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H^p SPACES ASSOCIATED WITH SCHRÖDINGER OPERATORS WITH POTENTIALS FROM REVERSE HÖLDER CLASSES

BY

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Abstract. Let $A = -\Delta + V$ be a Schrödinger operator on $\mathbb{R}^d$, $d \geq 3$, where V is a nonnegative potential satisfying the reverse Hölder inequality with an exponent $q > d/2$. We say that f is an element of H^p_A if the maximal function $\sup_{t>0} |T_t f(x)|$ belongs to $L^p(\mathbb{R}^d)$, where $\{T_t\}_{t>0}$ is the semigroup generated by $-A$. It is proved that for $d/(d+1) < p \leq 1$ the space H^p_A admits a special atomic decomposition.

1. Introduction. Let $k_t(x,y)$ be the integral kernels of the semigroup of linear operators $\{T_t\}_{t>0}$ generated by a Schrödinger operator $-A = \Delta - V$ on $\mathbb{R}^d$, $d \geq 3$.

Throughout this paper we assume that V is a nonnegative potential on $\mathbb{R}^d$ that belongs to the reverse Hölder class RH^q, $q > d/2$, that is, there exists a constant $C > 0$ such that

$$\left(\frac{1}{|B|}\int_B V(y)^q\,dy\right)^{1/q} \leq \frac{C}{|B|}\int_B V(y)\,dy \quad \text{for every ball } B. \tag{1.1}$$

Since V is nonnegative and belongs to $L^q_{\mathrm{loc}}(\mathbb{R}^d)$ the Feynman–Kac formula implies that

$$0 \leq k_t(x,y) \leq (4\pi t)^{-d/2} e^{-|x-y|^2/(4t)} = p_t(x-y). \tag{1.2}$$

For $0 < p < 1$ we define the space H^p_A as the completion of the space of compactly supported $L^1(\mathbb{R}^d)$-functions in the quasi-norm $\|f\|^p_{H^p_A} = \|\mathcal{M}f\|^p_{L^p}$, where

$$\mathcal{M}f(x) = \sup_{t>0} |T_t f(x)| = \sup_{t>0} \left|\int k_t(x,y) f(y)\,dy\right|. \tag{1.3}$$

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H_A^p spaces associated with Schrödinger operators with potentials from reverse Hölder classes were studied in [DZ2] and [DZ4]. It was proved there that for $d/(d+\min(1,2-d/q)) < p \le 1$ the space H_A^p admits an atomic decomposition. The main purpose of the present paper is to prove that if $d/2 < q < d$, then also for

$$\frac{d}{d+1} < p \le \frac{d}{d+\min(1,2-d/q)} = \frac{d}{d+2-d/q}$$

the elements of H_A^p can be decomposed into special atoms, but for this range of p's different type cancellation conditions for the atoms may occur.

The auxiliary function

$$m(x,V) = \left(\sup\left\{r>0 : \frac{1}{r^{d-2}} \int_{B(x,r)} V(y)\,dy \le 1\right\}\right)^{-1} \tag{1.4}$$

will play a crucial role in the paper. The function $m(x,V)$ is well defined, and $0 < m(x,V) < \infty$ (cf. [Sh]). We set

$$R(x) = R(x,V) = m(x,V)^{-1}. \tag{1.5}$$

For a positive ε (small) we define

$$G_\varepsilon(x) = ((\mathrm{Id}+A_\varepsilon^*)^{-1}\mathbf{1})(x),$$

where $\mathbf{1}(x)=1$ for $x\in\mathbb{R}^d$,

$$A_\varepsilon f(x) = V(x)\int_0^{(\varepsilon R(x))^2} p_s * f(x)\,ds,$$

and $(\mathrm{Id}+A_\varepsilon^*)^{-1}$ is the inverse operator to $\mathrm{Id}+A_\varepsilon^*$.

We have

LEMMA 1.6.

$$\lim_{\varepsilon\to 0^+} \|G_\varepsilon - \mathbf{1}\|_\infty = 0,$$

Let $\delta = 2-d/q$ and $\delta_0 = \min(1,\delta)$.

LEMMA 1.7. *For every $\delta' < \delta_0$ there exists a constant $C>0$ such that*

$$|G_\varepsilon(x)-G_\varepsilon(y)| \le C((m(x,V)+m(y,V))|x-y|)^{\delta'}.$$

The constant C is independent of ε provided $\varepsilon<\varepsilon_0$, with $\varepsilon_0>0$ sufficiently small.

REMARK. For $\delta_0=\delta<1$ the conclusion of Lemma 1.7 holds with $\delta'=\delta$.

The proofs of Lemmas 1.6 and 1.7 are provided in Section 4.

We are now in a position to define a notion of H_A^p-atom. Fix a small real number $\varepsilon>0$. A function b is an H_A^p-*atom* associated with a ball $B(x_0,r)$ if

$$\operatorname{supp} b \subset B(x_0,r), \tag{1.8}$$

$$\|b\|_\infty \le |B(x_0,r)|^{-1/p}, \tag{1.9}$$

$$r \le R(x_0), \tag{1.10}$$

$$\text{if} \quad r \le \tfrac{1}{4}R(x_0) \quad \text{then} \quad \int b(x)G_\varepsilon(x)\,dx = 0 \tag{1.11}$$

The *atomic quasi-norm* of an element $f \in H^p_A$ is given by

$$\|f\|^p_{H^p_A\text{-atom}} = \inf\Big\{\sum_j |\lambda_j|^p\Big\}, \tag{1.12}$$

where the infimum is taken over all decompositions $f = \sum_j \lambda_j b_j$, where λ_j are scalars and b_j are H^p_A-atoms. The main result of the paper is the following theorem:

THEOREM 1.13. *Let $d/(d+1) < p \le 1$. There exists a constant C such that for every compactly supported function $f \in L^1(\mathbb{R}^d)$ we have*

$$C^{-1}\|f\|_{H^p_A} \le \|f\|_{H^p_A\text{-atom}} \le C\|f\|_{H^p_A}. \tag{1.14}$$

REMARK. We point out that the notion of H^p_A-atom, and, in consequence, the norm $\|f\|_{H^p_A\text{-atom}}$ depend on ε (see (1.11)). However, we shall prove that (1.14) holds for any fixed $\varepsilon > 0$ provided ε is small enough.

It follows from Lemma 1.7 that for $p \in (p_0, 1]$, where $p_0 = d/(d+\delta_0)$, the condition (1.11) in the definition of H^p_A-atoms can be replaced by

$$\text{if} \quad r \le \tfrac{1}{4}R(x_0) \quad \text{then} \quad \int b(x)\,dx = 0. \tag{1.15}$$

In this case the atoms are appropriately scaled local atoms in the sense of Goldberg (cf. [G]).

For $p = 1$ the above result was obtained in [DZ2]. Therefore we shall restrict our attention to the case where $p \in (d/(d+1), 1)$.

2. Auxiliary definitions. A function a is said to be an $(\mathbf{h}^p_\varepsilon(m), \infty)$-*atom* associated with a ball $B(x_0, r)$ if

$$r \le \varepsilon R(x_0), \tag{2.1}$$

$$\operatorname{supp} a \subset B(x_0, r), \tag{2.2}$$

$$\|a\|_\infty \le |B(x_0,r)|^{-1/p}, \tag{2.3}$$

$$\text{if} \quad r \le \tfrac{1}{4}\varepsilon R(x_0), \quad \text{then} \quad \int a(x)\,dx = 0. \tag{2.4}$$

We say that a function b is an $(H^p_A, \infty, \varepsilon)$-*atom* associated with a ball $B(x_0, r)$ if (2.1)–(2.3) hold for b instead of a, and the condition (2.4) is replaced by

$$\text{if} \quad r \le \tfrac{1}{4}\varepsilon R(x_0), \quad \text{then} \quad \int b(x)G_\varepsilon(x)\,dx = 0. \tag{2.4$'$}$$

Let $M \geq 0$ and $d/(d+1) < p < 1$. A function a is called a *generalized* $(\mathbf{h}^p_\varepsilon(m), 1, M)$*-atom* associated with a ball $B(x_0, r)$ if

$$r \leq \varepsilon R(x_0), \tag{2.5}$$

$$\int |a(x)| \left(1 + \frac{|x - x_0|}{r}\right) \left(1 + \frac{|x - x_0|}{\varepsilon R(x_0)}\right)^M dx \leq |B(x_0, r)|^{1-1/p}, \tag{2.6}$$

$$\text{if} \quad r \leq \tfrac{1}{4}\varepsilon R(x_0), \quad \text{then} \quad \int a(x)\, dx = 0. \tag{2.7}$$

Similarly, b is said to be a *generalized* $(H^p_A, 1, \varepsilon, M)$*-atom* associated with a ball $B(x_0, r)$ if (2.5)–(2.6) are satisfied for b instead of a and (2.7) is replaced by

$$\text{if} \quad r \leq \tfrac{1}{4}\varepsilon R(x_0), \quad \text{then} \quad \int b(x) G_\varepsilon(x)\, dx = 0. \tag{2.7$'$}$$

Let us note that every $(\mathbf{h}^p_\varepsilon(m), \infty)$-atom is also a generalized $(\mathbf{h}^p_\varepsilon(m), 1, M)$-atom. It is not difficult to prove the following lemma, using the properties of the function m stated in Lemma 4.3 and Corollary 4.6.

LEMMA 2.8. *If* $d/(d+1) < p < 1$ *then there is a constant* $C > 0$ *such that if* a *is a generalized* $(\mathbf{h}^p_\varepsilon(m), 1, M)$*-atom, then there is a sequence* a_j *of* $(\mathbf{h}^p_\varepsilon(m), \infty)$*-atoms and a sequence of scalars* λ_j *such that*

$$a = \sum \lambda_j a_j, \qquad \sum |\lambda_j|^p \leq C.$$

The constant C *depends on* m *and* p, *but it is independent of* ε.

The norm in the space $\mathbf{h}^p_\varepsilon(m)$ is defined by

$$\|f\|^p_{\mathbf{h}^p_\varepsilon(m)} = \inf\Big\{\sum_j |\lambda_j|^p\Big\},$$

where the infimum is taken over all decompositions $f = \sum_j \lambda_j a_j$, where a_j are $(\mathbf{h}^p_\varepsilon(m), \infty)$-atoms and λ_j are scalars.

LEMMA 2.9. *There exists* $\varepsilon_0 > 0$ *such that for every* $0 < \varepsilon < \varepsilon_0$ *if* a *is a generalized* $(\mathbf{h}^p_\varepsilon(m), 1)$*-atom associated with a ball* $B(x_0, r)$ *then*

$$(\mathrm{Id} + A_\varepsilon) a$$

is (*up to a multiplicative constant independent of* ε) *a generalized* $(H^p_A, 1, \varepsilon)$*-atom associated with the ball* $B(x_0, r)$.

Conversely, $(\mathrm{Id} + A_\varepsilon)^{-1} b$ *is up to a multiplicative constant a generalized* $(\mathbf{h}^p_\varepsilon(m), 1)$*-atom associated with a ball* $B(x_0, r)$, *provided* b *is a generalized* $(H^p_A, 1, \varepsilon)$*-atom associated with the same ball.*

Proof. See Section 5.

Corollary 2.10. *There exists a constant $C > 0$ such that*

$$\|G_\varepsilon(\mathrm{Id} + A_\varepsilon)\|_{\mathbf{h}^p_\varepsilon(m)\to\mathbf{h}^p_\varepsilon(m)} \le C$$

provided $0 < \varepsilon < \varepsilon_0$.

It is not difficult to prove the following proposition.

