COLLOQUIUM MATHEMATICUM

VOL. 98 2003 NO. 1

FACTORIZATION OF MATRICES ASSOCIATED WITH CLASSES OF ARITHMETICAL FUNCTIONS

BY

SHAOFANG HONG (Chengdu and Haifa)

Abstract. Let f be an arithmetical function. A set $S = \{x_1, \ldots, x_n\}$ of n distinct positive integers is called multiple closed if $y \in S$ whenever $x \mid y \mid \mathrm{lcm}(S)$ for any $x \in S$, where $\mathrm{lcm}(S)$ is the least common multiple of all elements in S. We show that for any multiple closed set S and for any divisor chain S (i.e. $x_1 \mid \ldots \mid x_n$), if f is a completely multiplicative function such that $(f * \mu)(d)$ is a nonzero integer whenever $d \mid \mathrm{lcm}(S)$, then the matrix $(f(x_i, x_j))$ having f evaluated at the greatest common divisor (x_i, x_j) of x_i and x_j as its i, j-entry divides the matrix $(f[x_i, x_j])$ having f evaluated at the least common multiple $[x_i, x_j]$ of x_i and x_j as its i, j-entry in the ring $M_n(\mathbb{Z})$ of $n \times n$ matrices over the integers. But such a factorization is no longer true if f is multiplicative.

1. Introduction. Let n be a positive integer and let $((i, j))$ be the $n \times n$ matrix having the greatest common divisor (i, j) of i and j as its (i, j)-entry. In 1876, H. J. S. Smith [17] published his celebrated results by showing that the determinant of the $n \times n$ matrix $((i, j))$ is the product $\prod_{k=1}^{n} \varphi(k)$, where φ is Euler's totient function. Let f be an arithmetical function. For any positive integers x and y, we let $f(x, y)$ and $f[x, y]$ denote, for brevity, $f((x, y))$ and $f([x, y])$, respectively. Here $[x, y]$ means the least common multiple of x and y. Smith also proved that if f is an arithmetical function and $(f(i, j))$ is the $n \times n$ matrix having f evaluated at the greatest common divisor (i, j) of i and j as its (i, j)-entry, then $\det(f(i, j)) = \prod_{k=1}^{n} (f * \mu)(k)$, where μ is the Möbius function and $f * \mu$ is the Dirichlet convolution of f and μ. In 1972, Apostol [2] extended Smith's result. In 1988, McCarthy [16] generalized Smith's and Apostol's results to the class of even functions (mod r). In 1993, Bourque and Ligh [6] extended the results of Smith, Apostol, and McCarthy. In 1999, Hong [9] improved the lower bounds for the determinants of the matrices considered by Bourque and Ligh [6]. In 2002, Hong [11] generalized the results

2000 *Mathematics Subject Classification*: Primary 11C20, 11A25.

Key words and phrases: factorization, multiplicative function, completely multiplicative function, multiple-closed set, divisor chain.

Supported partially by an NNSF of China (Grant No. 10101015) and the Lady Davis Fellowship at the Technion.

of Smith, Apostol, McCarthy, Bourque and Ligh to certain classes of arithmetical functions.

Let $S = \{x_1, \dots, x_n\}$ be a set of n distinct positive integers. Denote by $(f(x_i, x_j))$ the $n \times n$ matrix having f evaluated at the greatest common divisor (x_i, x_j) of x_i and x_j as its i, j-entry, and by $(f[x_i, x_j])$ the $n \times n$ matrix having f evaluated at the least common multiple $[x_i, x_j]$ of x_i and x_j as its i, j-entry. The set S is said to be *factor closed* if it contains every divisor of x for any $x \in S$. From Bourque and Ligh's result [7, Theorem 4], we can see that if S is a factor closed set and f is a multiplicative function such that $f \in \mathcal{L}_S$, where $\mathcal{L}_S$ is the class of arithmetical functions defined by

$$\mathcal{L}_S := \{f : (f * \mu)(d) \in \mathbb{Z}^* \text{ whenever } d \,|\, \mathrm{lcm}(S)\},$$

where $\mathbb{Z}^* := \mathbb{Z}\backslash\{0\}$ denotes the set of nonzero integers and $\mathrm{lcm}(S)$ means the least common multiple of all elements in S, then the matrix $(f(x_i, x_j))$ divides the matrix $(f[x_i, x_j])$ in the ring $M_n(\mathbb{Z})$ of $n \times n$ matrices over the integers. Observe that the condition $f \in \mathcal{L}_S$ of [7, Theorem 4] was stated as $f \in \mathcal{T}_S := \{f : (f * \mu)(x) \in \mathbb{Z}^* \text{ for any } x \in S\}$. In fact, we can easily show that if S is factor closed and f is multiplicative, then $f \in \mathcal{L}_S$ if and only if $f \in \mathcal{T}_S$.

