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ON STABLE CURRENTS IN POSITIVELY PINCHED CURVED HYPERSURFACES

BY

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Abstract. Let M^n $(n \geq 3)$ be an n-dimensional complete hypersurface in a real space form $N(c)$ $(c \geq 0)$. We prove that if the sectional curvature K_M of M satisfies the following pinching condition: $c + \delta < K_M \leq c + 1$, where $\delta = \frac{1}{5}$ for $n \geq 4$ and $\delta = \frac{1}{4}$ for $n = 3$, then there are no stable currents (or stable varifolds) in M. This is a positive answer to the well-known conjecture of Lawson and Simons.

1. Introduction. The following conjecture is well known:

CONJECTURE. There are no stable currents (or stable varifolds) in a compact, simply connected $\frac{1}{4}$-pinched Riemannian manifold.

In connection with this conjecture, Y. B. Shen and Q. He proved the following:

THEOREM A ([3]). *Let $N(c)$ be a real space form with constant sectional curvature c $(c \geq 0)$ and $M \hookrightarrow N(c)$ be an n-dimensional $(n \geq 3)$ complete hypersurface immersed in $N(c)$. If the sectional curvature K_M of M satisfies the following pinching condition*:

$$c + \delta < K_M \leq c + 1,$$

where $\delta = \frac{1}{5}$ for $n \geq 7$, $\delta = \frac{1}{4}$ for $n = 5, 6$ and $\delta = \frac{1}{3}$ for $n = 3, 4$, then there are no stable currents (*or stable varifolds*) *in M.*

In this paper, we prove the following theorem which is a positive answer to the above conjecture on complete pinched hypersurfaces immersed in a real space form.

THEOREM. *Let $N(c)$ be a real space form with constant sectional curvature c $(c \geq 0)$ and $M \hookrightarrow N(c)$ be an n-dimensional $(n \geq 3)$ complete hypersurface immersed in $N(c)$. If the sectional curvature K_M of M satisfies the following pinching condition*:

$$c + \delta < K_M \leq c + 1,$$

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where $\delta = \frac{1}{5}$ *for* $n \geq 4$ *and* $\delta = \frac{1}{4}$ *for* $n = 3$, *then there are no stable currents* (*or stable varifolds*) *in* M.

2. Preliminaries. From now on we make use of the following convention on ranges of indices unless otherwise stated:

$$1 \leq \alpha, \beta, \ldots \leq n; \quad 1 \leq i, j, \ldots \leq p; \quad p+1 \leq r, s, \ldots \leq n.$$

The following proposition is well known from [1]:

PROPOSITION 2.1. *Let* $N(c)$ *be a real space form with constant sectional curvature* c $(c \geq 0)$ *and* $M \hookrightarrow N(c)$ *an* n*-dimensional compact submanifold with the second fundamental form* B *in* $N(c)$. *If for any point* $x \in M$ *and any local orthonormal frame field* $\{e_i, e_r\}$ *at* $x \in M$,

$$F(n,p) = \sum_{i,r} \{2\|B(e_i, e_r)\|^2 - \langle B(e_i, e_i), B(e_r, e_r)\rangle\} < p(n-p)c, \tag{2.2}$$

where $0 < p < n$, *then there are no stable* p*-currents* (*or stable* p*-varifolds*) *in* M.

Let $x \in M$ be an arbitrary point and let $\{\lambda_\alpha\}$ be the principal curvatures of M corresponding to the principal direction vectors $\{\overline{e}_\alpha\}$ which form an orthonormal basis at x. For any local orthonormal frame field $\{e_\alpha\}$ at $x \in M$, there is an orthogonal matrix (a_α^β) such that

$$e_\alpha = \sum_\beta a_\alpha^\beta \overline{e}_\beta. \tag{2.3}$$

In the following, all calculations will be made at x. It can be seen from (2.2) and (2.3) that

$$F(n,p) = \sum_{\alpha,i,r} (\lambda_\alpha a_i^\alpha a_r^\alpha)^2 - F_1 - F_2, \tag{2.4}$$

where

$$F_1 = \sum_{\alpha \neq \beta} \lambda_\alpha \lambda_\beta \Big\{ \Big(\sum_i a_i^\alpha a_i^\beta \Big)^2 + \Big(\sum_r a_r^\alpha a_r^\beta \Big)^2 \Big\}, \tag{2.5}$$

$$F_2 = \sum_{\substack{\alpha \neq \beta \\ i,r}} \lambda_\alpha \lambda_\beta (a_i^\alpha a_r^\beta)^2. \tag{2.6}$$

