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CONVERGENCE TO STABLE LAWS AND A LOCAL LIMIT THEOREM FOR STOCHASTIC RECURSIONS

BY

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Dedicated to the memory of Andrzej Hulanicki—my advisor

Abstract. We consider the random recursion $X_n^x = M_n X_{n-1}^x + Q_n + N_n(X_{n-1}^x)$, where $x \in \mathbb{R}$ and (M_n, Q_n, N_n) are i.i.d., Q_n has a heavy tail with exponent $\alpha > 0$, the tail of M_n is lighter and $N_n(X_{n-1}^x)$ is smaller at infinity, than $M_n X_{n-1}^x$. Using the asymptotics of the stationary solutions we show that properly normalized Birkhoff sums $S_n^x = \sum_{k=0}^n X_k^x$ converge weakly to an α-stable law for $\alpha \in (0, 2]$. The related local limit theorem is also proved.

1. Introduction. We assume that $(M_n, Q_n, N_n)_{n\in\mathbb{N}}$ with $M_n, Q_n > 0$ and $N_n : \mathbb{R} \to \mathbb{R}_+$ is a sequence of independent random triples identically distributed according to the measure μ. Moreover, we assume that

$$\psi_n(x) = M_n x + Q_n + N_n(x) \quad \text{for } x \in \mathbb{R},$$

is Lipschitz with Lipschitz constant L_n and we consider the stochastic recursion

$$X_n^x = \psi_n(X_n^x) = M_n X_{n-1}^x + Q_n + N_n(X_{n-1}^x), \tag{1.1}$$

where $X_0^x = x \in \mathbb{R}$. We are interested in the asymptotic behaviour of the Birkhoff sums $S_n^x = \sum_{k=0}^n X_k^x$. We are going to show that S_n^x normalized appropriately converge to an α-stable random variable (see Theorem 1.7). We also prove a related local limit theorem (see Theorem 1.12). Throughout the paper we will assume that the sequence $(M_n, Q_n, N_n)_{n\in\mathbb{N}}$ satisfies the hypotheses of the theorem stated below.

THEOREM 1.2 (Grey [5]). *Let $(M, Q, N) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+$ be a generic triple of the sequence above. Let ψ be a random nondecreasing Lipschitz function with Lipschitz constant $L < \infty$ and let*

$$\psi(x) = Mx + Q + N(x) \quad \textit{for } x \in \mathbb{R}. \tag{1.3}$$

Assume that

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(1) $\mathbb{E}(L^\alpha) < 1$, $\mathbb{E}(L^\beta) < \infty$ *and* $\mathbb{E}(N^\beta) < \infty$ *for some* $0 < \alpha < \beta$.
(2) *The tails of* Q *satisfy*

$$\mathbb{P}(\{Q > t\}) \sim ct^{-\alpha} \quad \text{as } t \to \infty, \text{ for some constant } c > 0.$$

(3) $N(x) \le N\phi(x)$ *for every* $x \in \mathbb{R}$, *where* ϕ *is a fixed nondecreasing nonnegative function such that* $\phi(x) = o(x)$ *as* $x \to \pm\infty$.

Then there exists a unique stationary solution S *of the above equation with law* ν *such that*

$$\mathbb{P}(\{S > t\}) \sim \frac{c}{1 - \mathbb{E}(M^\alpha)}\, t^{-\alpha} \quad \text{as } t \to \infty. \tag{1.4}$$

The proof goes along the same lines as in [5] and we write here only what is specific to the current situation.

It is easy to see that $\mathbb{E}(L^\alpha) < 1$ and $\mathbb{E}(L^\beta) < \infty$ imply respectively $\mathbb{E}(M^\alpha) < 1$ and $\mathbb{E}(M^\beta) < \infty$. (1.4) implies immediately that for every bounded continuous function f,

$$\lim_{t\to\infty} t^\alpha \int_{\mathbb{R}} f(t^{-1}x)\, \nu(dx) = \int_{\mathbb{R}} f(x)\, \Lambda(dx), \quad \text{where} \tag{1.5}$$

$$\Lambda(dx) = C_- \mathbf{1}_{(-\infty,0)}(x)\, \frac{dx}{|x|^{\alpha+1}} + C_+ \mathbf{1}_{(0,\infty)}(x)\, \frac{dx}{x^{\alpha+1}}, \tag{1.6}$$

and $C_- = 0$ and $C_+ = \alpha c/(1 - \mathbb{E}(M^\alpha))$.

One of our main results is the following:

THEOREM 1.7. *Assume that the random variables* M, N *and* Q *satisfy the hypotheses of the previous theorem and* S *is the stationary solution of* (1.1) *with law* ν. *Additionally, assume that the function* ϕ *of Theorem* 1.2 *is bounded. Let* $S_n^x = \sum_{k=0}^n X_k^x$ *for* $n \in \mathbb{N}$, $m = \int_{\mathbb{R}} x\, \nu(dx)$ *and* $W = \sum_{k=1}^\infty M_1 \cdot \ldots \cdot M_k$ *with law* η.

- *If* $0 < \alpha < 1$ *and* Δ_α^n *is the characteristic function of the random variable* $n^{-1/\alpha} S_n^x$ *for* $n \in \mathbb{N}$, *then*

$$\lim_{n\to\infty} \Delta_\alpha^n(t) = \Upsilon_\alpha(t) = \exp(t^\alpha C_\alpha), \tag{1.8}$$

where

$$C_\alpha = \alpha c \vartheta_\alpha \mathbb{E}((W+1)^\alpha) \quad \text{and} \quad \vartheta_\alpha = -\frac{\Gamma(1-\alpha)}{\alpha}\, e^{-i\alpha\pi/2}.$$

- *If* $\alpha = 1$ *and* Δ_1^n *is the characteristic function of the random variable* $n^{-1} S_n^x - n\xi(n^{-1})$ *for* $n \in \mathbb{N}$, *then*

$$\lim_{n\to\infty} \Delta_1^n(t) = \Upsilon_1(t) = \exp(tC_1 - iC_+ t \log t), \tag{1.9}$$

where

$$\xi(t) = \int_{\mathbb{R}} \frac{tx}{1+t^2x^2}\,\nu(dx),$$

$$C_1 = C_+\vartheta_1 - iC_+\mathbb{E}((W+1)\log(W+1) - W\log W)$$

and $\vartheta_1 = -\pi/2 + i\kappa$ *where* $\kappa > 0$.

