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POSITIVE L^1 OPERATORS ASSOCIATED WITH NONSINGULAR MAPPINGS AND AN EXAMPLE OF E. HILLE

BY

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Abstract. E. Hille [Hi1] gave an example of an operator in $L^1[0,1]$ satisfying the mean ergodic theorem (MET) and such that $\sup_n \|T^n\| = \infty$ (actually, $\|T^n\| \sim n^{1/4}$). This was the first example of a non-power bounded mean ergodic L^1 operator. In this note, the possible rates of growth (in n) of the norms of T^n for such operators are studied. We show that, for every $\gamma > 0$, there are positive L^1 operators T satisfying the MET with $\lim_{n\to\infty} \|T^n\|/n^{1-\gamma} = \infty$. In the class of positive operators these examples are the best possible in the sense that for every such operator T there exists a $\gamma_0 > 0$ such that $\limsup_{n\to\infty} \|T^n\|/n^{1-\gamma_0} = 0$.

A class of numerical sequences $\{\alpha_n\}$, intimately related to the problem of the growth of norms, is introduced, and it is shown that for every sequence $\{\alpha_n\}$ in this class one can get $\|T^n\| \geq \alpha_n$ $(n = 1, 2, \ldots)$ for some T. Our examples can be realized in a class of positive L^1 operators associated with piecewise linear mappings of $[0, 1]$.

0. Introduction. The mean ergodic theorem (MET) was originally proved by von Neumann for unitary operators in Hilbert spaces. This theorem triggered a huge number of results, including those extending it to various classes of spaces and operators (see, e.g., [Kr]). We say that a bounded linear operator T in a Banach space X *satisfies the MET* (or is *mean ergodic*) if $\lim_{n\to\infty} n^{-1}\sum_{k=1}^n T^k f$ exists for all $f \in X$. An obvious necessary condition for T to satisfy the MET comes from the classical Banach–Steinhaus theorem; namely, one must have $\sup_{n\geq1} \|A_n\| < \infty$, where $A_n = A_n(T) = n^{-1}\sum_{k=1}^n T^k$. Such operators T are called *Cesàro bounded*. A stronger condition, $\sup_{n\geq1} \|T^n\| < \infty$, which is called *power boundedness* of T, turns out not to be necessary for the mean ergodicity of T. The first, and nontrivial, example in this direction was given in an old paper of E. Hille [Hi1]. He proved that the operator T defined on $L^1[0,1]$ by $Tf(x) = f(x) - \int_0^x f(y)\,dy$ is mean ergodic, but the norms of the T^n grow as $n^{1/4}$. This rate of growth ($n^{1/4}$) is, of course, related to the concrete analytical nature of Hille's example (more precisely, it is connected with the asymptotics of the Laguerre polynomials, which appear in the kernels of the

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iterations of the integral operator T). Concerning the question about the (highest) possible rates of growth, it was only noted in [Hi1, p. 247] that "it is still a far cry from $O(n^{1/4})$ to $o(n)$". As far as we know, this question remained unanswered.

In this note we show that for any $\gamma > 0$, the rate of growth (in n) of $\|T^n\|$ for the L^1 operators T satisfying the MET can actually be faster than $n^{1-\gamma}$.

Also, we will be concerned with *positive* L^1 operators (Hille's operator is clearly nonpositive). We will show that the above estimate cannot be improved in the class of positive operators.

For any fixed p, $1 \le p \le \infty$, an L^p operator T is called *positive* if T preserves the cone $L^p_+ = \{f \in L^p : f \ge 0\}$. If $1 < p < \infty$ (but not for $p = 1$), the existence of positive mean ergodic L^p operators which are not power bounded follows from the results of [É]. The Cesàro bounded, but not power bounded operators have also been studied in [N] and [DM]. Y. Derriennic [D] constructed a (*nonpositive*) mean ergodic operator T in L^2 with the highest possible rate of growth, namely, $\limsup n^{-1}\|T^n\| > 0$. Recently, Yu. Tomilov and J. Zemanek [TZ] suggested another, simpler way of constructing L^2 operators having the same properties as Derriennic's example.

The L^1 case, originally considered by Hille, requires different methods. Our L^1 examples can be realized in a class of operators which appears naturally in dynamics, in particular, in questions about cocycles for nonsingular transformations (see, e.g., [KKry], [KL]).

Concerning the estimates from above for the rates of growth of $\|T^n\|$, it was shown in [BHL, Lemma 5] that $\|T^n\|/n \to 0$ as $n \to \infty$ for every positive Cesàro bounded L^1 operator. Actually, if one looks at their argument, based on an observation made in [DL, Theorem 2.1(v)], one can see that it gives a better estimate

$$\sup_n \frac{\|T^n\| \ln n}{n} < \infty.$$

It turns out that this estimate can be further strenghtened. Actually, we strenghen it twice. First, in §1 (Theorem 1) we give a simple inductive argument to show that, for every N, the sequence $\|T^n\|$ must satisfy the condition $\sup_n \|T^n\|(\ln n)^N/n < \infty$. Then, in §2 (Theorem 2), we give a more analytical argument for the above-mentioned power estimate, which is the best in the asymptotic sense. Formally, Theorem 1 is not a direct consequence of Theorem 2, because both give not only the asymptotic statements, but more concrete estimates. In §3 we give two examples of operators which show that the rate of growth of $\|T^n\|$ given by Theorem 2 cannot be further improved.