Many generalizations of Smith's result in various directions have been published [2–14, 16]. Our main interest in the present paper is in the divisibility of the matrix $(f[x_i, x_j])$ by $(f(x_i, x_j))$. We introduce the following concept: The set S is said to be *multiple closed* if $y \in S$ whenever $x \,|\, y \,|\, \mathrm{lcm}(S)$ for any $x \in S$. For example, $S = \{2, 3, 6, 10, 15, 30\}$ is multiple closed. It is obvious that if S is multiple closed, then $\max(S) = \mathrm{lcm}(S)$ and so $x \,|\, \max(S)$ for any $x \in S$, where $\max(S)$ denotes the largest element in S. We have the following natural and interesting question.

PROBLEM 1.1. Let $S = \{x_1, \dots, x_n\}$ be a multiple closed set and let f be a multiplicative function such that $f \in \mathcal{L}_S$. Does the matrix $(f(x_i, x_j))$ divide $(f[x_i, x_j])$ in the ring $M_n(\mathbb{Z})$?

In this paper, we will associate a class $\mathcal{C}_S$ of arithmetical functions with any set S of distinct positive integers (see Definition 4.1 below; note that $\mathcal{L}_S \subseteq \mathcal{C}_S$) and show that for $f \in \mathcal{C}_S$ the matrices $(f(x_i, x_j))$ and $(f[x_i, x_j])$ are integral. We find, surprisingly, that the answer to Problem 1.1 is negative. We will construct a counterexample in Section 2. However, for f completely multiplicative, the answer is affirmative (see Theorem 4.5 below).

The set $S = \{x_1, \dots, x_n\}$ is said to be a *divisor chain* if $x_i \,|\, x_j$ for all $1 \le i \le j \le n$. We will show that for any arithmetical function $f \in \mathcal{C}_S$ such that there exists an integer z_i satisfying $f(x_i) = z_i f(x_1)$ for all $2 \le i \le n$, if S is a divisor chain, then the matrix $(f(x_i, x_j))$ divides $(f[x_i, x_j])$ in $M_n(\mathbb{Z})$. As a corollary, we show that for any completely multiplicative function f

with $f \in \mathcal{C}_S$, if S is a divisor chain, then $(f(x_i, x_j))$ divides $(f[x_i, x_j])$ in $M_n(\mathbb{Z})$. But such a factorization is no longer true if f is just multiplicative.

Throughout this paper, given any set S of distinct positive integers let $m = \operatorname{lcm}(S)$. Then $m = \max(S)$ if S is multiple closed. We let $\mathbb{Z}$ and $\mathbb{Z}^+$ denote the sets of integers and of positive integers, respectively. As usual, for $x \in \mathbb{Z}^+$ and a prime p, let $v_p(x)$ denote the *p-adic valuation* of x, i.e. $v_p(x)$ is the largest integer such that $p^{v_p(x)}$ divides x.

2. A counterexample to Problem 1.1. In this section, we give an example to show that the answer to Problem 1.1 is negative. Define

$$S = \{6, 8, 12, 24\}. \tag{1}$$

Then S is clearly multiple closed. Note that it is not factor closed. For any $x \in \mathbb{Z}^+$, let $\sigma(x)$ denote the sum of the positive divisors of x. It is well known that σ is multiplicative but not completely multiplicative. The equality $(\sigma * \mu)(x) = x$ implies $\sigma \in \mathcal{L}_S$. One can easily calculate that the product $(\sigma[x_i, x_j]) \cdot (\sigma(x_i, x_j))^{-1}$ does not lie in $M_4(\mathbb{Z})$. So the 4×4 matrix $(\sigma(x_i, x_j))$ does not divide $(\sigma[x_i, x_j])$ in $M_4(\mathbb{Z})$. This answers negatively Problem 1.1.

3. Inverse of $(f(x_i, x_j))$. In 1993, Bourque and Ligh gave a formula for the inverse of the matrix $(f(x_i, x_j))$ when S is factor closed as follows.

LEMMA 3.1 ([5]). *Let f be an arithmetical function and $S = \{x_1, \dots, x_n\}$ be factor closed. If $(f * \mu)(x) \neq 0$ for all $x \in S$, then $(f(x_i, x_j))^{-1} = (a_{ij})$, where*

$$a_{ij} = \sum_{\substack{x_i \mid x_l \\ x_j \mid x_l}} \frac{\mu\big(\frac{x_l}{x_i}\big)\mu\big(\frac{x_l}{x_j}\big)}{(f * \mu)(x_l)}.$$

In what follows we calculate the inverse of the matrix $(f(x_i, x_j))$ when S is a multiple closed set. First we need the following definition.

DEFINITION 3.2 ([13]). Let $S = \{x_1, \dots, x_n\}$ be a set of n distinct positive integers. Then the *reciprocal set* of S, denoted by mS^{-1}, is defined by $mS^{-1} = \{m/x_1, \dots, m/x_n\}$.