- *If* $1 < \alpha < 2$ *and* Δ_α^n *is the characteristic function of the random variable* $n^{-1/\alpha}(S_n^x - nm)$ *for* $n \in \mathbb{N}$, *then*

$$\lim_{n\to\infty} \Delta_\alpha^n(t) = \Upsilon_\alpha(t) = \exp(t^\alpha C_\alpha), \tag{1.10}$$

where

$$C_\alpha = \alpha c\vartheta_\alpha \mathbb{E}((W+1)^\alpha) \quad \textit{and} \quad \vartheta_\alpha = \frac{\Gamma(2-\alpha)}{\alpha(\alpha-1)}\,e^{-i\alpha\pi/2}.$$

- *If* $\alpha = 2$ *and* Δ_2^n *is the characteristic function of the random variable* $(n\log n)^{-1/2}(S_n^x - nm)$ *for* $n \in \mathbb{N}$, *then*

$$\lim_{n\to\infty} \Delta_2^n(t) = \Upsilon_2(t) = \exp(t^2 C_2), \tag{1.11}$$

where $C_2 = -(c/2)\mathbb{E}((W+1)^2)$. *Moreover,* $\Re C_\alpha < 0$ *for all* $\alpha \in (0,2]$.

Similar problems have recently been investigated in the context of affine recursion by Guivarc'h and LePage [6] (one-dimensional case), by Buraczewski, Damek and Guivarc'h [2] (multidimensional matrix case) and by Mirek [11] (for some class of Lipschitz maps close to affine at infinity) when Kesten's conditions are satisfied [10], i.e. $\mathbb{E}(|M|^\alpha) = 1$ and $\mathbb{E}(|Q|^\alpha) < \infty$ for some $\alpha > 0$ (see also [4] for simplifications and [3] for generalizations). The proof of the theorem stated above is based on spectral properties of the transition operator P and its Fourier perturbations P_t (see Section 2.1 for precise definitions). The spectral method was initiated by Nagaev [12] and then used and developed by many authors (for more references see especially [7] and [8]; see also [2], [6] and [11]). The most important tool in the proof is the perturbation theorem of Keller and Liverani [9], which allows us to show that the operators P_t have similar spectral properties to the operator P for sufficiently small values of $|t|$ (see Proposition 2.6). We also conclude that the behaviour of the large powers of the operator P_t is determined by the peripheral eigenvalue $k(t)$ associated with this operator. The eigenvalues $k(t)$ appear naturally in the expansions of the characteristic functions of appropriately normalized Birkhoff sums. The asymptotic behaviour (1.4) of the stationary measure ν allows us to expand the dominant eigenvalue $k(t)$ at 0, which is crucial for Theorem 1.7.

Now we have the following

THEOREM 1.12. *Assume that* $|\mathbb{E}(e^{itS})| < 1$ *for every* $t \neq 0$. *Suppose that the hypotheses of the previous theorem are satisfied. Then*

$$\lim_{n\to\infty} n^{1/\alpha}\mathbb{P}(\{S_n^x \in I\}) = \frac{|I|}{2\pi}\int_{\mathbb{R}} \Upsilon_\alpha(t)\,dt \qquad \textit{if } \alpha \in (0,1),$$

$$\lim_{n\to\infty} n^{1/\alpha}\mathbb{P}(\{S_n^x - nm \in I\}) = \frac{|I|}{2\pi}\int_{\mathbb{R}} \Upsilon_\alpha(t)\,dt \qquad \textit{if } \alpha \in (1,2),$$

for every bounded interval $I \subseteq \mathbb{R}$, where $|I|$ denotes the Lebesgue measure of I.

The proof of the above theorem is based on the ideas from [2]. Theorem 3.1 below, which says that the spectral radius of the operators P_t for $t \neq 0$ is strictly smaller than 1, plays a crucial role in the proof. It also shows an interesting connection between the operators P_t and the stationary solution S of (1.1). The rest of the proof of Theorem 1.12 strongly uses the asymptotic properties and expansions of the dominant eigenvalues $k(t)$.

2. Limit theorem

2.1. Fourier operators. We start by introducing two Banach spaces $\mathcal{C}_\rho(\mathbb{R})$ and $\mathcal{B}_{\rho,\epsilon,\lambda}(\mathbb{R})$ cointained in the space $\mathcal{C}(\mathbb{R})$ of continuous functions:

$$\mathcal{C}_\rho = \mathcal{C}_\rho(\mathbb{R}) = \left\{ f \in \mathcal{C}(\mathbb{R}) : |f|_\rho = \sup_{x\in\mathbb{R}} \frac{|f(x)|}{(1+|x|)^\rho} < \infty \right\},$$

$$\mathcal{B}_{\rho,\epsilon,\lambda} = \mathcal{B}_{\rho,\epsilon,\lambda}(\mathbb{R}) = \{ f \in \mathcal{C}(\mathbb{R}) : \|f\|_{\rho,\epsilon,\lambda} = |f|_\rho + [f]_{\epsilon,\lambda} < \infty \},$$

where

$$[f]_{\epsilon,\lambda} = \sup_{x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\epsilon (1+|x|)^\lambda (1+|y|)^\lambda}.$$

Remark 2.1. If $\epsilon + \lambda < \rho$, then by the Arzelà–Ascoli theorem the injection operator $\mathcal{B}_{\rho,\epsilon,\lambda} \hookrightarrow \mathcal{C}_\rho$ is compact.

On $\mathcal{C}_\rho$ and $\mathcal{B}_{\rho,\epsilon,\lambda}$ we consider the transition operator

$$Pf(x) = \mathbb{E}(f(Mx + Q + N(x)))$$

and its perturbations

$$P_t f(x) = \mathbb{E}(e^{it(Mx+Q+N(x))} f(Mx + Q + N(x))),$$

where $x \in \mathbb{R}$ and $t \in [-1,1]$. We will also use the Fourier operators

$$T_t f(x) = \mathbb{E}\big(e^{i(Mx+tQ+tN(t^{-1}x))} f(Mx + tQ + tN(t^{-1}x))\big)$$

for $x \in \mathbb{R}$, where $t \in [-1,1]$. Denote $T = T_0 = \mathbb{E}(e^{iMx} f(Mx)))$. Recall that for every $n \in \mathbb{N}$,

$$T^n f(x) = \mathbb{E}(e^{i\sum_{k=1}^n M_k\cdot\ldots\cdot M_1 x} f(M_n \cdot \ldots \cdot M_1 x)),$$

The lemma below shows a connection between the operators P_t and T_t.

LEMMA 2.2. *If $f \in \mathcal{C}_\rho$, then for every $n \in \mathbb{N}$, $x \in \mathbb{R}$ and $t \in [-1, 1]$,*

$$P_t^n(f \circ \delta_t)(x) = T_t^n f(tx). \tag{2.3}$$

Moreover, if $f \in \mathcal{C}_\rho$ is an eigenfunction of T_t with eigenvalue $k(t)$, then $f \circ \delta_t$ is an eigenfunction of P_t with the same eigenvalue.

