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MEAN VALUE DENSITIES FOR TEMPERATURES

BY

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Abstract. A positive measurable function K on a domain D in $\mathbb{R}^{n+1}$ is called a mean value density for temperatures if $u(0,0) = \iint_D K(x,t)u(x,t)\,dx\,dt$ for all temperatures u on $\overline{D}$. We construct such a density for some domains. The existence of a bounded density and a density which is bounded away from zero on D is also discussed.

1. Let D be a bounded domain in $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1} = \{(x,t);\ x \in \mathbb{R}^n, t \in \mathbb{R}\}$. Suppose that $(0,0) \in \overline{D}$. We say that a measurable function $K(x,t)$ on D is a *mean value density* (at the origin with respect to the heat equation) if $K > 0$ a.e. on D and

$$u(0,0) = \iint_D K(x,t)u(x,t)\,dx\,dt \tag{1}$$

for every temperature u on $\overline{D}$, that is, for every function u which satisfies the heat equation on a neighborhood of $\overline{D}$.

An interesting example of such a density is the following function K on $\Omega(c)$:

$$K(x,t) := \frac{1}{2^{n+2}(\pi c)^{n/2}} \frac{\|x\|^2}{t^2} \tag{2}$$

(see [5]). Here $\Omega(c)$ is the heat ball defined by a level surface of the Gauss–Weierstrass kernel W, that is,

$$\Omega(c) := \{(x,t) \in \mathbb{R}^{n+1};\ W(x,-t) > (4\pi c)^{-n/2}\}$$

with

$$W(x,t) := \begin{cases} (4\pi t)^{-n/2}\exp(-\|x\|^2/4t) & \text{if } t > 0, \\ 0 & \text{if } t \le 0, \end{cases}$$

and $\|x\| = (x_1^2 + \ldots + x_n^2)^{1/2}$.

In this paper, we consider the following problems:

(i) Which domains have a mean value density?
(ii) Which domains have a bounded mean value density?

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(iii) Does there exist a mean value density that is bounded away from zero?

For the harmonic case, similar problems were discussed by Hansen and Netuka in [2]. They showed that, for every bounded domain U in $\mathbb{R}^n$ that contains 0, there exists a bounded function $K > 0$ on U such that

$$h(0) = \int_U K(x)h(x)\,dx$$

for every bounded harmonic function h on U. Furthermore, for smooth domains they constructed such functions K with $\inf_{x\in U} K(x) > 0$. In our parabolic case the situation is more complicated. It is easily seen that if

$$\sup\{t;\ (x,t) \in D\} > 0 \tag{3}$$

then D does not have a mean value density. Furthermore, there is no mean value density on a cone $\{\|x\| < -ct;\ -1 < t < 0\}$ (see Corollary 7(a) below). On the other hand, every rectangle $\{(x,t);\ |x_i| < c$ for all i, $-c^2 < t < 0\}$ has a bounded mean value density (see [1, p. 276]). A heat ball has a mean value density as above, but we shall see later that there is no bounded density there. Another example of a domain that has bounded mean value density is a modified heat ball, defined in [6]. Bounded mean value densities are useful for the monotone approximation of subtemperatures by smooth subtemperatures.

In Section 2 we construct mean value densities for certain domains. The argument is based on that in [2], but considerable modification of the details is necessary. In Section 3, we discuss the above problems (i)–(iii) for special domains.

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2. For a domain D in $\mathbb{R}^{n+1}$, we denote by $\partial_{\mathrm{p}} D$ the parabolic boundary of D, that is, the set of boundary points which can be connected to some point of D by a curve in D having strictly increasing t-coordinate. Also for $(x_0, t_0) \in D$, $\Lambda(x_0, t_0; D)$ is the set of all points $(x,t) \in D \setminus \{(x_0,t_0)\}$ which can be connected to (x_0,t_0) by a polygonal line in D having strictly increasing t-coordinate. We write $\Omega(y,s;c)$ for the heat ball with centre (y,s) and radius $c > 0$, that is,

$$\Omega(y,s;c) := \{(x,t) \in \mathbb{R}^{n+1};\ W(y-x, s-t) > (4\pi c)^{-n/2}\}. \tag{4}$$