LEMMA 3.3. *Let $S = \{x_1, \dots, x_n\}$ be a set of n distinct positive integers. Then S is multiple closed if and only if the reciprocal set mS^{-1} is factor closed.*

Proof. Assume that S is multiple closed. For any given $1 \le i \le n$, let $d \mid \frac{m}{x_i}$. One then deduces that $x_i \mid \frac{m}{d} \mid m$. Since S is multiple closed, there exists a $1 \le j \le n$ such that $m/d = x_j$. So $d = m/x_j$. That is, $d \in mS^{-1}$. Hence mS^{-1} is factor closed. The converse is proved similarly. ∎

Consequently, we can give the following structure theorem.

LEMMA 3.4. *Let $S = \{x_1, \ldots, x_n\}$ be a set of n distinct positive integers. Let f be a completely multiplicative function such that $f(m) \neq 0$. Then*

$$(f(x_i, x_j)) = \frac{1}{f(m)} \cdot \operatorname{diag}(f(x_1), \ldots, f(x_n)) \cdot \left(f\left(\frac{m}{x_i}, \frac{m}{x_j}\right)\right) \cdot \operatorname{diag}(f(x_1), \ldots, f(x_n)).$$

Proof. First we have

$$(x_i, x_j) = \frac{m}{\left[\frac{m}{x_i}, \frac{m}{x_j}\right]} = \frac{m \cdot \left(\frac{m}{x_i}, \frac{m}{x_j}\right)}{\frac{m}{x_i} \cdot \frac{m}{x_j}} = \frac{x_i x_j}{m} \cdot \left(\frac{m}{x_i}, \frac{m}{x_j}\right).$$

Since f is completely multiplicative and $f(m) \neq 0$, it follows that

$$f(x_i, x_j) = \frac{f(x_i) f(x_j)}{f(m)} \cdot f\left(\frac{m}{x_i}, \frac{m}{x_j}\right).$$

Therefore the result follows immediately. ∎

REMARK 1. Lemma 3.4 is not true if f is not completely multiplicative.

Now we can give the main result of this section, which will be needed in the next section.

THEOREM 3.5. *Let $S = \{x_1, \ldots, x_n\}$ be multiple closed and f a completely multiplicative function such that $f(m) \neq 0$ and $(f * \mu)(d) \neq 0$ for any divisor d of m. Then $(f(x_i, x_j))^{-1} = (b_{ij})$, where*

$$b_{ij} = \frac{f(m)}{f(x_i) f(x_j)} \sum_{x_l \mid (x_i, x_j)} \frac{\mu\left(\frac{x_i}{x_l}\right) \mu\left(\frac{x_j}{x_l}\right)}{(f * \mu)\left(\frac{m}{x_l}\right)}.$$

Proof. Define a set $T = \{y_1, \ldots, y_n\}$ as follows: $x_i y_i = m$ for all $1 \leq i \leq n$. Then $T = mS^{-1}$. Since S is multiple closed, T is factor closed by Lemma 3.3. On the other hand, by Lemma 3.1 we have

$$((f(y_i, y_j))^{-1})_{ij} = \sum_{\substack{y_i \mid y_l \\ y_j \mid y_l}} \frac{\mu\left(\frac{y_l}{y_i}\right) \mu\left(\frac{y_l}{y_j}\right)}{(f * \mu)(y_l)}. \tag{2}$$

Let $\Lambda = \operatorname{diag}(f(x_1), \ldots, f(x_n))$. Since f is a completely multiplicative function such that $f(m) \neq 0$ and each x_i divides m, it follows that $f(x_i) \neq 0$ for all $1 \leq i \leq n$. It then follows from Lemma 3.4 and (2) that

$$\begin{aligned}b_{ij} &= (f(m)\cdot \Lambda^{-1}\cdot (f(y_i,y_j))^{-1}\cdot \Lambda^{-1})_{ij}\\ &= \frac{f(m)}{f(x_i)f(x_j)}\cdot ((f(y_i,y_j))^{-1})_{ij}\\ &= \frac{f(m)}{f(x_i)f(x_j)}\cdot \sum_{\substack{y_i\mid y_l\\ y_j\mid y_l}} \frac{\mu\left(\frac{y_l}{y_i}\right)\mu\left(\frac{y_l}{y_j}\right)}{(f*\mu)(y_l)}\\ &= \frac{f(m)}{f(x_i)f(x_j)}\cdot \sum_{\substack{x_l\mid x_i\\ x_l\mid x_j}} \frac{\mu\left(\frac{x_i}{x_l}\right)\mu\left(\frac{x_j}{x_l}\right)}{(f*\mu)\left(\frac{m}{x_l}\right)}\end{aligned}$$

as desired. ■

4. The multiple closed case. In this section we will first associate a class $\mathcal{C}_S$ of arithmetical functions with any set S of distinct positive integers and show that for $f\in\mathcal{C}_S$ the matrices $(f(x_i,x_j))$ and $(f[x_i,x_j])$ are integral.

