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POINTWISE LIMITS FOR SEQUENCES OF ORBITAL INTEGRALS

BY

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Dedicated to the memory of Professor Andrzej Hulanicki

Abstract. In 1967, Ross and Stromberg published a theorem about pointwise limits of orbital integrals for the left action of a locally compact group G on (G, ρ), where ρ is the right Haar measure. We study the same kind of problem, but more generally for left actions of G on any measure space (X, μ), which leave the σ-finite measure μ relatively invariant, in the sense that $s\mu = \Delta(s)\mu$ for every $s \in G$, where Δ is the modular function of G. As a consequence, we also obtain a generalization of a theorem of Civin on one-parameter groups of measure preserving transformations.

The original motivation for the circle of questions treated here dates back to classical problems concerning pointwise convergence of Riemann sums of Lebesgue integrable functions.

1. Introduction. The study of almost everywhere convergence of Riemann sums is an old problem, with many ramifications (see [RW06] for a recent survey). Let us consider the interval $[0,1[$, identified with the torus $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$, and let ρ be the normalized Lebesgue measure on $\mathbb{T}$. Let $f : [0,1[\to \mathbb{C}$ be a measurable function. For $n \in \mathbb{N}^*$ and $x \in \mathbb{T}$, the corresponding Riemann sum of f is defined by

$$R_n f(x) = \frac{1}{n}\sum_{j=0}^{n-1} f(x + j/n).$$

When f is Riemann integrable, for any $x \in \mathbb{T}$ we have

$$\lim_n R_n(f)(x) = \int_0^1 f(t)\, d\rho(t). \tag{1.1}$$

When f is Lebesgue integrable, it is easily seen that $(R_n(f))$ converges in mean to $\int_0^1 f\, d\rho$ (see for instance [RW06, §2]). On the other hand, the study of almost everywhere convergence is much more subtle. The first result on

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this subject seems to date back to the paper [Jes34] of Jessen. Jessen's theorem states that if (n_k) is a sequence of positive integers such that n_k divides n_{k+1} for every k then, for any $f \in L^1(\mathbb{T})$, we have, for almost every $x \in \mathbb{T}$,

$$\lim_k R_{n_k} f(x) = \int_0^1 f(t)\, d\rho(t). \tag{1.2}$$

Soon after, Marcinkiewicz and Zygmund [MZ37] on one hand, and Ursell [Urs37] independently, gave examples of functions $f \in L^1(\mathbb{T})$ for which (1.1) fails to hold almost everywhere. For instance, given $1/2 < \delta < 1$,

$$f : x \in]0,1] \mapsto |x|^{-\delta}$$

is such an example (see [Urs37, Rud64]). Rudin [Rud64] was even able to provide many examples of bounded measurable functions f (characteristic functions indeed) such that, for almost every $x \in \mathbb{T}$, the sequence $(R_n f(x))$ diverges. Moreover, Rudin's paper highlighted deep connections between pointwise convergence of Riemann sums along a given subsequence (n_k) of integers and arithmetical properties of the subsequence, a question now widely developed. The following different important question has also been considered by many authors: under which kind of regularity conditions on f does the associated sequence $(R_n(f))$ of Riemann sums converge a.e.? (see [RW06] for these questions and many related ones).

In this paper, we deal with another sort of problem, namely we study possible extensions of Jessen's result to general locally compact groups and dynamical systems.

Let us first return to Jessen's theorem and give another formulation of it. Denote by G the group $\mathbb{T}$ and set $G_k = \mathbb{Z}/(n_k\mathbb{Z})$. Then (G_k) is an increasing sequence of closed subgroups of G, whose union is dense in G. If ρ_k is the Haar probability measure on G_k, Jessen's result reads as follows: for every $f \in L^1(\mathbb{T})$,

$$\lim_{k\to\infty} \int_{G_k} f(t+x)\, d\rho_k(t) = \int_G f\, d\rho \qquad \text{a.e.}$$

Under this form, this theorem has been extended by Ross and Stromberg to locally compact groups. An assumption about the behaviour of the modular functions of the subgroups is needed (see [RS67] and Corollary 3.12 below). It is automatically satisfied in the abelian case.

More generally, we are interested in the following questions. *Let $G \curvearrowright (X,\mu)$ be an action of a locally compact group G on a measure space (X,μ), where μ is σ-finite, and let $(G_n)_{n\in\mathbb{N}}$ be an increasing sequence of closed subgroups of G with dense union.*

(i) *Find conditions on the action and on the (right) Haar measures ρ_n of G_n and ρ of G so that for every $f \in L^1(X,\mu)$ and every n, $t \in G_n \mapsto f(tx)$ is ρ_n-integrable for almost every $x \in X$.*

(ii) *If* (i) *holds, study the pointwise convergence of the sequence of orbital integrals*

$$\int_{G_n} f(tx)\, d\rho_n(t).$$

(iii) *Identify the pointwise limit, in case it exists.*

A necessary condition for (i) to be satisfied is that, for every Borel subset B of X with $\mu(B) < +\infty$, every $n \in \mathbb{N}$, and almost every $x \in X$,

$$\rho_n(\{t \in G_n\,;\, x \in t^{-1}B\}) = \int_{G_n} \mathbf{1}_B(tx)\, d\rho_n(t) < +\infty.$$

When μ is finite, say a probability measure, condition (i) implies that the groups G_n are compact. If we normalize their Haar measures by $\rho_n(G_n) = 1$, the reversed martingale theorem implies that, for $f \in L^1(X,\mu)$, the sequence of orbital integrals converges pointwise and in mean (see [Tem92, Cor. 3.5, p. 219] and Proposition 3.1 below). This fact is well-known. Moreover, the limit is the conditional expectation of f with respect to the σ-field of G-invariant Borel subsets of X. Of course, Jessen's theorem is a particular case.