For our proofs we introduce a class of numerical sequences, which we call the sequences of sublinear Cesàro growth (SCG-sequences), and estimate their possible rates of growth. In §4 we show that the SCG-sequences are intimately related to the question about the growth of $\|T^n\|$. Namely, we prove (Theorem 3) that for every SCG-sequence $\{\alpha_n\}$ there is a positive L^1 operator T satisfying the MET with $\|T^n\| \geq \alpha_n$ for all $n \geq 1$.

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1. An upper estimate. We start with the following definition.

DEFINITION. Let $\{\alpha_n\}_{n\geq 1}$ be a sequence of positive numbers. We say that this sequence has *sublinear Cesàro growth* (is a *SCG-sequence*) if there is a constant K such that for any positive integers n, p we have

$$\frac{1}{p}\sum_{k=0}^{p-1} \alpha_{n+k} \leq K\alpha_n. \tag{1}$$

A constant K in (1) is called an *SCG-constant* for $\{\alpha_n\}$.

It is easy to check that if T is a positive Cesàro bounded L^1 operator, and $f \in L^1$, then $\alpha_n = \|T^n f\|$ is a SCG-sequence. Indeed, let $K = \sup_{n\geq 0} \|A_n(T)\|$ (here we adopt, for convenience, that $A_0(T) = \mathrm{Id}$; this implies, in particular, that $K \geq 1$). For $f \in L^1_+$ we have

$$\begin{aligned}
\frac{1}{p}\sum_{k=0}^{p-1} \alpha_{n+k} &= \frac{1}{p}\sum_{k=0}^{p-1} \|T^{n+k} f\| = \frac{1}{p}\,\|T^n f\| + \left\|\frac{1}{p}\sum_{k=1}^{p-1} T^{n+k} f\right\| \\
&= \frac{1}{p}\,\|T^n f\| + \left\|\frac{p-1}{p}\,A_{p-1}(T^n f)\right\| \\
&\leq \frac{1}{p}\,\|T^n f\| + \frac{p-1}{p}\,K\|T^n f\| = \left(\frac{1}{p} + \frac{p-1}{p}\,K\right)\alpha_n \leq K\alpha_n.
\end{aligned}$$

For general $f \in L^1$ we split it into its positive and negative parts and use the fact that the class of SCG-sequences is linear. The above calculation shows that, for a given T, all sequences $\alpha_n = \|T^n f\|$ admit the same SCG-constant.

In §3, after Example 1, we will see that, for a positive Cesàro bounded operator T, the sequence $\beta_n = \|T^n\|$ (as opposed to the sequences $\|T^n f\|$ for $f \in L^1$) need not be an SCG-sequence.

LEMMA 1. *For any $N, K \geq 0$ there is a constant $C_{N,K}$ such that if a sequence $\{\alpha_n\}_{n=1}^{\infty}$ has sublinear Cesàro growth, with an SCG-constant K, then for any $n = 1, 2, \ldots$ and any $p = 1, \ldots, n$,*

$$\alpha_n \,\frac{(\ln p)^N}{p} \leq C_{N,K}\alpha_{n-p+1}. \tag{2}$$

Proof. For $N = 0$ the necessary inequality follows immediately from the SCG-property. Indeed, for any $n \geq 1$ and p, $1 \leq p \leq n$, we get

$$\frac{\alpha_n}{p} \leq \frac{1}{p} \sum_{k=0}^{p-1} \alpha_{n-p+k+1} \leq K\alpha_{n-p+1}.$$

Let us assume by induction that the inequality (2) is true for some N with some constant $C_{N,K}$. Then, using (1), we get

$$\alpha_n \sum_{k=1}^{p} \frac{(\ln k)^N}{k} \leq C_{N,K} \sum_{k=1}^{p} \alpha_{n-k+1} \leq C_{N,K} K p \alpha_{n-p+1}. \tag{3}$$

A routine argument shows that, for some constant $D > 0$,

$$\sum_{k=1}^{p} \frac{(\ln k)^N}{k} \geq D(\ln p)^{N+1} \tag{4}$$

for all $p \geq 1$. Putting this into (3), we obtain the desired inequality (2) for $N + 1$ with $C_{N+1,K} = D^{-1} C_{N,K} K$.

For completeness, we include an elementary argument for (4). Since the function $g(x) = (\ln x)^N / x$ is decreasing for $x \geq L := e^N$, for any $p > L$ we can write

$$\begin{aligned} \sum_{k=1}^{p} \frac{(\ln k)^N}{k} &\geq \sum_{k=L}^{p} \frac{(\ln k)^N}{k} \\ &\geq \int_L^p \frac{(\ln x)^N}{x} \, dx = \frac{1}{N+1} \left[(\ln p)^{N+1} - N^{N+1}\right]. \end{aligned}$$

Therefore, for sufficiently large p, say, for $p \geq M$, we get

$$\sum_{k=1}^{p} \frac{(\ln k)^N}{k} \geq \frac{1}{2(N+1)} (\ln p)^{N+1}.$$

Since there are only finitely many values $p < M$, the proof of (4) is complete.

