COLLOQUIUM MATHEMATICUM

VOL. 118 2010 NO. 2

ERGODIC THEOREM, REVERSIBILITY AND THE FILLING SCHEME

BY

YVES DERRIENNIC (Brest)

Dedicated to the memory of Andrzej Hulanicki

Abstract. The aim of this short note is to present in terse style the meaning and consequences of the "filling scheme" approach for a probability measure preserving transformation. A cohomological equation encapsulates the argument. We complete and simplify Woś' study (1986) of the reversibility of the ergodic limits when integrability is not assumed. We give short and unified proofs of well known results about the behaviour of ergodic averages, like Kesten's lemma (1975). The strikingly simple proof of the ergodic theorem in one dimension given by Neveu (1979), without any maximal inequality nor clever combinatorics, followed this approach and was the starting point of the present study.

1. Introduction. In the proof of their famous ergodic theorem for positive L^1-contractions Chacon and Ornstein (1960) introduced the "filling scheme" method. During subsequent years it became fashionable and was analysed and developed by several authors. Garsia proved in ten lines Hopf's maximal inequality [3]. By a remarkable insight into the method, in 1979, Neveu gave a direct and very simple proof of Chacon–Ornstein's theorem, without using any maximal inequality [9]. During the eighties Woś used this method to specify Birkhoff's ergodic theorem on several points, giving new proofs of results of Kesten and Tanny [12].

The aim of this short note is to present in terse style the meaning and consequences of the "filling scheme" approach in the basic situation of ergodic theory, that is, for a probability measure preserving transformation. Motivations to write this note are several. We complete and simplify Woś' study of the reversibility of the ergodic limits when integrability is not assumed. We give short and unified proofs of well known results about the behaviour of ergodic averages, which have never been gathered together. We try to explain the origin of the strikingly simple proof of the ergodic theorem in one dimension given by Neveu, without any maximal inequality

2010 *Mathematics Subject Classification*: 37A05, 28D.
Key words and phrases: Birkhoff's theorem, filling scheme.

DOI: 10.4064/cm118-2-16

or clever combinatorics, following his work on the filling scheme. Since its appearance in the book [5], this proof has become popular. It shows how special are one-dimensional actions with respect to the ergodic theorem; for a view on actions of more general groups we refer to [11].

2. The filling scheme equation. Let T be a measure preserving transformation of a probability space $(X, \mathfrak{X}, \mu)$. The filling scheme idea can be expressed in a compact form by one cohomological equation that we present now.

Given a real measurable function f put

$$F_n = \max_{1 \le k \le n} \sum_{j=0}^{k-1} f \circ T^j.$$

Since T commutes with the lattice operations we have $F_{n+1}(x) - F_n \circ T(x) = f(x)$ if $F_{n+1}(x) > f(x)$, that is, if $F_n \circ T(x) > 0$. Therefore

$$F_{n+1}(x) - F_n \circ T(x) = f(x) + F_n^- \circ T(x).$$

We recall the notations $f^+ = \max(f, 0)$ and $f^- = (-f)^+$. Using the decomposition of F_{n+1} into positive and negative parts, this relation becomes

$$f = -F_{n+1}^- + F_{n+1}^+ - TF_n^+$$

where we use the operator notation Tf instead of $f \circ T$, as we shall do from now on. The expression $F_{n+1}^+ - TF_n^+$ can be seen as an approximate coboundary. Since $f^- \ge F_n^-$ the set where $\lim_{n\to\infty} \uparrow F_n = +\infty$ is T-invariant.

Up to now the measure played no role and equalities or inequalities were valid everywhere. If the invariant measure μ is ergodic, the invariant set where $F = \lim_{n\to\infty} \uparrow F_n = +\infty$ has measure 0 or 1.