We may always assume that at $x \in M$,

$$0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n. \tag{2.7}$$

We need the following lemma:

LEMMA 2.2.

$$F_1 \geq 2\lambda_1 \sum_{\substack{\alpha \neq 1 \\ i,r}} \lambda_\alpha (a_i^\alpha a_r^\alpha)^2 + 2\lambda_1 \lambda_2 \sum_{i,r} (a_i^1 a_r^1)^2.$$

Proof. By using (2.7) and the fact that the matrix (a_α^β) is orthogonal, we have

$$\text{(2.8)}\quad \sum_{\beta\neq 1}\lambda_1\lambda_\beta\Big\{\Big(\sum_i a_i^1 a_i^\beta\Big)^2+\Big(\sum_r a_r^1 a_r^\beta\Big)^2\Big\}$$
$$\geq \sum_{\beta\neq 1}\lambda_1\lambda_2\Big\{\Big(\sum_i a_i^1 a_i^\beta\Big)^2+\Big(\sum_r a_r^1 a_r^\beta\Big)^2\Big\}=2\lambda_1\lambda_2\sum_{i,r}(a_i^1a_r^1)^2.$$

For all $\alpha\neq 1$,

$$\text{(2.9)}\quad \sum_{\beta\neq \alpha}\lambda_\alpha\lambda_\beta\Big\{\Big(\sum_i a_i^\alpha a_i^\beta\Big)^2+\Big(\sum_r a_r^\alpha a_r^\beta\Big)^2\Big\}$$
$$\geq \sum_{\beta\neq \alpha}\lambda_1\lambda_\alpha\Big\{\Big(\sum_i a_i^\alpha a_i^\beta\Big)^2+\Big(\sum_r a_r^\alpha a_r^\beta\Big)^2\Big\}=2\lambda_1\lambda_\alpha\sum_{i,r}(a_i^\alpha a_r^\alpha)^2.$$

The assertion follows from (2.5), (2.8) and (2.9) immediately.

Lemma 2.3.

$$F_2\geq \frac{n-1}{n}\Big\{\lambda_1\lambda_2+\lambda_1\sum_{\alpha\neq 1}\lambda_\alpha\Big\}-\lambda_1\lambda_2\sum_{i,r}(a_i^1a_r^1)^2-\lambda_1\sum_{\substack{\alpha\neq 1\\ i,r}}\lambda_\alpha(a_i^\alpha a_r^\alpha)^2.$$

Proof. For all $\alpha\neq 1$, we have

$$\text{(2.10)}\quad \sum_{\substack{\beta\neq\alpha\\ i,r}}\lambda_\alpha\lambda_\beta(a_i^\alpha a_r^\beta)^2\geq \sum_{\substack{\beta\neq\alpha\\ i,r}}\lambda_\alpha\lambda_1(a_i^\alpha a_r^\beta)^2$$
$$=\lambda_1\lambda_\alpha(n-p)\sum_i(a_i^\alpha)^2-\lambda_1\lambda_\alpha\sum_{i,r}(a_i^\alpha a_r^\alpha)^2,$$

and

$$\text{(2.11)}\quad \sum_{\substack{\beta\neq 1\\ i,r}}\lambda_1\lambda_\beta(a_i^1 a_r^\beta)^2\geq \lambda_1\lambda_2\sum_{\substack{\beta\neq 1\\ i,r}}(a_i^1 a_r^\beta)^2$$
$$=\lambda_1\lambda_2(n-p)\sum_i(a_i^1)^2-\lambda_1\lambda_2\sum_{i,r}(a_i^1 a_r^1)^2.$$

It follows from (2.6), (2.10) and (2.11) that

$$\text{(2.12)}\quad F_2\geq (n-p)\Big\{\lambda_1\lambda_2\sum_i(a_i^1)^2+\lambda_1\sum_{\substack{\alpha\neq 1\\ i}}\lambda_\alpha(a_i^\alpha)^2\Big\}$$
$$-\lambda_1\lambda_2\sum_{i,r}(a_i^1a_r^1)^2-\lambda_1\sum_{\substack{\alpha\neq 1\\ i,r}}\lambda_\alpha(a_i^\alpha a_r^\alpha)^2.$$