When μ is only σ-finite, we cannot use the reversed martingale theorem any longer and we shall need other arguments. As already said, Ross and Stromberg studied the case of the left action $G \curvearrowright (G,\rho)$. They normalized the right Haar measures ρ_n and ρ in such a way that for every $f \in \mathcal{C}_c(G)$ (the space of continuous functions with compact support on G), we have $\lim_n \rho_n(f) = \rho(f)$. This is always possible, due to a result of Fell (see [Bou63, Chap. VIII, §5], and [RS67] for more references). Morever, Ross and Stromberg assumed that for every n, the modular function of G_n is the restriction of the modular function of G. We shall name this property *the modular condition* (MC). Under these assumptions, Ross and Stromberg proved that for every $f \in L^1(G,\rho)$, one has, for almost every $x \in G$,

$$\lim_n \int_{G_n} f(tx)\, d\rho_n(t) = \int_G f(tx)\, d\rho(t) = \int_G f(t)\, d\rho(t)\ (^1).$$

Later, Ross and Willis [RW97] provided an example showing that the modular condition does not always hold. They also proved that the Ross–Stromberg theorem always fails when the modular condition is not satisfied.

(1) We adopt the following convention: each time we write an integral $\int f$, either f is non-negative, or it is implicitly assumed that f is integrable.

Therefore, in our paper *we shall always assume that the increasing sequence* (G_n) *of closed subgroups of* G *has a dense union and satisfies the modular condition* (MC). We shall denote by Δ the modular function of G, so that $\lambda = \Delta\rho$, where λ is the left Haar measure of G. *We also assume that* G *acts on* (X,μ) *in such a way that* μ *is* Δ*-relatively invariant under the action*, in the sense that $s\mu = \Delta(s)\mu$ for $s \in G$. An important example is the left action $G \curvearrowright (G,\rho)$.

Under these assumptions, we give a necessary and sufficient condition for the pointwise limit theorem to be satisfied.

Theorem (3.13). *The following two properties are equivalent:*

(a) X *is a countable union of Borel subsets* B_k *of finite measure such that for every* k *and almost every* $x \in X$, *we have*

$$\rho(\{t \in G\,;\, x \in t^{-1}B_k\}) = \int_G \mathbf{1}_{B_k}(tx)\,d\rho(t) < +\infty.$$

(b) *For every* $f \in L^1(X,\mu)$ *and for almost every* $x \in X$,

$$\lim_n \int_{G_n} f(tx)\,d\rho_n(t) = \int_G f(tx)\,d\rho(t).$$

A crucial intermediate step is the above mentioned Ross–Stromberg theorem. For completeness, we provide a proof of this result, partly based on one of the ideas contained in [RS67].

We give examples where our Theorem 3.13 applies. In particular, as an easy consequence we get our second main result:

Theorem (3.18). *Consider* $G \curvearrowright (X,\mu)$, *where now the* σ*-finite measure* μ *is invariant. Let* $(G_n)_{n\in\mathbb{N}}$ *be an increasing sequence of lattices in* G *whose union is dense. Fix a Borel fundamental domain* D *for* G_0 *and normalize* ρ *by* $\rho(D) = 1$ (2). *Then, for every* G_0*-invariant function* $f \in L^1(X,\mu)$ *and for almost every* $x \in X$, *we have*

$$\lim_n \frac{1}{|G_n \cap D|} \sum_{t \in G_n \cap D} f(tx) = \int_D f(tx)\,d\rho(t),$$

where $|G_n \cap D|$ *is the cardinality of* $G_n \cap D$.

This gives a simple way to extend a result of Civin [Civ55], who treated the case $G = \mathbb{R}$ by a different method, apparently not directly adaptable to more general locally compact groups G.

2. Notation and conventions. In this paper, locally compact spaces are implicitly assumed to be Hausdorff and σ-compact. A *measure space*

(2) Note that our assumptions imply the unimodularity of G.

(X, μ) is a Borel standard space equipped with a (non-negative) σ-finite measure μ.

Let G be a locally compact group. We denote by Δ its modular function, and by λ and ρ respectively its left and right Haar measures, so that $\lambda = \Delta\rho$. By an action of G on a measure space (X, μ) we mean a Borel map $G \times X \to X$, $(t, x) \mapsto tx$, which is a left action and leaves μ quasi-invariant. In fact, we shall need the following stronger property:

Definition 2.1. Given an action $G \curvearrowright (X, \mu)$, we say that μ is Δ-*relatively invariant* if $s\mu = \Delta(s)\mu$ for all $s \in G$.

Note that this property is satisfied for the left action $G \curvearrowright (G, \rho)$.

In all our statements, we shall consider an increasing sequence (G_n) of closed subgroups of G with dense union. Then Δ_n, ρ_n, $\lambda_n = \Delta_n\rho_n$ will be the modular function and Haar measures of G_n, respectively. We shall have to choose appropriate normalizations of ρ and ρ_n, $n \in \mathbb{N}$. For a compact group, we usually choose its Haar measure to have total mass one (but see Remark 2.3 below).

As already mentioned in the introduction, there is also a natural normalization of the Haar measures as follows (see [Bou63, Chap. VIII, §5] and [RS67]).

Definition 2.2. Let G be a locally compact group and ρ a right Haar measure on G. Let (G_n) be an increasing sequence of closed subgroups whose union is dense in G. There is an essentially unique normalization of the right Haar measures ρ_n of the G_n such that, for every continuous function f on G with compact support, we have $\lim_n \rho_n(f) = \rho(f)$. In this case, we shall say that the sequence (ρ_n) of right Haar measures is *normalized with respect to* ρ. We shall also say that it is a *Fell normalization*.

Remark 2.3. When G is compact, the Fell normalization is the classical normalization, where the Haar measures are probability measures. On the other hand, when the G_n are compact whereas G is not, it is easily seen that the Fell normalization implies that $\lim_n \rho_n(G_n) = +\infty$. For instance, let G be a countable discrete group which is the union of an increasing sequence (G_n) of finite subgroups (e.g. the group S_∞ of finite permutations of the integers). Then, if ρ is the counting measure on G, the normalization of the sequence ρ_n with respect to ρ is the sequence of counting measures, for which we have $\rho_n(G_n) = |G_n|$, the cardinality of G_n.

Finally, another property of the sequence (G_n) will be fundamental in this paper. It was already present in the work of Ross and Stromberg [RS67] and later shown to be crucial (see [RW97]).

Definition 2.4. Let G be a locally compact group and (G_n) an increasing sequence of closed subgroups of G. We say that (G_n) satisfies the *modular condition* (MC) if, for every n, the modular function Δ_n of G_n is the restriction to G_n of the modular function Δ of G.

