notin [x, y)$ for all $u = 1, \dots, (2C + 1)(k + 1)$, $v \in \operatorname{int} I_u$ and $i = 0, \dots, k$. One of them, say $I =$

([10]) K is a constant such that $m_i \leq K$ for every $i \in \mathbb{N}$.

([11]) Note that here κ does not depend on $\eta > 0$.

([12]) Note that $q_s \geq q_{s_1}$ by (23).

([13]) Here by clopen intervals we mean those of the form $[A, B) \cap \mathbb{Z}$, where $A, B \in \mathbb{N}$.

$[M_0, K_0]$, has length at least $\frac{1}{(2C+1)(k+1)}(q_{s+1} - q_s)$. We have $M_0 < q_{s+1}$ (since $[M_0, K_0] \subset [q_s, q_{s+1}]$). Let

$$L_0 := \kappa M_0 < \kappa q_{s+1} \leq \kappa C q_s \leq \frac{C_2 q_s}{(k_1 \dots k_{n_0})^2}. \tag{25}$$

Moreover, by (23) and (24),

$$M_0 \geq q_s \geq q_{s_1} \geq \frac{1}{\delta} \geq \frac{N}{\kappa} \geq N \quad \text{and} \quad L_0 \geq \kappa M_0 \geq \kappa q_{s_1} \geq \kappa \frac{N}{\kappa} \geq N. \tag{26}$$

We define $p(x, y) := f^{(M_0)}(y) - f^{(M_0)}(x)$. Then $|p| < 2C \operatorname{Var} f$ (by Lemma 5.1). Denote by $n := n_{xy}$ the unique natural number such that

$$\frac{1}{k_1 \dots k_n} \leq \|x - y\| < \frac{1}{k_1 \dots k_{n-1}}. \tag{27}$$

If $f_{\mathrm{s}}(x) \neq f_{\mathrm{s}}(y)$ then $\partial A_n \cap [x, y] \neq \emptyset$. Recall that $|\partial A_n| = 2m_1 \dots m_n$. Take $z \in \partial A_n$. It follows from Remark 2.3 with $z_1 = z_2$, $r = 0$, $q = 1$ and $I = [x, y]$ and from (25) that there exists at most one $v_0 \in [M_0, M_0 + L_0]$ such that $z + v_0\alpha \in [x, y]$. Note that by the choice of the interval $[M_0, M_0 + L_0]$ ($\beta_i - v\alpha \notin [x, y)$ for $i = 0, \dots, k$ and $v \in [M_0, M_0 + L_0]$), for every $v \in [M_0, M_0 + L_0]$ we have $\{-v\alpha\} \notin [x, y]$ (taking $i = 0$, recall that $\beta_0 = 0$). Therefore, by (27) and (17), for every $v \in [M_0, M_0 + L_0]$ we have

$$|f_{\mathrm{s}}(y + v\alpha) - f_{\mathrm{s}}(x + v\alpha)| < \frac{1}{m_1 \dots m_{n-1}}.$$

Consider now Remark 3.4 with $i_0 = n_0$ and the corresponding equivalence relation $\backsim$ for n. Note that by (27) and (23) we have

$$\frac{1}{k_1 \dots k_n} \leq \|x - y\| \leq \delta \leq \frac{1}{k_1 \dots k_{n_0}},$$

and therefore $n \geq n_0$.

Note that in each equivalence class there is at most one point $z_0 \in \partial A_n$ for which there exists $w_0 \in [M_0, M_0 + L_0]$ such that $z_0 - w_0\alpha \in [x, y]$. Indeed, if for some z_0, z_1 in the same equivalence class, there exist $w_0, w_1 \in [M_0, M_0 + L_0]$ such that $z_0 - w_0\alpha, z_1 - w_1\alpha \in [x, y]$, then, by (5) with $I = [x, y]$, (25) and (24), we have

$$\frac{C_2 q_s}{(k_1 \dots k_{n_0})^2} > |L_0| \geq |w_0 - w_1| \geq \frac{C_2}{(k_1 \dots k_{n_0})^2 \|x - y\|} \geq \frac{C_2 q_s}{(k_1 \dots k_{n_0})^2},$$

a contradiction. So there are at most $2m_{n_0+1} \dots m_n - 1$ (the number of equivalence classes by Remark 3.4) points $z \in \partial A_n$ for which there exists a (unique) $w_z \in [M_0, M_0 + L_0]$ such that $z - w_z\alpha \in [x, y]$.

