COLLOQUIUM MATHEMATICUM

VOL. 118 2010 NO. 2

LONG TIME BEHAVIOR OF RANDOM WALKS ON ABELIAN GROUPS

BY

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To the memory of Andrzej Hulanicki

Abstract. Let $\mathbb{G}$ be a locally compact non-compact metric group. Assuming that $\mathbb{G}$ is abelian we construct symmetric aperiodic random walks on $\mathbb{G}$ with probabilities $n \mapsto \mathbb{P}(S_{2n} \in V)$ of return to any neighborhood V of the neutral element decaying at infinity almost as fast as the exponential function $n \mapsto \exp(-n)$. We also show that for some discrete groups $\mathbb{G}$, the decay of the function $n \mapsto \mathbb{P}(S_{2n} \in V)$ can be made as slow as possible by choosing appropriate aperiodic random walks S_n on $\mathbb{G}$.

1. Introduction. Let $\{X_k\}$ be a sequence of independent, identically distributed real-valued random variables with common distribution $\mathbb{P}_{X_1} := \mu$. Assume that μ is symmetric and belongs to the domain of attraction of a stable law with exponent $0 < \alpha \leq 2$. Then, by a local limit theorem (see [8], [11], [16], [18]),

$$\mathbb{P}(S_n \in I) \sim c_{\alpha,\mu}|I|n^{-1/\alpha} \quad \text{as } n \to \infty.$$

This shows that as $\alpha \to 0$ the decay of the function $n \mapsto \mathbb{P}(S_n \in I)$ becomes faster than that of any given function $n \mapsto n^{-k}$, $k > 0$.

To put our observations in perspective let us replace the group $\mathbb{R}$ by a more general group. Namely, let $\mathbb{G}$ be a locally compact non-compact metric group. Let ν be a left Haar measure on $\mathbb{G}$ and $L^2 = L^2(\nu)$. Let μ be a symmetric probability measure on $\mathbb{G}$ such that $\operatorname{supp}\mu$ generates a dense subgroup of $\mathbb{G}$. Let $\mathfrak{L}_\mu : L^2 \to L^2$ be the corresponding left-convolution operator $h \mapsto \mu * h$. In general, $\|\mathfrak{L}_\mu\|_{L^2\to L^2} \leq 1$ and it is equal to 1 if and only if the group $\mathbb{G}$ is amenable (see e.g. [4]). On the other hand, let $\{X_k\}$ be i.i.d. on $\mathbb{G}$ with the law $\mathbb{P}_{X_1} = \mu$ and let $S_n = X_1 \cdot \ldots \cdot X_n$ be the corresponding random walk on $\mathbb{G}$. According to [5] the following characterization of S_n via the norm of the convolution operator $\mathfrak{L}_\mu$ holds: For all relatively compact

2010 *Mathematics Subject Classification*: 60-02, 60B15, 62E10, 43A05.

Key words and phrases: random walk, locally compact abelian group, infinitely divisible distribution, Laplace transform, Köhlbecker transform, Legendre transform, return probability, heat kernel.

DOI: 10.4064/cm118-2-6

neighborhoods V of the neutral element $e \in \mathbb{G}$,

$$\lim_{n\to\infty} \mathbb{P}(S_{2n} \in V)^{1/2n} = \|\mathfrak{L}_\mu\|_{L^2\to L^2}.$$

In particular, if $\mathbb{G}$ is amenable, $\|\mathfrak{L}_\mu\|_{L^2\to L^2} = 1$ and therefore

$$\mathbb{P}(S_{2n} \in V) = \exp(-n \cdot o(1)) \quad \text{as } n \to \infty.$$

If the group $\mathbb{G}$ is not amenable, then $\|\mathfrak{L}_\mu\|_{L^2\to L^2} < 1$. This implies that the decay at infinity of the function $n \mapsto \mathbb{P}(S_{2n} \in V)$ is always exponential.

In what follows we call a measure μ *admissible* if it is absolutely continuous with respect to the measure ν and admits a bounded and strictly positive density $x \mapsto \mu(x)$ in some neighborhood of the identity.

All the above leads us to the following question: *Is it true that for any non-compact amenable group $\mathbb{G}$ the decay of the function $n \mapsto \mathbb{P}(S_{2n} \in V)$ can be made as close as possible to the exponential one by an appropriate choice of a symmetric admissible probability measure $\mu = \mathbb{P}_{X_1}$?*

Any abelian group is amenable. In this paper we prove the following theorem.

THEOREM 1.1. *Let $\mathbb{G}$ be a locally compact non-compact metric abelian group. Let $F : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-decreasing function such that $F(t) = o(t)$ at infinity. There exists a symmetric admissible probability measure μ on $\mathbb{G}$ such that*

$$-\log \mu^{*n}(e)/F(n) \to \infty \quad \textit{at } \infty.$$

Observe that $\mathbb{P}(S_{2n} \in V) \le \mu^{*2n}(e)\nu(V)$, hence for abelian groups Theorem 1.1 brings a positive answer to the above question.

To prove Theorem 1.1 we consider the following three cases (Sections 2, 3 and 4): $\mathbb{G} = \mathbb{R}$, $\mathbb{G} = \mathbb{Z}$ and $\mathbb{G}$ is a countable periodic group, and prove our claim for these special groups. In the final Section 5, using the structure theory of locally compact abelian groups [13], [14], and our knowledge of the result for special groups, we construct probability measures on $\mathbb{G}$ with the desired properties.

Section 4 is of independent interest. The underlying group $\mathbb{G}$ is a union of finite subgroups $\mathbb{G}_k \subset \mathbb{G}$. This group is not compactly generated. The special structure of $\mathbb{G}$ allows us to introduce a class of probabilities on $\mathbb{G}$ of the form $\mu = \sum_k c_k m_k$, where m_k is the normalized Haar measure on $\mathbb{G}_k$. Each $\mu = \mu(c)$ is infinitely divisible and hence can be embedded in a weakly continuous convolution semigroup $\mu_t = \mu(c(t))$. In particular, $\mu^{*n} = \mu(c(n))$. Thanks to this fact our computations become very precise. In particular,

$$\mu^{*n}(e) \asymp \int_0^\infty e^{-n\lambda} d\mathbb{N}(\lambda) \quad \text{at } \infty,$$

where the function $\lambda \mapsto \mathbb{N}(\lambda) = \mathbb{N}(c, \lambda)$ has a very precise form. As an application, we show (Theorem 4.3) that the decay of the function $n \mapsto \mu^{*n}(e)$ can be made as slow as possible by an appropriate choice of the measure $\mu = \mu(c)$ (cf. Theorem 1.1). In this connection observe that any compactly generated abelian group is of the form $\mathbb{R}^l \times \mathbb{Z}^m \times K$, where K is a compact group. It follows that for any admissible symmetric probability μ on this group we must have $\mu^{*2n}(e) \preceq n^{-(l+m)/2}$ at ∞. See [21].

Notation. For any two functions f and g defined in a neighborhood of infinity we will write $f \preceq g$ at ∞ if there exists a constant $c > 0$ such that $f(x) \le cg(x)$ for all x large enough. If $f \preceq g$ and $g \preceq f$ we will write $f \asymp g$. We also write $f \sim g$ if $f/g \to 1$ at ∞.

2. The case of the group $\mathbb{G} = \mathbb{R}$. In this section we give a proof of Theorem 1.1 assuming that $\mathbb{G} = \mathbb{R}$. We let $|A|$ be the Lebesgue measure of a Borel set $A \subset \mathbb{R}$. Let us choose a probability measure $\mu = \mathbb{P}_{X_1}$ which is symmetric and infinitely divisible. This implies that there exists a one-parameter convolution semigroup $(\mu_t)_{t>0}$ of symmetric probability measures on $\mathbb{G}$ such that:

- $\mu = \mu_t$ for $t = 1$. In particular, $\mu^{*n} = \mu_n$.
- $\mu_t \to \varepsilon_0$ weakly as $t \to \infty$, where ε_0 is the Dirac measure concentrated at 0.