Hence $\Omega(c) = \Omega(0,0;c)$. Further, for $a > 0$ we put

$$K_a(x,t) := \frac{\|x\|^2}{(-t)^{(n+4-2a)/2}} \exp\left(\frac{(2a-n)\|x\|^2}{4n(-t)}\right) \qquad (t < 0) \tag{5}$$

and define the constant $p(a,c)$ by

$$p(a,c) := \frac{a}{2^{n+1} n \pi^{n/2} c^a}. \tag{6}$$

Note that $p(n/2,c)K_{n/2}$ is the function K in (2). In view of [5], the functions $p(a,c)K_a$ are also mean value densities on $\Omega(c)$.

Regarding the existence of mean value densities, we have the following result.

THEOREM 1. *Let D be a bounded domain in $\mathbb{R}^{n+1}$ such that $\Omega(c_0) \subset D$ for some $c_0 > 0$. Suppose that there exists a family $\{E_\alpha\}_{\alpha \in A}$ of subdomains satisfying the following conditions*:

(a) *For each $\alpha \in A$, $\Omega(c_0/2) \subset E_\alpha \subset D$, $\overline{E}_\alpha^\circ = E_\alpha$, and for every $(y,s) \in E_\alpha$ there exists $(z,r) \in \Omega(c_0/2)$ such that $(y,s) \in \Lambda(z,r;E_\alpha)$.*

(b) *$\bigcup_{\alpha \in A} \overline{\partial_{\mathrm{p}} E_\alpha} \supset D \setminus \Omega(2c_0/3)$.*

Then there is a mean value density on D.

Proof. Fix a nonnegative, continuous function η on $[0,\infty)$ such that $\{t;\, \eta(t) > 0\} = [0,1)$ and

$$\int_0^1 (4\pi t)^{n/2} \eta(t)\, dt = 1.$$

For each $(y,s) \in D$, put

$$\gamma(y,s) := \frac{1}{2} \sup\{c;\, \Omega(y,s;c) \subset D\},$$

and define

$$\begin{aligned}\tau_{(y,s)}(x,t) &:= \frac{1}{2n} K_{(n+2)/2}(y-x, s-t) \\ &\qquad \times \gamma(y,s)^{-(n+2)/2} \eta\left(\frac{s-t}{\gamma(y,s)} \exp\left(\frac{\|y-x\|^2}{2n(s-t)}\right)\right)\end{aligned}$$

whenever $t < s$, and $\tau_{(y,s)}(x,t) := 0$ whenever $t \ge s$. Then $\tau_{(y,s)}$ is continuous on $\mathbb{R}^n \times (-\infty, s)$, and

$$\begin{aligned}&\{(x,t);\, \tau_{(y,s)}(x,t) > 0\} \\ &\quad = \{(x,t);\, 0 < (s-t)\exp(\|y-x\|^2/(2n(s-t))) < \gamma(y,s),\ x \ne y\} \\ &\quad = \Omega(y,s;\gamma(y,s)) \setminus (\{y\} \times \mathbb{R}).\end{aligned}$$

For every $\alpha \in A$, let $\mu_\alpha^{(z,r)}$ denote the parabolic measure at (z,r) for E_α, and put

$$w_\alpha(x,t) := p(n/2, c_0/2) \int_{\Omega(c_0/2)} \left(\int_{\partial E_\alpha} \tau_{(y,s)}(x,t)\, d\mu_\alpha^{(z,r)}(y,s)\right) \frac{\|z\|^2}{r^2}\, dz\, dr$$

for every $(x,t) \in \mathbb{R}^{n+1}$. By the minimum principle for temperatures, if $(y,s) \in \varLambda(z,r;E_\alpha)$ then $\mu_\alpha^{(y,s)}$ is absolutely continuous with respect to $\mu_\alpha^{(z,r)}$, so that condition (a) implies that

$$\bigcup\{\operatorname{supp}(\mu_\alpha^{(y,s)});\ (y,s)\in E_\alpha\} = \bigcup\{\operatorname{supp}(\mu_\alpha^{(z,r)});\ (z,r)\in \varOmega(c_0/2)\}.$$