For each such w_z, we have

$$[x + w_z\alpha, y + w_z\alpha] \cap \partial A_n \neq \emptyset,$$

and therefore, by (16) and (17),

$$0 \neq f_{\mathrm{s}}(y + w_z\alpha) - f_{\mathrm{s}}(x + w_z\alpha) < \frac{1}{m_1 \dots m_{n-1}}. \tag{28}$$

There are at most $2m_{n_0+1} \dots m_n - 1 < 2m_{n_0+1} \dots m_n$ such $w = w_z \in [M_0, M_0 + L_0]$ and for $r \neq w_z$ (z as above),

$$[x + r\alpha, y + r\alpha] \cap \partial A_n = \emptyset.$$

Therefore, by (15), $f_{\mathrm{s}}(x + r\alpha) = f_{\mathrm{s}}(y + r\alpha)$ for $r \neq w_z$. Hence, for every $r \in [M_0, M_0 + L_0]$ (remembering that $f_{\mathrm{s}}(T^{M_0+j}y) - f_{\mathrm{s}}(T^{M_0+j}x) = 0$ for $M_0 + j \neq w_z$),

$$\begin{aligned} &|f_{\mathrm{s}}^{(r)}(y) - f_{\mathrm{s}}^{(r)}(x) - (f_{\mathrm{s}}^{(M_0)}(y) - f_{\mathrm{s}}^{(M_0)}(x))| \\ &= |f_{\mathrm{s}}^{(r-M_0)}(T^{M_0}y) - f_{\mathrm{s}}^{(r-M_0)}(T^{M_0}x)| \\ &\leq \sum_{j=0}^{r-M_0-1} |f_{\mathrm{s}}(T^{M_0+j}y) - f_{\mathrm{s}}(T^{M_0+j}x)| \\ &\leq \sum_{j=0}^{L_0-1} |f_{\mathrm{s}}(T^{M_0+j}y) - f_{\mathrm{s}}(T^{M_0+j}x)| \\ &\overset{(28)}{\leq} 2m_{n_0+1} \dots m_n \frac{1}{m_1 \dots m_{n-1}} = \frac{2m_n}{m_1 \dots m_{n_0}} \overset{(21)}{\leq} \frac{\epsilon}{8} < \frac{\epsilon}{4}. \end{aligned} \tag{29}$$

REMARK 6.3. Note that if $I = [A, B]$ is an interval such that $|I| \leq C_2 q_s/(k_1 \dots k_{n_0})^2$ and for any $v \in [A, B]$ we have $\{-v\alpha\} \notin [x, y)$, then for every $r \in [A, B]$,

$$|f_{\mathrm{s}}^{(r)}(y) - f_{\mathrm{s}}^{(r)}(x) - (f_{\mathrm{s}}^{(A)}(y) - f_{\mathrm{s}}^{(A)}(x))| \leq \epsilon/4.$$

Indeed, these are the only assumptions that were used to prove (29).

Moreover, it is easy to see that (because f_{pl} is piecewise linear and there is no discontinuity point of f in $[T^r x, T^r y]$ for $r \in [M_0, M_0 + L_0]$) for $r \in [M_0, M_0 + L_0]$,

$$\begin{aligned} &|(f_{\mathrm{pl}}^{(r)}(y) - f_{\mathrm{pl}}^{(r)}(x)) - (f_{\mathrm{pl}}^{(M_0)}(y) - f_{\mathrm{pl}}^{(M_0)}(x))| \\ &\leq \sum_{j=0}^{r-M_0-1} (f_{\mathrm{pl}}(T^{M_0+j}(y)) - f_{\mathrm{pl}}(T^{M_0+j}(x))) \\ &\leq (r - M_0)\tilde{S}\|y - x\| \leq L_0\tilde{S}\|y - x\| \leq \tilde{S}\kappa M_0 \frac{1}{q_s} \leq \kappa\tilde{S}C \overset{(22)}{\leq} \frac{\epsilon}{4}. \end{aligned} \tag{30}$$