On the other hand, we also have, for all $\beta \neq 1$,

$$\sum_{\substack{\alpha\neq\beta\\ i,r}} \lambda_\alpha\lambda_\beta (a_i^\alpha a_r^\beta)^2 \geq \lambda_1\lambda_\beta p \sum_r (a_r^\beta)^2 - \lambda_1\lambda_\beta \sum_{i,r} (a_i^\beta a_r^\beta)^2, \tag{2.13}$$

$$\sum_{\substack{\alpha\neq 1\\ i,r}} \lambda_\alpha\lambda_1 (a_i^\alpha a_r^1)^2 \geq \lambda_1\lambda_2 p \sum_r (a_r^1)^2 - \lambda_1\lambda_2 \sum_{i,r} (a_i^1 a_r^1)^2. \tag{2.14}$$

Substituting (2.13) and (2.14) into (2.6),we get

$$\begin{aligned} F_2 \geq p\Big\{\lambda_1\lambda_2 \sum_r (a_r^1)^2 + \lambda_1 \sum_{\substack{\beta\neq 1\\ r}} \lambda_\beta (a_r^\beta)^2\Big\} \\ - \lambda_1\lambda_2 \sum_{i,r} (a_i^1 a_r^1)^2 - \lambda_1 \sum_{\substack{\beta\neq 1\\ i,r}} \lambda_\beta (a_i^\beta a_r^\beta)^2. \end{aligned} \tag{2.15}$$

By (2.12)$\times p$+(2.15)$\times(n-p)$, we obtain

$$\begin{aligned} F_2 \geq \frac{p(n-p)}{n}\Big\{\lambda_1\lambda_2 + \lambda_1 \sum_{\alpha\neq 1} \lambda_\alpha\Big\} \\ - \lambda_1\lambda_2 \sum_{i,r} (a_i^1 a_r^1)^2 - \lambda_1 \sum_{\substack{\alpha\neq 1\\ i,r}} \lambda_\alpha (a_i^\alpha a_r^\alpha)^2. \end{aligned}$$

Lemma 2.4.

$$F(n,p) \leq -\frac{n-1}{n}\lambda_1\lambda_2 + \frac{1}{4}\sum_{\alpha\neq 1}\Big(\lambda_\alpha^2 - \frac{5n-4}{n}\lambda_1\lambda_\alpha\Big).$$

Proof. By (2.4) and Lemmas 2.2 and 2.3, we have

$$\begin{aligned} F(n,p) \leq -\frac{n-1}{n}\Big\{\lambda_1\lambda_2 + \lambda_1 \sum_{\alpha\neq 1}\lambda_\alpha\Big\} + \sum_{i,r}(\lambda_1^2 - \lambda_1\lambda_2)(a_i^1 a_r^1)^2 \\ + \sum_{\substack{\alpha\neq 1\\ i,r}} (\lambda_\alpha^2 - \lambda_1\lambda_\alpha)(a_i^\alpha a_r^\alpha)^2. \end{aligned} \tag{2.16}$$

Since the matrix (a_α^β) is orthogonal, we obtain

$$\sum_{i,r}(a_i^\alpha a_r^\alpha)^2 \leq \frac{1}{4}\Big[\sum_i (a_i^\alpha)^2 + \sum_r (a_r^\alpha)^2\Big]^2 = \frac{1}{4}, \quad \forall\alpha. \tag{2.17}$$

From (2.16) and (2.17), the conclusion follows.

Lemma 2.5 ([3]). *If $\varepsilon^2 = \delta$, then*

(1) $\lambda_\alpha > \varepsilon$ *for* $\alpha \neq 1$,
(2) $\lambda_\alpha \leq 1$ *for* $\alpha \neq n$,
(3) $\lambda_n < \varepsilon^{-1}$ *and* $\lambda_1 > \varepsilon^2$ *if* $n \geq 3$.

3. Proof of the Theorem. The proof is divided into two cases.

The first case. Suppose that $\lambda_1 < \varepsilon$. Since $\varepsilon < \lambda_n < \varepsilon^{-1}$, we consider six subcases separately.

Subcase (1): $\dfrac{2\sqrt{5}}{5\varepsilon} \le \lambda_n < \dfrac{1}{\varepsilon}$.