Let $\hat{\mu}_t$ be the Fourier transform of the probability measure μ_t. Then

$$\hat{\mu}_t(\xi) = \exp(-t\Psi(\xi)), \quad \xi \in \mathbb{R},$$

where $\xi \mapsto \Psi(\xi)$ is an *even non-negative definite* function on $\mathbb{R}$ ([6, Thm. 8.3]).

Assumption 1. We assume that *for any $t > 0$, the function $\xi \mapsto e^{-t\Psi(\xi)}$ is in L^1.* This implies that μ_t is absolutely continuous with respect to the Lebesgue measure, admits a continuous bounded density $x \mapsto \mu_t(x)$, and

$$\mu_t(0) = \int_{\mathbb{R}} e^{-t\Psi(\xi)}\, d\xi = 2\int_0^\infty e^{-ts}\, d\mathcal{F}(s),$$

where $\mathcal{F}(s) = |\{\tau > 0 : \Psi(\tau) \le s\}|$.

Assumption 2. We assume that *there exists a function $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that f is increasing,* $\log f(t) = o(t)$ *at ∞ and*

$$\mathcal{F}(s) = \int_0^s f(t)\, dt, \quad s \ge 0. \tag{2.1}$$

Assumptions 1 and 2 imply the following identity, crucial for our purpose:

$$\mu_t(0) = 2\int_0^\infty e^{-ts} f(s)\, ds, \quad t > 0. \tag{2.2}$$

Thus, in order to prove Theorem 1.1 with $\mathbb{G} = \mathbb{R}$ we are left to investigate the asymptotic behavior of the Laplace integral of the function f. See Theorem 2.1 below.

REMARK 2.1. 1) Observe that if μ is a symmetric stable distribution of index $0 < \alpha \leq 2$, that is, $\hat{\mu}_t(\xi) = \exp(-|\xi|^\alpha)$, then it is easy to see that the representation (2.1) is possible only if $0 < \alpha \leq 1$.

2) That for any increasing function $f \geq 0$ the equality (2.1) indeed gives rise to an infinitely divisible distribution follows from the celebrated Pólya theorem (see, e.g., [8], [17]): Let $\Psi \geq 0$ be an even continuous function such that $\Psi(0) = 0$. Assume that Ψ restricted to $\mathbb{R}_+$ is increasing and concave. Then the function $x \mapsto e^{-t\Psi(x)}$ restricted to $\mathbb{R}_+$ is decreasing, takes the value 1 at 0, and is convex. By the Pólya theorem, it coincides with the characteristic function of some probability measure μ_t on $\mathbb{R}$. In particular, an even function Ψ defined on $\mathbb{R}_+$ as the inverse of the function $s \mapsto \int_0^s f(t)\,dt$ satisfies the hypotheses above. Hence there exists a symmetric convolution semigroup $(\mu_t)_{t>0}$ such that $\hat{\mu}_t = \exp(-t\Psi)$.

Thanks to our choice (Assumptions 1 and 2) the semigroup $(\mu_t)_{t>0}$ has the following important properties:

(1) For each $t > 0$, the density $x \mapsto \mu_t(x)$ is a strictly positive C^∞-function. In particular, μ_t is admissible.
(2) If $1/f^2$ is convex, then $x \mapsto \mu_t(x)$ is a unimodal function, i.e. has a strict maximum (at $x = 0$).

The first property is a consequence of the following two facts:

- $\Psi(s)/\log s \to \infty$ at ∞,
- $x \mapsto e^{-t\Psi(x)}$ is decreasing and strictly convex.

The second property is an application of the non-trivial criteria of unimodality due to Askey [1].

To investigate the Laplace integral (2.2) we introduce two auxiliary transforms. Let $M : \mathbb{R}_+ \to \mathbb{R}_+$ be a right-continuous decreasing function such that $M(0) = +\infty$. Define two transforms:

- The *Köhlbecker transform* of M:

$$\mathcal{K}(M)(x) := -\log\Big(\int_0^\infty e^{-xt}\,de^{-M(t)}\Big), \qquad x > 0.$$

- The *Legendre transform* of M:

$$\mathcal{L}(M)(x) := \inf_{\tau>0}\{x\tau + M(\tau)\}, \quad x > 0.$$

The following theorem is crucial in our computations. See [2, Lemma 3.2].

THEOREM 2.1. *In the notation above,*

$$\mathcal{K}(M)(x) \sim \mathcal{L}(M)(x) \quad \text{as } x \to \infty.$$

For completeness we give a short proof of this result: For fixed $x > 0$ consider a positive function $m_x(t) = xt + M(t)$ on $(0, \infty)$. The function m_x tends to ∞ at 0 and at ∞. Let t_x be the smallest t at which m_x almost attains its infimum, so that $(1+\epsilon)\mathcal{L}(\mathcal{M})(x) \geq m_x(t_x)$. We have

$$\begin{aligned}\int_0^\infty e^{-xt}\, de^{-M(t)} &= x\int_0^\infty e^{-(xt+M(t))}\, dt \geq x\int_{t_x}^\infty e^{-(xt+M(t))}\, dt\\ &\geq xe^{-M(t_x)}\int_{t_x}^\infty e^{-xt}\, dt = e^{-(xt_x+M(t_x))} \geq e^{-(1+\epsilon)\mathcal{L}(M)(x)}.\end{aligned}$$

This proves the desired lower bound. For the upper bound, write

$$\begin{aligned}\int_0^\infty e^{-xt} de^{-M(t)} &= x\int_0^\infty e^{-(xt+M(t))}\, dt\\ &\leq x\Big(\int_0^{\mathcal{L}(M)(x)/x} e^{-(xt+M(t))}\, dt + \int_{\mathcal{L}(M)(x)/x}^\infty e^{-xt}\, dt\Big)\\ &\leq x\int_0^{\mathcal{L}(M)(x)/x} e^{-\mathcal{L}(M)(x)}\, dt + \int_{\mathcal{L}(M)(x)}^\infty e^{-u} du\\ &= \mathcal{L}(M)(x)e^{-\mathcal{L}(M)(x)} + e^{-\mathcal{L}(M)(x)}.\end{aligned}$$

That $\mathcal{K}(M) \sim \mathcal{L}(M)$ at infinity follows easily from these two bounds. ∎

EXAMPLE 2.1. Let $g : \mathbb{R}_+ \to \mathbb{R}_+$ be a decreasing function, $g(0) = +\infty$. Put $f = e^{-g}$ and define $\mathcal{F}(t) = \int_0^t f(\tau)\, d\tau$. Let $(\mu_t)_{t>0}$ be the corresponding convolution semigroup. We have

$$\mu_t(0) = 2\int_0^\infty e^{-st}\, d\mathcal{F}(s) = 2\int_0^\infty e^{-st} f(s)\, ds = \frac{2}{t}\int_0^\infty e^{-st}\, de^{-g(s)}.$$

This gives

$$-\log \mu_t(0) = \log\frac{t}{2} + \mathcal{K}(g)(t). \tag{2.3}$$

Choose $g(s)$ such that $g(s)/\log(1/s) \to \infty$ at zero. Then $\mathcal{F}(s) = o(s^A)$ at zero, for any $A > 1$. It follows that $-\log \mu_t(0)/\log t \to \infty$ at ∞. Hence applying Theorem 2.1 and the equality (2.3), we obtain the following asymptotic relation:

$$-\log \mu_t(0) \sim \mathcal{K}(g)(t) \sim \mathcal{L}(g)(t) \quad \text{at } \infty.$$

Some particular results based on the direct computation of $\mathcal{L}(g)$ are presented in the table below, where we use the notation

$$\mu_t(0) = \exp\left\{ -t\left[\frac{-\log \mu_t(0)}{t}\right]\right\} := \exp\{-t\cdot o(1)\}.$$

Table 1. Some examples of fast decaying functions $t \mapsto \mu_t(0)$

	$g(s) \asymp$ at zero	$-\log \mu_t(0) \asymp$ at infinity	$o(1) \asymp$ at infinity
1	$(\log \frac{1}{s})^\alpha$, $\alpha > 1$	$(\log t)^\alpha$	$\frac{(\log t)^\alpha}{t}$
2	$s^{-\beta}$, $\beta > 0$	t^{β_0}, $\beta_0 := \frac{\beta}{\beta+1}$	$(\frac{1}{t})^{1-\beta_0}$
3	$\exp\{s^{-\gamma}\}$, $\gamma > 0$	$\frac{t}{(\log t)^{1/\gamma}}$	$\frac{1}{(\log t)^{1/\gamma}}$
4	$\exp_{(k)}\{s^{-\nu}\}$, $\nu > 0$ $^{(*)}$	$\frac{t}{(\log_{(k)} t)^{1/\nu}}$ $^{(**)}$	$\frac{1}{(\log_{(k)} t)^{1/\nu}}$

$(*)$ $\exp_{(k)}(t) = \underbrace{\exp(\exp(\dots\exp(t)))}_{k \text{ times}}$, $(**)$ $\log_{(k)}(t) = \underbrace{\log(\log(\dots\log(t)))}_{k \text{ times}}$.