Proof. A straightforward application of the definitions of P_t and T_t. ∎

LEMMA 2.4. *The unique eigenvalue of modulus 1 for the operator P acting on $\mathcal{C}_\rho$ is 1 and the eigenspace is one-dimensional. The corresponding projection on $\mathbb{C} \cdot 1$ is given by the map $f \mapsto \nu(f)$.*

Proof. See the proof of the lemma below. ∎

LEMMA 2.5. *The unique eigenvalue of modulus 1 for the operator T acting on $\mathcal{C}_\rho$ is 1 with the eigenspace $\mathbb{C} \cdot h(x)$, where $h(x) = \mathbb{E}(e^{iWx})$, and $W = \sum_{k=1}^\infty M_1 \cdot \ldots \cdot M_k$ has law η.*

Proof. Observe that the random variables $\sum_{k=1}^\infty M_k \cdot \ldots \cdot M_1 x$ and $\sum_{k=2}^\infty M_k \cdot \ldots \cdot M_2 x$ have the same law, hence

$$\begin{aligned} Th(x) &= \mathbb{E}(e^{iM_1x} h(M_1x)) = \mathbb{E}(e^{iM_1x} e^{i\sum_{k=2}^\infty M_k \cdot \ldots \cdot M_2(M_1x)}) \\ &= \mathbb{E}(e^{iM_1x} e^{i\sum_{k=2}^\infty M_k \cdot \ldots \cdot M_2 \cdot M_1 x}) = h(x). \end{aligned}$$

This proves that 1 is an eigenvalue for T. Let $f \in \mathcal{C}_\rho$ be such that $T^n f(x) = f(x)$. Since $h(0) = 1$ and $\lim_{n\to\infty} M_n \cdot \ldots \cdot M_1 x = 0$ a.e. we have

$$|f(x) - f(0)h(x)| \xrightarrow{n\to\infty} 0.$$

Hence $f(x) = f(0)h(x)$. Now assume that for a z of modulus 1 and a nontrivial $f \in \mathcal{C}_\rho$ we have $Tf(x) = zf(x)$. Then for every x such that $f(x) \neq 0$ we have $\lim_{n\to\infty} z^n = (f(0)/f(x))h(x)$, which is impossible. ∎

The following proposition summarizes the necessary properties of the operators P_t and T_t.

PROPOSITION 2.6. *Assume that $0 < \epsilon < 1$, $\lambda > 0$, $\lambda + 2\epsilon < \rho = 2\lambda$ and $2\lambda + \epsilon < \alpha$. Then there exist $0 < \varrho < 1$, $\delta > 0$ and $t_0 > 0$ such that $\varrho < 1 - \delta$ and for every $|t| \leq t_0$:*

- $\sigma(P_t), \sigma(T_t) \subset \mathcal{D} = \{z \in \mathbb{C} : |z| \leq \varrho\} \cup \{z \in \mathbb{C} : |z - 1| \leq \delta\}$.
- *The sets $\sigma(P_t) \cap \{z \in \mathbb{C} : |z-1| \leq \delta\}$ and $\sigma(T_t) \cap \{z \in \mathbb{C} : |z-1| \leq \delta\}$ consist of exactly one eigenvalue $k(t)$, the corresponding eigenspace is one-dimensional and $\lim_{t\to 0} k(t) = 1$.*
- *For all $z \in \mathcal{D}^c$ and $f \in \mathcal{B}_{\rho,\epsilon,\lambda}$,*

  $$\|(z - P_t)^{-1} f\|_{\rho,\epsilon,\lambda} \leq D\|f\|_{\rho,\epsilon,\lambda}, \qquad \|(z - T_t)^{-1} f\|_{\rho,\epsilon,\lambda} \leq D\|f\|_{\rho,\epsilon,\lambda},$$

  where $D > 0$ is a universal constant which does not depend on $|t| \leq t_0$.

- *Moreover, for every* $n \in \mathbb{N}$,

$$P_t^n = k(t)^n \Pi_{P,t} + Q_{P,t}^n, \qquad T_t^n = k(t)^n \Pi_{T,t} + Q_{T,t}^n,$$

where $\Pi_{P,t}$ *and* $\Pi_{T,t}$ *are projections onto the above mentioned one-dimensional eigenspaces.* $Q_{P,t}$ *and* $Q_{T,t}$ *are complementary operators to* $\Pi_{P,t}$ *and* $\Pi_{T,t}$ *respectively, such that* $\Pi_{P,t}Q_{P,t} = Q_{P,t}\Pi_{P,t} = 0$ *and* $\Pi_{T,t}Q_{T,t} = Q_{T,t}\Pi_{T,t} = 0$. *Furthermore,* $\|Q_{P,t}\|_{\mathcal{B}_{\rho,\epsilon,\lambda}} \le \varrho$ *and* $\|Q_{T,t}\|_{\mathcal{B}_{\rho,\epsilon,\lambda}} \le \varrho$.
- *The above operators can be expressed in the terms of the resolvents of* P_t *and* T_t. *Indeed, for appropriate parameters* $\xi_1, \xi_2 > 0$,

$$k(t)\Pi_{F,t} = \frac{1}{2\pi i} \int_{|z-1|=\xi_1} z(z-F_t)^{-1}\,dz,$$

$$\Pi_{F,t} = \frac{1}{2\pi i} \int_{|z-1|=\xi_1} (z-F_t)^{-1}\,dz,$$

$$Q_{F,t} = \frac{1}{2\pi i} \int_{|z|=\xi_2} z(z-F_t)^{-1}\,dz,$$

where $F_t = P_t$ *or* $F_t = T_t$.

Proof. A direct application of the perturbation theorem of Keller and Liverani [9], Lemmas 2.4, 2.5 and arguments from [11]. ∎

2.2. Rate of convergence of projections. The main goal of this section is to prove the following

THEOREM 2.7. *Assume that the hypotheses of Proposition* 2.6 *hold and let* h *be the eigenfunction of the operator* T *defined as in Lemma* 2.5. *Then for any* $0 < \delta \le 1$ *and* $\epsilon < \delta < \alpha$ *there exists* $C > 0$ *such that*

$$\|((\Pi_{T,t} - \Pi_{T,0})h) \circ \delta_t\|_{\rho,\epsilon,\lambda} \le C|t|^{\delta} \quad \text{for every } |t| \le t_0. \tag{2.8}$$

We start with

LEMMA 2.9. *Assume that the hypotheses of Proposition* 2.6 *hold and let* h *be the eigenfunction of* T *defined as in Lemma* 2.5. *Then for any* $0 < \delta \le 1$ *and* $\epsilon < \delta < \alpha$ *we have*

$$[(T_t - T)h]_{\epsilon,\lambda} \le C_1|t|^{\delta-\epsilon}, \tag{2.10}$$

$$|(T_t - T)h|_{\rho} \le C_2|t|^{\delta}, \tag{2.11}$$

where $C_1, C_2 > 0$ *do not depend on* t.