Proposition 2.11. *For every $\varepsilon' > \varepsilon > 0$ there exists a constant $C_{\varepsilon',\varepsilon}$ such that*

$$\|f\|_{\mathbf{h}^p_\varepsilon(m)} \le \|f\|_{\mathbf{h}^p_{\varepsilon'}(m)} \le C_{\varepsilon',\varepsilon}\|f\|_{\mathbf{h}^p_\varepsilon(m)}.$$

3. Idea of the proof of atomic decomposition. In order to prove the second inequality in (1.14) it suffices to show that there are constants $C, \varepsilon_0 > 0$ such that for every $0 < \varepsilon < \varepsilon_0$ if

$$\mathcal{K}^* f(x) = \sup_{0<t<(\varepsilon R(x))^2} |T_t f(x)| \in L^p$$

then

$$f(x)G_\varepsilon(x) \in \mathbf{h}^p_\varepsilon(m),$$

and

$$\|f(x)G_\varepsilon(x)\|_{\mathbf{h}^p_\varepsilon(m)} \le C\|\mathcal{K}^* f\|_{L^p}. \tag{3.1}$$

To prove this we consider the following identity based on the perturbation formula:

$$\begin{aligned} p_t(x,y) &= k_t(x,y) + \int_0^t \int k_{t-s}(x,z)V(z)p_s(z,y)\,dz\,ds \\ &= (T_t(\mathrm{Id}+A_\varepsilon))(x,y) + H_t(x,y) + E_t(x,y) + Z_{(\varepsilon),t}(x,y), \end{aligned}$$

where

$$H_t(x,y) = \int_{t/2}^t \int k_{t-s}(x,z)V(z)p_s(z-y)\,dz\,ds,$$

$$E_t(x,y) = \int_0^{t/2} \int (k_{t-s} - k_t)(x,z)V(z)p_s(z-y)\,dz\,ds,$$

$$Z_{(\varepsilon),t}(x,y) = \int k_t(x,z)V(z)W_{(\varepsilon),t}(z,y)\,dz,$$

with

$$W_{(\varepsilon),t}(z,y) = \begin{cases} -\displaystyle\int_{t/2}^{(\varepsilon R(z))^2} p_s(z-y)\,ds & \text{if } (\varepsilon R(z))^2 > t/2, \\ \displaystyle\int_{(\varepsilon R(z))^2}^{t/2} p_s(z-y)\,ds & \text{if } (\varepsilon R(z))^2 \le t/2. \end{cases}$$

Let $f \in L^1_{\rm c}(\mathbb{R}^d)$. Set $g = (\mathrm{Id} + A_\varepsilon)^{-1} f$. We have

$$P_t g = T_t f + E_t g + H_t g + Z_{(\varepsilon),t} g,$$

where P_t, E_t, H_t, $Z_{(\varepsilon),t}$ are the operators with the integral kernels $p_t(x-y)$, $E_t(x,y)$, $H_t(x,y)$, $Z_{(\varepsilon),t}(x,y)$ respectively. Set

$$\mathcal{P}^*_\varepsilon g(x) = \sup_{0<t<(\varepsilon R(x))^2} |P_t g(x)|, \qquad \mathcal{H}^*_\varepsilon g(x) = \sup_{0<t<(\varepsilon R(x))^2} |H_t g(x)|,$$
$$\mathcal{E}^*_\varepsilon g(x) = \sup_{0<t<(\varepsilon R(x))^2} |E_t g(x)|, \qquad \mathcal{Z}^*_\varepsilon g(x) = \sup_{0<t<(\varepsilon R(x))^2} |Z_{(\varepsilon),t} g(x)|.$$

We shall show that the following two lemmas hold:

LEMMA 3.2. *There exists a constant $C > 0$ independent of ε such that*

$$C^{-1} \|\mathcal{P}^*_\varepsilon g\|_{L^p} \leq \|g\|_{\mathbf{h}^p_\varepsilon(m)} \leq C \|\mathcal{P}^*_\varepsilon g\|_{L^p}. \tag{3.3}$$

The proof of the lemma is given in Section 8.

LEMMA 3.4.

$$\lim_{\varepsilon\to 0^+} \|\mathcal{E}^*_\varepsilon\|_{\mathbf{h}^p_\varepsilon(m)\to L^p} = 0, \tag{3.5}$$

$$\lim_{\varepsilon\to 0^+} \|\mathcal{H}^*_\varepsilon\|_{\mathbf{h}^p_\varepsilon(m)\to L^p} = 0, \tag{3.6}$$

$$\lim_{\varepsilon\to 0^+} \|\mathcal{Z}^*_\varepsilon\|_{\mathbf{h}^p_\varepsilon(m)\to L^p} = 0. \tag{3.7}$$

See Section 6 for the proofs of (3.5), (3.6), and Section 7 for the proof of (3.7).

Having these, we obtain

$$\begin{aligned}\|g\|_{\mathbf{h}^p_\varepsilon(m)} &\leq C \|\mathcal{P}^*_\varepsilon g\|_{L^p} \\ &\leq C \|\mathcal{K}^*_\varepsilon f\|_{L^p} + C \|\mathcal{E}^*_\varepsilon\|_{\mathbf{h}^p_\varepsilon(m)\to L^p} \|g\|_{\mathbf{h}^p_\varepsilon(m)} \\ &\quad + C \|\mathcal{H}^*_\varepsilon\|_{\mathbf{h}^p_\varepsilon(m)\to L^p} \|g\|_{\mathbf{h}^p_\varepsilon(m)} + C \|\mathcal{Z}^*_\varepsilon\|_{\mathbf{h}^p_\varepsilon(m)\to L^p} \|g\|_{\mathbf{h}^p_\varepsilon(m)}.\end{aligned}$$

As a consequence of Lemma 2.9 and the fact that every compactly supported L^1-function is an element of $H^p_{A,\varepsilon}$ we have $\|g\|_{\mathbf{h}^p_\varepsilon(m)} < \infty$. Thus, by Lemma 3.4, we get

$$\|g\|_{\mathbf{h}^p_\varepsilon(m)} \leq C \|\mathcal{K}^*_\varepsilon f\|_{L^p}$$

provided ε is close to 0. Applying Corollary 2.10 we get (3.1).

The paper is organized as follows. In Section 4 we provide the proofs of Lemmas 1.6 and 1.7. The proof of Lemma 2.9 is presented in Section 5. Section 6 is devoted to the proofs of (3.5) and (3.6), whereas the proof of (3.7) is given in Section 7. The proof of Lemma 3.2 occupies Section 8. Finally, in Section 9 we show the first inequality in (1.14).

4. Auxiliary estimates. In the present section we state some result concerning the estimates of the kernels associated with the semigroup $\{T_t\}_{t>0}$. At the end of the section we prove Lemmas 1.6 and 1.7.

LEMMA 4.1 (see [Sh, Lemma 1.2]). *For every nonnegative potential $V \in RH^q$, $q > d/2$, there exists a constant $C > 0$ such that for every $0 < r < R$ we have*

$$\frac{1}{r^{d-2}} \int_{B(x,r)} V(y)\,dy \le C\left(\frac{r}{R}\right)^{\delta} \frac{1}{R^{d-2}} \int_{B(x,R)} V(y)\,dy.$$

COROLLARY 4.2. *If $r < R(x) = m(x,V)^{-1}$ then*

$$\int_{B(x,r)} V(y)\,dy \le C(rm(x,V))^{\delta} r^{d-2}.$$

LEMMA 4.3 (see [Sh, Lemma 1.4]). *There exist constants $C, k_0 > 0$ such that*

$$m(y,V) \le C(1 + |x-y|m(x,V))^{k_0} m(x,V), \tag{4.4}$$

$$m(y,V) \ge \frac{m(x,V)}{C(1+|x-y|m(x,V))^{k_0/(1+k_0)}}. \tag{4.5}$$

COROLLARY 4.6. *For every $C_1 > 0$ there exists a constant $C_2 > 0$ such that if $|x-y|m(x,V) \le C_1$ then*

$$C_2^{-1} \le \frac{m(x,V)}{m(y,V)} \le C_2.$$

LEMMA 4.7 (cf. [Sh, Lemma 1.8]). *There exist constants $C_0, C > 0$ such that if $r > R(x) = m(x,V)^{-1}$ then*

$$\int_{B(x,r)} V(y)\,dy \le C(rm(x,V))^{C_0} m(x,V)^{2-d}.$$

We say that a function ψ defined on $\mathbb{R}^d$ is *rapidly decaying* if for every $N > 0$ there exists a constant C_N such that

$$|\psi(x)| \le C_N(1+|x|)^{-N}.$$

COROLLARY 4.8. *If ψ is a rapidly decaying nonnegative function, then there exists a constant $C > 0$ such that*

$$\int V(y)\psi_t(x-y)\,dy \le \begin{cases} Ct^{-1}(m(x,V)t^{1/2})^{\delta} & \text{for } t \le R(x)^2, \\ Ct^{-d/2}(\sqrt{t}\,m(x,V))^{C_0} m(x,V)^{2-d} & \text{for } t > R(x)^2, \end{cases}$$

where $\psi_t(x) = t^{-d/2}\psi(t^{-1/2}x)$.

The Kato–Trotter formula asserts that

$$k_t(x,y) = p_t(x-y) - \int_0^t \int p_s(x-z)V(z)k_{t-s}(z,y)\,dz\,ds \tag{4.9}$$
$$= p_t(x-y) - \int_0^t \int k_{t-s}(x,z)V(z)p_s(z-y)\,dz\,ds.$$

A proof of the theorem below can be found in [K] (see also [DZ4]).

THEOREM 4.10. *For every $M>0$ there exists a constant C_M such that*

$$k_t(x,y) \le C_M t^{-d/2}(1+\sqrt{t}(m(x,V)+m(y,V)))^{-M} e^{-|x-y|^2/(5t)}.$$

PROPOSITION 4.11. *For every $0<\delta'<\delta_0$ there exists a constant $c>0$ such that for every $M>0$ there exists a constant $C>0$ such that for $|h|<\sqrt{t}$, we have*

$$|k_t(x,y+h)-k_t(x,y)| \le C\Big(\frac{|h|}{\sqrt{t}}\Big)^{\delta'} t^{-d/2} e^{-c|x-y|^2/t}\Big(1+\frac{\sqrt{t}}{R(x)}+\frac{\sqrt{t}}{R(y)}\Big)^{-M}. \tag{4.12}$$

Proof. Obviously, using Theorem 4.10 and Lemma 4.3, we see that (4.12) holds for $\sqrt{t/2}\le|h|\le\sqrt{t}$. We first prove (4.12) under the assumption $|h|\le|x-y|/4$. Theorem 4.10 combined with Lemma 4.3 implies that for $|h|<|x-y|/4$ one has

$$|k_t(x,y+h)-k_t(x,y)| \le Ct^{-d/2}e^{-|x-y|^2/(5t)}\Big(1+\frac{\sqrt{t}}{R(x)}+\frac{\sqrt{t}}{R(y)}\Big)^{-3M} \tag{4.13}$$
$$\le Ct^{-d/2}e^{-|x-y|^2/(5t)}\Big(1+\frac{\sqrt{t}}{R(x)}+\frac{\sqrt{t}}{R(y)}\Big)^{-2M}\Big(\frac{R(y)}{\sqrt{t}}\Big)^{M}.$$

Therefore it suffices to verify (4.12) for $|h|\le R(y)$. Let $q_t(x,y)=p_t(x,y)-k_t(x,y)$. One can prove (see [DZ4, Proposition 2.17]) that for every $0<\delta''<\delta_0$ there is a constant $c>0$ such that for $|h|\le|x-y|/4$, $|h|\le R(y)$, we have

$$|q_t(x,y+h)-q_t(x,y)| \le C\Big(\frac{|h|}{\sqrt{t}}\Big)^{\delta''}\Big(\frac{\sqrt{t}}{R(x)}\Big)^{\delta''} t^{-d/2}e^{-c|x-y|^2/t}.$$

Thus

$$|k_t(x,y+h)-k_t(x,y)| \le C\Big(\frac{|h|}{\sqrt{t}}\Big)^{\delta''}\Big(1+\frac{\sqrt{t}}{R(x)}\Big)^{\delta''} t^{-d/2}e^{-c|x-y|^2/t},$$

which combined with (4.13) gives (4.12).

To complete the proof, we have to consider $|x-y|/4<|h|\le\sqrt{t/2}$. By the semigroup property,

$$|k_t(x,y+h)-k_t(x,y)|\le\int k_{t/2}(x,z)|k_{t/2}(z,y+h)-k_{t/2}(z,y)|\,dz$$
$$=\int\limits_{|z-y|\le 4|h|}+\int\limits_{|z-y|>4|h|}=S_1+S_2.$$

Obviously, by Theorem 4.10,

$$S_1\le Ct^{-d/2}\left(\frac{|h|}{\sqrt{t}}\right)^d(1+\sqrt{t}\,m(x,V))^{-M}.$$

Since $|z-y|>4|h|$, we apply (4.12) and obtain

$$S_2\le C\int\limits_{|z-y|>4|h|}k_t(x,z)\left(\frac{|h|}{\sqrt{t}}\right)^{\delta''}t^{-d/2}e^{-c|x-y|^2/t}\,dz$$
$$\le C(1+\sqrt{t}\,m(x,V))^{-M}t^{-d/2}e^{-c|x-y|^2/t}\left(\frac{|h|}{\sqrt{t}}\right)^{\delta''}.$$

Hence, by the assumption $|x-y|/4<|h|\le\sqrt{t/2}$, we have

$$S_1+S_2\le C(1+\sqrt{t}\,m(x,V))^{-M}\left(\frac{|h|}{\sqrt{t}}\right)^{\delta''}t^{-d/2}e^{-c|x-y|^2/t}.$$

Applying Lemma 4.3, we get (4.12) for $|x-y|<4|h|$. ■

Let $A_\varepsilon(x,y)$ denote the integral kernel of the operator A_ε. Then

$$A_\varepsilon(x,y)=V(x)\Gamma_\varepsilon(x,y),\qquad \Gamma_\varepsilon(x,y)=\int\limits_0^{(\varepsilon R(x))^2}p_s(x-y)\,ds. \tag{4.14}$$

It follows from (4.14) that there exist constants $C,c>0$ such that

$$\Gamma_\varepsilon(x,y)\le\frac{C}{|x-y|^{d-2}}\exp(-c|x-y|^2/(\varepsilon R(x))^2). \tag{4.15}$$

For a fixed nonnegative M we set $w_M(x)=(1+|x|/R(0))^M$.

Proposition 4.16. $\lim_{\varepsilon\to0^+}\|A_\varepsilon\|_{L^1(w_M(x)\,dx)\to L^1(w_M(x)\,dx)}=0$.