COROLLARY 1. *If $\{\alpha_n\}_{n \geq 1}$ is an SCG-sequence with a constant K, then for any $n, N \geq 1$,*

$$\alpha_n (\ln n)^N / n \leq C_{N,K} \alpha_1. \tag{5}$$

Proof. Take $p = n$ in (2).

Now we can prove the first of the promised upper estimates for the norm growth. It is true not only for the operators satisfying the MET, but for all positive Cesàro bounded operators.

THEOREM 1. *Let T be a positive Cesàro bounded operator in L^1. Then*:

(i) *For every $f \in L^1$ and every natural N,*

$$\|T^n f\|(\ln n)^N/n \le C_{N,K}\|T\|\|f\|, \tag{6}$$

where $C_{N,K}$ is the constant from Lemma 1 for the sequence $\{\|T^n f\|\}$.

(ii) *For every natural N we have* $\sup_n \|T^n\|(\ln n)^N/n < \infty$.

Proof. Fix $N \ge 1$, take $f \in L^1$, and define $\alpha_n = \|T^n f\|$, $n \ge 1$. As was earlier mentioned, the sequence $\{\alpha_n\}$ is an SCG-sequence. Hence, the above corollary gives us

$$\|T^n f\|(\ln n)^N/n \le C_{N,K}\|Tf\| \le C_{N,K}\|T\|\|f\|.$$

This proves (i). To prove (ii), we just take the supremum over all f with $\|f\| = 1$ of both parts of (6) and keep in mind that $C_{N,K}$ can be chosen the same for all such f's, because $K = \sup_n \|A_n(T)\|$ can be used as an SCG-constant for every such sequence.

2. The power upper estimate. The power estimate of the rate of growth of $\|T^n\|$ is a direct consequence of the following lemma about the SCG-sequences.

LEMMA 2. *If a sequence $\{\alpha_n\}_{n=1}^{\infty}$ is an SCG-sequence with an SCG-constant K, then for any $n = 1, 2, \ldots$ and any $p = 1, \ldots, n-1$,*

$$\alpha_{n-p} \ge \frac{\alpha_n}{K}\,\frac{p!}{\prod_{j=1}^{p}(j+1-1/K)}. \tag{7}$$

Proof. Fix n. To see that inequality (7) is true for $p = 1$, let us notice that

$$(\alpha_n + \alpha_{n-1})/2 \le K\alpha_{n-1},$$

hence

$$\alpha_{n-1} \ge \alpha_n\,\frac{1}{2K-1} = \frac{\alpha_n}{K}\,\frac{1}{2-1/K}.$$

It is also true that

$$\alpha_n + \alpha_{n-1} \ge \alpha_n(1 + 1/(2K-1)) = \alpha_n(2K/(2K-1)).$$

Let us now assume, by induction, that for $j = p$ the inequality (7) is true, and also that

$$\sum_{j=0}^{p}\alpha_{n-j} \ge \frac{\alpha_n K^p (p+1)!}{\prod_{j=1}^{p}[K(j+1)-1]}. \tag{8}$$

We have

$$\frac{1}{p+2}\sum_{j=0}^{p+1}\alpha_{n-j} \le K\alpha_{n-p-1}.$$

Therefore, by (8),

$$\alpha_{n-p-1} \geq \frac{1}{K(p+2)-1}\sum_{j=0}^{p}\alpha_{n-j} \geq \frac{1}{K(p+2)-1}\,\frac{\alpha_n K^p (p+1)!}{\prod_{j=1}^{p}[K(j+1)-1]}$$
$$= \alpha_n \frac{K^p(p+1)!}{\prod_{j=1}^{p+1}[K(j+1)-1]} = \frac{\alpha_n}{K}\,\frac{(p+1)!}{\prod_{j=1}^{p+1}[(j+1)-1/K]},$$

and also

$$\sum_{j=0}^{p+1}\alpha_{n-j} \geq \alpha_{n-p-1} + \sum_{j=0}^{p}\alpha_{n-j}$$
$$\geq \alpha_n \frac{K^p(p+1)!}{\prod_{j=1}^{p+1}[K(j+1)-1]} + \frac{\alpha_n K^p(p+1)!}{\prod_{j=1}^{p}[K(j+1)-1]}$$
$$= \frac{\alpha_n K^p(p+1)!}{\prod_{j=1}^{p}[K(j+1)-1]}\left(\frac{1}{K(p+2)-1}+1\right)$$
$$= \frac{\alpha_n K^p(p+1)!}{\prod_{j=1}^{p}[K(j+1)-1]}\,\frac{K(p+2)}{K(p+2)-1} = \frac{\alpha_n K^{p+1}(p+2)!}{\prod_{j=1}^{p+1}[K(j+1)-1]}.$$

This completes the inductive step and proves the lemma.