Proof. We will estimate the seminorm $[(T_t - T)h]_{\epsilon,\lambda}$. Notice that

$$[(T_t - T)h]_{\epsilon,\lambda} \leq \sup_{x\neq y,\, |x-y|\leq t} \frac{|(T_t - T)h(x) - (T_t - T)h(y)|}{|x-y|^\epsilon(1+|x|)^\lambda(1+|y|)^\lambda} + \sup_{x\neq y,\, |x-y|> t} \frac{|(T_t - T)h(x) - (T_t - T)h(y)|}{|x-y|^\epsilon(1+|x|)^\lambda(1+|y|)^\lambda}. \tag{2.12}$$

For the first term in (2.12) ($|x-y|\leq t$) we observe that

$$\begin{aligned}
&(T_t - T)h(x) - (T_t - T)h(y)\\
&= \mathbb{E}((e^{i(Mx+tQ+tN(t^{-1}x))} - e^{i(My+tQ+tN(t^{-1}y))})h(Mx+tQ+tN(t^{-1}x))) && (2.13)\\
&\quad + \mathbb{E}(e^{i(My+tQ+tN(t^{-1}y))}(h(Mx+tQ+tN(t^{-1}x)) - h(My+tQ+tN(t^{-1}y)))) && (2.14)\\
&\quad - \mathbb{E}((e^{iMx} - e^{iMy})h(Mx)) && (2.15)\\
&\quad - \mathbb{E}(e^{iMy}(h(Mx) - h(My))). && (2.16)
\end{aligned}$$

We will estimate (2.13), (2.14), (2.15) and (2.16) separately. Observe that for every $0<\delta\leq 1$ such that $\epsilon<\delta<\alpha$ we have

$$\mathbb{E}\Big(\frac{|e^{i(Mx+tQ+tN(t^{-1}x))} - e^{i(My+tQ+tN(t^{-1}y))}|\,|h(Mx+tQ+tN(t^{-1}x))|}{|x-y|^\epsilon(1+|x|)^\lambda(1+|y|)^\lambda}\Big) \leq 2\mathbb{E}(L^\delta)|x-y|^{\delta-\epsilon} \leq 2\mathbb{E}(L^\delta)|t|^{\delta-\epsilon}. \tag{2.17}$$

Similarly we estimate the second term:

$$\mathbb{E}\Big(\frac{|e^{i(My+tQ+tN(t^{-1}y))}(h(Mx+tQ+tN(t^{-1}x)) - h(Mx+tQ+tN(t^{-1}y)))|}{|x-y|^\epsilon(1+|x|)^\lambda(1+|y|)^\lambda}\Big) \leq 2\mathbb{E}(L^\delta)\mathbb{E}(W^\delta)|x-y|^{\delta-\epsilon} \leq 2\mathbb{E}(L^\delta)\mathbb{E}(W^\delta)|t|^{\delta-\epsilon}. \tag{2.18}$$

The terms (2.15) and (2.16) are estimated in a similar way. Now consider the second term of (2.12) ($|x-y|>t$) and notice that

$$\begin{aligned}
&(T_t - T)h(x) - (T_t - T)h(y)\\
&= \mathbb{E}\big((e^{i(Mx+tQ+tN(t^{-1}x))} - e^{iMx})h(Mx+tQ+tN(t^{-1}x))\big) && (2.19)\\
&\quad + \mathbb{E}(e^{iMx}(h(Mx+tQ+tN(t^{-1}x)) - h(Mx))\big) && (2.20)\\
&\quad - \mathbb{E}((e^{My+tQ+tN(t^{-1}y)} - e^{iMy})h(My+tQ+tN(t^{-1}y))) && (2.21)\\
&\quad - \mathbb{E}(e^{iMy}(h(My+tQ+tN(t^{-1}y)) - h(My))). && (2.22)
\end{aligned}$$

As before we will estimate (2.19), (2.20), (2.21) and (2.22) separately using $|tQ+tN(t^{-1}x)|\leq |t|\,|R|$ for some random variable R such that $\mathbb{E}(|R|^\beta)<\infty$ where $\beta>0$ is as in Theorem 1.2. Indeed, for every $0<\delta\leq 1$ such that

$\epsilon < \delta < \alpha$ we have

$$\mathbb{E}\biggl(\frac{|e^{i(Mx+tQ+tN(t^{-1}x))} - e^{iMx}|\,|h(Mx+tQ+tN(t^{-1}x))|}{|x-y|^{\epsilon}(1+|x|)^{\lambda}(1+|y|)^{\lambda}}\biggr) \le 2\mathbb{E}\biggl(\frac{|t|^{\delta}|R|^{\delta}}{|x-y|^{\epsilon}}\biggr) \le 2\mathbb{E}(|R|^{\delta})|t|^{\delta-\epsilon}. \tag{2.23}$$

Similarly we estimate the second term:

$$\begin{aligned}&\mathbb{E}\biggl(\frac{|e^{iMx}(h(Mx+tQ+tN(t^{-1}x)) - h(Mx))|}{|x-y|^{\epsilon}(1+|x|)^{\lambda}(1+|y|)^{\lambda}}\biggr)\\ &\le 2\mathbb{E}\biggl(\frac{|Mx+tQ+tN(t^{-1}x)-Mx|^{\delta}W^{\delta}}{|x-y|^{\epsilon}}) \le 2\mathbb{E}\biggl(\frac{|t|^{\delta}|R|^{\delta}W^{\delta}}{|x-y|^{\epsilon}}\biggr)\\ &\le 2\mathbb{E}(|R|^{\delta})\mathbb{E}(W^{\delta})|t|^{\delta-\epsilon}.\end{aligned} \tag{2.24}$$

Also (2.21) and (2.22) can be estimated similarly. Hence, in view of (2.17), (2.18), (2.23) and (2.24), we obtain (2.10). For (2.11) notice that

$$\begin{aligned}(T_t - T)h(x) &= \mathbb{E}\bigl((e^{iMx+tQ+tN(t^{-1}x)} - e^{iMx})h(Mx+tQ+tN(t^{-1}x))\bigr)\\ &\quad + \mathbb{E}\bigl(e^{iMx}(h(Mx+tQ+tN(t^{-1}x)) - h(Mx))\bigr),\end{aligned} \tag{2.25}$$

and arguing as above we obtain the assertion. ∎

Proof of Theorem 2.7. It is easy to see that for $f \in \mathcal{B}_{\rho,\epsilon,\lambda}$, $x \in \mathbb{R}$ and $|t| \le t_0$ we have

$$((z - P_{t,v})^{-1}(f \circ \delta_t))(x) = ((z - T_{t,v})^{-1}f)(tx), \tag{2.26}$$

$$(\varPi_{T,t} - \varPi_T)h = \frac{1}{2\pi}\int_0^{2\pi} (\xi e^{is} + 1 - T_{t,v})^{-1}((T_t - T)h)\,ds. \tag{2.27}$$

