Introduction

It is well known that the spaces of homogeneous type introduced by Coifman and Weiss in [4] include $\mathbb{R}^n$, the surface of the unit ball and the n-torus in $\mathbb{R}^n$, the C^∞ compact Riemannian manifolds, and in particular, the d-sets in $\mathbb{R}^n$ as special models. It has been proved by Triebel in [33] that these d-sets in $\mathbb{R}^n$ include various kinds of fractals.

Homogeneous Besov and Triebel–Lizorkin spaces on spaces of homogeneous type have been studied in [23]. In [20], inhomogeneous Besov and Triebel–Lizorkin spaces on spaces of homogeneous type were introduced via generalized Littlewood–Paley g-functions when $p, q \geq 1$. In [21], inhomogeneous Triebel–Lizorkin spaces were generalized to the cases where $p_0 < p \leq 1 \leq q < \infty$ via generalized Littlewood–Paley S-functions, where p_0 is a positive number. In the case of d-sets, $p_0 = 1/2$.

The motivation for this paper is to answer a question posed by Triebel in [34]. Let Γ be a compact d-set in $\mathbb{R}^n$ with $0 < d < n$; see [33] for the definition. Triebel has introduced the spaces $B^s_{pq}(\Gamma)$ for $s > 0$ by use of two different but equivalent methods, namely, traces in [33] and quarkonial decompositions in [34]. He asked in [34] if these spaces $B^s_{pq}(\Gamma)$ are the same as those defined by regarding Γ as a space of homogeneous type. In [36], we answered this question. Moreover, our methods can be used to introduce new spaces $B^0_{pq}(\Gamma)$ with $1 < q \leq \infty$ and $1 \leq p \leq \infty$ or $q = 1$ and $p = 1, \infty$, and new spaces $F^s_{pq}(\Gamma)$ with $s \in (-1, 1)$, $1 < p < \infty$ and $1 < q \leq \infty$, which cannot be defined by the trace method or quarkonial method. We point out that the spaces $B^0_{p1}(\Gamma)$ for $1 < p < \infty$ are introduced by quarkonial decompositions; see Definition 9.29(ii) in [34]. One of the main purposes of this paper is to obtain some estimates of the entropy numbers of compact embeddings between these spaces. To do this, we first need some frame characterizations for these function spaces. It is well known that the atomic decomposition characterizations of these spaces are not enough to obtain estimates of the entropy numbers, since atoms depend on functions; see [34]. We will do this in the setting of general homogeneous type spaces. We have given some applications of these estimates for entropy numbers to estimates of the eigenvalues of some fractal differential operators on d-sets in [36] and Riesz potentials on quasi-metric spaces in [35]. Another main purpose of this paper is to show that the fractional integrals and derivatives can be used as a lifting tool in these function spaces on homogeneous type spaces.

We begin with briefly reviewing the definition of spaces of homogeneous type. A *quasi-metric* ϱ on a set X is a function $\varrho : X \times X \to [0, \infty)$ satisfying

(i) $\varrho(x, y) = 0$ if and only if $x = y$.
(ii) $\varrho(x, y) = \varrho(y, x)$ for all $x, y \in X$.

(iii) There exists a constant $A \in [1, \infty)$ such that for all $x, y, z \in X$,

$$\varrho(x, y) \leq A[\varrho(x, z) + \varrho(z, y)].$$

Any quasi-metric defines a topology, for which the balls $B(x, r) = \{y \in X : \varrho(y, x) < r\}$ for all $x \in X$ and all $r > 0$ form a basis.

The following spaces of homogeneous type are variants of those introduced by Coifman and Weiss in [4]. In what follows, we set $\operatorname{diam} X = \sup\{\varrho(x, y) : x, y \in X\}$, and $A \sim B$ means that there are two constants $C_1 > 0$ and $C_2 > 0$ independent of the main parameters such that $C_1 < A/B < C_2$.

Definition 0.1. Let $d > 0$ and $0 < \theta \leq 1$. A *space of homogeneous type* $(X, \varrho, \mu)_{d,\theta}$ is a set X together with a quasi-metric ϱ and a nonnegative Borel measure μ on X with $\operatorname{supp} \mu = X$ such that there exists a constant $0 < C < \infty$ such that for all $0 < r < \operatorname{diam} X$ and all $x, x', y \in X$,

$$\mu(B(x, r)) \sim r^d, \tag{0.1}$$

$$|\varrho(x, y) - \varrho(x', y)| \leq C\varrho(x, x')^\theta [\varrho(x, y) + \varrho(x', y)]^{1-\theta}. \tag{0.2}$$

Remark 0.1. It is easy to see that if $\operatorname{diam} X < \infty$, then (0.1) holds for all $0 < r < \operatorname{diam} X$ if and only if it holds for all $0 < r < 1$.

Remark 0.2. From (0.1), it is easy to deduce $\mu(\{x\}) = 0$ for all $x \in X$. This means that spaces of homogeneous type defined by Definition 0.1 are atomless measure spaces.

Macias and Segovia [26] have proved that our spaces $(X, \varrho, \mu)_{d,\theta}$ for $d = 1$ are just the spaces of homogeneous type in the sense of Coifman and Weiss, whose definitions only require that ϱ is a quasi-metric without (0.2) and μ satisfies the following doubling condition weaker than (0.1): there is a constant $0 < A' < \infty$ such that for all $x \in X$ and all $r > 0$,

$$\mu(B(x, 2r)) \leq A'\mu(B(x, r)). \tag{0.3}$$

However, in [26], Macias and Segovia have shown that for spaces of homogeneous type in the sense of Coifman and Weiss, one can replace the original quasi-metric ϱ by another quasi-metric $\bar{\varrho}$, which yields the same topology on X as ϱ, such that there exist $C > 0$ and some $\bar{\theta} \in (0, 1]$ satisfying

$$\bar{\varrho}(x, y) \sim \inf\{\mu(B) : B \text{ is a ball containing } x \text{ and } y\}$$

and (0.2) with ϱ and θ replaced, respectively, by $\bar{\varrho}$ and $\bar{\theta}$, and that μ satisfies (0.1) with $d = 1$ for balls corresponding to this new quasi-metric. Moreover, there is a positive constant C_0 such that $\bar{\varrho}(x, y)^{1/C_0}$ is equivalent to a metric on $X \times X$. It is easy to see that the set X with this new quasi-metric $\bar{\varrho}$, the original measure μ and the balls corresponding to the new quasi-metric satisfies (0.1) with $d = 1$ and (0.2).

The above definition of spaces of homogeneous type turns out to be convenient for our purposes. In fact, $(\mathbb{R}^n, \varrho, m)_{n,1}$ is just the usual $\mathbb{R}^n$, where ϱ is the standard Euclidean metric and m is the n-dimensional Hausdorff measure, or, equivalently, the n-dimensional Lebesgue measure. Moreover, it is also easy to see that any bounded d-set Γ in $\mathbb{R}^n$ with $0 \leq d \leq n$ is just $(\Gamma, \varrho, \mu)_{d,1}$, where ϱ is again the standard Euclidean metric and μ is a Radon measure on Γ with $\operatorname{supp} \mu = \Gamma$; see [33] and [36]. We remark that in some cases, the Borel measure μ appearing in Definition 0.1 can be proved to be actually a Radon measure. In fact, in Definition 3.1 of d-sets in [33, p. 5], Γ is not necessarily bounded and the Borel measure μ in $\mathbb{R}^n$ satisfies (0.1) only for $0 < r < 1$. However, Triebel [33] has shown that this Borel measure is actually a Radon measure by using some results of [27].

In addition, we also point out that the θ in (0.2) is crucial to us. In fact, the spaces $B^s_{pq}(X)$ and $F^s_{pq}(X)$ introduced in [20] have the restriction $|s| < \theta$. In particular, when $X = \mathbb{R}^n$ for $n \in \mathbb{N}$, if we take $d = n$, μ the n-dimensional Hausdorff measure and $\varrho(x, y) = |x - y|$ for any $x, y \in \mathbb{R}^n$, then we have $\theta = 1$ and all the spaces $B^s_{pq}(X)$ and $F^s_{pq}(X)$ with $|s| < 1$; and if we take $\bar{d} = 1$, μ the n-dimensional Hausdorff measure and $\bar{\varrho}(x, y) = |x - y|^n$ for any $x, y \in \mathbb{R}^n$, then we have $\bar{\theta} = 1/n$ and all the spaces $\overline{B}{}^{\overline{s}}_{pq}(X)$ and $\overline{F}{}^{\overline{s}}_{pq}(X)$ with $|\overline{s}| < 1/n$. In the next section, we will show that $\overline{B}{}^{\overline{s}}_{pq}(X) = B^{n\overline{s}}_{pq}(X)$ and $\overline{F}{}^{\overline{s}}_{pq}(X) = F^{n\overline{s}}_{pq}(X)$. Note that $|\overline{s}| < 1/n$ if and only if $n|\overline{s}| < 1$. We see that we still obtain all the spaces $B^s_{pq}(X)$ and $F^s_{pq}(X)$ for all $|s| < 1$. However, if we choose $\widetilde{d} = n^2$, μ the n-dimensional Hausdorff measure and $\widetilde{\varrho}(x, y) = |x - y|^{1/n}$ for any $x, y \in \mathbb{R}^n$, then we have $\widetilde{\theta} = 1/n$ and all the spaces $\widetilde{B}{}^{\widetilde{s}}_{pq}(X)$ and $\widetilde{F}{}^{\widetilde{s}}_{pq}(X)$ with $|\widetilde{s}| < 1/n$. We will also show that in this case, $\widetilde{B}{}^{ns}_{pq}(X) = B^s_{pq}(X)$ and $\widetilde{F}{}^{ns}_{pq}(X) = F^s_{pq}(X)$ for $|s| < 1/n^2$. From this, we can see that if we take $\widetilde{d} = n^2$, μ the n-dimensional Hausdorff measure and $\widetilde{\varrho}(x, y) = |x - y|^{1/n}$ for any $x, y \in \mathbb{R}^n$, then, by our method, we cannot obtain all the spaces $B^s_{pq}(X)$ and $F^s_{pq}(X)$ for all $|s| < 1$. Thus, in any case, by suitably choosing ϱ such that we can take a maximum corresponding θ in (0.2), we can obtain more spaces by the procedure in [20]; see also [23]. This reflects the flexibility of the above definition of spaces of homogeneous type.

Let $\varepsilon > 0$. By (0.1), it is easy to deduce that

$$\int_{B(x,r)} \varrho(z, x)^{\varepsilon - d}\, d\mu(z) \simeq r^{\varepsilon} \quad \text{and} \quad \int_{X \setminus B(x,r)} \varrho(z, x)^{-d-\varepsilon}\, d\mu(z) \simeq r^{-\varepsilon}.$$

In this paper, we assume that the total measure of X can be finite or infinite. But, in some places, we make the restriction $\mu(X) < \infty$, which will be explicitly indicated. Also, we let

$$L^p(X) = \{f : X \to \mathbb{C} \text{ is a } \mu\text{-measurable function and } \|f\|_{L^p(X)} < \infty\}$$

for $p \in (0, \infty]$, where

$$\|f\|_{L^p(X)} = \Big\{ \int_X |f(x)|^p\, d\mu(x) \Big\}^{1/p} \quad \text{for } p \in (0, \infty), \qquad \|f\|_{L^\infty(X)} = \operatorname*{ess\,sup}_{x \in X} |f(x)|.$$

The organization of this paper is as follows. In the next section, we will recall all the related theory of spaces of homogeneous type. Most of it is known and will be used in the later sections. In particular, we will show the independence of the spaces $B^s_{pq}(X)$ and $F^s_{pq}(X)$ from the equivalent quasi-metrics satisfying (0.2), and the above two claims. We will also give a new characterization for $B^s_{pq}(X)$ and $F^s_{pq}(X)$ in terms of smooth blocks when $s > 0$.

In Section 2, we will introduce fractional integrals and derivatives on spaces of homogeneous type, which are just the discrete and inhomogeneous versions of the fractional integrals and derivatives introduced by Gatto, Segovia and Vági in [11]; see [11, Theorem 1.6]. Such discrete and inhomogeneous fractional integrals and derivatives were also considered by Nahmod in [28] and [29]. We will show that they can be used as lifting tools. Using them, we will show that $B^s_{pq}(X)$ and $F^s_{pq}(X)$ have the lifting properties when $|s| < \theta$; see also [31] for the lifting property of these spaces on $\mathbb{R}^n$. Thus, our results give a new characterization of these spaces.

In Section 3, we will give explicit representations for the left and right inverses of fractional integrals and derivatives introduced in Section 2 for $\mu(X) \le \infty$. The left inverses and right inverses of fractional integrals and derivatives on spaces of homogeneous type are not the same, in contrast to the case of Euclidean spaces. By using these explicit representations, we show that the fractional integrals and derivatives are independent of the choices of approximations to the identity. These results are new even when $X = \mathbb{R}^n$. If $\mu(X) < \infty$, we then establish some basic properties of these left and right inverses. In particular, we are able to introduce fractional Sobolev spaces for all $|s| < \alpha_0 < \varepsilon$ and $\mu(X) \le \infty$, which complete and generalize those fractional Sobolev spaces for $\mu(X) = \infty$ introduced by Gatto and Vági in [12] when s is positive and small; see Theorem 2.1 in [22] and Theorem 6 in [10]. For Sobolev functions in $F^s_{p2}(X)$ with $s > 0$ small enough, $1 < p < \infty$ and $\mu(X) < \infty$, by using the above fractional derivatives and their left inverses, we also obtain some Poincaré-type inequalities. We remark that our results in this section and Section 2 have homogeneous versions. We will discuss that in another paper.

In Section 4, we will establish frame decomposition characterizations for $B^s_{pq}(X)$ and $F^s_{pq}(X)$ by using the discrete Calderón reproducing formulae established in [22]. Such frames are called Banach frames in [13] and [8]. These frame characterizations will play a key role in estimates of entropy numbers for compact embeddings between these spaces and they are new even when $X = \mathbb{R}^n$ or X is a d-set in $\mathbb{R}^n$.

In Section 5, by applying the frame characterizations, we will obtain estimates for entropy numbers of compact embeddings between $B^s_{pq}(X)$ or $F^s_{pq}(X)$ when $\mu(X) < \infty$. Part of these results is new even when X is a d-set in $\mathbb{R}^n$. We also consider some limiting embeddings between these spaces; see also [17] for homogeneous versions. By considering the spaces $L^p(\log L)_a(X)$ for $p \in (0, \infty)$ and $a \in \mathbb{R}$, which were first introduced by Haroske in [24] in terms of an equivalent norm (see [6, Theorem 2.6.2/1] and its proof), we then establish some limiting compact embeddings when $\mu(X) \le \infty$ and obtain some estimates of entropy numbers for these embeddings when $\mu(X) < \infty$.

In metric spaces with doubling Borel measures, the Sobolev spaces of order 1 were introduced by Hajłasz in [14]; see also [16], [15] and [25]. We recall that if X is a metric space admitting a Borel regular measure μ such that (0.1) holds, then X is called an *Ahlfors d-regular metric measure space*; see [25, p. 62]. If X is just a subset of $\mathbb{R}^n$, then X is also called *strictly d-regular*; see [14]. In all these cases, the θ in (0.2) equals 1. In Section 6, for any Ahlfors d-regular metric measure space, we will establish the connection between the Sobolev spaces of order 1 defined by Hajłasz in [14] and the spaces defined by our methods.

Finally, in Section 7, by using Carl's well known inequality (see [2], [6] and [33]), which connects spectral properties of compact operators with their geometry described in terms of entropy numbers, and the estimates of entropy numbers in Section 5, we obtain estimates of eigenvalues of some positive-definite self-adjoint operators related to quadratic forms in $L^2(X)$, which is a version of Theorem 25.2 in [33] in spaces of homogeneous type.

More applications can be found in [35] and [36]; see also [6] and [33].

We now make some conventions. Throughout the paper, if X_1 and X_2 are two Banach spaces, $X_1 \subset X_2$ means that there is a constant $C > 0$ such that for all $x \in X_1$,

$$\|x\|_{X_2} \le C\|x\|_{X_1},$$

where $\|x\|_X$ is the norm of x in the Banach space X. In what follows, we will use C to denote a positive constant which is independent of the main parameters, but may vary from line to line.

1. Preliminaries

In this section, we consider spaces of homogeneous type $(X, \varrho, \mu)_{d,\theta}$, as defined in Definition 0.1. Most of these results are well known when $d = 1$ or when $X = \mathbb{R}^n$ $(d = n)$. Generalizations to general $(X, \varrho, \mu)_{d,\theta}$ are obvious. We will omit all the details. Moreover, we will show the independence of our spaces from the equivalent quasi-metrics satisfying (0.2), and we prove two claims stated in the introduction. Finally we will also give a new characterization for the spaces $B^s_{pq}(X)$ and $F^s_{pq}(X)$ in terms of smooth blocks when $s > 0$.

Let us first recall the definition of spaces of test functions on X in [23]; see also [18].

Definition 1.1. Fix $\gamma > 0$ and $\theta \ge \beta > 0$. A function f defined on X is said to be a *test function of type* (x_0, r, β, γ) with $x_0 \in X$ and $r > 0$ if:

(i) $|f(x)| \le C\dfrac{r^\gamma}{(r + \varrho(x, x_0))^{d+\gamma}}$;

(ii) $|f(x)-f(y)| \le C\Big(\dfrac{\varrho(x, y)}{r + \varrho(x, x_0)}\Big)^\beta \dfrac{r^\gamma}{(r + \varrho(x, x_0))^{d+\gamma}}$ for $\varrho(x, y) \le \dfrac{1}{2A}[r+\varrho(x, x_0)]$.

If f is a test function of type (x_0, r, β, γ), we write $f \in \mathcal{G}(x_0, r, \beta, \gamma)$, and the norm of f in $\mathcal{G}(x_0, r, \beta, \gamma)$ is defined by

$$\|f\|_{\mathcal{G}(x_0,r,\beta,\gamma)} = \inf\{C : \text{(i) and (ii) hold}\}.$$

Here and in what follows, θ is the same as in (0.2).

Now fix $x_0 \in X$ and let $\mathcal{G}(\beta, \gamma) = \mathcal{G}(x_0, 1, \beta, \gamma)$. It is easy to see that

$$\mathcal{G}(x_1, r, \beta, \gamma) = \mathcal{G}(\beta, \gamma)$$

with equivalent norms for all $x_1 \in X$ and $r > 0$. Furthermore, it is easy to check that $\mathcal{G}(\beta, \gamma)$ is a Banach space with respect to the norm in $\mathcal{G}(\beta, \gamma)$. Also, let

$$\mathcal{G}_0(x_0, r, \beta, \gamma) = \Big\{f \in \mathcal{G}(x_0, r, \beta, \gamma) : \int_X f(x)\, d\mu(x) = 0\Big\}$$

and let the dual space $(\mathcal{G}(\beta,\gamma))'$ be all linear functionals $\mathcal{L}$ from $\mathcal{G}(\beta,\gamma)$ to $\mathbb{C}$ with the property that there exists a finite constant $C \ge 0$ such that for all $f \in \mathcal{G}(\beta,\gamma)$,

$$|\mathcal{L}(f)| \le C\|f\|_{\mathcal{G}(\beta,\gamma)}.$$

We denote by $\langle h, f\rangle$ the natural pairing of $h \in (\mathcal{G}(\beta,\gamma))'$ and $f \in \mathcal{G}(\beta,\gamma)$. It is easy to see that $f \in \mathcal{G}(x_0, r, \beta, \gamma)$ with $x_0 \in X$ and $r > 0$ if and only if $f \in \mathcal{G}(\beta,\gamma)$. Thus, for all $h \in (\mathcal{G}(\beta,\gamma))'$, $\langle h, f\rangle$ is well defined for all $f \in \mathcal{G}(x_0, r, \beta, \gamma)$ with $x_0 \in X$ and $r > 0$.

To state the definition of the inhomogeneous Besov and Triebel–Lizorkin spaces $B^s_{pq}(X)$ and $F^s_{pq}(X)$ studied in [20], we need the following approximations to the identity which were first introduced in [18].

Definition 1.2. A sequence $\{S_k\}_{k\ge 0}$ of linear operators is said to be an *approximation to the identity* if there exist $\varepsilon \in (0,\theta]$ and $0 < C < \infty$ such that for all $k \ge 0$ and all $x, x', y, y' \in X$, the kernel $S_k(x,y)$ of S_k is a function from $X \times X$ into $\mathbb{C}$ satisfying

(i) $S_k(x,y) = 0$ if $\varrho(x,y) \ge C2^{-k}$ and $\|S_k\|_{L^\infty(X)} \le C2^{dk}$;

(ii) $|S_k(x,y) - S_k(x',y)| \le C2^{k(d+\varepsilon)}\varrho(x,x')^\varepsilon$;

(iii) $|S_k(x,y) - S_k(x,y')| \le C2^{k(d+\varepsilon)}\varrho(y,y')^\varepsilon$;

(iv) $|[S_k(x,y) - S_k(x,y')] - [S_k(x',y) - S_k(x',y')]| \le C2^{k(d+2\varepsilon)}\varrho(x,x')^\varepsilon\varrho(y,y')^\varepsilon$;

(v) $\int_X S_k(x,y)\,d\mu(y) = 1$;

(vi) $\int_X S_k(x,y)\,d\mu(x) = 1$.

Here, that $S_k(x,y)$ is the kernel of S_k means that for suitable functions f,

$$S_k f(x) = \int_X S_k(x,y)f(y)\,d\mu(y).$$

Remark 1.1. The approximation to the identity can be defined in a more general form as follows. A sequence $\{S_k\}_{k\ge 0}$ of linear operators is said to be an *approximation to the identity* if there exist $\beta,\gamma \in (0,\theta]$, $\varepsilon, \sigma > 0$ and $0 < C < \infty$ such that for all $k \ge 0$ and all $x, x', y, y' \in X$, the kernel $S_k(x,y)$ of S_k is a function from $X \times X$ into $\mathbb{C}$ satisfying

(i) $|S_k(x,y)| \le C\dfrac{2^{-k\varepsilon}}{(2^{-k} + \varrho(x,y))^{d+\varepsilon}}$;

(ii) $|S_k(x,y) - S_k(x',y)| \le C\left(\dfrac{\varrho(x,x')}{2^{-k} + \varrho(x,y)}\right)^\beta \dfrac{2^{-k\varepsilon}}{(2^{-k} + \varrho(x,y))^{d+\varepsilon}}$ for $\varrho(x,x') \le \dfrac{1}{2A}(2^{-k} + \varrho(x,y))$;

(iii) $|S_k(x,y) - S_k(x,y')| \le C\left(\dfrac{\varrho(y,y')}{2^{-k} + \varrho(x,y)}\right)^\beta \dfrac{2^{-k\varepsilon}}{(2^{-k} + \varrho(x,y))^{d+\varepsilon}}$ for $\varrho(y,y') \le \dfrac{1}{2A}(2^{-k} + \varrho(x,y))$;

(iv) $|[S_k(x,y)-S_k(x,y')]-[S_k(x',y)-S_k(x',y')]|$

$$\le C\Big(\frac{\varrho(x,x')}{2^{-k}+\varrho(x,y)}\Big)^{\gamma}\Big(\frac{\varrho(y,y')}{2^{-k}+\varrho(x,y)}\Big)^{\gamma}\frac{2^{-k\sigma}}{(2^{-k}+\varrho(x,y))^{d+\sigma}}$$

for $\varrho(x,x')\le \frac{1}{2A}(2^{-k}+\varrho(x,y))$ and $\varrho(y,y')\le \frac{1}{2A}(2^{-k}+\varrho(x,y))$;

(v) $\int_X S_k(x,y)\,d\mu(y)=1$;

(vi) $\int_X S_k(x,y)\,d\mu(x)=1$.

Moreover, as pointed out in [19], in the above, we can take $\beta=\varepsilon\in(0,\theta]$, $\gamma=\varepsilon'$ and $\sigma=\varepsilon-\varepsilon'$, where ε' can be any positive number less than ε. Also, $1/2$ can be replaced by any $\delta\in(0,1)$. See also [5].

Remark 1.2. By Coifman's similar construction in [5], one can construct an approximation to the identity with compact supports as in Definition 1.2 for spaces of homogeneous type from Definition 0.1. Furthermore, one can show that for such an approximation to the identity, $\{S_k\}_{k=0}^{\infty}$, $\lim_{k\to\infty}S_k=I$, the identity operator on $L^2(X)$, in the strong operator topology on $L^2(X)$; see [5] or [23, p. 11]. By this fact, it is easy to show that any space of test functions, $\mathcal{G}(\beta,\gamma)$, with $0<\beta,\gamma\le\theta$, is a dense subset of $L^2(X)$.

The following inhomogeneous Calderón reproducing formulae established in [18] play an important role in the whole paper.

Lemma 1.1. *Suppose that $\{S_k\}_{k\ge0}$ is an approximation to the identity as in Definition* 1.2. *Let $D_k=S_k-S_{k-1}$ for $k\ge1$ and $D_0=S_0$. Then there exist families of linear operators $\widetilde{D}_k$ and $\widetilde{E}_k$ for $k\in\mathbb{N}\cup\{0\}$ and a fixed large integer $N\in\mathbb{N}$ such that for $f\in\mathcal{G}(\beta_1,\gamma_1)$ with $0<\beta_1,\gamma_1<\varepsilon$,*

$$f=\sum_{k=0}^{\infty}\widetilde{D}_kD_k(f)=\sum_{k=0}^{\infty}D_k\widetilde{E}_k(f), \tag{1.1}$$

where the series converge in the norm of $\mathcal{G}(\beta_1',\gamma_1')$ for $0<\beta_1'<\beta_1$ and $0<\gamma_1'<\gamma_1$. Moreover, the kernels of the operators $\widetilde{D}_k$ satisfy conditions (i) *and* (ii) *of Remark* 1.1 *with ε replaced by ε' for $0<\varepsilon'<\varepsilon$, and*

$$\int_X\widetilde{D}_k(x,y)\,d\mu(y)=\int_X\widetilde{D}_k(x,y)\,d\mu(x)=\begin{cases}1, & k=0,1,\dots,N,\\ 0, & k\ge N+1,\end{cases}$$

and the kernels of the operators $\widetilde{E}_k$ have the same properties.

Remark 1.3. By a similar argument to the proof of Theorem 3.9 in [23], one can also show that (1.1) holds for all $f\in L^p(X)$ with $1<p<\infty$ with the series converging in $L^p(X)$. Moreover, $\mathcal{G}(\beta,\gamma)$, with $0<\beta,\gamma\le\theta$, is a dense subset of $L^p(X)$ for $1<p<\infty$.

The next lemma was obtained in [18] by a duality argument from Lemma 1.1.

Lemma 1.2. *With the notation of Lemma* 1.1, *for all $f\in(\mathcal{G}(\beta_1,\gamma_1))'$ with $0<\beta_1,\gamma_1<\varepsilon$,* (1.1) *holds with the series converging in $(\mathcal{G}(\beta_1',\gamma_1'))'$ for $\varepsilon>\beta_1'>\beta_1$ and $\varepsilon>\gamma_1'>\gamma_1$.*

Now, we can introduce the spaces $B^s_{pq}(X)$ and $F^s_{pq}(X)$ via approximations to the identity; these spaces were first studied in [20].

DEFINITION 1.3. Let $\varepsilon \in (0,\theta]$, $s \in (-\varepsilon,\varepsilon)$ and $\{S_k\}_{k=0}^\infty$ be an approximation to the identity and let $E_k = S_k - S_{k-1}$ for $k \geq 1$ and $E_0 = S_0$. The *inhomogeneous Besov space* $B^s_{pq}(X)$ for $1 \leq p,q \leq \infty$ is the collection of $f \in (\mathcal{G}(\beta,\gamma))'$ for $\max(0,-s) < \beta < \varepsilon$ and $0 < \gamma < \varepsilon$ such that

$$\|f\|_{B^s_{pq}(X)} = \Big\{\sum_{k=0}^\infty [2^{ks}\|E_k(f)\|_{L^p(X)}]^q\Big\}^{1/q} < \infty.$$

The *inhomogeneous Triebel–Lizorkin* space $F^s_{pq}(X)$ for $1 < p < \infty$ and $1 < q \leq \infty$ is the collection of $f \in (\mathcal{G}(\beta,\gamma))'$ for $\max(0,-s) < \beta < \varepsilon$ and $0 < \gamma < \varepsilon$ such that

$$\|f\|_{F^s_{pq}(X)} = \Big\|\Big\{\sum_{k=0}^\infty [2^{ks}|E_k(f)|]^q\Big\}^{1/q}\Big\|_{L^p(X)} < \infty.$$

It was proved in [20] that the above definitions are independent of the choices of approximations to the identity and the pair (β,γ) with $\max(0,-s) < \beta < \varepsilon$ and $0 < \gamma < \varepsilon$. Moreover, by a similar argument, we can show that the above definitions are also independent of taking equivalent quasi-metrics satisfying (0.2). We say that a quasi-metric ϱ is *equivalent* to another quasi-metric ϱ' if there is a constant $C > 0$ such that for all $x, y \in X$,

$$C^{-1}\varrho'(x,y) \leq \varrho(x,y) \leq C\varrho'(x,y).$$

PROPOSITION 1.1. *Let ϱ and ϱ' be two equivalent quasi-metrics satisfying* (0.2) *with θ and θ', respectively. Suppose $\varepsilon \in (0,\theta]$, $\varepsilon' \in (0,\theta']$ and $|s| \leq \min(\varepsilon,\varepsilon')$. Let $\{S_k\}_{k=0}^\infty$ and $\{S'_k\}_{k=0}^\infty$ be two approximations to the identity with respect to ϱ, ε and ϱ', ε', respectively, as in Definition* 1.2 (*or Remark* 1.1). *Let $\{E_k\}_{k\in\mathbb{N}\cup\{0\}}$ be as in Definition* 1.3, *$E'_k = S'_k - S'_{k-1}$ for $k \in \mathbb{N}$ and $E'_0 = S'_0$. Then there is a constant $C > 0$ such that for all $f \in (\mathcal{G}(\beta,\gamma))'$ with $0 < \beta,\gamma < \varepsilon$, if*

$$\Big\{\sum_{k=0}^\infty [2^{ks}\|E_k(f)\|_{L^p(X)}]^q\Big\}^{1/q} < \infty$$

for $1 \leq p,q \leq \infty$, or

$$\Big\|\Big\{\sum_{k=0}^\infty [2^{ks}|E_k(f)|]^q\Big\}^{1/q}\Big\|_{L^p(X)} < \infty$$

for $1 < p < \infty$ and $1 < q \leq \infty$, then

$$\Big\{\sum_{k=0}^\infty [2^{ks}\|E'_k(f)\|_{L^p(X)}]^q\Big\}^{1/q} \leq C\Big\{\sum_{k=0}^\infty [2^{ks}\|E_k(f)\|_{L^p(X)}]^q\Big\}^{1/q}$$

for $1 \leq p,q \leq \infty$, or

$$\Big\|\Big\{\sum_{k=0}^\infty [2^{ks}|E'_k(f)|]^q\Big\}^{1/q}\Big\|_{L^p(X)} \leq C\Big\|\Big\{\sum_{k=0}^\infty [2^{ks}|E_k(f)|]^q\Big\}^{1/q}\Big\|_{L^p(X)}$$

for $1 < p < \infty$ and $1 < q \leq \infty$.

The converses are also true.