Hence, by [4, Theorem 1],

$$\overline{\partial_{\mathrm{p}} E_\alpha} = \overline{\bigcup\{\operatorname{supp}(\mu_\alpha^{(z,r)});\ (z,r)\in \varOmega(c_0/2)\}}.$$

We claim that

$$\{(x,t);\ w_\alpha(x,t)>0\} = \bigcup\{\varOmega(y,s;\ \gamma(y,s))\setminus(\{y\}\times\mathbb{R});(y,s)\in\overline{\partial_{\mathrm{p}}E_\alpha}\}. \tag{7}$$

To prove (7), we first show that $w_\alpha(x,t)>0$ if $(x,t)\in\varOmega(y_0,s_0;\gamma(y_0,s_0))\setminus(\{y_0\}\times\mathbb{R})$ for some $(y_0,s_0)\in\overline{\partial_{\mathrm{p}}E_\alpha}$. Since $\tau_{(y_0,s_0)}(x,t)>0$, there is an open neighbourhood B of (y_0,s_0) such that $\tau_{(y,s)}(x,t)>0$ for all $(y,s)\in B$. In particular $\tau_{(y,s)}(x,t)>0$ on $B\cap\partial E_\alpha$.

We consider two cases. First suppose that

$$\mu_\alpha^{(z,r)}(B\cap\partial E_\alpha)=0$$

for all $(z,r)\in\varOmega(c_0/2)$. Then $\operatorname{supp}(\mu_\alpha^{(z,r)})$ is contained in $\partial E_\alpha\setminus B$, so that since $B\cap\partial E_\alpha$ is open in ∂E_α,

$$(y_0,s_0)\in\overline{\partial_{\mathrm{p}}E_\alpha} = \overline{\bigcup\{\operatorname{supp}(\mu_\alpha^{(z,r)});\ (z,r)\in\varOmega(c_0/2)\}} \subset \partial E_\alpha\setminus B.$$

This contradicts the fact that $(y_0,s_0)\in B$.

Second, suppose that

$$\mu_\alpha^{(z_0,r_0)}(B\cap\partial E_\alpha)>0$$

for some $(z_0,r_0)\in\varOmega(c_0/2)$. Then, by the minimum principle,

$$\mu_\alpha^{(z,r)}(B\cap\partial E_\alpha)>0$$

for all $(z,r)\in\varOmega(c_0/2)$ with $r>r_0$. Since $\tau_{(y,s)}(x,t)>0$ for $(y,s)\in B\cap\partial E_\alpha$, we have

$$\int_{\partial E_\alpha}\tau_{(y,s)}(x,t)\,d\mu_\alpha^{(z,r)}(y,s)>0$$

for all $(z,r)\in\varOmega(c_0/2)$ with $r>r_0$. This implies that $w_\alpha(x,t)>0$.

Conversely, if $w_\alpha(x,t)>0$, then

$$\int_{\partial E_\alpha}\tau_{(y,s)}(x,t)\,d\mu_\alpha^{(z_0,r_0)}(y,s)>0$$

for some (z_0,r_0), and hence

$$\tau_{(y_0,s_0)}(x,t)>0$$

for some $(y_0,s_0)\in\operatorname{supp}(\mu_\alpha^{(z_0,r_0)})$. Thus $(x,t)\in\Omega(y_0,s_0;\gamma(y_0,s_0))\backslash(\{y_0\}\times\mathbb{R})$. Since

$$\operatorname{supp}(\mu_\alpha^{(z_0,r_0)})\subset\overline{\partial_{\mathrm{p}}E_\alpha}$$

we have $(y_0,s_0)\in\overline{\partial_{\mathrm{p}}E_\alpha}$. Thus (7) is established.

