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A NOTE ON INTEGER TRANSLATES OF A SQUARE INTEGRABLE FUNCTION ON $\mathbb{R}$

BY

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Abstract. We consider the subspace of $L^2(\mathbb{R})$ spanned by the integer shifts of one function ψ, and formulate a condition on the family $\{\psi(\cdot - n)\}_{n=-\infty}^{\infty}$, which is equivalent to the weight function $\sum_{n=-\infty}^{\infty} |\hat{\psi}(\cdot + n)|^2$ being > 0 a.e.

When I first met Andrzej he was just renovating his beautiful house, and everywhere there was a smell of wood, lacquers and the hide glue. It was clear at the first glance that Andrzej was an uncommon person. Brilliant, lively, worldly, with a keen sense of humor, energetic and passionate. His passion was mathematics. He just could not prevent himself from engaging and solving mathematical problems, big and small, theoretic and practical. To further mathematical research he did everything that he felt was necessary. He built houses, laid floors, ran a hotel, confronted the communists, and then the anti-communists. He once told me how he lost the tip of his index finger, cut off by a woodworking machine. This was also a partly mathematical accident. His story began like that: "There was this faculty meeting, and we had a heated discussion. I shouldn't have planed that plank right after that meeting." As an accomplished mathematician he remained modest and humble. I remember I once teased him: "Well, I heard that in your famous paper there are three serious errors". "That might be true", Andrzej replied, "but only one of them was catastrophic". Ever since our first encounter, to me this scent of Andrzej's house from many years ago remains a symbol of dedication, competence and accomplishment.

Introduction. Let us consider the space $L^2(\mathbb{R})$, a function $\psi \in L^2(\mathbb{R})$, and the shift-invariant subspace generated by ψ:

$$<\psi> = \overline{\operatorname{span}\{\psi_n = \psi(\,\cdot - n) : n \in \mathbb{Z}\}}.$$

It is well known (see, for example, [1]–[4]) that properties of the family $\{\psi_n\}$ as the generating set of the subspace $<\psi>$ can be naturally expressed in

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terms of the weight function

$$p_\psi(\xi) = \sum_{n=-\infty}^{\infty} |\hat{\psi}(\xi+n)|^2. \tag{1}$$

Namely, the family $\{\psi_n\}$ is:

(a) an orthonormal basis for $<\psi>$ if and only if $p_\psi \equiv 1$ a.e.,

(b) a Riesz basis for $<\psi>$ if and only if $A \le p_\psi \le B$ a.e. for some positive constants A, B,

(c) a Parseval frame for $<\psi>$ (that is, $\|f\|^2 = \sum |\langle f, \psi_n\rangle|^2$ for all $f \in\, <\psi>$) if and only if $p_\psi = \chi_\Omega$ a.e. for some 1-periodic set $\Omega \subset \mathbb{R}$,

(d) a frame for $<\psi>$ (that is, $A\|f\|^2 \le \sum |\langle f, \psi_n\rangle|^2 \le B\|f\|^2$ for some positive constants A, B and all $f \in\, <\psi>$) if and only if for some positive constants A, B and a 1-periodic set $\Omega \subset \mathbb{R}$,

$$A\chi_\Omega \le p_\psi \le B\chi_\Omega \quad \text{a.e.}$$

(e) a Schauder basis for $<\psi>$ if and only if p_ψ is an $\mathcal{A}_2(\mathbb{T})$ weight.

It is not unreasonable to expect that the condition $p_\psi > 0$ a.e. is in some way related to the linear independence of the family $\{\psi_n\}$, perhaps even equivalent to it. The aim of this note is to formulate a condition in terms of the functions ψ_n, which is equivalent to $p_\psi > 0$ a.e.

The result. Let $\mathcal{B} = \{\psi_n = \psi(\cdot - n) : n \in \mathbb{Z}\}$ for some $\psi \in L^2(\mathbb{R})$. Let us introduce the following definition:

DEFINITION. We say that $\mathcal{B}$ is L^2-*Cesàro linearly independent* if

$$S_n = \sum_{|k|\le n} \alpha_k \psi_k \xrightarrow{n\to\infty} 0$$

in $L^2(\mathbb{R})$ as Cesàro averages (for some sequence $\{\alpha_n\}_{n=-\infty}^{\infty} \in \ell^2$) implies $\alpha_n \equiv 0$ for all $n \in \mathbb{Z}$.

REMARKS. (a) The convergence as Cesàro averages means

$$\frac{1}{n}\sum_{k=0}^{n-1} S_k \xrightarrow{n\to\infty} 0 \quad \text{in } L^2(\mathbb{R}).$$

(b) If $\mathcal{B}$ is an orthonormal family, or a Riesz family, then clearly it is L^2-Cesàro linearly independent. On the other hand, if it is L^2-Cesàro linearly independent, then it is linearly independent in the usual sense. Hence this condition is placed somewhere in between.

(c) A disappointing feature of the above definition is that the notion of L^2-Cesàro linear independence depends on the ordering of the set $\mathcal{B}$.

(d) The definition can be clearly carried over to the setting of an abstract Hilbert space, and a countable set of vectors. However, because of its dependence on a particular ordering of the family, it is probably of little use in such an abstract setting.