COROLLARY 2. *If a sequence* $\{\alpha_n\}_{n=1}^{\infty}$ *has sublinear Cesàro growth with an SCG-constant* K, *then*

$$\limsup_{n} \frac{\alpha_n}{n^{1-1/K}} \leq \alpha_1 \frac{K}{\Gamma(2-1/K)}, \tag{9}$$

where Γ *stands for the Gamma function.*

Proof. First take $p = n-1$ in Lemma 2, which gives

$$\alpha_1 \geq \frac{\alpha_n}{K}\,\frac{(n-1)!}{\prod_{j=1}^{n-1}(j+1-1/K)}.$$

This is equivalent to

$$\alpha_n \leq \alpha_1 K\,\frac{\prod_{j=1}^{n-1}(j+1-1/K)}{(n-1)!} = \alpha_1 \frac{K}{\Gamma(2-1/K)}\,\frac{\Gamma(n+1-1/K)}{\Gamma(n)}.$$

Using the formula

$$\lim_{x\to\infty}\frac{\Gamma(x+a)}{x^a\Gamma(x)} = 1 \tag{$*$}$$

(see, for example, [Hi2, p. 238]) we obtain

$$\lim_{n\to\infty}\frac{\Gamma(n+1-1/K)}{\Gamma(n)}\,n^{1/K-1} = 1,$$

from which (9) follows.

THEOREM 2. *Let T be a positive Cesàro bounded L^1 operator with $\sup_n \|A_n(T)\| = K$. Then* $\limsup \|T^n\|/n^{1-1/K} < \infty$.

Proof. As in the proof of Theorem 1, take $f \in L^1_+$. The sequence $\{\alpha_n\}$ defined by $\alpha_n = \|T^n f\|$ is an SCG-sequence with an SCG-constant K. Hence Corollary 2 gives us

$$\limsup_n \frac{\|T^n f\|}{n^{1-1/K}} \le \|Tf\| \frac{K}{\Gamma(2-1/K)} \le \frac{K\|T\|}{\Gamma(2-1/K)} \|f\|.$$

Again, for arbitrary $f \in L^1$ we get the same inequality by splitting f into positive and negative parts. By taking the supremum over all f's with $\|f\| \le 1$ we obtain

$$\limsup_n \frac{\|T^n\|}{n^{1-1/K}} \le \frac{K\|T\|}{\Gamma(2-1/K)}.$$

3. The examples. We give two examples which show that the estimate of the rate of growth of $\|T^n\|$ given in Theorem 2 is the best possible. Example 1 is shorter, but we also include Example 2 to motivate the block-type construction in the proof of Theorem 3 in §4.

We will look for our examples in the following class of operators. Consider a nonsingular invertible transformation τ of a probability space (Ω, μ) (nonsingularity of τ means that the measure $\mu \circ \tau$ is equivalent to μ). Let $w \in L^\infty(\Omega)$. For all $f \in L^1(\Omega)$ define an operator T by

$$Tf(x) = w(x) \frac{d\mu \circ \tau}{d\mu}(x) f(\tau x).$$

This is a bounded L^1 operator, whose powers T^n, $n \ge 1$, are given by

$$T^n f(x) = w(x,n) \frac{d\mu \circ \tau^n}{d\mu}(x) f(\tau^n x),$$

where $w(x,n) = \prod_{k=0}^{n-1} w(\tau^k x)$ is the multiplicative cocycle for τ generated by w (see, e.g., [S]). The abundance of nontrivial cocycles makes the class of such operators T a natural source for producing examples of L^1 operators with nontrivial properties.

We start with a "discrete" version of the first example, for the space $L^1(\mathbb{Z})$. The construction in the "continuous" case, for $L^1[0,1]$, can be done in essentially the same way. We prefer, however, (to simplify the exposition and notation) to obtain the continuous versions of both examples by "transferring" the discrete ones.

Let $X = L^1(\mathbb{Z})$ be the space of all doubly infinite sequences $\{x_j\}_{j=-\infty}^{\infty}$ of real numbers for which $\|x\| = \sum_{j=-\infty}^{\infty} |x_j| < \infty$.

Every bounded sequence $w = \{w_j\}_{-\infty}^{\infty}$ gives rise to a bounded linear operator $T = T_w$ in X (weighted shift operator). Namely, for $x \in X$ we put $Tx = y$, where $y_j = w_j x_{j+1}$ for all j. If $w_j \geq 0$ for every j, the operator T_w is positive.

We want to construct a nonnegative $\{w_j\}$ which will guarantee that the norms of T^n will grow fast enough as $n \to \infty$. In addition, we want to ensure that

$$\lim_{n\to\infty} A_n x = 0 \quad \text{for all } x \in X. \tag{10}$$

Here, as before, A_n is the nth average operator, i.e., $A_n = n^{-1}\sum_{k=1}^{n} T^k$.

Note that the operator T^n $(n > 0)$ is given by $(T^n x)_j = w_j^{(n)} x_{j+n}$, where $w_j^{(n)} = \prod_{s=j}^{j+n-1} w_s$. The formula for A_n is $(A_n x)_j = n^{-1}\sum_{k=1}^{n} w_j^{(k)} x_{j+k}$.