Finally, by (18), (20), (29), (30) we get, for $r \in [M_0, M_0 + L_0]$,

$$\begin{aligned}(31)\quad & |f^{(r)}(y) - f^{(r)}(x) - p| = |(f^{(r)}(y) - f^{(r)}(x)) - (f^{(M_0)}(y) - f^{(M_0)}(x))| \\ &= |(f_{\mathrm{ac}}^{(r)}(y) - f_{\mathrm{ac}}^{(r)}(x)) - (f_{\mathrm{ac}}^{(M_0)}(y) - f_{\mathrm{ac}}^{(M_0)}(x)) + (f_{\mathrm{s}}^{(r)}(y) - f_{\mathrm{s}}^{(r)}(x)) \\ &\quad - (f_{\mathrm{s}}^{(M_0)}(y) - f_{\mathrm{s}}^{(M_0)}(x)) + (f_{\mathrm{pl}}^{(r)}(y) - f_{\mathrm{pl}}^{(r)}(x)) - (f_{\mathrm{pl}}^{(M_0)}(y) - f_{\mathrm{pl}}^{(M_0)}(x))| \\ &\le \frac{\epsilon}{8} + \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{5}{8}\epsilon < \epsilon.\end{aligned}$$

REMARK 6.4. Notice that if $M' < q_{s+1}$ is such that for all $v' \in [M', M' + \kappa M']$ we have $\beta_i - v'\alpha \notin [x, y)$ for $i = 0, \ldots, k$, and if we set $p' := f^{(M')}(y) - f^{(M')}(x)$, then for any $r' \in [M', M' + \kappa M']$ we have

$$|f^{(r')}(y) - f^{(r')}(x) - p'| \le \frac{5}{8}\epsilon.$$

Indeed, these are the only assumptions on M_0 that were used to prove (31).

Consider again the interval $I = [M_0, K_0]$. We recall that none of the points of the form $\beta_i - v\alpha$ with $i = 0, \ldots, k$ and $v \in [M_0, K_0]$ belongs to $[x, y]$. Therefore, using the fact that $\tilde{S} > 0$ and f_{s} is non-decreasing, for every $r \in [M_0, K_0]$, by (17), we have (using $n \ge n_0$)

$$\begin{aligned}(32)\qquad \tilde{S}\|y - x\| &\le (f_{\mathrm{pl}} + f_{\mathrm{s}})(T^r y) - (f_{\mathrm{pl}} + f_{\mathrm{s}})(T^r x) \\ &\le \tilde{S}\|y - x\| + \frac{1}{m_1 \ldots m_{n-1}} \\ &\le \tilde{S}\delta + \frac{K}{m_1 \ldots m_{n_0}} \overset{(23),(21)}{\le} \frac{\epsilon}{16} + \frac{\epsilon}{16} = \frac{\epsilon}{8}.\end{aligned}$$

The length of I is at least $\frac{1}{(2C+1)(k+1)}(q_{s+1} - q_s)$, so by (20) and (32), for every $r > M_0$ we have

$$\begin{aligned}(33)\quad & |(f^{(r)}(y) - f^{(r)}(x)) - (f^{(M_0)}(y) - f^{(M_0)}(x))| \\ &= |(f_{ac}^{(r-M_0)}(T^{M_0}y) - f_{ac}^{(r-M_0)}(T^{M_0}x)) \\ &\quad + ((f_{\mathrm{pl}} + f_{\mathrm{s}})^{(r-M_0)}(T^{M_0}y) - (f_{\mathrm{pl}} + f_{\mathrm{s}})^{(r-M_0)}(T^{M_0}x))| \\ &\ge \left| -\frac{\epsilon}{4} + \sum_{i=M_0}^{r} (f_{\mathrm{pl}} + f_{\mathrm{s}})(T^{M_0+i}y) - (f_{\mathrm{pl}} + f_{\mathrm{s}})(T^{M_0+i}x) \right| \\ &\ge (r - M_0)\tilde{S}\|y - x\| - \epsilon/4.\end{aligned}$$

In particular, for $r_0 = M_0 + [K_0/2]$, by (24) and the fact that

$$q_{s+1} \ge q_s + q_{s-1} \ge q_s + q_s/c,$$

we get

$$
\begin{aligned}
&|(f^{(r_0)}(y) - f^{(r_0)}(x)) - (f^{(M_0)}(y) - f^{(M_0)}(x))| \\
&\quad > \frac{\tilde{S}}{2(2C+1)(k+1)}(q_{s+1} - q_s)\|x-y\| \\
&\quad \geq \frac{\tilde{S}}{2(2C+1)(k+1)}\left(1 - \frac{q_s}{q_{s+1}}\right) \geq \frac{\tilde{S}}{2(2C+1)(k+1)}\left(1 - \frac{C}{C+1}\right) \overset{(19)}{\geq} \eta.
\end{aligned}
$$