Proof. The proofs of these inequalities are similar, and are also similar to the proof of Lemma 1.3 in [20] and the proof of Proposition 4.1 in [23] by using Lemma 1.2. Let us just give an outline for the proof of the first inequalities. Let $f \in (\mathcal{G}(\beta,\gamma))'$ with $0<\beta,\gamma<\varepsilon$ and

$$\Big\{\sum_{k=0}^{\infty}[2^{ks}\|E_k(f)\|_{L^p(X)}]^q\Big\}^{1/q}<\infty.$$

By Lemma 1.2, there is a family of linear operators $\widetilde{D}_k$ and a large $N\in\mathbb{N}$ satisfying the conditions in Lemma 1.2 such that

$$f=\sum_{k=0}^{\infty}\widetilde{D}_kE_k(f), \tag{1.2}$$

where the series converges in $(\mathcal{G}(\beta',\gamma'))'$ with $\varepsilon>\beta'>\beta$ and $\varepsilon>\gamma'>\gamma$. Moreover, the kernels of $\widetilde{D}_k$'s satisfy the conditions (i) and (ii) of Remark 1.1 with any $\sigma_1\in(0,\varepsilon)$:

$$|S_k(x,y)|\le C\frac{2^{-k\sigma_1}}{(2^{-k}+\varrho(x,y))^{d+\sigma_1}}; \tag{1.3}$$

$$|S_k(x,y)-S_k(x',y)|\le C\bigg(\frac{\varrho(x,x')}{2^{-k}+\varrho(x,y)}\bigg)^{\sigma_1}\frac{2^{-k\sigma_1}}{(2^{-k}+\varrho(x,y))^{d+\sigma_1}} \tag{1.4}$$
$$\text{for } \varrho(x,x')\le\frac{1}{2A}(2^{-k}+\varrho(x,y)).$$

We now claim that for any $\sigma_2\in(s,\min(\varepsilon,\varepsilon'))$, there is a constant $C>0$ such that for all $k,l\in\mathbb{N}\cup\{0\}$ and all $x,y\in X$,

$$|(E'_k\widetilde{D}_l)(x,y)|\le C2^{-|k-l|\sigma_2}\frac{2^{-(k\wedge l)\sigma_2}}{(2^{-(k\wedge l)\sigma}+\varrho(x,y))^{d+\sigma_2}}, \tag{1.5}$$

where $k\wedge l=\min(k,l)$. The proof of (1.5) is completely similar to the proof of (1.6) in [20] and (3.9) in [18]; see also (2.15) below. For the convenience of the reader, we give the details by assuming $\{S'_k\}_{k=0}^{\infty}$ is an approximation to the identity as in Remark 1.1 with ϱ and ε replaced by ϱ' and ε', respectively. We recall that $E'_0=S'_0$ and $E'_k=S'_k-S'_{k-1}$ for $k\in\mathbb{N}$. For a given $\sigma_2\in(s,\min(\varepsilon,\varepsilon'))$, we choose $\sigma_1\in(0,\varepsilon)$ satisfying $\sigma_1>\sigma_2$. Suppose $l>k\ge0$. By (1.3), and (i) and (ii) of Remark 1.1, we have

$$\begin{aligned}
|(E'_k\widetilde{D}_l)(x,y)| &= \bigg|\int_X E'_k(x,z)\widetilde{D}_l(z,y)\,d\mu(z)\bigg| = \bigg|\int_X [E'_k(x,z)-E'_k(x,y)]\widetilde{D}_l(z,y)\,d\mu(z)\bigg|\\
&\le \int_{\{z:\,\varrho'(z,y)\le\frac{1}{2A}(2^{-k}+\varrho'(x,y))\}}|E'_k(x,z)-E'_k(x,y)||\widetilde{D}_l(z,y)|\,d\mu(z)\\
&\quad+\int_{\{z:\,\varrho'(z,y)>\frac{1}{2A}(2^{-k}+\varrho'(x,y))\}}|E'_k(x,z)||\widetilde{D}_l(z,y)|\,d\mu(z)\\
&\quad+\int_{\{z:\,\varrho'(z,y)>\frac{1}{2A}(2^{-k}+\varrho'(x,y))\}}|E'_k(x,y)||\widetilde{D}_l(z,y)|\,d\mu(z)\\
&\le \frac{C}{(2^{-k}+\varrho(x,y))^{d+\sigma_2}}\int_X\varrho(z,y)^{\sigma_2}\frac{2^{-l\sigma_1}}{(2^{-l}+\varrho(z,y))^{d+\sigma_1}}\,d\mu(z)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{C2^{-l\sigma_2}}{(2^{-k}+\varrho(x,y))^{d+\sigma_2}}\int\limits_X |E_k'(x,z)|\,d\mu(z)\\
&+C2^{-(l-k)\sigma_2}\frac{2^{-k\sigma_2}}{(2^{-k}+\varrho(x,y))^{d+\sigma_2}}\int\limits_X \frac{2^{-l(\sigma_1-\sigma_2)}}{(2^{-l}+\varrho(z,y))^{d+\sigma_1-\sigma_2}}\,d\mu(z)\\
\le{}& C2^{-(l-k)\sigma_2}\frac{2^{-k\sigma_2}}{(2^{-k}+\varrho(x,y))^{d+\sigma_2}}.
\end{aligned}
$$

Thus, (1.5) is true in this case. If $k > l \ge 0$, by (1.4) and (i), we can also show (1.5) in a similar way. The proof of (1.5) for $l = k = 0$ is trivial.

From (1.2), (1.5), and the Hölder inequality, it follows that for $1 \le p \le \infty$ and $1 \le q < \infty$,

$$
\begin{aligned}
\Big\{\sum_{k=0}^{\infty}[2^{ks}\|E_k'(f)\|_{L^p(X)}]^q\Big\}^{1/q} &\le C\Big\{\sum_{k=0}^{\infty}\Big[2^{ks}\sum_{l=0}^{\infty}2^{-|k-l|\sigma_2}\|E_l(f)\|_{L^p(X)}\Big]^q\Big\}^{1/q}\\
&\le C\Big\{\sum_{k=0}^{\infty}\Big[\sum_{l=0}^{\infty}2^{(k-l)s-|k-l|\sigma_2}\Big]^{q/q'}\Big[\sum_{l=0}^{\infty}2^{(k-l)s-|k-l|\sigma_2}(2^{ls}\|E_l(f)\|_{L^p(X)})^q\Big]\Big\}^{1/q}\\
&\le C\Big\{\sum_{l=0}^{\infty}[2^{ls}\|E_l(f)\|_{L^p(X)}]^q\Big\}^{1/q}.
\end{aligned}
$$

When $q = \infty$, the proof is trivial.

This finishes the proof of Proposition 1.1.

In [20], the atomic decompositions for these spaces were also given. To state these, we need the following construction of Christ [3], which provides an analogue of the grid of Euclidean dyadic cubes on a space of homogeneous type.

LEMMA 1.3. *Let* $(X,\varrho,\mu)_{d,\theta}$ *be a space of homogeneous type. Then there exists a collection* $\{Q_\alpha^k \subset X : k \in \mathbb{N}\cup\{0\},\ \alpha \in M_k\}$ *of open subsets, where* M_k *is some (possibly finite) index set, and constants* $\delta \in (0,1)$, $a_0 > 0$ *and* $0 < C < \infty$ *such that*

(i) $\mu(X \setminus \bigcup_\alpha Q_\alpha^k) = 0$ *for each fixed* k *and* $Q_\beta^k \cap Q_\alpha^k = \emptyset$ *if* $\alpha \ne \beta$;
(ii) *for any* α,β,k,l *with* $l \ge k$, *either* $Q_\beta^l \subset Q_\alpha^k$ *or* $Q_\beta^l \cap Q_\alpha^k = \emptyset$;
(iii) *for each* (k,α) *and each* $l < k$ *there is a unique* β *such that* $Q_\alpha^k \subset Q_\beta^l$;
(iv) $\operatorname{diam}(Q_\alpha^k) \le C\delta^k$;
(v) *each* Q_α^k *contains some ball* $B(z_\alpha^k, a_0\delta^k)$, *where* $z_\alpha^k \in X$.

In fact, we can think of Q_α^k as being essentially a dyadic cube with diameter roughly δ^k and center z_α^k.

The following (dyadic) smooth atoms on a space of homogeneous type were introduced in [23].

DEFINITION 1.4. Fix $\delta \in (0,1)$ and a collection $\{Q_\tau^k \subset X : k \in \mathbb{N}\cup\{0\},\ \tau \in M_k\}$ of open subsets satisfying the conditions of Lemma 1.3. A function $a_{Q_\tau^k}$ defined on X is said to be a γ-*smooth atom for* Q_τ^k if

(i) $\operatorname{supp} a_{Q_\tau^k} \subset B(z_\tau^k, 3AC\delta^k)$;

(ii) $\int_X a_{Q_\tau^k}(x)\, d\mu(x) = 0$;

(iii) $|a_{Q_\tau^k}(x)| \le \mu(Q_\tau^k)^{-1/2}$ and $|a_{Q_\tau^k}(x) - a_{Q_\tau^k}(y)| \le \mu(Q_\tau^k)^{-1/2-\gamma/d} \varrho(x,y)^\gamma$.

A function $a_{Q_\tau^k}$ defined on X is said to be a *γ-smooth block for Q_τ^k* if $a_{Q_\tau^k}$ satisfies only (i) and (iii) above.

As in the case $X = \mathbb{R}^n$ (see [9]), we also define certain inhomogeneous spaces of sequences indexed by "dyadic cubes" $\{Q_\tau^k\}_{\tau\in M_k,\, k\in\mathbb{N}\cup\{0\}} \equiv \mathcal{J}$ in X, which will characterize the coefficients in atomic and molecular decompositions of $B_{pq}^s(X)$ and $F_{pq}^s(X)$. For $-\varepsilon < s < \varepsilon$, $1 \le p, q \le \infty$, we let $b_{pq}^s(X)$ be the collection of all sequences $\lambda = \{\lambda_Q\}_{Q\in\mathcal{J}}$ such that

$$\|\lambda\|_{b_{pq}^s(X)} = \Big\{ \sum_{k=0}^{\infty} \Big[\sum_{\tau\in M_k} (\mu(Q_\tau^k)^{-s/d-1/2+1/p} |\lambda_{Q_\tau^k}|)^p \Big]^{q/p} \Big\}^{1/q}$$

is finite; and, for $-\varepsilon < s < \varepsilon$, $1 < p < \infty$, $1 < q \le \infty$, let $f_{pq}^s(X)$ be the collection of all sequences $\lambda = \{\lambda_Q\}_{Q\in\mathcal{J}}$ such that

$$\|\lambda\|_{f_{pq}^s(X)} = \Big\| \Big\{ \sum_{k=0}^{\infty} \sum_{\tau\in M_k} (\mu(Q_\tau^k)^{-s/d-1/2} |\lambda_{Q_\tau^k}| \chi_{Q_\tau^k})^q \Big\}^{1/q} \Big\|_{L^p(X)}$$

is finite, where $\chi_{Q_\tau^k}$ is the characteristic function of Q_τ^k.

We have the following atomic decompositions for $B_{pq}^s(X)$ and $F_{pq}^s(X)$, which were proved in [20].

LEMMA 1.4. *Suppose $-\varepsilon < s < \varepsilon$.*

(i) *If $1 \le p, q \le \infty$ and $f \in B_{pq}^s(X) \cap (\mathcal{G}(\beta,\gamma))'$ with $0 < \beta, \gamma < \varepsilon$, then there exist a sequence $\lambda = \{\lambda_{Q_\tau^k}\}_{Q_\tau^k\in\mathcal{J}} \in b_{pq}^s(X)$, ε-smooth atoms $\{a_{Q_\tau^k}\}_{k\in\mathbb{N},\,\tau\in M_k}$ and ε-smooth blocks $\{a_{Q_\tau^0}\}_{\tau\in M_0}$ such that*

$$f = \sum_{k=0}^{\infty} \sum_{\tau\in M_k} \lambda_{Q_\tau^k} a_{Q_\tau^k}$$

with convergence both in the norm of $B_{pq}^s(X)$ and in $(\mathcal{G}(\beta,\gamma))'$ when $1 \le p, q < \infty$ and only in $(\mathcal{G}(\beta,\gamma))'$ when $1 \le p, q \le \max(p,q) = \infty$, and

$$\|\lambda\|_{b_{pq}^s(X)} \le C\|f\|_{B_{pq}^s(X)}.$$

Similarly, if $1 < p < \infty$, $1 < q \le \infty$ and $f \in F_{pq}^s(X) \cap (\mathcal{G}(\beta,\gamma))'$ with $0 < \beta, \gamma < \varepsilon$, then there exist a sequence $\lambda = \{\lambda_{Q_\tau^k}\}_{Q_\tau^k\in\mathcal{J}} \in f_{pq}^s(X)$, ε-smooth atoms $\{a_{Q_\tau^k}\}_{k\in\mathbb{N},\,\tau\in M_k}$ and ε-smooth blocks $\{a_{Q_\tau^0}\}_{\tau\in M_0}$ such that

$$f = \sum_{k=0}^{\infty} \sum_{\tau\in M_k} \lambda_{Q_\tau^k} a_{Q_\tau^k}$$

with convergence both in the norm of $F_{pq}^s(X)$ and in $(\mathcal{G}(\beta,\gamma))'$ when $1 < p, q < \infty$ and only in $(\mathcal{G}(\beta,\gamma))'$ when $1 < p < \infty$ and $q = \infty$, and

$$\|\lambda\|_{f_{pq}^s(X)} \le C\|f\|_{F_{pq}^s(X)}.$$

(ii) *Conversely, suppose*

$$f=\sum_{k=0}^{\infty}\sum_{\tau\in M_k}\lambda_{Q_\tau^k}a_{Q_\tau^k}$$

in $(\mathcal{G}(\beta,\gamma))'$ *with* $\max(0,-s)<\beta<\varepsilon$ *and* $0<\gamma<\varepsilon$, *where* $a_{Q_\tau^0}$*'s for* $\tau\in M_0$ *are* ε*-smooth blocks and* $a_{Q_\tau^k}$*'s for* $k\in\mathbb{N}$ *and* $\tau\in M_k$ *are* ε*-smooth atoms. Then*

$$\|f\|_{B^s_{pq}(X)}\le C\|\lambda\|_{b^s_{pq}(X)}\quad \textit{for } 1\le p,q\le\infty,$$
$$\|f\|_{F^s_{pq}(X)}\le C\|\lambda\|_{f^s_{pq}(X)}\quad \textit{for } 1<p<\infty,\ 1<q\le\infty.$$

Characterizations of "smooth molecules" for $B^s_{pq}(X)$ and $F^s_{pq}(X)$ are also important in applications. In fact, we will use them and Lemma 1.4 to obtain the boundedness of fractional integrals and derivatives in the next section. See [20] for the proof of Lemma 1.5 below.

Definition 1.5. Fix $\delta\in(0,1)$ and a collection $\{Q_\tau^k\subset X: k\in\mathbb{N}\cup\{0\},\ \tau\in M_k\}$ of open subsets as in Lemma 1.3. A function $m_{Q_\tau^k}$ defined on X is said to be a (β,γ)-*smooth molecule for* Q_τ^k if

(i) $\int_X m_{Q_\tau^k}(x)\,d\mu(x)=0$;

(ii) $|m_{Q_\tau^k}(x)|\le\mu(Q_\tau^k)^{-1/2}(1+\delta^{-k}\varrho(x,z_\tau^k))^{-(d+\gamma)}$;

(iii) $$|m_{Q_\tau^k}(x)-m_{Q_\tau^k}(x')|\le\mu(Q_\tau^k)^{-1/2-\beta/d}\varrho(x,x')^\beta\left\{\frac{1}{(1+\delta^{-k}\varrho(x,z_\tau^k))^{d+\gamma}}+\frac{1}{(1+\delta^{-k}\varrho(x',z_\tau^k))^{d+\gamma}}\right\}.$$

A function $m_{Q_\tau^k}$ defined on X is said to be a (β,γ)-*smooth unit for* Q_τ^k if $m_{Q_\tau^k}$ satisfies only (ii) and (iii) above.

Lemma 1.5. *Suppose* $\{Q_\tau^k\}_{k\in\mathbb{N}\cup\{0\},\,\tau\in M_k}$ *are dyadic cubes in* X *as in Lemma* 1.3 *and that* $m_{Q_\tau^0}$ *is a* (β,γ)*-smooth unit for* Q_τ^0 *and* $\tau\in M_0$ *and* $m_{Q_\tau^k}$ *is a* (β,γ)*-smooth molecule for* Q_τ^k, $k\in\mathbb{N}$ *and* $\tau\in M_k$ *with* $\max(0,-s)<\beta<\varepsilon$ *and* $0<\gamma<\varepsilon$. *Then for* $\lambda=\{\lambda_{Q_\tau^k}\}_{Q_\tau^k\in\mathcal{J}}$,

$$\Big\|\sum_{k=0}^{\infty}\sum_{\tau}\lambda_{Q_\tau^k}m_{Q_\tau^k}\Big\|_{B^s_{pq}(X)}\le C\|\lambda\|_{b^s_{pq}(X)}\quad \textit{for } -\varepsilon<s<\varepsilon \textit{ and } 1\le p,q\le\infty,$$
$$\Big\|\sum_{k=0}^{\infty}\sum_{\tau}\lambda_{Q_\tau^k}m_{Q_\tau^k}\Big\|_{F^s_{pq}(X)}\le C\|\lambda\|_{f^s_{pq}(X)}\quad \textit{for } -\varepsilon<s<\varepsilon \textit{ and } 1<p<\infty,\ 1<q\le\infty.$$

In [22], inhomogeneous discrete Calderón reproducing formulae on X were established. We will use these formulae to establish frame characterizations of $B^s_{pq}(X)$ and $F^s_{pq}(X)$ in Section 5. To state these results, we need more notation. In the following, we will denote by $Q_\tau^{k,\nu}$, $\nu=1,2,\dots,N(k,\tau)$, the set of all cubes $Q_{\tau'}^{k+j}\subset Q_\tau^k$, where j is a fixed large positive integer. Denote by $y_\tau^{k,\nu}$ a point in $Q_\tau^{k,\nu}$.

The following discrete Calderón reproducing formulae are the main results in [22].

LEMMA 1.6. *Suppose that* $\{S_k\}_{k\ge 0}$ *is an approximation to the identity as in Definition 1.2. Let* $\{D_k\}_{k\in\mathbb{N}\cup\{0\}}$ *be as in Lemma 1.1. Then there exist families of linear operators* $\widetilde{D}_k$ *and* $\widetilde{E}_k$ *for* $k\in\mathbb{N}$, *and* $\widetilde{E}^{0,\nu}_\tau$ *for* $\tau\in M_0$ *and* $\nu=1,\dots,N(0,\tau)$, *and a fixed large integer* $N\in\mathbb{N}$ *such that for any fixed* $y^{k,\nu}_\tau\in Q^{k,\nu}_\tau$ *with* $k\in\mathbb{N}$, $\tau\in M_k$ *and* $\nu\in\{1,\dots,N(k,\tau)\}$ *and all* $f\in\mathcal{G}(\beta_1,\gamma_1)$ *with* $0<\beta_1,\gamma_1<\varepsilon$,

$$\begin{aligned}(1.6)\quad f&=\sum_{\tau\in M_0}\sum_{\nu=1}^{N(0,\tau)}\mu(Q^{0,\nu}_\tau)\widetilde{D}^{0,\nu}_\tau(x)D^{0,\nu}_{\tau,1}(f)+\sum_{k=1}^{N}\sum_{\tau\in M_k}\sum_{\nu=1}^{N(k,\tau)}\mu(Q^{k,\nu}_\tau)\widetilde{D}_k(x,y^{k,\nu}_\tau)D^{k,\nu}_{\tau,1}(f)\\&\quad+\sum_{k=N+1}^{\infty}\sum_{\tau\in M_k}\sum_{\nu=1}^{N(k,\tau)}\mu(Q^{k,\nu}_\tau)\widetilde{D}_k(x,y^{k,\nu}_\tau)D_k(f)(y^{k,\nu}_\tau)\\&=\sum_{\tau\in M_0}\sum_{\nu=1}^{N(0,\tau)}\mu(Q^{0,\nu}_\tau)D^{0,\nu}_{\tau,2}(x)\widetilde{E}^{0,\nu}_\tau(f)+\sum_{k=1}^{N}\sum_{\tau\in M_k}\sum_{\nu=1}^{N(k,\tau)}\mu(Q^{k,\nu}_\tau)D^{k,\nu}_{\tau,2}(x)\widetilde{E}_k(f)(y^{k,\nu}_\tau)\\&\quad+\sum_{k=N+1}^{\infty}\sum_{\tau\in M_k}\sum_{\nu=1}^{N(k,\tau)}\mu(Q^{k,\nu}_\tau)D_k(x,y^{k,\nu}_\tau)\widetilde{E}_k(f)(y^{k,\nu}_\tau),\end{aligned}$$

where the series converge in the norms of both $L^p(X)$, $1<p<\infty$, *and* $\mathcal{G}(\beta_1',\gamma_1')$ *for* $0<\beta_1'<\beta_1$ *and* $0<\gamma_1'<\gamma_1$; $\widetilde{D}^{0,\nu}_\tau(x)$ *for* $\tau\in M_0$ *and* $\nu=1,\dots,N(0,\tau)$ *is a function satisfying*

(i) $\int_X \widetilde{D}^{0,\nu}_\tau(x)\,d\mu(x)=1$,

(ii) *for any given* $\varepsilon'\in(0,\varepsilon)$, *there is a constant* $C>0$ *such that*

$$|\widetilde{D}^{0,\nu}_\tau(x)|\le C\frac{1}{(1+\varrho(x,y))^{d+\varepsilon'}}$$

for all $x\in X$ *and* $y\in Q^{0,\nu}_\tau$,

(iii) $$|\widetilde{D}^{0,\nu}_\tau(x)-\widetilde{D}^{0,\nu}_\tau(z)|\le C\left(\frac{\varrho(x,z)}{1+\varrho(x,y)}\right)^{\varepsilon'}\frac{1}{(1+\varrho(x,y))^{d+\varepsilon'}}$$

for all $x,z\in X$ *and all* $y\in Q^{0,\nu}_\tau$ *satisfying* $\varrho(x,z)\le\frac{1}{2A}(1+\varrho(x,y))$; *and*

$$\widetilde{E}^{0,\nu}_\tau(f)=\int_X \widetilde{E}^{0,\nu}_\tau(y)f(y)\,d\mu(y)$$

for $\tau\in M_0$ *and* $\nu=1,\dots,N(0,\tau)$, *and* $\widetilde{E}^{0,\nu}_\tau(x)$ *satisfies the same conditions as* $\widetilde{D}^{0,\nu}_\tau(x)$; *for* $k=0,1,\dots,N$, $\tau\in M_k$ *and* $\nu=1,\dots,N(k,\tau)$,

$$D^{k,\nu}_{\tau,1}(f)=\int_X D^{k,\nu}_{\tau,1}(y)f(y)\,d\mu(y),$$

and $D^{k,\nu}_{\tau,1}(y)$ *is defined by*

$$D^{k,\nu}_{\tau,1}(y)=\frac{1}{\mu(Q^{k,\nu}_\tau)}\int_{Q^{k,\nu}_\tau}D_k(z,y)\,d\mu(z);$$

and the function $D^{k,\nu}_{\tau,2}(x)$ *is defined by*

$$D^{k,\nu}_{\tau,2}(x) = \frac{1}{\mu(Q^{k,\nu}_\tau)} \int_{Q^{k,\nu}_\tau} D_k(x,z)\, d\mu(z).$$

Moreover, the kernels of the linear operators $\widetilde{D}_k$ *and* $\widetilde{E}_k$ *satisfy the same conditions as in Lemma* 1.1.

The following lemma was obtained in [22] by a dual argument.

LEMMA 1.7. *With the notation of Lemma* 1.6, *for all* $f \in (\mathcal{G}(\beta_1,\gamma_1))'$ *with* $0 < \beta_1, \gamma_1 < \varepsilon$, (1.6) *holds with the series converging in* $(\mathcal{G}(\beta_1',\gamma_1'))'$ *for* $\varepsilon > \beta_1' > \beta_1$ *and* $\varepsilon > \gamma_1' > \gamma_1$.

In [20], the following dual spaces of the spaces $B^s_{pq}(X)$ and $F^s_{pq}(X)$ were established.

LEMMA 1.8. *Suppose* $-\varepsilon < s < \varepsilon$.

(A) $(B^s_{pq}(X))^* = B^{-s}_{p'q'}(X)$ *for* $1 \le p,q < \infty$ *with* $1/p + 1/p' = 1/q + 1/q' = 1$. *More precisely, given* $g \in B^{-s}_{p'q'}(X)$, *then* $\mathcal{L}_g(f) = \langle f, g\rangle$ *defines a linear functional on* $\mathcal{G}(\varepsilon',\varepsilon') \cap B^s_{pq}(X)$ *with* $0 < \varepsilon' < \varepsilon$ *such that*

$$|\mathcal{L}_g(f)| \le C\|f\|_{B^s_{pq}(X)} \|g\|_{B^{-s}_{p'q'}(X)},$$

and this linear functional can be extended to $B^s_{pq}(X)$ *with norm at most* $C\|g\|_{B^{-s}_{p'q'}(X)}$.

Conversely, if $\mathcal{L}$ *is a linear functional on* $B^s_{pq}(X)$, *then there exists a unique* $g \in B^{-s}_{p'q'}(X)$ *such that* $\mathcal{L}_g(f) = \langle f, g\rangle$ *defines a linear functional on* $\mathcal{G}(\varepsilon',\varepsilon') \cap B^s_{pq}(X)$, *and* $\mathcal{L}$ *is the extension of* $\mathcal{L}_g$ *with*

$$\|g\|_{B^{-s}_{p'q'}(X)} \le C\|\mathcal{L}\|.$$

(B) $(F^s_{pq}(X))^* = F^{-s}_{p'q'}(X)$ *for* $1 < p,q < \infty$ *with* $1/p + 1/p' = 1/q + 1/q' = 1$. *More precisely, given* $g \in F^{-s}_{p'q'}(X)$, *then* $\mathcal{L}_g(f) = \langle f, g\rangle$ *defines a linear functional on* $\mathcal{G}(\varepsilon',\varepsilon') \cap F^s_{pq}(X)$ *with* $0 < \varepsilon' < \varepsilon$ *such that*

$$|\mathcal{L}_g(f)| \le C\|f\|_{F^s_{pq}(X)} \|g\|_{F^{-s}_{p'q'}(X)},$$

and this linear functional can be extended to $F^s_{pq}(X)$ *with norm at most* $C\|g\|_{F^{-s}_{p'q'}(X)}$.

Conversely, if $\mathcal{L}$ *is a linear functional on* $F^s_{pq}(X)$, *then there exists a unique* $g \in F^{-s}_{p'q'}(X)$ *such that* $\mathcal{L}_g(f) = \langle f, g\rangle$ *defines a linear functional on* $\mathcal{G}(\varepsilon',\varepsilon') \cap F^s_{pq}(X)$, *and* $\mathcal{L}$ *is the extension of* $\mathcal{L}_g$ *with*

$$\|g\|_{F^{-s}_{p'q'}(X)} \le C\|\mathcal{L}\|.$$

REMARK 1.4. We first remark that by Proposition 3.3 in [20], we know that for $0 < \varepsilon' < \varepsilon$, $\mathcal{G}(\varepsilon',\varepsilon') \cap B^s_{pq}(X)$ and $\mathcal{G}(\varepsilon',\varepsilon') \cap F^s_{pq}(X)$ are dense, respectively, in $B^s_{pq}(X)$ with $1 \le p,q < \infty$ and $F^s_{pq}(X)$ with $1 < p,q < \infty$; see also Proposition 4.11 in [23].

REMARK 1.5. We point out that Lemma 1.8(A) in [20] has the restriction $\min(p,q) > 1$. But, by a similar proof to that of Theorem 7.1 in [23], one can show that Lemma 1.8(A) holds even when $\min(p,q) = 1$. This is still true for Theorem 7.1 in [23]. Moreover, let us now define $\overset{\circ}{B}{}^s_{pq}(X)$ with $-\varepsilon < s < \varepsilon$ and $1 \le p,q \le \infty$ as the completion of

$\bigcup_{0<\varepsilon'<\varepsilon} \mathcal{G}(\varepsilon', \varepsilon')$ in $B^s_{pq}(X)$ endowed with the quasi-norm of $B^s_{pq}(X)$. Then, in the sense of Lemma 1.8, we have

$$(\overset{\circ}{B}{}^s_{pq}(X))^* = B^{-s}_{p'q'}(X) \tag{1.7}$$

with p' and q' as in Lemma 1.8. (1.7) is new only for the case $\max(p,q)=\infty$ in comparison with Lemma 1.8(A). This fact can be easily proved by combining the argument in [23, pp. 116–120] with that in [31, p. 180]; see also [30, pp. 121–122]. We omit the details.

We also need the following lemma which can be found in [23, p. 93]; see also [9].

LEMMA 1.9. *Let* $1 \le p \le \infty, \mu, \eta \in \mathbb{N} \cup \{0\}$ *with* $\eta \le \mu$ *and for "dyadic cubes"* Q^μ_τ,

$$|f_{Q^\mu_\tau}(x)| \le (1 + 2^\eta \varrho(x, z^\mu_\tau))^{-d-\sigma},$$

where z^μ_τ *is the "center" of* Q^μ_τ *as in Lemma* 1.3 *and* $\sigma > 0$ *(recall that* $\mu(Q^\mu_\tau) \approx 2^{-\mu d}$*). Then*

$$\Big\| \sum_\tau \lambda_{Q^\mu_\tau} f_{Q^\mu_\tau} \Big\|_{L^p(X)} \le C 2^{(\mu-\eta)d} 2^{-\mu d/p} \Big(\sum_\tau |\lambda_{Q^\mu_\tau}|^p \Big)^{1/p}$$

and

$$\sum_\tau |\lambda_{Q^\mu_\tau}| |f_{Q^\mu_\tau}(x)| \le C 2^{(\mu-\eta)d} M \Big(\sum_\tau |\lambda_{Q^\mu_\tau}| \chi_{Q^\mu_\tau} \Big)(x),$$

where C *is independent of* x, μ *and* η, *and* M *is the Hardy–Littlewood maximal operator on* X.

The following lemma was established in [22].

LEMMA 1.10. *For* $1 < p < \infty$, $F^0_{p2}(X) = L^p(X)$ *with equivalent norms.*

The following trivial properties of the spaces $B^s_{pq}(X)$ and $F^s_{pq}(X)$ can be obtained by combining their definitions with Hölder's inequality; see Proposition 2 in [31, p. 47]. We omit the details.

PROPOSITION 1.2. *Let* $-\varepsilon < s < \varepsilon$ *and* $-\varepsilon < s_1 < s_2 < \varepsilon$. *Then*

(i) $B^{s_2}_{p,q_2}(X) \subset B^{s_1}_{p,q_1}(X)$ *for* $1 \le p, q_1, q_2 \le \infty$;
(ii) $B^{s}_{p,q_2}(X) \subset B^{s}_{p,q_1}(X)$ *for* $1 \le p \le \infty$ *and* $1 \le q_2 \le q_1 \le \infty$;
(iii) $F^{s_2}_{p,q_2}(X) \subset F^{s_1}_{p,q_1}(X)$ *for* $1 < p < \infty$ *and* $1 < q_1, q_2 \le \infty$;
(iv) $F^{s}_{p,q_2}(X) \subset F^{s}_{p,q_1}(X)$ *for* $1 < p < \infty$ *and* $1 < q_2 \le q_1 \le \infty$.