By (2.7) and the Gauss equation, we have

$$\lambda_1 \ge \varepsilon^2 \lambda_n \ge \frac{2\sqrt{5}}{5}\,\varepsilon. \tag{3.1}$$

It can be seen from (3.1) that

$$\frac{5n-4}{2n}\,\lambda_1 \ge \frac{1}{2}\left(\frac{1}{\lambda_n} + \frac{\varepsilon^2}{\lambda_1}\right). \tag{3.2}$$

For $\alpha \ne 1, n$, we think of $\lambda_\alpha^2 - \frac{5n-4}{n}\lambda_1\lambda_\alpha$ as a function of λ_α. By (3.2) we obtain

$$\lambda_\alpha^2 - \frac{5n-4}{n}\,\lambda_1\lambda_\alpha \le \frac{\varepsilon^4}{\lambda_1^2} - \frac{5n-4}{n}\,\varepsilon^2. \tag{3.3}$$

Using (2.7), (3.3) and Lemma 2.4, we get

$$\begin{aligned} F(n,p) \le\; & -\frac{n-1}{n}\,\varepsilon^2 + \frac{1}{4}\,\lambda_n^2 - \frac{5n-4}{4n}\,\lambda_1\lambda_n \\ & + \frac{1}{4}\sum_{\alpha \ne 1,n}\left\{\frac{\varepsilon^4}{\lambda_1^2} - \frac{5n-4}{n}\,\varepsilon^2\right\} \le \frac{1}{4n\lambda_1^2}\, f(\lambda_1), \end{aligned} \tag{3.4}$$

where

$$f(\lambda_1) = \frac{n\lambda_1^2}{\varepsilon^2} - \frac{5n-4}{\varepsilon}\,\lambda_1^3 - (5n^2-10n+4)\varepsilon^2\lambda_1^2 + n(n-2)\varepsilon^4. \tag{3.5}$$

When $\lambda_1 \in \left[\frac{2\sqrt{5}}{5}\varepsilon, \varepsilon\right)$, $f(\lambda_1)$ is a decreasing function, so we have

$$\begin{aligned} F(n,p) \le \frac{1}{4n\lambda_1^2}\Big\{ & \frac{4}{5}n - \frac{8}{25}\sqrt{5}(5n-4)\varepsilon^2 \\ & - \frac{4}{5}(5n^2-10n+4)\varepsilon^4 + n(n-2)\varepsilon^4 \Big\} < 0. \end{aligned} \tag{3.6}$$

Subcase (2): $\dfrac{\sqrt{15}}{5\varepsilon} \le \lambda_n < \dfrac{2\sqrt{5}}{5\varepsilon}$.

It is easy to see that

$$\lambda_1 \ge \varepsilon^2 \lambda_n \ge \frac{\sqrt{15}}{5}\,\varepsilon, \tag{3.7}$$

and hence

$$\frac{5n-4}{2n}\,\lambda_1 \ge \frac{1}{2}\left(\frac{1}{\lambda_n} + \frac{\varepsilon^2}{\lambda_1}\right). \tag{3.8}$$

From (2.7), (3.8) and Lemma 2.4, we obtain

$$\begin{aligned}(3.9)\quad & F(n,p)\\ &\le \frac{1}{4n\lambda_1^2}\{n\lambda_n^2\lambda_1^2-(5n-4)\lambda_1^3\lambda_n-(5n^2-10n+4)\varepsilon^2\lambda_1^2+n(n-2)\varepsilon^4\}\\ &\le \frac{1}{4n\lambda_1^2}\, g(\lambda_1),\end{aligned}$$

where

$$(3.10)\qquad g(\lambda_1)=\frac{4n}{5\varepsilon^2}\lambda_1^2-\frac{2\sqrt{5}}{5\varepsilon}(5n-4)\lambda_1^3-(5n^2-10n+4)\varepsilon^2\lambda_1^2+n(n-2)\varepsilon^4.$$

When $\lambda_1 \in [\frac{\sqrt{15}}{5}\,\varepsilon,\varepsilon)$, $g(\lambda_1)$ is a decreasing function,so we have

$$\begin{aligned}(3.11)\quad F(n,p) \le \frac{1}{4n\lambda_1^2}\Big\{&\frac{12}{25}\,n-\frac{6\sqrt{3}}{25}\,(5n-4)\varepsilon^2\\ &-\frac{3}{5}\,(5n^2-10n+4)\varepsilon^4+n(n-2)\varepsilon^4\Big\}<0.\end{aligned}$$

Subcase (3): $\frac{x}{\varepsilon} \le \lambda_n < \frac{\sqrt{15}}{5\varepsilon}$, where $x=\frac{\sqrt{2}}{2}$ for $n \ge 4$, $x=\frac{\sqrt{66}}{11}$ for $n=3$.