Taking the limit in the preceding equation we expect a similar relation with an exact coboundary (a function of the type $g - Tg$). That is the crucial point, but we have to avoid $+\infty - \infty$. Thus we define the *filling scheme equation* to be

$$f = -F^- + F^+ - TF^+ \qquad \mu\text{-a.e.},$$

which is valid if

$$F = \sup_{n \ge 1} \sum_{j=0}^{n-1} T^j f = \lim_{n\to\infty} \uparrow F_n < \infty \qquad \mu\text{-}a.e.$$

Let us emphasize that under this condition each of the three terms appearing in the equation is finite. Moreover, $F_n^+ \uparrow F^+$ and $f^- \ge F_n^- \downarrow F^-$ μ-a.e.

3. First consequences of the filling scheme argument. To make ideas transparent we shall assume *ergodicity* throughout, and we shall use the notation

$$A_n(T,f) = \frac{1}{n}\sum_{j=0}^{n-1} T^j f.$$

For any measurable f, the functions $\limsup_n A_n(T,f), \limsup_n |A_n(T,f)|$, $\limsup_n \frac{T^n f}{n}$ are invariant, and therefore constant, finite or not (*idem* with $\liminf$).

(a) (Ergodic theorem) *If* $f \in L^1$ *then* $\lim_n A_n(T,f) = \int f\,d\mu$ μ*-a.e.*

Proof. If $F_n \uparrow +\infty$ then $F_n^- \downarrow 0$, hence $f + TF_n^- = F_{n+1} - TF_n \downarrow f$ μ-a.e. and

$$0 \le \int (F_{n+1} - F_n)\,d\mu = \int (F_{n+1} - TF_n)\,d\mu \xrightarrow[n\to\infty]{} \int f\,d\mu$$

because the measure is invariant.

Therefore $\int f\,d\mu < 0$ implies $F_n \uparrow F < +\infty$ μ-a.e., hence

$$\limsup_n A_n(T,f) \le \limsup_n \frac{1}{n} F_n \le 0 \qquad \mu\text{-a.e.}$$

Now assume $\int f\,d\mu = 0$; for any $\varepsilon > 0$, $\int (f-\varepsilon)\,d\mu < 0$, so it follows that $\limsup_n A_n(T,f) \le \varepsilon$. Reversing signs we get as well $\liminf_n A_n(T,f) \ge -\varepsilon$, hence $\lim_n A_n(T,f) = 0$ μ-a.e. ∎

Comment and remark. This very simple proof is due to Neveu; it is a byproduct of his work on Chacon–Ornstein's theorem [9] and it was included in a lecture note of a course given in Paris in 1981 [10].

For a measurable function f that is bounded below (or above) by a constant, $\lim_n A_n(T,f)$ exists μ-a.e.; it is finite if and only if $f \in L^1$, otherwise it is equal to $+\infty$ (or $-\infty$). To see this, just apply the ergodic theorem to the functions $\min(f,a)$ and let $a \to \infty$.

(b) (Maximal ergodic inequality) *If* $f \in L^1$ *then for every* n,

$$\int_{\{F_n > 0\}} f\,d\mu \ge 0.$$

Proof. Using the equation given above,

$$f = -F_{n+1}^- + F_{n+1}^+ - TF_n^+,$$

we get

$$\int_{\{F_{n+1}>0\}} f = \int F_{n+1}^+ - \int_{\{F_{n+1}>0\}} TF_n^+ \ge \int (F_{n+1}^+ - TF_n^+) = \int (F_{n+1}^+ - F_n^+) \ge 0$$

because the measure is invariant. ∎

Comment. This is essentially Garsia's proof [3] with the simplification due to the commutativity of T with lattice operations. The maximal inequality was not used above in the proof of the ergodic theorem and will not be used in what follows.