Now let us use Lemma 1.4 to show our two claims in the introduction. Let $|\overline{s}| < 1/n$. We first show that $\overline{B}{}^{\overline{s}}_{pq}(X) = B^{n\overline{s}}_{pq}(X)$ for $1 \le p, q \le \infty$ and $\overline{F}{}^{\overline{s}}_{pq}(X) = F^{n\overline{s}}_{pq}(X)$ for $1 < p < \infty$ and $1 < q \le \infty$. We only show the first equality; the proof of the second is similar. Obviously, we can take $\{Q^k_\tau : k \in \mathbb{N} \cup \{0\},\ \tau \in M_k\}$ in Lemma 1.3 corresponding to $d = n$ and $\varrho(x,y) = |x-y|$ for all $x, y \in \mathbb{R}^n$ to be the usual dyadic cubes, that is, $Q^k_\tau = \{x \in \mathbb{R}^n : 2^{-k}\tau_i \le x_i < 2^{-k}(\tau_i + 1),\ i = 1, \ldots, n\}$, where we let $\tau \in M_k \equiv \mathbb{Z}^n$. We then take $\{\overline{Q}{}^{nk}_\tau : k \in \mathbb{N} \cup \{0\},\ \tau \in \overline{M}_{nk}\}$ in Lemma 1.3 corresponding to $d = 1$ and $\overline{\varrho}(x,y) = |x-y|^n$ for all $x, y \in \mathbb{R}^n$ to be $\overline{Q}{}^{nk}_\tau \equiv Q^k_\tau$ and $\overline{M}_{nk} \equiv M_k$. By Lemma 1.4, we then have

$$f = \sum_{k=0}^\infty \sum_{\tau \in \overline{M}_{nk}} \lambda_{\overline{Q}{}^{nk}_\tau} a_{\overline{Q}{}^{nk}_\tau} = \sum_{k=0}^\infty \sum_{\tau \in M_k} \lambda_{Q^k_\tau} a_{Q^k_\tau}$$

and

$$\|f\|_{\overline{B}^{\,\overline{s}}_{pq}(X)} \sim \Big\{ \sum_{k=0}^{\infty} \Big[\sum_{\tau \in \overline{M}_{nk}} (\mu(\overline{Q}^{\,nk}_{\tau})^{-\overline{s}-1/2+1/p} |\lambda_{\overline{Q}^{\,nk}_{\tau}}|)^p \Big]^{q/p} \Big\}^{1/p}$$

$$\sim \Big\{ \sum_{k=0}^{\infty} \Big[\sum_{\tau \in M_k} (\mu(Q^k_\tau)^{-\overline{s}-1/2+1/p} |\lambda_{Q^k_\tau}|)^p \Big]^{q/p} \Big\}^{1/p} \sim \|f\|_{B^{n\overline{s}}_{pq}(X)},$$

since $d = n$ and $\bar{d} = 1$.

Let $|s| < 1/n^2$. Now let us show $\widetilde{B}^{ns}_{pq}(X) = B^s_{pq}(X)$ for $1 \le p, q \le \infty$ and $\widetilde{F}^{ns}_{pq}(X) = F^s_{pq}(X)$ for $1 < p < \infty$ and $1 < q \le \infty$. As above, we only show the first equality. We now take $\{\widetilde{Q}^k_\tau : k \in \mathbb{N} \cup \{0\},\ \tau \in \widetilde{M}_k\}$ in Lemma 1.3 corresponding to $d = n^2$ and $\widetilde{\varrho}(x, y) = |x - y|^{1/n}$ for all $x, y \in \mathbb{R}^n$ to be $\widetilde{Q}^k_\tau \equiv Q^{nk}_\tau$ and $\widetilde{M}_k \equiv M_{nk}$, and we take $\{Q^{nk}_\tau : k \in \mathbb{N} \cup \{0\},\ \tau \in M_{nk}\}$ as those cubes in Lemma 1.3 corresponding to $d = 1$ and $\varrho(x, y) = |x - y|$ for all $x, y \in \mathbb{R}^n$. By Lemma 1.4, we then have

$$f = \sum_{k=0}^{\infty} \sum_{\tau \in \widetilde{M}_k} \lambda_{\widetilde{Q}^k_\tau} a_{\widetilde{Q}^k_\tau} = \sum_{k=0}^{\infty} \sum_{\tau \in M_{nk}} \lambda_{Q^{nk}_\tau} a_{Q^{nk}_\tau}$$

and

$$\|f\|_{\widetilde{B}^{ns}_{pq}(X)} \sim \Big\{ \sum_{k=0}^{\infty} \Big[\sum_{\tau \in \widetilde{M}_k} (\mu(\widetilde{Q}^k_\tau)^{-s/n-1/2+1/p} |\lambda_{\widetilde{Q}^k_\tau}|)^p \Big]^{q/p} \Big\}^{1/p}$$

$$\sim \Big\{ \sum_{k=0}^{\infty} \Big[\sum_{\tau \in M_{nk}} (\mu(Q^{nk}_\tau)^{-s/n-1/2+1/p} |\lambda_{Q^{nk}_\tau}|)^p \Big]^{q/p} \Big\}^{1/p} \sim \|f\|_{B^s_{pq}(X)},$$

since $\widetilde{d} = n^2$ and $d = n$.

In fact, by a technical modification of the above proofs, we can prove a more general result, where the quasi-metric is $|x - y|^\kappa$ for any given $\kappa > 0$ and all $x, y \in \mathbb{R}^n$, and μ is the n-dimensional Lebesgue measure. In this case, $d = n/\kappa$ and $\theta = \kappa$ if $\kappa \le 1$ or $\theta = 1/\kappa$ if $\kappa > 1$. We omit the details.

Finally, we establish a generalization of Lemma 1.4 which will be used in Section 6. In the following, we say a function $a_{Q^k_\tau}$ is a γ-*smooth block for* Q^k_τ if $a_{Q^k_\tau}$ only satisfies (i) and (iii) in Definition 1.4.

THEOREM 1.1. *Suppose* $0 < s < \varepsilon$.

(i) *If* $1 \le p, q \le \infty$ *and* $f \in B^s_{pq}(X) \cap (\mathcal{G}(\beta, \gamma))'$ *with* $0 < \beta, \gamma < \varepsilon$, *then there exist a sequence* $\lambda = \{\lambda_{Q^k_\tau}\}_{Q^k_\tau \in \mathcal{J}} \in b^s_{pq}(X)$ *and* ε-*smooth blocks* $\{a_{Q^k_\tau}\}_{k \in \mathbb{N} \cup \{0\},\, \tau \in M_k}$ *such that*

$$f = \sum_{k=0}^{\infty} \sum_{\tau \in M_k} \lambda_{Q^k_\tau} a_{Q^k_\tau}$$

with convergence both in the norm of $B^s_{pq}(X)$ *and in* $(\mathcal{G}(\beta, \gamma))'$ *when* $1 \le p, q < \infty$ *and only in* $(\mathcal{G}(\beta, \gamma))'$ *when* $1 \le p, q \le \max(p, q) = \infty$, *and*

$$\|\lambda\|_{b^s_{pq}(X)} \le C \|f\|_{B^s_{pq}(X)}.$$

Similarly, if $1 < p < \infty$, $1 < q \le \infty$ *and* $f \in F^s_{pq}(X) \cap (\mathcal{G}(\beta, \gamma))'$ *with* $0 < \beta,\ \gamma < \varepsilon$, *then there exist a sequence* $\lambda = \{\lambda_{Q^k_\tau}\}_{Q^k_\tau \in \mathcal{J}} \in f^s_{pq}(X)$ *and* ε-*smooth blocks* $\{a_{Q^k_\tau}\}_{k \in \mathbb{N} \cup \{0\},\, \tau \in M_k}$

such that

$$f = \sum_{k=0}^{\infty} \sum_{\tau \in M_k} \lambda_{Q_\tau^k} a_{Q_\tau^k}$$

with convergence both in the norm of $F_{pq}^s(X)$ *and in* $(\mathcal{G}(\beta,\gamma))'$ *when* $1 < p, q < \infty$ *and only in* $(\mathcal{G}(\beta,\gamma))'$ *when* $1 < p < \infty$ *and* $q = \infty$, *and*

$$\|\lambda\|_{f_{pq}^s(X)} \le C\|f\|_{F_{pq}^s(X)}.$$

(ii) *Conversely, suppose*

$$f = \sum_{k=0}^{\infty} \sum_{\tau \in M_k} \lambda_{Q_\tau^k} a_{Q_\tau^k}$$

in $(\mathcal{G}(\beta,\gamma))'$ *with* $0 < \beta, \gamma < \varepsilon$, *where* $a_{Q_\tau^k}$ *for* $k \in \mathbb{N} \cup \{0\}$ *are* ε*-smooth blocks. Then*

$$\|f\|_{B_{pq}^s(X)} \le C\|\lambda\|_{b_{pq}^s(X)} \quad \text{for } 1 \le p, q \le \infty,$$
$$\|f\|_{F_{pq}^s(X)} \le C\|\lambda\|_{f_{pq}^s(X)} \quad \text{for } 1 < p < \infty \text{ and } 1 < q \le \infty.$$

Proof. (i) is just a corollary of Lemma 1.4. To show (ii), let $\{S_k\}_{k=0}^{\infty}$ be an approximation to the identity, $E_k = S_k - S_{k-1}$ for $k \in \mathbb{N}$ and $E_0 = S_0$. We need to establish that for $k, l \in \mathbb{N} \cup \{0\}$, $k \le l$, $\tau \in M_l$ and all $x \in X$,

$$|E_k(a_{Q_\tau^l})(x)| \le C\mu(Q_\tau^l)^{-1/2} 2^{-(l-k)d} (1 + 2^k \varrho(x, z_\tau^l))^{-(d+\varepsilon)} \tag{1.8}$$

and that for $k, l \in \mathbb{N} \cup \{0\}$, $k \ge l$, $\tau \in M_l$ and all $x \in X$,

$$|E_k(a_{Q_\tau^l})(x)| \le C\mu(Q_\tau^l)^{-1/2} 2^{-(k-l)\varepsilon} (1 + 2^l \varrho(x, z_\tau^l))^{-(d+\varepsilon)}, \tag{1.9}$$

where C is independent of k, l, τ and x.

(1.9) is just (2.10) in [20]; see also (6.16) in [23]. To show (1.8), by Definitions 1.2 and 1.4, we have $\operatorname{supp} E_k(a_{Q_\tau^l}) \subset \{x \in X : \varrho(x, z_\tau^l) \le 4A^2C2^{-k}\}$. Thus,

$$\begin{aligned}
|E_k(a_{Q_\tau^l})(x)| &= \Big| \int_X E_k(x,y) a_{Q_\tau^l}(y)\, d\mu(y) \Big| \chi_{\{x \in X:\, \varrho(x, z_\tau^l) \le 4A^2C2^{-k}\}}(x) \\
&\le C\mu(Q_\tau^l)^{-1/2} 2^{-(l-k)d} \chi_{\{x \in X:\, \varrho(x, z_\tau^l) \le 4A^2C2^{-k}\}}(x) \\
&\le C\mu(Q_\tau^l)^{-1/2} 2^{-(l-k)d} (1 + 2^k \varrho(x, z_\tau^l))^{-(d+\varepsilon)}.
\end{aligned}$$

Thus, (1.8) holds.

Using (1.8), (1.9) and the fact that $s > 0$, together with an argument similar to [23, pp. 94–96] or [20], we can prove (ii).

This finishes the proof of Theorem 1.1.

2. Fractional integrals and derivatives

In this section, we work on spaces of homogeneous type, $(X, \varrho, \mu)_{d,\theta}$, as defined in Definition 0.1. We introduce fractional integrals and derivatives by means of approximations to the identity and then by using atomic and molecular decomposition characterizations, we establish their invertibility on $B_{pq}^s(X)$ and $F_{pq}^s(X)$.

DEFINITION 2.1. Let $(X, \varrho, \mu)_{d,\theta}$ be a space of homogeneous type as in Definition 0.1. Let $\{S_l\}_{l=0}^{\infty}$ be an approximation to the identity as in Definition 1.2 and let $E_l = S_l - S_{l-1}$ for $l \geq 1$ and $E_0 = S_0$. Let $\alpha \in \mathbb{R}$. Then the operator I_α for $f \in \mathcal{G}(\beta, \gamma)$ with $0 < \beta \leq \theta$ and $0 < \gamma$ is defined by

$$I_\alpha(f)(x) = \sum_{l=0}^{\infty} 2^{-l\alpha} E_l(f)(x),$$

where $x \in X$.

Obviously, when $\alpha > 0$, I_α is the discrete and inhomogeneous version of the fractional integrals introduced in [11] and [12]; while when $\alpha < 0$, I_α is the discrete and inhomogeneous version of the fractional derivatives introduced there. When $\alpha = 0$, I_α is just the identity. We also mention that in [28] and [29], Nahmod has considered some discrete and inhomogeneous fractional integrals and derivatives similar to the above.

THEOREM 2.1. *Let $\varepsilon \in (0, \theta]$, $\alpha \in \mathbb{R}$, $\theta \geq \beta > 0$, $\varepsilon > \alpha + \beta > 0$ and $\gamma > \max(\alpha, 0)$. Then I_α maps $\mathcal{G}(\beta, \gamma)$ continuously into $\mathcal{G}(\beta + \alpha, \gamma - \max(\alpha, 0))$, namely, there is a constant $C > 0$ independent of f such that for all $f \in \mathcal{G}(\beta, \gamma)$,*

$$\|I_\alpha(f)\|_{\mathcal{G}(\beta+\alpha, \gamma - \max(\alpha,0))} \leq C\|f\|_{\mathcal{G}(\beta,\gamma)}.$$

Proof. Let $f \in \mathcal{G}(\beta, \gamma)$. We have

$$\begin{aligned} |E_0(f)(x)| &= \Big| \int_X E_0(x,y) f(y)\, d\mu(y) \Big| \\ &\leq \|f\|_{\mathcal{G}(\beta,\gamma)} \int_{\{x: \varrho(x,y) \leq C\}} |E_0(x,y)| \frac{1}{(1+\varrho(y,x_0))^{d+\gamma}}\, d\mu(y) \\ &\leq C\|f\|_{\mathcal{G}(\beta,\gamma)} \frac{1}{(1+\varrho(x,x_0))^{d+\gamma}}, \end{aligned} \tag{2.1}$$

since $1 + \varrho(x, x_0) \leq A(1 + C)(1 + \varrho(y, x_0))$.

For $l \in \mathbb{N}$, we then have

$$\begin{aligned} |E_l(f)(x)| &= \Big| \int_X E_l(x,y) f(y)\, d\mu(y) \Big| = \Big| \int_X E_l(x,y)(f(y) - f(x))\, d\mu(y) \Big| \\ &\leq \frac{\|f\|_{\mathcal{G}(\beta,\gamma)}}{(1+\varrho(x,x_0))^{d+\gamma+\beta}} \int_{\{x:\, \varrho(x,y) \leq C2^{-l}\}} |E_l(x,y)| \varrho(x,y)^\beta\, d\mu(y) \\ &\leq C2^{-l\beta} \|f\|_{\mathcal{G}(\beta,\gamma)} \frac{1}{(1+\varrho(x,x_0))^{d+\gamma+\beta}}. \end{aligned} \tag{2.2}$$

By (2.1) and (2.2), we obtain

$$\begin{aligned} |I_\alpha(f)(x)| &= \Big| \sum_{l=0}^{\infty} 2^{-l\alpha} E_l(f)(x) \Big| \leq C\|f\|_{\mathcal{G}(\beta,\gamma)} \frac{1}{(1+\varrho(x,x_0))^{d+\gamma}} \sum_{l=0}^{\infty} 2^{-l(\alpha+\beta)} \\ &\leq C\|f\|_{\mathcal{G}(\beta,\gamma)} \frac{1}{(1+\varrho(x,x_0))^{d+\gamma}}, \end{aligned} \tag{2.3}$$

since $\alpha + \beta > 0$.

Now, if $\frac{1}{4A^2} < \varrho(x,x') \le \frac{1}{2A}(1+\varrho(x,x_0))$, by (2.2) for $l \in \mathbb{N}$, we obtain

$$
\begin{aligned}
(2.4)\qquad |I_\alpha(f)(x)-I_\alpha(f)(x')| &\le |E_0(f)(x)-E_0(f)(x')|+\sum_{l=1}^{\infty}2^{-l\alpha}[|E_l(f)(x)|+|E_l(f)(x')|]\\
&\le \Big|\int_X (E_0(x,y)-E_0(x',y))(f(y)-f(x))\,d\mu(y)\Big|\\
&\quad + C\|f\|_{\mathcal{G}(\beta,\gamma)}\sum_{l=1}^{\infty}2^{-l(\alpha+\beta)}\Big\{\frac{1}{(1+\varrho(x,x_0))^{d+\gamma+\beta}}+\frac{1}{(1+\varrho(x',x_0))^{d+\gamma+\beta}}\Big\}\\
&\le C\frac{\|f\|_{\mathcal{G}(\beta,\gamma)}}{(1+\varrho(x,x_0))^{d+\gamma+\beta}}\Big\{1+\int_X[|E_0(x,y)|+|E_0(x',y)|]\varrho(y,x)^\beta\,d\mu(y)\Big\}\\
&\le C\frac{\|f\|_{\mathcal{G}(\beta,\gamma)}}{(1+\varrho(x,x_0))^{d+\gamma+\beta}}\{1+\varrho(x,x')^\beta\}\\
&\le C\|f\|_{\mathcal{G}(\beta,\gamma)}\Big(\frac{\varrho(x,x')}{1+\varrho(x,x_0)}\Big)^{\alpha+\beta}\frac{1}{(1+\varrho(x,x_0))^{d+\gamma-\max(\alpha,0)}},
\end{aligned}
$$

since $1+\varrho(x,x_0)\le 2A(1+\varrho(x',x_0))$ and $\alpha+\beta>0$, where for the term $l=0$, we used the fact that $\varrho(y,x)\le AC+A\varrho(x,x')$ if $\varrho(x',y)\le C$.

Now, we suppose that there is an $l_1\in\mathbb{N}$ such that

$$\frac{2^{-l_1}}{4A^2} < \varrho(x,x') \le \frac{2^{1-l_1}}{4A^2}.$$

For the terms with $l \ge l_1$, by (2.2), we obtain

$$
\begin{aligned}
(2.5)\qquad |I_\alpha(f)(x)-I_\alpha(f)(x')| &= \Big|\sum_{l=0}^{\infty}2^{-l\alpha}[E_l(f)(x)-E_l(f)(x')]\Big|\\
&\le \sum_{l=0}^{l_1}2^{-l\alpha}|E_l(f)(x)-E_l(f)(x')|+\sum_{l=l_1+1}^{\infty}2^{-l\alpha}[|E_l(f)(x)|+|E_l(f)(x')|]\\
&= \sum_{l=0}^{l_1}2^{-l\alpha}\Big|\int_X[E_l(x,y)-E_l(x',y)][f(y)-f(x)]\,d\mu(y)\Big|\\
&\quad + \sum_{l=l_1+1}^{\infty}2^{-l\alpha}[|E_l(f)(x)|+|E_l(f)(x')|]\\
&\le C\|f\|_{\mathcal{G}(\beta,\gamma)}\frac{1}{(1+\varrho(x,x_0))^{d+\gamma+\beta}}\Big\{\varrho(x,x')^\varepsilon\sum_{l=0}^{l_1}2^{l(\varepsilon-\alpha-\beta)}+\sum_{l=l_1+1}^{\infty}2^{-l(\alpha+\beta)}\Big\}\\
&\le C\|f\|_{\mathcal{G}(\beta,\gamma)}\frac{\varrho(x,x')^{\alpha+\beta}}{(1+\varrho(x,x_0))^{d+\gamma+\beta}},
\end{aligned}
$$

since $\alpha+\beta\in(0,\varepsilon)$.

Thus, if $\varrho(x,x') \le \frac{1}{2A}(1+\varrho(x,x_0))$, by (2.4) and (2.5), we obtain

$$(2.6)\quad |I_\alpha(f)(x)-I_\alpha(f)(x')| \le C\|f\|_{\mathcal{G}(\beta,\gamma)}\left(\frac{\varrho(x,x')}{1+\varrho(x,x_0)}\right)^{\alpha+\beta}\frac{1}{(1+\varrho(x,x_0))^{d+\gamma-\max(\alpha,0)}}.$$

By (2.3) and (2.6), the proof of Theorem 2.1 is complete.

Remark 2.1. We remark that in the proof of Theorem 2.1, only reguality in the first variable of the kernels of E_l is necessary; this fact will be used in Section 3.

Remark 2.2. By a similar argument, we can show Theorem 2.1 is still true if I_α is defined by use of approximations to the identity without compact supports as in Remark 1.1.

Let $\{E_k\}_{k\in\mathbb{N}\cup\{0\}}$ be as in Definition 2.1. We define a new family of linear operators $\{E_k^t\}_{k\in\mathbb{N}\cup\{0\}}$ by letting their kernels $E_k^t(x,y)$ be $E_k(y,x)$ for all $k\in\mathbb{N}\cup\{0\}$ and all $x,y\in X$. For $\alpha\in\mathbb{R}$, we define

$$I_\alpha^t(f)(x)=\sum_{k=0}^{\infty}2^{-k\alpha}E_k^t(f)(x)$$

for all test functions f. We now generalize the fractional integrals to the dual spaces by use of this operator.

Definition 2.2. Let $\alpha\in(-\varepsilon,\varepsilon)$, $0<\beta\le\theta$, $0<\beta+\alpha<\varepsilon$ and $\gamma>\max(\alpha,0)$. We define I_α on $(\mathcal{G}(\beta+\alpha,\gamma-\max(\alpha,0)))'$ by

$$\langle I_\alpha(f),\varphi\rangle=\langle f,I_\alpha^t(\varphi)\rangle\quad\text{for } f\in(\mathcal{G}(\beta+\alpha,\gamma-\max(\alpha,0)))' \text{ and } \varphi\in\mathcal{G}(\beta,\gamma).$$

We will use the atomic and molecular characterizations of $B_{pq}^s(X)$ and $F_{pq}^s(X)$ to establish the boundedness of I_α on these spaces.

Theorem 2.2. *Let $s,\alpha\in(-\varepsilon,\varepsilon)$ be such that $\alpha+s\in(-\varepsilon,\varepsilon)$. Then I_α maps $B_{pq}^s(X)$ continuously into $B_{pq}^{s+\alpha}(X)$ for $1\le p,q\le\infty$ and $F_{pq}^s(X)$ continuously into $F_{pq}^{s+\alpha}(X)$ for $1<p<\infty$ and $1<q\le\infty$, namely, there is a constant $C>0$ independent of f such that*

$$\|I_\alpha(f)\|_{B_{pq}^{s+\alpha}(X)}\le C\|f\|_{B_{pq}^s(X)}\quad\text{for all } f\in B_{pq}^s(X),$$
$$\|I_\alpha(f)\|_{F_{pq}^{s+\alpha}(X)}\le C\|f\|_{F_{pq}^s(X)}\quad\text{for all } f\in F_{pq}^s(X).$$

Proof. Let $0<\gamma<\varepsilon$ and $\max(0,-s-\alpha)<\beta<\varepsilon$. Let $\{a_{Q_\tau^k}\}_{k\in\mathbb{N},\tau\in M_k}$ be ε-smooth atoms and $\{a_{Q_\tau^0}\}_{\tau\in M_0}$ be ε-smooth blocks as in Definition 1.4 with $\delta=1/2$. In the rest of the paper, we suppose $\delta=1/2$; see [23, pp. 96–98] for how to remove this restriction. For $k\in\mathbb{N}\cup\{0\}$ and $\tau\in M_k$, we define

$$m_{Q_\tau^k}(x)=2^{k\alpha}I_\alpha(a_{Q_\tau^k})(x).$$

By Lemmas 1.4 and 1.5, we only need to verify that $m_{Q_\tau^k}$ is a (β,γ)-smooth molecule for Q_τ^k, $k\in\mathbb{N}$ and $\tau\in M_k$ and that $m_{Q_\tau^0}$ is a (β,γ)-smooth unit for Q_τ^0 and $\tau\in M_0$. Let us begin with the latter. Obviously, we can suppose $\alpha \neq 0$. We have

$$\begin{aligned}|m_{Q_\tau^0}(x)| &\le |E_0(a_{Q_\tau^0})(x)| + \sum_{l=1}^{\infty} 2^{-l\alpha}|E_l(a_{Q_\tau^0})(x)| \\ &\le C + \sum_{l=1}^{\infty} 2^{-l\alpha}\Big|\int_X E_l(x,y)(a_{Q_\tau^0}(x) - a_{Q_\tau^0}(y))\,d\mu(y)\Big| \\ &\le C+\sum_{l=1}^{\infty} 2^{-l\alpha}\int_X |E_l(x,y)|\mu(Q_\tau^0)^{-1/2-\varepsilon/d}\varrho(x,y)^{\varepsilon}\,d\mu(y) \le C+\sum_{l=1}^{\infty} 2^{-l(\alpha+\varepsilon)} \le C,\end{aligned}$$

since $\alpha > -\varepsilon$. Noting that $\operatorname{supp} m_{Q_\tau^0} \subset \{x \in X : \varrho(x, z_\tau^0) \le 4A^2C\}$, we have

$$|m_{Q_\tau^0}(x)| \le C\mu(Q_\tau^0)^{-1/2}(1 + \varrho(x, z_\tau^0))^{-(d+\gamma)}. \tag{2.7}$$

Now we claim that there are β and γ satisfying $0 < \gamma < \varepsilon$ and $\max(0, -s-\alpha) < \beta < \varepsilon$ such that

$$\begin{aligned}&|m_{Q_\tau^0}(x) - m_{Q_\tau^0}(x')| \\ &\quad\le C\mu(Q_\tau^0)^{-1/2-\beta/d}\varrho(x,x')^{\beta}\Big\{\frac{1}{(1+\varrho(x,z_\tau^0))^{d+\gamma}} + \frac{1}{(1+\varrho(x',z_\tau^0))^{d+\gamma}}\Big\}.\end{aligned} \tag{2.8}$$

We consider three cases.

Case 1: $\varrho(x, x') \ge 6A^2C$. In this case, since $m_{Q_\tau^0}$ satisfies (2.7), we have

$$\begin{aligned}|m_{Q_\tau^0}(x) - m_{Q_\tau^0}(x')| &\le C\mu(Q_\tau^0)^{-1/2}\Big\{\frac{1}{(1+\varrho(x,z_\tau^0))^{d+\gamma}} + \frac{1}{(1+\varrho(x',z_\tau^0))^{d+\gamma}}\Big\} \\ &\le C\mu(Q_\tau^0)^{-1/2-\beta/d}\varrho(x,x')^{\beta}\Big\{\frac{1}{(1+\varrho(x,z_\tau^0))^{d+\gamma}} + \frac{1}{(1+\varrho(x',z_\tau^0))^{d+\gamma}}\Big\}.\end{aligned}$$

Thus, (2.8) holds in this case.

Case 2: $\varrho(x, x') < 6A^2C$ and $\varrho(x, z_\tau^0) > 12A^3C$. In this case, it is easy to see that

$$\varrho(x', z_\tau^0) > 6A^2C.$$

Thus, $m_{Q_\tau^0}(x) = m_{Q_\tau^0}(x') = 0$ and (2.8) holds.

Case 3: $\varrho(x, x') < 6A^2C$ and $\varrho(x, z_\tau^0) < 12A^3C$. In this case, we also have

$$\varrho(x', z_\tau^0) < 18A^4C.$$

We further suppose that there is an $l_1 \in \mathbb{N}$ such that

$$6A^2C2^{-l_1} \le \varrho(x, x') < 6A^2C2^{-l_1+1}.$$

We then write

$$\begin{aligned}|m_{Q_\tau^0}(x) - m_{Q_\tau^0}(x')| &= \Big|\sum_{l=0}^{\infty} 2^{-l\alpha}(E_l(a_{Q_\tau^0})(x) - E_l(a_{Q_\tau^0})(x'))\Big| \\ &\le |E_0(a_{Q_\tau^0})(x) - E_0(a_{Q_\tau^0})(x')| + \Big|\sum_{l=1}^{l_1} 2^{-l\alpha}(E_l(a_{Q_\tau^0})(x) - E_l(a_{Q_\tau^0})(x'))\Big| \\ &\quad + \sum_{l=l_1+1}^{\infty} 2^{-l\alpha}(|E_l(a_{Q_\tau^0})(x)| + |E_l(a_{Q_\tau^0})(x')|)\end{aligned}$$

$$\begin{aligned}
&\le \Big|\int_X (E_0(x,y)-E_0(x',y))a_{Q^0_\tau}(y)\,d\mu(y)\Big| \\
&\quad + \sum_{l=1}^{l_1} 2^{-l\alpha}\Big|\int_X (E_l(x,y)-E_l(x',y))(a_{Q^0_\tau}(y)-a_{Q^0_\tau}(x))\,d\mu(y)\Big| \\
&\quad + \sum_{l=l_1+1}^{\infty} 2^{-l\alpha}\Big[\Big|\int_X E_l(x,y)(a_{Q^0_\tau}(y)-a_{Q^0_\tau}(x))\,d\mu(y)\Big| \\
&\quad + \Big|\int_X E_l(x',y)(a_{Q^0_\tau}(y)-a_{Q^0_\tau}(x'))\,d\mu(y)\Big|\Big] \\
&\le C\varrho(x,x')^\varepsilon + C\varrho(x,x')^\varepsilon \sum_{l=1}^{l_1} 2^{-l\alpha} + C\sum_{l=l_1+1}^{\infty} 2^{-l(\alpha+\varepsilon)} \\
&\le \begin{cases} C\varrho(x,x')^\varepsilon, & \alpha>0, \\ C\varrho(x,x')^{\varepsilon+\alpha}, & -\varepsilon<\alpha<0. \end{cases}
\end{aligned}$$

Thus, if we take $\beta=\varepsilon$ for $\alpha>0$ and $\beta=\varepsilon+\alpha$ for $-\varepsilon<\alpha<0$, then (2.8) also holds in this case.

From (2.7) and (2.8), we deduce that $m_{Q^0_\tau}$ is a (β,γ)-smooth unit for Q^0_τ and $\tau\in M_0$, multiplied with a normalizing constant.

Let $k\in\mathbb{N}$ and $\tau\in M_k$. We intend to show that there are β and γ satisfying $0<\gamma<\varepsilon$ and $\max(0,-s-\alpha)<\beta<\varepsilon$ such that $m_{Q^k_\tau}$ is a (β,γ)-smooth molecule for Q^k_τ. We first write

$$m_{Q^k_\tau}(x) = \sum_{l=0}^{\infty} 2^{(k-l)\alpha}E_l(a_{Q^k_\tau})(x) = \sum_{l=0}^{k} 2^{(k-l)\alpha}E_l(a_{Q^k_\tau})(x) + \sum_{l=k+1}^{\infty}\ldots = G_1+G_2.$$

For $0\le l\le k$, we have

$$\begin{aligned}
|E_l(a_{Q^k_\tau})(x)| &= \Big|\int_X E_l(x,y)a_{Q^k_\tau}(y)\,d\mu(y)\Big|\chi_{\{x:\,\varrho(x,z^k_\tau)\le 4A^2C2^{-l}\}}(x) \\
&= \Big|\int_X (E_l(x,y)-E_l(x,z^k_\tau))a_{Q^k_\tau}(y)\,d\mu(y)\Big|\chi_{\{x:\,\varrho(x,z^k_\tau)\le 4A^2C2^{-l}\}}(x) \\
&\le C\mu(Q^k_\tau)^{-1/2}(1+2^k\varrho(x,z^k_\tau))^{-(d+\gamma)}2^{(k-l)(\gamma-\varepsilon)}.
\end{aligned}$$

From this, it follows that

$$\begin{aligned}
|G_1| &\le C\mu(Q^k_\tau)^{-1/2}(1+2^k\varrho(x,z^k_\tau))^{-(d+\gamma)}\sum_{l=0}^{k} 2^{(k-l)(\gamma-\varepsilon+\alpha)} \\
&\le C\mu(Q^k_\tau)^{-1/2}(1+2^k\varrho(x,z^k_\tau))^{-(d+\gamma)},
\end{aligned}$$

if we choose $\gamma<\varepsilon-\alpha$.