Recall the weight function (1),

$$p_\psi(\xi) = \sum_{n=-\infty}^{\infty} |\hat{\psi}(\xi+n)|^2,$$

and let

$$\Omega_\psi = \{\xi \in [0,1] : p_\psi(\xi) > 0\}.$$

THEOREM. *$\mathcal{B}$ is L^2-Cesàro linearly independent if and only if $|\Omega_\psi| = 1$.*

Proof. ($\Rightarrow$) Suppose $|\Omega_\psi| < 1$, that is, $|\Omega_\psi^c| > 0$. Let $\chi(\xi) = \chi_{\Omega_\psi^c}(\xi)$, the characteristic function of the complement in $[0,1]$ of Ω_ψ, and let α_k be its Fourier coefficients. Further, let

$$S_n(\xi) = \sum_{|k|\le n} \alpha_k e^{2\pi i\, k\xi}$$

be the partial sum of the Fourier series of χ, and let

$$F_n(\xi) = \frac{1}{n}\sum_{k=0}^{n-1} S_n(\xi)$$

be the Fejér mean. We thus have

$$F_n \to \chi \quad \text{a.e.}$$

by the Fejér–Lebesgue theorem (see, for example, [5]), and

$$|F_n(\xi)| \le 1,$$

by the properties of the Fejér kernel. Recall that $p_\psi \in L^1([0,1])$, and thus we can apply the dominated convergence theorem to obtain

$$\int_0^1 |F_n(\xi)|^2 p_\psi(\xi)\, d\xi \to \int_0^1 \chi(\xi) p_\psi(\xi)\, d\xi = 0. \tag{2}$$

Now it is sufficient to "de-Fourier" these integrals:

$$\begin{aligned}\int_0^1 |F_n(\xi)|^2\, p_\psi(\xi)\, d\xi &= \int_{-\infty}^{\infty} |F_n(\xi)\hat{\psi}(\xi)|^2\, d\xi \\ &= \int_{-\infty}^{\infty} \left| \frac{1}{n}\sum_{k=0}^{n-1}\sum_{|l|\le k} \alpha_l e^{2\pi i\, l\xi}\hat{\psi}(\xi)\right|^2 d\xi \\ &= \int_{-\infty}^{\infty} \left| \frac{1}{n}\sum_{k=0}^{n-1}\sum_{|l|\le k} \alpha_l (\psi_{-l})\widehat{\ }(\xi)\right|^2 d\xi\end{aligned} \tag{3}$$

$$= \int_{-\infty}^{\infty} \left| \frac{1}{n} \sum_{k=0}^{n-1} \sum_{|l| \le k} \alpha_l \psi_{-l}(x) \right|^2 dx$$

$$= \left\| \frac{1}{n} \sum_{k=0}^{n-1} \sum_{|l| \le k} \alpha_l \psi_{-l} \right\|^2 \xrightarrow{n \to \infty} 0,$$

by (2). We have thus found a non-zero sequence $\{\alpha_n\} \in \ell^2$ of coefficients with which the ψ_l's are Cesàro summable to 0 in $L^2(\mathbb{R})$. This violates the definition of L^2-Cesàro linear independence.

($\Leftarrow$) Now, suppose $|\Omega_\psi| = 1$, that is,

$$p_\psi(\xi) > 0 \quad \text{a.e.}$$

Let $\{\alpha_n\}_{n=-\infty}^{\infty}$ be any sequence of coefficients in ℓ^2, satisfying

$$\frac{1}{n} \sum_{k=0}^{n-1} \sum_{|l| \le k} \alpha_l \psi_{-l} \xrightarrow{n \to \infty} 0 \quad \text{in } L^2(\mathbb{R}). \tag{4}$$

Applying the Fourier transform to (4), as in (3), we obtain

$$\int_0^1 |F_n(\xi)|^2 p_\psi(\xi)\, d\xi \to 0 \quad \text{as } n \to \infty,$$

where $F_n(\xi)$ is the Fejér mean

$$F_n(\xi) = \frac{1}{n} \sum_{k=0}^{n-1} \sum_{|l| \le k} \alpha_l e^{2\pi i l \xi}.$$

Since $\{\alpha_n\}_{n=-\infty}^{\infty} \in \ell^2$ these means converge a.e. to some $f \in L^2(\mathbb{T})$:

$$F_n(\xi) \xrightarrow{n \to \infty} f(\xi) \quad \text{a.e., } f \in L^2(\mathbb{T}).$$

We can now use Fatou's lemma:

$$\int_0^1 |f(x)|^2 p_\psi(\xi)\, d\xi \le \liminf_{n \to \infty} \int_0^1 |F_n(\xi)|^2 p_\psi(\xi)\, d\xi = 0.$$

Thus, $|f(\xi)|^2\, p_\psi(\xi) = 0$ a.e. By the assumption $p_\psi(\xi) > 0$ a.e., we must have $f(\xi) = 0$ a.e., and so $\alpha_n = 0$ for all $n = 0, \pm 1, \pm 2, \ldots$, since these are the Fourier coefficients of f. ∎

Remark. A more natural extension of the notion of linear independence would be the following:

$$\sum_n \alpha_n \psi_n = 0 \text{ in } L^2(\mathbb{R}) \ \Rightarrow\ \forall n,\ \alpha_n = 0.$$

The reason that we have settled on an (apparently) stronger condition is purely technical.

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