(c) *Let f be measurable, not necessarily integrable. If* $\limsup_n A_n(T,f) < \infty$ *μ-a.e. then there exist two measurable functions u and v such that $f = u + v - Tv$ with u bounded above by a constant and $v \geq 0$.*

Proof. If $\limsup_n A_n(T,f) = c \in \mathbb{R}$, put $h = f - (c+\varepsilon)$ with $\varepsilon > 0$. Since $\limsup_n A_n(T,h) = -\varepsilon < 0$ we have

$$H = \sup_{n\geq 1} \sum_{j=0}^{n-1} T^j h < \infty,$$

so the filling scheme equation for h reads

$$h = -H^- + H^+ - TH^+ \quad \text{or} \quad f = (c+\varepsilon) - H^- + H^+ - TH^+.$$

In other words, $f = u+v-Tv$ with $u = (c+\varepsilon) - H^- \leq c+\varepsilon$ and $v = H^+ \geq 0$.

If $\limsup_n A_n(T,f) = -\infty$ μ-a.e., such a decomposition holds as well since the filling scheme equation applies to f itself and reads $f = -F^- + F^+ - TF^+$. ■

Comment. This decomposition will be a main tool in the following. It is implicit in the work of Halász [4] and appears in Woś [12].

4. Reversibility and ergodic limits for non-integrable functions. In this part the transformation T will be assumed to be *invertible* and ergodic. We shall see that for a measurable not necessarily integrable function, the forward and backward ergodic averages converge a.e. simultaneously when one limit is finite and then both limits are equal. When the a.e. limit is infinite on one side, the forward and backward ergodic averages may have different behaviours. Of course, for integrable or one-side-bounded functions, the forward and backward limits are the same by the ergodic theorem.

(a) *Let f be measurable. If* $\limsup_n A_n(T,f) < \infty$ *μ-a.e. then*

$$\limsup_n A_n(T,f) = \liminf_n A_n(T^{-1},f) \quad \mu\text{-a.e.},$$

this value being finite or $-\infty$.

Proof. Write the decomposition $f = u + v - Tv$ with u bounded above by a constant and $v \geq 0$ according to 3(c).

Since u is bounded above, $\lim_n A_n(T,u)$ exists μ-a.e. If $u \in L^1$, this limit is $\int u\, d\mu$; otherwise it is $-\infty$. For any measurable $v \geq 0$, $\liminf_n \frac{T^n v}{n} = 0$ μ-a.e. by ergodicity since almost every orbit will infinitely often visit a set where v is bounded. Thus, if $u \in L^1$, we get

$$\limsup_n A_n(T,f) = \int u\,d\mu - \liminf_n \frac{T^n v}{n} = \int u\,d\mu \qquad \mu\text{-a.e.}$$

Taking backward sums with respect to T^{-1} we have

$$\sum_{j=0}^{n-1} T^{-j} f = \sum_{j=0}^{n-1} T^{-j} u + (T^{-n+1} v - Tv).$$

Since $\liminf_n \frac{T^{-n}v}{n} = 0$ as well, when $u \in L^1$ we get

$$\liminf_n A_n(T^{-1},f) = \int u\,d\mu + \liminf_n \frac{T^{-n+1}v}{n} = \int u\,d\mu \qquad \mu\text{-a.e.}$$

On the other hand, if $u \notin L^1$ we get $\limsup_n A_n(T,f) \le \lim_n A_n(T,u) = -\infty$ and $\liminf_n A_n(T^{-1},f) = -\infty$. ∎

Remark. Changing signs in (a) we see that $\liminf_n A_n(T,f) > -\infty$ implies $\liminf_n A_n(T,f) = \limsup_n A_n(T^{-1},f)$, this value being finite or $+\infty$.

(b) *If $\limsup_n |A_n(T,f)| < \infty$ μ-a.e. then the forward and backward ergodic limits exist, are finite and are equal μ-a.e.:*

$$\lim_n A_n(T,f) = \lim_n A_n(T^{-1},f) \qquad \mu\text{-a.e.}$$

Proof. By application of (a) to f and $-f$ we get

$$\limsup_n A_n(T,f) = \liminf_n A_n(T^{-1},f) \le \limsup_n A_n(T^{-1},f) = \liminf_n A_n(T,f). \ ∎$$

Comment. This result was first proved by Woś in [13] but his argument was different. He used an analysis of trajectories which excluded infinite limits.