Let us show for instance how to compute the Legendre transform of the function $g : \tau \mapsto \exp_{(k)}\{\tau^{-\nu}\}$ for $k > 1$ and $\nu > 0$. Set $R(\tau) := t\tau + g(\tau)$. The function $R(\tau)$ is strictly convex and tends to ∞ at 0 and at ∞. Let τ_* be the (unique!) value of τ at which $R(\tau)$ attains its minimum, so that $R(\tau_*) = \mathcal{L}(g)(t)$. Since $\tau \mapsto R(\tau)$ is smooth, we obtain the equation

$$0 = R'(\tau_*) = t + g'(\tau_*) = t - \frac{\nu}{\tau_*^{\nu+1}}\, g(\tau_*) \log g(\tau_*) \log_{(2)} g(\tau_*) \cdots \log_{(k-1)} g(\tau_*),$$

which, in turn, implies the following two crucial properties:

$$(1) \qquad \log_{(k)} t \sim \tau_*^{-\nu} \quad \text{as } t \to \infty, \quad \text{in particular,} \quad \tau_* \to 0 \quad \text{as } t \to \infty,$$

$$(2) \qquad \frac{g(\tau_*)}{\tau_* t} = \frac{\tau_*^\nu}{\nu \log g(\tau_*) \log\log g(\tau_*) \cdots \log_{(k-1)} g(\tau_*)} \to 0 \quad \text{as } t \to \infty.$$

Finally, we arrive at the desired conclusion

$$\mathcal{L}(g)(t) = R(\tau_*) = t\tau_*\left(1 + \frac{g(\tau_*)}{\tau_* t}\right) \sim t\tau_* \sim \frac{t}{(\log_{(k)} t)^{1/\nu}} \quad \text{as } t \to \infty. \ \blacksquare$$

Remark 2.2. The same method works also in a slightly more general setting: Let $r : \mathbb{R}_+ \to \mathbb{R}_+$ be a strictly increasing function with $r(+\infty) = +\infty$. Assume that $\lambda r'(\lambda) \asymp r(\lambda)$ at ∞. Let $g(\tau) = \exp_{(k)}(r(1/\tau))$, $\tau > 0$. Then

$$\mathcal{L}(g)(t) \asymp \frac{t}{r^{-1}(\log_{(k)}(t))} \quad \text{at } \infty.$$

Theorem 2.2. *For any non-decreasing function $F : \mathbb{R}_+ \to \mathbb{R}_+$ which is $o(t)$ at ∞, there exists a symmetric admissible probability measure μ on $\mathbb{R}$ such that*

$$-\log \mu^{*n}(e)/F(n) \to \infty \quad \textit{at } \infty.$$

Proof. Choose a concave function $x \mapsto \widetilde{F}(x)$ such that $\widetilde{F}(x) = o(x)$ and $\widetilde{F}/F \to \infty$ at infinity (see below for the existence of such a function). Define the conjugate Legendre transform $\mathcal{L}^*(\widetilde{F})$ as

$$\mathcal{L}^*(\widetilde{F})(x) = \sup_{t>0}\{-tx + \widetilde{F}(t)\}, \quad x > 0, \tag{2.4}$$

and put $f = \exp(-\mathcal{L}^*(\widetilde{F}))$. Let $\mathcal{F}(t) = \int_0^t f(x)\,dx$, $\Psi = \mathcal{F}^{-1}$ and let μ_t be a probability density such that $\hat{\mu}_t = \exp(-t\Psi)$. By Theorem 2.1,

$$-\log \mu_t(0) \sim \mathcal{K}(\mathcal{L}^*(\widetilde{F}))(t) \sim \mathcal{L}(\mathcal{L}^*(\widetilde{F}))(t) \quad \text{at } \infty.$$

Since $\widetilde{F}$ is concave, $\mathcal{L}(\mathcal{L}^*(\widetilde{F})) = \widetilde{F}$. It follows that

$$-\log \mu_t(0)/F(t) \sim \widetilde{F}(t)/F(t) \to +\infty \quad \text{at } \infty.$$

Construction of the function $\widetilde{F}$: Since $F(t) = o(t)$ at ∞, we can choose a decreasing sequence $\varepsilon_k \downarrow 0$ and an increasing sequence $t_n \uparrow \infty$ such that

$$F(t) < \varepsilon_0 t \quad \text{for } t \in [t_0, t_1],$$

and

$$F(t) < \varepsilon_k t + \sum_{i=1}^{k} t_i(\varepsilon_{i-1} - \varepsilon_i) \quad \text{for } t \in [t_k, t_{k+1}],\ k \geq 1.$$

Finally, we let $\widetilde{F}$ be a piecewise linear function defined by the right-hand sides of the inequalities above. Evidently $t \mapsto \widetilde{F}(t)$ is a concave function. The proof is finished. ∎

3. The case of the group $\mathbb{G} = \mathbb{Z}$. The aim of this section is to prove Theorem 1.1 assuming that $\mathbb{G} = \mathbb{Z}$. This can be done by reducing the problem to the one on $\mathbb{R}$.

Reduction to the group $\mathbb{R}$. Let μ be a symmetric probability measure on $\mathbb{Z}$ and $\Phi = \hat{\mu}$ be its characteristic function. We have

$$\mu^{*2n}(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} [\Phi(x)]^{2n}\,dx = \frac{1}{\pi}\int_{0}^{\pi} [\Phi(x)]^{2n}\,dx.$$

We are looking for Φ supported in $[-\epsilon, \epsilon] \subset [-\pi, \pi]$ and having the form $\Phi = e^{-g}$ near zero. Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing function such that $f(0) = 0$. Define g and Φ_0 by the equalities

$$g = \Big(\lambda \mapsto \int_0^\lambda f(\tau)\,d\tau\Big)^{-1}, \quad \Phi_0 = e^{-g}.$$

Then, by the Pólya theorem, Φ_0 is the characteristic function of some probability measure μ_0 on $\mathbb{R}$, that is, $\Phi_0 = \hat{\mu}_0$. Next define Φ as in Figure 1.