DEFINITION 4.1. Given any set S of distinct positive integers define the class of arithmetical functions $\mathcal{C}_S=\{f:(f*\mu)(d)\in\mathbb{Z}$ whenever $d\mid m\}$.

Clearly $\mathcal{L}_S\subset\mathcal{C}_S$. Therefore $\mathcal{C}_S$ is not empty.

LEMMA 4.2. *Let $S=\{x_1,\ldots,x_n\}$ be a set of n distinct positive integers and $f\in\mathcal{C}_S$. Then each of the following is true*:

(i) *For every divisor d of m, $f(d)$ is an integer.*

(ii) *The matrices $(f(x_i,x_j))$ and $(f[x_i,x_j])$ are integral matrices of order n.*

Proof. This lemma is a simple consequence of the Möbius inversion formula. ■

Now let $f,g\in\mathcal{C}_S$ and $d_1\mid m$. Then $f(d_1)\in\mathbb{Z}$ by Lemma 4.2(i). This implies that $((f*g)*\mu)(d)=\sum_{d_1\mid d}f(d_1)(g*\mu)(d/d_1)\in\mathbb{Z}$ whenever $d\mid m$. Therefore $f*g\in\mathcal{C}_S$ and thus the class $\mathcal{C}_S$ is closed with respect to Dirichlet convolution.

Next we prove two lemmas on completely multiplicative functions.

LEMMA 4.3. *Let b be a positive integer. If f is a completely multiplicative function, then for every $a\ge 2$ at which f does not vanish, we have*

$$g(a):=\sum_{d\mid a}\frac{\mu(d)}{f(d)f(b,a/d)}=\frac{(f*\mu)(a)}{f(a)f(a,b)}\cdot\delta_{a,b},$$

where

$$\delta_{a,b}=\begin{cases}0 & \text{if } v_p(a)\le v_p(b) \text{ for some prime } p\mid a,\\ 1 & \text{if } v_p(a)> v_p(b) \text{ for all primes } p\mid a.\end{cases}$$

Proof. Since $g(xy) = g(x)g(y)$ for all co-prime integers x, y at which f does not vanish, it suffices to establish the assertion in the case of $a = p^r$ with p prime, $r \in \mathbb{Z}^+$, $f(a) = f(p)^r \neq 0$. Then

$$g(p^r) = \frac{1}{f(b, p^r)} - \frac{1}{f(p)f(b, p^{r-1})}.$$

If $v_p(b) \geq v_p(a)$, then $p^r \mid b$, thus $f(b, p^r) = f(p)^r$ and $f(b, p^{r-1}) = f(p)^{r-1}$, implying $g(p^r) = 0$. If $v_p(b) < v_p(a)$, then $f(b, p^r) = f(b, p^{r-1})$, and since f is completely multiplicative we deduce

$$g(p^r) = \frac{1 - 1/f(p)}{f(b, p^r)} = \frac{f(p^r)(1 - 1/f(p))}{f(p^r)f(b, p^r)} = \frac{(f * \mu)(p^r)}{f(p^r)f(b, p^r)}$$

as required. ■

Lemma 4.4. *Let f be a completely multiplicative function. Let $x, y, z \in \mathbb{Z}^+$ be such that $[x, y] \mid z$. Then $f(x, y)f(z) = f(x)f(y)f(z/x, z/y)$.*

Proof. Since $x \mid z$ and $y \mid z$, we have $(x, y)z = xy(z/x, z/y)$. But f is completely multiplicative, and so the result follows immediately. ■

Since $\mathcal{L}_S \subset \mathcal{C}_S$, it follows immediately from Lemma 4.2(ii) that for any set S and any $f \in \mathcal{L}_S$, we have $(f(x_i, x_j)) \in M_n(\mathbb{Z})$ and $(f[x_i, x_j]) \in M_n(\mathbb{Z})$, so we can consider the divisibility of the two matrices in the ring $M_n(\mathbb{Z})$. Now we are in a position to give the first main result of this paper.

Theorem 4.5. *Let $S = \{x_1, \ldots, x_n\}$ be a multiple closed set. Let f be a completely multiplicative function such that $f(m) \neq 0$ and $f \in \mathcal{L}_S$. Then the matrix $(f(x_i, x_j))$ divides $(f[x_i, x_j])$ in $M_n(\mathbb{Z})$.*