Let us explain the interest of this condition. An action $G \curvearrowright (X,\mu)$ leaving the measure μ Δ-relatively invariant has the following property: for every Borel function $f : X \times G \to \mathbb{R}_+^*$ (or any $\mu \otimes \lambda$-integrable function $f : X \times G \to \mathbb{C}$), we have

$$\int_{X\times G} f(tx,t)\,d\mu(x)\,d\rho(t) = \int_{X\times G} f(x,t)\,d\mu(x)d\lambda(t) = \int_{X\times G} f(x,t^{-1})\,d\mu(x)\,d\rho(t). \tag{2.1}$$

This property will be essential throughout this paper, and we shall need it to remain satisfied for all the restricted actions $G_n \curvearrowright (X,\mu)$. This requires that the restriction of Δ to G_n is the modular function of G_n.

3. Limit theorems

3.1. Compactness assumptions. In this section, the Haar measure of every compact group will have total mass one.

PROPOSITION 3.1. *Let G be a locally compact group, acting in a measure preserving way on a probability space (X,μ). Let (G_n) be an increasing sequence of compact subgroups of G whose union is dense in G. Let $f \in L^1(X,\mu)$.*

(a) *For a.e. $x \in X$, we have $\lim_{n\to\infty} \int_{G_n} f(tx)\,d\rho_n(t) = \mathbb{E}(f\,|\,\mathcal{I})(x)$, where $\mathbb{E}(f\,|\,\mathcal{I})$ is the conditional expectation of f with respect to the σ-field $\mathcal{I}$ of G-invariant Borel subsets of X.*

(b) *If moreover G is compact, then we have $\lim_{n\to\infty} \int_{G_n} f(tx)\,d\rho_n(t) = \int_G f(tx)\,d\rho(t)$ for almost every $x \in X$.*

Proof. (a) Observe first that $t \in G_n \mapsto f(tx)$ is ρ_n-integrable for almost every x, since

$$\int_X \Big(\int_{G_n} |f(tx)|\,d\rho_n(t)\Big)\,d\mu(x) = \int_X |f(x)|\,d\mu(x) < +\infty.$$

We set $R_n(f)(x) = \int_{G_n} f(tx)\,d\rho_n(t)$. Obviously, this function is G_n-invariant and μ-integrable. Moreover, let A be a G_n-invariant Borel subset of X. Then

$$\begin{aligned}\int_A R_n(f)(x)\,d\mu(x) &= \int_{X\times G_n} \mathbf{1}_A(x)f(tx)\,d\mu(x)\,d\rho_n(t)\\ &= \int_{X\times G_n} \mathbf{1}_A(t^{-1}x)f(x)\,d\mu(x)\,d\rho_n(t) = \int_A f(x)\,d\mu(x),\end{aligned}$$

since A is G_n-invariant. Therefore $R_n(f)$ is the conditional expectation of f with respect to the σ-field $\mathcal{B}_n$ of Borel G_n-invariant subsets of X. The sequence $(\mathcal{B}_n)$ of σ-fields is decreasing. The reversed martingale theorem [Nev72, p. 119] states that the sequence $(R_n(f))$ converges μ-a.e. and in mean to the conditional expectation of f with respect to the σ-field $\mathcal{B}_\infty = \bigcap_n \mathcal{B}_n$ of $(\bigcup_n G_n)$-invariant Borel subsets of X. Since $\bigcup_n G_n$ is dense in G and μ is finite, $\mathcal{B}_\infty$ is also the σ-field of G-invariant Borel subsets of X.

(b) is obvious since $\mathbb{E}(f \mid \mathcal{I})(x) = \int_G f(tx)\, d\rho(t)$ for almost every $x \in X$ when G is compact. ∎

REMARK 3.2. Let μ be a σ-finite measure on a Borel space $(X, \mathcal{B})$, and let $(\mathcal{B}_n)$ be a decreasing sequence of σ-fields. Assume that $(X, \mathcal{B}_n, \mu)$ is σ-finite for every n. For $f \in L^1(X, \mathcal{B}, \mu)$, the conditional expectation $\mathbb{E}(f \mid \mathcal{B}_n)$ is well defined, as a Radon–Nikodým derivative. In [Jer59], Jerison has proved that $\lim_{n\to\infty} \mathbb{E}(f \mid \mathcal{B}_n)(x)$ exists almost everywhere. Moreover, if $(X, \mathcal{B}_\infty, \mu)$ is σ-finite, the limit is $\mathbb{E}(f \mid \mathcal{B}_\infty)$. Otherwise, one may write X as the disjoint union of two elements V, W of $\mathcal{B}_\infty$, where V is a countable union of elements of $\mathcal{B}_\infty$ of finite measure while any subset of W that belongs to $\mathcal{B}_\infty$ has measure 0 or ∞. By [Jer59, §2.6], $\lim_{n\to\infty} \mathbb{E}(f \mid \mathcal{B}_n)(x) = 0$ a.e. on W.

This observation can be used to prove that Proposition 3.1 still holds under the weaker assumption that μ is σ-finite. Indeed, it is enough to prove that when H is any compact group acting in measure preserving way on a σ-finite measure space $(X, \mathcal{B}, \mu)$, then $(X, \mathcal{B}_H, \mu)$ is still σ-finite, where $\mathcal{B}_H$ is the σ-field of Borel H-invariant subsets. To show this, consider a strictly positive function $f \in L^1(X, \mathcal{B}, \mu)$. As seen in the proof of Proposition 3.1, the function $x \mapsto R(f)(x) = \int_H f(tx)\, d\rho(t)$ is H-invariant and μ-integrable, and we deduce that $(X, \mathcal{B}_H, \mu)$ is σ-finite from the fact that $R(f)$ is strictly positive everywhere. Finally, for every $f \in L^1(X, \mathcal{B}, \mu)$, the conditional expectation $\mathbb{E}(f \mid \mathcal{B}_H)$ may be defined and we obviously have $R(f)(x) = \mathbb{E}(f \mid \mathcal{B}_H)$.

We shall give another proof of Proposition 3.1(b), for a compact group G and a σ-finite measure μ, in Corollary 3.14.

In the rest of the paper we are interested in the more general situation where μ is a σ-finite measure on X and the subgroups G_n are not assumed to be compact. In particular, G is not always unimodular.

3.2. General case: local results. Let $G \curvearrowright (X, \mu)$ be a measure G-space. When the measure μ is not finite, it may be useful to study the restriction of the action to every Borel subset B of X such that $\mu(B) < +\infty$, even if B is not G-invariant ([3]). We extend to X every function defined on B, by giving it the value 0 on $X \setminus B$. In particular, we have $L^1(B, \mu) \subset L^1(X, \mu)$.