Proof. It suffices to show that

$$I=\int V(x)\Gamma_\varepsilon(x,y)w_M(x)\,dx\le c(\varepsilon)w_M(y), \tag{4.17}$$

where $\lim_{\varepsilon\to0^+}c(\varepsilon)=0$. Split

$$I=\int V(x)\Gamma_\varepsilon(x,y)w_M(x)\,dx=\int\limits_{|x-y|\le 2R(y)}+\int\limits_{|x-y|>2R(y)}=I_1+I_2.$$

By (4.15) and Corollary 4.6 we have

$$I_1 \leq C \sum_{j=-1}^{\infty} \int_{2^{-j-1}R(y)\leq|x-y|\leq 2^{-j}R(y)} V(x)2^{j(d-2)}R(y)^{2-d} \times \exp(-c'2^{-j}/\varepsilon)\left(1+\frac{|x|}{R(0)}\right)^M dx.$$

Applying Corollaries 4.6 and 4.2, and the fact that $1+|x|/R(0) \sim 1+|y|/R(0)$ for $|x-y| \leq 2R(y)$ (cf. Lemma 4.3), we obtain

$$I_1 \leq C\left(1+\frac{|y|}{R(0)}\right)^M \sum_{j=-1}^{\infty} (2^{-j})^{\delta} \exp(-c'2^{-j}/\varepsilon). \tag{4.18}$$

Now we estimate I_2. By (4.15),

$$I_2 \leq C \sum_{j=1}^{\infty} \int_{2^{j}R(y)\leq|x-y|\leq 2^{j+1}R(y)} V(x)(2^jR(y))^{2-d} \times \exp\left(\frac{-c'|x-y|}{\varepsilon R(x)}\right)\left(1+\frac{|x|}{R(0)}\right)^M dx.$$

It follows from (4.4) that

$$|x|m(0,V) \leq C(1+|y|m(0,V))(1+|x-y|m(x,V))^{k_0+1}.$$

Thus, using Lemma 4.7, we have

$$I_2 \leq C \sum_{j=1}^{\infty} \int_{2^{j}R(y)\leq|x-y|\leq 2^{j+1}R(y)} V(x)(2^jR(y))^{2-d} \times \exp\left(\frac{-c_1|x-y|}{\varepsilon R(x)}\right)\left(1+\frac{|y|}{R(0)}\right)^M dx.$$

Observe that, by (4.5), $R(x)^{-1} \geq cR(y)^{-1}(1+2^j)^{-k_0/(1+k_0)}$ for $|x-y| \sim 2^jR(y)$. Hence, by Lemma 4.7, we obtain

$$I_2 \leq C\left(1+\frac{|y|}{R(0)}\right)^M \sum_{j=1}^{\infty} 2^{Cj} \exp(-c_2 2^{j/k_0}/\varepsilon). \tag{4.19}$$

Now (4.17) follows from (4.18) and (4.19). ∎

Setting $M=0$ we get

Corollary 4.20.

$$\sup_{y\in\mathbb{R}^d} \int V(x)|x-y|^{2-d} \exp(-c|x-y|^2/(\varepsilon R(x))^2)\, dx \leq c(\varepsilon),$$

where $\lim_{\varepsilon\to 0^+} c(\varepsilon) = 0$.

Proof of Lemma 1.6. Applying Proposition 4.16 with $M = 0$, we obtain $\|A_\varepsilon^*\|_{L^\infty\to L^\infty} \le c(\varepsilon)$, where $\lim_{\varepsilon\to 0^+} c(\varepsilon) = 0$. Since $G_\varepsilon(x) - \mathbf{1} = \sum_{n=1}^{\infty}((-A_\varepsilon^*)^n\mathbf{1})(x)$, we get

$$\lim_{\varepsilon\to 0^+} \|G_\varepsilon - \mathbf{1}\|_{L^\infty} \le \lim_{\varepsilon\to 0^+} \sum_{n=1}^{\infty} c(\varepsilon)^n = 0. \ \blacksquare$$

Proof of Lemma 1.7. We shall show that for every $\delta' < \delta_0$ there exist constants $C_{\delta'}$ and $\varepsilon_0 > 0$ such that

$$|G_\varepsilon(x+h) - G_\varepsilon(x)| \le C_{\delta'}(|h|m(x,V))^{\delta'} \tag{4.21}$$

for $0 < \varepsilon < \varepsilon_0$. Let $A_\varepsilon^*(x,y) = A_\varepsilon(y,x) = V(y)\varGamma_\varepsilon(y,x)$ be the kernels of the operators A_ε^*. We are going to prove that

$$I = \int |A_\varepsilon^*(x+h,y) - A_\varepsilon^*(x,y)|\,dy \le C_{\delta'}(|h|m(x,V))^{\delta'}. \tag{4.22}$$

It suffices to show (4.21) for $|h|m(x,V) \le 1/4$. We have

$$I = \int\limits_{|x-y|\le 4|h|} + \int\limits_{4|h|<|x-y|\le R(x)} + \int\limits_{|x-y|>R(x)} = I_1 + I_2 + I_3.$$

Applying (4.15) and Corollary 4.2 we get

$$\begin{aligned}
I_1 &\le C \int\limits_{|x-y|\le 4|h|} (A_\varepsilon^*(x+h,y) + A_\varepsilon^*(x,y))\,dy \\
&\le C \int\limits_{|x-y|\le 4|h|} V(y)|x-y|^{2-d}\,dy \\
&\quad + C \int\limits_{|x+h-y|\le 5|h|} V(y)|x+h-y|^{2-d}\,dy \\
&\le C \sum_{j\ge 0} \int\limits_{2^{-j+1}|h|<|x-y|<2^{-j+2}|h|} V(y)(2^{-j}|h|)^{2-d}\,dy \\
&\quad + C \sum_{j\ge 0} \int\limits_{2^{-j+2}|h|<|x+h-y|<2^{-j+3}|h|} V(y)(2^{-j}|h|)^{2-d}\,dy \\
&\le C(|h|m(x,V))^{\delta} + C(|h|m(x+h,V))^{\delta}.
\end{aligned}$$

Hence, by Corollary 4.6,

$$I_1 \le C(|h|m(x,V))^{\delta}.$$

Note that for $|h| < |x-y|/4$ we have

$$|A_\varepsilon^*(x+h,y) - A_\varepsilon^*(x,y)| \le CV(y)\,\frac{|h|}{|x-y|^{d-1}}\,e^{-c|x-y|^2/(\varepsilon^2 R(y)^2)}.$$

Application of Lemma 4.3 leads to

$$|A_\varepsilon^*(x+h,y) - A_\varepsilon^*(x,y)| \le CV(y)\,\frac{|h|}{|x-y|^{d-1}}\,e^{-c|x-y|^\gamma/(\varepsilon^2 R(x)^\gamma)}, \tag{4.23}$$

with a constant $\gamma > 0$. Therefore setting $n = [\log_2(R(x)/|h|)] + 1$, and using (4.23) and Corollary 4.2, we obtain

$$\begin{aligned} I_2 &\le C \int_{4|h|<|x-y|\le R(x)} V(y) \frac{|h|}{|x-y|^{d-1}}\, dy \\ &\le C \sum_{j=2}^{n} \int_{2^j|h|<|x-y|\le 2^j|h|} V(y) \frac{|h|}{(2^j|h|)^{d-1}}\, dy \\ &\le C \sum_{j=1}^{n} 2^{-j}(2^j m(x,V)|h|)^{\delta} \le C(m(x,V)|h|)^{\delta'}. \end{aligned}$$

Finally, by (4.23) and Lemma 4.7, we get

$$\begin{aligned} I_3 &\le C \sum_{j\ge 0} \int_{2^j R(x)<|x-y|<2^{j+1}R(x)} V(y) \frac{|h|}{(2^j R(x))^{d-1}} e^{-c(2^j R(x)/(\varepsilon^2 R(x)))^{\gamma}} \\ &\le C \sum_{j\ge 0} \frac{|h|}{R(x)} 2^{jC} e^{-c(2^j/\varepsilon^2)^{\gamma}} \le C(m(x,V)|h|), \end{aligned}$$

which completes the proof of (4.22). It follows from (4.22) that

$$|A_\varepsilon^* f(x+h) - A_\varepsilon^* f(x)| \le C(|h| m(x,V))^{\delta'} \|f\|_{L^\infty}. \tag{4.24}$$

Now (4.21) is a consequence of (4.24). Indeed,

$$\begin{aligned} |G_\varepsilon(x+h) - G_\varepsilon(x)| &= \Big| \sum_{n=1}^{\infty} ((-A_\varepsilon^*)^n \mathbf{1}(x+h) - (-A_\varepsilon^*)^n \mathbf{1}(x)) \Big| \\ &= \Big| \sum_{n=0}^{\infty} -A_\varepsilon^*((-A_\varepsilon^*)^n \mathbf{1})(x+h) + A_\varepsilon^*((-A_\varepsilon^*)^n \mathbf{1})(x) \Big| \\ &\le \sum_{n=0}^{\infty} C(|h| m(x,V))^{\delta'} \|(-A_\varepsilon^*)^n \mathbf{1}\|_{L^\infty} \\ &\le C(|h| m(x,V))^{\delta'} \sum_{n=0}^{\infty} \|A_\varepsilon^*\|^n_{L^\infty\to L^\infty} \\ &\le C(|h| m(x,V))^{\delta'} \sum_{n=0}^{\infty} c(\varepsilon)^n \le C(|h| m(x,V))^{\delta'}. \quad \blacksquare \end{aligned}$$

5. Proof of Lemma 2.9. For $\varepsilon > 0$, $y_0 \in \mathbb{R}^d$, $0 < r \le \varepsilon R(y_0)$, and $M \ge 0$ we define the space $L^1_{\varepsilon,r,y_0,M}$ by

$$L^1_{\varepsilon,r,y_0,M} = \Big\{ f : \int |f(x)| \Big(1 + \frac{|x-y_0|}{r}\Big)\Big(1 + \frac{|x-y_0|}{\varepsilon R(y_0)}\Big)^M dx = \|f\|_{L^1_{\varepsilon,r,y_0,M}} < \infty \Big\}.$$

Let $L^1_{\varepsilon,r,y_0,M,0} = \{f \in L^1_{\varepsilon,r,y_0,M} : \int f(x)\,dx = 0\}$. Set

$$\mathcal{G}_\varepsilon f(x) = (G_\varepsilon(x) - \mathbf{1})f(x) + G_\varepsilon(x)A_\varepsilon f(x). \tag{5.1}$$

LEMMA 5.2. *For every $M \geq 0$ we have*

$$\lim_{\varepsilon\to 0^+} \|\mathcal{G}_\varepsilon\|_{L^1_{\varepsilon,r,y_0,M,0}\to L^1_{\varepsilon,r,y_0,M,0}} = 0$$

uniformly with respect to y_0 and r.

Proof. Note that $\int \mathcal{G}_\varepsilon f(x)\,dx = 0$. Indeed, by the definition of G_ε,

$$\begin{aligned}\int \mathcal{G}_\varepsilon f(x)\,dx &= \int (G_\varepsilon(x)(\mathrm{Id} + A_\varepsilon)f(x) - f(x))\,dx \\ &= \int ((\mathrm{Id} + A^*_\varepsilon)G_\varepsilon(x))f(x)\,dx = \int f(x)\,dx = 0.\end{aligned}$$

Therefore, by Lemma 1.6, it suffices to show that

$$\lim_{\varepsilon\to 0^+} \|A_\varepsilon\|_{L^1_{\varepsilon,r,y_0,M,0}\to L^1_{\varepsilon,r,y_0,M}} = 0 \quad \text{uniformly with respect to } y_0 \text{ and } r.$$

There is no loss of generality in assuming that $y_0 = 0$. Since

$$\begin{aligned}A_\varepsilon f(x) &= \int V(x)\Gamma_\varepsilon(x,y)f(y)\,dy \\ &= \int V(x)(\Gamma_\varepsilon(x,y) - \Gamma_\varepsilon(x,0))f(y)\,dy,\end{aligned}$$

we need only show that

$$\begin{aligned}J_1 &= \int V(x)|\Gamma_\varepsilon(x,y) - \Gamma_\varepsilon(x,0)|\left(1 + \frac{|x|}{r}\right)\left(1 + \frac{|x|}{\varepsilon R(0)}\right)^M dx \\ &\leq c(\varepsilon)\left(1 + \frac{|y|}{r}\right)\left(1 + \frac{|y|}{\varepsilon R(0)}\right)^M,\end{aligned}$$

with $c(\varepsilon) \to 0$ as $\varepsilon \to 0$. Note that there is a constant $C > 0$ such that

$$|\Gamma_\varepsilon(x,y) - \Gamma_\varepsilon(x,0)| \leq C\,\frac{|y|}{|x|^{d-1}}\,\exp(-c|x|^2/(\varepsilon R(x))^2) \quad \text{for } 4|y| < |x|.$$

Thus

$$\begin{aligned}J_1 &\leq \int_{|x|>4|y|} + \int_{|x|\leq 4|y|} \\ &\leq C\int_{|x|>4|y|} V(x)\frac{|y|}{|x|^{d-1}}\exp(-c|x|^2/(\varepsilon R(x))^2)\left(1 + \frac{|x|}{\varepsilon R(0)}\right)^M dx \\ &\quad + C\int_{|x|>4|y|} V(x)\frac{|y|}{|x|^{d-1}}\exp(-c|x|^2/(\varepsilon R(x))^2)\frac{|x|}{r}\left(1 + \frac{|x|}{\varepsilon R(0)}\right)^M dx \\ &\quad + C\int_{|x|\leq 4|y|} V(x)(\Gamma_\varepsilon(x,y) + \Gamma_\varepsilon(x,0))\left(1 + \frac{|x|}{r}\right)\left(1 + \frac{|x|}{\varepsilon R(0)}\right)^M dx \\ &= J_1^{(1)} + J_1^{(2)} + J_1^{(3)}.\end{aligned}$$