Proof. Since $f(m) \neq 0$ and f is completely multiplicative, it follows that $f(d) \neq 0$ for any divisor d of m. Let $C = (f[x_i, x_j]) \cdot (f(x_i, x_j))^{-1}$. Write $C = (c_{ij})$. Clearly we need to show $c_{ij} \in \mathbb{Z}$ for all $1 \leq i, j \leq n$. By Theorem 3.5, for $1 \leq i, j \leq n$ we have

$$c_{ij} = \sum_{k=1}^{n} f[x_i, x_k] \cdot \frac{f(m)}{f(x_k)f(x_j)} \sum_{\substack{x_l \mid x_k \\ x_l \mid x_j}} \frac{\mu\left(\frac{x_k}{x_l}\right)\mu\left(\frac{x_j}{x_l}\right)}{(f * \mu)\left(\frac{m}{x_l}\right)} \tag{3}$$

$$= \frac{1}{f(x_j)} \sum_{x_l \mid x_j} \frac{\mu\left(\frac{x_j}{x_l}\right)}{(f * \mu)\left(\frac{m}{x_l}\right)} \sum_{x_l \mid x_k} \frac{f(m)}{f(x_k)} \cdot f[x_i, x_k] \cdot \mu\left(\frac{x_k}{x_l}\right)$$

$$= \frac{f(x_i)}{f(x_j)} \sum_{x_l \mid x_j} \frac{\mu\left(\frac{x_j}{x_l}\right)}{(f * \mu)\left(\frac{m}{x_l}\right)} \sum_{x_l \mid x_k} \frac{f(m)}{f(x_i, x_k)} \cdot \mu\left(\frac{x_k}{x_l}\right).$$

Fix l with $1 \leq l \leq n$ and $x_l \mid x_j$. For $x_l \mid x_k$, let $d = x_k/x_l$. Since $x_k \mid m$, we deduce $d \mid \frac{m}{x_l}$. So by Lemma 4.4 we have

$$\text{(4)} \quad \sum_{x_l \mid x_k} \frac{f(m)}{f(x_i, x_k)} \cdot \mu\left(\frac{x_k}{x_l}\right) = \sum_{d \mid \frac{m}{x_l}} \frac{f(m)}{f(x_i, dx_l)} \cdot \mu(d)$$

$$= \sum_{d \mid \frac{m}{x_l}} \frac{(f(m))^2}{f(x_i)f(x_l)} \cdot \frac{\mu(d)}{f(d)f\left(\frac{m}{x_i}, \frac{m}{dx_l}\right)} = \frac{(f(m))^2}{f(x_i)f(x_l)} \sum_{d \mid \frac{m}{x_l}} \frac{\mu(d)}{f(d)f\left(\frac{m}{x_i}, \frac{m/x_l}{d}\right)}.$$

Since f is completely multiplicative, by Lemma 4.3 applied to the last sum in (4), it follows from (3) and (4) and Lemma 4.4 that

$$c_{ij} = \frac{f(x_i)}{f(x_j)} \sum_{x_l \mid x_j} \frac{\mu\left(\frac{x_j}{x_l}\right)}{(f * \mu)\left(\frac{m}{x_l}\right)} \cdot \frac{(f(m))^2}{f(x_i)f(x_l)} \cdot \frac{f(x_l) \cdot (f * \mu)\left(\frac{m}{x_l}\right)}{f(m)f\left(\frac{m}{x_i}, \frac{m}{x_l}\right)} \cdot \delta'_{l,i}$$

$$= \frac{f(x_i)}{f(x_j)} \sum_{x_l \mid x_j} \frac{f(x_l)}{f(x_i, x_l)} \cdot \mu\left(\frac{x_j}{x_l}\right) \cdot \delta'_{l,i},$$

where

$$\delta'_{l,i} := \delta_{m/x_l, m/x_i} = \begin{cases} 0 & \text{if } v_p\left(\frac{m}{x_l}\right) \le v_p\left(\frac{m}{x_i}\right) \text{ for some prime } p \mid \frac{m}{x_l}, \\ 1 & \text{if } v_p\left(\frac{m}{x_l}\right) > v_p\left(\frac{m}{x_i}\right) \text{ for all primes } p \mid \frac{m}{x_l}. \end{cases}$$

Obviously the terms corresponding to x_l for which x_j/x_l is not square-free vanish. Define an index set I_j as follows:

$$I_j = \{l : 1 \le l \le n,\ x_l < x_j,\ x_l \mid x_j \text{ and } x_j/x_l \text{ is square-free}\}.$$

Then

$$\text{(5)} \quad c_{ij} = \frac{f(x_i)}{f(x_i, x_j)} \cdot \delta'_{j,i} + \sum_{l \in I_j} \frac{f(x_i)}{f(x_i, x_l)} \cdot \frac{f(x_l)}{f(x_j)} \cdot \mu\left(\frac{x_j}{x_l}\right) \cdot \delta'_{l,i}.$$

Assume first that $I_j = \emptyset$. Then $c_{ij} = \frac{f(x_i)}{f(x_i,x_j)} \cdot \delta'_{j,i}$. But $\frac{x_i}{(x_i,x_j)} \mid x_i \mid m$. It follows from Lemma 4.2(i) that $f(x_i)/f(x_i, x_j) = f(x_i/(x_i, x_j)) \in \mathbb{Z}$. So $c_{ij} \in \mathbb{Z}$ as desired. Now assume that $I_j \neq \emptyset$. Let

$$I'_j = \{l \in I_j : v_p(x_i) = v_p((x_i, x_l)) \text{ for some prime divisor } p \text{ of } x_j/x_l\},$$
$$I''_j = \{l \in I_j : v_p(x_i) > v_p((x_i, x_l)) \text{ for all prime divisors } p \text{ of } x_j/x_l\}.$$