(c) *If $\limsup_n A_n(T,f) = c \in \mathbb{R}$ and $\liminf_n A_n(T,f) = -\infty$ then*

$$\liminf_n A_n(T^{-1},f) = c \quad \text{and} \quad \limsup_n A_n(T^{-1},f) = +\infty \quad \mu\text{-a.e.}$$

Proof. Application of 4(a) and 4(b) to T^{-1}. ∎

(d) *Let v be measurable and $v \ge 0$. Then either $\lim_n \frac{T^n v}{n} = 0$ μ-a.e. or $\limsup_n \frac{T^n v}{n} = +\infty$ μ-a.e. Moreover, the backward and forward limits are the same:*

$$\lim_n \frac{T^{-n}v}{n} = 0 \ \mu\text{-a.e.} \quad \text{if and only if} \quad \lim_n \frac{T^n v}{n} = 0 \ \mu\text{-a.e.}$$

Proof. Assume $\limsup_n \frac{T^n v}{n} < \infty$ μ-a.e. Then with $f = v - Tv$, we have $\limsup_n |A_n(T,f)| < \infty$. By (b),

$$\lim_n \frac{1}{n}(v - T^n v) = \lim_n \frac{1}{n}(T^{-n+1}v - Tv),$$

which is finite μ-a.e. Since $\liminf_n \frac{T^n v}{n} = 0$, we get $\lim_n \frac{T^n v}{n} = \lim_n \frac{T^{-n} v}{n} = 0$ μ-a.e. ∎

(e) (Reversibility for $f \notin L^1$, the last two cases) *If* $\limsup_n A_n(T, f) = +\infty$ *and* $\liminf_n A_n(T, f) = -\infty$ *then two cases are possible: either*

$$\limsup_n A_n(T^{-1}, f) = +\infty \quad \textit{and} \quad \liminf_n A_n(T^{-1}, f) = -\infty \quad \mu\textit{-a.e.}$$

or $\lim_n A_n(T^{-1}, f)$ *exists* μ*-a.e. and is infinite* ($+\infty$ or $-\infty$).

Proof. By contraposition, using (b) or (c) for T^{-1} and f or $-f$, the sequence $A_n(T^{-1}, f)$ cannot have a finite limsup or liminf. So we just have to show that both cases are actually possible.

Taking two non-negative functions u and v with disjoint supports such that $\limsup_n \frac{T^n u}{n} = \limsup_n \frac{T^n v}{n} = +\infty$, examples of which are easy to build, we get the first case with $f = u - Tu + Tv - v$.

The second case requires a little more attention. First we observe that two non-negative functions u and v with $\int u \, d\mu = +\infty$, $\limsup_n \frac{T^n v}{n} = +\infty$ and moreover $\limsup_n (\frac{T^n v}{n} - A_n(T, u)) = +\infty$, will lead to a function $f = u + v - Tv$ that is an example of the second case.

Now consider a sequence of i.i.d. positive random variables $(X_n)_{n\geq 1}$ with a stable law of index $0 < \alpha < 1$. Put $S_n = \sum_{k=1}^n X_k$ and $M_n = \max(X_1, \ldots, X_n)$. It is known ([12, Chap. 13, Ex. 20]) that

$$\liminf_n \frac{S_n}{M_n} \leq \frac{1}{1-\alpha} \quad \text{a.s.} \quad \text{and} \quad \lim_n \frac{M_n}{n} = +\infty \quad \text{a.s.}$$

This implies

$$\limsup_n \frac{M_n}{n}\left(c - \frac{S_n}{M_n}\right) = +\infty \quad \text{a.s.} \quad \text{if } c > \frac{1}{1-\alpha}.$$