Obviously, by (4.4), since $0<\varepsilon<1$, we have

$$1+\frac{|x|}{\varepsilon R(0)}\le C\Big(1+\frac{|x|}{\varepsilon R(x)}\Big)^{k_0+1}. \tag{5.3}$$

Therefore, applying Corollary 4.20, we get

$$\begin{aligned}J_1^{(1)}&\le C\int_{|x|>4|y|}V(x)\frac{1}{|x|^{d-2}}\exp\Big(\frac{-c|x|^2}{(\varepsilon R(x))^2}\Big)\Big(1+\frac{|x|}{\varepsilon R(x)}\Big)^{M(k_0+1)}dx\\ &\le c(\varepsilon)\le c(\varepsilon)\Big(1+\frac{|y|}{r}\Big)\Big(1+\frac{|y|}{\varepsilon R(0)}\Big)^{M}.\end{aligned}$$

Similarly

$$\begin{aligned}J_1^{(2)}&\le C\int_{|x|>4|y|}V(x)\,\frac{|y|}{r}\,\frac{1}{|x|^{d-2}}\,\exp\Big(\frac{-c|x|^2}{(\varepsilon R(x))^2}\Big)\Big(1+\frac{|x|}{\varepsilon R(x)}\Big)^{M(k_0+1)}dx\\ &\le c(\varepsilon)\,\frac{|y|}{r}\le c(\varepsilon)\Big(1+\frac{|y|}{r}\Big)\Big(1+\frac{|x|}{\varepsilon R(0)}\Big)^{M}.\end{aligned}$$

In order to estimate $J_1^{(3)}$ we use again (4.15) and Corollary 4.20 to obtain

$$\begin{aligned}J_1^{(3)}&\le C\int_{|x|\le 4|y|}V(x)(\varGamma_\varepsilon(x,y)+\varGamma_\varepsilon(x,0))\Big(1+\frac{|y|}{r}\Big)\Big(1+\frac{|y|}{\varepsilon R(0)}\Big)^{M}dx\\ &\le c(\varepsilon)\Big(1+\frac{|y|}{r}\Big)\Big(1+\frac{|y|}{\varepsilon R(0)}\Big)^{M}. \ \blacksquare\end{aligned}$$

Lemma 5.4. *Fix $M\ge 0$. If $\varepsilon R(y_0)/4<r\le\varepsilon R(y_0)$ then*

$$\lim_{\varepsilon\to 0^+}\|\mathcal{G}_\varepsilon\|_{L^1(\varepsilon,r,y_0,M)\to L^1(\varepsilon,r,y_0,M)}=0$$

uniformly with respect to y_0 and r.

Proof. By Lemma 1.6 it is enough to show that

$$\begin{aligned}&\int V(x)\varGamma_\varepsilon(x,y)\Big(1+\frac{|x-y_0|}{r}\Big)\Big(1+\frac{|x-y_0|}{\varepsilon R(y_0)}\Big)^{M}dx\\ &\qquad\le c(\varepsilon)\Big(1+\frac{|y-y_0|}{r}\Big)\Big(1+\frac{|y-y_0|}{\varepsilon R(y_0)}\Big)^{M}.\end{aligned}$$

We shall prove this for $y_0=0$. The proof for arbitrary y_0 is identical. By

(5.3), (4.15), and Corollary 4.20, we get

$$\begin{aligned}\int V(x)\Gamma_\varepsilon(x,y)\Big(1+\frac{|x|}{r}\Big)\Big(1+\frac{|x|}{\varepsilon R(0)}\Big)^M dx \\ \le C\int_{|x|\le 4|y|} + C\int_{|x|>4|y|} \\ \le C\int_{|x|\le 4|y|} V(x)\Gamma_\varepsilon(x,y)\Big(1+\frac{|y|}{r}\Big)\Big(1+\frac{|y|}{\varepsilon R(0)}\Big)^M dx \\ + C\int_{|x|>4|y|} V(x)\Gamma_\varepsilon(x,y)\Big(1+\frac{|x-y|}{\varepsilon R(x)}\Big)^{(k_0+1)(M+1)} dx \\ \le c(\varepsilon)\Big(1+\frac{|y|}{r}\Big)\Big(1+\frac{|y|}{\varepsilon R(0)}\Big)^M + c(\varepsilon). \ \blacksquare\end{aligned}$$

Proof of Lemma 2.9. Since $G_\varepsilon(\mathrm{Id}+A_\varepsilon) = \mathrm{Id}+\mathcal{G}_\varepsilon$, Lemma 2.9 follows from Lemma 5.2, Lemma 5.4, and the equality

$$(\mathrm{Id}+A_\varepsilon)^{-1}f = \Big(\sum_{n=0}^{\infty}(-\mathcal{G}_\varepsilon)^n\Big)(G_\varepsilon f). \ \blacksquare$$

6. Estimates of the kernels E_t, H_t and related maximal functions

LEMMA 6.1. *There exist constants $C, c > 0$ such that for every $\eta > 0$ and every $y \in \mathbb{R}^d$ we have*

$$\|T_t\|_{L^2(e^{\eta|x-y|}\,dx)\to L^2(e^{\eta|x-y|}\,dx)} \le Ce^{ct\eta^2}.$$

Proof. This is a direct consequence of (1.2). ■

COROLLARY 6.2. *The semigroup T_t has the (unique) extension to a holomorphic semigroup T_ζ on $L^2(e^{\eta|x-y|}\,dx)$ in the sector $\Delta_{\pi/4} = \{\zeta : |\mathrm{Arg}\,\zeta| < \pi/4\}$. Moreover, there exist constants $C, c' > 0$ such that for every $\eta > 0$ we have*

$$\|T_\zeta\|_{L^2(e^{\eta|x-y|}\,dx)\to L^2(e^{\eta|x-y|}\,dx)} \le Ce^{c'\eta^2\Re\zeta}.$$

Proof. See the proof of Proposition 3.2 in [DZ3]. ■

Let $k_\zeta(x,y)$ be the integral kernel of the operator T_ζ.

LEMMA 6.3. *There exists a constant $c > 0$ such that for every $M > 0$ there exists a constant $C > 0$ such that for every $\eta > 0$ and every $y \in \mathbb{R}^d$ we have*

$$\int |k_\zeta(x,y)|^2 e^{\eta|x-y|}\,dx \le Ce^{c\eta^2\Re\zeta}(\Re\zeta)^{-d/2}\Big(1+\frac{\Re\zeta}{R(y)^2}\Big)^{-M} \quad \textit{for } \zeta\in\Delta_{\pi/5}.$$

Proof. Let $t = \Re\zeta$. Since $k_\zeta(x,y) = [T_{\zeta-t/10}k_{t/10}(\,\cdot\,,y)](x)$, using Corollary 6.2, we obtain

$$\int |k_\zeta(x,y)|^2 e^{\eta|x-y|}\,du \le Ce^{c\eta^2 t}\int |k_{t/10}(u,y)|^2 e^{\eta|u-y|}\,du.$$

Applying Theorem 4.10 we get

$$\begin{aligned}\int |k_{t/10}(u,y)|^2 e^{\eta|u-y|}\,du &\le C\int \left(1+\frac{\sqrt{t}}{R(y)}\right)^{-2M} t^{-d}e^{-c|u-y|^2/t}e^{\eta|u-y|}\,du\\ &\le Ct^{-d/2}e^{2c\eta^2 t}\left(1+\frac{t}{R(y)^2}\right)^{-M}. \ \blacksquare\end{aligned}$$

Corollary 6.4. *There exists a constant $c>0$ such that for every $M\ge 0$ there is a constant C_M such that*

$$|k_\zeta(x,y)| \le C_M(\Re\zeta)^{-d/2}\left(1+\frac{\Re\zeta}{R(y)^2}\right)^{-M}\left(1+\frac{\Re\zeta}{R(x)^2}\right)^{-M}e^{-c|x-y|^2/\Re\zeta}$$

for $\zeta \in \Delta_{\pi/5}$.

Proof. We have

$$\begin{aligned}|k_\zeta(x,y)|e^{\eta|x-y|} &= \left|\int k_{\zeta/2}(x,u)k_{\zeta/2}(u,y)\,du\right|e^{\eta|x-y|}\\ &\le \left(\int |k_{\zeta/2}(x,u)|^2 e^{2\eta|x-u|}\,du\right)^{1/2}\left(\int |k_{\zeta/2}(u-y)|^2 e^{2\eta|u-y|}\,du\right)^{1/2}\\ &\le C_M(\Re\zeta)^{-d/2}e^{c\eta^2\Re\zeta}\left(1+\frac{\Re\zeta}{R(y)^2}\right)^{-M}.\end{aligned}$$

Setting $\eta = c''|x-y|(\Re\zeta)^{-1}$ (with $c''>0$ small enough) and using the fact that $|k_\zeta(x,y)| = |k_{\bar\zeta}(y,x)|$ we get the required estimate. ■

Proposition 6.5. *There exists a constant $c>0$ such that for every $M>0$ there exists a constant $C>0$ such that*

$$|k_{t+s}(x,y)-k_t(x,y)| \le C\,\frac{s}{t}\,t^{-d/2}e^{-c|x-y|^2/t}\left(1+\frac{t}{R(y)^2}\right)^{-M}\left(1+\frac{t}{R(x)^2}\right)^{-M}$$

for $0<s<t$.

Proof. By Corollary 6.4 it suffices to prove the estimate for $0<s<t/20$. Using the Cauchy integral formula and Corollary 6.4 we get

$$\begin{aligned}|k_{t+s}(x,y)-k_t(x,y)| &= \left|\int_0^s \frac{d}{dt}k_{t+\tau}(x,y)\,d\tau\right|\\ &= C\left|\int_0^s \int_{|\zeta-t|=t/10} \frac{k_\zeta(x,y)}{(\zeta-t-\tau)^2}\,d\zeta\,d\tau\right|\end{aligned}$$

$$\leq C \int_0^s \int_{|\zeta - t| = t/10} \frac{|k_\zeta(x,y)|}{|\zeta - t - \tau|^2}\, d|\zeta|\, d\tau$$

$$\leq Cs\,\frac{t}{t^2}\, t^{-d/2} e^{-c|x-y|^2/t} \left(1 + \frac{t}{R(y)^2}\right)^{-M} \left(1 + \frac{t}{R(x)^2}\right)^{-M}. \quad \blacksquare$$

LEMMA 6.6. *There exists a rapidly decaying function* ψ *such that*

$$H_t(x,y) \leq \begin{cases} (\sqrt{t}\, m(x,V))^\delta \psi_t(x-y) & \text{for } t < m(x,V)^{-2}, \\ \psi_t(x-y) & \text{for } t \geq m(x,V)^{-2}. \end{cases} \tag{6.7}$$

Proof. From Theorem 4.10 we conclude

$$H_t(x,y) \leq C \int_{t/2}^t \int (t-s)^{-d/2} e^{-c|z|/\sqrt{t-s}} V(z+x)$$
$$\times\, t^{-d/2} e^{-c|z+x-y|/\sqrt{t}} \left(1 + \frac{t-s}{R(x)^2}\right)^{-M} dz\, ds$$
$$\leq C \int_{t/2}^t \int_{|z| \leq |x-y|/4} + C \int_{t/2}^t \int_{|z| > |x-y|/4}.$$

We note that for $|z| > |x-y|/4$ we have

$$(t-s)^{-d/2} e^{-c|z|/\sqrt{t-s}} \leq C(t-s)^{-d/2} e^{-c'|x-y|/\sqrt{t}} e^{-c'|z|/\sqrt{t-s}}.$$

Thus

$$H_t(x,y) \leq C_M \int_{t/2}^t \int (t-s)^{-d/2} e^{-c'|z|/\sqrt{t-s}} V(z+x)$$
$$\times\, t^{-d/2} e^{-c'|x-y|/\sqrt{t}} \left(1 + \frac{t-s}{R(x)^2}\right)^{-M} dz\, ds.$$

Set $\psi_t(x) = t^{-d/2} e^{-c'|x|/\sqrt{t}}$. If $t < m(x,V)^{-2}$ then, by Corollary 4.8, we obtain

$$H_t(x,y) \leq C\psi_t(x-y) \int_{t/2}^t (t-s)^{-1} (m(x,V)\sqrt{t-s})^\delta\, ds$$
$$\leq C\psi_t(x-y)(m(x,V)\sqrt{t})^\delta.$$

If $t \geq m(x,V)^{-2}$ then

$$H_t(x,y) \leq \psi_t(x-y) \int_{t/2}^{t} \int \frac{e^{-c'|z|/\sqrt{t-s}}}{(t-s)^{d/2}} V(z+x)\left(1+\frac{t-s}{R(x)^2}\right)^{-M} dz\, ds$$
$$\leq \psi_t(x-y) \int_0^{t/2} \frac{e^{-c'|z|/\sqrt{s}}}{s^{d/2}} V(z+x)\left(1+\frac{s}{R(x)^2}\right)^{-M} dz\, ds.$$

Applying again Corollary 4.8 we get

$$H_t(x,y) \leq \psi_t(x-y)\Big(\int_0^{R(x)^2} s^{-1}(m(x,V)\sqrt{s})^{\delta}\, ds$$
$$+ \int_{R(x)^2}^{t} s^{-d/2}(\sqrt{s}\, m(x,V))^{-M+C} m(x,V)^{2-d}\, ds\Big)$$
$$\leq C\psi_t(x-y). \ \blacksquare$$

LEMMA 6.8. *There exists a rapidly decaying function ψ such that*

$$|H_t(x,y+h) - H_t(x,y)| \leq \frac{|h|}{\sqrt{t}}\,(m(x,V)\sqrt{t})^{\delta}\psi_t(x-y) \tag{6.9}$$

for $t \leq Cm(x,V)^{-2}$, $|h| \leq |x-y|/8$, and

$$|H_t(x,y+h) - H_t(x,y)| \leq \frac{|h|}{\sqrt{t}}\,\psi_t(x-y) \tag{6.10}$$

for $t \geq Cm(x,V)^{-2}$, $|h| < |x-y|/8$.