Notice that for every $f \in \mathcal{B}_{\rho,\epsilon,\lambda}$ we have

$$\|f \circ \delta_t\|_{\rho,\epsilon,\lambda} \le \begin{cases} |f|_{\rho} + |t|^{\epsilon}[f]_{\epsilon,\lambda} & \text{if } |t| \le 1,\\ |t|^{\rho}|f|_{\rho} + |t|^{2\lambda+\epsilon}[f]_{\epsilon,\lambda} & \text{if } |t| > 1.\end{cases} \tag{2.28}$$

In view of (2.27) and (2.26) we have

$$\begin{aligned}((\varPi_{T,t} - \varPi_T)h)(tx) &= \frac{1}{2\pi}\int_0^{2\pi} \bigl((\xi e^{is} + 1 - T_t)^{-1}(T_t - T)h\bigr)(tx)\,ds\\ &= \frac{1}{2\pi}\int_0^{2\pi} \bigl((\xi e^{is} + 1 - P_t)^{-1}(((T_t - T)h) \circ \delta_t)\bigr)(x)\,ds.\end{aligned} \tag{2.29}$$

A straightforward application of (2.29), Proposition 2.6, and inequalities (2.28), (2.10) and (2.11) yields

$$\begin{aligned}
&\|((\Pi_{T,t} - \Pi_T)h) \circ \delta_t\|_{\rho,\epsilon,\lambda} \\
&\qquad \le \frac{1}{2\pi} \int_0^{2\pi} \|((\xi e^{is} + 1 - P_{t,v})^{-1}(((T_t - T)h) \circ \delta_t))\|_{\rho,\epsilon,\lambda}\, ds \\
&\qquad \le D(|((T_t - T)h) \circ \delta_t|_\rho + [((T_t - T)h) \circ \delta_t]_{\epsilon,\lambda}) \\
&\qquad \le D(|(T_t - T)h|_\rho + |t|^\epsilon [(T_t - T)h]_{\epsilon,\lambda}) \\
&\qquad \le D(C_2|t|^\delta + |t|^\epsilon C_1 |t|^{\delta-\epsilon}) \le C|t|^\delta
\end{aligned}$$

for every $|t| \le t_0$, and the proof is finished. ∎

2.3. Proof of the limit theorem

CONDITION 2.30. Assume that $0 < \epsilon < 1$, $\lambda > 0$, $\lambda + 2\epsilon < \rho = 2\lambda$ and $2\lambda + \epsilon < \alpha$ as in Proposition 2.6 and additionally

- If $0 < \alpha \le 1$, take any $0 < \beta < 1/2$ such that $\rho + 2\beta < \alpha$.
- If $1 < \alpha \le 2$, take any $\lambda > 0$ such that $\rho = 2\lambda < 1$ and $\rho + 1 < \alpha$.

THEOREM 2.31. *Let h be the eigenfunction of the operator T defined as in Lemma 2.5. If $0 < \alpha < 1$, then*

$$\lim_{t\to 0} \frac{k(t) - 1}{|t|^\alpha} = C_\alpha = \int_{\mathbb{R}} (e^{ix} - 1)h(x)\, \Lambda(dx). \tag{2.32}$$

- *If $\alpha = 1$ and $\xi(t) = \int_{\mathbb{R}} \frac{tx}{1+t^2x^2}\, \nu(dx)$, then*

$$\lim_{t\to 0} \frac{k(t) - 1 - i\xi(t)}{|t|} = C_1 = \int_{\mathbb{R}} \left((e^{ix} - 1)h(x) - \frac{ix}{1+x^2} \right) \Lambda(dx). \tag{2.33}$$

- *If $1 < \alpha < 2$ and $m = \int_{\mathbb{R}} x\, \nu(dx)$, then*

$$\lim_{t\to 0} \frac{k(t) - 1 - itm}{|t|^\alpha} = C_\alpha = \int_{\mathbb{R}} ((e^{ix} - 1)h(x) - ix)\, \Lambda(dx). \tag{2.34}$$

- *If $\alpha = 2$ and $m = \int_{\mathbb{R}} x\, \nu(dx)$, then*

$$\lim_{t\to 0} \frac{k(t) - 1 - itm}{|t|^2 \big|\log |t|\big|} = 2C_2 = -\frac{1}{2} \int_{\{\pm 1\}} (1 + 2\mathbb{E}(W))\, \sigma_\Lambda(dw). \tag{2.35}$$

Proof. Notice that $\Pi_{T,t}(h) \circ \delta_t$ is an eigenfunction of P_t corresponding to the eigenvalue $k(t)$ and we have

$$(k(t) - 1) \cdot \nu(\Pi_{T,t}(h) \circ \delta_t) = \nu((e^{it(\cdot)} - 1) \cdot (\Pi_{T,t}(h) \circ \delta_t)), \tag{2.36}$$

where ν is the stationary measure for P. In view of Condition 2.30 and Theorem 2.7, for $0 < \alpha < 2$ we have

$$\lim_{t\to 0} \frac{1}{|t|^\alpha} \int_{\mathbb{R}^d} (e^{itx} - 1)(\Pi_{T,t}(h)(tx) - \Pi_T(h)(tx))\, \nu(dx) = 0. \tag{2.37}$$

If $\alpha = 2$, then

$$\lim_{t\to 0} \frac{1}{t^2\big|\log|t|\big|} \int_{\mathbb{R}^d} (e^{itx}-1)(\varPi_{T,t}(h)(tx) - \varPi_T(h)(tx))\,\nu(dx) = 0. \tag{2.38}$$