For $k+1\le l<\infty$, we have

$$\begin{aligned}
|E_l(a_{Q^k_\tau})(x)| &= \Big|\int_X E_l(x,y)a_{Q^k_\tau}(y)\,d\mu(y)\Big|\chi_{\{x:\,\varrho(x,z^k_\tau)\le 4A^2C2^{-k}\}}(x) \\
&= \Big|\int_X E_l(x,y)(a_{Q^k_\tau}(y)-a_{Q^k_\tau}(x))\,d\mu(y)\Big|\chi_{\{x:\,\varrho(x,z^k_\tau)\le 4A^2C2^{-k}\}}(x) \\
&\le C\mu(Q^k_\tau)^{-1/2}(1+2^k\varrho(x,z^k_\tau))^{-(d+\gamma)}2^{(k-l)\varepsilon}.
\end{aligned}$$

From this, it follows that

$$|G_2| \le C\mu(Q^k_\tau)^{-1/2}(1+2^k\varrho(x,z^k_\tau))^{-(d+\gamma)} \sum_{l=k+1}^{\infty} 2^{(k-l)(\varepsilon+\alpha)} \le C\mu(Q^k_\tau)^{-1/2}(1+2^k\varrho(x,z^k_\tau))^{-(d+\gamma)},$$

since $\varepsilon+\alpha>0$. Thus, we have

$$|m_{Q^k_\tau}(x)| \le C\mu(Q^k_\tau)^{-1/2}(1+2^k\varrho(x,z^k_\tau))^{-(d+\gamma)}. \tag{2.9}$$

Now we claim that there are β and γ satisfying $0<\gamma<\varepsilon$ and $\max(0,-s-\alpha)<\beta<\varepsilon$ such that

$$|m_{Q^k_\tau}(x) - m_{Q^k_\tau}(x')| \le C\mu(Q^k_\tau)^{-1/2-\beta/d}\varrho(x,x')^\beta \times \left\{\frac{1}{(1+2^k\varrho(x,z^k_\tau))^{d+\gamma}} + \frac{1}{(1+2^k\varrho(x',z^k_\tau))^{d+\gamma}}\right\}. \tag{2.10}$$

To do this, we consider two cases.

Case 1: $\varrho(x,x')\ge 6A^2C2^{-k}$. In this case, by (2.9), it is easy to obtain (2.10).

Case 2: $\varrho(x,x')< 6A^2C2^{-k}$. In this case, we write

$$\begin{aligned}|m_{Q^k_\tau}(x) - m_{Q^k_\tau}(x')| &\le \sum_{l=0}^{\infty} 2^{(k-l)\alpha}|E_l(a_{Q^k_\tau})(x) - E_l(a_{Q^k_\tau})(x')| \\ &= \sum_{l=0}^{k} 2^{(k-l)\alpha}|E_l(a_{Q^k_\tau})(x) - E_l(a_{Q^k_\tau})(x')| + \sum_{l=k+1}^{\infty}\ldots = H_1+H_2.\end{aligned}$$

Then, for H_1, we have

$$\begin{aligned}H_1 &= \sum_{l=0}^{k} 2^{(k-l)\alpha}|E_l(a_{Q^k_\tau})(x) - E_l(a_{Q^k_\tau})(x')| \\ &\quad\times [\chi_{\{x:\,\varrho(x,z^k_\tau)\le 4A^2C2^{-l}\}}(x) + \chi_{\{x':\,\varrho(x',z^k_\tau)\le 4A^2C2^{-l}\}}(x')] \\ &= \sum_{l=0}^{k} 2^{(k-l)\alpha}\Big|\int_X([E_l(x,y)-E_l(x',y)] - [E_l(x,z^k_\tau)-E_l(x',z^k_\tau)])a_{Q^k_\tau}(y)\,d\mu(y)\Big| \\ &\quad\times [\chi_{\{x:\,\varrho(x,z^k_\tau)\le 4A^2C2^{-l}\}}(x) + \chi_{\{x':\,\varrho(x',z^k_\tau)\le 4A^2C2^{-l}\}}(x')] \\ &\le C\Big[\sum_{l=0}^{k} 2^{(k-l)(\alpha+\gamma-2\varepsilon)}\Big]\mu(Q^k_\tau)^{-1/2-\beta/d}\varrho(x,x')^\beta \\ &\quad\times\left\{\frac{1}{(1+2^k\varrho(x,z^k_\tau))^{d+\gamma}} + \frac{1}{(1+2^k\varrho(x',z^k_\tau))^{d+\gamma}}\right\} \\ &\le C\mu(Q^k_\tau)^{-1/2-\beta/d}\varrho(x,x')^\beta\left\{\frac{1}{(1+2^k\varrho(x,z^k_\tau))^{d+\gamma}} + \frac{1}{(1+2^k\varrho(x',z^k_\tau))^{d+\gamma}}\right\},\end{aligned}$$

since $\alpha+\gamma-2\varepsilon<0$.

For H_2, if $\varrho(x,z^k_\tau)\ge 12A^3C2^{-k}$, we have $\varrho(x',z^k_\tau)\ge 6A^2C2^{-k}$ since $\varrho(x,x')< 6A^2C2^{-k}$. Thus, in this case, $H_2=0$. Now we suppose that $\varrho(x,z^k_\tau)<12A^3C2^{-k}$ and there is an $l_1\in\mathbb{N}$ such that

$$6A^2C2^{-(k+l_1)} \le \varrho(x,x') < 6A^2C2^{-(k+l_1-1)}.$$

We then write

$$
\begin{aligned}
H_2 &= \sum_{l=k+1}^{k+l_1} 2^{(k-l)\alpha} |E_l(a_{Q_\tau^k})(x) - E_l(a_{Q_\tau^k})(x')| + \sum_{l=k+l_1+1}^{\infty} \cdots \\
&= \sum_{l=k+1}^{k+l_1} 2^{(k-l)\alpha} \Big| \int_X [E_l(x,y) - E_l(x',y)][a_{Q_\tau^k}(y) - a_{Q_\tau^k}(x)]\, d\mu(y) \Big| \\
&\quad + \sum_{l=k+l_1+1}^{\infty} 2^{(k-l)\alpha} \Big[\Big| \int_X E_l(x,y)[a_{Q_\tau^k}(y) - a_{Q_\tau^k}(x)]\, d\mu(y) \Big| \\
&\quad + \Big| \int_X E_l(x',y)[a_{Q_\tau^k}(y) - a_{Q_\tau^k}(x')]\, d\mu(y) \Big| \Big] \\
&\le C\mu(Q_\tau^k)^{-1/2-\varepsilon/d} \Big\{ \varrho(x,x')^\varepsilon \sum_{l=k+1}^{k+l_1} 2^{(k-l)\alpha} + \sum_{l=k+l_1+1}^{\infty} 2^{(k-l)\alpha - l\varepsilon} \Big\} \\
&\le \begin{cases} C\mu(Q_\tau^k)^{-1/2-\varepsilon/d} \varrho(x,x')^\varepsilon, & \alpha > 0, \\ C\mu(Q_\tau^k)^{-1/2-(\varepsilon+\alpha)/d} \varrho(x,x')^{\varepsilon+\alpha}, & \alpha < 0. \end{cases}
\end{aligned}
$$

Thus, if we choose $\beta = \varepsilon$ for $\alpha > 0$ and $\beta = \varepsilon + \alpha$ for $\alpha < 0$, then (2.10) also holds in this case.

By (2.9) and (2.10), we know that $m_{Q_\tau^k}$ is a (β, γ)-smooth molecule for Q_τ^k, $k \in \mathbb{N}$ and $\tau \in M_k$, multiplied with a normalizing constant.

The proof of Theorem 2.2 is finished.

The converse of Theorem 2.2 is also true, that is, $B_{pq}^s(X)$ and $F_{pq}^s(X)$ have the lifting properties by using I_α as a lifting tool; see [31].

THEOREM 2.3. *Let $s, \alpha \in (-\varepsilon, \varepsilon)$ be such that $\alpha + s \in (-\varepsilon, \varepsilon)$. Let $\alpha < s + \varepsilon$ when $s < 0$ and $\alpha > s - \varepsilon$ when $s > 0$. Then there exists $\alpha_0(s) \in (0, \varepsilon)$ and a constant $C > 0$ independent of f such that if $-\alpha_0(s) < \alpha < \alpha_0(s)$, then*

$$
\begin{aligned}
\|f\|_{B_{pq}^s(X)} &\le C\|I_\alpha(f)\|_{B_{pq}^{s+\alpha}(X)} \quad \text{for } 1 \le p, q \le \infty, \\
\|f\|_{F_{pq}^s(X)} &\le C\|I_\alpha(f)\|_{F_{pq}^{s+\alpha}(X)} \quad \text{for } 1 < p < \infty \text{ and } 1 < q \le \infty.
\end{aligned}
$$

The key point to show Theorem 2.3 is to prove the invertibility of $I_\alpha I_{-\alpha}$ on $B_{pq}^s(X)$ and $F_{pq}^s(X)$. To do this, we will use a similar idea to that used in [18] to establish inhomogeneous Calderón reproducing formulae on spaces of homogeneous type. Let I be the identity operator on $B_{pq}^s(X)$ or $F_{pq}^s(X)$ and let $E_l = 0$ for $l < 0$. For any given $N \in \mathbb{N}$, we write

$$
\text{(2.11)} \qquad I - I_\alpha I_{-\alpha} = \sum_{k=0}^{\infty} \sum_{|l| \le N} (1 - 2^{l\alpha}) E_k E_{k+l} + \sum_{k=0}^{\infty} \sum_{|l| > N} (1 - 2^{l\alpha}) E_k E_{k+l} = T_N + R_N.
$$

We will show that if N is sufficiently large and if $|\alpha|$ is small enough, then the operators T_N and R_N are bounded on $B_{pq}^s(X)$ and $F_{pq}^s(X)$ with small operator norms. To do that, we need some properties of the operators $E_k E_{k+l}$. In what follows, we denote the kernels of the operators $E_k E_{k+l}$ just by $(E_k E_{k+l})(x, y)$ for $x, y \in X$. All the estimates in the following lemma are special cases of (3.9)–(3.12) in [18]. Moreover, estimates similar to those in Lemma 2.1 still hold if $\{S_k\}_{k=0}^{\infty}$ and $\{\widetilde{S}_k\}_{k=0}^{\infty}$ are two approximations to the identity as in Remark 1.1 with kernels not having compact supports; see [18]. But, for completeness, we will give a proof of the following lemma. Recall that for $a, b \in \mathbb{R}$, we denote the minimum of a and b by $a \wedge b$.

Lemma 2.1. *Let $\{S_k\}_{k=0}^{\infty}$ and $\{\widetilde{S}_k\}_{k=0}^{\infty}$ be two approximations to the identity as in Definition 1.2. Let $E_k = S_k - S_{k-1}$ and $\widetilde{E}_k = \widetilde{S}_k - \widetilde{S}_{k-1}$ for $k \in \mathbb{N}$, $E_0 = S_0$, $\widetilde{E}_0 = \widetilde{S}_0$, and $E_l = 0 = \widetilde{E}_l$ for $l \in \mathbb{Z} \setminus (\mathbb{N} \cup \{0\})$. Then $(E_k\widetilde{E}_{k+l})(x,y)$, the kernels of the operators $E_k\widetilde{E}_{k+l}$, have the following basic properties:*

(2.12) $\operatorname{supp}(E_k\widetilde{E}_{k+l}) \subset \{(x,y) \in X \times X : \varrho(x,y) \le AC2^{-k}\}$ *for* $k, l \ge 0$;

(2.13) $\operatorname{supp}(E_k\widetilde{E}_{k+l}) \subset \{(x,y) \in X \times X : \varrho(x,y) \le AC2^{-k-l}\}$ *for* $k \ge 0$, $l \le 0$ *and* $k + l \ge 0$;

(2.14) $\int_X (E_k\widetilde{E}_{k+l})(x,y)\,d\mu(x) = 0 = \int_X (E_k\widetilde{E}_{k+l})(x,y)\,d\mu(y)$ *for* $l \ne 0$, $k \ge 0$ *and* $k + l \ge 0$, *and for* $l = 0$ *and* $k > 0$.

Moreover, for any given $\sigma \in (0,1)$, there exists a constant $C > 0$ such that for $k \ge 0$, $l \in \mathbb{N}$ and $k + l \ge 0$,

(2.15) $|(E_k\widetilde{E}_{k+l})(x,y)| \le C2^{-|l|\varepsilon}2^{(k\wedge(k+l))d}$;

(2.16) $|(E_k\widetilde{E}_{k+l})(x,y) - (E_k\widetilde{E}_{k+l})(x,y')| \le C2^{-|l|\sigma\varepsilon}\varrho(y,y')^{(1-\sigma)\varepsilon}2^{(k\wedge(k+l))(d+(1-\sigma)\varepsilon)}$;

(2.17) $|(E_k\widetilde{E}_{k+l})(x,y) - (E_k\widetilde{E}_{k+l})(x',y)| \le C2^{-|l|\sigma\varepsilon}\varrho(x,x')^{(1-\sigma)\varepsilon}2^{(k\wedge(k+l))(d+(1-\sigma)\varepsilon)}$;

(2.18)
$$|[(E_k\widetilde{E}_{k+l})(x,y) - (E_k\widetilde{E}_{k+l})(x,y')] - [(E_k\widetilde{E}_{k+l})(x',y) - (E_k\widetilde{E}_{k+l})(x',y')]| \le C2^{-|l|\sigma\varepsilon}\varrho(x,x')^{(1-\sigma)\varepsilon}\varrho(y,y')^{(1-\sigma)\varepsilon}2^{(k\wedge(k+l))(d+2(1-\sigma)\varepsilon)}.$$

Proof. (2.12)–(2.14) are obvious. Without loss of generality, we suppose $l > 0$ in the following. Let us first show (2.15). We write

$$|(E_k\widetilde{E}_{k+l})(x,y)| = \Big|\int_X E_k(x,z)\widetilde{E}_{k+l}(z,y)\,d\mu(z)\Big| = \Big|\int_X [E_k(x,z) - E_k(x,y)]\widetilde{E}_{k+l}(z,y)\,d\mu(z)\Big|$$
$$\le C2^{k(d+\varepsilon)}\int_X \varrho(z,y)^{\varepsilon}|\widetilde{E}_{k+l}(z,y)|\,d\mu(z) \le C2^{-l\varepsilon}2^{kd}.$$

This is (2.15).

To show (2.16), we first note that if $\varrho(y,y') \le 3AC2^{-(k+l)}$ and $\varrho(z,y') \le C2^{-(k+l)}$, then $\varrho(z,y) \le 4A^2C2^{-(k+l)}$, and

$$(2.19)\quad |(E_k\widetilde{E}_{k+l})(x,y) - (E_k\widetilde{E}_{k+l})(x,y')| = \Big|\int_X E_k(x,z)[\widetilde{E}_{k+l}(z,y) - \widetilde{E}_{k+l}(z,y')]\,d\mu(z)\Big|$$
$$= \Big|\int_X [E_k(x,z) - E_k(x,y)][\widetilde{E}_{k+l}(z,y) - \widetilde{E}_{k+l}(z,y')]\,d\mu(z)\Big|$$
$$\le C\varrho(y,y')^{\varepsilon}2^{k(d+\varepsilon)}2^{(k+l)(d+\varepsilon)}\int_{\{z:\,\varrho(z,y)\le 4A^2C2^{-(k+l)}\}} \varrho(y,z)^{\varepsilon}\,d\mu(z)$$
$$\le C\varrho(y,y')^{\varepsilon}2^{k(d+\varepsilon)}.$$

Note that if $\varrho(y,y') > 3AC2^{-(k+l)}$ and either $\varrho(z,y) \le C2^{-(k+l)}$ or $\varrho(z,y') \le C2^{-(k+l)}$, then $\varrho(z,y') \le C2^{-(k+l)}$ or $\varrho(z,y) \le C2^{-(k+l)}$, respectively. From this, it is easy to deduce that if $\varrho(y,y') > 3AC2^{-(k+l)}$, then

$$
\begin{aligned}
(2.20)\quad &|(E_k\widetilde{E}_{k+l})(x,y) - (E_k\widetilde{E}_{k+l})(x,y')|\\
&= \Big|\int_X [E_k(x,z) - E_k(x,y)]\widetilde{E}_{k+l}(z,y)\,d\mu(z) - \int_X [E_k(x,z) - E_k(x,y')]\widetilde{E}_{k+l}(z,y')\,d\mu(z)\Big|\\
&= \Big|\int_{\{z:\,\varrho(z,y)\le C2^{-(k+l)}\}} [E_k(x,z) - E_k(x,y)][\widetilde{E}_{k+l}(z,y) - \widetilde{E}_{k+l}(z,y')]\,d\mu(z)\\
&\quad - \int_{\{z:\,\varrho(z,y')\le C2^{-(k+l)}\}} [E_k(x,z) - E_k(x,y')][\widetilde{E}_{k+l}(z,y) - \widetilde{E}_{k+l}(z,y')]\,d\mu(z)\Big|\\
&\le C\varrho(y,y')^{\varepsilon}2^{k(d+\varepsilon)}.
\end{aligned}
$$

For any $\sigma\in(0,1)$, by the geometric mean of (2.15), (2.19) and (2.20), we obviously have

$$
\begin{aligned}
&|(E_k\widetilde{E}_{k+l})(x,y) - (E_k\widetilde{E}_{k+l})(x,y')|\\
&\quad = |(E_k\widetilde{E}_{k+l})(x,y) - (E_k\widetilde{E}_{k+l})(x,y')|^{\sigma}|(E_k\widetilde{E}_{k+l})(x,y) - (E_k\widetilde{E}_{k+l})(x,y')|^{1-\sigma}\\
&\quad \le C2^{-|l|\sigma\varepsilon}\varrho(y,y')^{(1-\sigma)\varepsilon}2^{(k\wedge(k+l))(d+(1-\sigma)\varepsilon)}.
\end{aligned}
$$

Thus (2.16) holds. The proof of (2.17) is similar.

We now show (2.18). Similarly to (2.19), we find that if $\varrho(y,y')\le 3AC2^{-(k+l)}$, then

$$
\begin{aligned}
(2.21)\quad &|[(E_k\widetilde{E}_{k+l})(x,y) - (E_k\widetilde{E}_{k+l})(x,y')] - [(E_k\widetilde{E}_{k+l})(x',y) - (E_k\widetilde{E}_{k+l})(x',y')]|\\
&= \Big|\int_X [E_k(x,z) - E_k(x',z)][\widetilde{E}_{k+l}(z,y) - \widetilde{E}_{k+l}(z,y')]\,d\mu(z)\Big|\\
&= \Big|\int_X \{[E_k(x,z) - E_k(x',z)] - [E_k(x,y) - E_k(x',y)]\}[\widetilde{E}_{k+l}(z,y) - \widetilde{E}_{k+l}(z,y')]\,d\mu(z)\Big|\\
&\le C\varrho(x,x')^{\varepsilon}\varrho(y,y')^{\varepsilon}2^{k(d+2\varepsilon)}2^{(k+l)(d+\varepsilon)}\int_{\{z:\,\varrho(z,y)\le 4A^2C2^{-(k+l)}\}} \varrho(y,z)^{\varepsilon}\,d\mu(z)\\
&\le C\varrho(x,x')^{\varepsilon}\varrho(y,y')^{\varepsilon}2^{k(d+2\varepsilon)}.
\end{aligned}
$$

If $\varrho(y,y') > 3AC2^{-(k+l)}$, then similarly to (2.20), we have

$$
\begin{aligned}
(2.22)\quad &|[(E_k\widetilde{E}_{k+l})(x,y) - (E_k\widetilde{E}_{k+l})(x,y')] - [(E_k\widetilde{E}_{k+l})(x',y) - (E_k\widetilde{E}_{k+l})(x',y')]|\\
&\le \Big|\int_X \{[E_k(x,z) - E_k(x',z)] - [E_k(x,y) - E_k(x',y)]\}\widetilde{E}_{k+l}(z,y)\,d\mu(z)\Big|\\
&\quad + \Big|\int_X \{[E_k(x,z) - E_k(x',z)] - [E_k(x,y') - E_k(x',y')]\}\widetilde{E}_{k+l}(z,y')]\,d\mu(z)\Big|\\
&\le \Big|\int_{\{z:\,\varrho(z,y)\le C2^{-(k+l)}\}} \{[E_k(x,z) - E_k(x',z)] - [E_k(x,y) - E_k(x',y)]\}\\
&\quad \times [\widetilde{E}_{k+l}(z,y) - \widetilde{E}_{k+l}(z,y')]\,d\mu(z)\Big|
\end{aligned}
$$

$$+\Big|\int_{\{z:\,\varrho(z,y')\le C2^{-(k+l)}\}}\{[E_k(x,z)-E_k(x',z)]$$

$$-[E_k(x,y')-E_k(x',y')]\}[\widetilde{E}_{k+l}(z,y)-\widetilde{E}_{k+l}(z,y')]\,d\mu(z)\Big|$$

$$\le C\varrho(x,x')^{\varepsilon}\varrho(y,y')^{\varepsilon}2^{k(d+2\varepsilon)}.$$

Now, by the geometric mean of (2.15), (2.21) and (2.22), we obtain (2.18).

This finishes the proof of Lemma 2.1.

Proof of Theorem 2.3. As pointed out above, we need to show that the operators T_N and R_N are bounded in $B^s_{pq}(X)$ and $F^s_{pq}(X)$ with small operator norms when N is large enough and s is small enough. We do this by using Lemmas 1.4 and 1.5. Let us first consider R_N. Let $0<\gamma<\varepsilon$ and $\max(0,-s)<\beta<\varepsilon$ and $\{a_{Q^k_\tau}\}_{k\in\mathbb{N},\tau\in M_k}$ be ε-smooth atoms and $\{a_{Q^0_\tau}\}_{\tau\in M_0}$ be ε-smooth blocks as in Definition 1.4 with $\delta=1/2$. For $\tau\in M_0$, we verify that

$$R_N(a_{Q^0_\tau})(x)=\sum_{k=0}^{\infty}\sum_{|l|>N,\,k+l\ge0}(1-2^{l\alpha})E_kE_{k+l}(a_{Q^0_\tau})(x)$$

is a (β,γ)-smooth unit for Q^0_τ, multiplied with a small normalizing constant, when N is large enough. We write

$$\begin{aligned}R_N(a_{Q^0_\tau})(x)&=\sum_{k=0}^{\infty}\sum_{|l|>N,\,k+l\ge0}(1-2^{l\alpha})E_kE_{k+l}(a_{Q^0_\tau})(x)\\&=\Big(\sum_{k=0}^{\infty}\sum_{l>N}+\sum_{k=0}^{\infty}\sum_{l<-N,\,k+l\ge0}\Big)(1-2^{l\alpha})E_kE_{k+l}(a_{Q^0_\tau})(x)\\&=J_1+J_2.\end{aligned}$$

For J_1, by (2.14), (2.15) and (2.12), we have

$$\begin{aligned}|J_1|&=\Big|\sum_{k=0}^{\infty}\sum_{l>N}(1-2^{l\alpha})\int_X(E_kE_{k+l})(x,y)(a_{Q^0_\tau}(y)-a_{Q^0_\tau}(x))\,d\mu(y)\Big|\\&\le C\sum_{k=0}^{\infty}\sum_{l>N}(1+2^{l\alpha})2^{-l\varepsilon}2^{-k\varepsilon}\le C2^{-\delta N},\end{aligned}$$

where C is independent of N and $\delta=\min(\varepsilon,\varepsilon-\alpha)$. Moreover, since $\operatorname{supp}J_1\subset\{x\in X:\varrho(x,z^0_\tau)\le 4A^2C\}$, we have $|J_1|\le C2^{-\delta N}\mu(Q^0_\tau)^{-1/2}(1+\varrho(x,z^0_\tau))^{-(d+\gamma)}$.

For J_2, by (2.14), (2.15) and (2.13), we have

$$\begin{aligned}|J_2|&=\Big|\sum_{k=0}^{\infty}\sum_{l<-N,\,k+l\ge0}(1-2^{l\alpha})\int_X(E_kE_{k+l})(x,y)(a_{Q^0_\tau}(y)-a_{Q^0_\tau}(x))\,d\mu(y)\Big|\\&\le C\sum_{k=0}^{\infty}\sum_{l<-N,\,k+l\ge0}(1+2^{l\alpha})2^{l\varepsilon}2^{-(k+l)\varepsilon}\le C2^{-\delta N},\end{aligned}$$

where C is independent of N and $\delta=\min(\varepsilon,\varepsilon+\alpha)$. Since $\operatorname{supp}J_2\subset\{x\in X:\varrho(x,z^0_\tau)\le$

$4A^2C\}$, we have

$$|J_2| \le C2^{-\delta N}\mu(Q_\tau^0)^{-1/2}(1+\varrho(x,z_\tau^0))^{-(d+\gamma)}.$$

Thus, if we choose $\delta = \min(\varepsilon+\alpha, \varepsilon-\alpha)$, then

$$|R_N(a_{Q_\tau^0})(x)| \le C2^{-\delta N}\mu(Q_\tau^0)^{-1/2}(1+\varrho(x,z_\tau^0))^{-(d+\gamma)}. \tag{2.23}$$

Now we claim that there are $\delta > 0$ independent of N, and β and γ satisfying $0 < \gamma < \varepsilon$ and $\max(0,-s) < \beta < \varepsilon$ such that

$$\begin{aligned}&|R_N(a_{Q_\tau^0})(x) - R_N(a_{Q_\tau^0})(x')|\\ &\quad\le C2^{-\delta N}\mu(Q_\tau^0)^{-1/2-\beta/d}\varrho(x,x')^\beta\Big\{\frac{1}{(1+\varrho(x,z_\tau^0))^{d+\gamma}} + \frac{1}{(1+\varrho(x',z_\tau^0))^{d+\gamma}}\Big\}.\end{aligned} \tag{2.24}$$

Similarly to the proof of (2.8), we also have three cases.

Case 1: $\varrho(x,x') \ge 6A^2C$. In this case, (2.24) can be deduced easily by (2.23).

Case 2: $\varrho(x,x') < 6A^2C$ and $\varrho(x,z_\tau^0) > 12A^3C$. In this case, it is easy to see that $\varrho(x',z_\tau^0) > 6A^2C$. Thus, $R_N(a_{Q_\tau^0})(x) = R_N(a_{Q_\tau^0})(x') = 0$ and (2.24) holds.

Case 3: $\varrho(x,x') < 6A^2C$ and $\varrho(x,z_\tau^0) < 12A^3C$. In this case, we also have $\varrho(x',z_\tau^0) < 18A^4C$. We further suppose that there is an $l_1 \in \mathbb{N}$ such that

$$6A^2C2^{-l_1} \le \varrho(x,x') < 6A^2C2^{-l_1+1}.$$

We then write

$$\begin{aligned}&|R_N(a_{Q_\tau^0})(x) - R_N(a_{Q_\tau^0})(x')|\\ &\quad= \Big|\sum_{k=0}^\infty \sum_{|l|>N,\, k+l\ge 0}(1-2^{l\alpha})[E_kE_{k+l}(a_{Q_\tau^0})(x) - E_kE_{k+l}(a_{Q_\tau^0})(x')]\Big|\\ &\quad\le C\sum_{k=0}^\infty\sum_{l>N}(1+2^{l\alpha})|E_kE_{k+l}(a_{Q_\tau^0})(x) - E_kE_{k+l}(a_{Q_\tau^0})(x')| + \sum_{k=0}^\infty \sum_{l<-N,\, k+l\ge 0}\cdots\\ &\quad= K_1 + K_2.\end{aligned}$$

By (2.14), (2.17) and (2.12), we have

$$\begin{aligned}K_1 &\le C\sum_{k=0}^{l_1}\sum_{l>N}(1+2^{l\alpha})\Big|\int_X[(E_kE_{k+l})(x,y) - (E_kE_{k+l})(x',y)](a_{Q_\tau^0}(y) - a_{Q_\tau^0}(x))\,d\mu(y)\Big|\\ &\quad + C\sum_{k=l_1+1}^\infty\sum_{l>N}(1+2^{l\alpha})\Big[\Big|\int_X(E_kE_{k+l})(x,y)(a_{Q_\tau^0}(y) - a_{Q_\tau^0}(x))\,d\mu(y)\Big|\\ &\quad + \Big|\int_X(E_kE_{k+l})(x',y)(a_{Q_\tau^0}(y) - a_{Q_\tau^0}(x'))\,d\mu(y)\Big|\Big]\\ &\le C\varrho(x,x')^{(1-\sigma)\varepsilon}\sum_{k=0}^{l_1}\sum_{l>N}(1+2^{l\alpha})2^{-l\sigma\varepsilon}2^{-k\sigma\varepsilon} + C\sum_{k=l_1+1}^\infty\sum_{l>N}(1+2^{l\alpha})2^{-l\varepsilon}2^{-k\varepsilon}\\ &\le C2^{-\delta N}\varrho(x,x')^{(1-\sigma)\varepsilon},\end{aligned}$$

where we choose $\sigma \in (0,1)$ such that $(1-\sigma)\varepsilon > \max(0,s)$, $\sigma\varepsilon > \alpha$ and $\delta = \sigma\varepsilon - \alpha$.

For K_2, by (2.14), (2.17) and (2.13), we have

$$\begin{aligned}
K_2 &\le C \sum_{l<-N} \sum_{0\le k\le l_1-l,\, k+l\ge 0} (1+2^{l\alpha}) \\
&\quad \times \Big| \int_X [(E_kE_{k+l})(x,y) - (E_kE_{k+l})(x',y)](a_{Q^0_\tau}(y) - a_{Q^0_\tau}(x))\, d\mu(y)\Big| \\
&\quad + \sum_{l<-N} \sum_{l_1-l<k} (1+2^{l\alpha}) \Big[\Big| \int_X (E_kE_{k+l})(x,y)(a_{Q^0_\tau}(y) - a_{Q^0_\tau}(x))\, d\mu(y)\Big| \\
&\quad + \Big| \int_X (E_kE_{k+l})(x',y)(a_{Q^0_\tau}(y) - a_{Q^0_\tau}(x'))\, d\mu(y)\Big|\Big] \\
&\le C\varrho(x,x')^{(1-\sigma)\varepsilon} \sum_{l<-N} (2^{l(\sigma\varepsilon+(1-\sigma)\varepsilon-\nu)} + 2^{l(\sigma\varepsilon+\alpha+(1-\sigma)\varepsilon-\nu)}) \sum_{k=0}^{l_1-l} 2^{k((1-\sigma)\varepsilon-\nu)} \\
&\quad + C \sum_{l<-N} (1+2^{l\alpha}) 2^{l\varepsilon} \sum_{k=l_1-l+1}^{\infty} 2^{-(k+l)\varepsilon} \\
&\le C2^{-\delta N} \varrho(x,x')^{(1-\sigma)\varepsilon},
\end{aligned}$$

where we choose $\sigma \in (0,1)$ and $\nu \in (0,\varepsilon)$ such that $(1-\sigma)\varepsilon > \max(0,s)$, $(1-\sigma)\varepsilon < \nu < \min(\varepsilon, \varepsilon+\alpha)$ and $\delta = \min(\varepsilon-\nu, \varepsilon+\alpha-\nu)$. Here we used the fact that

$$|a_{Q^0_\tau}(y) - a_{Q^0_\tau}(x')| \le C\varrho(y,x')^\nu.$$

This can be easily proved by the definition of the blocks.

Thus, (2.24) holds. From (2.23) and (2.24), we see that $R_N(a_{Q^0_\tau})$ is a (β,γ)-smooth unit for Q^0_τ, multiplied with a normalizing constant which can be estimated from above by $C2^{-\delta N}$ for some $\delta > 0$, where $\max(0,-s) < \beta < \varepsilon$, $0 < \gamma < \varepsilon$, and C is independent of N.