Proof. It suffices to show (6.9) and (6.10) for $2|h| \leq \sqrt{t}$. We have

$$|H_t(x,y+h) - H_t(x,y)|$$
$$= \Big| \int_{t/2}^{t} \int k_{t-s}(x,z)V(z)(p_s(z-y-h) - p_s(z-y))\, dz\, ds \Big|.$$

Since $2|h| \leq \sqrt{t}$ and $t/2 \leq s \leq t$, we have

$$|p_s(z-y-h) - p_s(z-y)| \leq C\,\frac{|h|}{\sqrt{t}}\, t^{-d/2} e^{-c|z-y|/\sqrt{t}}.$$

Therefore

$$|H_t(x,y+h) - H_t(x,y)|$$
$$\leq C \int_{t/2}^{t} \int k_{t-s}(z)V(z+x)\,\frac{|h|}{\sqrt{t}}\, t^{-d/2} e^{-c|z+x-y|/\sqrt{t}}\, dz\, ds.$$

Using the same arguments as in the proof of Lemma 6.6 we get (6.9) and (6.10). $\blacksquare$

LEMMA 6.11. *There exists a rapidly decaying function φ such that for every $M > 0$ there is a constant C_M such that*

$$|E_t(x,y)| \le C_M(\sqrt{t}\, m(x,V))^{\delta}\varphi_t(x-y) \times \left(1+\frac{t}{R(x)^2}\right)^{-M}\left(1+\frac{t}{R(y)^2}\right)^{-M}. \tag{6.12}$$

Proof. Applying Proposition 6.5 and (4.5), we obtain

$$|E_t(x,y)| \le C\int_0^{t/2}\int \frac{s}{t}\, t^{-d/2}e^{-c|x-y-z|/\sqrt{t}}V(z+y) \times s^{-d/2}e^{-c|z|/\sqrt{s}}\left(1+\frac{t}{R(x)^2}\right)^{-M}\left(1+\frac{t}{R(y)^2}\right)^{-M} dz\,ds.$$

Now splitting the integral on the right-hand side into two integrals, we get

$$\begin{aligned}
|E_t(x,y)| &\le C\int_0^{t/2}\int_{|z|\le|x-y|/4} + C\int_0^{t/2}\int_{|z|>|x-y|/4}\\
&\le C_M\phi_t(x-y)\left(1+\frac{t}{R(x)^2}\right)^{-M}\left(1+\frac{t}{R(y)^2}\right)^{-M}\\
&\quad\times\int_0^{t/2}\int \frac{s}{t}\, V(z+y)s^{-d/2}e^{-c|z|/\sqrt{s}}\,dz\,ds\\
&\quad + C_M\left(1+\frac{t}{R(x)^2}\right)^{-M}\left(1+\frac{t}{R(y)^2}\right)^{-M}\\
&\quad\times\int_0^{t/2}\int_{|z|>|x-y|/4}\frac{s}{t}\, t^{-d/2}V(z+y)s^{-d/2}e^{-c'|z|/\sqrt{s}}e^{-c'|z|/\sqrt{s}}\,dz\,ds\\
&\le C_M\phi_t(x-y)\left(1+\frac{t}{R(x)^2}\right)^{-M}\left(1+\frac{t}{R(y)^2}\right)^{-M}\\
&\quad\times\int_0^{t/2}\int \frac{s}{t}\, V(z+y)s^{-d/2}e^{-c'|z|/\sqrt{s}}\,dz\,ds\\
&= C_M\phi_t(x-y)\left(1+\frac{t}{R(x)^2}\right)^{-M}\left(1+\frac{t}{R(y)^2}\right)^{-M}\\
&\quad\times\int_0^{\min(t/2,R(y)^2)}\int \frac{s}{t}\, V(z+y)\psi_s(z)\,dz\,ds
\end{aligned}$$

$$+ C_M\phi_t(x-y)\Big(1+\frac{t}{R(x)^2}\Big)^{-M}\Big(1+\frac{t}{R(y)^2}\Big)^{-M}$$
$$\times \int\limits_{\min(t/2,R(y)^2)}^{t/2} \int \frac{s}{t} V(z+y)\psi_s(z)\,dz\,ds,$$

where ϕ and ψ are rapidly decaying functions. By Corollary 4.8 we have

$$|E_t(x,y)| \le C_M t^{-1}\phi_t(x-y)\Big(1+\frac{t}{R(x)^2}\Big)^{-M}\Big(1+\frac{t}{R(y)^2}\Big)^{-M}$$
$$\times \int\limits_0^{\min(t/2,R(y)^2)} (\sqrt{s}\,m(y,V))^{\delta}\,ds$$
$$+ C_M t^{-1}\phi_t(x-y)\Big(1+\frac{t}{R(x)^2}\Big)^{-M}\Big(1+\frac{t}{R(y)^2}\Big)^{-M}$$
$$\times \int\limits_{\min(t/2,R(y)^2)}^{t/2} s s^{-d/2}\frac{(\sqrt{s}\,m(y,V))^{C_0}}{m(y,V)^{d-2}}\,ds$$
$$\le C_M\phi_t(x-y)(\sqrt{t}\,m(y,V))^{\delta}\Big(1+\frac{t}{R(x)^2}\Big)^{-M}\Big(1+\frac{t}{R(y)^2}\Big)^{-M}$$
$$+ C_M\phi_t(x-y)\Big(1+\frac{t}{R(x)^2}\Big)^{-M}\Big(1+\frac{t}{R(y)^2}\Big)^{-M}\Big(\frac{\sqrt{t}}{R(y)}\Big)^{C_0+2-d}.$$

Applying Lemma 4.3, we get

$$|E_t(x,y)| \le C_M\phi_t(x-y)\Big(1+\frac{|x-y|}{\sqrt{t}}\sqrt{t}\,m(x,V)\Big)^{k_0\delta}$$
$$\times (\sqrt{t}\,m(x,V))^{\delta}\Big(1+\frac{t}{R(x)^2}\Big)^{-M}\Big(1+\frac{t}{R(y)^2}\Big)^{-M}$$
$$+ C_M + \phi_t(x-y)\Big(1+\frac{|x-y|}{\sqrt{t}}\sqrt{t}\,m(x,V)\Big)^{k_0(2-d+C_0)}$$
$$\times (\sqrt{t}\,m(x,V))^{2-d+C_0}\Big(1+\frac{t}{R(x)^2}\Big)^{-M}\Big(1+\frac{t}{R(y)^2}\Big)^{-M}$$
$$\le C_M\varphi_t(x-y)\Big(\frac{\sqrt{t}}{R(x)}\Big)^{\delta}$$
$$\times \Big(1+\frac{t}{R(x)^2}\Big)^{-M+k_0(2-d+C_0+\delta)/2}\Big(1+\frac{t}{R(y)^2}\Big)^{-M}. \ \blacksquare$$

Using the same method as in the proofs of Lemmas 6.6, 6.8, 6.11 one can prove

LEMMA 6.13. *For every $M \geq 0$ there exists a rapidly decaying function φ such that*

$$(6.14)\qquad |E_t(x,y+h)-E_t(x,y)| \leq \frac{|h|}{\sqrt{t}}(\sqrt{t}\,m(x,V))^\delta \varphi_t(x-y)\left(1+\frac{t}{R(x)^2}\right)^{-M}\left(1+\frac{t}{R(y)^2}\right)^{-M}$$

provided $2|h|<\sqrt{t}$, $8|h|\leq |x-y|$.

Proof of (3.5) and (3.6). First we prove (3.6). Assume that a is an $(\mathbf{h}^p_\varepsilon(m),\infty)$-atom associated with a ball $B(x_0,r)$. Then, by the definition, $r\leq \varepsilon R(x_0)$. By Lemma 6.6, if $t<\varepsilon^2R(x)^2$ and $x\in B(x_0,8r)$, then

$$|H_ta(x)| = \left|\int H_t(x,y)a(y)\,dy\right| \leq C\varepsilon^\delta\|a\|_\infty \leq C\varepsilon^\delta r^{-d/p}.$$

Therefore

$$\int_{B(x_0,8r)} (\mathcal{H}^*_\varepsilon a(x))^p \leq C\varepsilon^{p\delta}.$$

In order to prove the required estimate on $B(x_0,8r)^{\mathrm{c}}$ we consider two cases.

CASE 1: $\frac{1}{4}\varepsilon R(x_0)<r\leq \varepsilon R(x_0)$. Then, by Lemma 6.6, for $t<\varepsilon^2R(x)^2$ and $x\in B(x_0,8r)^{\mathrm{c}}$, we have

$$\begin{aligned}|H_ta(x)| &\leq \varepsilon^\delta \int_{B(x_0,r)} |\psi_t(x-y)a(y)|\,dy\\ &\leq C_N\varepsilon^\delta\|a\|_{L^1}t^{-d/2}\left(1+\frac{|x-x_0|}{\sqrt{t}}\right)^{-2N}\\ &\leq C\varepsilon^\delta r^{-d/p+d}t^{-d/2}\left(1+\frac{|x-x_0|}{\sqrt{t}}\right)^{-2N}.\end{aligned}$$

It follows from (4.5) that $R(x)^2\leq C(1+|x-x_0|/R(x_0))^{2k_0/(1+k_0)}R(x_0)^2 = \tau(x,x_0)$. Thus

$$\begin{aligned}&\int_{B(x_0,8r)^{\mathrm{c}}} (\mathcal{H}^*_\varepsilon a(x))^p\,dx\\ &\quad\leq C_N\varepsilon^{p\delta}r^{-d+dp}\int_{B(x_0,8r)^{\mathrm{c}}} \sup_{0<t<\varepsilon^2\tau(x,x_0)} t^{-dp/2}\left(1+\frac{|x-x_0|}{\sqrt{t}}\right)^{-2Np}dx\\ &\quad\leq C_N\varepsilon^{p\delta}.\end{aligned}$$

CASE 2: $r\leq \frac{1}{4}\varepsilon R(x_0)$. Then $\int a=0$. Therefore, by Lemma 6.8, for $|x-x_0|>8r$ and $t<\varepsilon^2R(x)^2$, we have

$$|H_t a(x)| = \Big| \int_{B(x_0,r)} (H_t(x,y) - H_t(x,x_0))a(y)\,dy \Big|$$
$$\le C\varepsilon^\delta \int_{B(x_0,r)} \frac{|y-x_0|}{\sqrt{t}}\,\psi_t(x-x_0)|a(y)|\,dy.$$

This leads to

$$\int_{B(x_0,8r)^c} (\mathcal{H}^*_\varepsilon a(x))^p\,dx \le C\varepsilon^{p\delta}.$$

The proof of (3.5) is identical and uses Lemmas 6.11 and 6.13. ∎

7. Maximal functions $\mathcal{Z}^*_\varepsilon$. Our goal in the present section is to prove (3.7). In order to do this it suffices to show that there exists a function $c(\varepsilon)$ satisfying $\lim_{\varepsilon\to0^+} c(\varepsilon) = 0$ such that

$$\|\mathcal{Z}^*_\varepsilon a\|_{L^p} \le c(\varepsilon) \tag{7.1}$$

for every $(\mathbf{h}^p_\varepsilon(m),\infty)$-atom a. There is no loss of generality in assuming that if a is an $(\mathbf{h}^p_\varepsilon(m),\infty)$-atom associated with $B(x_0,r)$, and if $r < \frac14\varepsilon R(x_0)$, then

$$\int x^\alpha a(x)\,dx = 0 \quad \text{for } |\alpha| \le C_0 + d + 4, \tag{7.2}$$

where C_0 is a constant from Corollary 4.8. Indeed, every $(\mathbf{h}^p_\varepsilon(m),\infty)$-atom a satisfying (2.4) can be decomposed as $a = \sum c_j a'_j$, where a'_j satisfies (2.1), (2.2), (2.3) and (7.2) in such a way that $\sum_j |c_j|^p \le C$.

The following lemma can be easily proved.

LEMMA 7.3. *Assume that a is an $(\mathbf{h}^p_\varepsilon(m),\infty)$-atom associated with a ball $B = B(x_0,r)$, where $r < \varepsilon R(x_0)$. Then*

$$\Big|\int_\alpha^\beta a * p_s(z)\,ds\Big| \le C\,\frac{e^{-c|z-x_0|^2/\beta}}{|z-x_0|^{d-2+M} + \alpha^{(d-2+M)/2}}\,|B|^{1-1/p+M/d} \tag{7.4}$$

for $|z-x_0| > 2r$, where $M = C_0 + d + 4$ if $r \le \frac14\varepsilon R(x_0)$, and $M = 0$ if $\frac14\varepsilon R(0) < r < \varepsilon R(x_0)$.