Furthermore, if $0 < \delta \le 1$ with $\delta < \alpha$ then in view of Theorem 2.7 we obtain

$$\nu(\varPi_{T,t}(h)\circ\delta_t - 1) \le D|t|^{\delta}. \tag{2.39}$$

A straightforward application of an argument from [3] extends the convergence in (1.5) to measurable functions f such that $\varLambda(\mathrm{Dis}(f)) = 0$ and

$$\sup_{x\in\mathbb{R}^d} |x|^{-\alpha}\big|\log|x|\big|^{1+\varepsilon}|f(x)| < \infty \quad \text{for some } \varepsilon > 0, \tag{2.40}$$

where $\mathrm{Dis}(f)$ is the set of all discontinuities of f. To prove (2.32), write

$$\begin{aligned}\frac{1}{|t|^{\alpha}} \int_{\mathbb{R}^d} &(e^{itx}-1)\varPi_{T,t}(h)(tx)\,\nu(dx)\\ &= \frac{1}{|t|^{\alpha}} \int_{\mathbb{R}^d} (e^{itx}-1)\cdot(\varPi_{T,t}(h)(tx) - \varPi_T(h)(tx))\,\nu(dx)\\ &\quad + \frac{1}{|t|^{\alpha}} \int_{\mathbb{R}^d} (e^{itx}-1)\varPi_T(h)(tx)\,\nu(dx).\end{aligned}$$

The first summand above tends to 0, by (2.37). Since $f(x) = (e^{ix}-1)h(x)$ satisfies (2.40) the second term tends to $C_\alpha = \int_{\mathbb{R}}(e^{ix}-1)h(x)\,\varLambda(dx)$, hence in view of (2.39) we obtain

$$\lim_{t\to 0}\frac{k(t)-1}{|t|^{\alpha}} = \lim_{t\to 0}\frac{1}{\nu(\varPi_{T,t}(h)\circ\delta_t)|t|^{\alpha}} \int_{\mathbb{R}^d} (e^{itx}-1)h(tx)\,\nu(dx) = C_\alpha.$$

This finishes the proof of (2.32). In a similar way we can show (2.33)–(2.35); for more details we refer to [2] and [11]. ∎

Proof of Theorem 1.7. *Case* $0 < \alpha < 1$. In order to prove (1.8) notice that by Proposition 2.6 we have

$$\varDelta_\alpha^n(t) = \mathbb{E}(e^{it_nS_n^x}) = (P_{t_n}^n(1))(x) = k_v^n(t_n)(\varPi_{P,t_n}(1))(x) + (Q_{P,t_n}^n(1))(x),$$

where $t_n = tn^{-1/\alpha}$ for $n\in\mathbb{N}$. Again Proposition 2.6 ensures that $\|Q_{P,t_n}^n\|_{\mathcal{B}_{\rho,\epsilon,\lambda}} \to 0$ as $n\to\infty$ because $\|Q_{P,t}\|_{\mathcal{B}_{\rho,\epsilon,\lambda}} < 1$. By Theorem 2.31 we have

$$\lim_{n\to\infty} n\cdot(k(t_n)-1) = \lim_{n\to\infty} t^{\alpha}\cdot\frac{k(t_n)-1}{t_n^{\alpha}} = t^{\alpha}C_\alpha,$$

hence

$$\lim_{n\to\infty} k^n(t_n) = \lim_{n\to\infty}(1+k(t_n)-1)^{\frac{n}{k(t_n)-1}\cdot(k(t_n)-1)} = \exp(t^{\alpha}C_\alpha).$$

This proves the pointwise convergence of $\varDelta_\alpha^n$ to $\varUpsilon_\alpha$. Continuity of $\varUpsilon_\alpha$ at 0 follows from the Lebesgue dominated convergence theorem. Now we give an explicit formula for C_α. First notice that

$$\int_0^\infty \frac{e^{itx}-1}{t^{\alpha+1}}\,dt = x^\alpha \vartheta_\alpha \quad \text{for } x>0,$$

where

$$\vartheta_\alpha = \int_0^\infty \frac{e^{it}-1}{t^{\alpha+1}}\,dt = -\frac{\Gamma(1-\alpha)}{\alpha}e^{-i\alpha\pi/2}.$$

Then

$$\begin{aligned} C_\alpha &= \int_{\mathbb{R}} (e^{ix}-1)h(x)\,\Lambda(dx) = C_+ \int_{\mathbb{R}}\int_0^\infty (e^{ix(y+1)}-e^{ixy})\,\frac{dx}{x^{\alpha+1}}\,\eta(dy) \\ &= C_+\vartheta_\alpha \mathbb{E}((W+1)^\alpha - W^\alpha) = C_+\vartheta_\alpha(1-\mathbb{E}(M^\alpha))\mathbb{E}((W+1)^\alpha) \\ &= \alpha c \vartheta_\alpha \mathbb{E}((W+1)^\alpha) \neq 0. \end{aligned}$$

In all cases below convergence is obtained as in the first case. We only give formulas for the constants C_α.

Case $\alpha = 1$. Convergence in (1.9) is obtained as in the previous case (see also [11]). Now we give a formula for C_1. Observe that

$$\begin{aligned} C_1 &= \int_{\mathbb{R}} \left((e^{ix}-1)h(x) - \frac{ix}{1+x^2}\right)\Lambda(dx) \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} \left[\left(e^{ix(y+1)} - 1 - \frac{ix(y+1)}{1+x^2(y+1)^2}\right) - \left(e^{ixy}-1-\frac{ixy}{1+x^2y^2}\right)\right. \\ &\qquad \left. + i\left(\frac{x(y+1)}{1+x^2(y+1)^2} - \frac{xy}{1+x^2y^2} - \frac{x}{1+x^2}\right)\right]\eta(dy)\,\Lambda(dx), \end{aligned}$$

and

$$\begin{aligned} &\int_{\mathbb{R}}\int_{\mathbb{R}} \left[\left(e^{ix(y+1)} - 1 - \frac{ix(y+1)}{1+x^2(y+1)^2}\right) - \left(e^{ixy}-1-\frac{ixy}{1+x^2y^2}\right)\right]\eta(dy)\,\Lambda(dx) \\ &= C_+\int_{\mathbb{R}}\int_0^\infty \left[\left(e^{ix(y+1)} - 1 - \frac{ix(y+1)}{1+x^2(y+1)^2}\right) - \left(e^{ixy}-1-\frac{ixy}{1+x^2y^2}\right)\right]\frac{dx}{x^2}\,\eta(dy) \\ &= C_+\vartheta_1\mathbb{E}((W+1)-W) = C_+\vartheta_1, \end{aligned}$$

where $\vartheta_1 = \int_{\mathbb{R}} \left(e^{ix}-1-\frac{ix}{1+x^2}\right)\frac{dx}{x^2} = -\frac{\pi}{2} + i\kappa$ for some $\kappa > 0$. Moreover,

$$\begin{aligned} &i\int_{\mathbb{R}}\int_{\mathbb{R}} \left(\frac{x(y+1)}{1+x^2(y+1)^2} - \frac{xy}{1+x^2y^2} - \frac{x}{1+x^2}\right)\eta(dy)\,\Lambda(dx) \\ &\qquad = -iC_+\mathbb{E}((W+1)\log(W+1) - W\log W). \end{aligned}$$