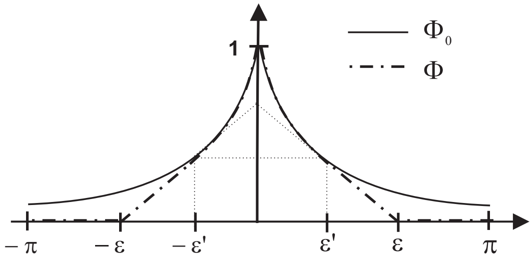

Fig. 1. Construction of the function $\mathit{\Phi}$

By construction, $\mathit{\Phi}$ restricted to $\mathbb{R}_+$ is a continuous, decreasing and convex function. The Pólya theorem implies that there exists a probability measure μ_1 on $\mathbb{R}$ such that $\hat{\mu}_1 = \mathit{\Phi}$. Since $\mathit{\Phi} \in L^1$, μ_1 is absolutely continuous with respect to the Lebesgue measure, and its density $x \mapsto \mu_1(x)$ can be expressed as the inverse Fourier transform of $\mathit{\Phi}$. Next we apply the Poisson summation formula to $(\mathit{\Phi}, \mu_1)$ (see [10]):

$$\sum_{k\in\mathbb{Z}} \mathit{\Phi}(\xi + 2k\pi) = \sum_{n\in\mathbb{Z}} \mu_1(n) e^{in\xi}, \qquad \xi \in \mathbb{R}. \tag{3.1}$$

Since $\mathit{\Phi}$ is supported in the interval $[-\epsilon, \epsilon] \subset [-\pi, \pi]$, the equation (3.1) shows that for $|\xi| < \pi$,

$$\mathit{\Phi}(\xi) = \sum_{n\in\mathbb{Z}} \mu_1(n) e^{in\xi}. \tag{3.2}$$

In particular, for $\xi = 0$, (3.2) gives

$$1 = \mathit{\Phi}(0) = \sum_{n\in\mathbb{Z}} \mu_1(n). \tag{3.3}$$

The equality (3.3) implies that the distribution μ on $\mathbb{Z}$ defined as $\mu(\{n\}) = \mu_1(n)$ is a probability distribution. Its characteristic function $\mathit{\Phi}$ coincides with $\mathit{\Phi}_0 = e^{-g}$ on the interval $(-\epsilon', \epsilon')$.

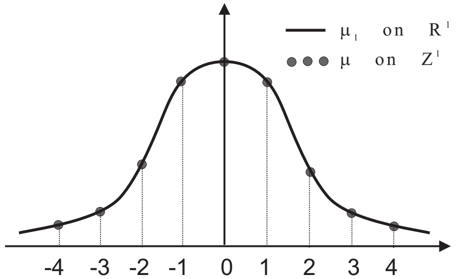

Fig. 2. Construction of the probability measure μ on $\mathbb{Z}$

These observations show that for some $\lambda > 0$,

$$\mu^{*2n}(0) = \frac{1}{\pi}\int_0^{\epsilon} [\varPhi(x)]^{2n}\,dx = \frac{1}{\pi}\int_0^{\epsilon'} [\varPhi_0(x)]^{2n}\,dx + O(e^{-\lambda n})$$
$$\sim \frac{1}{\pi}\int_0^{\epsilon'} e^{-2ng(x)}\,dx \sim \frac{1}{\pi}\int_0^{\infty} e^{-2ns} f(s)\,ds \quad \text{at } \infty,$$

and therefore we can proceed as in Section 2 to prove the following theorem.

THEOREM 3.1. *For any non-decreasing function $F : \mathbb{R}_+ \to \mathbb{R}_+$ which is $o(t)$ at ∞ there exists a symmetric admissible probability measure μ on $\mathbb{Z}$ such that*

$$-\log \mu^{*n}(e)/F(n) \to \infty \quad \textit{at } \infty.$$

4. The case when $\mathbb{G}$ is a countable periodic group. Let $\mathbb{G}$ be a countable periodic abelian group, that is, each element $g \in \mathbb{G}$ has a finite order. Then $\mathbb{G}$ can be represented as the union $\bigcup_{k=0}^{\infty} \mathbb{G}_k$ of an increasing sequence of finite subgroups $\mathbb{G}_k$. Indeed, let $\mathbb{G} = \{\mathrm{id}, a_1, a_2, \ldots\}$, $\mathbb{G}_0 = \{\mathrm{id}\}$ and let $\mathbb{G}_k = \langle a_1, \ldots, a_k\rangle$ be the group generated by the first k elements $a_1, \ldots, a_k$. By construction, every $a \in \mathbb{G}_k$ is of the form $a_1^{m_1} \cdot \ldots \cdot a_k^{m_k}$, where $m_i \le \max\{\text{order } a_i\}$. We have

$$\mathbb{G}_k \subseteq \mathbb{G}_{k+1} \subseteq \mathbb{G}, \quad k = 0, 1, 2, \ldots.$$

Next we can renumber the sequence $\{\mathbb{G}_k\}$ so that

$$\mathbb{G}_k \subset \mathbb{G}_{k+1} \subset \mathbb{G}.$$

Clearly all $\mathbb{G}_k$ are finite groups and, in fact, by structure theory [13, §A.27], each $\mathbb{G}_k$ is a finite product of cyclic groups $\mathbb{Z}(n_i)$.

EXAMPLE 4.1. Let $\mathbb{Z}(2)^{\infty} = \mathbb{Z}(2) \times \mathbb{Z}(2) \times \cdots$, where $\mathbb{Z}(2) \cong \{1, 0\}$ with addition mod 2. Then all elements $\xi = (\xi_0, \xi_1, \ldots) \in \mathbb{Z}(2)^{\infty}$ have order 1 or 2. We define the infinite countable periodic group $\mathbb{G} = \mathbb{Z}(2)^{(\infty)} \subset \mathbb{Z}(2)^{\infty}$ as the set of all sequences $\xi = (\xi_k)$ which are eventually zero. For $i \in \mathbb{N}$, let $\overline{\xi}_i$ be the sequence (ξ_k) with $\xi_i = 1$ and $\xi_k = 0$ for $k \ne i$. Then clearly

$$\mathbb{G}_k = \langle \overline{\xi}_1, \ldots, \overline{\xi}_k\rangle \cong \mathbb{Z}(2)^k \quad \text{and} \quad \mathbb{G} = \bigcup_{k=0}^{\infty} \mathbb{G}_k.$$

EXAMPLE 4.2. Let $\mathbb{G} = \mathbb{Z}(p^{\infty})$ be the group of all p^k-roots of unity,

$$\mathbb{Z}(p^{\infty}) = \{\xi = \exp(2\pi m i/p^k) : 0 \le m \le p^k - 1,\ k = 1, 2, \ldots\}.$$

Clearly $\mathbb{Z}(p^k) \subset \mathbb{Z}(p^{k+1})$ and $\mathbb{G} = \bigcup_{k=1}^{\infty} \mathbb{Z}(p^k)$.

PROPOSITION 4.1. *Let $\{d_k\}$ be a sequence of natural numbers such that d_{k+1}/d_k is an integer equal to 2 or greater. Then there exists a countable periodic group $\mathbb{G}$ and an increasing sequence of groups $\mathbb{G}_k \subset \mathbb{G}$ such that $\mathbb{G} = \bigcup_{k=0}^{\infty} \mathbb{G}_k$ and d_k is the cardinality of $\mathbb{G}_k$.*

Proof. Define $c_k := d_{k+1}/d_k$, $k = 0, 1, 2, \ldots$. Then $d_n = d_0 \cdot c_0 \cdot \ldots \cdot c_{n-1}$, $n = 1, 2, \ldots$. Put $\tilde{\mathbb{G}}_0 = \mathbb{Z}(d_0)$, $\tilde{\mathbb{G}}_n = \mathbb{Z}(d_0) \times \mathbb{Z}(c_0) \times \cdots \times \mathbb{Z}(c_{n-1})$, $n \geq 1$. We have $|\tilde{\mathbb{G}}_n| = d_0 \cdot c_0 \cdot \ldots \cdot c_{n-1} = d_n$. Let now $\mathbb{G}_0 = \{(e_0, 1, 1, \ldots) : e_0 \in \tilde{\mathbb{G}}_0\}, \ldots, \mathbb{G}_n = \{(e_0, e_1, \ldots, e_n, 1, 1, \ldots) : (e_0, e_1, \ldots, e_n) \in \tilde{\mathbb{G}}_n\}$. Clearly $\{\mathbb{G}_k\}$ increases and $\mathbb{G} = \bigcup_{k=0}^{\infty} \mathbb{G}_k$. Also $|\mathbb{G}_k| = |\tilde{\mathbb{G}}_k| = d_k$. The group $\mathbb{G}$ is a countable periodic group. ∎

Let $H = \widehat{\mathbb{G}}$ be the dual group of $\mathbb{G}$, that is, the group of all characters of $\mathbb{G}$ (see [13], [14]). According to the structure theory of abelian groups, H is a compact totally disconnected group. Some examples which are basic for our purpose are given below.