Let a be as in Lemma 7.3 and let $K = B(x_0, R(x_0))$. We define

$$\begin{aligned}\mathcal{Z}^*_{\varepsilon,0}a(x) &= \sup_{0<t<(\varepsilon R(x))^2} |Z^0_{(\varepsilon),t}a(x)| \\ &= \sup_{0<t<(\varepsilon R(x))^2} \Big|\int_K k_t(x,z)V(z)W_{(\varepsilon),t}a(z)\,dz\Big|,\end{aligned} \tag{7.5}$$

$$\mathcal{Z}^*_{\varepsilon,\infty}a(x) = \sup_{0<t<(\varepsilon R(x))^2} \Big|\int_{K^c} k_t(x,z)V(z)W_{(\varepsilon),t}a(z)\,dz\Big|, \tag{7.6}$$

where $W_{(\varepsilon),t}a(z) = \int W_{(\varepsilon),t}(z,y)a(y)\,dy$ (cf. Section 3).

LEMMA 7.7. *There exists a function* $c(\varepsilon)$ *with* $\lim_{\varepsilon\to 0^+} c(\varepsilon) = 0$ *such that for every* $(\mathbf{h}^p_\varepsilon(m),\infty)$*-atom* a *associated with a ball* $B(x_0,r)$ *we have*

$$\|\mathcal{Z}^*_{\varepsilon,\infty} a\|^p_{L^p} \le c(\varepsilon). \tag{7.8}$$

Proof. There is no loss of generality in assuming that $x_0 = 0$. Then

$$\begin{aligned} \mathcal{Z}^*_{\varepsilon,\infty} a(x) &\le \sum_{j=0}^{\infty} \sup_{0<t<(\varepsilon R(x))^2} \int k_t(x,z) V(z) |W_{(\varepsilon),t} a(z)| \chi_{U_j}(z)\, dz \\ &= \sum_{j=0}^{\infty} f_j^*(x), \end{aligned} \tag{7.9}$$

where $U_j = B(0, 2^{j+1}R(0)) \setminus B(0, 2^j R(0))$. It follows from Lemma 4.3 that if $|x| < 2^{j+2}R(0)$, then $R(x) \le C 2^{jk_0/(1+k_0)} R(0)$. Therefore, by Lemma 7.3, there exists $\gamma > 0$ such that

$$\begin{aligned} V(z)|W_{(\varepsilon),t} a(z)| \chi_{U_j}(z) &\le C V(z) e^{-c(|z|/\varepsilon R(0))^\gamma} \frac{|B(0,r)|^{1-1/p+M/d}}{|z|^{d+M-2}} \chi_{U_j}(z) \\ &= f_j(z). \end{aligned}$$

One can check using Lemma 4.7 that

$$\|f_j\|_{L^1} \le e^{-c'(2^j/\varepsilon)^\gamma} |B(0, 2^j R(0))|^{1-1/p}.$$

This gives

$$\|f_j^*\|^p_{L^p(B(0,2^{j+2}R(0)))} \le c(\varepsilon) 2^{-j}. \tag{7.10}$$

We now turn to estimating f_j^* on the set $|x| > 2^{j+2}R(0)$. In this case

$$\begin{aligned} &V(z)|W_{(\varepsilon),t} a(z)| \chi_{U_j}(z) \\ &\quad \le \begin{cases} C V(z) e^{-c(|z|/\varepsilon R(0))^\gamma} \dfrac{|B(0,r)|^{1-1/p+M/d}}{|z|^{d+M-2}} \chi_{U_j}(z) & \text{if } t/2 \le (\varepsilon R(z))^2 \\[2ex] C V(z) e^{-c|z|^2/t} \dfrac{|B(0,r)|^{1-1/p+M/d}}{|z|^{d+M-2}} \chi_{U_j}(z) & \text{if } t/2 > (\varepsilon R(z))^2 \end{cases} \\ &\quad = f_j^{(x,t)}(z). \end{aligned}$$

Thus

$$\int k_t(x,z) V(z) |W_{(\varepsilon),t} a(z)| \chi_{U_j}(z)\, dz \le \varphi_t(x) \|f_j^{(x,t)}\|_{L^1},$$

where φ is a rapidly decaying function. Therefore, for $|x| > 2^{j+2}R(0)$, we have

$$f_j^*(x) \le \sup_{0<t<(\varepsilon R(x))^2} \varphi_t(x) \|f_j^{(x,t)}\|_{L^1}.$$

It is not difficult to verify using Lemmas 4.3 and 4.7 that

$$\|f_j^{(x,t)}\|_{L^1} \le C2^{Cj}|B(0,2^jR(0))|^{1-1/p}(e^{-c(2^j/\varepsilon)^\gamma} + e^{-c2^{2j}(R(0)/(\varepsilon|x|))^{\frac{2k_0}{k_0+1}}}).$$

Consequently,

$$f_j^*(x) \le c(\varepsilon)2^{-j}|B(0,2^jR(0))|^{1-1/p}(R(0)^{-d+N}|x|^{-N} + R(0)^{-d+L}|x|^{-L}).$$

This leads to

$$\|f_j^*\|^p_{L^p(B(0,2^{j+2}R(0))^c)} \le c(\varepsilon)^p 2^{-jp},$$

which combined with (7.9) and (7.10) completes the proof of the lemma. ∎

LEMMA 7.11. *There exists a function $c(\varepsilon)$ with $\lim_{\varepsilon\to 0^+} c(\varepsilon) = 0$ such that for every $(\mathbf{h}^p_\varepsilon(m),\infty)$-atom a associated with a ball $B(x_0,r)$, where $r < \frac{1}{4}\varepsilon R(x_0)$, we have*

$$\|\mathcal{Z}^*_{\varepsilon,0}a\|_{L^p} \le c(\varepsilon).$$

Proof. Similarly to the proof of Lemma 7.7 we assume that $x_0 = 0$. Let $C_1 > 4$ be such that $C_1^{-1/2} < m(x,V)/m(y,V) < C_1^{1/2}$ for $|x-y| < 16m(x,V)^{-1}$ (cf. Corollary 4.6).

CASE 1: $r^2 < t/2$. We have

$$\begin{aligned} |Z^0_{(\varepsilon),t}a(x)| &\le \Big|\int_{K_1} k_t(x,z)V(z)\int_{t/2}^{(\varepsilon R(z))^2} p_s * a(z)\,ds\,dz\Big| \\ &\quad + \Big|\int_{K_2} k_t(x,z)V(z)\int_{(\varepsilon R(z))^2}^{t/2} p_s * a(z)\,ds\,dz\Big| \\ &= J_{K_1}(x) + J_{K_2}(x), \end{aligned}$$

where $K_1 = \{z \in K : t/2 < (\varepsilon R(z))^2\}$ and $K_2 = K\setminus K_1$. From Corollary 4.6 we conclude

$$\begin{aligned} J_{K_1}(x) &\le \Big|\int_{K_1} k_t(x,z)V(z)\int_{t/2-t/4}^{(\varepsilon R(z))^2-t/4} p_{t/4} * p_s * a(z)\,ds\,dz\Big| \\ &\le \int_{K_1} k_t(x,z)V(z)\int_{t/4}^{C_1(\varepsilon R(0))^2}\int p_{t/4}(z-y)|p_s * a(y)|\,dy\,ds\,dz. \end{aligned}$$

Since $\int_{K_1} k_t(x,z)V(z)p_{t/4}(z-y)\,dz \le t^{-1}\phi_t(x-y)(t^{1/2}m(x,V))^\delta$, where ϕ is a rapidly decaying function, we get

$$J_{K_1}(x) \le \int t^{-1}\phi_t(x-y)c(\varepsilon)\int_{t/4}^{C_1(\varepsilon R(0))^2} |a * p_s(y)|\,ds\,dy.$$

Now using (7.2) we obtain

$$\begin{aligned}
J_{K_1}(x) \le & \int t^{-1}\phi_t(x-y) \\
& \times c(\varepsilon) \sum_{j\ge 0,\, 2^j t/4<2C_1(\varepsilon R(0))^2} \left(\frac{r}{2^{j/2}t^{1/2}}\right)^M \phi_{2^j t}(y) 2^j t \|a\|_{L^1} dy \\
\le & \sum_{j\ge 0} D_j(x),
\end{aligned}$$

where

$$\begin{aligned}
D_j(x) = & \sup_{r^2<t<2^{-j+1}C_1(\varepsilon R(0))^2} 2^j c(\varepsilon)\left(\frac{r}{2^{j/2}t^{1/2}}\right)^{M/2}\left(\frac{r}{2^{j/2}t^{1/2}}\right)^{M/2} \\
& \times \phi_{2^j t}(x)\|a\|_{L^1} \\
\le & \begin{cases} 2^j c(\varepsilon) 2^{-jM/2}(2^j r^2)^{-d/2}\|a\|_{L^1} & \text{for } |x|\le 2r, \\ 2^j c(\varepsilon) r^{M/2}|x|^{-d-M/2}2^{-jM/4}\|a\|_{L^1} & \text{for } |x|>2r. \end{cases}
\end{aligned}$$

This leads to

$$\int \sup_{0<t<(\varepsilon R(x))^2,\, 2r^2<t} |J_{K_1}(x)|^p\,dx \le c(\varepsilon)^p.$$

In order to estimate $J_{K_2}(x)$ we first consider $|x|>3R(0)$. There are rapidly decaying functions ϕ and ψ such that

$$\begin{aligned}
J_{K_2}(x) \le & \int_{K_2} k_t(x,z)V(z) \int_{(\varepsilon R(0)/C_4)^2}^{t/2} |p_s * a(z)|\,ds\,dz \\
\le & \int_{K_2} \phi_t(x)V(z)\|a\|_{L^1}\Bigg(\int_{(\varepsilon R(0)/C_4)^2}^{R(0)^2} \left(\frac{r}{\sqrt{s}}\right)^M \psi_s(z)\,ds \\
& + \int_{R(0)^2}^{\max(R(0)^2,t/2)} \left(\frac{r}{\sqrt{s}}\right)^M \psi_s(z)\,ds\Bigg)\,dz.
\end{aligned}$$

Applying Corollaries 4.6 and 4.8, we have

$$\begin{aligned}
J_{K_2}(x) \le & C\|a\|_{L^1}\phi_t(x)\Bigg(\int_{(\varepsilon R(0)/C_4)^2}^{R(0)^2} r^M s^{-M/2}(\sqrt{s}/R(0))^{\delta} s^{-1}\,ds \\
& + \int_{R(0)^2}^{\max(R(0)^2,t/2)} r^M s^{-M/2} s^{-d/2} R(0)^{d-2}\left(\frac{\sqrt{s}}{R(0)}\right)^{C_0} ds\Bigg) \\
\le & Cr^M\|a\|_{L^1}\phi_t(x)((\varepsilon R(0))^{-M} + R(0)^{-M}).
\end{aligned}$$

Since

$$\sup_{0<t<(\varepsilon R(x))^2} \phi_t(x) \le C\varepsilon^{-d+L}R(0)^{(-d+L)/(1+k_0)}|x|^{-L+k_0(-d+L)/(1+k_0)},$$

we get

$$\int_{|x|>3R(0)} \Big(\sup_{0<t<(\varepsilon R(x))^2,\, 2r^2<t} J_{K_2}(x)\Big)^p\,dx$$
$$\leq C\varepsilon^{(-d+L)p}R(0)^{-pd+d}\bigg(\Big(\frac{r}{\varepsilon R(0)}\Big)^{Mp}+\Big(\frac{r}{R(0)}\Big)^{Mp}\bigg)\|a\|_{L^1}^p \leq c(\varepsilon).$$

If $|x|\leq 3R(0)$ and $z\in K_2$, then $R(x)\sim R(0)\sim R(z)$. Therefore

$$J_{K_2}(x)=\Big|\int_{K_2} k_t(x,z)V(z)\int_{(\varepsilon R(z))^2}^{t/2} p_s*a(z)\,ds\,dz\Big|$$
$$\leq \int_{K_2} k_t(x,z)V(z)\int_{(\varepsilon R(0)/2C_4)^2}^{(C_4\varepsilon R(0))^2}\int p_{t/C_4}(z-y)|p_s*a(y)|\,dy\,ds\,dz.$$

Moreover, there exist rapidly decaying functions ϕ and ψ such that

$$\int_{K_2} k_t(x,z)V(z)p_{t/C_4}(z-y)\,dz \leq t^{-1}\phi_t(x-y)(\sqrt{t}\,m(x,V))^{\delta},$$

$$|p_s*a(y)|\leq\Big(\frac{r}{\varepsilon R(0)}\Big)^{M}\psi_{(\varepsilon R(0))^2}(y)\|a\|_{L^1}.$$

Hence

$$J_{K_2}(x)\leq C\varepsilon^{\delta}\Big(\frac{r}{\varepsilon R(0)}\Big)^{M-1}\psi_{(\varepsilon R(0))^2}(x)\|a\|_{L^1}.$$