Next, for every temperature u on $\overline{D}$, we have

$$\begin{aligned}
&\iint u(x,t)\tau_{(y,s)}(x,t)\,dx\,dt\\
&\quad=\iint\limits_{\Omega(y,s;\gamma(y,s))} u(x,t)K_{(n+2)/2}(y-x,s-t)\\
&\qquad\times\gamma(y,s)^{-(n+2)/2}\eta\biggl(\frac{s-t}{\gamma(y,s)}\exp\biggl(\frac{\|y-x\|^2}{2n(s-t)}\biggr)\biggr)\,dx\,dt\\
&\quad=\int\limits_0^{\gamma(y,s)} d\ell\int\limits_{\partial\Omega(y,s;\ell)} Q(y-\xi,s-\tau)u(\xi,\tau)\\
&\qquad\times\gamma(y,s)^{-(n+2)/2}\eta\biggl(\frac{\ell}{\gamma(y,s)}\biggr)\,d\sigma(\xi,\tau)\\
&\quad=\int\limits_0^{\gamma(y,s)}\gamma(y,s)^{-(n+2)/2}\eta\biggl(\frac{\ell}{\gamma(y,s)}\biggr)(4\pi\ell)^{n/2}u(y,s)\,d\ell\\
&\quad=u(y,s)\int\limits_0^1\gamma(y,s)^{-(n+2)/2}\eta(t)(4\pi t\gamma(y,s))^{n/2}\gamma(y,s)\,dt\\
&\quad=u(y,s)\int\limits_0^1(4\pi t)^{n/2}\eta(t)\,dt=u(y,s),
\end{aligned}$$

because $(s-\tau)\exp(\|y-\xi\|^2/2n(s-\tau))=\ell$ on $\partial\Omega(y,s;\ell)$; here $Q(x,t)=\|x\|^2(4\|x\|^2t^2+(\|x\|^2-2nt)^2)^{-1/2}$ and σ is the surface area measure on $\partial\Omega((y,s);\ell)$ (see [6]). Hence

$$\begin{aligned}
&\iint\limits_D u(x,t)w_\alpha(x,t)\,dx\,dt\\
&=p(n/2,c_0/2)\\
&\quad\times\int\limits_{\Omega(c_0/2)}\biggl(\int\limits_{\partial E_\alpha}\biggl(\iint\limits_D u(x,t)\tau_{(y,s)}(x,t)\,dx\,dt\biggr)\,d\mu_\alpha^{(z,r)}(y,s)\biggr)\frac{\|z\|^2}{r^2}\,dz\,dr\\
&=p(n/2,c_0/2)\int\limits_{\Omega(c_0/2)}\biggl(\int\limits_{\partial E_\alpha}u(y,s)\,d\mu_\alpha^{(z,r)}(y,s)\biggr)\frac{\|z\|^2}{r^2}\,dz\,dr
\end{aligned}$$

$$= p(n/2, c_0/2) \int_{\Omega(c_0/2)} u(z,r)\, \frac{\|z\|^2}{r^2}\, dz\, dr = u(0,0).$$

If $(x,t) \in D \setminus \Omega(c_0)$, then there is $(y,s) \in D \setminus \Omega(2c_0/3)$ such that $(x,t) \in \underline{\Omega(y,s}; \gamma(y,s)) \setminus (\{y\} \times \mathbb{R})$. By condition (b), there is α such that $(y,s) \in \overline{\partial_{\mathrm{p}} E_\alpha}$. So, by (7), $w_\alpha(x,t) > 0$ and $\{w_\alpha > 0\}$ is open. Thus the sets $\{w_\alpha > 0\}_{\alpha \in A}$ form an open cover for $D \setminus \Omega(c_0)$, so that the Lindelöf property ensures that we can choose a countable subcover $\{w_{\alpha_k} > 0\}_{k=1}^{\infty}$. Put