The norms of the operators T^n and A_n can be expressed in terms of $\{w_j\}$. We will assume that $\{w_j\}$ is nonnegative. Then, for $x \in X$, $\|T^n x\| = \sum_j w_j^{(n)} |x_{j+n}|$, and we have

$$\|T^n\| = \sup_{\|x\|\leq 1} \sum_j w_j^{(n)} |x_{j+n}| = \sup_j w_j^{(n)}. \tag{11}$$

Similarly,

$$\|A_n x\| \leq \frac{1}{n}\sum_j \sum_{k=1}^{n} w_j^{(k)} |x_{j+k}| = \frac{1}{n}\sum_{s=-\infty}^{\infty} (w_{s-1}^{(1)} + w_{s-2}^{(2)} + \ldots + w_{s-n}^{(n)})|x_s|,$$

and for nonnegative $x \in X$ the inequality becomes equality. This yields

$$\|A_n\| = \sup_s \frac{1}{n}\,(w_{s-1}^{(1)} + \ldots + w_{s-n}^{(n)}). \tag{12}$$

Let us notice that if $w_j = 0$ for $j \leq 0$, then, in order to prove that $A_n x \to 0$ for every $x \in X$, it is enough to show that the norms of $\|A_n\|$ are uniformly bounded. Indeed, assume that $\|A_n\| \leq K$, take $x \in X$ and fix $\varepsilon > 0$. Choose N such that $\sum_{j>N} |x_j| < \varepsilon/(2K)$. Let

$$x_n^{(1)} = \begin{cases} x_n & \text{for } 1 \leq n \leq N,\\ 0 & \text{otherwise,} \end{cases}$$

and $x^{(2)} = x - x^{(1)}$. Since $w_j = 0$ for $j \leq 0$, we have $T^n x^{(1)} = 0$ for $n \geq N$. Let $S = \max_{0\leq n<N}\{\|T^n x^{(1)}\|\}$. Then for every $n \geq 2NS/\varepsilon$ we have

$$\|A_n x\| \leq \|A_n x^{(1)}\| + \|A_n x^{(2)}\| \leq \varepsilon/2 + K\varepsilon/(2K) = \varepsilon.$$

EXAMPLE 1. The operator T will be a weighted shift $T = T_w$, and the corresponding sequence $w = \{w_n\}_{n=-\infty}^{\infty}$ will have the properties that $w_n = 0$ for $n \leq 0$ and $\{w_n\}_{n=1}^{\infty}$ is decreasing. It follows from (11) and (12) that for the operators $T = T_w$ corresponding to such sequences we have

$$\|T^n\| = \prod_{j=1}^{n} w_j, \tag{13}$$

$$\|A_n(T)\| = \frac{1}{n}\sum_{j=1}^{n} w_{n-j+1}^{(j)} = \frac{w_1w_2\dots w_n + w_2w_3\dots w_n + \dots + w_n}{n}. \tag{14}$$

Fix a number $K > 1$. The sequence $\{w_n\}_{n=1}^{\infty}$ will be constructed with the goal of having $\|A_n(T)\| = K$ for every n. For the decreasing sequences $\{w_n\}_{n=1}^{\infty}$ this requirement, due to (14), is equivalent to the conditions

$$w_1 = K, \quad \frac{w_1w_2 + w_2}{2} = K, \quad \frac{w_1w_2w_3 + w_2w_3 + w_3}{3} = K, \quad \dots,$$

which define $\{w_n\}_{n=1}^{\infty}$ uniquely. In fact, these conditions give us $w_2 = 2K/(K-1)$, $w_3 = 3K/(2K-1)$, ..., and (as can be easily checked by induction) lead to the formula

$$w_n = \begin{cases} \dfrac{nK}{(n-1)K+1} & \text{for } n \ge 1, \\ 0 & \text{for } n \le 0. \end{cases}$$

It is elementary to check that the sequence $\{w_n\}_{n=1}^{\infty}$ given by this formula is decreasing. Therefore, (14) for $\|A_n(T_w)\|$ can be applied, and by the construction of w, we have $\|A_n(T_w)\| = K$ for all n (this can be formally checked by induction on n). Since $w_n = 0$ for $n \le 0$, by the observation just before Example 1, the uniform boundedness of the norms $\|A_n\|$ implies that $A_nx \to 0$ for all $x \in X$.

It remains to calculate the norms of T^n. We have

$$\|T^n\| = \prod_{j=1}^{n} w_j = \prod_{j=1}^{n} \frac{jK}{(j-1)K+1} = \frac{n!}{\prod_{j=1}^{n}((j-1)+1/K)}$$
$$= \frac{n!}{K\prod_{q=1}^{n-1}(q+1/K)} = \frac{\Gamma(1+1/K)}{K}\,\frac{\Gamma(n+1)}{\Gamma(n+1/K)}.$$

Using again the formula (*) for the Γ function, we get

$$\lim_{n\to\infty} \frac{\|T^n\|}{n^{1-1/K}} = \frac{\Gamma(1+1/K)}{K}. \tag{15}$$

This completes the construction for the "discrete case", $X = L^1(\mathbb{Z})$.