Notice that if we set

$$H(r) := (f^{(r)}(y) - f^{(r)}(x)) - (f^{(M_0)}(y) - f^{(M_0)}(x)),$$

then for every $r \in [M_0, M_0 + r_0]$,

$$|H(r+1) - H(r)| < \epsilon/4. \tag{34}$$

Indeed, $|H(r+1) - H(r)| = |f(T^r y) - f(T^r x)|$, and (34) follows from (32) and (20). By (33) and (34) (by considering $r = M_0, \dots, M_0 + r_0$) we see that there exists $R_0 \in [M_0, r_0]$ such that

$$\eta + \epsilon/4 \geq |(f^{(R_0)}(y) - f^{(R_0)}(x)) - (f^{(M_0)}(y) - f^{(M_0)}(x))| \geq \eta. \tag{35}$$

Define $M_1 := R_0$ and $L_1 := \kappa M_1$. Then, from $M_1 \geq M_0$ and (26), we have $M_1 > q_s > N$ and $L_1 \geq \kappa M_1 > N$. Hence, by (35),

$$|f^{(M_1)}(x) - f^{(M_1)}(y) - p - \eta| < \epsilon/4. \tag{36}$$

It follows from Remark 6.4 and (36) that for any $r \in [M_1, M_1 + L_1]$,

(37)

$$
\begin{aligned}
|f^{(r)}(x) - f^{(r)}(y) - p - \eta| &\leq |(f^{(r)}(x) - f^{(r)}(y)) - (f^{(M_1)}(x) - f^{(M_1)}(y))| \\
&\quad + |f^{(M_1)}(x) - f^{(M_1)}(y) - p - \eta| \\
&< \frac{5}{8}\epsilon + \frac{1}{4}\epsilon < \epsilon.
\end{aligned}
$$

(ii) First note that there exist $b \in \mathbb{N}$, $b < C$, such that for every $s \geq 1$,

$$\left[\frac{q_{s+b}}{q_{s+1}}\right] \geq 6C + 4. \tag{38}$$

By Remark 6.2, there exists s_0 such that for every $s \geq s_0$ we have

$$\sup_{0 \leq n < q_{s+b}} \sup_{\|y-x\| < 1/q_s} |f_{\mathrm{ac}}^{(n)}(y) - f_{\mathrm{ac}}^{(n)}(x)| < \epsilon/8. \tag{39}$$

Define

$$\kappa = \kappa(\epsilon) := \min\left(\frac{C_2}{C^b(k_1 \dots k_{n_0})^2}, \frac{1}{(2C+1)^b}\right), \tag{40}$$

with n_0 coming from (21). Let $s_1 \in \mathbb{N}$ be smallest with

$$q_{s_1} \geq \max(q_{s_0}, N/\kappa) \quad \text{and let} \quad \delta(\epsilon, N) := 1/q_{s_1+1}. \tag{41}$$

Take any $x, y \in \mathbb{T}$ with $\|x-y\| < \delta$. Let $s \in \mathbb{N}$ be unique such that

$$\frac{1}{q_{s+1}} \le \|x-y\| < \frac{1}{q_s}. \tag{42}$$

Consider the time interval $[q_s, q_{s+b}]$. Denote by

$$(q_s \le)\ R_0 < \cdots < R_t\ (\le q_{s+b})$$

all natural numbers in $[q_s, q_{s+b}]$ for which $\{-R_i\alpha\} \in [x, y)$. These numbers divide $[q_s, q_s + b]$ into some clopen subintervals ([14]) $I_0, \ldots, I_t$ ($I_i = [R_i, R_{i+1})$). By Remark 2.3 with $z_1 = z_2 = 0$ and $I = [x, y)$, using (42) we get, for $i = 0, \ldots, t-1$,

$$|I_i| = R_{i+1} - R_i \ge \frac{C_2}{\|x-y\|} \ge C_2 q_s, \tag{43}$$

and similarly, by (42),

$$|I_i| \le q_{s+1} \le C q_s. \tag{44}$$

Hence, by (43) and (44),

$$\frac{C^b}{C_2} \ge \frac{q_{s+b}}{q_s C_2} \ge t \ge \left[\frac{q_{s+b}}{q_{s+1}}\right] \overset{(38)}{\ge} 6C + 4. \tag{45}$$