$$K(x,t) := \frac{p(n/2, c_0)}{2}\, \frac{\|x\|^2}{t^2}\, \chi_{\Omega(c_0)}(x,t) + \sum_{k=1}^{\infty} 2^{-k-1} w_{\alpha_k}(x,t),$$

where $\chi_{\Omega(c_0)}$ is the characteristic function of $\Omega(c_0)$. Then $K > 0$ a.e. on D. Also, for every temperature u on $\overline{D}$, we have

$$\begin{aligned}\iint_D u(x,t) K(x,t)\, dx\, dt &= \frac{p(n/2, c_0)}{2} \iint_{\Omega(c_0)} u(x,t)\, \frac{\|x\|^2}{t^2}\, dx\, dt \\ &\quad + \sum_{k=1}^{\infty} 2^{-k-1} \iint_D u(x,t) w_{\alpha_k}(x,t)\, dx\, dt \\ &= \frac{u(0,0)}{2} + \sum_{k=1}^{\infty} 2^{-k-1} u(0,0) = u(0,0).\end{aligned}$$

This completes the proof of Theorem 1.

The class of domains which have bounded mean value densities is more restricted. The closure of such a domain contains every truncated heat ball, as we now show.

THEOREM 2. *Assume that there is a bounded mean value density K on a domain D. Then for every $c > 0$, there exists $t_c < 0$ such that*

(8) $$\overline{D} \supset \Omega(c) \cap \{t > t_c\}.$$

Proof. Consider the function

(9) $$v(y,s) := \iint_D K(x,t) W(x-y, t-s)\, dx\, dt.$$

Suppose that the assertion does not hold for some $c > 0$. Then we can choose points $\{(y_k, s_k)\}$ in $\Omega(c) \setminus \overline{D}$ such that $s_k > -1/k$ for all $k \geq 1$. Note that $(y_k, s_k) \to (0,0)$ as $k \to \infty$. Since $(y_k, s_k) \notin \overline{D}$, we have

(10) $$W(y_k, -s_k) = v(y_k, s_k)$$

by (1). Then $\liminf_{k\to\infty} W(y_k, -s_k) \geq (4\pi c)^{-n/2}$ because $(y_k, s_k) \in \Omega(c)$. On the other hand, the right hand side of (10) tends to zero as $k \to \infty$, because the boundedness of K ensures that v is continuous on $\mathbb{R}^{n+1}$ and $v(0,0) = 0$. This is a contradiction.

3. In this section we discuss mean value densities on domains of the form

$$D(\varphi) := \{(x,t) \in \mathbb{R}^{n+1};\ \|x\| < \varphi(t),\ -1 < t < 0\},$$

where φ is a continuous function on $[-1,0]$ with $\varphi > 0$ on $(-1,0)$. For simplicity, we also assume that

$(*)$ there is $t_0 \in [-1,0]$ such that φ is strictly decreasing on $[t_0,0]$ and strictly increasing on $[-1,t_0]$.

The following remark will be useful below.

REMARK 3. If $D(\varphi)$ has a mean value density K, then whenever $(y,s) \notin \overline{D(\varphi)}$,

$$W(y,-s) = \iint_{D(\varphi)} K(x,t) W(x-y, t-s)\, dx\, dt. \tag{11}$$

Hence letting $(y,s) \to (y_0,s_0) \in \partial D(\varphi)$, we deduce from Fatou's lemma that

$$W(y_0,-s_0) \geq \iint_{D(\varphi)} K(x,t) W(x-y_0, t-s_0)\, dx\, dt. \tag{12}$$

Regarding the nonexistence of mean value densities, we have the following result.