([3]) The restriction of μ to the Borel subspace B will be denoted by the same letter.

Note that Borel functions of the form

$$x \mapsto \rho(\{t \in G\,;\, tx \in B\}) = \int_G \mathbf{1}_B(tx)\, d\rho(t)$$

are G-invariant. This crucial fact will be used repeatedly and without mention.

THEOREM 3.3. *Let (G_n) be an increasing sequence of closed subgroups of a locally compact group G with dense union and satisfying condition* (MC). *Let $G \curvearrowright (X,\mu)$ be an action leaving the measure Δ-relatively invariant. Let B be a Borel subset of X with $\mu(B) < +\infty$. For a.e. $x \in B$ and every $n \in \mathbb{N}$, assume that $0 < \rho_n(\{t \in G_n\,;\, tx \in B\}) < +\infty$, where ρ_n is a right Haar measure on G_n. Then, for every $f \in L^1(B,\mu)$, the averaging sequence of functions*

$$x \in B \mapsto \frac{1}{\rho_n(\{t \in G_n\,;\, tx \in B\})} \int_{G_n} f(tx)\, d\rho_n(t)$$

converges, almost everywhere on B and in L^1 norm, to an element of $L^1(B,\mu)$.

Proof. We first check that for every n and almost every $x \in B$, the function $t \in G_n \mapsto f(tx)$ is ρ_n-integrable. This is a consequence of the following computation (where we use equality (2.1) as well as the right-invariance of ρ_n):

$$\begin{aligned}
\int_B \frac{1}{\rho_n(\{s \in G_n\,;\, sx \in B\})}&\Big(\int_{G_n} |f(tx)|\, d\rho_n(t)\Big)\, d\mu(x)\\
&= \int_{X\times G_n} \mathbf{1}_B(x)\, \frac{|f(tx)|}{\rho_n(\{s \in G_n\,;\, sx \in B\})}\, d\mu(x)\, d\rho_n(t)\\
&= \int_{X\times G_n} \mathbf{1}_B(tx)\, \frac{|f(x)|}{\rho_n(\{s \in G_n\,;\, sx \in B\})}\, d\mu(x)\, d\rho_n(t)\\
&= \int_B |f(x)|\, \frac{1}{\rho_n(\{s \in G_n\,;\, sx \in B\})}\Big(\int_{G_n} \mathbf{1}_B(tx)\, d\rho_n(t)\Big)\, d\mu(x)\\
&= \int_B |f(x)|\, d\mu(x) < +\infty.
\end{aligned}$$

In addition, we see that $x \in B \mapsto \int_{G_n} f(tx)\, d\rho_n(t)/\rho_n(\{s \in G_n\,;\, sx \in B\})$ is μ-integrable on B. Denote by $R_n(f)$ this function defined on B.

Let $\mathcal{O}_n(B)$ be the equivalence relation on B induced by the G_n-action: for $x, y \in B$, $x \sim_{\mathcal{O}_n(B)} y$ if there exists $t \in G_n$ with $x = ty$. We denote by $\mathcal{B}_n(B)$ the σ-field of Borel subsets of B invariant under this equivalence relation. Observe that $R_n(f)$ is invariant under $\mathcal{O}_n(B)$. It is also straightforward to check that $R_n(f)$ is the conditional expectation of f with respect to $\mathcal{B}_n(B)$. Then again the conclusion follows from the reversed martingale theorem. ∎

Concerning the pointwise convergence of sequences of orbital integrals, we immediately get:

COROLLARY 3.4. *Under the assumptions of the previous theorem, the following conditions are equivalent:*

(a) $\lim_{n\to\infty} \rho_n(\{t \in G_n\,;\, tx \in B\})$ *exists a.e. on* B;
(b) *for every* $f \in L^1(B,\mu)$, $\lim_{n\to\infty} \int_{G_n} f(tx)\,d\rho_n(t)$ *exists a.e. on* B.

3.3. General case: global results. In order to study the problem globally, we shall need the following lemma, ensuring the integrability of orbital functions.

LEMMA 3.5. *Let* $G \curvearrowright (X,\mu)$ *be an action leaving the measure* Δ*-relatively invariant. The following two conditions are equivalent:*

(i) $X = \bigcup B_k$, *where every* B_k *is a Borel subset of* X, *with* $\mu(B_k) < +\infty$, *such that for almost every* $x \in X$,
$$\rho(\{t \in G\,;\, tx \in B_k\}) < +\infty.$$
(ii) *For every* $f \in L^1(X,\mu)$, $f_x : t \mapsto f(tx)$ *belongs to* $L^1(G,\rho)$ *for a.e.* $x \in X$.

Proof. (ii)⇒(i) is obvious: write X as a countable union of Borel subsets B_k of finite measure and take $f = \mathbf{1}_{B_k}$.

(i)⇒(ii). Let $f \in L^1(X,\mu)_+$. It suffices to show that for every k and a.e. $x \in B_k$, the function $t \mapsto f(tx)$ is ρ-integrable. We set
$$A = \Big\{x \in X\,;\, \int_G \mathbf{1}_{B_k}(tx)\,d\rho(t) = 0\Big\}.$$
Note that A is G-invariant.

We first check that for a.e. $x \in B_k \cap A$, we have $\int_G f(tx)\,d\rho(t) = 0$. Indeed,
$$\int_{B_k\cap A}\int_G f(tx)\,d\rho(t)\,d\mu(x) = \int_{X\times G} \mathbf{1}_{B_k\cap A}(tx) f(x)\,d\rho(t)\,d\mu(x)$$
$$= \int_X f(x)\Big(\int_G \mathbf{1}_{B_k\cap A}(tx)\,d\rho(t)\Big)\,d\mu(x) = 0.$$
The first equality uses relation (2.1) and the last one follows from the observation that $\mathbf{1}_{B_k\cap A}(tx) \neq 0$ implies $x \in A$ since A is G-invariant. Hence, $\int_G f(tx)\,d\rho(t) = 0$ a.e. on $B_k \cap A$.