Remark. The sequence $\|T^n\|$ for the above example is clearly not an SCG-sequence, simply because for every SCG-sequence $\{\alpha_n\}$ one must have $\liminf \alpha_n < \infty$.

Now consider the "continuous case", $X = L^1[0,1]$. Choose an increasing sequence $\{t_j\}_{j=-\infty}^{\infty}$ of points in $(0,1)$ with $\lim_{j\to-\infty} t_j = 0$, $\lim_{j\to\infty} t_j = 1$, and represent $[0,1]$ (modulo a countable set) as the disjoint union of the intervals $I_j = (t_j, t_{j+1})$, $-\infty < j < \infty$.

Let $\lambda_j = t_{j+1} - t_j$ be the length of I_j. Let $\tau : [0,1] \to [0,1]$ be the piecewise linear transformation which maps each I_j to the next one, I_{j+1}, linearly. Define $w : [0,1] \to \mathbb{R}_+$ by $w(x) = w_j$ for $x \in I_j$, where $\{w_j\}$ is the same sequence as before. Similarly to the discrete case, we define a bounded linear operator $\widetilde{T} = \widetilde{T}_w$ in the following way: for $f \in X$ we put $\widetilde{T} f = g$, where

$$g(x) = w(x)\,\frac{\lambda_{j+1}}{\lambda_j}\,f(\tau x)$$

for $x \in I_j$. Note that the operator $\widetilde{T}$ is of the form described at the beginning of this section.

To show that the analogs of properties (10) and (15) hold for $\widetilde{T}$, one can repeat, with simple modifications, the argument in the discrete case. A shorter way of dealing with the continuous case is to introduce a map $\theta : L^1[0,1] \to L^1(\mathbb{Z})$ which takes $\widetilde{T}$ to T. Namely, for $f \in L^1[0,1]$ we put $\theta f = x$, where $x = \{x_j\}$ with $x_j = \int_{I_j} f$. It is clear that θ is a positive operator, i.e., $f \ge 0$ implies $\theta f \ge 0$, and θ is an isometry on the positive cone $L^1_+[0,1] = \{f \in L^1[0,1] : f \ge 0\}$. One can also easily check that θ conjugates $\widetilde{T}$ and T, i.e., $T \circ \theta = \theta \circ \widetilde{T}$. These properties imply that $\|\widetilde{T}^n f\| = \|T^n(\theta f)\|$ for all $f \in L^1_+[0,1]$ and all $n \ge 0$. Hence, we get the analog of (15) for $\widetilde{T}$.

Finally, take any $f \in L^1[0,1]$ and split it into its positive and negative parts: $f = f_+ - f_-$. Then, if $\widetilde{A}_n := n^{-1} \sum_{k=1}^n \widetilde{T}^k$, we can use property (10) for T to conclude that

$$\begin{aligned}\|\widetilde{A}_n f\| = \|\widetilde{A}_n(f_+ - f_-)\| &= \|\widetilde{A}^n f_+\| + \|\widetilde{A}^n f_-\| \\ &= \|A_n(\theta f_+)\| + \|A_n(\theta f_-)\| \to 0\end{aligned}$$

as $n \to \infty$.

Example 2. The operator T is again a weighted shift $T = T_w$. Let δ, $0 < \delta < 1$, be fixed, and let $\gamma = \delta/2$. Define

$$w_j = \begin{cases} 0 & \text{for } j \le 0, \\ 0 & \text{for } j = 2^p,\ p = 0, 1, \ldots, \\ 2^{1-\gamma} & \text{for } j = 2^p + 2^q,\ p = 1, 2, \ldots,\ q = 0, 1, \ldots, p-1, \\ 1 & \text{otherwise.} \end{cases} \tag{16}$$

Less formally, one can visualize the part of $\{w_j\}$ for $j > 1$ as consisting of blocks Δ_p, $p = 0, 1, \ldots$, where $\Delta_p = \{w_j\}_{j=2^p+1}^{2^{p+1}}$. Slightly abusing terminology, we will also denote by Δ_p the interval $\{j \in \mathbb{Z} : 2^p + 1 \le j \le 2^{p+1}\}$. The block Δ_p has length 2^p and contains p coordinates which are equal to

$2^{1-\gamma}$ (at the places $2^p + 2^0, \dots, 2^p + 2^{p-1}$), one coordinate (the last one) which is equal to 0; all other coordinates of the block Δ_p are equal to 1.

Since $w_j \geq 0$ for all j, the operator T_w is positive. Also, using (11) and (16), for $n = 2^p - 1$ we get

$$\|T^n\| \geq \prod_{s=2^p+1}^{2^{p+1}-1} w_s = 2^{p(1-\gamma)} \geq n^{1-\gamma}.$$

The proof of (10) requires more work. First, we will show that the norms $\|A_n\|$ are uniformly bounded, which is a necessary condition for (10).