THEOREM 4. *If the origin is a regular boundary point of $D(\varphi)$ with respect to the Dirichlet problem for the heat equation, then there is no mean value density on $D(\varphi)$.*

Proof. Under this hypothesis $t_0 < 0$. Let f be a continuous function on $\partial D(\varphi)$ such that $f(0,0) = 0$, $f(x,t) > 0$ if $t > t_0$, and $f(x,t) = 0$ if $t \leq t_0$. Let v be the solution of the Dirichlet problem on $D(\varphi)$ with boundary function f, and $v = f$ on $\partial D(\varphi)$. Then $v \geq 0$ and $v \not\equiv 0$. For $k \in \mathbb{N}$ such that $-1/k > t_0$, put

$$u_k(x,t) := \begin{cases} v(x, t-1/k) & \text{if } t > t_0 + 1/k, \\ 0 & \text{if } t \leq t_0 + 1/k. \end{cases}$$

Now suppose that there is a mean value density K on $D(\varphi)$. Since u_k is a temperature on $\overline{D(\varphi)}$, we have

$$u_k(0,0) = \iint_{D(\varphi)} K(x,t) u_k(x,t)\, dx\, dt.$$

Since $(0,0)$ is regular, we have

$$\lim_{k\to\infty} u_k(0,0) = \lim_{k\to\infty} v(0,-1/k) = f(0,0) = 0.$$

On the other hand, Fatou's lemma implies that

$$\liminf_{k\to\infty} \iint\limits_{D(\varphi)} K(x,t)u_k(x,t)\,dx\,dt \geq \iint\limits_{D(\varphi)} K(x,t)v(x,t)\,dx\,dt > 0.$$

This is a contradiction.

REMARK 5. It is known that if φ satisfies

$$\varphi(t) < (4(-t)\log|\log(-t)|)^{1/2} \tag{13}$$

on a neighborhood of $t=0$, then the origin is a regular boundary point (see [1, p. 339]). On the other hand, $(0,0)$ is an irregular boundary point of $\Omega(c)$ (see [1, p. 340]). Let $m \geq 3$ be an integer. A modified heat ball $\Omega_m(c)$ is defined by

$$\Omega_m(c) := \{(x,t) \in \mathbb{R}^{n+1};\ \|x\| < (2(m+n)(-t)\log(c/(-t)))^{1/2},\ -c<t<0\}.$$

The function

$$K(x,t) := c_0(2(m+n)(-t)\log(c/(-t)) - \|x\|^2)^{m/2} \times \left(\frac{m(m+n)}{-t}\log(c/(-t)) + \frac{\|x\|^2}{t^2}\right) \tag{14}$$

is a bounded mean value density on $\Omega_m(c)$ (see [6]), where

$$c_0 := \frac{\omega_m}{2(m+2)(4\pi c)^{(m+n)/2}}$$

and ω_m is the volume of the unit ball in $\mathbb{R}^m$.

REMARK 6. In general, the regularity of the origin is not a sufficient condition for nonexistence of a mean value density. In fact, given an integer $m \geq 3$, let

$$D := \Omega_m(c) \setminus \{(x_1,\ldots,x_{n-1},0,t);\ x_i \in \mathbb{R}\ (i=1,\ldots,n-1),\ -c/2<t<0\}.$$

Then $(0,0)$ is a regular point of ∂D (see [3, p. 218] for the case $n=1$); but (14) is a bounded mean value density for D, because $\overline{D}^\circ = \Omega_m(c)$. This example also shows that we cannot replace $\overline{D}$ by D in (8).

For two functions φ and ψ on $[-1,0]$, we write $\varphi \approx \psi$ if there exist positive constants c_1, c_2 such that $c_1\psi(t) \leq \varphi(t) \leq c_2\psi(t)$ on a neighbourhood of $t=0$. We have the following results about the domains $D(\varphi)$.

COROLLARY 7. (a) *If $\varphi(t) \approx (-t)^\beta$ with $\beta \geq 1/2$, then there is no mean value density on $D(\varphi)$.*

(b) *If $\varphi(t) \approx (-t)^\beta$ with $\beta < 1/2$, then we can construct a mean value density on $D(\varphi)$.*

(c) *If there are $c_1 > 0$ and $t_1 < 0$ such that $(D(\varphi)\setminus\Omega(c_1)) \cap \{t > t_1\} = \emptyset$, then $D(\varphi)$ does not have a bounded mean value density. In particular, there is no bounded mean value density on a heat ball.*

(d) *If* $\varphi(t) \approx (-t)^\beta$, *then* $D(\varphi)$ *has no mean value density that is bounded away from zero.*

Proof. Part (a) follows from Theorem 4. To prove (b), we use Theorem 1. Choose $c_0 > 0$ such that $D(\varphi) \supset \Omega(c_0)$. For $0 < \alpha < 1$, put

$$E_\alpha := \{(x,t);\ \|x\| < \alpha\varphi(t),\ -1 < t < 0\} \cup \Omega(c_0/2).$$

Then $\{E_\alpha\}_{0<\alpha<1}$ satisfies the conditions of Theorem 1.