Now, we intend to show that for the above β and γ, $R_N(a_{Q^j_\tau})$ with $j \in \mathbb{N}$ and $\tau \in M_j$ is a (β,γ)-smooth molecule for Q^j_τ, multiplied with a normalizing constant which can be estimated from above by $C2^{-\delta N}$ for some $\delta > 0$. Obviously, we have

$$\int_X R_N(a_{Q^j_\tau})(x)\, d\mu(x) = 0. \tag{2.25}$$

To establish an estimate for $R_N(a_{Q^j_\tau})$, similar to (2.23), we first estimate

$$\begin{aligned}
L_1 &= \Big| \sum_{k=0}^{\infty} \sum_{l>N} (1-2^{l\alpha}) E_kE_{k+l}(a_{Q^j_\tau})(x)\Big| \\
&\le \sum_{k=0}^{j} \sum_{l>N} (1+2^{l\alpha}) |E_kE_{k+l}(a_{Q^j_\tau})(x)| \chi_{\{x:\, \varrho(x,z^j_\tau)\le 4A^2C2^{-k}\}}(x) \\
&\quad + \sum_{k=j+1}^{\infty} \sum_{l>N} (1+2^{l\alpha}) |E_kE_{k+l}(a_{Q^j_\tau})(x)| \chi_{\{x:\, \varrho(x,z^j_\tau)\le 4A^2C2^{-j}\}}(x) \\
&\le \sum_{k=0}^{j} \sum_{l>N} (1+2^{l\alpha}) \Big| \int_X [(E_kE_{k+l})(x,y) - (E_kE_{k+l})(x,z^j_\tau)] a_{Q^j_\tau}(y)\, d\mu(y)\Big|
\end{aligned}$$

$$
\begin{aligned}
&\quad\times \chi_{\{x:\, \varrho(x,z_\tau^j)\le 4A^2C2^{-k}\}}(x)\\
&\quad+\sum_{k=j+1}^{\infty}\sum_{l>N}(1+2^{l\alpha})\Big|\int_X (E_kE_{k+l})(x,y)[a_{Q_\tau^j}(y)-a_{Q_\tau^j}(x)]\,d\mu(y)\Big|\\
&\quad\times \chi_{\{x:\, \varrho(x,z_\tau^j)\le 4A^2C2^{-j}\}}(x)\\
&\le C\mu(Q_\tau^j)^{-1/2}(1+2^j\varrho(x,z_\tau^j))^{-(d+\gamma)}\Big\{\sum_{k=0}^{j}\sum_{l>N}(1+2^{l\alpha})2^{-l\sigma\varepsilon+(k-j)((1-\sigma)\varepsilon-\gamma)}\\
&\quad+\sum_{k=j+1}^{\infty}\sum_{l>N}(1+2^{l\alpha})2^{-l\varepsilon+(j-k)\varepsilon}\Big\}\\
&\le C2^{-\delta N}\mu(Q_\tau^j)^{-1/2}(1+2^j\varrho(x,z_\tau^j))^{-(d+\gamma)},
\end{aligned}
$$

where we choose $\sigma\in(0,1)$ such that $(1-\sigma)\varepsilon>\gamma>0$, $\sigma\varepsilon>\alpha$ and $\delta=\min(\sigma\varepsilon,\sigma\varepsilon-\alpha)$. Here we used the condition that $\alpha<s+\varepsilon$ if $s<0$.

We also write

$$
\begin{aligned}
L_2&=\Big|\sum_{k=0}^{\infty}\sum_{l<-N,\,k+l\ge 0}(1-2^{l\alpha})E_kE_{k+l}(a_{Q_\tau^j})(x)\Big|\\
&\le \sum_{l<-N}\sum_{0\le k\le j-l,\,k+l\ge 0}(1+2^{l\alpha})|E_kE_{k+l}(a_{Q_\tau^j})(x)|\chi_{\{x:\, \varrho(x,z_\tau^j)\le 4A^2C2^{-k-l}\}}(x)\\
&\quad+\sum_{l<-N}\sum_{k=j-l+1}^{\infty}(1+2^{l\alpha})|E_kE_{k+l}(a_{Q_\tau^j})(x)|\chi_{\{x:\, \varrho(x,z_\tau^j)\le 4A^2C2^{-j}\}}(x)\\
&=\sum_{l<-N}\sum_{0\le k\le j-l,\,k+l\ge 0}(1+2^{l\alpha})\Big|\int_X[(E_kE_{k+l})(x,y)-(E_kE_{k+l})(x,z_\tau^j)]a_{Q_\tau^j}(y)\,d\mu(y)\Big|\\
&\quad\times \chi_{\{x:\, \varrho(x,z_\tau^j)\le 4A^2C2^{-k-l}\}}(x)\\
&\quad+\sum_{l<-N}\sum_{k=j-l+1}^{\infty}(1+2^{l\alpha})\Big|\int_X (E_kE_{k+l})(x,y)[a_{Q_\tau^j}(y)-a_{Q_\tau^j}(x)]\,d\mu(y)\Big|\\
&\quad\times \chi_{\{x:\, \varrho(x,z_\tau^j)\le 4A^2C2^{-j}\}}(x)\\
&\le C\mu(Q_\tau^j)^{-1/2}(1+2^j\varrho(x,z_\tau^j))^{-(d+\gamma)}\Big\{\sum_{l<-N}\sum_{0\le k\le j-l,\,k+l\ge 0}(1+2^{l\alpha})2^{l\sigma\varepsilon+(k-j+l)((1-\sigma)\varepsilon-\gamma)}\\
&\quad+\sum_{l<-N}\sum_{k=j-l+1}^{\infty}(1+2^{l\alpha})2^{l\varepsilon-(k+l-j)\varepsilon}\Big\}\\
&\le C2^{-\delta N}\mu(Q_\tau^j)^{-1/2}(1+2^j\varrho(x,z_\tau^j))^{-(d+\gamma)},
\end{aligned}
$$

where we choose $\sigma\in(0,1)$ such that $\sigma\varepsilon>-\alpha$, $(1-\sigma)\varepsilon>\gamma>0$ and $\delta=\min(\sigma\varepsilon,\sigma\varepsilon+\alpha)$.

Thus, we can choose $\varepsilon>\gamma>0$ and $\delta>0$ such that

$$
|R_N(a_{Q_\tau^j})(x)|\le C2^{-\delta N}\mu(Q_\tau^j)^{-1/2}(1+2^j\varrho(x,z_\tau^j))^{-(d+\gamma)}. \tag{2.26}
$$

Now let us show that there are $\delta>0$ independent of N, and β and γ satisfying $0<\gamma<\varepsilon$ and $\max(0,-s)<\beta<\varepsilon$ such that

$$(2.27)\quad |R_N(a_{Q^j_\tau})(x) - R_N(a_{Q^j_\tau})(x')| \le C2^{-\delta N}\mu(Q^j_\tau)^{-1/2-\beta/d}\varrho(x,x')^\beta\Big\{\frac{1}{(1+2^j\varrho(x,z^j_\tau))^{d+\gamma}} + \frac{1}{(1+2^j\varrho(x',z^j_\tau))^{d+\gamma}}\Big\}.$$

We have two cases.

Case 1: $\varrho(x,x') \ge 6A^2C2^{-j}$. In this case, we can easily obtain (2.27) by (2.26).

Case 2: $\varrho(x,x') < 6A^2C2^{-j}$. In this case, we write

$$\begin{aligned}
|R_N(a_{Q^j_\tau})(x) - R_N(a_{Q^j_\tau})(x')| &= \Big|\sum_{k=0}^{\infty}\sum_{|l|>N,\, k+l\ge 0}(1-2^{l\alpha})[E_kE_{k+l}(a_{Q^j_\tau})(x) - E_kE_{k+l}(a_{Q^j_\tau})(x')]\Big| \\
&\le \sum_{k=0}^{\infty}\sum_{l>N}(1+2^{l\alpha})|E_kE_{k+l}(a_{Q^j_\tau})(x) - E_kE_{k+l}(a_{Q^j_\tau})(x')| + \sum_{l<-N}\sum_{0\le k,\, 0\le k+l}\cdots \\
&= O_1 + O_2.
\end{aligned}$$

For O_1, we further decompose it into

$$\begin{aligned}
O_1 &= \sum_{k=0}^{\infty}\sum_{l>N}(1+2^{l\alpha})|E_kE_{k+l}(a_{Q^j_\tau})(x) - E_kE_{k+l}(a_{Q^j_\tau})(x')| \\
&= \sum_{k=0}^{j}\sum_{l>N}(1+2^{l\alpha})|E_kE_{k+l}(a_{Q^j_\tau})(x) - E_kE_{k+l}(a_{Q^j_\tau})(x')| + \sum_{k=j+1}^{\infty}\sum_{l>N}\cdots \\
&= O_1^1 + O_1^2.
\end{aligned}$$

For O_1^1, we have

$$\begin{aligned}
O_1^1 &= \sum_{k=0}^{j}\sum_{l>N}(1+2^{l\alpha})|E_kE_{k+l}(a_{Q^j_\tau})(x) - E_kE_{k+l}(a_{Q^j_\tau})(x')| \\
&\quad \times [\chi_{\{x:\, \varrho(x,z^j_\tau)\le 4A^2C2^{-k}\}}(x) + \chi_{\{x':\, \varrho(x',z^j_\tau)\le 4A^2C2^{-k}\}}(x')] \\
&= \sum_{k=0}^{j}\sum_{l>N}(1+2^{l\alpha})\Big|\int_X\{[(E_kE_{k+l})(x,y) - (E_kE_{k+l})(x',y)] \\
&\quad - [(E_kE_{k+l})(x,z^j_\tau) - (E_kE_{k+l})(x',z^j_\tau)]\}a_{Q^j_\tau}(y)\,d\mu(y)\Big| \\
&\quad \times [\chi_{\{x:\, \varrho(x,z^j_\tau)\le 4A^2C2^{-k}\}}(x) + \chi_{\{x':\, \varrho(x',z^j_\tau)\le 4A^2C2^{-k}\}}(x')] \\
&\le C\mu(Q^j_\tau)^{-1/2}\varrho(x,x')^{(1-\sigma)\varepsilon}\Big\{\frac{1}{(1+2^j\varrho(x,z^j_\tau))^{d+\gamma}} + \frac{1}{(1+2^j\varrho(x',z^j_\tau))^{d+\gamma}}\Big\} \\
&\quad \times \sum_{k=0}^{j}\sum_{l>N}(1+2^{l\alpha})2^{-l\sigma\varepsilon + j(\gamma-(1-\sigma)\varepsilon) + k(2(1-\sigma)\varepsilon-\gamma)} \\
&\le C2^{-N\delta}\mu(Q^j_\tau)^{-1/2-(1-\sigma)\varepsilon/d}\varrho(x,x')^{(1-\sigma)\varepsilon}\Big\{\frac{1}{(1+2^j\varrho(x,z^j_\tau))^{d+\gamma}} + \frac{1}{(1+2^j\varrho(x',z^j_\tau))^{d+\gamma}}\Big\},
\end{aligned}$$

where we choose $\sigma\in(0,1)$ such that $\sigma\varepsilon > \alpha$, $0<\gamma<2(1-\sigma)\varepsilon$ and $\delta = \min(\sigma\varepsilon, \sigma\varepsilon-\alpha)$.

Now, if $\varrho(x, z^j_\tau) \geq 12A^3C2^{-j}$, then we also have $\varrho(x', z^j_\tau) \geq 6A^2C2^{-j}$. Thus, in this case, we have $O_1^2 = 0$ and (2.27) holds. Now we suppose that $\varrho(x, z^j_\tau) < 12A^3C2^{-j}$ and there is a $j_1 \in \mathbb{N}$ such that

$$6A^2C2^{-(j+j_1)} \leq \varrho(x, x') < 6A^2C2^{-(j+j_1-1)}.$$

We now write

$$\begin{aligned}
O_1^2 &\leq \sum_{k=j+1}^{j+j_1} \sum_{l>N} (1+2^{l\alpha})|E_kE_{k+l}(a_{Q^j_\tau})(x) - E_kE_{k+l}(a_{Q^j_\tau})(x')| \\
&\quad + \sum_{k=j+j_1+1}^{\infty} \sum_{l>N} (1+2^{l\alpha})(|E_kE_{k+l}(a_{Q^j_\tau})(x)| + |E_kE_{k+l}(a_{Q^j_\tau})(x')|) \\
&= \sum_{k=j+1}^{j+j_1} \sum_{l>N} (1+2^{l\alpha}) \Big| \int_X [(E_kE_{k+l})(x,y) - (E_kE_{k+l})(x',y)][a_{Q^j_\tau}(y) - a_{Q^j_\tau}(x)]\, d\mu(y) \Big| \\
&\quad + \sum_{k=j+j_1+1}^{\infty} \sum_{l>N} (1+2^{l\alpha}) \Big[\Big| \int_X (E_kE_{k+l})(x,y)[a_{Q^j_\tau}(y) - a_{Q^j_\tau}(x)]\, d\mu(y) \Big| \\
&\quad + \Big| \int_X (E_kE_{k+l})(x',y)[a_{Q^j_\tau}(y) - a_{Q^j_\tau}(x')]\, d\mu(y) \Big| \Big] \\
&\leq C\mu(Q^j_\tau)^{-1/2-\varepsilon/d} \Big\{ \varrho(x,x')^{(1-\sigma)\varepsilon} \sum_{k=j+1}^{j+j_1} \sum_{l>N} (1+2^{l\alpha}) 2^{-l\sigma\varepsilon + k((1-\sigma)\varepsilon - \varepsilon)} \\
&\quad + \sum_{k=j+j_1+1}^{\infty} \sum_{l>N} (1+2^{l\alpha}) 2^{-l\varepsilon - k\varepsilon} \Big\} \\
&\leq C2^{-\delta N} \mu(Q^j_\tau)^{-1/2-(1-\sigma)\varepsilon/d} \varrho(x,x')^{(1-\sigma)\varepsilon},
\end{aligned}$$

where we choose $\sigma \in (0,1)$ such that $\sigma\varepsilon > \alpha$, $(1-\sigma)\varepsilon > \max(0, -s)$ and $\delta = \min(\sigma\varepsilon, \sigma\varepsilon - \alpha)$.

We now estimate O_2. We first have

$$\begin{aligned}
O_2 &= \sum_{l<-N} \sum_{0\leq k,\, 0\leq k+l} (1+2^{l\alpha})|E_kE_{k+l}(a_{Q^j_\tau})(x) - E_kE_{k+l}(a_{Q^j_\tau})(x')| \\
&= \sum_{l<-N} \sum_{0\leq k\leq j-l,\, 0\leq k+l} (1+2^{l\alpha})|E_kE_{k+l}(a_{Q^j_\tau})(x) - E_kE_{k+l}(a_{Q^j_\tau})(x')| + \sum_{l<-N} \sum_{j-l<k} \cdots \\
&= O_2^1 + O_2^2.
\end{aligned}$$

The estimate for O_2^1 is similar to that for O_1^1. In fact, we have

$$\begin{aligned}
O_2^1 &= \sum_{l<-N} \sum_{0\leq k\leq j-l,\, 0\leq k+l} (1+2^{l\alpha}) \Big| \int_X \{[(E_kE_{k+l})(x,y) - (E_kE_{k+l})(x',y)] \\
&\quad - [(E_kE_{k+l})(x, z^j_\tau) - (E_kE_{k+l})(x', z^j_\tau)]\} a_{Q^j_\tau}(y)\, d\mu(y) \Big| \\
&\quad \times [\chi_{\{x:\, \varrho(x,z^j_\tau)\leq 4A^2C2^{-k-l}\}}(x) + \chi_{\{x':\, \varrho(x',z^j_\tau)\leq 4A^2C2^{-k-l}\}}(x')] \\
&\leq C\mu(Q^j_\tau)^{-1/2-(1-\sigma)\varepsilon/d} \varrho(x,x')^{(1-\sigma)\varepsilon} \Big\{ \frac{1}{(1+2^j\varrho(x,z^j_\tau))^{d+\gamma}} + \frac{1}{(1+2^j\varrho(x',z^j_\tau))^{d+\gamma}} \Big\}
\end{aligned}$$

$$\times \sum_{l<-N} \sum_{0\le k\le j-l,\, 0\le k+l} (1+2^{l\alpha}) 2^{l\sigma\varepsilon+(k+l-j)(2(1-\sigma)\varepsilon-\gamma)}$$

$$\le C 2^{-N\delta} \mu(Q_\tau^j)^{-1/2-(1-\sigma)\varepsilon/d} \varrho(x,x')^{(1-\sigma)\varepsilon} \bigg\{ \frac{1}{(1+2^j \varrho(x,z_\tau^j))^{d+\gamma}} + \frac{1}{(1+2^j \varrho(x',z_\tau^j))^{d+\gamma}} \bigg\},$$

where we take $\sigma\in(0,1)$ such that $\sigma\varepsilon>-\alpha$, $2(1-\sigma)\varepsilon>\gamma>0$ and $\delta=\min(\sigma\varepsilon,\alpha+\sigma\varepsilon)$.

The estimate for O_2^2 is similar to that for O_1^2. If $\varrho(x,z_\tau^j)\ge 12A^3C2^{-j}$, then we also have $\varrho(x',z_\tau^j)\ge 6A^2C2^{-j}$. Thus, in this case, we have $O_2^2=0$ and (2.27) holds. Now we suppose that $\varrho(x,z_\tau^j)<12A^3C2^{-j}$ and there is a $j_1\in\mathbb{N}$ such that

$$6A^2C2^{-(j+j_1)} \le \varrho(x,x') < 6A^2C2^{-(j+j_1-1)}.$$

We estimate O_2^2 by

$$\begin{aligned}
O_2^2 \le & \sum_{l<-N} \sum_{j-l<k\le j+j_1-l} (1+2^{l\alpha}) |E_kE_{k+l}(a_{Q_\tau^j})(x) - E_kE_{k+l}(a_{Q_\tau^j})(x')| \\
& + \sum_{l<-N} \sum_{k>j+j_1-l} (1+2^{l\alpha}) (|E_kE_{k+l}(a_{Q_\tau^j})(x)| + |E_kE_{k+l}(a_{Q_\tau^j})(x')|) \\
= & \sum_{l<-N} \sum_{j-l<k\le j+j_1-l} (1+2^{l\alpha}) \\
& \times \Big| \int_X [(E_kE_{k+l})(x,y)-(E_kE_{k+l})(x',y)][a_{Q_\tau^j}(y)-a_{Q_\tau^j}(x)]\,d\mu(y)\Big| \\
& + \sum_{l<-N} \sum_{k>j+j_1-l} (1+2^{l\alpha}) \Big[\Big| \int_X (E_kE_{k+l})(x,y)[a_{Q_\tau^j}(y)-a_{Q_\tau^j}(x)]\,d\mu(y)\Big| \\
& + \Big| \int_X (E_kE_{k+l})(x',y)[a_{Q_\tau^j}(y)-a_{Q_\tau^j}(x')]\,d\mu(y)\Big|\Big] \\
\le & C\mu(Q_\tau^j)^{-1/2-\varepsilon/d} \Big\{ \varrho(x,x')^{(1-\sigma)\varepsilon} \sum_{l<-N} \sum_{j-l<k\le j+j_1-l} (1+2^{l\alpha}) 2^{l\sigma\varepsilon-(k+l)\sigma\varepsilon} \\
& + \sum_{l<-N} \sum_{k>j+j_1-l} (1+2^{l\alpha}) 2^{l\varepsilon-(k+l)\varepsilon} \Big\} \\
\le & C2^{-\delta N} \mu(Q_\tau^j)^{-1/2-(1-\sigma)\varepsilon/d} \varrho(x,x')^{(1-\sigma)\varepsilon},
\end{aligned}$$

where we choose $\sigma\in(0,1)$ such that $\sigma\varepsilon>-\alpha$, $(1-\sigma)\varepsilon>\max(0,-s)$ and $\delta=\min(\sigma\varepsilon,\varepsilon\sigma+\alpha)$. Here we used the condition that $\alpha>s-\varepsilon$ if $s>0$.

Thus, (2.27) is true. From (2.25)–(2.27), we deduce that $R_N(a_{Q_\tau^j})$ is a (β,γ)-smooth molecule for Q_τ^j, multiplied with a normalizing constant which can be estimated from above by $C2^{-\delta N}$ for some $\delta>0$. By Lemmas 1.4 and 1.5, R_N is bounded in $B_{pq}^s(X)$ and $F_{pq}^s(X)$ with operator norms no more than $C_1 2^{-\delta N}$ for some $\delta>0$, where C_1 is independent of N. Moreover, if we take $\delta_0>0$ small enough and if $|\alpha|\le\delta_0$, then C_1 is independent of N and α, but it depends on δ_0. This is a desired estimate for R_N.

Now we show that T_N is bounded in $B_{pq}^s(X)$ and $F_{pq}^s(X)$ with small operator norms when $|\alpha|$ is small. We write

$$T_N = \sum_{|l|\le N} (1-2^{l\alpha}) \sum_{k=0}^{\infty} E_kE_{k+l} = \sum_{|l|\le N} (1-2^{l\alpha}) T_N^l.$$

For any given $N \in \mathbb{N}$, we will use Lemmas 1.4 and 1.5 to show that T_N^l is bounded in $B_{pq}^s(X)$ and $F_{pq}^s(X)$ uniformly in l with $|l| \le N$.

Let $\{a_{Q_\tau^0}\}_{\tau\in M_0}$ be ε-smooth blocks. For $0 \le l \le N$, we have

$$|T_N^l(a_{Q_\tau^0})(x)| = \Big|\sum_{k=0}^{\infty} E_k E_{k+l}(a_{Q_\tau^0})(x)\Big| \le \sum_{k=0}^{\infty}\Big|\int_X (E_k E_{k+l})(x,y)(a_{Q_\tau^0}(x) - a_{Q_\tau^0}(y))\, d\mu(y)\Big|$$
$$\le C2^{-l\varepsilon}\sum_{k=0}^{\infty} 2^{-k\varepsilon} \le C2^{-l\varepsilon}.$$

Noting that $\operatorname{supp} T_N^l(a_{Q_\tau^0}) \subset \{x \in X : \varrho(x, z_\tau^0) \le 4A^2C\}$, we have

$$|T_N^l(a_{Q_\tau^0})(x)| \le C2^{-l\varepsilon}\mu(a_{Q_\tau^0})^{-1/2}(1 + \varrho(x, z_\tau^0))^{-(d+\gamma)}.$$

For $-N \le l < 0$, we have

$$|T_N^l(a_{Q_\tau^0})(x)| = \Big|\sum_{k\ge 0,\, k+l\ge 0} E_k E_{k+l}(a_{Q_\tau^0})(x)\Big|$$
$$\le \sum_{k\ge 0,\, k+l\ge 0}\Big|\int_X (E_k E_{k+l})(x,y)(a_{Q_\tau^0}(x) - a_{Q_\tau^0}(y))\, d\mu(y)\Big|$$
$$\le C2^{l\varepsilon}\sum_{k+l=0}^{\infty} 2^{-(k+l)\varepsilon} \le C2^{l\varepsilon}.$$

By noting that $\operatorname{supp} T_N^l(a_{Q_\tau^0}) \subset \{x \in X : \varrho(x, z_\tau^0) \le 4A^2C\}$, we also have

$$|T_N^l(a_{Q_\tau^0})(x)| \le C2^{l\varepsilon}\mu(a_{Q_\tau^0})^{-1/2}(1 + \varrho(x, z_\tau^0))^{-(d+\gamma)}.$$

Thus, for $|l| \le N$, we have

$$|T_N^l(a_{Q_\tau^0})(x)| \le C2^{-|l|\varepsilon}\mu(a_{Q_\tau^0})^{-1/2}(1 + \varrho(x, z_\tau^0))^{-(d+\gamma)}, \tag{2.28}$$

where C is independent of l.

Let $\sigma \in (0,1)$ be such that $(1-\sigma)\varepsilon > \max(0, -s)$. We now show that for $|l| \le N$, there is a γ satisfying $0 < \gamma < \varepsilon$ such that

$$|T_N^l(a_{Q_\tau^0})(x) - T_N^l(a_{Q_\tau^0})(x')| \le C2^{-|l|\sigma\varepsilon}\mu(Q_\tau^0)^{-1/2-(1-\sigma)\varepsilon/d}\varrho(x,x')^{(1-\sigma)\varepsilon} \times \bigg\{\frac{1}{(1+\varrho(x, z_\tau^0))^{d+\gamma}} + \frac{1}{(1+\varrho(x', z_\tau^0))^{d+\gamma}}\bigg\}. \tag{2.29}$$

Similarly to the estimate for (2.24), we consider three cases.

Case 1: $\varrho(x,x') \ge 6A^2C$. In this case, it is easy to obtain (2.29) by (2.28).

Case 2: $\varrho(x,x') < 6A^2C$ and $\varrho(x, z_\tau^0) \ge 12A^3C$. In this case, it is easy to see that $\varrho(x', z_\tau^0) > 6A^2C$. Thus, $T_N^l(a_{Q_\tau^0})(x) = T_N^l(a_{Q_\tau^0})(x') = 0$. Therefore, in this case, we also have (2.29).

Case 3: $\varrho(x,x') < 6A^2C$ and $\varrho(x, z_\tau^0) < 12A^3C$. In this case, we also have $\varrho(x', z_\tau^0) < 18A^4C$. We further suppose that there is an $l_1 \in \mathbb{N}$ such that

$$6A^2C2^{-l_1} \le \varrho(x,x') < 6A^2C2^{-l_1+1}.$$

For $0 \le l \le N$, we have

$$|T_N^l(a_{Q_\tau^0})(x) - T_N^l(a_{Q_\tau^0})(x')| = \Big|\sum_{k=0}^{\infty}[E_kE_{k+l}(a_{Q_\tau^0})(x) - E_kE_{k+l}(a_{Q_\tau^0})(x')]\Big|$$

$$\le \sum_{k=0}^{l_1}\Big|\int_X[(E_kE_{k+l})(x,y) - (E_kE_{k+l})(x',y)][a_{Q_\tau^0}(y) - a_{Q_\tau^0}(x)]\,d\mu(y)\Big|$$

$$+ \sum_{k=l_1+1}^{\infty}\Big[\Big|\int_X(E_kE_{k+l})(x,y)[a_{Q_\tau^0}(y) - a_{Q_\tau^0}(x)]\,d\mu(y)\Big|$$

$$+ \Big|\int_X(E_kE_{k+l})(x',y)[a_{Q_\tau^0}(y) - a_{Q_\tau^0}(x')]\,d\mu(y)\Big|\Big]$$

$$\le C2^{-l\sigma\varepsilon}\varrho(x,x')^{(1-\sigma)\varepsilon}\sum_{k=0}^{l_1}2^{k((1-\sigma)\varepsilon-\varepsilon)} + C\sum_{k=l_1+1}^{\infty}2^{-l\varepsilon-k\varepsilon} \le C2^{-l\sigma\varepsilon}\varrho(x,x')^{(1-\sigma)\varepsilon}.$$

For $-N \le l < 0$, we have

$$|T_N^l(a_{Q_\tau^0})(x) - T_N^l(a_{Q_\tau^0})(x')| = \Big|\sum_{k\ge0,\,k+l\ge0}[E_kE_{k+l}(a_{Q_\tau^0})(x) - E_kE_{k+l}(a_{Q_\tau^0})(x')]\Big|$$

$$\le \sum_{0\le k\le l_1-l,\,k+l\ge0}\Big|\int_X[(E_kE_{k+l})(x,y) - (E_kE_{k+l})(x',y)][a_{Q_\tau^0}(y) - a_{Q_\tau^0}(x)]\,d\mu(y)\Big|$$

$$+ \sum_{k>l_1-l}\Big[\Big|\int_X(E_kE_{k+l})(x,y)[a_{Q_\tau^0}(y) - a_{Q_\tau^0}(x)]\,d\mu(y)\Big|$$

$$+ \Big|\int_X(E_kE_{k+l})(x',y)[a_{Q_\tau^0}(y) - a_{Q_\tau^0}(x')]\,d\mu(y)\Big|\Big]$$

$$\le C2^{l\sigma\varepsilon}\varrho(x,x')^{(1-\sigma)\varepsilon}\sum_{0\le k+l\le l_1}2^{-(k+l)\sigma\varepsilon} + C\sum_{k+l>l_1}2^{l\varepsilon-(k+l)\varepsilon} \le C2^{l\sigma\varepsilon}\varrho(x,x')^{(1-\sigma)\varepsilon}.$$

Thus (2.29) holds.

By (2.28) and (2.29), we see that for $|l| \le N$ and $\tau \in M_0$, $T_N^l(a_{Q_\tau^0})$ is an (ε',γ)-smooth unit for Q_τ^0, multiplied with a normalizing constant which can be estimated from above by $C2^{-|l|\sigma\varepsilon}$, where C is independent of l and τ.

Now for $j \in \mathbb{N}$ and $\tau \in M_j$, let $a_{Q_\tau^j}$ be an ε-smooth atom for Q_τ^j and let $|l| \le N$. We intend to show that $T_N^l(a_{Q_\tau^j})$ is a (β,γ)-smooth molecule for Q_τ^j, multiplied with some normalizing constant, where $\max(0,-s) < \beta < \varepsilon$ and $0 < \gamma < \varepsilon$.

For $0 \le l \le N$, we have

$$|T_N^l(a_{Q_\tau^j})(x)| = \Big|\sum_{k=0}^{\infty}E_kE_{k+l}(a_{Q_\tau^j})(x)\Big|$$

$$\le \sum_{k=0}^{j}|E_kE_{k+l}(a_{Q_\tau^j})(x)|\chi_{\{x:\,\varrho(x,z_\tau^j)\le 4A^2C2^{-k}\}}(x)$$

$$+\sum_{k=j+1}^{\infty}|E_kE_{k+l}(a_{Q_\tau^j})(x)|\chi_{\{x:\,\varrho(x,z_\tau^j)\le 4A^2C2^{-j}\}}(x)$$

$$\le \sum_{k=0}^{j}\Big|\int_X[(E_kE_{k+l})(x,y)-(E_kE_{k+l})(x,z_\tau^j)]a_{Q_\tau^j}(y)\,d\mu(y)\Big|\chi_{\{x:\,\varrho(x,z_\tau^j)\le 4A^2C2^{-k}\}}(x)$$

$$+\sum_{k=j+1}^{\infty}\Big|\int_X(E_kE_{k+l})(x,y)[a_{Q_\tau^j}(y)-a_{Q_\tau^j}(x)]\,d\mu(y)\Big|\chi_{\{x:\,\varrho(x,z_\tau^j)\le 4A^2C2^{-j}\}}(x)$$

$$\le C\mu(Q_\tau^j)^{-1/2}(1+2^j\varrho(x,z_\tau^j))^{-(d+\gamma)}\Big\{2^{-l\sigma\varepsilon}\sum_{k=0}^{j}2^{(k-j)((1-\sigma)\varepsilon-\gamma)}+2^{-l\varepsilon}\sum_{k=j+1}^{\infty}2^{-(k-j)\varepsilon}\Big\}$$

$$\le C2^{-l\sigma\varepsilon}\mu(Q_\tau^j)^{-1/2}(1+2^j\varrho(x,z_\tau^j))^{-(d+\gamma)}.$$

For $-N\le l<0$, we have

$$|T_N^l(a_{Q_\tau^j})(x)|=\Big|\sum_{k\ge0,\,k+l\ge0}E_kE_{k+l}(a_{Q_\tau^j})(x)\Big|$$

$$\le \sum_{0\le k\le j-l,\,k+l\ge0}|E_kE_{k+l}(a_{Q_\tau^j})(x)|\chi_{\{x:\,\varrho(x,z_\tau^j)\le 4A^2C2^{-k-l}\}}(x)$$

$$+\sum_{k=j-l+1}^{\infty}|E_kE_{k+l}(a_{Q_\tau^j})(x)|\chi_{\{x:\,\varrho(x,z_\tau^j)\le 4A^2C2^{-j}\}}(x)$$

$$\le \sum_{0\le k\le j-l,\,k+l\ge0}\Big|\int_X[(E_kE_{k+l})(x,y)-(E_kE_{k+l})(x,z_\tau^j)]a_{Q_\tau^j}(y)\,d\mu(y)\Big|$$

$$\times\chi_{\{x:\,\varrho(x,z_\tau^j)\le 4A^2C2^{-k-l}\}}(x)$$

$$+\sum_{k=j-l+1}^{\infty}\Big|\int_X(E_kE_{k+l})(x,y)[a_{Q_\tau^j}(y)-a_{Q_\tau^j}(x)]\,d\mu(y)\Big|\chi_{\{x:\,\varrho(x,z_\tau^j)\le 4A^2C2^{-j}\}}(x)$$

$$\le C\mu(Q_\tau^j)^{-1/2}(1+2^j\varrho(x,z_\tau^j))^{-(d+\gamma)}$$

$$\times\Big\{2^{l\sigma\varepsilon}\sum_{0\le k+l\le j}2^{(k+l-j)((1-\sigma)\varepsilon-\gamma)}+2^{l\varepsilon}\sum_{k=j-l+1}^{\infty}2^{-(k+l-j)\varepsilon}\Big\}$$

$$\le C2^{l\sigma\varepsilon}\mu(Q_\tau^j)^{-1/2}(1+2^j\varrho(x,z_\tau^j))^{-(d+\gamma)}.$$

Thus, for $|l|\le N$, we have

$$|T_N^l(a_{Q_\tau^j})(x)|\le C2^{-|l|\sigma\varepsilon}\mu(Q_\tau^j)^{-1/2}(1+2^j\varrho(x,z_\tau^j))^{-(d+\gamma)}, \tag{2.30}$$

where $\sigma\in(0,1)$ is such that $(1-\sigma)\varepsilon>\gamma>0$ and C is independent of l, N, j and τ.