Obviously, the following inequality holds:

$$(3.12)\qquad \lambda_1 \ge \varepsilon^2\lambda_n \ge x\varepsilon,$$

so we have

$$(3.13)\qquad \frac{5n-4}{2n}\,\lambda_1 \ge \frac{1}{2}\left(\frac{1}{\lambda_n}+\frac{\varepsilon^2}{\lambda_1}\right).$$

By (2.7), (3.13) and Lemma 2.4, we obtain

$$\begin{aligned}(3.14)\qquad F(n,p) &\le \frac{1}{4n\lambda_1^2}\,\{n\lambda_1^2\lambda_n^2-(5n-4)\lambda_1^3\lambda_n\\ &\quad -(5n^2-10n+4)\varepsilon^2\lambda_1^2+n(n-2)\varepsilon^4\}\\ &\le \frac{1}{4n\lambda_1^2}\,h(\lambda_1),\end{aligned}$$

where

$$\begin{aligned}(3.15)\qquad h(\lambda_1)=\frac{3}{5\varepsilon^2}\,n\lambda_1^2-\frac{\sqrt{15}}{5\varepsilon}\,(5n-4)\lambda_1^3\\ -(5n^2-10n+4)\varepsilon^2\lambda_1^2+n(n-2)\varepsilon^4.\end{aligned}$$

When $\lambda_1 \in [x\varepsilon,\varepsilon)$, $h(\lambda_1)$ is a decreasing function, so we have

$$F(n,p) \le \frac{1}{4n\lambda_1^2}\left\{\frac{3}{5}nx^2 - \frac{\sqrt{15}}{5}x^3(5n-4)\varepsilon^2 - [(5x^2-1)n^2-(10x^2-2)n+4x^2]\varepsilon^4\right\} < 0. \tag{3.16}$$

Subcase (4): $\sqrt{2} \le \lambda_n < \frac{x}{\varepsilon}$, where $x = \frac{\sqrt{2}}{2}$ for $n \ge 4$ or $x = \frac{\sqrt{66}}{11}$ for $n = 3$.

From (2.7) and Lemma 2.4, we obtain

$$F(n,p) \le -\frac{n-1}{n}\varepsilon^2 + \frac{1}{4}\lambda_n^2 - \frac{5n-4}{4n}\varepsilon^2\lambda_n + \frac{1}{4}\sum_{\alpha\ne 1,n}\left\{\frac{1}{\lambda_n^2} - \frac{5n-4}{n}\varepsilon^2\right\} < 0. \tag{3.17}$$

Subcase (5): $1 \le \lambda_n < \sqrt{2}$.

Similarly, we have

$$F(n,p) \le \{n\lambda_n^4 - (5n^2-5n)\varepsilon^2\lambda_n^2 + n(n-2)\} < 0. \tag{3.18}$$

Subcase (6): $\varepsilon < \lambda_n < 1$.

Obviously,

$$F(n,p) < -\frac{n-1}{n}\varepsilon^2 + \frac{1}{4}\sum_{\alpha\ne 1}\left(1 - \frac{5n-4}{n}\varepsilon^2\right) = \frac{n-1}{4}(1-5\varepsilon^2) \le 0. \tag{3.19}$$

The second case. Suppose that $\lambda_1 \ge \varepsilon$. By Lemma 2.4 and 2.5, we have

$$\begin{aligned} F(n,p) &\le -\frac{n-1}{n}\lambda_2\varepsilon + \frac{1}{4}\sum_{\alpha\ne 1}\left(\lambda_\alpha^2 - \frac{5n-4}{n}\varepsilon\lambda_\alpha\right) \\ &= \left(\frac{1}{4}\lambda_2^2 - \frac{9n-8}{4n}\lambda_2\varepsilon\right) + \left(\frac{1}{4}\lambda_n^2 - \frac{5n-4}{4n}\varepsilon\lambda_n\right) \\ &\quad + \frac{1}{4}\sum_{\alpha\ne 1,2,n}\left(\lambda_\alpha - \frac{5n-4}{n}\varepsilon\right)\lambda_\alpha \\ &\le \left(\frac{1}{4}\varepsilon^2 - \frac{9n-8}{4n}\varepsilon^2\right) + \left(\frac{1}{4\varepsilon^2} - \frac{5n-4}{4n}\right) < 0. \end{aligned} \tag{3.20}$$

In summary, $F(n,p) < 0$ at any point $x \in M$ and any local orthonormal frame field $\{e_\alpha\}$ at $x \in M$. By Proposition 2.1, the Theorem is proved completely.

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