EXAMPLE 4.3.

- $\mathbb{G} \cong \mathbb{Z}(p^\infty)$, $H \cong \Delta_p$, the group of p-adic integers,
- $\mathbb{G} \cong \mathbb{Z}(l)^{(\infty)}$, $H \cong \mathbb{Z}(l)^\infty$, $l \geq 2$.

More generally,

- $\mathbb{G} \cong (\prod_{k=0}^{\infty})^* \mathbb{Z}(l_k)$, $H \cong \prod_{k=0}^{\infty} \mathbb{Z}(l_k)$,

where $\prod^* X_k$ is the *weak product* of the groups X_k, that is, the set of all sequences $x = (x_i) \in \prod X_k$ which are eventually identities.

Let m_k be the uniform distribution on $\mathbb{G}_k$, i.e. for $A \subset \mathbb{G}_k$,

$$m_k(A) = \frac{|A|}{|\mathbb{G}_k|}.$$

Let $\{c_k\}_{k=0}^{\infty} \subset \mathbb{R}_+$ be a sequence of positive reals such that $\sum_{k=0}^{\infty} c_k = 1$. Define a probability measure $\mu = \mu(c)$ on $\mathbb{G}$ as follows:

$$\mu = c_0 m_0 + c_1 m_1 + \cdots .$$

Evidently μ is a symmetric admissible probability measure on $\mathbb{G}$. We want to find the Fourier transform $\hat{\mu}$ of the measure μ,

$$\hat{\mu}(y) = \int_{\mathbb{G}} \langle y, x \rangle \, d\mu(x), \quad y \in H.$$

Let $H_k = A(H, \mathbb{G}_k) = \{y \in H : \langle y, x \rangle = 1, \ \forall x \in \mathbb{G}_k\}$ be the annihilator of the group $\mathbb{G}_k$ in the group $H = \widehat{\mathbb{G}}$. In particular, $H_0 = H$, $H_{k+1} \subset H_k$ and

$$H = (H_0 \setminus H_1) \cup (H_1 \setminus H_2) \cup \cdots .$$

EXAMPLE 4.4. Let $\mathbb{G} = (\prod_{i=1}^{\infty})^* \mathbb{Z}(p_i)$. Then

$$\mathbb{G}_k = \prod_{i=1}^{k} \mathbb{Z}(p_i) \times \{\overline{e}\}, \quad H_0 = \prod_{i=1}^{\infty} \mathbb{Z}(p_i), \quad H_k = \{\underline{e}\} \times \prod_{i=k+1}^{\infty} \mathbb{Z}(p_i),$$

where $\overline{e} = (e_{k+1}, e_{k+2}, \ldots)$ and $\underline{e} = (e_1, \ldots, e_k)$ stand for identities.

PROPOSITION 4.2. *The Fourier transform* $\hat{\mu}$ *of the measure* μ *is of the form*

$$\hat{\mu}(y) = c_0 + c_1 + \cdots + c_k, \quad y \in H_k \setminus H_{k+1}, \quad k = 0, 1, \ldots.$$

Proof. Let $\mathbb{G}$ be a locally compact abelian group and $L \subset \mathbb{G}$ be a compact subgroup. Let m_L be the Haar measure of L regarded as a measure on $\mathbb{G}$. The Fourier transform $\widehat{m}_L$ of the measure m_L is of the form [9, 2.14]

$$\widehat{m}_L(y) = \begin{cases} 1 & \text{if } y \in A(H, L), \\ 0 & \text{if } y \notin A(H, L). \end{cases}$$

In particular, for $L = \mathbb{G}_k \subset \mathbb{G}$,

$$\widehat{m}_k(y) = \begin{cases} 1 & \text{if } y \in H_k, \\ 0 & \text{if } y \in H \setminus H_k. \end{cases} \tag{4.1}$$

Using (4.1) we compute the Fourier transform $\hat{\mu}$ of the measure μ

$$\hat{\mu} = \sum_{k=0}^{\infty} c_k \widehat{m}_k = \sum_{k=0}^{\infty} c_k 1_{H_k} = \sum_{k=0}^{\infty} \Big(\sum_{i=0}^{k} c_i \Big) 1_{H_k \setminus H_{k+1}}. \tag{4.2}$$

Clearly (4.2) gives the desired result. The proof is finished. ■

PROPOSITION 4.3. *Put* $\sigma_k := c_0 + c_1 + \cdots + c_k$ *for* $k \geq 0$ *and* $\sigma_{-1} := 0$. *Then*

$$\mu^{*n} = \sum_{k=0}^{\infty} (\sigma_k^n - \sigma_{k-1}^n) m_k, \quad n = 1, 2, \ldots.$$

Proof. Observe that $c_k = \sigma_k - \sigma_{k-1}$. Proposition 4.2 and the fact that $\widehat{\mu^{*n}} = (\hat{\mu})^n$ imply that

$$\widehat{\mu^{*n}}(y) = \sigma_k^n, \quad y \in H_k \setminus H_{k+1}, \quad k = 0, 1, \ldots.$$

Since for any $i > j$, $m_i * m_j = m_j$, the measure μ^{*n} has the same structure as μ, that is, $\mu^{*n} = \sum a_k m_k$. Observe that the sum converges in variation. Hence, by Proposition 4.2, for any $k = 0, 1, \ldots$,

$$\widehat{\mu^{*n}}(y) = \sum_{k=0}^{\infty} a_k \widehat{m}_k(y) = a_0 + a_1 + \cdots + a_k, \quad y \in H_k \setminus H_{k+1}.$$

It follows that for $k = 0, 1, 2, \ldots$, we must have $a_k := \sigma_k^n - \sigma_{k-1}^n$. The proof is finished. ■

Proposition 4.4. *The measure $\mu = \mu(c)$ defined on the group $\mathbb{G}$ is infinitely divisible. More precisely, for any $n = 2, 3, \dots$, $\mu = \mu^{*n}(a)$, where $a = (a_k)$ is the sequence with entries $a_k = \sigma_k^{1/n} - \sigma_{k-1}^{1/n}$, $k = 0, 1, \dots$.*

Proof. By Proposition 4.2, for any sequence $a = (a_i)$ with non-negative entries and for any $k = 0, 1, \dots$,

$$\widehat{\mu^{*n}}(a)(y) = (a_0 + a_1 + \dots + a_k)^n, \quad y \in H_k \setminus H_{k+1}.$$

We want to find $a = (a_i)$ such that $\mu(c) = \mu^{*n}(a)$. This gives an infinite system of algebraic equations

$$c_0 + c_1 + \dots + c_k = (a_0 + a_1 + \dots + a_k)^n, \quad k = 0, 1, \dots,$$

which has a unique solution $a = (a_k)$: $a_k = \sigma_k^{1/n} - \sigma_{k-1}^{1/n}$. The proof is finished. ∎

Proposition 4.5. *The Fourier transform $\hat{\mu}$ of the measure $\mu = \mu(c)$ can be represented in the form*

$$\hat{\mu}(\theta) = \exp(-\Psi(\theta)), \quad \theta \in H,$$

where the negative-definite function Ψ has the representation

$$\Psi(\theta) = \int_{\mathbb{G}} (1 - \langle x, \theta \rangle) \, d\Pi(x), \quad \theta \in H.$$

The measure Π on $\mathbb{G}$ is finite and can be written in the form

$$\Pi = \sum_{k=0}^{\infty} p_k m_k, \qquad p_k > 0, \, k = 0, 1, 2, \dots,$$

where

$$p_0 = \Pi(\mathbb{G}) - \log \frac{1}{c_0} \quad \text{and} \quad p_k = \log \left[1 + \frac{c_k}{\sigma_{k-1}} \right], \qquad k \geq 1.$$

Proof. By Proposition 4.4, the measure μ is infinitely divisible, hence by the representation formula valid for any locally compact abelian group (see [6, Thm. 8.3] and [15]) its Fourier transform $\hat{\mu}$ has the form

$$\hat{\mu}(\theta) = \exp\{-\Psi(\theta)\}, \quad \theta \in H,$$

where $\Psi : H \to \mathbb{C}$ is a negative-definite function on H. Since μ is symmetric, Ψ is real-valued. By the celebrated Lévy–Khinchin formula ([6, Thm. 18.19]),

$$\Psi(\theta) = \phi(\theta) + \int_{\mathbb{G} \setminus \{e\}} \operatorname{Re}(1 - \langle x, \theta \rangle) \, d\Pi(x),$$

where ϕ is a non-negative definite quadratic form on H and Π is a symmetric measure on $\mathbb{G} \setminus \{e\}$. Since the group $H = \widehat{\mathbb{G}}$ is totally disconnected, $\phi \equiv 0$. Since $\mathbb{G}$ is discrete, Π, by definition, is a finite symmetric measure on $\mathbb{G} \setminus \{e\}$. Extend the measure Π to the whole group $\mathbb{G}$ putting $\Pi(\{e\}) = \pi_0 > 0$.