Now we show that for $|l|\le N$, there is a γ satisfying $0<\gamma<\varepsilon$ such that

$$|T_N^l(a_{Q_\tau^j})(x)-T_N^l(a_{Q_\tau^j})(x')|\le C2^{-|l|\sigma\varepsilon}\mu(Q_\tau^j)^{-1/2-(1-\sigma)\varepsilon/d}\varrho(x,x')^{(1-\sigma)\varepsilon}$$
$$\times\Big\{\frac{1}{(1+\varrho(x,z_\tau^j))^{d+\gamma}}+\frac{1}{(1+\varrho(x',z_\tau^j))^{d+\gamma}}\Big\}, \tag{2.31}$$

where we take $\sigma \in (0,1)$ such that $(1-\sigma)\varepsilon > \max(0,-s)$, $2(1-\sigma)\varepsilon > \gamma > 0$ and C is independent of l, N, j and τ.

Similarly to the estimate for (2.27), we consider two cases.

Case 1: $\varrho(x,x') \geq 6A^2C2^{-j}$. In this case, it is easy to obtain (2.31) by (2.30).

Case 2: $\varrho(x,x') < 6A^2C2^{-j}$. In this case, we further suppose that there is a $j_1 \in \mathbb{N}$ such that $6A^2C2^{-j_1-j} \leq \varrho(x,x') < 6A^2C2^{-j_1-j+1}$.

Now if $0 \leq l \leq N$, we have

$$\begin{aligned}|T_N^l(a_{Q_\tau^j})(x) - T_N^l(a_{Q_\tau^j})(x')| &= \Big|\sum_{k=0}^{\infty}[E_kE_{k+l}(a_{Q_\tau^j})(x) - E_kE_{k+l}(a_{Q_\tau^j})(x')]\Big| \\ &\leq \sum_{k=0}^{j}|E_kE_{k+l}(a_{Q_\tau^j})(x) - E_kE_{k+l}(a_{Q_\tau^j})(x')| + \sum_{k=j+1}^{\infty}\cdots \\ &= P_1^1 + P_1^2.\end{aligned}$$

For P_1^1, we have

$$\begin{aligned}P_1^1 &= \sum_{k=0}^{j}|E_kE_{k+l}(a_{Q_\tau^j})(x) - E_kE_{k+l}(a_{Q_\tau^j})(x')| \\ &\quad \times (\chi_{\{x:\,\varrho(x,z_\tau^j)\leq 4A^2C2^{-k}\}}(x) + \chi_{\{x':\,\varrho(x',z_\tau^j)\leq 4A^2C2^{-k}\}}(x')) \\ &= \sum_{k=0}^{j}\Big|\int_X\{[(E_kE_{k+l})(x,y) - (E_kE_{k+l})(x',y)] \\ &\quad - [(E_kE_{k+l})(x,z_\tau^j) - (E_kE_{k+l})(x',z_\tau^j)]\}a_{Q_\tau^j}(y)\,d\mu(y)\Big| \\ &\quad \times (\chi_{\{x:\,\varrho(x,z_\tau^j)\leq 4A^2C2^{-k}\}}(x) + \chi_{\{x':\,\varrho(x',z_\tau^j)\leq 4A^2C2^{-k}\}}(x')) \\ &\leq C2^{-l\sigma\varepsilon}\mu(Q_\tau^j)^{-1/2-(1-\sigma)\varepsilon/d}\varrho(x,x')^{(1-\sigma)\varepsilon}\sum_{k=0}^{j}2^{(k-j)(2(1-\sigma)\varepsilon-\gamma)} \\ &\quad \times \bigg\{\frac{1}{(1+\varrho(x,z_\tau^j))^{d+\gamma}} + \frac{1}{(1+\varrho(x',z_\tau^j))^{d+\gamma}}\bigg\} \\ &\leq C2^{-l\sigma\varepsilon}\mu(Q_\tau^j)^{-1/2-(1-\sigma)\varepsilon/d}\varrho(x,x')^{(1-\sigma)\varepsilon}\bigg\{\frac{1}{(1+\varrho(x,z_\tau^j))^{d+\gamma}} + \frac{1}{(1+\varrho(x',z_\tau^j))^{d+\gamma}}\bigg\}.\end{aligned}$$

Now if $\varrho(x,z_\tau^j) \geq 12A^3C2^{-j}$, then $\varrho(x',z_\tau^j) \geq 6A^2C2^{-j}$. Thus $P_1^2 = 0$ in this case and we have (2.31). If $\varrho(x,z_\tau^j) < 12A^3C2^{-j}$, we also have $\varrho(x',z_\tau^j) < 18A^4C2^{-j}$. Thus, we obtain

$$\begin{aligned}P_1^2 &= \sum_{k=j+1}^{j+j_1}|E_kE_{k+l}(a_{Q_\tau^j})(x) - E_kE_{k+l}(a_{Q_\tau^j})(x')| + \sum_{k=j+j_1+1}^{\infty}\cdots \\ &= \sum_{k=j+1}^{j+j_1}\Big|\int_X[(E_kE_{k+l})(x,y) - (E_kE_{k+l})(x',y)][a_{Q_\tau^j}(y) - a_{Q_\tau^j}(x)]\,d\mu(y)\Big| \\ &\quad + \sum_{k=j+j_1+1}^{\infty}\Big[\Big|\int_X(E_kE_{k+l})(x,y)[a_{Q_\tau^j}(y) - a_{Q_\tau^j}(x)]\,d\mu(y)\Big|\end{aligned}$$

$$+\Big|\int_X (E_kE_{k+l})(x',y)[a_{Q^j_\tau}(y)-a_{Q^j_\tau}(x')]\,d\mu(y)\Big|\Big]$$

$$\le C\mu(Q^j_\tau)^{-1/2-(1-\sigma)\varepsilon/d}\Big[2^{-l\sigma\varepsilon}\varrho(x,x')^{(1-\sigma)\varepsilon}\sum_{k=j+1}^{j+j_1}2^{-(k-j)\sigma\varepsilon}+2^{-l\varepsilon+j\sigma\varepsilon}\sum_{k=j+j_1+1}^{\infty}2^{-k\varepsilon}\Big]$$

$$\le C2^{-l\sigma\varepsilon}\mu(Q^j_\tau)^{-1/2-(1-\sigma)\varepsilon/d}\varrho(x,x')^{(1-\sigma)\varepsilon}.$$

Now letting $-N\le l<0$, we write

$$|T^l_N(a_{Q^j_\tau})(x)-T^l_N(a_{Q^j_\tau})(x')|$$

$$=\Big|\sum_{k\ge 0,\,k+l\ge 0}[E_kE_{k+l}(a_{Q^j_\tau})(x)-E_kE_{k+l}(a_{Q^j_\tau})(x')]\Big|$$

$$\le\sum_{0\le k\le j-l,\,k+l\ge 0}|E_kE_{k+l}(a_{Q^j_\tau})(x)-E_kE_{k+l}(a_{Q^j_\tau})(x')|+\sum_{k=j-l+1}^{\infty}\cdots=P^1_2+P^2_2.$$

For P^1_2, we have

$$P^1_2=\sum_{k\ge 0,\,k+l\ge 0}|E_kE_{k+l}(a_{Q^j_\tau})(x)-E_kE_{k+l}(a_{Q^j_\tau})(x')|$$

$$\times(\chi_{\{x:\,\varrho(x,z^j_\tau)\le 4A^2C2^{-k-l}\}}(x)+\chi_{\{x':\,\varrho(x',z^j_\tau)\le 4A^2C2^{-k-l}\}}(x'))$$

$$=\sum_{k\ge 0,\,k+l\ge 0}\Big|\int_X\{[(E_kE_{k+l})(x,y)-(E_kE_{k+l})(x',y)]$$

$$-[(E_kE_{k+l})(x,z^j_\tau)-(E_kE_{k+l})(x',z^j_\tau)]\}a_{Q^j_\tau}(y)\,d\mu(y)\Big|$$

$$\times(\chi_{\{x:\,\varrho(x,z^j_\tau)\le 4A^2C2^{-k-l}\}}(x)+\chi_{\{x':\,\varrho(x',z^j_\tau)\le 4A^2C2^{-k-l}\}}(x'))$$

$$\le C2^{l\sigma\varepsilon}\mu(Q^j_\tau)^{-1/2-(1-\sigma)\varepsilon/d}\varrho(x,x')^{(1-\sigma)\varepsilon}\sum_{0\le k+l\le j}2^{(k+l-j)(2(1-\sigma)\varepsilon-\gamma)}$$

$$\times\Big\{\frac{1}{(1+\varrho(x,z^j_\tau))^{d+\gamma}}+\frac{1}{(1+\varrho(x',z^j_\tau))^{d+\gamma}}\Big\}$$

$$\le C2^{l\sigma\varepsilon}\mu(Q^j_\tau)^{-1/2-(1-\sigma)\varepsilon/d}\varrho(x,x')^{(1-\sigma)\varepsilon}\Big\{\frac{1}{(1+\varrho(x,z^j_\tau))^{d+\gamma}}+\frac{1}{(1+\varrho(x',z^j_\tau))^{d+\gamma}}\Big\}.$$

Now if $\varrho(x,z^j_\tau)\ge 12A^3C2^{-j}$, then it is easy to see that $\varrho(x',z^j_\tau)\ge 6A^2C2^{-j}$. Thus $P^2_2=0$ in this case and we have (2.31). If $\varrho(x,z^j_\tau)<12A^3C2^{-j}$, then $\varrho(x',z^j_\tau)<18A^4C2^{-j}$. Therefore, we have

$$P^2_2=\sum_{k=j-l+1}^{j+j_1-l}|E_kE_{k+l}(a_{Q^j_\tau})(x)-E_kE_{k+l}(a_{Q^j_\tau})(x')|+\sum_{k=j-l+j_1+1}^{\infty}\cdots$$

$$=\sum_{k=j-l+1}^{j-l+j_1}\Big|\int_X[(E_kE_{k+l})(x,y)-(E_kE_{k+l})(x',y)][a_{Q^j_\tau}(y)-a_{Q^j_\tau}(x)]\,d\mu(y)\Big|$$

$$+\sum_{k=j-l+j_1+1}^{\infty}\Big[\Big|\int_X(E_kE_{k+l})(x,y)[a_{Q^j_\tau}(y)-a_{Q^j_\tau}(x)]\,d\mu(y)\Big|$$

$$+\Big|\int\limits_X (E_kE_{k+l})(x',y)[a_{Q_\tau^j}(y)-a_{Q_\tau^j}(x')]\,d\mu(y)\Big|\Big]$$
$$\le C\mu(Q_\tau^j)^{-1/2-(1-\sigma)\varepsilon/d}$$
$$\times\Big[2^{l\sigma\varepsilon}\varrho(x,x')^{(1-\sigma)\varepsilon}\sum_{k=j-l+1}^{j-l+j_1}2^{-(k+l-j)\sigma\varepsilon}+2^{l\varepsilon+j\sigma\varepsilon}\sum_{k=j-l+j_1+1}^{\infty}2^{-(k+l)\varepsilon}\Big]$$
$$\le C2^{l\sigma\varepsilon}\mu(Q_\tau^j)^{-1/2-(1-\sigma)\varepsilon/d}\varrho(x,x')^{(1-\sigma)\varepsilon}.$$

Thus (2.31) holds. This means that $T_N^l(a_{Q_\tau^j})$ is a $((1-\sigma)\varepsilon,\gamma)$-smooth molecule for Q_τ^j, multiplied with a normalizing constant bounded above by $C2^{-|l|\sigma\varepsilon}$, where $0<\gamma<\varepsilon$ and C is independent of l, N, k and τ. Thus, by Lemmas 1.4 and 1.5, T_N^l is bounded in $B_{pq}^s(X)$ and $F_{pq}^s(X)$ with operator norms no more than $C_22^{-|l|\sigma\varepsilon}$, and thus T_N is bounded in $B_{pq}^s(X)$ and $F_{pq}^s(X)$ with operator norms no more than $C_2\sum_{|l|\le N}|1-2^{l\alpha}|2^{-|l|\sigma\varepsilon}$, where C_2 is independent of α and N. By combining the estimates for R_N and T_N, we find that $I-I_\alpha I_{-\alpha}$ is bounded in $B_{pq}^s(X)$ and $F_{pq}^s(X)$ with operator norms no more than $C_12^{-\delta N}+C_2\sum_{|l|\le N}|1-2^{l\alpha}|2^{-|l|\sigma\varepsilon}$, where C_1 is independent of N and α, provided $|\alpha|\le\delta_0$ with $\delta_0>0$ small enough. Now, obviously, we can choose $\alpha_0(s)\in(0,\delta_0]$ such that if $|\alpha|<\alpha_0(s)$, then $C_12^{-\delta N}+C_2\sum_{|l|\le N}|1-2^{l\alpha}|2^{-|l|\delta_1}<1$. Thus, when $|\alpha|<\alpha_0(s)$, $I_\alpha I_{-\alpha}$ and $I_{-\alpha}I_\alpha$ are invertible in $B_{pq}^s(X)$ and $F_{pq}^s(X)$. Therefore, for $f\in B_{pq}^s(X)$, by Theorem 2.2 and the above facts, we have

$$\|f\|_{B_{pq}^s(X)}=\|(I_{-\alpha}I_\alpha)^{-1}I_{-\alpha}I_\alpha(f)\|_{B_{pq}^s(X)}\le C\|I_{-\alpha}I_\alpha(f)\|_{B_{pq}^s(X)}\le C\|I_\alpha(f)\|_{B_{pq}^{s+\alpha}(X)},$$

where C is independent of f.

We can prove a similar conclusion for $F_{pq}^s(X)$.

This finishes the proof of Theorem 2.3.

From Theorems 2.2 and 2.3, we deduce the following corollary on the lifting property of the spaces $B_{pq}^s(X)$ and $F_{pq}^s(X)$, and the independence from the approximation to the identity in the definition of the fractional integrals and derivatives.

COROLLARY 2.1. *Let $s\in(-\varepsilon,\varepsilon)$ and $\alpha_0(s)$ be as in Theorem 2.3. Let $\alpha\in(-\varepsilon,\varepsilon)$ with $|\alpha|<\alpha_0(s)$ and $s+\alpha\in(-\varepsilon,\varepsilon)$. Let $\alpha<s+\varepsilon$ when $s<0$ and $\alpha>s-\varepsilon$ when $s>0$. Then there is a constant independent of f such that*

$$\frac{1}{C}\|f\|_{B_{pq}^s(X)}\le\|I_\alpha(f)\|_{B_{pq}^{s+\alpha}(X)}\le C\|f\|_{B_{pq}^s(X)}\quad\text{for } 1\le p,q\le\infty,$$
$$\frac{1}{C}\|f\|_{F_{pq}^s(X)}\le\|I_\alpha(f)\|_{F_{pq}^{s+\alpha}(X)}\le C\|f\|_{F_{pq}^s(X)}\quad\text{for } 1<p<\infty \text{ and } 1<q\le\infty.$$

Moreover, let $\{S_l\}_{l=0}^\infty$ and $\{\overline{S}_l\}_{l=0}^\infty$ be two approximations to the identity as in Definition 1.2 and let $E_l=S_l-S_{l-1}$ and $\overline{E}_l=\overline{S}_l-\overline{S}_{l-1}$ for $l\ge1$, $E_0=S_0$ and $\overline{E}_0=\overline{S}_0$. If we let

$$I_\alpha(f)=\sum_{l=0}^\infty 2^{-l\alpha}E_l(f)\quad\text{and}\quad \bar{I}_\alpha(f)=\sum_{l=0}^\infty 2^{-l\alpha}\overline{E}_l(f),$$

then there is a constant C independent of f such that

$$\frac{1}{C}\|\bar{I}_\alpha(f)\|_{B^{s+\alpha}_{pq}(X)} \le \|I_\alpha(f)\|_{B^{s+\alpha}_{pq}(X)} \le C\|\bar{I}_\alpha(f)\|_{B^{s+\alpha}_{pq}(X)} \quad \text{for } 1 \le p, q \le \infty,$$

$$\frac{1}{C}\|\bar{I}_\alpha(f)\|_{F^{s+\alpha}_{pq}(X)} \le \|I_\alpha(f)\|_{F^{s+\alpha}_{pq}(X)} \le C\|\bar{I}_\alpha(f)\|_{F^{s+\alpha}_{pq}(X)} \quad \text{for } 1<p<\infty \text{ and } 1<q\le\infty.$$

We remark that the independence from the approximations to the identity can also be seen from Theorem 1.6 of [11]. In [11], Gatto, Segovia and Vági first introduced their fractional integrals and derivatives by using some quasi-metrics related to the approximations to the identity which were proved to be equivalent to the original quasi-metric of the relevant space of homogeneous type. They then established some representation formulae for the fractional integrals and derivatives. Our definitions are just the discrete and inhomogeneous versions of their representation formulae. Thus, in some sense, the fractional integrals and derivatives are only related to the given quasi-metric of the relevant space of homogeneous type.

3. Explicit representations of inverses

In this section, we first establish explicit representation formulae in spaces of test functions for left and right inverses of fractional integrals and derivatives. The left and right inverses do not coincide, which contrasts with the case of spaces of homogeneous type and Euclidean spaces. We then give some basic properties of these inverses when $\mu(X) < \infty$. At the end of this section, we use the left inverses of fractional derivatives and Theorem 2.2 to establish some Poincaré-type inequalities for functions in $F^s_{p2}(X)$ when $\mu(X) < \infty$, $1 < p < \infty$ and $s > 0$ is small enough.

We have shown, in Section 2, that the fractional integrals and derivatives are invertible in $B^s_{pq}(X)$ and $F^s_{pq}(X)$ when $|s| < \varepsilon$ and $|\alpha|$ is small enough, where $\varepsilon \in (0, \theta]$. To do that, we used the well known atomic and molecular theories on these spaces for $|s| < \varepsilon$. Now, we are going to establish explicit representation formulae in spaces of test functions for the left and right inverses of fractional integrals and derivatives by using the theory of singular integrals in spaces of test functions; see Theorem 1 in [18]. This means that to show $I_\alpha I_{-\alpha}$ is invertible in spaces of test functions, we will show $I - I_\alpha I_{-\alpha}$ is a singular integral with a standard kernel, say $K(x, y)$, where I is the identity operator on these spaces. We will also show $K(x, y)$ has a "strong" weak boundedness property. Let $\|K\|$ be the smallest constant in all these estimates satisfied by $K(x, y)$. The key point here is that we will show that $\|K\|$ can be small if $|\alpha|$ is small. In fact, we will show that $\|K\|$ can go to 0 as $|\alpha| \to 0$. We point out that some ideas used here are similar to those used in [18] to establish the Calderón reproducing formulae on spaces of homogeneous type; see also the proof of Theorem 2.3. Also, in [11], Gatto, Segovia and Vági have shown that the homogeneous and continuous version of $I_\alpha I_{-\alpha}$ is a Calderón–Zygmund operator; see Theorems 1.4 and 1.5 in [11]. This means that $I_\alpha I_{-\alpha}$ is also bounded in $L^2(X)$ and therefore in $L^p(X)$ for $p \in (1, \infty)$, which can also be deduced from Theorem 2.3 and Lemma 1.10; see also Theorem 2.1 in [22].

Now let us recall some definitions. For $\theta \geq \eta > 0$, let $C_0^\eta(X)$ be the space of all continuous functions on X with compact support such that

$$\|f\|_{C_0^\eta(X)} = \|f\|_{L^\infty(X)} + \sup_{x \neq y} \frac{|f(x) - f(y)|}{\varrho(x,y)^\eta} < \infty.$$

We denote the dual space of $C_0^\eta(X)$ by $(C_0^\eta(X))'$.

By Remark 1.2, for spaces of homogeneous type as in Definition 0.1, one can construct an approximation to the identity, $\{S_k\}_{k\in\mathbb{N}\cup\{0\}}$, with compact supports as in Definition 1.2 such that $\lim_{k\to\infty} S_k = I$, the identity operator on $L^2(X)$, in the strong operator topology of $L^2(X)$. By using this fact, it is easy to show that for any $0 < \eta \leq \theta$, $C_0^\eta(X)$ is a dense subset of $L^2(X)$.

Definition 3.1. A continuous complex-valued function $K(x,y)$ defined on

$$\Omega = \{(x,y) \in X \times X : x \neq y\}$$

is called a *standard kernel* if there exist $\varepsilon \in (0,\theta]$ and $0 < C < \infty$ such that for all $x, y \in X$ with $x \neq y$,

$$|K(x,y)| \leq C\varrho(x,y)^{-d}, \tag{3.1}$$

$$|K(x,y) - K(x',y)| \leq C\varrho(x,x')^\varepsilon \varrho(x,y)^{-(d+\varepsilon)} \quad \text{for } \varrho(x,x') \leq \varrho(x,y)/(2A), \tag{3.2}$$

$$|K(x,y) - K(x,y')| \leq C\varrho(y,y')^\varepsilon \varrho(x,y)^{-(d+\varepsilon)} \quad \text{for } \varrho(y,y') \leq \varrho(x,y)/(2A). \tag{3.3}$$

Definition 3.2. A continuous linear operator $T : C_0^\eta(X) \to (C_0^\eta(X))'$ is a *singular integral operator* if there is a standard kernel K such that

$$\langle Tf, g\rangle = \int_X \int_X K(x,y) f(y) g(x)\, d\mu(y)\, d\mu(x)$$

for all $f, g \in C_0^\eta(X)$ whose supports are separated by a positive distance. We then write $T \in \mathrm{CZK}(\varepsilon)$.

We also need the following notion; see [23, p. 10].

Definition 3.3. A singular integral operator T is said to have the *"strong" weak boundedness property* if there exist $\eta > 0$ and a constant $0 < C < \infty$ such that for all $r > 0$,

$$|\langle K, f\rangle| \leq Cr^d \tag{3.4}$$

for all $r > 0$ and all continuous f on $X \times X$ with $\operatorname{supp} f \subseteq B(x_1, r) \times B(y_1, r)$, where $x_1, y_1 \in X$, $\|f\|_{L^\infty(X\times X)} \leq 1$,

$$\sup_{x \neq z} \frac{|f(x,y) - f(z,y)|}{\varrho(x,z)^\eta} \leq r^{-\eta}$$

for all $y \in X$ and

$$\sup_{y \neq z} \frac{|f(x,y) - f(x,z)|}{\varrho(y,z)^\eta} \leq r^{-\eta}$$

for all $x \in X$. We will denote this by $T \in \mathrm{SWBP}$.

To apply Theorem 1 in [18], we also need to verify that the kernel $K(x,y)$ satisfies

$$|[K(x,y) - K(x',y)] - [K(x,y') - K(x',y')]| \leq C\varrho(x,x')^\varepsilon \varrho(y,y')^\varepsilon \varrho(x,y)^{-(d+2\varepsilon)} \tag{3.5}$$

for $\varrho(x,x'), \varrho(y,y') \leq \frac{1}{3A^2}\varrho(x,y)$.

We will denote by $\|K\|$ the smallest constants appearing in (3.1)–(3.5).

We have the following estimate for the kernel $K(x,y)$ of $I - I_\alpha I_{-\alpha}$ which plays a crucial role in establishing explicit formulae for the inverses of fractional integrals and derivatives.

THEOREM 3.1. *Let $K(x,y)$ be the kernel of $I - I_\alpha I_{-\alpha}$ for $|\alpha| < \varepsilon$. There are $\alpha_1, \delta, \delta_1 \in (0,\varepsilon)$ and constants $C_1, C_2 > 0$ such that if $|\alpha| < \alpha_1$, then for any given $N \in \mathbb{N}$,*

$$\|K\| \le C_1 2^{-\delta N} + C_2 \sum_{|l|\le N} |1 - 2^{l\alpha}| 2^{-|l|\delta_1},$$

where C_1 and C_2 are independent of N and α, but C_1 may depend on α_1 and δ. Moreover, α_1 and δ can be any positive number less than ε.

Proof. For any given $N \in \mathbb{N}$, we write

$$\begin{aligned} T = I - I_\alpha I_{-\alpha} &= \sum_{|l|\le N} \sum_{k\ge 0,\, k+l\ge 0} (1 - 2^{l\alpha}) E_k E_{k+l} + \sum_{|l|>N} \sum_{k\ge 0,\, k+l\ge 0} (1 - 2^{l\alpha}) E_k E_{k+l} \\ &= \sum_{|l|\le N} (1 - 2^{l\alpha}) T_N^l + R_N = T_N + R_N. \end{aligned}$$

We denote the kernels of T_N, T_N^l, R_N, and R_N^l by $T_N(x,y)$, $T_N^l(x,y)$, $R_N(x,y)$ and $R_N^l(x,y)$, respectively.

Let us first establish (3.1). By (2.12), (2.13) and (2.15), we have

$$\begin{aligned} |T_N(x,y)| &\le C \sum_{0\le l\le N} |1 - 2^{l\alpha}| \sum_{k=0}^{[\log_2 \frac{AC}{\varrho(x,y)}]} 2^{-l\varepsilon} 2^{kd} + C \sum_{-N\le l<0} |1 - 2^{l\alpha}| \sum_{k+l=0}^{[\log_2 \frac{AC}{\varrho(x,y)}]} 2^{l\varepsilon} 2^{(k+l)d} \\ &\le \frac{C}{\varrho(x,y)^d} \sum_{|l|\le N} |1 - 2^{l\alpha}| 2^{-|l|\varepsilon}, \end{aligned}$$

where $[a]$ is the maximum integer no more than a, and C is independent of α and N.

By (2.12), (2.13) and (2.15), we have

$$\begin{aligned} |R_N(x,y)| &\le C \sum_{l>N} (1 + 2^{l\alpha}) \sum_{k=0}^{[\log_2 \frac{AC}{\varrho(x,y)}]} 2^{-l\varepsilon} 2^{kd} + C \sum_{l<-N} (1 + 2^{l\alpha}) \sum_{k+l=0}^{[\log_2 \frac{AC}{\varrho(x,y)}]} 2^{l\varepsilon} 2^{(k+l)d} \\ &\le \frac{C}{\varrho(x,y)^d} \Big\{ \sum_{l>N} (2^{-l\varepsilon} + 2^{-l(\varepsilon-\alpha)}) + \sum_{l<-N} (2^{l\varepsilon} + 2^{l(\varepsilon+\alpha)}) \Big\} \le C 2^{-\delta N} \frac{1}{\varrho(x,y)^d}, \end{aligned}$$

where we choose $|\alpha| < \varepsilon$, $\delta = \min\{\varepsilon - \alpha, \varepsilon + \alpha\}$, and C is independent of N. Moreover, if $|\alpha| < \alpha_1 \le \varepsilon$, then C is also independent of α, but it may depend on α_1. Thus, (3.1) holds.

Now let us prove (3.2). Let $\varrho(x,x') \le \varrho(x,y)/(2A)$. Then by (2.12), (2.13) and (2.17), we have

$$\begin{aligned} |K(x,y) - K(x',y)| \le &\sum_{|l|\le N} \sum_{k\ge 0,\, k+l\ge 0} |1 - 2^{l\alpha}| |(E_k E_{k+l})(x,y) - (E_k E_{k+l})(x',y)| \\ &+ \sum_{|l|>N} \sum_{k\ge 0,\, k+l\ge 0} (1 + 2^{l\alpha}) |(E_k E_{k+l})(x,y) - (E_k E_{k+l})(x',y)| \end{aligned}$$

$$
\begin{aligned}
&\leq \varrho(x,x')^{(1-\sigma)\varepsilon}\Big\{C\sum_{0\leq l\leq N}|1-2^{l\alpha}|\sum_{k=0}^{[\log_2\frac{2A^2C}{\varrho(x,y)}]}2^{-l\sigma\varepsilon}2^{k(d+(1-\sigma)\varepsilon)}\\
&\quad+C\sum_{-N\leq l<0}|1-2^{l\alpha}|\sum_{k+l=0}^{[\log_2\frac{2A^2C}{\varrho(x,y)}]}2^{l\sigma\varepsilon}2^{(k+l)(d+(1-\sigma)\varepsilon)}\\
&\quad+C\sum_{l>N}(1+2^{l\alpha})\sum_{k=0}^{[\log_2\frac{2A^2C}{\varrho(x,y)}]}2^{-l\sigma\varepsilon}2^{k(d+(1-\sigma)\varepsilon)}\\
&\quad+C\sum_{l<-N}(1+2^{l\alpha})\sum_{k+l=0}^{[\log_2\frac{2A^2C}{\varrho(x,y)}]}2^{l\sigma\varepsilon}2^{(k+l)(d+(1-\sigma)\varepsilon)}\Big\}\\
&\leq \frac{\varrho(x,x')^{(1-\sigma)\varepsilon}}{\varrho(x,y)^{d+(1-\sigma)\varepsilon}}\Big\{C_2\sum_{|l|\leq N}|1-2^{l\alpha}|2^{-|l|\sigma\varepsilon}+C_1[2^{-N\sigma\varepsilon}+2^{-N(\sigma\varepsilon-\alpha)}+2^{-N(\sigma\varepsilon+\alpha)}]\Big\}\\
&\leq \Big\{C_12^{-\delta N}+C_2\sum_{|l|\leq N}|1-2^{l\alpha}|2^{-|l|\sigma\varepsilon}\Big\}\frac{\varrho(x,x')^{(1-\sigma)\varepsilon}}{\varrho(x,y)^{d+(1-\sigma)\varepsilon}},
\end{aligned}
$$

where we choose $\sigma\in(0,1)$ such that $|\alpha|<\sigma\varepsilon$, $\delta=\min\{\sigma\varepsilon-\alpha,\sigma\varepsilon+\alpha\}$, and C_1 and C_2 are as in the theorem. Thus, (3.2) holds. The proof of (3.3) is similar.