Then $I'_j \cap I''_j = \emptyset$ and $I_j = I'_j \cup I''_j$. It follows from (5) that

$$\text{(6)} \quad c_{ij} = \frac{f(x_i)}{f(x_i, x_j)} \cdot \delta'_{j,i} + \sum_{l \in I'_j} \frac{f(x_i)}{f(x_i, x_l)} \cdot \frac{f(x_l)}{f(x_j)} \cdot \mu\left(\frac{x_j}{x_l}\right) \cdot \delta'_{l,i}$$
$$+ \sum_{l \in I''_j} \frac{f(x_i)}{f(x_i, x_l)} \cdot \frac{f(x_l)}{f(x_j)} \cdot \mu\left(\frac{x_j}{x_l}\right) \cdot \delta'_{l,i}.$$

We claim that $\delta'_{l,i} = 0$ for $l \in I'_j$. In fact, if $l \in I'_j$, then there exists a prime divisor p of x_j/x_l such that $v_p(x_i) = v_p((x_i, x_l))$. Hence $v_p(x_i) \le v_p(x_l)$. This implies that $v_p(m/x_i) \ge v_p(m/x_l)$. It follows that $\delta'_{l,i} = 0$, proving the claim. Then from (6) we deduce

$$c_{ij} = \frac{f(x_i)}{f(x_i, x_j)} \cdot \delta'_{j,i} + \sum_{l \in I''_j} \frac{f(x_i)}{f(x_i, x_l)} \cdot \frac{f(x_l)}{f(x_j)} \cdot \mu\left(\frac{x_j}{x_l}\right) \cdot \delta'_{l,i}. \tag{7}$$

Now let $l \in I''_j$. Let p be any prime divisor of x_j/x_l. Then $v_p(x_i) > v_p((x_i, x_l))$. Hence $v_p(x_i/(x_i, x_l)) \ge 1$. On the other hand, since x_j/x_l is square-free, $v_p(x_j/x_l) = 1$. Therefore

$$v_p\left(\frac{x_i}{(x_i, x_l)} \cdot \frac{x_l}{x_j}\right) \ge 0. \tag{8}$$

By the arbitrariness of p, (8) implies that the rational number $\frac{x_i}{(x_i,x_l)} \cdot \frac{x_l}{x_j}$ has no primes in its denominator, i.e. $\frac{x_i}{(x_i,x_l)} \cdot \frac{x_l}{x_j} \in \mathbb{Z}$. Since f is a completely multiplicative function with $f \in \mathcal{L}_S$ and $\frac{x_i}{(x_i,x_l)} \cdot \frac{x_l}{x_j}$ is a factor of m, by Lemma 4.2(i) we have $\frac{f(x_i)}{f(x_i,x_l)} \cdot \frac{f(x_l)}{f(x_j)} \in \mathbb{Z}$. It then follows from (7) that $c_{ij} \in \mathbb{Z}$. Thus $C \in M_n(\mathbb{Z})$ and this concludes the proof of Theorem 4.5. ∎

Example 4.6. To illustrate Theorem 4.5, let S be as in (1) and let λ be the Liouville function which is defined for positive integers x by $\lambda(x) = (-1)^{\alpha_1 + \ldots + \alpha_t}$ if $x = p_1^{\alpha_1} \ldots p_t^{\alpha_t}$, where $p_1, \ldots, p_t$ are distinct prime numbers and $\alpha_1, \ldots, \alpha_t \in \mathbb{Z}^+$. Then λ is a completely multiplicative function. It is easy to show that for any $x \in \mathbb{Z}^+$, $(\lambda * \mu)(x) = \lambda(x) \cdot 2^{\nu(x)}$, where $\nu(x)$ denotes the number of distinct prime factors of x. Hence $\lambda \in \mathcal{L}_S$ and $\lambda(m) \ne 0$. Let $D = ([x_i, x_j]) \cdot ((x_i, x_j))^{-1}$ and $E = (\lambda[x_i, x_j]) \cdot (\lambda(x_i, x_j))^{-1}$. We can easily check that D and E lie in $M_4(\mathbb{Z})$. Therefore $((x_i, x_j)) \mid ([x_i, x_j])$ and $(\lambda(x_i, x_j)) \mid (\lambda[x_i, x_j])$ in $M_4(\mathbb{Z})$.