Almost surely there is an increasing sequence of integers n_i along which $M(n_i) = X(n_i)$; for $n_i < n < n_{i+1}$ we have

$$\frac{M_n}{n}\left(c - \frac{S_n}{M_n}\right) \leq \frac{M(n_i)}{n_i}\left(c - \frac{S(n_i)}{M(n_i)}\right),$$

thus

$$\limsup_n \frac{X_n}{n}\left(c - \frac{S_n}{X_n}\right) = +\infty \quad \text{a.s.}$$

Therefore on the Bernoulli scheme built with this i.i.d. sequence the functions $u = X_1$ and $v = cX_1$ achieve the needed conditions. ∎

Comment. In his work about ergodic limits without integrability assumption and reversibility [13], Woś did not treat the case where the limits are infinite so the existence of these two cases stayed unnoticed.

It is not quite clear how to build an example of the second case on any ergodic dynamical system with zero entropy. In [1] by a direct method it is shown that the preceding example still holds for a set of random variables including the domain of attraction of a stable law of index $0 < \alpha < 1$.

5. Other consequences of the filling scheme argument. In this part we do not assume the invertibility of the transformation T any more, but keep ergodicity. We shall use the filling scheme argument to give short proofs of results which were originally proved by different methods.

(a) (Cohomology class of a measurable function whose ergodic averages converge) *Let f be measurable. If $\lim_n A_n(T,f) = c \in \mathbb{R}$ μ-a.e. then for every $\varepsilon > 0$ there exist two measurable functions g and h such that*

$$f = g + h - Th, \quad |g - c| \le \varepsilon, \quad \textit{and} \quad \lim_n \frac{T^n h}{n} = 0 \quad \mu\text{-a.e.}$$

Proof. Assume $c = 0$. It follows from the proof of 3(c) that any measurable function f for which $\lim_n A_n(T,f) = 0$ μ-a.e. admits a decomposition $f = u + v - Tv$ with $u = \varepsilon - H^- \le \varepsilon$, $H^- \le (f-\varepsilon)^-$ and $v \ge 0$. Moreover, we necessarily have $u \in L^1$ since otherwise the limit would be $-\infty$. Then $\int u\,d\mu = 0$ and $\lim_n \frac{T^n v}{n} = 0$ μ-a.e.

Now write the analogous decomposition for the function $-u$. It reads $-u = (\varepsilon - J^-) + w - Tw$ where $J^- \le (-u-\varepsilon)^- = \max(u+\varepsilon, 0) \le 2\varepsilon$. It just remains to put $g = -\varepsilon + J^-$ and $h = v - w$. ∎

Comment. When $f \in L^1$ this is a well known result due to Kochergin [7]. When f is only measurable it appears in Woś [13], without the estimate on the size of g. Note that the sign of the second function h cannot be chosen.

(b) (Sums of stationary sequences cannot grow slower than linearly; Kesten [6]) *Let f be measurable. If on a set of positive measure we have the strict inequality $\sum_{j=0}^{n-1} T^j f > 0$ for all n large enough, then $\liminf_n A_n(T,f) > 0$ μ-a.e. (this liminf is $c > 0$ or $+\infty$).*

Proof. To use the filling scheme equation, we change signs. If $\sum_{j=0}^{n-1} T^j f < 0$ for all n large enough on a set of positive measure, then on this set

$$F = \sup_n \sum_{j=0}^{n-1} T^j f < \infty.$$

By ergodicity we get $F < \infty$ μ-a.e. on X. Then we can write

$$f = -F^- + F^+ - TF^+ \quad \text{and} \quad A_n(T,f) = -A_n(T,F^-) + \frac{1}{n}(F^+ - T^n F^+).$$