Now let us consider the integral
$$\int_{B_k\setminus A} \frac{1}{\rho(\{s \in G\,;\, sx \in B_k\})}\Big(\int_G f(tx)\,d\rho(t)\Big)\,d\mu(x).$$

It is equal to

$$\begin{aligned}\int_{X\times G} \mathbf{1}_{B_k\setminus A}(x)&\frac{1}{\rho(\{s\in G\,;\,sx\in B_k\})}\,f(tx)\,d\mu(x)\,d\rho(t)\\ &= \int_{X\times G} \mathbf{1}_{B_k\setminus A}(tx)\,\frac{1}{\rho(\{s\in G\,;\,sx\in B_k\})}\,f(x)\,d\mu(x)\,d\rho(t)\\ &= \int_{X\setminus A} f(x)\Big(\int_G \frac{\mathbf{1}_{B_k\setminus A}(tx)}{\rho(\{s\in G\,;\,sx\in B_k\})}\,d\rho(t)\Big)\,d\mu(x)\\ &\le \int_{X\setminus A} f(x)\,d\mu(x) < +\infty,\end{aligned}$$

since for every x such that $tx\in B_k\setminus A$ we have $x\in G(B_k\setminus A) = GB_k\setminus A$, and

$$\int_G \mathbf{1}_{B_k\setminus A}(tx)\,d\rho(t) \le \int_G \mathbf{1}_{B_k}(tx)\,d\rho(t) = \rho(\{s\in G\,;\,sx\in B_k\})$$

with

$$0<\rho(\{s\in G\,;\,sx\in B_k\})<+\infty \quad \text{a.e. on } X\setminus A.$$

It follows that $\int_G f(tx)\,d\rho(t)<+\infty$ for almost every $x\in B_k\setminus A$. ■

Remark 3.6. The assumption of this lemma holds for instance when G acts on (X,μ), where μ is a Δ-relatively invariant Radon measure on a locally compact space X, the action being continuous with closed orbits and compact stabilizers. Indeed, in this situation, for $x\in X$, the natural map $G/G_x\to Gx$, where G_x is the stabilizer of x, is a homeomorphism. Then if B is an open relatively compact subset of X, the set $\{t\in G\,;\,tx\in B\}$ is open and relatively compact in G and the conclusion follows. Particular cases are proper actions, and more generally integrable actions [Rie04].

We shall now give a condition sufficient to guarantee the pointwise convergence of sequences of orbital integrals for every $f\in L^1(X,\mu)$. Note that the integrability of the functions appearing in the statement below follows from Lemma 3.5.

Theorem 3.7. *Let (G_n) be an increasing sequence of closed subgroups in G with dense union and satisfying condition (MC). Let $G\curvearrowright(X,\mu)$ be an action leaving the measure Δ-relatively invariant. Assume that $X=\bigcup X_k$ where (X_k) is an increasing sequence of Borel subspaces such that for all k:*

(i) $\mu(X_k)<+\infty$;
(ii) $\rho_n(\{t\in G_n\,;\,tx\in X_k\})>0$ *for almost every $x\in X_k$ and every n;*
(iii) *there exists $c_k>0$ such that for almost every $x\in X$,*

$$\sup_n \rho_n(\{t\in G_n\,;\,tx\in X_k\})\le c_k.$$

The following conditions are equivalent:

(a) *for every k, $\lim_n \rho_n(\{t \in G_n\,;\, tx \in X_k\})$ exists for a.e. $x \in X_k$;*
(b) *the pointwise limit $\lim_n \int_{G_n} f(tx)\,d\rho_n(t)$ exists a.e. on X, for every $f \in L^1(X,\mu)$;*
(c) *there exists a dense subset $\mathcal{D}$ of $L^1(X,\mu)$ such that the pointwise limit $\lim_n \int_{G_n} f(tx)\,d\rho_n(t)$ exists a.e. on X, for every f in $\mathcal{D}$.*

Remark 3.8. Assume that for every $x \in X_k$ and every n,

$$\rho_n(\{t \in G_n\,;\, tx \in X_k\}) \le c_k. \tag{3.1}$$

Then assumption (iii) of the above theorem is fulfilled. Indeed, by invariance, if (3.1) holds for $x \in X_k$, it also holds for $x \in G_n X_k$. On the other hand, if $x \notin G_n X_k$ then

$$\{t \in G_n\,;\, tx \in X_k\} = \emptyset,$$

and therefore $\rho_n(\{t \in G_n\,;\, tx \in X_k\}) = 0 \le c_k$.

For the proof of Theorem 3.7, we need the following lemma which repeats arguments from [RS67, Lemma 3].

Lemma 3.9. *Let (G_n) be an increasing sequence of closed subgroups in G, with dense union and satisfying condition* (MC). *Let $G \curvearrowright (X,\mu)$ be an action leaving the measure Δ-relatively invariant. Let $X_k \subset X$ satisfy conditions* (i) *and* (iii) *of the previous theorem and let $f : X \to \mathbb{R}_+$ be a Borel function. For $x \in X$, set*

$$f^\star(x) = \sup_n \int_{G_n} f(tx)\,d\rho_n(t),$$

and for $\alpha > 0$, set $Q_\alpha = \{x \in X\,;\, f^\star(x) > \alpha\}$. Then

$$\alpha\mu(Q_\alpha \cap X_k) \le c_k \int_{Q_\alpha} f\,d\mu. \tag{3.2}$$

Proof. For $n \in \mathbb{N}$, we set $\phi_n(x) = \int_{G_n} f(tx)\,d\rho_n(t)$ and we introduce the subsets

$$E_n = \{x \in X\,;\, \phi_n(x) > \alpha\}, \qquad D_n = \{x \in X\,;\, \sup_{1\le l\le n} \phi_l(x) > \alpha\}.$$

We fix an integer N. It is enough to show that

$$\alpha\mu(D_N \cap X_k) \le c_k \int_{D_N} f\,d\mu,$$

since Q_α is the increasing union of the sets D_N, $N \ge 1$.