Evidently this does not change the value of the function $\Psi(\theta)$, $\theta \in H$. After these preparations we can write the following equality:

$$\Psi(\theta) = \int_{\mathbb{G}} (1 - \langle x, \theta \rangle)\, d\Pi(x) = \Pi(\mathbb{G}) - \widehat{\Pi}(\theta).$$

On the other hand, we must have

$$\Psi(\theta) = -\log \hat{\mu}(\theta) = \log \frac{1}{\sigma_k} \quad \text{if } \theta \in H_k \setminus H_{k+1},\ k = 0, 1, \ldots.$$

Put $\lambda = \Pi(\mathbb{G})$; then $\widehat{\Pi}(\theta) = \lambda - \Psi(\theta)$. It follows that

$$\widehat{\Pi}(\theta) = \lambda - \log \frac{1}{\sigma_k} \quad \text{if } \theta \in H_k \setminus H_{k+1},\ k = 0, 1, \ldots. \tag{4.3}$$

Clearly we can choose the value $\pi_0 = \Pi(\{e\})$ large enough so that $\Pi(\mathbb{G}) > \log(1/c_0) > 0$. The equality (4.3) shows that Π has the same structure as μ,

$$\Pi = \sum_{k=0}^{\infty} p_k m_k.$$

To find $\{p_k\}$ we solve the system of algebraic equations

$$\lambda - \log \frac{1}{\sigma_k} = p_0 + p_1 + \cdots + p_{k-1} + p_k, \quad k = 0, 1, \ldots.$$

The desired result follows. ∎

Notation. For any finite measure $\mathbb{P}$ on $\mathbb{G}$ we define

$$e(\mathbb{P}) := e^{-\mathbb{P}(\mathbb{G})} \Big\{ m_0 + \mathbb{P} + \frac{1}{2!} \mathbb{P}^{*2} + \cdots \Big\},$$

and call this measure the *compound Poisson measure.*

PROPOSITION 4.6. *The measure $\mu = \mu(c)$ can be embedded in a weakly continuous convolution semigroup $(\mu_t)_{t>0}$ of symmetric probability measures on $\mathbb{G}$. Moreover, the following properties hold:*

(1) *Each measure μ_t has a representation*

$$\mu_t = e(t\Pi), \quad t > 0,$$

where Π is a finite measure on $\mathbb{G}$ (see Proposition 4.5).

(2) *In particular, $\mu_t = \sum_{k=0}^{\infty} c_k(t) m_k$, where $c_k(t) = \sigma_k^t - \sigma_{k-1}^t$, $k = 0, 1, \ldots$.*

Proof. Let Ψ be the negative-definite function defined by μ. For each $t > 0$ we define the probability measure μ_t by its Fourier transform

$$\hat{\mu}_t(\theta) = \exp\{-t\Psi(\theta)\}, \quad \theta \in H.$$

That this equation defines μ_t as a probability on $\mathbb{G}$ follows from the celebrated theorem of Bochner valid on any locally compact abelian group (see [6, Thm. 8.3]). Evidently (μ_t) is a weakly continuous convolution semigroup.

The equation $\widehat{e(t\Pi)} = \exp(-t\Psi)$ follows by inspection. Hence the equality $\mu = \mu_1$ follows from Proposition 4.5. The second statement for rational $t = m/n$ is a consequence of Propositions 4.3 and 4.4. Then for any real $t > 0$ it follows by continuity. ∎

PROPOSITION 4.7. *Let m be the Haar measure on $\mathbb{G}$ such that $m(\{x\})=1$ for any $x \in \mathbb{G}$. For any $t > 0$ the measure μ_t is absolutely continuous with respect to m and has a density $x \mapsto \mu_t(x)$ given by*

$$\mu_t(x) = \sum_{k=0}^{\infty} \frac{c_k(t)}{|\mathbb{G}_k|} 1_{\mathbb{G}_k}(x) = \sum_{k=0}^{\infty} \left(\sum_{n=k}^{\infty} \frac{c_n(t)}{|\mathbb{G}_n|} \right) 1_{\mathbb{G}_k \setminus \mathbb{G}_{k-1}}(x), \quad x \in \mathbb{G}.$$

In particular, for any finite set $\mathbb{F} \subset \mathbb{G}$, $\mu_t(\mathbb{F}) \sim \mu_t(e)|\mathbb{F}|$ at ∞.

Proof. Since μ_t is symmetric we must have

$$\mu_t(x) \le \mu_t(e), \quad x \in \mathbb{G}.$$

On the other hand, for $x \in \mathbb{G}_n \setminus \mathbb{G}_{n-1}$,

$$\begin{aligned} \mu_t(x) &= \sum_{k=n}^{\infty} \frac{\sigma_k^t - \sigma_{k-1}^t}{|\mathbb{G}_k|} = \sum_{k=0}^{\infty} \frac{\sigma_k^t - \sigma_{k-1}^t}{|\mathbb{G}_k|} - \sum_{k=0}^{n-1} \frac{\sigma_k^t - \sigma_{k-1}^t}{|\mathbb{G}_k|} \\ &= \mu_t(e) - \sum_{k=0}^{n-1} \frac{\sigma_k^t - \sigma_{k-1}^t}{|\mathbb{G}_k|}. \end{aligned}$$

Since each term $\sigma_k^t/|\mathbb{G}_k|$, as a function of $t > 0$, has an exponential decay and the function $t \mapsto \mu_t(e)$ has subexponential decay (this property holds for any amenable group!) we must have

$$\mu_t(x) \sim \mu_t(e) \quad \text{at } \infty,$$

which is true for any $x \in \mathbb{F} \cap (\mathbb{G}_n \setminus \mathbb{G}_{n-1})$. Since we assume that $\mathbb{F}$ is finite, this gives the result. ∎

Next, we want to investigate the asymptotic properties of the function $t \mapsto \mu_t(e)$ at infinity. Let $d_k := |\mathbb{G}_k|$ and $\sigma(k) := \sum_{i=k+1}^{\infty} c_i = 1 - \sigma_k$. Define a step function $x \mapsto \mathbb{N}(x)$ as follows: It has jumps at the points $\lambda_k = \sigma(k)$ and the values of the jumps are $1/d_k$. We also assume that $x \mapsto \mathbb{N}(x)$ is right-continuous and $\mathbb{N}(0) = 0$.