Corollary 4.7. *Let $S = \{x_1, \ldots, x_n\}$ be a multiple closed set. Let f be a completely multiplicative function such that $f(m) \ne 0$ and $f \in \mathcal{L}_S$. Then the matrix $((-1)^{i+j} \cdot f(x_i, x_j))$ divides $((-1)^{i+j} \cdot f[x_i, x_j])$ in $M_n(\mathbb{Z})$.*

Proof. Let Γ be the $n \times n$ diagonal matrix with the diagonal elements $(-1)^i$, $i = 1, \ldots, n$. Let $F = \Gamma \cdot C \cdot \Gamma$, where C is as in the proof of Theorem 4.5. It follows from Theorem 4.5 that $F \in M_n(\mathbb{Z})$. We can easily check that $((-1)^{i+j} \cdot f[x_i, x_j]) = F \cdot ((-1)^{i+j} \cdot f(x_i, x_j))$. So the result follows immediately. ∎

Remark 2. Corollary 4.7 is not true if f is not completely multiplicative.

Furthermore, from Theorem 4.5, letting $f(n) = n^{\varepsilon}$ gives the following consequence.

COROLLARY 4.8. *Let $S = \{x_1, \ldots, x_n\}$ be a multiple closed set and let ε be a positive integer. Then the matrix $((x_i, x_j)^\varepsilon)$ divides $([x_i, x_j]^\varepsilon)$ in $M_n(\mathbb{Z})$.*

In particular, we have the following consequence.

COROLLARY 4.9. *Let $S = \{x_1, \ldots, x_n\}$ be a multiple closed set. Then the GCD matrix $((x_i, x_j))$ divides the LCM matrix $([x_i, x_j])$ in $M_n(\mathbb{Z})$.*

5. The divisor chain case. By Lemma 4.2(ii), for any set S and for any $f \in \mathcal{L}_S$, the matrices $(f(x_i, x_j))$ and $(f[x_i, x_j])$ are integral. In this section, we consider the divisor chain case. Now we prove the second main result of this paper.

THEOREM 5.1. *Let $S = \{x_1, \ldots, x_n\}$ be a divisor chain and $f \in \mathcal{C}_S$. If there exists an integer z_i such that $f(x_i) = z_i f(x_1)$ for all $2 \le i \le n$, then the matrix $(f(x_i, x_j))$ divides $(f[x_i, x_j])$ in $M_n(\mathbb{Z})$.*

Proof. First it follows from Lemma 4.2(ii) together with $f \in \mathcal{C}_S$ that the matrices $(f(x_i, x_j))$ and $(f[x_i, x_j])$ are integral. Since S is a divisor chain, $m = x_n$. For $1 \le i \le j \le n$, since $x_i \mid x_j$, we have $f(x_i, x_j) = f(x_i)$ and $f[x_i, x_j] = f(x_j)$. If $f(x_1) = 0$, from the assumption we then deduce $f(x_i) = 0$ for all $2 \le i \le n$. So $(f(x_i, x_j)) = (f[x_i, x_j]) = O_{n,n}$, the zero matrix of order n. Now let $f(x_1) \ne 0$. Define an $n \times n$ matrix G as follows:

$$G = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 & 1 \\ f(x_2)/f(x_1) & -1 & 0 & \ldots & 0 & 1 \\ f(x_3)/f(x_1) & 0 & -1 & \ldots & 0 & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ f(x_{n-1})/f(x_1) & 0 & 0 & \ldots & -1 & 1 \\ f(x_n)/f(x_1) & 0 & 0 & \ldots & 0 & 0 \end{pmatrix}.$$

By assumption we have $f(x_i)/f(x_1) \in \mathbb{Z}$ for $2 \le i \le n$. Thus $G \in M_n(\mathbb{Z})$. On the other hand, we can easily check that

$$G \cdot (f(x_i, x_j)) = (f[x_i, x_j]).$$

Therefore the result in this case follows immediately. ∎

COROLLARY 5.2. *Let $S = \{x_1, \ldots, x_n\}$ be a divisor chain and $f \in \mathcal{C}_S$. If there exists an integer z_i such that $f(x_i) = z_i f(x_1)$ for all $2 \le i \le n$, then the matrix $((-1)^{i+j} \cdot f(x_i, x_j))$ divides $((-1)^{i+j} \cdot f[x_i, x_j])$ in $M_n(\mathbb{Z})$.*

COROLLARY 5.3. *Let $S = \{x_1, \ldots, x_n\}$ be a divisor chain and f a completely multiplicative function such that $f \in \mathcal{C}_S$. Then the matrix $(f(x_i, x_j))$ divides $(f[x_i, x_j])$ in $M_n(\mathbb{Z})$.*

Proof. Since f is completely multiplicative, we have $f(x_i) = f(x_1) f(x_i/x_1)$ for $2 \le i \le n$. Since $f \in \mathcal{C}_S$, Lemma 4.2(i) together with the fact $\frac{x_i}{x_1} \mid m$

implies $f(x_i/x_1) \in \mathbb{Z}$. So Corollary 5.3 follows immediately from Theorem 5.1. ∎