Part (c) follows from Theorem 2.

To show (d), we use the following assertion: There is a positive integer N, which depends only on the dimension n, and points $\{x_i\}_{i=1}^N$ in the unit sphere of $\mathbb{R}^n$ such that for every $r > 0$,

$$B(0,r/2) \setminus \{0\} \subset \bigcup_{i=1}^N B(rx_i, r), \tag{15}$$

where $B(x,r)$ is the usual ball in $\mathbb{R}^n$ with centre x and radius $r > 0$. The existence of N and $\{x_i\}_{i=1}^N$ is not difficult, because $B(0,1/2) \cap B(x_1,1)$ contains a truncated cone at the origin which has a positive aperture.

Now suppose that there is a mean value density K such that $K \geq c_0 > 0$ on $D(\varphi)$. Then $\beta < 1/2$. For $\{x_i\}_{i=1}^N$as above and $-1 < s < 0$, we put

$$u_s(x,t) := \sum_{i=1}^N W(x - x_i\varphi(s), t - s).$$

Then u_s is a nonnegative temperature on $D(\varphi)$, and by (12) we have

$$u_s(0,0) \geq \iint_{D(\varphi)} K(x,t)u_s(x,t)\,dx\,dt \geq c_0 \iint_{D(\varphi)} u_s(x,t)\,dx\,dt. \tag{16}$$

Note that

$$u_s(0,0) = \sum_{i=1}^N W(x_i\varphi(s), -s) = \frac{N}{(4\pi(-s))^{n/2}} \exp\left(-\frac{\varphi(s)^2}{-4s}\right).$$

On the other hand, since

$$B(0,\varphi(s)/2) \setminus \{0\} \subset \bigcup_{i=1}^N B(x_i\varphi(s), \varphi(s))$$

by (15), we see that

$$\begin{aligned}\iint_{D(\varphi)} u_s(x,t)\,dx\,dt &= \int_s^0 \Big(\int_{\|x\|<\varphi(s)} \sum_{i=1}^N W(x - x_i\varphi(s), t - s)\,dx \Big)\,dt \\ &\geq \int_s^0 \Big(\int_{\|x\|<\varphi(s)/2} W(x, t-s)\,dx \Big)\,dt\end{aligned}$$

$$= n\omega_n \int_s^0 \bigg(\int_0^{\varphi(s)/2} \frac{1}{(4\pi(t-s))^{n/2}} \exp\bigg(-\frac{r^2}{4(t-s)}\bigg) r^{n-1}\,dr\bigg)\,dt$$

$$= n\omega_n \pi^{-n/2} \int_s^0 \bigg(\int_0^{\varphi(s)/4\sqrt{t-s}} \tau^{n-1} \exp(-\tau^2)\,d\tau\bigg)\,dt \ge c_3(-s)$$

for all sufficiently small s, because $\varphi(s) \ge c_1(-s)^\beta$ and $\beta < 1/2$. Thus (16) implies that

$$\frac{N}{(4\pi(-s))^{n/2}} \exp(-c_1(-s)^{2\beta-1}) \ge \frac{N}{(4\pi(-s))^{n/2}} \exp\bigg(-\frac{\varphi(s)^2}{-s}\bigg) \ge c_0 c_3(-s).$$

This is a contradiction.

Remark 8. We conjecture that there is no domain which has a mean value density bounded away from zero. Assertion (d) in Corollary 7 supports our conjecture.

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