We decompose D_N into the disjoint union $\bigcup_{n=1}^N F_n$, where

$$F_n = E_n \cap \bigcup_{l=n+1}^{N} E_l^c.$$

Observe that $G_n E_n = E_n$, and therefore $G_n F_n = F_n$ for every n. We have

$$\begin{aligned}\alpha\mu(F_n \cap X_k) &\le \int\limits_{F_n\cap X_k} \phi_n(x)\,d\mu(x) \le \int\limits_{X\times G_n} \mathbf{1}_{F_n\cap X_k}(x) f(tx)\,d\mu(x)\,d\rho_n(t)\\ &\le \int\limits_{X\times G_n} \mathbf{1}_{F_n\cap X_k}(tx) f(x)\,d\mu(x)\,d\rho_n(t)\\ &\le \int\limits_X f(x)\Big(\int\limits_{G_n} \mathbf{1}_{F_n\cap X_k}(tx)\,d\rho_n(t)\Big)\,d\mu(x).\end{aligned}$$

For $\mathbf{1}_{F_n\cap X_k}(tx)$ to be non-zero, it is necessary that $x \in G_n(X_k \cap F_n) = G_n X_k \cap F_n$. It follows that

$$\alpha\mu(F_n \cap X_k) \le \int\limits_{F_n} f(x)\Big(\int\limits_{G_n} \mathbf{1}_{F_n\cap X_k}(tx)\,d\rho_n(t)\Big)\,d\mu(x) \le c_k \int\limits_{F_n} f(x)\,d\mu(x).$$

The conclusion is then an immediate consequence of the fact that D_N is the disjoint union of the subsets F_n, $1 \le n \le N$. ∎

Remark 3.10. As a particular case, we shall use the following assertion. Let $G \curvearrowright (X,\mu)$ be a G-action such that μ is Δ-relatively invariant. Let X_k be a Borel subset of X with $\mu(X_k) < +\infty$. Assume the existence of c_k such that for almost every $x \in X$, $\rho(\{t \in G\,;\, tx \in X_k\}) \le c_k$. Let $f : X \to \mathbb{R}_+$ be a Borel function. For $\alpha > 0$, set $\tilde{Q}_\alpha = \{x \in X\,;\, \int_G f(tx)\,d\rho(t) > \alpha\}$. Then

$$\alpha\mu(\tilde{Q}_\alpha \cap X_k) \le c_k \int\limits_{\tilde{Q}_\alpha} f\,d\mu.$$

Proof of Theorem 3.7. (b)⇒(a) is obvious. Let us show that (a)⇒(c). Let $f \in L^1(X,\mu)$ be null outside X_k. Fix $p > k$. By Theorem 3.3, we know that

$$\lim_n \frac{\int_{G_n} f(tx)\,d\rho_n(t)}{\rho_n(\{s \in G_n\,;\, sx \in X_p\})} \quad \text{exists a.e. on } X_p.$$

If (a) holds, we immediately get the existence of $\lim_n \int_{G_n} f(tx)\,d\rho_n(t)$ almost everywhere on X_p and therefore on the union X of the X_p. Now, observe that such functions f, supported in some X_k, form a dense subspace of $L^1(X,\mu)$.

Finally, let us prove that (c) implies (b). We introduce

$$\Lambda(f)(x) = \lim_{N\to+\infty}\big(\sup_{n,m\ge N} |\phi_n(f)(x) - \phi_m(f)(x)|\big).$$

We fix an integer p, and we shall show that $\Lambda(f)(x) = 0$ for almost every $x \in X_p$. This will end the proof. Take $g \in \mathcal{D}$. We have $\Lambda(g) = 0$ a.e. on X and

$$\Lambda(f) = \Lambda(f) - \Lambda(g) \le \Lambda(f-g) \le 2|f-g|^\star.$$

Given $\alpha > 0$, it follows from Lemma 3.9, that

$$\mu(\{x \in X_p\,;\, \Lambda(f)(x) > \alpha\}) \leq \mu(\{x \in X_p\,;\, |f - g|^\star > \alpha/2\}) \leq \frac{2c_p}{\alpha}\,\|f - g\|_1.$$

Since we can choose g so that $\|f - g\|_1$ is as close to 0 as we wish, we see that $\mu(\{x \in X_p\,;\, \Lambda(f)(x) > \alpha\}) = 0$, from which we get $\mu(\{x \in X_p\,;\, \Lambda(f)(x) > 0\}) = 0$. ■

We apply Theorem 3.7 to the following situation where, in addition, it is possible to identify the limit.

Theorem 3.11. *Let G act properly on a locally compact σ-compact space X and let μ be a Δ-relatively invariant Radon measure on X. Let (G_n) be an increasing sequence of closed subgroups, whose union is dense in G and which satisfies the modular condition* (MC). *Assume that the sequence (ρ_n) of Haar measures is normalized with respect to ρ. Then, for every $f \in L^1(X,\mu)$ and a.e. $x \in X$,*

$$\lim_{n\to\infty} \int_{G_n} f(tx)\,d\rho_n(t) = \int_G f(tx)\,d\rho(t). \tag{3.3}$$

Proof. Let (X_k) be an increasing sequence of open relatively compact subspaces of X with $X = \bigcup X_k$. Of course, since the action is proper, we have

$$0 < \rho_n(\{t \in G_n\,;\, tx \in X_k\}) < +\infty$$

for every $x \in X_k$. Let us show that condition (iii) of Theorem 3.7 is also fulfilled. Set $K_k = \{t \in G\,;\, tX_k \cap X_k \neq \emptyset\}$. This set is relatively compact. We choose a continuous function φ on G, with compact support such that $\mathbf{1}_{K_k} \leq \varphi$. We have

$$\forall n \in \mathbb{N},\ \forall x \in X_k, \quad \rho_n(\{t \in G_n\,;\, tx \in X_k\}) \leq \rho_n(\varphi).$$

Since $\lim_n \rho_n(\varphi) = \rho(\varphi) < +\infty$, there exists a constant c_k such that

$$\forall n \in \mathbb{N},\ \forall x \in X_k, \quad \rho_n(\{t \in G_n\,;\, tx \in X_k\}) \leq c_k.$$

Now, (iii) of Theorem 3.7 is satisfied, by Remark 3.8. Clearly, we may also choose c_k such that, as well, $\rho(\{t \in G\,;\, tx \in X_k\}) \leq c_k$ for all $x \in X$.

The required integrability conditions for (3.3) follow from Lemma 3.5. The existence of the limit is an immediate consequence of Theorem 3.7, applied to the space $\mathcal{D} = \mathcal{C}_c(X)$ of continuous functions with compact support in X. We use the fact that for every $x \in X$ and $f \in \mathcal{C}_c(X)$, the function $t \mapsto f(tx)$ is continuous with compact support. Hence, due to the normalization of the ρ_n, we have the existence of $\lim_n \int_{G_n} f(tx)\,d\rho_n(t)$. Here, we even know that the limit is $\int_G f(tx)\,d\rho(t)$ for every x.