Now let us prove (3.5). By (2.18), (2.12) and (2.13), for $\varrho(x,x')\leq\varrho(x,y)/(3A^2)$ and $\varrho(y,y')\leq\varrho(x,y)/(2A^2)$, or $\varrho(x,x')\leq\varrho(x,y)/(2A^2)$ and $\varrho(y,y')\leq\varrho(x,y)/(3A^2)$, we have

$$
\begin{aligned}
&|[K(x,y)-K(x',y)]-[K(x,y')-K(x',y')]|\\
&\qquad\leq\sum_{|l|\leq N}\sum_{k\geq0,\,k+l\geq0}|1-2^{l\alpha}||[(E_kE_{k+l})(x,y)-(E_kE_{k+l})(x',y)]\\
&\qquad\quad-[(E_kE_{k+l})(x,y')-(E_kE_{k+l})(x',y')]|\\
&\qquad\quad+\sum_{|l|>N}\sum_{k\geq0,\,k+l\geq0}(1+2^{l\alpha})|[(E_kE_{k+l})(x,y)-(E_kE_{k+l})(x',y)]\\
&\qquad\quad-[(E_kE_{k+l})(x,y')-(E_kE_{k+l})(x',y')]|\\
&\qquad\leq\varrho(x,x')^{(1-\sigma)\varepsilon}\varrho(y,y')^{(1-\sigma)\varepsilon}\Big\{C\sum_{0\leq l\leq N}|1-2^{l\alpha}|\sum_{k=0}^{[\log_2\frac{6A^3C}{\varrho(x,y)}]}2^{-l\sigma\varepsilon}2^{k(d+2(1-\sigma)\varepsilon)}\\
&\qquad\quad+C\sum_{-N\leq l<0}|1-2^{l\alpha}|\sum_{k+l=0}^{[\log_2\frac{6A^3C}{\varrho(x,y)}]}2^{l\sigma\varepsilon}2^{(k+l)(d+2(1-\sigma)\varepsilon)}\\
&\qquad\quad+C\sum_{l>N}(1+2^{l\alpha})\sum_{k=0}^{[\log_2\frac{6A^3C}{\varrho(x,y)}]}2^{-l\sigma}2^{k(d+2(1-\sigma)\varepsilon)}
\end{aligned}
$$

$$+C\sum_{l<-N}(1+2^{l\alpha})\sum_{k+l=0}^{[\log_2\frac{6A^3C}{\varrho(x,y)}]}2^{l\sigma}2^{(k+l)(d+2(1-\sigma)\varepsilon)}\Big\}$$

$$\leq\frac{\varrho(x,x')^{(1-\sigma)\varepsilon}\varrho(y,y')^{(1-\sigma)\varepsilon}}{\varrho(x,y)^{d+2(1-\sigma)\varepsilon}}\Big\{C_1[2^{-N\sigma\varepsilon}+2^{-N(\sigma\varepsilon-\alpha)}+2^{-N(\sigma\varepsilon+\alpha)}]$$

$$+C_2\sum_{|l|\leq N}|1-2^{l\alpha}|2^{-|l|\sigma\varepsilon}\Big\}$$

$$\leq\Big\{C_1 2^{-\delta N}+C_2\sum_{|l|\leq N}|1-2^{l\alpha}|2^{-|l|\sigma\varepsilon}\Big\}\frac{\varrho(x,x')^{(1-\sigma)\varepsilon}\varrho(y,y')^{(1-\sigma)\varepsilon}}{\varrho(x,y)^{d+2(1-\sigma)\varepsilon}},$$

where we choose $\sigma\in(0,1)$ such that $|\alpha|<\sigma\varepsilon$, $\delta=\min\{\sigma\varepsilon-\alpha,\sigma\varepsilon+\alpha\}$, and C_1 and C_2 are as in the theorem. Thus, (3.5) holds for $\varrho(x,x')\leq\varrho(x,y)/(3A^2)$ and $\varrho(y,y')\leq\varrho(x,y)/(2A^2)$, or $\varrho(x,x')\leq\varrho(x,y)/(2A^2)$ and $\varrho(y,y')\leq\varrho(x,y)/(3A^2)$. If $\varrho(x,y)/(3A^2)\leq\varrho(x,x')$ and $\varrho(y,y')\leq\varrho(x,y)/(2A^2)$, then (3.5) can be deduced from (3.2) and (3.3). Thus, in any case, (3.5) holds.

Finally, let us show (3.4). Let f be a continuous function on $X\times X$ with

$$\operatorname{supp} f\subseteq B(x_1,r)\times B(y_1,r),$$

where x_1 and $y_1\in X$, $\|f\|_{L^\infty(X\times X)}\leq 1$,

$$\sup_{x\neq z}\frac{|f(x,y)-f(z,y)|}{\varrho(x,z)^\eta}\leq r^{-\eta}\qquad\text{for all } y\in X,$$

$$\sup_{y\neq z}\frac{|f(x,y)-f(x,z)|}{\varrho(y,z)^\eta}\leq r^{-\eta}\qquad\text{for all } x\in X.$$

We first establish some estimates on $|(E_kE_{k+l})(f)|$ whose proofs are similar to those of (3.18), (3.23), (3.24) and (3.25) in [18]. For $k\geq 0$, $l\in\mathbb{N}$ and $k+l\geq 0$, by (2.15), we have

$$\begin{aligned}(3.6)\qquad |\langle E_kE_{k+l},f\rangle|&=\Big|\int_X\int_X(E_kE_{k+l})(x,y)f(x,y)\,d\mu(x)\,d\mu(y)\Big|\\&\leq C2^{-|l|\varepsilon}\|f\|_{L^\infty(X\times X)}r^d\leq C2^{-|l|\varepsilon}r^d,\end{aligned}$$

where C is independent of l and r.

If $k\geq 0$, $l\in\mathbb{N}$ and $k+l>0$, by (2.14) and (2.15), we have

$$\begin{aligned}(3.7)\qquad |\langle E_kE_{k+l},f\rangle|&=\Big|\int_X\int_X\int_X E_k(x,z)E_{k+l}(z,y)f(x,y)\,d\mu(z)\,d\mu(x)\,d\mu(y)\Big|\\&=\Big|\int_X\int_X\int_X E_k(x,z)E_{k+l}(z,y)[f(x,y)-f(x,z)]\,d\mu(z)\,d\mu(x)\,d\mu(y)\Big|\\&\leq r^d\int_X\int_X\int_X|E_k(x,z)E_{k+l}(z,y)|\varrho(z,y)^{-\eta}\,d\mu(z)\,d\mu(x)\,d\mu(y)\\&\leq C2^{-(k+l)\eta}r^{-\eta}r^d,\end{aligned}$$

where C is independent of l and r.

If $k \geq 0$, $l \in \mathbb{N}$ and $k+l \geq 0$, we also have the following trivial estimate:

$$|\langle E_k E_{k+l}, f\rangle| \leq C 2^{(k+l)d} r^{2d}, \tag{3.8}$$

where C is independent of l and r.

Now by (3.6) and (3.7), for $k \geq 0$, $l \in \mathbb{N}$ and $k+l > 0$, we have

$$|\langle E_k E_{k+l}, f\rangle| \leq C 2^{-|l|\varepsilon\sigma} 2^{-(k+l)\eta(1-\sigma)} r^{-\eta(1-\sigma)} r^d, \tag{3.9}$$

where σ can be any number in (0,1), and C is independent of l, σ and r.

By (3.6) and (3.8), for $k \geq 0$, $l \in \mathbb{N}$ and $k+l > 0$, we have

$$|\langle E_k E_{k+l}, f\rangle| \leq C 2^{-|l|\varepsilon\sigma} 2^{(k+l)d(1-\sigma)} r^{d(1-\sigma)} r^d, \tag{3.10}$$

where σ can be any number in (0,1), and C is independent of l, σ and r.

Now, by (3.6), (3.9) and (3.10), we have

$$\begin{aligned}
|\langle R_N, f\rangle| &= \Big| \sum_{|l|>N} \sum_{k\geq 0,\, k+l\geq 0} (1-2^{l\alpha})\langle E_k E_{k+l}, f\rangle \Big| \\
&\leq \sum_{l<-N} (1+2^{l\alpha})|\langle E_{-l}E_0, f\rangle| + \sum_{|l|>N} \sum_{k+l>0,\, 2^{-(k+l)}<r} (1+2^{l\alpha})|\langle E_k E_{k+l}, f\rangle| \\
&\quad + \sum_{|l|>N} \sum_{k+l>0,\, 2^{-(k+l)}\geq r} (1+2^{l\alpha})|\langle E_k E_{k+l}, f\rangle| \\
&\leq C \sum_{l<-N} (1+2^{l\alpha}) 2^{l\varepsilon} r^d \\
&\quad + C \sum_{|l|>N} \sum_{k+l>0,\, 2^{-(k+l)}<r} (1+2^{l\alpha}) 2^{-|l|\varepsilon\sigma} 2^{-(k+l)\eta(1-\sigma)} r^{-\eta(1-\sigma)} r^d \\
&\quad + C \sum_{|l|>N} \sum_{k+l>0,\, 2^{-(k+l)}\geq r} (1+2^{l\alpha}) 2^{-|l|\varepsilon\sigma} 2^{(k+l)d(1-\sigma)} r^{d(1-\sigma)} r^d \\
&\leq C(2^{-N\varepsilon} + 2^{-N(\alpha+\varepsilon)} + 2^{-N\varepsilon\sigma} + 2^{-N(\varepsilon\sigma+\alpha)} + 2^{-N(\varepsilon\sigma-\alpha)}) r^d \leq C_1 2^{-\delta N} r^d,
\end{aligned}$$

where we take $\sigma \in (0,1)$ such that $|\alpha| < \varepsilon\sigma$, $\delta = \min(\varepsilon\sigma+\alpha, \varepsilon\sigma-\alpha)$ and C_1 is as in the theorem.

For $0 < l \leq N$, by (3.6), (3.9) and (3.10), we have

$$\begin{aligned}
|\langle T_N^l, f\rangle| &\leq \sum_{k+l>0} |\langle E_k E_{k+l}, f\rangle| \\
&\leq C \sum_{k+l>0,\, 2^{-(k+l)}<r} 2^{-l\varepsilon\sigma} 2^{-(k+l)\eta(1-\sigma)} r^{-\eta(1-\sigma)} r^d \\
&\quad + C \sum_{k+l>0,\, 2^{-(k+l)}\geq r} 2^{-l\varepsilon\sigma} 2^{(k+l)d(1-\sigma)} r^{d(1-\sigma)} r^d \\
&\leq C 2^{-l\varepsilon\sigma} r^d,
\end{aligned}$$

where $\sigma \in (0,1)$ and C is independent of r and l.

For $l = 0$, by (3.6), (3.9) and (3.10), we have

$$|\langle T_N^l, f\rangle| \le \sum_{k\ge 0} |\langle E_k E_k, f\rangle| = |\langle E_0 E_0, f\rangle| + \sum_{k>0} |\langle E_k E_k, f\rangle|$$
$$\le Cr^d\Big\{1 + \sum_{k\in\mathbb{N},\, 2^{-k}<r} 2^{-k\eta(1-\sigma)} r^{-\eta(1-\sigma)} + \sum_{k\in\mathbb{N},\, 2^{-k}\ge r} 2^{kd(1-\sigma)} r^{d(1-\sigma)}\Big\} \le Cr^d,$$

where $\sigma \in (0,1)$ and C is independent of r.

For $-N \le l < 0$, by (3.6), (3.9) and (3.10), we have

$$|\langle T_N^l, f\rangle| \le \sum_{k\ge 0,\, k+l\ge 0} |\langle E_k E_{k+l}, f\rangle|$$
$$= |\langle E_{-l} E_0, f\rangle| + \sum_{k+l>0,\, 2^{-(k+l)}<r} |\langle E_k E_{k+l}, f\rangle| + \sum_{k+l>0,\, 2^{-(k+l)}\ge r} |\langle E_k E_{k+l}, f\rangle|$$
$$\le C2^{l\varepsilon} r^d + C \sum_{k+l>0,\, 2^{-(k+l)}<r} 2^{l\varepsilon\sigma} 2^{-(k+l)\eta(1-\sigma)} r^{-\eta(1-\sigma)} r^d$$
$$+ \sum_{k+l>0,\, 2^{-(k+l)}\ge r} 2^{l\varepsilon\sigma} 2^{(k+l)d(1-\sigma)} r^{d(1-\sigma)} r^d$$
$$\le C2^{l\varepsilon\sigma} r^d,$$

where $\sigma \in (0,1)$ and C is independent of r and l.

By summing up all the estimates on $\langle T_N^l, f\rangle$ and $\langle R_N, f\rangle$, we conclude that (3.4) holds.

This finishes the proof of Theorem 3.1.

In the following, for $|\alpha| < \varepsilon$, we define the left inverse, $(I_\alpha)_l^{-1}$, and the right inverse, $(I_\alpha)_r^{-1}$, respectively, by

$$(I_\alpha)_l^{-1} I_\alpha = I = I_\alpha (I_\alpha)_r^{-1}$$

in $\mathcal{G}(\beta,\gamma)$ for $0 < \beta, \gamma < \varepsilon$. The following theorem guarantees the existence of $(I_\alpha)_l^{-1}$ and $(I_\alpha)_r^{-1}$. Moreover, we have their obvious expressions.

THEOREM 3.2. *Let $0 < \beta, \gamma < \varepsilon$. There exists an $\alpha_0(\beta,\gamma) \in (0,\varepsilon)$ such that if $|\alpha| < \alpha_0(\beta,\gamma)$, then $(I_\alpha)_l^{-1}$ and $(I_\alpha)_r^{-1}$ exist in $\mathcal{G}(\beta,\gamma)$. Moreover,*

$$(I_\alpha)_l^{-1} = \sum_{k=0}^{\infty} 2^{k\alpha} \widetilde{E}_k \quad \textit{and} \quad (I_\alpha)_r^{-1} = \sum_{k=0}^{\infty} 2^{k\alpha} \widetilde{D}_k,$$

where $\widetilde{E}_k$ and $\widetilde{D}_k$ are linear operators whose kernels, $\widetilde{E}_k(x,y)$ and $\widetilde{D}_k(x,y)$, have the following properties:

(i) $\displaystyle\int_X \widetilde{E}_k(x,y)\, d\mu(x) = \int_X \widetilde{E}_k(x,y)\, d\mu(y) = \begin{cases} 1 & \textit{for } k = 0, \\ 0 & \textit{for } k \in \mathbb{N}; \end{cases}$

(ii) $\displaystyle|\widetilde{E}_k(x,y)| \le C \frac{2^{-k\gamma'}}{(2^{-k} + \varrho(x,y))^{d+\gamma'}}$ *for $k \in \mathbb{N} \cup \{0\}$;*

(iii) $\displaystyle|\widetilde{E}_k(x,y) - \widetilde{E}_k(x',y)| \le C\Big(\frac{\varrho(x,x')}{2^{-k} + \varrho(x,y)}\Big)^{\varepsilon'} \frac{2^{-k\gamma'}}{(2^{-k} + \varrho(x,y))^{d+\gamma'}}$

for $\varrho(x,x') \le \frac{1}{2A}(2^{-k} + \varrho(x,y))$ and $k \in \mathbb{N} \cup \{0\}$;

(iv) $\int_X \widetilde{D}_k(x,y)\,d\mu(x) = \int_X \widetilde{D}_k(x,y)\,d\mu(y) = \begin{cases} 1 & \textit{for } k=0, \\ 0 & \textit{for } k\in\mathbb{N}; \end{cases}$

(v) $|\widetilde{D}_k(x,y)| \le C \dfrac{2^{-k\gamma'}}{(2^{-k}+\varrho(x,y))^{d+\gamma'}}$ *for* $k\in\mathbb{N}\cup\{0\}$;

(vi) $|\widetilde{D}_k(x,y) - \widetilde{D}_k(x,y')| \le C\left(\dfrac{\varrho(y,y')}{2^{-k}+\varrho(x,y)}\right)^{\varepsilon'} \dfrac{2^{-k\gamma'}}{(2^{-k}+\varrho(x,y))^{d+\gamma'}}$

for $\varrho(y,y') \le \frac{1}{2A}(2^{-k}+\varrho(x,y))$ *and* $k\in\mathbb{N}\cup\{0\}$.

Here $\beta\le\varepsilon'<\varepsilon$ *and* $\gamma\le\gamma'<\varepsilon$. *Moreover, if* $0<\beta_1\le\beta\le\beta_2<\varepsilon$ *and* $0<\gamma_1\le\gamma\le\gamma_2<\varepsilon$, *then* $\alpha_0(\beta,\gamma)$ *can be independent of* β *and* γ, *but it may depend on* β_1, β_2, γ_1 *and* γ_2.

Proof. Let us first establish the representation formula for $(I_\alpha)_l^{-1}$. Let $T = I - I_{-\alpha}I_\alpha$ and K be its kernel, where I is the identity in the space $\mathcal{G}(\beta,\gamma)$. Then, obviously, $T(1) = T^*(1) = 0$. Let us first show

$$|(TE_0)(x,y)| \le \Big(C_1 2^{-\delta N} + C_2 \sum_{|l|\le N} |1-2^{-l\alpha}| 2^{-|l|\varepsilon}\Big) \frac{1}{(1+\varrho(x,y))^{d+\gamma'}} \tag{3.11}$$

and

$$|(TE_0)(x,y) - (TE_0)(x',y)| \le \Big(C_1 2^{-\delta N} + C_2 \sum_{|l|\le N} |1-2^{-l\alpha}| 2^{-|l|\sigma\varepsilon}\Big) \times \left(\frac{\varrho(x,x')}{1+\varrho(x,y)}\right)^{(1-\sigma)\varepsilon} \frac{1}{(1+\varrho(x,y))^{d+\gamma'}} \tag{3.12}$$

for $\varrho(x,x') \le \frac{1}{2A}(1+\varrho(x,y))$, where C_1, C_2 and δ are as in Theorem 3.1 and $\sigma\in(0,1)$.

For any given $N\in\mathbb{N}$, we write

$$\begin{aligned} T &= I - I_{-\alpha}I_\alpha \\ &= \sum_{|l|\le N} \sum_{k\ge0,\,k+l\ge0} (1-2^{-l\alpha}) E_k E_{k+l} + \sum_{|l|>N} \sum_{k\ge0,\,k+l\ge0} (1-2^{-l\alpha}) E_k E_{k+l} = \overline{T}_N + \overline{R}_N. \end{aligned}$$

Similarly to (2.23), by (2.12)–(2.15), we have

$$\begin{aligned} &|(\overline{R}_N E_0)(x,y)| \\ &\quad = \Big| \sum_{l>N} (1-2^{-l\alpha}) \sum_{k=0}^{\infty} (E_k E_{k+l} E_0)(x,y) + \sum_{l<-N} (1-2^{-l\alpha}) \sum_{k\ge0,\,k+l\ge0} (E_k E_{k+l} E_0)(x,y) \Big| \\ &\quad \le \sum_{l>N} (1+2^{-l\alpha}) \chi_{\{(x,y):\,\varrho(x,y)\le 2A^2C\}}(x,y) \\ &\qquad \times \sum_{k=0}^{\infty} \Big| \int_X (E_k E_{k+l})(x,z)(E_0(z,y)-E_0(x,y))\,d\mu(z) \Big| \\ &\qquad + \sum_{l<-N} (1+2^{-l\alpha}) \sum_{k\ge0,\,k+l\ge0} \Big| \int_X (E_k E_{k+l})(x,z)(E_0(z,y)-E_0(x,y))\,d\mu(z) \Big| \\ &\qquad \times \chi_{\{(x,y):\,\varrho(x,y)\le 2A^2C\}}(x,y) \end{aligned}$$

$$
\begin{aligned}
&\le \sum_{l>N}(1+2^{-l\alpha})\sum_{k=0}^{\infty}2^{-l\varepsilon}2^{-k\varepsilon}\chi_{\{(x,y):\,\varrho(x,y)\le 2A^2C\}}(x,y)\\
&\quad+\sum_{l<-N}(1+2^{-l\alpha})\sum_{k+l\ge 0}2^{l\varepsilon}2^{-(k+l)\varepsilon}\chi_{\{(x,y):\,\varrho(x,y)\le 2A^2C\}}(x,y)\\
&\le C2^{-\delta N}\frac{1}{(1+\varrho(x,y))^{d+\gamma'}},
\end{aligned}
$$

where $\delta=\min(\varepsilon+\alpha,\varepsilon-\alpha)$.

Similarly to (2.28), by (2.12)–(2.15) we have

$$
\begin{aligned}
|(\overline{T}_N E_0)(x,y)| &= \Big|\sum_{0\le l\le N}(1-2^{-l\alpha})\sum_{k=0}^{\infty}(E_kE_{k+l}E_0)(x,y)\\
&\quad+\sum_{-N\le l<0}(1-2^{-l\alpha})\sum_{k\ge 0,\,k+l\ge 0}(E_kE_{k+l}E_0)(x,y)\Big|\\
&\le \sum_{0\le l\le N}|1-2^{-l\alpha}|\sum_{k=0}^{\infty}\Big|\int_X (E_kE_{k+l})(x,z)(E_0(z,y)-E_0(x,y))\,d\mu(z)\Big|\\
&\quad\times\chi_{\{(x,y):\,\varrho(x,y)\le 2A^2C\}}(x,y)\\
&\quad+\sum_{-N\le l<0}|1-2^{-l\alpha}|\sum_{k\ge 0,\,k+l\ge 0}\Big|\int_X (E_kE_{k+l})(x,z)(E_0(z,y)-E_0(x,y))\,d\mu(z)\Big|\\
&\quad\times\chi_{\{(x,y):\,\varrho(x,y)\le 2A^2C\}}(x,y)\\
&\le \sum_{0\le l\le N}|1-2^{-l\alpha}|\sum_{k=0}^{\infty}2^{-l\varepsilon}2^{-k\varepsilon}\chi_{\{(x,y):\,\varrho(x,y)\le 2A^2C\}}(x,y)\\
&\quad+\sum_{-N\le l<0}|1-2^{-l\alpha}|\sum_{k+l\ge 0}2^{l\varepsilon}2^{-(k+l)\varepsilon}\chi_{\{(x,y):\,\varrho(x,y)\le 2A^2C\}}(x,y)\\
&\le C\sum_{|l|\le N}|1-2^{-l\alpha}|2^{-|l|\varepsilon}\frac{1}{(1+\varrho(x,y))^{d+\gamma'}}.
\end{aligned}
$$

Thus, (3.11) holds.

Now, let us show (3.12). Similarly to (2.24) and (2.29), we also consider three cases.

Case 1: $6A^2C\le\varrho(x,x')\le\frac{1}{2A}(1+\varrho(x,y))$. In this case, (3.12) can be deduced easily from (3.11).

Case 2: $\varrho(x,x')<6A^2C$ and $\varrho(x,y)>12A^3C$. In this case, it is easy to deduce $\varrho(x',y)>6A^2C$. Thus, $TE_0(x,y)=TE_0(x',y)=0$ and (3.12) holds.

Case 3: $\varrho(x,x')<6A^2C$ and $\varrho(x,y)<12A^3C$. We further suppose that there is an $l_1\in\mathbb{N}$ such that $6A^2C2^{-l_1}\le\varrho(x,x')<6A^2C2^{-l_1+1}$. We then write

$$
\begin{aligned}
&|(TE_0)(x,y)-(TE_0)(x',y)|\\
&\qquad\le\sum_{0\le l\le N}|1-2^{-l\alpha}|\Big\{\sum_{k=0}^{l_1}|(E_kE_{k+l}E_0)(x,y)-(E_kE_{k+l}E_0)(x',y)|
\end{aligned}
$$

$$+\sum_{k=l_1+1}^{\infty}[|(E_kE_{k+l}E_0)(x,y)|+|(E_kE_{k+l}E_0)(x',y)|]\Big\}$$

$$+\sum_{-N\le l<0}|1-2^{-l\alpha}|\Big\{\sum_{0\le k\le l_1-l,\,k+l\ge 0}|(E_kE_{k+l}E_0)(x,y)-(E_kE_{k+l}E_0)(x',y)|$$

$$+\sum_{k>l_1-l}[|(E_kE_{k+l}E_0)(x,y)|+|(E_kE_{k+l}E_0)(x',y)|]\Big\}$$

$$+\sum_{l>N}(1+2^{-l\alpha})\Big\{\sum_{k=0}^{l_1}|(E_kE_{k+l}E_0)(x,y)-(E_kE_{k+l}E_0)(x',y)|$$

$$+\sum_{k=l_1+1}^{\infty}[|(E_kE_{k+l}E_0)(x,y)|+|(E_kE_{k+l}E_0)(x',y)|]\Big\}$$

$$+\sum_{l<-N}(1+2^{-l\alpha})\Big\{\sum_{0\le k\le l_1-l,\,k+l\ge 0}|(E_kE_{k+l}E_0)(x,y)-(E_kE_{k+l}E_0)(x',y)|$$

$$+\sum_{k>l_1-l}[|(E_kE_{k+l}E_0)(x,y)|+|(E_kE_{k+l}E_0)(x',y)|]\Big\}$$

$$\le\sum_{0\le l\le N}|1-2^{-l\alpha}|\Big\{\sum_{k=0}^{l_1}\Big|\int_X[(E_kE_{k+l})(x,z)-(E_kE_{k+l})(x',z)][E_0(z,y)-E_0(x,y)]\,d\mu(z)\Big|$$

$$+\sum_{k=l_1+1}^{\infty}\Big[\Big|\int_X(E_kE_{k+l})(x,z)[E_0(z,y)-E_0(x,y)]\,d\mu(z)\Big|$$

$$+\Big|\int_X(E_kE_{k+l})(x',z)[E_0(z,y)-E_0(x',y)]\,d\mu(z)\Big|\Big]\Big\}$$

$$+\sum_{-N\le l<0}|1-2^{-l\alpha}|\Big\{\sum_{0\le k\le l_1-l,\,k+l\ge 0}\Big|\int_X[(E_kE_{k+l})(x,z)-(E_kE_{k+l})(x',z)]$$

$$\times[E_0(z,y)-E_0(x,y)]\,d\mu(z)\Big|$$

$$+\sum_{k>l_1-l}\Big[\Big|\int_X(E_kE_{k+l})(x,z)[E_0(z,y)-E_0(x,y)]\,d\mu(z)\Big|$$

$$+\Big|\int_X(E_kE_{k+l})(x',z)[E_0(z,y)-E_0(x',y)]\,d\mu(z)\Big|\Big]\Big\}$$

$$+\sum_{l>N}(1+2^{-l\alpha})\Big\{\sum_{k=0}^{l_1}\Big|\int_X[(E_kE_{k+l})(x,z)-(E_kE_{k+l})(x',z)]$$

$$\times[E_0(z,y)-E_0(x,y)]\,d\mu(z)\Big|$$

$$+\sum_{k=l_1+1}^{\infty}\Big[\Big|\int_X(E_kE_{k+l})(x,z)[E_0(z,y)-E_0(x,y)]\,d\mu(z)\Big|$$

$$+\Big|\int_X(E_kE_{k+l})(x',z)[E_0(z,y)-E_0(x',y)]\,d\mu(z)\Big|\Big]\Big\}$$

$$+\sum_{l<-N}(1+2^{-l\alpha})\Big\{\sum_{0\le k\le l_1-l,\,k+l\ge 0}\Big|\int_X[(E_kE_{k+l})(x,z)-(E_kE_{k+l})(x',z)]$$

$$\times[E_0(z,y)-E_0(x,y)]\,d\mu(z)\Big|$$

$$
\begin{aligned}
&+\sum_{k>l_1-l}\bigg[\Big|\int_X (E_kE_{k+l})(x,z)[E_0(z,y)-E_0(x,y)]\,d\mu(z)\Big|\\
&+\Big|\int_X (E_kE_{k+l})(x',z)[E_0(z,y)-E_0(x',y)]\,d\mu(z)\Big|\bigg]\bigg\}\\
&\le C\sum_{0\le l\le N}|1-2^{-l\alpha}|\Big\{2^{-l\sigma\varepsilon}\varrho(x,x')^{(1-\sigma)\varepsilon}\sum_{k=0}^{l_1}2^{-k\sigma\varepsilon}+\sum_{k=l_1+1}^{\infty}2^{-l\varepsilon-k\varepsilon}\Big\}\\
&+C\sum_{-N\le l<0}|1-2^{-l\alpha}|\Big\{2^{l\sigma\varepsilon}\varrho(x,x')^{(1-\sigma)\varepsilon}\sum_{0\le k+l\le l_1}2^{-(k+l)\sigma\varepsilon}+\sum_{k+l>l_1}2^{l\varepsilon-(k+l)\varepsilon}\Big\}\\
&+\sum_{l>N}(1+2^{-l\alpha})\Big\{2^{-l\sigma\varepsilon}\varrho(x,x')^{(1-\sigma)\varepsilon}\sum_{k=0}^{l_1}2^{-k\sigma\varepsilon}+\sum_{k=l_1+1}^{\infty}2^{-l\varepsilon-k\varepsilon}\Big\}\\
&+\sum_{l<-N}(1+2^{-l\alpha})\Big\{2^{l\sigma\varepsilon}\varrho(x,x')^{(1-\sigma)\varepsilon}\sum_{0\le k+l\le l_1}2^{-(k+l)\sigma\varepsilon}+\sum_{k+l>l_1}2^{l\varepsilon-(k+l)\varepsilon}\Big\}\\
&\le\Big(C_12^{-\delta N}+C_2\sum_{|l|\le N}|1-2^{-l\alpha}|2^{-|l|\sigma\varepsilon}\Big)\varrho(x,x')^{(1-\sigma)\varepsilon},
\end{aligned}
$$

where we choose $\sigma\in(0,1)$ such that $|\alpha|<\sigma\varepsilon$, and δ, C_1 and C_2 are the same constants as in Theorem 3.1. Thus, (3.12) holds.

Obviously we have

$$
\int_X (TE_0)(x,y)\,d\mu(y)=0=\int_X (TE_0)(x,y)\,d\mu(x). \tag{3.13}
$$

By Theorem 3.1, T satisfies all the conditions of Theorem 1 in [18]. Thus, by that theorem, T maps $\mathcal{G}_0(x_1,r,\beta,\gamma)$ with $x_1\in X$, $r>0$ and $0<\beta,\gamma<\varepsilon$ continuously into $\mathcal{G}_0(x_1,r,\beta,\gamma)$. That is, there is a constant C_3 independent of x_1 and r such that for all $f\in\mathcal{G}_0(x_1,r,\beta,\gamma)$,

$$
\|Tf\|_{\mathcal{G}_0(x_1,r,\beta,\gamma)}\le C_3\Big(C_12^{-\delta N}+C_2\sum_{|l|\le N}|1-2^{-l\alpha}|2^{-|l|\delta_1}\Big)\|f\|_{\mathcal{G}_0(x_1,r,\beta,\gamma)}.
$$

Now we choose $\alpha_0(\beta,\gamma)\in(0,\varepsilon)$ such that if $|\alpha|<\alpha_0(\beta,\gamma)$, then

$$
C_4\equiv C_3\Big(C_12^{-\delta N}+C_2\sum_{|l|\le N}|1-2^{-l\alpha}|2^{-|l|\delta_1}\Big)<1. \tag{3.14}
$$

Since C_1, C_2 and C_3 are independent of x_1 and r, obviously, $\alpha_0(\beta,\gamma)$ is also independent of x_1 and r. Moreover, by all the above proofs and the proof of Theorem 1 in [18], we can see that C_1 and C_2 are independent of β, γ, $\varepsilon-\beta$ and $\varepsilon-\gamma$ and at most C_3 is the linear combination of $1/\beta$, $1/\gamma$, $1/(\varepsilon-\beta)$ and $1/(\varepsilon-\gamma)$. Thus, if $0<\beta_1\le\beta\le\beta_2<\varepsilon$ and $0<\gamma_1\le\gamma\le\gamma_2<\varepsilon$, we can then easily control C_3 by the linear combination of $1/\beta_1$, $1/\gamma_1$, $1/(\varepsilon-\beta_2)$ and $1/(\varepsilon-\gamma_2)$. Therefore, in this case, we can choose $\alpha_0(\beta,\gamma)$ independent of β and γ (but depending on β_1, β_2, γ_1 and γ_2) such that when $|\alpha|<\alpha_0(\beta,\gamma)$, (3.14) holds.

Now, let $|\alpha|<\alpha_0(\beta,\gamma)$. Since $\beta\le\varepsilon'$ and $\gamma\le\gamma'$, we have $\mathcal{G}(\varepsilon',\gamma')\subset\mathcal{G}(\beta,\gamma)$ and therefore,

$$
I=(I_{-\alpha}I_\alpha)^{-1}I_{-\alpha}I_\alpha
$$

in $\mathcal{G}(\varepsilon', \gamma')$. Thus, we see that

$$(I_\alpha)_l^{-1} = (I_{-\alpha} I_\alpha)^{-1} I_{-\alpha} = \sum_{k=0}^{\infty} 2^{k\alpha} \Big\{ \sum_{m=0}^{\infty} T^m E_k \Big\},$$

where $T^0 = I$, the identity operator, and for $m \in \mathbb{N}$, $T^m = TT \dots T$ (m times). Thus,

$$\widetilde{E}_k = \sum_{m=0}^{\infty} T^m E_k.$$

Obviously, the kernel $\widetilde{E}_k(x, y)$ of $\widetilde{E}_k$ satisfies (i) of the theorem.