It is convenient to introduce some notation. For a finite set M we denote by $|M|$ its cardinality. Let $J = [a, b]$ be an interval in $\mathbb{Z}$. For any r, $0 \leq r < |J|$, we denote by J^r the interval $[a+r, b]$. We define $w_J := \prod_{j\in J} w_j$ and put

$$\sigma_J := \frac{1}{|J|} \sum_{r=0}^{|J|-1} w_{J^r}. \tag{17}$$

In view of (12), we can write

$$\|A_n\| = \sup_{J\colon |J|=n} \sigma_J, \tag{18}$$

and the condition of the uniform boundedness of the norms $\|A_n\|$ can now be written in the form

$$\sup_J \sigma_J < \infty, \tag{19}$$

where the supremum is taken over all intervals $J \subset \mathbb{Z}$.

Note that $w_J = 0$ unless J is a subinterval of some interval $J_p := [2^p + 1, 2^{p+1} - 1]$ (because $w_{2^p} = 0$). This shows that (19) follows from the following lemma.

LEMMA 3. *Suppose $J \subseteq J_p$ for some p. Then there exists a constant $C = C_\gamma$ depending on γ, but not on p, such that $\sigma_J \leq C$.*

Proof. First, we consider the interval $J = J_p$ and estimate σ_{J_p}. Let us partition J_p into $p + 1$ disjoint intervals $J_{p,q}$, $0 \leq q \leq p$, where $J_{p,0}$ is a singleton, $J_{p,0} = \{2^p + 1\}$, $J_{p,q} = [2^p + 2^{q-1} + 1, 2^p + 2^q]$, $1 \leq q < p$, and $J_{p,p} = [2^p + 2^{p-1} + 1, 2^{p+1} - 1]$. Note that $|J_{p,0}| = 1$, $|J_{p,q}| = 2^{q-1}$ for $1 \leq q < p$, and $|J_{p,p}| = 2^{p-1} - 1$.

Now we split the sum (17) for σ_{J_p} accordingly:

$$\sigma_{J_p} = \frac{1}{2^p - 1} \sum_{q=0}^{p} S_q,$$

where $S_q = \sum_r w_{J_p^r}$, and the last summation is taken over r satisfying $2^p + 1 + r \in J_{p,q}$.

By (16), each term of the sum S_q is equal to $2^{(p-q)(1-\gamma)}$ $(0 \le q \le p)$. Therefore,

$$\begin{aligned} \sigma_{J_p} &\le \frac{1}{2^p - 1}\Big(2^{p(1-\gamma)} + \sum_{q=1}^{p} 2^{q-1}\cdot 2^{(p-q)(1-\gamma)}\Big) \\ &\le \frac{1}{2^{p-1}}\Big(2^{p(1-\gamma)} + 2^{p-1}\sum_{q=1}^{p} 2^{-(p-q)\gamma}\Big) \le C_\gamma, \end{aligned} \tag{20}$$

where $C_\gamma = 1/(1-2^{-\gamma}) + 2$.

The next step is to show that the same estimate $w_J \le C_\gamma$ holds not only for $J = J_p$, but for all $J \subseteq J_p$.

Let J, $\widetilde{J}$ be two subintervals of J_p with the same right endpoint, i.e., $J = [a, b]$, $\widetilde{J} = [\widetilde{a}, b]$, and let $a \le \widetilde{a}$. Then $\sigma_{\widetilde{J}} \le \sigma_J$. To see this, it is enough to observe that the sequence $\{w_{J^r}\}_{r=0}^{|J|-1}$, whose average is σ_J, is decreasing: $w_J \ge w_{J^1} \ge \ldots \ge w_{J^{|J|-1}}$; this is simply because $w_j \ge 1$ for all $j \in J_p$. Hence,

$$\sigma_{\widetilde{J}} = \frac{1}{|\widetilde{J}|}\sum_{r=0}^{|\widetilde{J}|-1} w_{\widetilde{J}^r} = \frac{1}{|\widetilde{J}|}\sum_{r=\widetilde{a}-a}^{|J|-1} w_{J^r} \le \frac{1}{|J|}\sum_{r=0}^{|J|-1} w_{J^r} = \sigma_J. \tag{21}$$

(In the last inequality we are using the following obvious fact: the average value of a finite collection of numbers cannot increase if one removes several biggest numbers of the collection.)

It follows from (21) that one can consider only the intervals $[a, b] \subseteq J_p$ whose left endpoints coincide with the left endpoint of J_p, i.e., the intervals of the form $[2^p + 1, b]$. The same monotonicity argument shows that one can consider only the intervals J of the form $J = [2^p + 1, b_q]$ with $b_q = 2^p + 2^q$, $0 \le q \le p - 1$. Indeed, for any other interval $J = [2^p + 1, b] \subseteq J_p$, by moving its right endpoint to the left until it reaches one of the b_q's, one can only increase the value of σ_J, since all terms w_{J^r} which are removed in this process are equal to $\min_{J \subseteq J_p} w_J = 1$.

Finally, for the interval $J = [2^p + 1, 2^p + 2^q]$ with $0 \le q \le p - 1$, one can estimate σ_J exactly as in (20), by partitioning J into $q + 1$ subintervals $J_{p,0}, J_{p,1}, \ldots, J_{p,q}$. This gives

$$\sigma_J \le \frac{1}{2^{q-1}}\Big(2^{q(1-\gamma)} + 2^{q-1}\sum_{m=1}^{q} 2^{-(q-m)\gamma}\Big) \le C_\gamma,$$

which proves Lemma 3.