Now it is easy to see that

$$C_1 = C_+\vartheta_1 - iC_+\mathbb{E}((W+1)\log(W+1) - W\log W).$$

Case $1 < \alpha < 2$. As in the first case, we obtain

$$C_\alpha = C_+ \vartheta_\alpha (1 - \mathbb{E}(M^\alpha))\mathbb{E}((W+1)^\alpha) = \alpha c \vartheta_\alpha \mathbb{E}((W+1)^\alpha) \neq 0,$$

where

$$\vartheta_\alpha = \int_0^\infty \frac{e^{it} - 1 - it}{t^{\alpha+1}}\, dt = \frac{\Gamma(2-\alpha)}{\alpha(\alpha-1)}\, e^{-i\alpha\pi/2}.$$

Case $\alpha = 2$. Observe that

$$\begin{aligned} C_2 &= -\frac{1}{4} \int_{\{\pm 1\}} (1 + 2\mathbb{E}(W))\, \sigma_\Lambda(dw) = -\frac{1}{4}\, C_+(1 + 2\mathbb{E}(W)) \\ &= -\frac{1}{4}\, C_+ \mathbb{E}((W+1)^2 - W^2) \\ &= -\frac{1}{4}\, C_+(1 - \mathbb{E}(M^2))\mathbb{E}((W+1)^2) = -\frac{c}{2}\, \mathbb{E}((W+1)^2) \neq 0. \ \blacksquare \end{aligned}$$

3. Local limit theorem. Recall that S is the stationary solution of the recursion (1.1). We prove the following

THEOREM 3.1. *Assume that* $|\mathbb{E}(e^{itS})| = \xi < 1$ *for all* $t \neq 0$. *Then the spectral radius satisfies* $r(P_t) < 1$.

Proof. Suppose for a contradiction that $r(P_t) = 1$ for some $t \neq 0$. This implies that for some $f \in \mathcal{B}_{\rho,\epsilon,\lambda}$ and $z \in \mathbb{C}$ such that $|z| = 1$ we have $P_t f(x) = z f(x)$, since the essential spectral radius satisfies $r_e(P_t) \leq \varrho < 1$, where $\varrho > 0$ was defined in Proposition 2.6. We will show that $f = 0$, which gives a contradiction.

First notice that f is bounded. Indeed, $|f(x)| \leq \lim_{n\to\infty} P^n(|f|)(x) = \nu(|f|)$. Suppose that $f \neq 0$. By the previous inequality we can assume that $f \neq 0$ on $\operatorname{supp}\nu$ and $|f| = 1$. Observe that for all $x \in \operatorname{supp}\nu$ and $n \in \mathbb{N}$ we have $z^n f(x) = e^{itS_n^x} f(X_n^x)$ $\mathbb{P}$-a.s., hence

$$\begin{aligned} z^n f(x) &= e^{itS_{n-1}^x} e^{itX_n^x} f(X_n^x) = e^{itS_{n-1}^x} f(X_{n-1}^x) e^{itX_n^x}\, \frac{f(X_n^x)}{f(X_{n-1}^x)} \\ &= z^{n-1} f(x) e^{itX_n^x}\, \frac{f(X_n^x)}{f(X_{n-1}^x)}. \end{aligned}$$

This implies that

$$\begin{aligned} P_t^n f(x) &= \mathbb{E}(e^{itS_n^x} f(X_n^x)) \\ &= z^{n-1} f(x) \mathbb{E}(e^{itX_n^x}) + z^{n-1} f(x) \mathbb{E}\biggl(e^{itX_n^x}\biggl(\frac{f(X_n^x)}{f(X_{n-1}^x)} - 1\biggr)\biggr). \end{aligned} \tag{3.2}$$

Now we obtain

$$\tag{3.3} \begin{aligned} &\left|\mathbb{E}\left(\frac{f(X_n^x)}{f(X_{n-1}^x)}-1\right)\right| \\ &\quad\le [f]_{\epsilon,\lambda}\mathbb{E}\big(|X_n^x-X_{n-1}^x|^\epsilon\big(1+|X_n^x|\big)^\lambda(1+|X_{n-1}^x|)^\lambda\big) \\ &\quad\le [f]_{\epsilon,\lambda}\mathbb{E}\big((L_1\cdot\ldots\cdot L_{n-1})^\epsilon|\psi_n(x)-x|^\epsilon(1+|X_n^x|)^\lambda(1+|X_{n-1}^x|)^\lambda\big) \\ &\quad\le C[f]_{\epsilon,\lambda}(1+|x|)^{2\lambda+\epsilon}\theta^n \end{aligned}$$

for some $0<\theta<1$. Fix $\varepsilon>0$ such that $1-\xi-\varepsilon>0$ and observe that $|\mathbb{E}(e^{itX_n^x}-e^{itS})|<\varepsilon<1-\xi$ for sufficiently large $n\in\mathbb{N}$. Now using (3.2) and (3.3), for fixed $x\in\operatorname{supp}\nu$ we have

$$\begin{aligned} 1=|z^nf(x)| &\le |\mathbb{E}(e^{itX_n^x}-e^{itS})|+|\mathbb{E}(e^{itS})|+\left|\mathbb{E}\left(\frac{f(X_n^x)}{f(X_{n-1}^x)}-1\right)\right| \\ &\le \varepsilon+\xi+C[f]_{\epsilon,\lambda}(1+|x|)^{2\lambda+\epsilon}\theta^n, \end{aligned}$$

so

$$1\le\frac{C}{1-\xi-\varepsilon}\,[f]_{\epsilon,\lambda}(1+|x|)^{2\lambda+\epsilon}\theta^n<1$$

for sufficiently large $n\in\mathbb{N}$, and this contradiction shows that f has to be zero. ■

Let

$$\begin{aligned} \Theta=\{\psi:\mathbb{R}\to\mathbb{R}:\psi(x)=mx+q+n(x)& \\ \text{for some } (m,q,n)\in\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}_+ &\text{ and } n(x)\le n\phi(x) \text{ for every } x\in\mathbb{R}\}, \end{aligned}$$

where ϕ is a fixed nondecreasing nonnegative function such that $\phi(x)=o(x)$ as $x\to\pm\infty$. The measure μ is a probability measure on Θ. We give a criterion for the stationary solution S for (1.1) to satisfy $|\mathbb{E}(e^{itS})|<1$.

PROPOSITION 3.4. *Assume that* $\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}_+\ni(m,q,n)\mapsto\psi(x)=mx+q+n(x)\in\mathbb{R}$ *is continuous for every* $x\in\mathbb{R}$ *and the functions* ψ *are invertible on* $\operatorname{supp}\nu$. *Then* $|\mathbb{E}(e^{itS})|<1$ *for all* $t\neq0$.