It is not difficult to check that

$$\int_{B(0,3R(0))} \Big(\sup_{0<t<(\varepsilon R(x))^2,\, 2r^2<t} J_{K_2}(x)\Big)^p\,dx$$
$$\leq C\int_{B(0,3R(0))}\varepsilon^{\delta p}\Big(\frac{r}{\varepsilon R(0)}\Big)^{(M-1)p}\psi_{(\varepsilon R(0))^2}(x)^p\|a\|_{L^1}^p\,dx\leq c(\varepsilon).$$

CASE 2: $t/2\leq r^2$. Then

$$|Z^0_{(\varepsilon),t}a(x)|\leq\Big|\int_{K_3} k_t(x,z)V(z)\int_{t/2}^{\min(r^2,(\varepsilon R(z))^2)} p_s*a(z)\,ds\,dz\Big|$$
$$+\Big|\int_{K_3} k_t(x,z)V(z)\int_{\min(r^2,(\varepsilon R(z))^2)}^{(\varepsilon R(z))^2} p_s*a(z)\,ds\,dz\Big|$$
$$+\Big|\int_{K_4} k_t(x,z)V(z)\int_{(\varepsilon R(z))^2}^{t/2} p_s*a(z)\,ds\,dz\Big|,$$

where $K_3 = \{z \in K : \ t/2 < (\varepsilon R(z))^2\}$ and $K_4 = K \setminus K_3$. By (7.4) and Corollary 4.6 we get

$$V(z)\chi_{K_3}(z)\Big|\int_{t/2}^{\min(r^2,(\varepsilon R(z)))^2} p_s * a(z)\,ds\Big|$$

$$+ V(z)\chi_{K_3}(z)\Big|\int_{\min(r^2,(\varepsilon R(z))^2)}^{(\varepsilon R(z))^2} p_s * a(z)\,ds\Big|$$

$$+ V(z)\chi_{K_4}(z)\Big|\int_{(\varepsilon R(z))^2}^{t/2} p_s * a(z)\,ds\Big|$$

$$\le \begin{cases} CV(z)(r^2\|a\|_{L^\infty} + r^{-d+2}\|a\|_{L^1}) & \text{for } |z| < 2r, \\ CV(z)e^{-c|z|^2/(\varepsilon R(0))^2}|z|^{2-d-M}|B(0,r)|^{1-1/p+M/d} & \text{for } 2r < |z| \le R(0). \end{cases}$$

It is not difficult to check that this is a multiple of $c(\varepsilon)$ and a generalized $(\mathbf{h}^p_{r/R(0)}(m), 1, M-1)$-atom associated with the ball $B(0,r)$. Thus

$$\|\sup_{0<t<2r^2} |Z^0_{(\varepsilon),t}a(x)|\,\|^p_{L^p(dx)} \le c(\varepsilon).$$

This completes the proof of the lemma. ■

Lemma 7.12. *There exists a function $c(\varepsilon)$ with $\lim_{\varepsilon\to 0^+} c(\varepsilon) = 0$ such that for every $(\mathbf{h}^p_\varepsilon(m), \infty)$-atom a associated with a ball $B(x_0, r)$, where $r \sim \varepsilon R(x_0)$, we have*

$$\|\mathcal{Z}^*_{\varepsilon,0}a\|_{L^p} \le c(\varepsilon).$$

Proof. As above we assume that $x_0 = 0$.

Case 1: $C_1(\varepsilon R(0))^2 < t/2 < (\varepsilon R(x))^2$. Then it suffices to consider $|x| > 3R(0)$. Therefore applying Lemma 4.7 and Corollary 4.8 we have

$$\begin{aligned}|Z^0_{(\varepsilon),t}a(x)| &\le \int_K \phi_t(x)V(z)\int_{(\varepsilon R(0))^2/C_5}^{t/2}\int p_s(z-y)|a(y)|\,dy\,ds\,dz \\ &\le \|a\|_{L^1}\phi_t(x)\Big[\int_{(\varepsilon R(0))^2/C_5}^{R(0)^2} s^{-1}\Big(\frac{\sqrt{s}}{R(0)}\Big)^\delta ds \\ &\quad + \int_{R(0)^2}^{\max(t/2,R(0)^2)} s^{-d/2}R(0)^{d-2}\,ds\Big] \\ &\le \|a\|_{L^1}\phi_t(x).\end{aligned}$$

Applying Lemma 4.3 we obtain

$$\Big\| \sup_{C_1(\varepsilon R(0))^2<t/2<(\varepsilon R(x))^2} |Z^0_{(\varepsilon),t}a(x)| \Big\|_{L^p(B(0,3R(0))^c,dx)} \le c(\varepsilon).$$

CASE 2: $t/2 < C_1(\varepsilon R(0))^2 \sim r^2$. Then

$$|Z^0_{(\varepsilon),t}a(x)| \le \int_K k_t(x,z)V(z) \int_{t/C_5}^{C_5(\varepsilon R(0))^2} |a * p_s(z)|\,ds\,dz.$$

Observe that

$$V(z)\int_{t/C_5}^{C_5r^2} |p_s * a(z)|\,ds \le C \begin{cases} V(z)r^2\|a\|_{L^\infty} & \text{for } |z|\le 2r,\\ \dfrac{V(z)}{|z|^{d-2}}\, e^{-c|z|^2/r^2}\|a\|_{L^1} & \text{for } 2r<|z|\le R(0).\end{cases}$$

Now the same argument as in the proof of Lemma 7.11 (Case 2) can be used. ∎

8. Proof of Lemma 3.2. First we prove that there is a constant $C>0$ such that

$$\|\mathcal{P}^*_\varepsilon g\|_{L^p} \le C\|g\|_{\mathbf{h}^p_\varepsilon(m)}. \tag{8.1}$$

Let a be an $(\mathbf{h}^p_\varepsilon(m),\infty)$-atom associated with a ball $B(y_0,r)$. If $\int a = 0$ then $\|\mathcal{P}^*_\varepsilon a\|_{L^p} \le C$. If $\int a \ne 0$ then, by definition, $r \sim \varepsilon R(y_0)$. Obviously, by Corollary 4.6 and [G], $\|\mathcal{P}^*_\varepsilon a\|_{L^p(B(y_0,R(y_0))^*)} \le C$. Here and subsequently, for any ball B we define B^* to be the ball that has the same center as B but whose radius is 4 times that of B. If $x \notin B(y_0,R(y_0))^*$, then, by (4.5), $R(x) \le C|x-y_0|^{k_0/(k_0+1)}R(y_0)^{1/(k_0+1)}$. Therefore for $0<t<(\varepsilon R(x))^2$ we have

$$|p_t * a(x)| \le C\|a\|_{L^1}\varepsilon^{M-d}R(y_0)^{(M-d)/(1+k_0)}|x-y_0|^{-(M+dk_0)/(1+k_0)}.$$

This leads to $\int_{|x-y_0|>2R(y_0)}(\mathcal{P}^*_\varepsilon a(x))^p\,dx \le C$, and (8.1) is proved.

Let $\varphi^{(\alpha)}$ be C^∞-functions on $\mathbb{R}^d$ such that $0\le\varphi^{(\alpha)}\le 1$, $\sum_\alpha \varphi^{(\alpha)}(x)=1$ for every $x\in\mathbb{R}^d$, $\operatorname{supp}\varphi^{(\alpha)} \subset B_\alpha = B(y_\alpha,R(y_\alpha))$, and the family of the balls B_α has the finite covering property.

LEMMA 8.2. *There exists a function* $c(\varepsilon)$ *with* $\lim_{\varepsilon\to 0^+} c(\varepsilon)=0$ *such that for every* α,

$$\Big\| \sup_{0<t<(\varepsilon\max(R(y_\alpha),R(x)))^2} |(g\varphi^{(\alpha)}) * p_t(x)| \Big\|^p_{L^p(B^{*c}_\alpha)} \le c(\varepsilon)\|g\varphi^{(\alpha)}\|^p_{\mathbf{h}^p_\varepsilon(m)}. \tag{8.3}$$

Proof. It suffices to prove (8.3) if $g\varphi^{(\alpha)}$ is replaced by an $(\mathbf{h}^p_\varepsilon(m),\infty)$-atom a associated with a ball $B(y_0,r)$, where $B(y_0,r)\cap B_\alpha \ne \emptyset$. Obviously $R(y_0)\sim R(y_\alpha)$. Note that for $x\in B^{*c}_\alpha$, we have

$$\max(R(y_\alpha),R(x)) \le C|x-y_0|^{k_0/(1+k_0)}R(y_0)^{1/(1+k_0)}.$$

Therefore if $r \sim \varepsilon R(y_\alpha)$ then

$$|a * p_t(x)| \le C_M \varepsilon^{M-d} \|a\|_{L^1} R(y_0)^{(M-d)/(1+k_0)} |x - y_0|^{-(M+dk_0)/(1+k_0)}$$

for $0 < t \le (\varepsilon \max(R(y_\alpha), R(x)))^2$, and consequently, the left-hand side of (8.3) is estimated by $C_M \varepsilon^{Mp-d}$.

If $r < \varepsilon R(y_0)/4$ then, by (2.4),

$$|a * p_t(x)| \le C r^{d+1-d/p} |x - y_0|^{-d-1}.$$

Thus the left-hand side of (8.3) is bounded by $C\varepsilon^{dp+p-d}$. ∎

COROLLARY 8.4. *There exists a constant $C > 0$ such that for every α and every $\varepsilon > 0$ small enough we have*

$$\|g\varphi^{(\alpha)}\|^p_{\mathbf{h}^p_\varepsilon(m)} \le C \|\mathcal{P}^*_\varepsilon (g\varphi^{(\alpha)})\|^p_{L^p}. \tag{8.5}$$

Proof. Applying results of Goldberg [G] and Lemma 8.2, we have

$$\begin{aligned}
\|g\varphi^{(\alpha)}\|^p_{\mathbf{h}^p_\varepsilon(m)} &\le C \Big\| \sup_{0<t<(\varepsilon R(y_\alpha))^2} |(g\varphi^{(\alpha)}) * p_t(x)| \Big\|^p_{L^p} \\
&\le C \Big\| \sup_{0<t<(\varepsilon R(y_\alpha))^2} |(g\varphi^{(\alpha)}) * p_t(x)| \Big\|^p_{L^p(B^*_\alpha)} \\
&\quad + C \Big\| \sup_{0<t<(\varepsilon R(y_\alpha))^2} |(g\varphi^{(\alpha)}) * p_t(x)| \Big\|^p_{L^p(B^{*c}_\alpha)} \\
&\le C \|\mathcal{P}^*_\varepsilon (g\varphi^{(\alpha)})\|^p_{L^p} + C c(\varepsilon) \|g\varphi^{(\alpha)}\|^p_{\mathbf{h}^p_\varepsilon(m)}. \ \blacksquare
\end{aligned}$$

LEMMA 8.6. *There exists a function $c(\varepsilon)$ with $\lim_{\varepsilon \to 0^+} c(\varepsilon) = 0$ such that*

$$\sum_\alpha \int \Big(\sup_{0<t<(\varepsilon R(x))^2} |\varphi^{(\alpha)}(x) P_t g(x) - P_t(\varphi^{(\alpha)} g)(x)|^p \Big)\, dx \le c(\varepsilon) \|g\|^p_{\mathbf{h}^p_\varepsilon(m)}. \tag{8.7}$$

Proof. Define $\mathcal{J}^*_{\alpha,\varepsilon} g(x) = \sup_{0<t<(\varepsilon R(x))^2} |\mathcal{J}_{\alpha,t} g(x)|$, where

$$\begin{aligned}
\mathcal{J}_{\alpha,t} g(x) &= \varphi^{(\alpha)}(x) P_t g(x) - P_t(\varphi^{(\alpha)} g)(x) \\
&= \int (\varphi^{(\alpha)}(x) - \varphi^{(\alpha)}(y)) p_t(x-y) g(y)\, dy.
\end{aligned}$$

Let a be an $(\mathbf{h}^p_\varepsilon(m), \infty)$-atom associated with a ball $B(y_0, r)$. Let $\mathcal{I}_1 = \{\alpha : y_0 \notin B^{**}_\alpha\}$ and $\mathcal{I}_2 = \{\alpha : y_0 \in B^{**}_\alpha\}$. We note that the number of elements in $\mathcal{I}_2$ is bounded by a constant independent of a. We may assume that ε is small. Therefore if $\alpha \in \mathcal{I}_1$, then $\mathcal{J}_{\alpha,t} a(x) = \int \varphi^{(\alpha)}(x) p_t(x-y) a(y)\, dy$. Thus, by Lemma 8.2, we get

$$\sum_{\alpha \in \mathcal{I}_1} \int \sup_{0<t<(\varepsilon R(x))^2} |\mathcal{J}_{\alpha,t} a(x)|^p\, dx \le c(\varepsilon).$$

Let now $\alpha \in \mathcal{I}_2$. If $x \notin B(y_\alpha, R(y_\alpha))^*$, then

$$\mathcal{J}_{\alpha,t} a(x) = \int p_t(x-y) \varphi^{(\alpha)}(y) a(y)\, dy.$$

Since $\|\varphi^{(\alpha)}a\|_{\mathbf{h}^p_\varepsilon(m)} \le C$, where the constant C is independent of ε, a and α, the same arguments as in the proof of Lemma 8.2 can be applied to obtain

$$\int_{B(y_\alpha,R(y_\alpha))^{*c}} \sup_{0<t<(\varepsilon R(x))^2} |\mathcal{J}_{\alpha,t}a(x)|^p\,dx \le c(\varepsilon).$$