If we had $\limsup_n A_n(T,f) = 0$ μ-a.e. we would get $F^- \in L^1$ with $\int F^-\,d\mu = 0$, therefore $F^- = 0$ μ-a.e. and $f = F^+ - TF^+$. But it is an easy consequence of Poincaré's recurrence theorem that the strict inequality $F^+ < T^n F^+$ cannot be true for all n large enough on a set of positive measure. So the assumption on the signs of $\sum_{j=0}^{n-1} T^j f$ implies $\limsup_n A_n(T,f) < 0$ and

$F^- \neq 0$ μ-a.e. If $F^- \in L^1$, $\limsup_n A_n(T,f) = -\int F^- \, d\mu < 0$, otherwise $\limsup_n A_n(T,f) = -\infty$. ■

Comment. The L^1-version, "if $f \in L^1$ and $\sum_{j=0}^{n-1} T^j f \to +\infty$ μ-a.e. then $\int f \, d\mu > 0$" is sometimes called the lemma of Guivarc'h and Raugi.

(c) (Infinite oscillations of ergodic averages toward their limit; Halász [4], Marcus and Petersen [8]) *Let f be measurable. If $\limsup_n A_n(T,f) = c \in \mathbb{R}$ then almost surely the sequence $A_n(T,f) - c$ changes sign infinitely often in the wide sense (that is, it cannot be ultimately strictly positive or negative).*

Proof. Corollary of the preceding.

(d) (Another proof of the statement 4(d)) *Let v be measurable and $v \geq 0$. If $\limsup_n \frac{T^n v}{n} < \infty$ μ-a.e., then there exists a measurable function w such that $w \geq v$ and $Tw - w \in L^1$. Hence either $\lim_n \frac{T^n v}{n} = 0$ μ-a.e. or $\limsup_n \frac{T^n v}{n} = +\infty$ μ-a.e.*

Proof. Using invertibility this conclusion was already obtained in 4(d) Here we do not assume invertibility.

If $\limsup_n \frac{T^n v}{n} = c \in \mathbb{R}$ μ-a.e., put $h = Tv - v - (c + \varepsilon)$ with $\varepsilon > 0$. The filling scheme equation for h reads $h = -H^- + H^+ - TH^+$ since $\limsup_n A_n(T,h) = -\varepsilon$ implies $H = \sup_{n\geq 1} \sum_{j=0}^{n-1} T^j h < \infty$ μ-a.e. Moreover, $H^- \in L^1$ because otherwise we would get $\lim_n A_n(T,h) = -\infty$. Therefore the function $w = v + H^+$ satisfies $Tw - w = -H^- + (c+\varepsilon) \in L^1$.

The conclusion is now easy since $T^n v \leq T^n w = \sum_{j=0}^{n-1} T^j (Tw - w) + w$; by the ergodic theorem, $\lim_n A_n(T, Tw - w)$ exists μ-a.e., and it is zero because this limit is obviously zero in probability. ■

Remark. This is due to Woś with essentially the same proof [12].

6. Final comments and remarks. In the preceding study the ergodicity assumption is just a convenience. To drop it is an exercise.

The filling scheme method was conceived to deal with positive contractions of L^1 that do not commute with lattice operations; in that general situation several equalities are replaced by inequalities and the heart of the method cannot be reduced to just one equation as is the case for a pointwise transformation. For more details see [9] or [12] and their references. For a pointwise transformation the idea to look at $\sup_{n\geq 1} \sum_{j=0}^{n-1} T^j f$ is, of course, quite old. For instance the classical theorem of Gottschalk and Hedlund asserting that if $\sup_{n\geq 1} |\sum_{j=0}^{n-1} T^j f|$ is bounded then f is a coboundary is much older than Chacon–Ornstein's theorem (see [5, p. 102, Th. 2.9.3], whose proof is defective, but can be easily fixed by observing that if $L = \limsup_n \sum_{j=0}^{n-1} T^j f$ satisfies $-\infty < L < \infty$ then $f = L - TL$). Yet the direct proof of Birkhoff's theorem given above in 3(a) was found by Neveu as a byproduct of a study of the general filling scheme. Since it appeared in the book [5] (p. 136) it has been reproduced in several other publications without proper reference.