Proof. It suffices to show that the measure ν has no atoms. Suppose that the set X of atoms of ν is not empty. Let $A=\{x\in X:\nu(\{x\})=\max_{z\in X}\nu(\{z\})=a\}$. The set $A=\{x_1,\ldots,x_n\}$ is finite because ν is a probability measure. Since the measure ν is μ-stationary, we have

$$na=\nu(A)=\int_\Theta\int_\mathbb{R}\mathbf{1}_A(\psi(x))\,\nu(dx)\,\mu(d\psi)=\sum_{k=1}^n\int_\Theta\int_\mathbb{R}\mathbf{1}_{\psi^{-1}(x_k)}(x)\,\nu(dx)\,\mu(d\psi).$$

Notice that for every $\psi\in\Theta$ and $x\in\mathbb{R}$, $\nu(\{\psi^{-1}(x)\})\le a$ and

$$\sum_{k=1}^n\int_\Theta(a-\nu(\{\psi^{-1}(x_k)\}))\,\mu(d\psi)=0.$$

Hence $\nu(\{\psi^{-1}(x_k)\}) = a$ μ-a.e. for all $1 \le k \le n$ and so $\psi(A) = A$ μ-a.e. But we want more. We prove that $\psi(A) = A$ for every $\psi \in \operatorname{supp}\mu$. It is enough to show that $\psi(A) \subseteq A$ for every $\psi \in \operatorname{supp}\mu$. Suppose that there exist $\psi_0 \in \operatorname{supp}\mu$ and $x_0 \in A$ such that $\psi_0(x_0) \notin A$. Let $A_1 = \{\psi \in \Theta : \psi(x_0) \in A^c\}$, so that $\psi_0 \in A_1$. Moreover, A_1 is open in Θ (since A is closed) and $\mu(A_1) = 0$, contrary to $\psi_0 \in \operatorname{supp}\mu$. Now let $\mathcal{L}_\Theta^\mu$ be the closed semigroup generated by $\operatorname{supp}\mu$. Observe that A is $\mathcal{L}_\Theta^\mu$-invariant. On the other hand, $\operatorname{supp}\nu \subseteq A$ (see Theorem 1.7 of [11]), but A is finite, which contradicts (1.4). This finishes the proof. ∎

Proof of Theorem 1.12. Let $\kappa = 1/\alpha$. We have $d_n = 0$ if $\alpha \in (0,1)$, and $d_n = mn$ if $\alpha \in (1,2)$. A straightforward application of Theorem 10.7 of [1] allows us to check only that

$$\lim_{n\to\infty} n^\kappa \mathbb{E}(h(S_n^x - d_n)) = \frac{1}{2\pi}\int_{\mathbb{R}} h(t)\,dt \cdot \int_{\mathbb{R}} \Upsilon_\alpha(t)\,dt$$

for every integrable function h whose Fourier transform is compactly supported. By the Fourier inversion formula we have

$$\mathbb{E}(h(S_n^x - d_n)) = \frac{1}{2\pi}\int_{\mathbb{R}} \mathbb{E}(e^{it(S_n^x - d_n)})\widehat{h}(t)\,dt = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-itd_n} P_t^n(1)(0)\widehat{h}(t)\,dt.$$

Now take $J = \operatorname{supp}\widehat{h}$ and $N = [-\delta, \delta]$. By Theorem 3.1, $r(P_t) < 1$ for $t \neq 0$ by Lemma 3.19 of [2] with $f = 1$ there exists $\beta > 0$ such that $r(P_t) < 1 - \beta$ for $t \in J \setminus N$, hence

$$\lim_{n\to\infty} n^\kappa \Big| \int_{J\setminus N} e^{-itd_n} P_t^n(1)(0)\widehat{h}(t)\,dt \Big| \le \lim_{n\to\infty} Cn^\kappa (1-\beta)^n = 0.$$

Notice that

$$\begin{aligned}(3.5)\qquad \lim_{n\to\infty} \frac{n^\kappa}{2\pi} &\int_N e^{-itd_n} P_t^n(1)(0)\widehat{h}(t)\,dt \\ &= \lim_{n\to\infty} \frac{n^\kappa}{2\pi} \int_N e^{-itd_n} (k^n(t)\Pi_{P,t}(1)(0) + Q_{P,t}^n(1)(0))\widehat{h}(t)\,dt \\ &= \lim_{n\to\infty} \frac{n^\kappa}{2\pi} \int_N e^{-itd_n} k^n(t)\Pi_{P,t}(1)(0)\widehat{h}(t)\,dt.\end{aligned}$$

To get the last equality observe that by Proposition 2.6 there exists $0 < \varrho < 1$ such that $\|Q_{P,t}\|_{\mathcal{B}_{\rho,\epsilon,\lambda}} \le \varrho$ for $t \in N$, so

$$\lim_{n\to\infty} \frac{n^\kappa}{2\pi} \Big| \int_N e^{-itd_n} Q_{P,t}^n(1)(0)\widehat{h}(t)\,dt \Big| \le \lim_{n\to\infty} Cn^\kappa \varrho^n = 0.$$

To compute the limit in (3.5) we change the variable $t \mapsto n^{-\kappa}t$ in (3.5) to obtain

$$\begin{aligned}(3.6)\qquad &\lim_{n\to\infty}\frac{n^\kappa}{2\pi}\int_N (e^{-itm}k(t))^n \varPi_{P,t}(1)(0)\widehat{h}(t)\,dt\\ &= \lim_{n\to\infty}\frac{1}{2\pi}\int_{\{t\in\mathbb{R}\,:\,|t|<\delta n^\kappa\}} (e^{-in^{-\kappa}tm}k(n^{-\kappa}t))^n \varPi_{P,n^{-\kappa}t}(1)(0)\widehat{h}(n^{-\kappa}t)\,dt.\end{aligned}$$

By Theorem 2.31 for $\alpha\in(0,1)\cup(1,2)$ we have $k(t)=1+itm+|t|^\alpha(C_\alpha+o(1))$ with $\Re C_\alpha<1$. Therefore it is easy to see that there exists $D>0$ such that $|e^{-itm}k(t)|\le e^{-D|t|^\alpha}$. This inequality and the Lebesgue dominated convergence theorem allow us to pass to the limit in the integrand of (3.6). Hence the limit in (3.6) is equal

$$\frac{1}{2\pi}\int_{\mathbb{R}} h(t)\,dt\cdot\int_{\mathbb{R}}\varUpsilon_\alpha(t)\,dt=\frac{1}{2\pi}\widehat{h}(0)\cdot\int_{\mathbb{R}}\varUpsilon_\alpha(t)\,dt.\ \blacksquare$$

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