If $x \in B(y_\alpha, R(y_\alpha))^*$, then $R(x) \sim R(y_0) \sim R(y_\alpha)$. Thus

$$|J_{\alpha,t}a(x)| = \left|\int \frac{\sqrt{t}}{R(y_0)}\,\Psi_t(x,y)a(y)\,dy\right| \le C\varepsilon\left|\int \Psi_t(x,y)a(y)\,dy\right|,$$

where $\Psi_t(x,y) = R(y_0)t^{-1/2}(\varphi^{(\alpha)}(x)-\varphi^{(\alpha)}(y))p_t(x-y)$. Clearly, $|\nabla_x\Psi_t(x,y)| \le t^{-1/2}\psi_t(x-y)$ for $0<t<CR(y_0)^2$ with ψ being a rapidly decaying function. Therefore standard arguments can be used in order to show that

$$\sum_{\alpha\in\mathcal{I}_2} \int_{B(y_\alpha,R(y_\alpha))^*} \sup_{0<t<(\varepsilon R(x))^2} |J_{\alpha,t}a(x)|^p\,dx \le c(\varepsilon). \ \blacksquare$$

We are now in a position to finish the proof of the second inequality in (3.3). Indeed, by Corollary 8.4 and Lemma 8.6, we obtain

$$\begin{aligned}\|g\|^p_{\mathbf{h}^p_\varepsilon(m)} &\le C\sum_\alpha \|\varphi^{(\alpha)}g\|^p_{\mathbf{h}^p_\varepsilon(m)} \le C\sum_\alpha \|\mathcal{P}^*_\varepsilon(\varphi^{(\alpha)}g)\|^p_{L^p}\\ &\le C\|\mathcal{P}^*_\varepsilon g\|^p_{L^p} + C\sum_\alpha \|\mathcal{J}^*_{\alpha,\varepsilon}g\|^p_{L^p} \le C\|\mathcal{P}^*_\varepsilon g\|^p_{L^p} + Cc(\varepsilon)\|g\|^p_{\mathbf{h}^p_\varepsilon(m)}.\end{aligned}$$

Taking ε_0 sufficiently small we get the required estimates for $0<\varepsilon<\varepsilon_0$. $\blacksquare$

9. Proof of the first inequality of (1.14). Fix $\varepsilon > 0$ (small). According to Lemma 2.9 it suffices to show that for every b of the form

$$b = (\mathrm{Id} + A_\varepsilon)a, \tag{9.1}$$

where a is an $(\mathbf{h}^p_\varepsilon(m),\infty)$-atom we have

$$\|\mathcal{M}b\|^p_{L^p} \le C \tag{9.2}$$

with C independent of a. Assume that a is an $(\mathbf{h}^p_\varepsilon(m),\infty)$-atom associated with a ball $B(x_0,r)$, $r \le \varepsilon R(x_0)$. Then, by Lemma 2.9,

$$\int |b(x)|\,dx \le \int |b(x)|\left(1+\frac{|x-x_0|}{r}\right)^M dx \le C|B(x_0,r)|^{1-1/p}. \tag{9.3}$$

Since $\mathcal{M}$ is of weak type $(1,1)$, we have

$$\begin{aligned}\int_{|x-x_0|<4r} (\mathcal{M}b(x))^p\,dx &= p\int_0^\infty |\{x\in B(x_0,4r) : \mathcal{M}b(x) > \lambda\}|\lambda^{p-1}\,d\lambda\\ &\le C\int_0^{r^{-d/p}} r^d\lambda^{p-1}\,d\lambda + C\int_{r^{-d/p}}^\infty \|b\|_{L^1}\lambda^{p-2}\,d\lambda \le C.\end{aligned} \tag{9.4}$$

Therefore it remains to show that

$$\int_{B(x_0,4r)^c} (\mathcal{M}b(x))^p\,dx \le C. \tag{9.5}$$

CASE 1: $\varepsilon R(x_0)/4 \le r \le \varepsilon R(x_0)$. Then we set $b(x) = \sum_{j=0}^{\infty} b_j(x)$, where $b_0(x) = b(x)\chi_{B(x_0,r)}(x)$ and $b_j(x) = b(x)\chi_{B(x_0,2^j r)\setminus B(x_0,2^{j-1}r)}(x)$. Obviously

$$\|b_j\|_{L^1} \le C|B(x_0,2^j r)|^{1-1/p}2^{-j(N+d-d/p)}. \tag{9.6}$$

Hence

$$\|\mathcal{M}b_j\|^p_{L^p(B(x_0,2^{j+2}r))} \le C2^{-j(N+d-d/p)p}. \tag{9.7}$$

If $x \notin B(x_0,2^{j+2}r)$ then using Corollary 6.4 and Lemma 4.3, we have

$$\begin{aligned} \mathcal{M}b_j(x) &\le C_L \sup_{t>0}\int |b_j(y)|t^{-d/2}e^{-c|x-y|/\sqrt{t}}\left(1+\frac{t}{R(x)^2}\right)^{-L} dy \\ &\le C_L \sup_{t>0} \|b_j\|_{L^1} t^{-d/2-L}e^{-c|x-x_0|/\sqrt{t}}R(x)^{2L} \\ &\le C_L\|b_j\|_{L^1}R(x_0)^{2L/(1+k_0)}|x-x_0|^{-d-2L/(1+k_0)} \end{aligned} \tag{9.8}$$

Applying (9.6)–(9.8) we obtain (9.5).

CASE 2: $r < \varepsilon R(x_0)/4$. It follows from Lemma 2.9 and Proposition 2.11 that $a = (\mathrm{Id} + A_\varepsilon)^{-1}b \in \mathbf{h}^p_\varepsilon(m)$. We have

$$T_t b(x) = p_t * a(x) - H_t a(x) - E_t a(x) - Z_{(\varepsilon),t}a(x),$$

and consequently

$$\mathcal{M}b(x) \le \mathcal{P}^*a(x) + \mathcal{H}^*a(x) + \mathcal{E}^*a(x) + \mathbf{Z}^*_\varepsilon a(x),$$

where

$$\mathcal{P}^*a(x) = \sup_{t>0}|p_t * a(x)|, \qquad \mathcal{H}^*a(x) = \sup_{t>0}|H_t a(x)|,$$
$$\mathcal{E}^*a(x) = \sup_{t>0}|E_t a(x)|, \qquad \mathbf{Z}^*_\varepsilon a(x) = \sup_{t>0}|Z_{(\varepsilon),t}a(x)|.$$

The estimates for $\|\mathcal{P}^*a\|_{L^p}$, $\|\mathcal{H}^*a\|_{L^p}$, $\|\mathcal{E}^*a\|_{L^p}$ follow from Lemmas 6.6, 6.8, 6.11, 6.13. Therefore it remains to prove the following proposition.

PROPOSITION 9.9. *For every $\varepsilon > 0$ (sufficiently small) there exists a constant $C_\varepsilon > 0$ such that for every $(\mathbf{h}^p_\varepsilon(m),\infty)$-atom a associated with a ball $B(x_0,r)$ with $r < \varepsilon R(x_0)/4$ we have*

$$\|\mathbf{Z}^*_\varepsilon a\|^p_{L^p} \le C_\varepsilon. \tag{9.10}$$

Proof. There is no loss of generality in assuming that a is an $(\mathbf{h}^p_\varepsilon(m),\infty)$-atom associated with a ball $B(0,r)$, where $r < \varepsilon R(0)/4$. By definition (cf. (2.4)), $\int a = 0$. We have

$$\mathbf{Z}_\varepsilon^* a(x) \le \sum_{j=0}^\infty \sup_{t>0} \Big| \int_{U_j} k_t(x,z)V(z)W_{(\varepsilon),t}a(z)\,dz \Big| = \sum_{j=0}^\infty \mathbf{Z}^*_{(\varepsilon),j} a(x),$$

where $U_0 = B(0, 2\varepsilon R(0))$ and $U_j = \{z : 2^j\varepsilon R(0) < |z| \le 2^{j+1}\varepsilon R(0)\}$ for $j = 1, 2, \ldots$

For $z \in U_j$, $j \ge 1$, by Lemmas 7.3, 4.3 and Corollary 4.6, we have

$$|W_{(\varepsilon),t}a(z)| \le \begin{cases} Ce^{-c2^{\gamma j}/\varepsilon^2}|B(0,r)|^{1-1/p-1/d}(2^j\varepsilon R(0))^{1-d} & \text{if } t/2 < (\varepsilon R(z))^2, \\ Ce^{-c2^j\varepsilon R(0)/\sqrt{t}}|B(0,r)|^{1-1/p-1/d}(2^j\varepsilon R(0))^{1-d} & \text{if } t/2 \ge (\varepsilon R(z))^2. \end{cases}$$

Applying Corollary 6.4 and the fact that $k_t(x,y) = k_t(y,x)$, we obtain

$$\mathbf{Z}^*_{(\varepsilon),j} a(x) \le \sup_{t>0} C \int_{U_j} t^{-d/2}e^{-c|x-z|^2/t}\left(1 + \frac{\sqrt{t}}{R(x)}\right)^{-M} \times \left(1 + \frac{\sqrt{t}}{R(z)}\right)^{-2L} V(z)|B(0,r)|^{1-1/p+1/d}(2^j\varepsilon R(0))^{1-d} \times (e^{-c2^j\varepsilon R(0)/\sqrt{t}} + e^{-c2^{\gamma j}/\varepsilon^2})\,dz.$$

Since $R(z) \le C(1 + |z|/R(0))^{k_0/(k_0+1)}R(0)$ (cf. Lemma 4.3), we have

$$\mathbf{Z}^*_{(\varepsilon),j} a(x) \le \sup_{t>0} C_\varepsilon \int_{U_j} t^{-d/2}e^{-c|x-z|^2/t}\left(1 + \frac{\sqrt{t}}{R(x)}\right)^{-M}\left(1 + \frac{\sqrt{t}}{R(z)}\right)^{-L} \times \left(1 + \frac{\sqrt{t}}{2^{jk_0/(1+k_0)}R(0)}\right)^{-L} V(z)|B(0,r)|^{1-1/p+1/d} \times (2^j\varepsilon R(0))^{1-d}(e^{-c2^jR(0)/\sqrt{t}} + e^{-c2^{\gamma j}})\,dz.$$

Since

$$\sup_{t>0}\left(1 + \frac{\sqrt{t}}{2^{jk_0/(1+k_0)}\varepsilon R(0)}\right)^{-L} e^{-c2^j\varepsilon R(0)/\sqrt{t}} \le C_{N,\varepsilon}2^{-Nj},$$

we get

$$\mathbf{Z}^*_{(\varepsilon),j} a(x) \le \sup_{t>0} C_\varepsilon \int_{U_j} t^{-d/2}e^{-c|x-z|^2/t}\left(1 + \frac{\sqrt{t}}{R(x)}\right)^{-M}\left(1 + \frac{\sqrt{t}}{R(z)}\right)^{-L} \times |B(0,r)|^{1-1/p+1/d}(2^jR(0))^{1-d}V(z)2^{-Nj}\,dz.$$

Note that the function $|B(0,r)|^{1-1/p+1/d}(2^jR(0))^{1-d}V(z)2^{-Nj}\chi_{U_j}(z)$ is supported by the ball $B(0, 2^jR(0))$ and its L^1-norm is bounded by

$C2^{-jN'}|B(0,2^jR(0))|^{1-1/p}$. Therefore

$$\sum_{j=1}^{\infty} \|\mathbf{Z}^*_{(\varepsilon),j}a\|^p_{L^p} \le C_\varepsilon.$$

In order to estimate $\mathbf{Z}^*_{(\varepsilon),0}a$ we consider two cases.

CASE 1: $t > 2C(\varepsilon R(0))^2$. Then

$$|W_{(\varepsilon),t}a(z)| \le \int\limits_{(\varepsilon R(0))^2/C_0}^{\infty} |p_s * a(z)|\,ds \le C_\varepsilon \int\limits_{(\varepsilon R(0))^2/C_0}^{\infty} s^{-(d+1)/2} r\|a\|_{L^1}\,ds \le C_\varepsilon R(0)^{1-d} r\|a\|_{L^1}.$$

Thus

$$\sup_{t>2(\varepsilon R(0))^2} \int\limits_{U_0} k_t(x,z)V(z)|W_{(\varepsilon),t}a(z)|\,dz \le \sup_{t>2(\varepsilon R(0))^2} C_\varepsilon \int\limits_{U_0} k_t(x,z)V(z)R(0)^{1-d} r\|a\|_{L^1}\,dz.$$

Observe that the function $V(z)R(0)^{1-d}r\|a\|_{L^1}\chi_{U_0}(z)$ is supported by the ball $B(0,2\varepsilon R(0))$ and its L^1-norm is bounded by $C(\varepsilon R(0))^{d-d/p}$. Therefore

$$\Big\| \sup_{t>2(\varepsilon R(0))^2} \int\limits_{U_0} k_t(x,z)V(z)|W_{(\varepsilon),t}a(z)|\,dz \Big\|^p_{L^p} \le C_\varepsilon.$$

CASE 2: $t < 2C(\varepsilon R(0))^2$. In this case we may apply the same arguments as in the proof of Lemma 7.11. ∎

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