Already the famous little book "Lectures on Ergodic Theory" by Halmos included an example of a measurable non-integrable function whose ergodic averages converge to a finite limit a.e. (p. 32). Yet it is well known that L^1 gives the best possible assumption for Birkhoff's theorem, in the sense of distributions of the functions. The description of the set of measurable functions whose ergodic averages converge to a finite limit a.e. was found by Woś (see 5(a)) who also observed that the forward and backward ergodic averages both converge a.e. and the limits are equal when they are finite. The approach used above is somewhat simpler than Woś' and allows the treatment of infinite limits, thanks to the systematic use of the filling scheme equation.

Some complementary remarks are in order concerning the convergence in probability. For any measurable f it is obvious that $\lim_n \frac{T^n f}{n} = 0$ in probability. It is also obvious that the forward and backward limits are the same for this mode of convergence. A direct byproduct of the preceding study is that $A_n(T, f)$ may not converge in probability (to a finite or infinite limit) only when $\limsup_n A_n(T, f) = \limsup_n A_n(T^{-1}, f) = +\infty$ μ-a.e. and $\liminf_n A_n(T, f) = \liminf_n A_n(T^{-1}, f) = -\infty$ μ-a.e. Yet we do not know the exact characterization of the set of measurable functions whose ergodic averages converge in probability.

Acknowledgements. The author thanks M. Boshernitzan, M. Hochman, E. Lesigne and B. Weiss for discussions and comments on the matter of this note.

REFERENCES

[1] J. P. Conze, Y. Guivarc'h and A. Raugi, *Dépassement des sommes partielles pour des v.a. indépendantes sans moment*, Publ. IRMAR, Univ. Rennes I, 1998.

[2] W. Feller, *An Introduction to Probability Theory and Its Applications*, Vol. 2, 2nd ed., Wiley, 1966.

[3] A. Garsia, *A simple proof of E. Hopf's maximal ergodic theorem*, J. Math. Mech. 14 (1965), 381–382.

[4] G. Halász, *Remarks on the remainder in Birkhoff's ergodic theorem*, Acta Math. Acad. Sci. Hungar. 28 (1976), 389–395.

[5] A. Katok and B. Hasselblatt, *Introduction to the Modern Theory of Dynamical Systems*, Cambridge Univ. Press, 1995.

[6] H. Kesten, *Sums of stationary sequences cannot grow slower than linearly*, Proc. Amer. Math. Soc. 49 (1975), 205–211.

[7] A. V. Kochergin, *The homology of functions over dynamical systems*, Dokl. Akad. Nauk SSSR 231 (1976), 795–798 (in Russian); English transl.: Soviet Math. Dokl. 17 (1976), 1637–1641.

[8] B. Marcus and K. Petersen, *Balancing ergodic averages*, in: Lecture Notes in Math. 729, Springer, Berlin, 1979, 126–143.

[9] J. Neveu, *The filling scheme and the Chacon–Ornstein theorem*, Israel J. Math. 33 (1979), 368–377 (1980).

[10] —, *Introduction aux processus aléatoires*, Labo. de Probabilités, Université P. et M. Curie, Paris, 1981–82.

[11] A. Nevo, *Pointwise ergodic theorems for actions of groups*, in: Handbook of Dynamical Systems, Vol. 1B, Elsevier, 2006, 871–982.

[12] J. Woś, *The filling scheme and the ergodic theorems of Kesten and Tanny*, Colloq. Math. 52 (1987), 263–276.

[13] —, *Approximate convergence in ergodic theory*, Proc. London Math. Soc. (3) 53 (1986), 65–84.

Yves Derriennic
U.E.B., Mathématiques (U.M.R. CNRS 6205)
Université de Brest
6 av. Le Gorgeu
29238 Brest, France
E-mail: derrienn@univ-brest.fr

Received 9 April 2009;
revised 26 November 2009 (5210)