REMARK 3. Corollary 5.3 is no longer true if f is just multiplicative. For instance, let $S = \{3, 9\}$. Then S is clearly a divisor chain. We calculate

$$(\sigma[x_i, x_j]) \cdot (\sigma(x_i, x_j))^{-1} = \begin{pmatrix} 4 & 13 \\ 13 & 13 \end{pmatrix} \cdot \begin{pmatrix} \frac{13}{36} & -\frac{1}{9} \\ -\frac{1}{9} & \frac{1}{9} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{13}{4} & 0 \end{pmatrix} \notin M_2(\mathbb{Z}).$$

So $(\sigma(x_i, x_j)) \nmid (\sigma[x_i, x_j])$ in $M_2(\mathbb{Z})$.

COROLLARY 5.4. *Let $S = \{x_1, \ldots, x_n\}$ be a divisor chain and f a completely multiplicative function such that $f \in \mathcal{C}_S$. Then the matrix $((-1)^{i+j} \cdot f(x_i, x_j))$ divides $((-1)^{i+j} \cdot f[x_i, x_j])$ in the ring $M_n(\mathbb{Z})$.*

REMARK 4. Corollary 5.4 is not true if f is not completely multiplicative.

Picking $f(n) = n^{\varepsilon}$, we can immediately deduce from Corollary 5.3 that the following result is true.

COROLLARY 5.5. *Let ε be a positive integer and let $S = \{x_1, \ldots, x_n\}$ be a divisor chain. Then the matrix $((x_i, x_j)^{\varepsilon})$ divides $([x_i, x_j]^{\varepsilon})$ in the ring $M_n(\mathbb{Z})$.*

REMARK 5. If we take $\varepsilon = 1$, then Corollary 5.5 becomes the result mentioned in [12] without proof. Note that by using and developing the method of [10], we proved [12] that there is a *gcd-closed set* $S = \{x_1, \ldots, x_n\}$ (i.e. $(x_i, x_j) \in S$ for all $1 \le i, j \le n$) such that the GCD matrix $((x_i, x_j))$ does not divide the LCM matrix $([x_i, x_j])$ in $M_n(\mathbb{Z})$.

Acknowledgments. The author would like to thank Professor Władysław Narkiewicz and the anonymous referee for their careful reading of the manuscript and very helpful comments which improved greatly the presentation.

REFERENCES

[1] T. M. Apostol, *Introduction to Analytic Number Theory*, Springer, New York, 1976.
[2] —, *Arithmetical properties of generalized Ramanujan sums*, Pacific J. Math. 41 (1972), 281–293.
[3] S. Beslin, *Reciprocal GCD matrices and LCM matrices*, Fibonacci Quart. 29 (1991), 271–274.
[4] S. Beslin and S. Ligh, *Another generalization of Smith's determinant*, Bull. Austral. Math. Soc. 40 (1989), 413–415.
[5] K. Bourque and S. Ligh, *Matrices associated with arithmetical functions*, Linear and Multilinear Algebra 34 (1993), 261–267.
[6] —, —, *Matrices associated with classes of arithmetical functions*, J. Number Theory 45 (1993), 367–376.

[7] K. Bourque and S. Ligh, *Matrices associated with classes of multiplicative functions*, Linear Algebra Appl. 216 (1995), 267–275.
[8] P. Haukkanen and J. Sillanpaa, *Some analogues of Smith's determinant*, Linear and Multilinear Algebra 41 (1996), 233–244.
[9] S. F. Hong, *Lower bounds for determinants of matrices associated with classes of arithmetical functions*, ibid. 45 (1999), 349–358.
[10] —, *On the Bourque–Ligh conjecture of least common multiple matrices*, J. Algebra 218 (1999), 216–228.
[11] —, *Gcd-closed sets and determinants of matrices associated with arithmetical functions*, Acta Arith. 101 (2002), 321–332.
[12] —, *On the factorization of LCM matrices on gcd-closed sets*, Linear Algebra Appl. 345 (2002), 225–233.
[13] —, *Notes on power LCM matrices*, to appear in Acta Arith.
[14] S. Ligh, *Generalized Smith's determinant*, Linear and Multilinear Algebra 22 (1988), 305–306.
[15] P. J. McCarthy, *Introduction to Arithmetical Functions*, Springer, New York, 1986.
[16] —, *A generalization of Smith's determinant*, Canad. Math. Bull. 29 (1988), 109–113.
[17] H. J. S. Smith, *On the value of a certain arithmetical determinant*, Proc. London Math. Soc. 7 (1875–1876), 208–212.

Mathematical College
Sichuan University
Chengdu 610064
P.R. China
E-mail: s-f.hong@163.net
hongsf02@yahoo.com

Department of Mathematics
Technion-Israel Institute of Technology
Haifa 32000, Israel

Received 13 June 2003;
revised 30 September 2003 (4354)