THEOREM 4.1. *The following inequality holds:*

$$\frac{1}{2} \int_0^{\infty} e^{-t\delta\lambda}\, d\mathbb{N}(\lambda) \le \mu_t(e) \le \int_0^{\infty} e^{-t\lambda}\, d\mathbb{N}(\lambda) \ (\exists \delta = \delta(c) > 1, \forall t > 0). \tag{4.4}$$

Proof. According to Proposition 4.7 we can write

$$\mu_t(e) = \sum_{k=0}^{\infty} \frac{\sigma_k^t - \sigma_{k-1}^t}{d_k} = \sum_{k=0}^{\infty} \frac{\sigma_k^t}{d_k} - \sum_{k=0}^{\infty} \frac{\sigma_{k-1}^t}{d_k}.$$

This gives an upper bound

$$\mu_t(e) \leq \sum_{k=0}^{\infty} \frac{\sigma_k^t}{d_k}.$$

Since $d_k \geq 2d_{k-1}$, we also have

$$\mu_t(e) \geq \sum_{k=0}^{\infty} \frac{\sigma_k^t}{d_k} - \frac{1}{2}\sum_{k=1}^{\infty} \frac{\sigma_{k-1}^t}{d_{k-1}} = \frac{1}{2}\sum_{k=0}^{\infty} \frac{\sigma_k^t}{d_k}.$$

Write $\sigma_k^t = (1-\sigma(k))^t = e^{t\cdot\log(1-\sigma(k))}$. Since all $c_k > 0$, we have $0 < \sigma(k) < \sigma(0) < 1$. It follows that for some $\delta > 1$, and all $k \geq 0$,

$$-\delta\sigma(k) < \log(1-\sigma(k)) < -\sigma(k),$$

and therefore

$$e^{-t\delta\sigma(k)} < \sigma_k^t < e^{-t\sigma(k)}, \quad k = 0, 1, 2, \ldots.$$

Altogether we get

$$\frac{1}{2}\sum_{k=0}^{\infty} \frac{1}{d_k}\, e^{-t\delta\sigma(k)} < \mu_t(e) < \sum_{k=0}^{\infty} \frac{1}{d_k}\, e^{-t\sigma(k)}.$$

Evidently we can write

$$\sum_{k=0}^{\infty} \frac{1}{d_k}\, e^{-t\sigma(k)} = \int_0^{\infty} e^{-t\lambda} d\mathbb{N}(\lambda),$$

which gives the result. ∎

Observe that for any non-decreasing continuous function f such that $f(t) \to 0$ as $t \to 0$ one can construct a right-continuous step function $\lambda \mapsto \mathbb{N}(\lambda)$ which has jumps $1/d_k$ at the points $\sigma(k)$, and such that $\mathbb{N} \leq f$. See Figure 3.

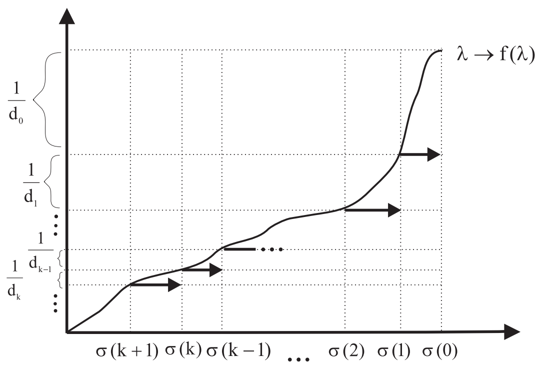

Fig. 3. Construction of the step function $\mathbb{N} \leq f$

Put $c_{k+1} = \sigma(k) - \sigma(k+1)$ and define a probability measure $\mu = \sum_{k=0}^{\infty} c_k m_k$. Let $(\mu_t)_{t\geq 0}$ be the convolution semigroup such that $\mu = \mu_1$. By (4.4),

$$\mu^{*n}(e) \leq \int_0^{\infty} e^{-n\lambda}\, df(\lambda).$$

With this bound in mind we can apply asymptotic properties of the Laplace integral (see Theorem 2.2) to get the following statement.

THEOREM 4.2. *Let $\mathbb{G}$ be a countable periodic abelian group. For any function $F: \mathbb{R}_+ \to \mathbb{R}_+$ such that $F(t) = o(t)$ at ∞, there exists a symmetric admissible probability measure μ on $\mathbb{G}$ such that*

$$-\log \mu^{*n}(e)/F(n) \to \infty \qquad \textit{at } \infty.$$

For any non-decreasing function $g(t)$ such that $\lim_{t\to 0} g(t) = 0$ one can construct a step function $\mathbb{N} \geq g$ with jumps $1/d_k$ at $\sigma(k)$. See Figure 4.

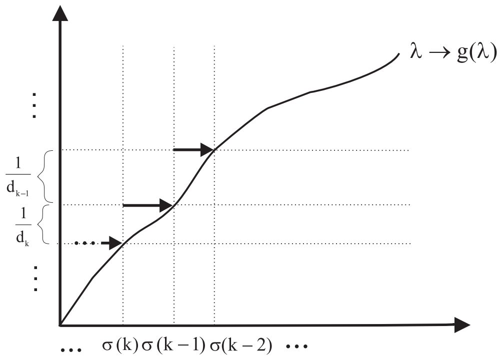

Fig. 4. Construction of the step function $\mathbb{N} \geq g$

Put $c_{k+1} = \sigma(k) - \sigma(k+1)$ and define a probability measure $\mu = \sum_{k=0}^{\infty} c_k m_k$. Let $(\mu_t)_{t\geq 0}$ be the convolution semigroup such that $\mu = \mu_1$. Applying the inequality (4.4) we obtain

$$\mu^{*n}(e) \geq \frac{1}{2}\int_0^{\infty} e^{-n\delta\lambda} d\mathbb{N}(\lambda) \geq \frac{1}{2}\int_0^{\infty} e^{-n\delta\lambda} dg(\lambda). \tag{4.5}$$

EXAMPLE 4.5. Assume that $t \mapsto g(t)$ is a non-decreasing function such that $\lim_{t\to 0} g(t) = 0$. Let $\mathbb{N} \geq g$ be as in Figure 4. Then

$$\begin{aligned}\mu_t(0) &\geq \frac{1}{2}\int_0^{\infty} e^{-t\delta\lambda}\, d\mathbb{N}(\lambda) \geq \frac{1}{2}\int_0^{\infty} e^{-t\delta\lambda}\, dg(\lambda) = \frac{\delta t}{2}\int_0^{\infty} e^{-t\delta\lambda} g(\lambda)\, d\lambda \\ &= \frac{\delta}{2}\int_0^{\infty} e^{-\delta s} g\left(\frac{s}{t}\right) ds = \frac{\delta}{2} g\left(\frac{1}{t}\right)\int_0^{\infty}\left[g\left(\frac{s}{t}\right)/g\left(\frac{1}{t}\right)\right] e^{-\delta s}\, ds.\end{aligned}$$

Assuming that $g(\lambda\tau)/g(\tau)$ has dominated convergence as $\tau \to 0$ to some integrable function (in fact, always to the function $\lambda \mapsto \lambda^\alpha$, $0 \le \alpha < \infty$, see [7]) we obtain

$$\liminf_{t\to\infty} \frac{\mu_t(0)}{g(1/t)} \ge \frac{\delta}{2}\int_0^\infty s^\alpha e^{-\delta s}\, ds = \frac{\delta^{-\alpha}}{2}\,\Gamma(1+\alpha).$$

This simple observation leads us to some examples presented in Table 2.

Table 2. Some examples of slowly decaying functions $t \mapsto \mu_t(0)$

	$g(t) \asymp$ at zero	$\mu_t(0) \succeq$ at infinity
1	$t^\alpha, \quad \alpha > 0$	$\frac{1}{t^\alpha}$
2	$\exp\{-(\log\frac{1}{t})^\alpha\}, \quad 0<\alpha<1$	$\exp\{-(\log t)^\alpha\}$
3	$(\log\frac{1}{t})^{-1/\alpha}, \quad \alpha>0$	$(\frac{1}{\log t})^{1/\alpha}$
4	$[\log(\log\frac{1}{t})]^{-1/\alpha}, \quad \alpha>0$	$[\frac{1}{\log(\log t)}]^{1/\alpha}$
5	$[\log_{(k)}\frac{1}{t}]^{-1/\alpha}, \quad \alpha>0$	$[\frac{1}{\log_{(k)} t}]^{1/\alpha}$

THEOREM 4.3. *Let $\mathbb{G}$ be a countable periodic abelian group. For any non-decreasing function $R : \mathbb{R}_+ \to \mathbb{R}_+$ such that $R(t) \to \infty$ at ∞ there exists a symmetric admissible probability measure μ on $\mathbb{G}$ such that*

$$-\log \mu^{*n}(e)/R(n) \to 0 \qquad \text{at } \infty.$$

Proof. Choose a concave increasing non-negative function $t \mapsto \widetilde{R}(t)$ such that $\widetilde{R}(t)/R(t) \to 0$, $\widetilde{R}(t) \to \infty$ and $\widetilde{R}(t) = o(t)$ as $t \to \infty$. That such a choice is possible follows from a simple geometric construction: see Figure 5.