It remains to identify the limit for every $f \in L^1(X,\mu)$. We set

$$\tilde{\Lambda}(f)(x) = \lim_{N\to\infty}\Big(\sup_{n\geq N}\Big|\int_{G_n} f(tx)\,d\rho_n(t) - \int_G f(tx)\,d\rho(t)\Big|\Big).$$

As in the proof of Theorem 3.7, we fix p, and we only need to show that $\tilde{\Lambda}(f)(x) = 0$ for almost every $x \in X_p$. Let g be a continuous function with compact support on X. We have

$$\begin{aligned}\tilde{\Lambda}(f)(x) &= \tilde{\Lambda}(f)(x) - \tilde{\Lambda}(g)(x) \leq \tilde{\Lambda}(f-g)(x)\\ &\leq \lim_N\Big(\sup_{n\geq N}\Big|\int_{G_n}(f(tx)-g(tx))\,d\rho_n(t)\Big|\Big) + \Big|\int_G (f(tx)-g(tx))\,d\rho(t)\Big|\\ &\leq |f-g|^\star + \int_G |f(tx)-g(tx)|\,d\rho(t).\end{aligned}$$

Given $\alpha > 0$, we have

$$\begin{aligned}\mu(\{x\in X_p\,;\,\tilde{\Lambda}(f)(x)>\alpha\}) &\leq \mu(\{x\in X_p\,;\,|f-g|^\star > \alpha/2\})\\ &\quad + \mu\Big(\{x\in X_p\,;\,\int_G |f(tx)-g(tx)|\,d\rho(t) > \alpha/2\}\Big)\\ &\leq \frac{4c_p}{\alpha}\,\|f-g\|_1.\end{aligned}$$

The last inequality follows from Lemma 3.9 and Remark 3.10. Now, we approximate f by a sequence (f_n) of continuous functions with compact support. This gives $\mu(\{x\in X_p\,;\,\tilde{\Lambda}(f)(x)>\alpha\}) = 0$. The conclusion is obtained by letting α go to 0. ∎

As a particular case, we obtain the following result of Ross and Stromberg. In contrast to their proof, we do not use the theorem of Edwards and Hewitt ([EH65, Theorem 1.6]) on pointwise limits of sublinear operators whose ranges are families of measurable functions.

Corollary 3.12 ([RS67]). *Let G be a locally compact group, and (G_n) be an increasing sequence of closed subgroups whose union is dense in G and which satisfies the modular condition* (MC). *Assume that the sequence (ρ_n) of Haar measures is normalized with respect to ρ. Then, for every $f \in L^1(G,\rho)$,*

$$\lim_n \int_{G_n} f(tx)\,d\rho_n(t) = \int_G f(t)\,d\rho(t) \qquad a.e.$$

Proof. We apply Theorem 3.11 to $G \curvearrowright (G,\rho)$. ∎

We can now state our main theorem.

Theorem 3.13. *Let $G \curvearrowright (X,\mu)$ be an action on a measure space, leaving the measure Δ-relatively invariant. Let (G_n) be an increasing sequence* *of closed subgroups whose union is dense in G and which satisfies the modular condition* (MC). *Assume that the sequence (ρ_n) of Haar measures is normalized with respect to ρ. The following properties are equivalent:*

(a) *X is a countable union of Borel subsets B_k of finite measure such that for every k and almost every $x \in X$,*

$$\rho(\{t \in G\,;\, tx \in B_k\}) = \int_G \mathbf{1}_{B_k}(tx)\, d\rho(t) < +\infty. \tag{3.4}$$

(b) *For every $f \in L^1(X,\mu)$ and almost every $x \in X$,*

$$\lim_n \int_{G_n} f(tx)\, d\rho_n(t) = \int_G f(tx)\, d\rho(t).$$

Proof. Assumption (b) contains the assertion that for every $f \in L^1(X,\mu)$ and almost every $x \in X$, the function $f_x : t \mapsto f(tx)$ is ρ-integrable. Thus, obviously (b) implies (a).

Let us show that (a) implies (b). Let $f \in L^1(X,\mu)_+$. By Lemma 3.5, there exists a conull subset $E \subset X$ such that for every $x \in E$, the function $f_x : t \mapsto f(tx)$ is in $L^1(G,\rho)$. We apply to f_x the previous corollary. There exists a conull subset A_x in G such that for every $s \in A_x$:

(i) for $n \in \mathbb{N}$, $t \in G_n \mapsto f_x(ts)$ is ρ_n-integrable;
(ii) $\lim_n \int_{G_n} f_x(ts)\, d\rho_n(t) = \int_G f_x(t)\, d\rho(t)$.

Denote by D the set of all $(s,x) \in G \times X$ for which:

- $t \in G_n \mapsto f(tsx) = f_x(ts)$ is ρ_n-integrable for all n,
- $t \in G \mapsto f(tsx) = f_x(ts)$ is ρ-integrable,
- $\lim \int_{G_n} f(tsx)\, d\rho_n(t) = \int_G f(tsx)\, d\rho(t)$.

Then D is a Borel subset of $G \times X$. Moreover,

$$D \supset \{(s,x)\,;\, x \in E,\ s \in A_x\}.$$

It follows, by using twice the Fubini–Tonelli theorem, that D is conull, and that for almost every $s \in G$, we have, for almost all $x \in X$:

(1) f_{sx} is ρ_n-integrable for $n \in \mathbb{N}$, and is ρ-integrable;
(2) $\lim_n \int_{G_n} f(tsx)\, d\rho_n(t) = \int_G f(tx)\, d\rho(t)$.

Choose such an s and let $C(s)$ be a conull subset of X for which (1) and (2) occur. Then for any $y \in sC(s)$, which is also conull, we have the required properties. ■

COROLLARY 3.14. *Let G be a compact group acting on a measure space (X,μ) in such a way that the σ-finite measure μ is invariant. Let (G_n) be an increasing sequence of closed subgroups of G whose union is dense in G. Choose the Haar measures to have total mass 1. Then for every*

$f \in L^1(X,\mu)$,

$$\lim_{n\to\infty} \int_{G_n} f(tx)\, d\rho_n(t) = \int_G f(tx)\, d\rho(t) \qquad a.e.$$

Theorem 3.13 also applies to several other situations. We have already mentioned in Remark 3.6 the case of continuous actions with closed orbits and compact stabilizers. We now give another example of application.