Let $j_t := q_{s+b} - R_t$. It follows that $j_t \le q_{s+1}$ and $R_0 \le 2q_{s+1}$. Using (42) and the fact that $\{-R_i\alpha\} \in [x, y)$ for $i = 0, \ldots, t$, we get (using $f_{\mathrm{s}}(0) = 0$, $f_{\mathrm{s}}(1) = 1$ and the construction of f_{s})

$$\begin{aligned} &|(f_{\mathrm{s}}^{(R_i+1)}(x) - f_{\mathrm{s}}^{(R_i+1)}(y)) - (f_{\mathrm{s}}^{(R_i)}(x) - f_{\mathrm{s}}^{(R_i)}(y))| \\ &\qquad = |f_{\mathrm{s}}(x + R_i\alpha) - f_{\mathrm{s}}(y + R_i\alpha)| > \frac{7}{8}, \end{aligned} \tag{46}$$

for $i = 0, \ldots, t$.

Moreover, there exists an $i \in \{0, \ldots, t-1\}$ such that

$$(f_{\mathrm{s}}^{(R_{i+1})}(x) - f_{\mathrm{s}}^{(R_{i+1})}(y)) - (f_{\mathrm{s}}^{(R_i)}(x) - f_{\mathrm{s}}^{(R_i)}(y)) \ge -3/4.$$

Indeed, if not, then by Lemma 5.1 with $D = C$ (using the fact that $j_t \le q_{s+1}$) and by (42), we get

$$\begin{aligned} |f_{\mathrm{s}}^{(-j_t)}(T^{q_{s+b}}x) - f_{\mathrm{s}}^{(-j_t)}(T^{q_{s+b}}y) &= |f_{\mathrm{s}}^{(j_t)}(T^{q_{s+b}-j_t}x) - f_{\mathrm{s}}^{(j_t)}(T^{q_{s+b}-j_t}y)| \\ &\le 2C \operatorname{Var} f. \end{aligned}$$

Therefore, by Proposition 2.4,

([14]) Of the form $[A, B) \cap \mathbb{Z}$, where $A, B \in \mathbb{Z}$.

$$
\begin{aligned}
2C+2 &= (2C+2)\operatorname{Var} f_{\mathrm{s}} = 2\operatorname{Var} f_{\mathrm{s}} + 2C\operatorname{Var} f_{\mathrm{s}} \\
&\geq |f_{\mathrm{s}}^{(q_{s+b})}(x) - f_{\mathrm{s}}^{(q_{s+b})}(y)| + |f_{\mathrm{s}}^{(-j_t)}(T^{q_{s+b}}x) - f_{\mathrm{s}}^{(-j_t)}(T^{q_{s+b}}y)| \\
&\geq |f_{\mathrm{s}}^{(q_{s+b}-j_t)}(x) - f_{\mathrm{s}}^{(q_{s+b}-j_t)}(y)| = |f_{\mathrm{s}}^{(R_t)}(x) - f_{\mathrm{s}}^{(R_t)}(y)| \\
&= \Big|\sum_{w=0}^{t-1}(f_{\mathrm{s}}^{(R_{w+1})}(x) - f_{\mathrm{s}}^{(R_{w+1})}(y)) - (f_{\mathrm{s}}^{(R_w)}(x) - f_{\mathrm{s}}^{(R_w)}(y)) \\
&\qquad + (f_{\mathrm{s}}^{(R_0)}(x) - f_{\mathrm{s}}^{(R_0)}(y))\Big| \\
&> t|-3/4| - 2C \overset{(45)}{>} 2C+2,
\end{aligned}
$$

a contradiction. Hence such an $i \in \{0,\dots,t-1\}$ exists. Then, by (46),

$$
(47)\quad (f_{\mathrm{s}}^{(R_{i+1})}(x) - f_{\mathrm{s}}^{(R_{i+1})}(y)) - (f_{\mathrm{s}}^{(R_i+1)}(x) - f_{\mathrm{s}}^{(R_i+1)}(y)) \geq \frac{7}{8} - \frac{3}{4} = \frac{1}{8}.
$$