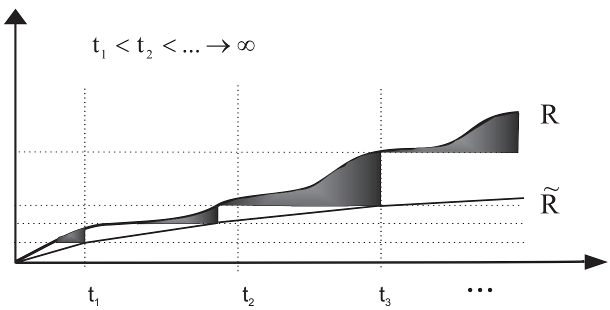

Fig. 5. Construction of the function $\widetilde{R}$

Define $g(x) = e^{-\mathcal{L}^*(\widetilde{R})(x)}$ and construct a step function $\mathbb{N} \ge g$ as in Figure 4. Applying (4.5) and Theorem 2.1 we obtain

$$\mu^{*n}(e) \ge \frac{1}{2}\int_0^\infty e^{-n\delta\lambda}\, dg(\lambda) = \frac{1}{2}\int_0^\infty e^{-n\delta\lambda}\, de^{-\mathcal{L}^*(\widetilde{R})(\lambda)} \succeq \frac{1}{2}\, e^{-\mathcal{L}(\mathcal{L}^*(\widetilde{R}))(n\delta)}.$$

Since $\widetilde{R}$ is concave, $\mathcal{L}(\mathcal{L}^*(\widetilde{R})) = \widetilde{R}$ and we get the desired result. ∎

REMARK 4.1. It is well known that if a locally compact non-compact group $\mathbb{G}$ is compactly generated (in particular, for discrete $\mathbb{G}$, finitely generated) the upper rate of decay of the function $n \mapsto \mu^{*n}(e)$ with symmetric admissible μ exists and is a geometric invariant of the group $\mathbb{G}$. See for instance [3], [19], [20]. In particular, let $\mathbb{G}$ be an abelian compactly generated group. By structure theory [13, Thm. 9.8],

$$\mathbb{G} \cong \mathbb{R}^l \times \mathbb{Z}^m \times K,$$

where K is a compact group. Then, for any symmetric admissible μ on $\mathbb{G}$, we must have

$$\mu^{*2n}(e) \preceq n^{-(l+m)/2} \quad \text{at } \infty.$$

Theorem 4.3 shows that if $\mathbb{G}$ is not compactly generated, the upper rate of decay of the function $n \mapsto \mu^{*2n}(e)$ may not exist in the sense explained above.

5. General case: proof of Theorem 1.1. Let $\mathbb{G}$ be a locally compact non-compact metric abelian group. According to structure theory [13, 24.30],

$$\mathbb{G} = \mathbb{R}^n \times \Gamma,$$

where $n \geq 0$ and the group Γ contains an open compact subgroup $\Gamma_0 \subset \Gamma$. In particular, Γ/Γ_0 is a countable abelian group. We shall consider the following two cases:

1. Assume that $n > 0$. Define a probability measure μ on $\mathbb{G}$ by

$$\mu = \mu_1 \otimes \mu_2,$$

where μ_1 and μ_2 are symmetric admissible probability measures on $\mathbb{R}$ and on $\mathbb{R}^{n-1} \times \Gamma$ respectively. Let $\mu_1(x)$ and $\mu_2(y)$ be their symmetric and continuous densities. Then $\mu(x, y) = \mu_1(x)\mu_2(y)$ is the density of μ. Theorem 2.2 and the equality above show that Theorem 1.1 is true in this case.

2. Assume that $n = 0$. Then $\mathbb{G} = \Gamma$ and Γ/Γ_0 is a countable group, say

$$\Gamma/\Gamma_0 = \{a_0 = \mathrm{id}, a_1, a_2, \ldots\}.$$

The following two cases are possible:

(a) Assume that Γ/Γ_0 contains an element a of infinite order, that is, $a^k \neq \mathrm{id}$ for $k = 1, 2, \ldots$. Let $\langle a \rangle$ be the subgroup of Γ/Γ_0 generated by a. Clearly the mapping

$$\gamma : \mathbb{Z} \to \langle a \rangle, \quad \gamma(n) = a^n,$$

is an isomorphism between the group $\mathbb{Z}$ and $\langle a \rangle$. Let $\pi : \Gamma \to \Gamma/\Gamma_0$ be the canonical homomorphism of Γ onto Γ/Γ_0. Evidently $\pi^{-1}(\langle a \rangle)$ is an open subgroup of Γ. It is clear that the group $\pi^{-1}(\langle a \rangle)$ is generated by the compact set $\Gamma_0 \cup \pi^{-1}(a)$. By structure theory [13, Thm. 9.8],

$$\pi^{-1}(\langle a\rangle) \cong \mathbb{R}^m \times \mathbb{Z}^l \times K,$$

where K is a compact group. Evidently $m = 0$ and since

$$\pi^{-1}(\langle a\rangle)/\Gamma_0 \cong \mathbb{Z},$$

we must have $l = 1$. Thus

$$\pi^{-1}(\langle a\rangle) \cong \mathbb{Z} \times \Gamma_0.$$

Let μ_1 be a symmetric probability measure on $\mathbb{Z}$ and μ_2 be the Haar measure on Γ_0. Consider the probability measure $\mu_1 \otimes \mu_2$ on $\mathbb{Z} \times \Gamma_0$ and lift it to the group $\pi^{-1}(\langle a\rangle)$. Call this lifting μ. Clearly μ will have the property claimed in Theorem 1.1 provided μ_1 is chosen as in Theorem 3.1.

(b) Assume that all elements of the group Γ/Γ_0 are of finite order, i.e. Γ/Γ_0 is a countable periodic group. Let μ be a probability measure on Γ/Γ_0 with values $\mu_i := \mu(\{a_i\})$, $i = 0, 1, 2, \ldots.$ Define a probability measure $\tilde{\mu}$ on Γ as follows: Set $\tilde{\mu}$ on each compact set $\pi^{-1}(a_i)$ to be a uniform distribution such that $\tilde{\mu}(\pi^{-1}(a_i)) = \mu_i$, $i = 0, 1, 2, \ldots.$ Since $\Gamma = \bigcup_{i\geq 0} \pi^{-1}(a_i)$ and the cosets $\pi^{-1}(a_i)$ do not intersect, the definition is correct. Evidently $\tilde{\mu}$ is a probability measure on Γ and $\pi(\tilde{\mu}) = \mu$. It follows (see [9, Proposition 2.4]) that for all $n = 1, 2, \ldots,$

$$\mu^{*n} = (\pi(\tilde{\mu}))^{*n} = \pi(\tilde{\mu}^{*n}).$$

In particular, since $\pi^{-1}(\mathrm{id}) = \Gamma_0$ we obtain

$$\tilde{\mu}^{*n}(\Gamma_0) = \mu^{*n}(\mathrm{id}), \quad n = 1, 2, \ldots.$$

It remains to choose the measure μ on Γ/Γ_0 as in Theorem 4.2 to get the desired result in this last case.

Acknowledgments. This work was started and finished at Bielefeld University (SFB 701). The authors would like to thank Alexander Grigor'yan for his kind invitation. They would like to thank Gennadiy Feldman, Christophe Pittet and Laurent Saloff-Coste for fruitful discussions and valuable comments. The authors are also grateful to the referee for valuable comments and sugestions.

Research of A. Bendikov and B. Bobikau was supported by the Polish Government Scientific Research Fund, Grant N N201 371736.

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Received 17 March 2009;
revised 31 August 2009 (5180)