COROLLARY 3.15. *Let G be a locally compact σ-compact group acting on a measure space (X,μ) in such a way that the σ-finite measure μ is invariant. Let (G_n) be an increasing sequence of closed subgroups of G whose union is dense in G and which satisfies the modular condition* (MC). *Assume that the sequence (ρ_n) of Haar measures is normalized with respect to ρ. Let $f \in L^1(G\times X, \rho\otimes\mu)$. Then for $\rho\otimes\mu$ almost every $(s,x)\in G\times X$,*

$$\lim_{n\to\infty} \int_{G_n} f(ts,tx)\, d\rho_n(t) = \int_G f(ts,tx)\, d\rho(t).$$

Proof. We apply Theorem 3.13 to $B_k = U_k \times V_k$, where $(U_k)_k$ is a sequence of relatively compact open subsets of G with $\bigcup U_k = G$, and where $(V_k)_k$ is a sequence of Borel subsets of X, of finite measure, with $\bigcup V_k = X$. It suffices to observe that

$$\{t\in G\,;\, t(s,x)\in U_k\times V_k\} \subset \{t\in G\,;\, ts\in U_k\}$$

and

$$\rho(\{t\in G, t(s,x)\in B_k\}) \le \rho(\{t\in G\,;\, ts\in U_k\}) < +\infty. \ \blacksquare$$

COROLLARY 3.16. *Let $G \curvearrowright (X,\mu)$ and (G_n) be as in the previous corollary. Let $h\in L^1(X,\mu)$ and let E be a Borel subset of G with $\rho(E) < +\infty$. Then, for almost every $s\in G$,*

$$\lim_{n\to\infty} \int_{Es\cap G_n} h(tx)\, d\rho_n(t) = \int_{Es} h(tx)\, d\rho(t) \qquad a.e.$$

Proof. We apply the previous corollary with $f(t,x) = \mathbf{1}_E(t)f(x)$. $\blacksquare$

EXAMPLE 3.17. Let $G=\mathbb{R}$ act on $\mathbb{R}$ by left translations, and for $n\in\mathbb{N}$, take $G_n = \mathbb{Z}/2^n$. The Haar measure on G_n is normalized by giving the weight $1/2^n$ to each point, and we take for μ the Lebesgue measure ρ on $\mathbb{R}$, normalized by $\rho([0,1]) = 1$. Let E be a Borel subset of $\mathbb{R}$ with $\rho(E) < +\infty$. Corollary 3.16 gives that for every $f\in L^1(\mathbb{R},\rho)$ and almost every $s\in\mathbb{R}$,

$$\lim_{n\to\infty} \frac{1}{2^n} \sum_{\{k\,;\,k/2^n\in E+s\}} f\left(\frac{k}{2^n}+x\right) = \int_{E+s} f(t+x)\, d\rho(t) \qquad \text{a.e.}$$

Let us take $E=\mathbb{Q}$ for example. For every s irrational, the above equality holds (both sides are 0). On the other hand, for $s\in\mathbb{Q}$ and $f=\mathbf{1}_{[0,1]}$, this equality is false for every x.

3.4. An extension of a theorem of Civin. We are now interested in the following problem: Let $G \curvearrowright (X,\mu)$ as before, that is, the σ-finite measure μ is Δ-relatively invariant. We are given an increasing sequence of closed subgroups of G with dense union and satisfying the modular condition. Let E be a Borel subset of X with $\rho(E) < +\infty$, and let $f \in L^1(X,\mu)$. Find conditions under which

$$\lim_n \int_{G_n \cap E} f(tx)\,d\rho_n(t) = \int_E f(tx)\,d\rho(t)$$

almost everywhere.

In [Civ55], Civin has considered the particular case where $G = \mathbb{R}$, $G_n = \mathbb{Z}/2^n$ as in Example 3.17. Let $(t,x) \mapsto t+x$ be a measure preserving action of $\mathbb{R}$ on a measure space (X,μ). Civin's result states that if $f \in L^1(X,\mu)$ satisfies $f(1+x) = f(x)$ a.e., then for almost every $x \in X$,

$$\lim_{n\to\infty} \int_{G_n \cap [0,1[} f(t+x)\,d\rho_n(t) = \lim_{n\to\infty} \frac{1}{2^n} \sum_{k=1}^{2^n} f\left(\frac{k}{2^n} + x\right) = \int_0^1 f(t+x)\,d\rho(t).$$

More generally, we have:

THEOREM 3.18. *Let $(G_n)_{n\in\mathbb{N}}$ be an increasing sequence of lattices of G (therefore G is unimodular) with dense union, and let D be a fundamental domain for G_0. Let $G \curvearrowright (X,\mu)$ be a measure preserving action. Assume that the Haar measure of G is normalized so that the volume of D is 1. Let $f \in L^1(X,\mu)$ be such that, for every $t \in G_0$, $f(tx) = f(x)$ almost everywhere. Then*

$$\lim_n \frac{1}{|G_n \cap D|} \sum_{t \in G_n \cap D} f(tx) = \int_D f(tx)\,d\rho(t) \qquad \textit{for a.e. } x \in X.$$

Proof. We normalize the Haar measure on G_n by giving each point the measure $1/|G_n \cap D|$. This gives a normalized sequence of Haar measures with respect to ρ. Corollary 3.16 shows that, for almost every $s \in G$,

$$\lim_n \frac{1}{|G_n \cap Ds|} \sum_{t \in G_n \cap Ds} f(tx) = \int_{Ds} f(tx)\,d\rho(t) \qquad \text{a.e.}$$

For every $t \in G_n \cap Ds$, there exists a unique $g \in G_0$ such that $gt \in D$. Due to the G_0-invariance of f, we have $f(tx) = f(gtx)$ and therefore

$$\frac{1}{|G_n \cap Ds|} \sum_{t \in G_n \cap Ds} f(tx) = \frac{1}{|G_n \cap D|} \sum_{t \in G_n \cap D} f(tx) \qquad \text{a.e.}$$

On the other hand, by [Bou63, Corollaire, p. 69], the G_0-invariance of f implies that the integral $\int_{Ds} f(tx)\,d\rho(t)$ does not depend on the choice of the fundamental domain: we have $\int_{Ds} f(tx)\,d\rho(t) = \int_D f(tx)\,d\rho(t)$. ∎

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