Let us define $M_0 := R_i + 1 \leq q_{s+b}$ ($R_i \in [q_s, q_{s+b})$ for every $i = \{0,\dots, t-1\}$) and

$$
(48)\qquad L_0 := \kappa M_0 < \kappa q_{s+b} \overset{(40)}{\leq} \frac{q_{s+b}}{C^b}\,\frac{C_2}{(k_1\dots k_{n_0})^2} \leq \frac{C_2 q_s}{(k_1\dots k_{n_0})^2}.
$$

Then $L_0 \geq \kappa M_0 \geq \kappa q_s \geq \kappa q_{s_1} \overset{(41)}{\geq} N$. It follows from Remark 6.3 that for every $r \in [M_0, M_0 + L_0]$,

$$
|(f_{\mathrm{s}}^{(r)}(x) - f_{\mathrm{s}}^{(r)}(y)) - (f_{\mathrm{s}}^{(M_0)}(x) - f_{\mathrm{s}}^{(M_0)}(y))| < \epsilon/4.
$$

Define $p = p(x,y) := f^{(M_0)}(x) - f^{(M_0)}(y)$. Then for $r \in [M_0, M_0 + L_0]$, by (39) and (48), we have

$$
|(f^{(r)}(x) - f^{(r)}(y)) - (f^{(M_0)}(x) - f^{(M_0)}(y))| \leq \epsilon/2.
$$

Set

$$
H(r) := (f^{(r)}(x) - f^{(r)}(y)) - (f^{(M_0)}(x) - f^{(M_0)}(y))
$$

for $r \in [M_0, R_{i+1}]$ (we have $H(R_i+1) = 0, H(R_{i+1}) \overset{(47)}{\geq} 1/8 \geq 2\eta$). By the choice of i and (32) (with $f_{\mathrm{pl}} = 0$, $\tilde{S} = 0$), there exists $R_\eta \in [M_0, R_{i+1}]$ such that $|f_{\mathrm{s}}^{(R_\eta)}(x) - f_{\mathrm{s}}^{(R_\eta)}(y) - p - \eta| < \epsilon/2$. Then defining $M_1 := R_\eta$, $L_1 := \kappa M_1$ ($M_1 > q_s > N$, $L_1 \geq \kappa M_1 > N$), we deduce (proceeding as in (37) and using (39)) that for every $r \in [M_1, M_1 + L_1]$,

$$
|f^{(r)}(x) - f^{(r)}(y) - p - \eta| < \epsilon.
$$

The proof of Theorem 1.1 is complete. ∎

7. Absence of partial rigidity and mild mixing. In this section, we will show the absence of partial rigidity of some special flows over the irrational rotation by α having bounded partial qoutients ($\sup_{n\geq 1} a_n + 1$

$< \infty$) and with roof functions of the form $f = f_{\mathrm{a}} + f_{\mathrm{j}} + f_{\mathrm{s}} + \tilde{S}\{\cdot\}$, where f_{j} has finitely many jumps, $f_{\mathrm{s}} = f(\mathcal{C})$ is non-decreasing and $\tilde{S} > 0$ ([15]). The proof of the theorem below is a repetition of the proof of Theorem 7.1 in [2].

THEOREM 7.1. *Assume $T : \mathbb{T} \to \mathbb{T}$ is an ergodic rotation by α having bounded partial quotients. Suppose $f = f_{\mathrm{a}}+f_{\mathrm{j}}+f_{\mathrm{s}}+\tilde{S}\{\cdot\}$, where f_{j} has finitely many jumps, $f_{\mathrm{s}} = f(\mathcal{C})$ is non-decreasing and $\tilde{S} > 0$. Then the special flow $(T_t^f)_{t\in\mathbb{R}}$ is not partially rigid.*

Combining Theorems 7.1 and 1.1 with the main argument used in [2] to prove Theorem 7.2 therein, we deduce the following result.

COROLLARY 7.2. *Suppose that $T : \mathbb{T} \to \mathbb{T}$ is the rotation by an irrational number α with bounded partial quotients and $f : \mathbb{T} \to \mathbb{R}$ is a positive function, bounded away from zero, of the form $f := f_{\mathrm{a}} + f_{\mathrm{s}} + f_{\mathrm{j}} + \tilde{S}\{\cdot\}$ with $\tilde{S} \neq 0$ and $f_{\mathrm{s}} = f_{\mathrm{s}}(\mathcal{C})$ for some quasi-similar Cantor set $\mathcal{C}$. Then (T_t^f) is mildly mixing.*

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