Since $w_j = 0$ for $j \le 0$ in this example, for the same reason as in Example 1, the uniform boundedness of the norms $\|A_n\|$ implies (10). This completes the construction of the example in $X = l^1(\mathbb{Z})$. The transition from $L^1(\mathbb{Z})$ to $L^1[0,1]$ is the same as in Example 1.

4. SCG-sequences and the norm growth. In this section we prove a theorem which gives one more connection between the SCG-sequences and the norm growth problem.

THEOREM 3. *If $\{\alpha_n\}$ is an SCG-sequence, then there exists a positive L^1 operator which satisfies the MET and such that $\|T^n\| \ge \alpha_n$ for all $n \ge 1$.*

Proof. As in §3, it is enough to define T as a weighted shift operator $T = T_w$ in the space $X = L^1(\mathbb{Z})$. We keep the notation of Example 2 in §3. For convenience, let $\alpha_0 = 1$. Define a sequence $\{v_n\}_{n\ge 1}$ by $v_n = \alpha_n/\alpha_{n-1}$.

Let

$$w_j = \begin{cases} 0 & \text{for } j \le 0, \\ 0 & \text{for } j = 2^p,\ p \ge 0, \\ v_k & \text{for } j = 2^{p+1} - k,\ p \ge 1,\ k = 1, 2, \ldots, 2^p - 1. \end{cases}$$

In other words, using the terminology of §2, we can say that the sequence $\{w_n\}$ consists of blocks Δ_p, and in the block Δ_p we put the numbers $v_1, v_2, \ldots$ in *reverse order.* By (11),

$$\|T^n\| = \sup_{-\infty<j<\infty} w_j^{(n)} = \sup_{j\ge 0} \prod_{k=1}^{n} v_{j+k} = \sup_{j\ge 0} \frac{\alpha_{j+n}}{\alpha_j}.$$

In particular, putting $j = 0$, we get $\|T^n\| \ge \alpha_n$.

Since $w_j = 0$ for $j \le 0$, the remark before Example 1 tells us that in order to prove that $A_n(x) \to 0$ for every $x \in X$, it is enough to show that the norms $\|A_n\|$ are bounded. By formula (18), we can write

$$\|A_n\| = \sup_{J\subset\mathbb{Z},\, |J|=n} \sigma_J = \sup_s \frac{1}{n} \left(w_s^{(1)} + \ldots + w_{s-n+1}^{(n)}\right).$$

Fix an interval $J = [s-n+1, s]$ of length n, and estimate σ_J. Without loss of generality we can assume that $s \ge 1$. Suppose that $s \in \Delta_p$, i.e., $2^p + 1 \le s \le 2^{p+1}$, and let $k = 2^{p+1} - s$ be the distance from s to the end of the block Δ_p. First, we consider the case when $J \subset \Delta_p$. In this case, since w_j's in Δ_p are just the numbers $v_1, v_2, \ldots$ in reverse order, we have

$$\begin{aligned}\sigma_J &= \frac{1}{n}\left(v_{k+1} + v_{k+1}v_{k+2} + \ldots + v_{k+1}v_{k+2}\ldots v_{k+n}\right)\\ &= \frac{1}{n}\left(\frac{\alpha_{k+1}}{\alpha_k} + \frac{\alpha_{k+2}}{\alpha_k} + \ldots + \frac{\alpha_{k+n}}{\alpha_k}\right)\\ &= \frac{1}{\alpha_k}\,\frac{\alpha_{k+1} + \alpha_{k+2} + \ldots + \alpha_{k+n}}{n} \le K\,\frac{\alpha_{k+1}}{\alpha_k},\end{aligned}$$

where K is an SCG-constant for $\{\alpha_n\}$.

By the SCG-property, $(\alpha_k + \alpha_{k+1})/2 \le K\alpha_k$ for $k \ge 1$, hence

$$\alpha_{k+1}/\alpha_k \le \max\{\alpha_1, 2K-1\} =: C.$$

Therefore, $\sigma_J \le CK$.

If the interval $J = [s-n+1, s]$ with the right endpoint $s \in \Delta_p$ is not entirely in Δ_p, we will show how to find another interval $J' = [s'-n+1, s']$ of length n with $\sigma_{J'} \ge \sigma_J$, and this will be enough to complete the proof. To do this, we simply consider the block Δ_m with $m > p$ large enough, so that if we take the right endpoint $s' = 2^{m+1} - k$ (the point at the same distance from the end of Δ_m as s is from the end of Δ_p), the entire interval J' will lie in Δ_m. Note again that the sequence $\{w_j\}$ consists of blocks of the sequence $\{v_j\}$, which are put in reverse order and separated by zeros. This implies that the expression for $\sigma_{J'}$ (formula (17) for $\sigma_{J'}$) consists of the same terms w_{J^r} as the expression for σ_J (as long as J^r is in Δ_p) and has additional nonnegative terms whose corresponding terms in the expression for σ_j are zeros. This shows that $\sigma_{J'} \ge \sigma_J$ and completes the proof.

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