We now verify (ii) and (iii). If $k \in \mathbb{N}$, since $E_k(x, y) \in \mathcal{G}_0(y, 2^{-k}, \varepsilon', \gamma')$, by Theorem 1 in [18] and (3.14), we know that $(R^m E_k)(x, y) \in \mathcal{G}_0(y, 2^{-k}, \varepsilon', \gamma')$ and

$$|\widetilde{E}_k(x, y)| \le \sum_{m=0}^{\infty} (C_4)^m \frac{2^{-k\gamma'}}{(2^{-k} + \varrho(x, y))^{d+\gamma'}} \le C \frac{2^{-k\gamma'}}{(2^{-k} + \varrho(x, y))^{d+\gamma'}}.$$

Thus, (ii) holds for $k \in \mathbb{N}$. Moreover, for $\varrho(x, x') \le \frac{1}{2A}(2^{-k} + \varrho(x, y))$, we have

$$\begin{aligned} |\widetilde{E}_k(x, y) - \widetilde{E}_k(x', y)| &\le \sum_{m=0}^{\infty} (C_4)^m \Big(\frac{\varrho(x, x')}{2^{-k} + \varrho(x, y)} \Big)^{\varepsilon'} \frac{2^{-k\gamma'}}{(2^{-k} + \varrho(x, y))^{d+\gamma'}} \\ &\le C \Big(\frac{\varrho(x, x')}{2^{-k} + \varrho(x, y)} \Big)^{\varepsilon'} \frac{2^{-k\gamma'}}{(2^{-k} + \varrho(x, y))^{d+\gamma'}}. \end{aligned}$$

That is, (iii) holds for $k \in \mathbb{N}$.

By (3.11)–(3.13), $(TE_0)(x, y) \in \mathcal{G}_0(y, 1, \varepsilon', \gamma')$. Thus, by Theorem 1 in [18] and (3.14) again, (ii) and (iii) also hold for $k = 0$. This establishes the representation formula for $(I_\alpha)_l^{-1}$.

To establish the representation formula for $(I_\alpha)_r^{-1}$, we need to replace the above operator T by $\overline{T} = I - I_\alpha I_{-\alpha}$ and we can then show that

$$(I_\alpha)_r^{-1} = I_{-\alpha}(I_\alpha I_{-\alpha})^{-1} = \sum_{k=0}^{\infty} 2^{k\alpha} \Big\{ \sum_{m=0}^{\infty} E_k \overline{T}^m \Big\}.$$

Then by Theorem 3.1 and Theorem 1 in [18] and a proof similar to the above, we can obtain the representation formula for $(I_\alpha)_r^{-1}$. We omit the details.

This finishes the proof of Theorem 3.2.

Now let us introduce the definition of the transpose, T^t, of an operator T which is defined on spaces of test functions or dual spaces.

Definition 3.4. Let $\theta \ge \beta > 0$ and $\gamma > 0$. Let T be an operator defined on $\mathcal{G}(\beta, \gamma)$. We then define the transpose, T^t, of T on $(\mathcal{G}(\beta, \gamma))'$ by $\langle T^t g, f \rangle = \langle g, Tf \rangle$ for all $f \in \mathcal{G}(\beta, \gamma)$ and all $g \in (\mathcal{G}(\beta, \gamma))'$. Let T be an operator defined on $(\mathcal{G}(\beta, \gamma))'$. We then define the transpose, T^t, of T on $\mathcal{G}(\beta, \gamma)$ by $\langle g, T^t f \rangle = \langle Tg, f \rangle$ for all $f \in \mathcal{G}(\beta, \gamma)$ and all $g \in (\mathcal{G}(\beta, \gamma))'$.

The left inverses and right inverses of fractional integrals and derivatives in dual spaces are defined as follows.

Definition 3.5. Let $|\alpha| < \theta$, $0 < \beta \le \theta$ and $\gamma > 0$. We say that $(I_\alpha)_l^{-1}$ and $(I_\alpha)_r^{-1}$ exist in $(\mathcal{G}(\beta,\gamma))'$ if $(I_\alpha^t)_l^{-1}$ and $(I_\alpha^t)_r^{-1}$ exist in $\mathcal{G}(\beta,\gamma)$. The transposes of the left inverse and right inverse of I_α^t in $\mathcal{G}(\beta,\gamma)$ are said to be, respectively, the right inverse and left inverse of I_α in $(\mathcal{G}(\beta,\gamma))'$, and we then write $(I_\alpha)_l^{-1} I_\alpha = I_\alpha (I_\alpha)_r^{-1} = I$ in $(\mathcal{G}(\beta,\gamma))'$.

In the rest of this section, we assume $\mu(X) < \infty$. But some of our results still hold for $\mu(X) = \infty$. We will indicate this in each case. Under this restriction, the γ in the space of test functions, $\mathcal{G}(\beta,\gamma)$, becomes unimportant. In fact, for all $\gamma > 0$, the $\mathcal{G}(\beta,\gamma)$ define the same space, $\mathrm{Lip}(\beta)$; see [11] for the definition of the latter. Based on this, we obtain the following improved version of Theorem 2.1 which has uniform forms for $\alpha > 0$ and $\alpha < 0$. Let us state it in a general form.

Theorem 3.3. *Let* $\mu(X) < \infty$, $\varepsilon \in (0,\theta]$, $\alpha \in \mathbb{R}$, $\theta \ge \beta > 0$, $\varepsilon > \alpha + \beta > 0$ *and* $\gamma > 0$. *Let*

$$I_\alpha = \sum_{l=0}^{\infty} 2^{-l\alpha} E_l,$$

where E_l*'s are linear operators for* $l \in \mathbb{N} \cup \{0\}$ *with kernels*, $E_l(x,y)$, *satisfying*

(i) $\displaystyle\int_X E_l(x,y)\, d\mu(y) = \begin{cases} 1 & \text{for } l = 0, \\ 0 & \text{for } l \in \mathbb{N}; \end{cases}$

(ii) $\displaystyle |E_l(x,y)| \le C \frac{2^{-l\varepsilon}}{(2^{-l} + \varrho(x,y))^{d+\varepsilon}}$ *for* $l \in \mathbb{N} \cup \{0\}$;

(iii) $\displaystyle |E_l(x,y) - E_l(x',y)| \le C \left(\frac{\varrho(x,x')}{2^{-l} + \varrho(x,y)} \right)^{\varepsilon} \frac{2^{-l\varepsilon}}{(2^{-l} + \varrho(x,y))^{d+\varepsilon}}$

$$\text{for } \varrho(x,x') \le \frac{1}{2A}(2^{-l} + \varrho(x,y)) \text{ and } k \in \mathbb{N} \cup \{0\}.$$

Then I_α *maps* $\mathcal{G}(\beta,\gamma)$ *continuously into* $\mathcal{G}(\beta+\alpha,\gamma)$, *namely, there is a constant* C *independent of* f *such that*

$$\|I_\alpha(f)\|_{\mathcal{G}(\beta+\alpha,\gamma)} \le C \|f\|_{\mathcal{G}(\beta,\gamma)}.$$

Proof. The proof is just a repeat of Theorem 2.1 by noting that $1 + \varrho(x,x_0) \sim 1$ due to $\mu(X) < \infty$; see also Remark 2.1. We omit the details.

From this theorem, we can obtain more information on the left inverses and right inverses in Theorem 3.2.

Corollary 3.1. *Let* $\mu(X) < \infty$, $0 < \beta < \varepsilon$ *and* $0 < \gamma$. *Let* $\alpha_0(\beta,\gamma)$ *be as in Theorem* 3.2. *Suppose* $|\alpha| < \min(\beta, \alpha_0(\beta,\gamma))$. *Let* $(I_\alpha)_l^{-1}$ *and* $(I_\alpha)_r^{-1}$ *be as in Theorem* 3.2. *Then:*

(i) $(I_\alpha)_l^{-1}$ *maps* $\mathcal{G}(\beta+\alpha,\gamma)$ *continuously into* $\mathcal{G}(\beta,\gamma)$, *namely, there is a constant* $C > 0$ *independent of* f *such that*

$$\|(I_\alpha)_l^{-1}(f)\|_{\mathcal{G}(\beta,\gamma)} \le \|f\|_{\mathcal{G}(\beta+\alpha,\gamma)};$$

(ii) $(I_\alpha)_r^{-1}$ *maps* $\mathcal{G}(\beta,\gamma)$ *continuously into* $\mathcal{G}(\beta-\alpha,\gamma)$, *namely, there is a constant* $C > 0$ *independent of* f *such that*

$$\|(I_\alpha)_r^{-1}(f)\|_{\mathcal{G}(\beta-\alpha,\gamma)} \le \|f\|_{\mathcal{G}(\beta,\gamma)};$$

(iii) *If* $\alpha > 0$, *then* $(I_\alpha)_l^{-1} = (I_\alpha)_r^{-1}|\mathcal{G}(\beta+\alpha,\gamma)$. *This means that when we restrict* $(I_\alpha)_l^{-1}$ *and* $(I_\alpha)_r^{-1}$ *to* $\mathcal{G}(\beta+\alpha,\gamma)$, *they are the same*;

(iv) *If* $\alpha < 0$, *then* $(I_\alpha)_r^{-1} = (I_\alpha)_l^{-1}|\mathcal{G}(\beta,\gamma)$. *This means that when we restrict* $(I_\alpha)_l^{-1}$ *and* $(I_\alpha)_r^{-1}$ *to* $\mathcal{G}(\beta,\gamma)$, *they are the same*;

(v) $(I_\alpha^t)_l^{-1,t} = (I_\alpha)_r^{-1}$ *holds in both* $\mathcal{G}(\beta,\gamma)$ *and* $(\mathcal{G}(\beta-\alpha,\gamma))'$;

(vi) $(I_\alpha^t)_r^{-1,t} = (I_\alpha)_l^{-1}$ *holds in both* $\mathcal{G}(\beta+\alpha,\gamma)$ *and* $(\mathcal{G}(\beta,\gamma))'$.

Proof. (i) is a simple corollary of Theorems 3.2 and 3.3; so is (ii). In fact, to see (ii), by the proof of Theorem 3.2, we have

$$(I_\alpha)_r^{-1} = I_{-\alpha}(I_\alpha I_{-\alpha})^{-1}$$

and $(I_\alpha I_{-\alpha})^{-1}$ is the inverse of the Calderón–Zygmund operator $I_\alpha I_{-\alpha}$ in $\mathcal{G}(\beta,\gamma)$. This means that there is a constant $C > 0$ such that for all $f \in \mathcal{G}(\beta,\gamma) = \mathcal{G}(x_0,1,\beta,\gamma)$, we have

$$\|(I_\alpha I_{-\alpha})^{-1}(f)\|_{\mathcal{G}(\beta,\gamma)} \le C\|f\|_{\mathcal{G}(\beta,\gamma)}.$$

To see this, let $\overline{T} = I - I_\alpha I_{-\alpha}$ be as in the proof of Theorem 3.2 and $\overline{K}$ be its kernel. By Theorem 3.1, $\overline{K}$ satisfies (3.1)–(3.5). Moreover, let C_3 be the constant appearing in Theorem 1 in [18]. By the proof of Theorem 3.2, we know that $C_3\|\overline{K}\| < 1$; see (3.14). Also, we can show that for any $f \in \mathcal{G}(\beta,\gamma)$ and this special $\overline{T}$, $\overline{T}f \in \mathcal{G}_0(\beta,\gamma)$ and

$$\|\overline{T}f\|_{\mathcal{G}(\beta,\gamma)} \le C_5\|\overline{K}\|\|f\|_{\mathcal{G}(\beta,\gamma)},$$

where C_5 is independent of f; see the proofs of (3.11) and (3.12). Thus, by Theorem 1 in [18], we have

$$\begin{aligned}\|(I_\alpha I_{-\alpha})^{-1}(f)\|_{\mathcal{G}(\beta,\gamma)} &\le \sum_{m=0}^{\infty} \|\overline{T}^m f\|_{\mathcal{G}(\beta,\gamma)}\\ &\le \Big\{1 + C_5\|\overline{K}\|\Big[1 + \sum_{m=2}^{\infty}(C_3\|\overline{K}\|)^{m-1}\Big]\Big\}\|f\|_{\mathcal{G}(\beta,\gamma)} \le C\|f\|_{\mathcal{G}(\beta,\gamma)}.\end{aligned}$$

Thus, our claim is true. Therefore, by Theorem 3.3, we obtain (ii).

Now let us show (iii). Since $\alpha > 0$, we have $\mathcal{G}(\beta+\alpha,\gamma) \subset \mathcal{G}(\beta,\gamma)$. By the proof of Theorem 3.2, we have

$$(I_\alpha)_l^{-1} = (I_{-\alpha}I_\alpha)^{-1}I_{-\alpha} \quad \text{and} \quad (I_\alpha)_r^{-1} = I_{-\alpha}(I_\alpha I_{-\alpha})^{-1},$$

where $(I_{-\alpha}I_\alpha)^{-1}$ and $(I_\alpha I_{-\alpha})^{-1}$ are respectively the inverse operators of the Calderón–Zygmund operators $I_{-\alpha}I_\alpha$ and $I_\alpha I_{-\alpha}$ in $\mathcal{G}(\beta,\gamma)$. Thus, $I = I_\alpha I_{-\alpha}(I_\alpha I_{-\alpha})^{-1}$ also holds in $\mathcal{G}(\beta+\alpha,\gamma)$. By multiplying this with $(I_\alpha)_l^{-1} = (I_{-\alpha}I_\alpha)^{-1}I_{-\alpha}$, we obtain

$$(I_\alpha)_l^{-1} = (I_{-\alpha}I_\alpha)^{-1}I_{-\alpha}I_\alpha I_{-\alpha}(I_\alpha I_{-\alpha})^{-1}.$$

By recombining them, we obtain $(I_\alpha)_l^{-1} = (I_\alpha)_r^{-1}$.

The proof of (iv) is similar. In fact, since $\alpha < 0$, we have $\mathcal{G}(\beta-\alpha,\gamma) \subset \mathcal{G}(\beta,\gamma)$. Thus, $I = (I_{-\alpha}I_\alpha)^{-1}I_{-\alpha}I_\alpha$ also holds in $\mathcal{G}(\beta-\alpha,\gamma)$. By multiplying this with $(I_\alpha)_r^{-1} = I_{-\alpha}(I_\alpha I_{-\alpha})^{-1}$, we obtain

$$(I_\alpha)_r^{-1} = (I_{-\alpha}I_\alpha)^{-1}I_{-\alpha}I_\alpha I_{-\alpha}(I_\alpha I_{-\alpha})^{-1}.$$

By recombining them, we obtain (iv).

The proofs of (v) and (vi) can be given by using definitions. We omit the details.

This finishes the proof of Corollary 3.1.

The theorem below yields the independence from the choices of approximations to the identity for fractional integrals and derivatives.

THEOREM 3.4. *Let* $\{S_k\}_{k=0}^{\infty}$ *and* $\{\overline{S}_k\}_{k=0}^{\infty}$ *be two approximations to the identity as in Definition* 1.2 *with* $\varepsilon \in (0, \theta]$. *Let* $E_k = S_k - S_{k-1}$ *and* $\overline{E}_k = \overline{S}_k - \overline{S}_{k-1}$ *for* $k \in \mathbb{N}$, $E_0 = S_0$ *and* $\overline{E}_0 = \overline{S}_0$. *For* $|\alpha| < \varepsilon$, *let*

$$I_\alpha = \sum_{k=0}^{\infty} 2^{-k\alpha} E_k \quad \text{and} \quad \bar{I}_\alpha = \sum_{k=0}^{\infty} 2^{-k\alpha} \overline{E}_k.$$

(i) *Let* $0 < s, \overline{s} < \varepsilon$ *and* $|\alpha|, |\overline{\alpha}| < \varepsilon$ *with* $s + \alpha = \overline{s} + \overline{\alpha} < \varepsilon$. *If* $(I_{-\alpha})_l^{-1}$ *and* $(\bar{I}_{-\overline{\alpha}})_l^{-1}$ *exist in* $(\mathcal{G}(\beta,\gamma))'$ *with* $0 < \beta, \gamma < \varepsilon$, *then for all* $f \in (\mathcal{G}(\beta,\gamma))'$,

$$\|I_{-\alpha} f\|_{B^s_{pq}(X)} \sim \|\bar{I}_{-\overline{\alpha}} f\|_{B^{\overline{s}}_{pq}} \quad \text{for } 1 \le p, q \le \infty, \tag{3.15}$$

$$\|I_{-\alpha} f\|_{F^s_{pq}(X)} \sim \|\bar{I}_{-\overline{\alpha}} f\|_{F^{\overline{s}}_{pq}} \quad \text{for } 1 < p < \infty, 1 < q \le \infty. \tag{3.16}$$

(ii) *Let* $-\varepsilon < s, \overline{s} < 0$ *and* $|\alpha|, |\overline{\alpha}| < \varepsilon$ *with* $s + \alpha = \overline{s} + \overline{\alpha} > -\varepsilon$. *If* $(I_{-\alpha})_r^{-1}$ *and* $(\bar{I}_{-\overline{\alpha}})_r^{-1}$ *exist in* $(\mathcal{G}(\beta,\gamma))'$ *with* $\max(-s, -\overline{s}) < \beta < \varepsilon$ *and* $0 < \gamma < \varepsilon$, *then for all* $f \in (\mathcal{G}(\beta,\gamma))'$,

$$\|(I_{-\alpha})_r^{-1} f\|_{B^s_{pq}(X)} \sim \|(\bar{I}_{-\overline{\alpha}})_r^{-1} f\|_{B^{\overline{s}}_{pq}} \quad \text{for } 1 \le p, q \le \infty, \tag{3.17}$$

$$\|(I_{-\alpha})_r^{-1} f\|_{F^s_{pq}(X)} \sim \|(\bar{I}_{-\overline{\alpha}})_r^{-1} f\|_{F^{\overline{s}}_{pq}} \quad \text{for } 1 < p < \infty, 1 < q \le \infty. \tag{3.18}$$

Proof. We only show (i). The proof of (ii) is similar. To do that, we only need to show that there is a constant $C > 0$ independent of f such that

$$\|I_{-\alpha} (\bar{I}_{-\overline{\alpha}})_l^{-1} f\|_{B^s_{pq}(X)} \le C \|f\|_{B^{\overline{s}}_{pq}} \quad \text{for } 1 \le p, q \le \infty, \tag{3.19}$$

$$\|I_{-\alpha} (\bar{I}_{-\overline{\alpha}})_l^{-1} f\|_{F^s_{pq}(X)} \le C \|f\|_{F^{\overline{s}}_{pq}} \quad \text{for } 1 < p < \infty, 1 < q \le \infty. \tag{3.20}$$

By Theorem 3.2, we have

$$(\bar{I}_{-\overline{\alpha}})_l^{-1} = \sum_{l=0}^{\infty} 2^{-l\overline{\alpha}} \widetilde{E}_l,$$

where $\widetilde{E}_l$'s satisfy (i)–(iii) of Theorem 3.2. Let $\{P_k\}_{k=0}^{\infty}$ be an approximation to the identity as in Definition 1.2. Let $D_k = P_k - P_{k-1}$ for $k \in \mathbb{N}$ and $D_0 = P_0$. To show (3.19) and (3.20), it suffices to establish the following estimates:

$$|[D_l I_{-\alpha} (\bar{I}_{-\overline{\alpha}})_l^{-1} D_k](x,y)| \le C 2^{l(\alpha - \overline{\alpha})} 2^{[(k-l)\wedge 0]\varepsilon_0} \frac{2^{-(k\wedge l)\sigma}}{(2^{-(k\wedge l)} + \varrho(x,y))^{d+\sigma}}, \tag{3.21}$$

where $\sigma > 0, \varepsilon > \varepsilon_0 > s + \alpha - \overline{\alpha}$ and $C > 0$ are independent of x, y, k and l; see [20] or [23, pp. 70–74].

Let $\varepsilon' \in (0, \varepsilon)$ be as in Theorem 3.2 and ε' can be any positive number close to ε. Then, for any $\varepsilon'' \in (0, \varepsilon')$ and $\delta \in (0,1)$, there is a constant $C > 0$ independent of $n \in \mathbb{N} \cup \{0\}$ and $m \in \mathbb{N} \cup \{0\}$ such that

$$|(E_n \widetilde{E}_m)(x,y)| \le C 2^{-|n-m|\varepsilon''} \frac{2^{-(n\wedge m)\varepsilon'}}{(2^{-(n\wedge m)} + \varrho(x,y))^{d+\varepsilon'}}, \tag{3.22}$$

and

$$(3.23)\quad |(E_n\widetilde{E}_m)(x,y)-(E_n\widetilde{E}_m)(x',y)| \le C2^{-|n-m|\delta\varepsilon''}\Big(\frac{\varrho(x,x')}{2^{-(n\wedge m)}+\varrho(x,y)}\Big)^{(1-\delta)\varepsilon'} \times \frac{2^{-(n\wedge m)\varepsilon'}}{(2^{-(n\wedge m)}+\varrho(x,y))^{d+\varepsilon'}}$$

for $\varrho(x,x') \le \frac{1}{4A^2}(2^{-(n\wedge m)}+\varrho(x,y))$, where C depends on δ and is independent of n, m, x and y. The proofs of (3.22) and (3.23) are, respectively, completely similar to those of (3.9) and (3.11) in [18]; see also Lemma 2.1. We omit the details.

Now let us show (3.21). We consider four cases. In the following, we always write, for $l \in \mathbb{N}\cup\{0\}$,

$$\begin{aligned}
[D_l I_{-\alpha}(\bar{I}_{-\overline{\alpha}})_l^{-1} D_k](x,y) &= \sum_{n,\,m\in\mathbb{N}\cup\{0\}} 2^{n\alpha-m\overline{\alpha}}(D_lE_n\widetilde{E}_mD_k)(x,y)\\
&= \sum_{0\le m\le n} 2^{n\alpha-m\overline{\alpha}}(D_lE_n\widetilde{E}_mD_k)(x,y) + \sum_{0\le n<m}\cdots\\
&= \sum_{0\le l\le m\le n} 2^{n\alpha-m\overline{\alpha}}(D_lE_n\widetilde{E}_mD_k)(x,y) + \sum_{0\le m\le l\le n}\cdots + \sum_{0\le m\le n\le l}\cdots\\
&\quad + \sum_{0\le l\le n<m}\cdots + \sum_{0\le n<l\le m}\cdots + \sum_{0\le n<m<l}\cdots\\
&= Q_1+Q_2+Q_3+Q_4+Q_5+Q_6.
\end{aligned}$$

Case 1: $0\le l\le k$ and $\varrho(x,y)\le 4A^2C2^{-l}$. In this case, for Q_1, if $\alpha<\overline{\alpha}$, by (3.22), we have

$$\begin{aligned}
|Q_1| &= \Big|\sum_{0\le l\le m\le n} 2^{n\alpha-m\overline{\alpha}}\int_X\int_X D_l(x,u)(E_n\widetilde{E}_m)(u,z)D_k(z,y)\,d\mu(u)\,d\mu(z)\Big|\\
&\le C\sum_{0\le l\le m\le n} 2^{n\alpha-m\overline{\alpha}-(n-m)\varepsilon''+ld}\int_X |D_k(z,y)|\Big\{\int_X \frac{2^{-m\varepsilon'}}{(2^{-m}+\varrho(u,z))^{d+\varepsilon'}}\,d\mu(u)\Big\}\,d\mu(z)\\
&\le C2^{ld}\sum_{m=l}^{\infty}\sum_{n=m}^{\infty} 2^{n(\alpha-\varepsilon'')}2^{m(\varepsilon''-\overline{\alpha})} \le C2^{ld}\sum_{m=l}^{\infty}2^{m(\alpha-\overline{\alpha})} \le C2^{l(\alpha-\overline{\alpha})}2^{ld},
\end{aligned}$$

which is a desired estimate.

Now, if $\alpha\ge\overline{\alpha}$ and $n=0$, then in this case we obviously have $l=m=n=0$ and by (3.22), it is easy to show

$$(3.24)\qquad |Q_1| = |(D_0E_0\widetilde{E}_0D_k)(x,y)| \le C,$$

which is a desired estimate.

If $\alpha\ge\overline{\alpha}$ and $n>0$, then, in this case, we choose $\nu\in[\varepsilon'',\varepsilon')$. Noting that

$$\int_X\int_X (E_n\widetilde{E}_m)(u,z)D_k(z,y)\,d\mu(u)\,d\mu(z) = 0,$$

by (3.22), we now have

$$\begin{aligned}|Q_1| &= \Big|\sum_{0\le l\le m\le n} 2^{n\alpha-m\overline{\alpha}}\int_X\int_X [D_l(x,u)-D_l(x,z)](E_n\widetilde{E}_m)(u,z)D_k(z,y)\,d\mu(u)\,d\mu(z)\Big|\\ &\le C\sum_{0\le l\le m\le n} 2^{n\alpha-m\overline{\alpha}-(n-m)\varepsilon''-m\varepsilon'+l(d+\nu)}\\ &\quad\times\int_X\bigg\{\int_X \frac{\varrho(u,z)^{\nu}}{(2^{-m}+\varrho(u,z))^{d+\varepsilon'}}\,d\mu(u)\bigg\}|D_k(z,y)|\,d\mu(z)\\ &\le C2^{l(d+\nu)}\sum_{m=l}^{\infty}2^{-m(\overline{\alpha}+\nu-\varepsilon'')}\sum_{n=m}^{\infty}2^{-n(\varepsilon''-\alpha)}\le C2^{l(\alpha-\overline{\alpha})}2^{ld},\end{aligned}$$

where we choose $\varepsilon''>\alpha$ and therefore, $\nu\ge\varepsilon''>\alpha-\overline{\alpha}$. This is also a desired estimate.

Now we estimate Q_2. By (3.22), we have

$$\begin{aligned}|Q_2| &= \Big|\sum_{0\le m\le l\le n} 2^{n\alpha-m\overline{\alpha}}(D_lE_n\widetilde{E}_mD_k)(x,y)\Big|\\ &\le C\sum_{0\le m\le l\le n} 2^{n\alpha-m\overline{\alpha}-(n-m)\varepsilon''+ld}\int_X |D_k(z,y)|\bigg\{\int_X \frac{2^{-m\varepsilon'}}{(2^{-m}+\varrho(u,z))^{d+\varepsilon'}}\,d\mu(u)\bigg\}\,d\mu(z)\\ &\le C2^{ld}\sum_{m=0}^{l}\sum_{n=l}^{\infty}2^{n(\alpha-\varepsilon'')}2^{m(\varepsilon''-\overline{\alpha})}\le C2^{l(\alpha-\overline{\alpha})}2^{ld},\end{aligned}$$

where we take $\varepsilon''>\max(\alpha,\overline{\alpha})$. This is a desired estimate for Q_2.

For Q_3, we have two cases. If $l=0$, then $l=m=n=0$ and by (3.24), we have a desired estimate for Q_3 in this case.

Now, if $l>0$, since

$$\int_X D_l(x,u)\,d\mu(u)=0, \tag{3.25}$$

by (3.23), we have

$$\begin{aligned}|Q_3| &= \Big|\sum_{0\le m\le n\le l} 2^{n\alpha-m\overline{\alpha}}(D_lE_n\widetilde{E}_mD_k)(x,y)\Big|\\ &\le \sum_{0\le m\le n\le l} 2^{n\alpha-m\overline{\alpha}}\Big|\int_X\int_X D_l(x,u)[(E_n\widetilde{E}_m)(u,z)-(E_n\widetilde{E}_m)(x,z)]D_k(z,y)\,d\mu(u)\,d\mu(z)\Big|\\ &\le C\sum_{0\le m\le n\le l} 2^{n\alpha-m\overline{\alpha}-(n-m)\delta\varepsilon''-m\varepsilon'}\\ &\quad\times\int_X\int_X |D_l(x,u)||D_k(z,y)|\frac{\varrho(u,x)^{(1-\delta)\varepsilon'}}{(2^{-m}+\varrho(u,z))^{d+\varepsilon'+(1-\delta)\varepsilon'}}\,d\mu(u)\,d\mu(z)\\ &\le C2^{-l(1-\delta)\varepsilon'+ld}\sum_{m=0}^{l}\sum_{n=m}^{l}2^{n(\alpha-\delta\varepsilon'')}2^{m(\delta\varepsilon''+(1-\delta)\varepsilon'-\overline{\alpha})}\le C2^{l(\alpha-\overline{\alpha})}2^{ld},\end{aligned}$$

where we choose $\delta\in(0,1)$ such that $\delta\varepsilon''>\alpha$ and $(1-\delta)\varepsilon'>\overline{\alpha}-\alpha$. This is a desired estimate for Q_3.

The estimates for Q_4, Q_5 and Q_6 are, respectively, similar to those for Q_1, Q_2 and Q_3. In fact, we only need to exchange the roles of n and m. This finishes the proof of the Case 1.

Case 2: $0 \le l \le k$ and $\varrho(x,y) > 4A^2C2^{-l}$. In this case, by (3.22), we have

$$\begin{aligned}|Q_1| &= \Big|\sum_{0\le l\le m\le n} 2^{n\alpha-m\overline{\alpha}} \int_X\int_X D_l(x,u)(E_n\widetilde{E}_m)(u,z)D_k(z,y)\,d\mu(u)\,d\mu(z)\Big|\\ &\le C\sum_{0\le l\le m\le n} 2^{n\alpha-m\overline{\alpha}-(n-m)\varepsilon''-m\varepsilon'}\\ &\quad\times \int_X\int_X |D_l(x,u)|\frac{1}{(2^{-m}+\varrho(u,z))^{d+\varepsilon'}}|D_k(z,y)|\,d\mu(u)\,d\mu(z)\\ &\le C\varrho(x,y)^{-(d+\varepsilon')}\sum_{m=l}^{\infty}\sum_{n=l}^{\infty} 2^{n(\alpha-\varepsilon'')}2^{-m(\overline{\alpha}+\varepsilon'-\varepsilon'')} \le C2^{l(\alpha-\overline{\alpha})}\frac{2^{-l\varepsilon'}}{\varrho(x,y)^{d+\varepsilon'}},\end{aligned}$$

where in the second step to the last, we use the fact that $\varrho(u,z) \ge \varrho(x,y)/(2A^2)$ and we take $\varepsilon' > \varepsilon'' > \alpha$. This is a desired estimate.

Now let us estimate Q_2 with $l = 0$. We then also have $m = 0$. Thus, in this case, similarly to the above estimate on Q_1, by (3.22), we have

$$\begin{aligned}(3.26)\quad |Q_2| &= \Big|\sum_{n=0}^{\infty} 2^{n\alpha}\int_X\int_X D_0(x,u)(E_n\widetilde{E}_0)(u,z)D_k(z,y)\,d\mu(u)\,d\mu(z)\Big|\\ &\le C\sum_{n=0}^{\infty}2^{n\alpha-n\varepsilon''}\int_X\int_X |D_0(x,u)|\frac{1}{(1+\varrho(u,z))^{d+\varepsilon'}}|D_k(z,y)|\,d\mu(u)\,d\mu(z)\\ &\le C\varrho(x,y)^{-(d+\varepsilon')}\sum_{n=0}^{\infty}2^{n(\alpha-\varepsilon'')} \le C\frac{1}{\varrho(x,y)^{d+\varepsilon'}},\end{aligned}$$

which is a desired estimate.

For Q_2 with $l > 0$, by (3.25) and (3.23), we have

$$\begin{aligned}|Q_2| &= \Big|\sum_{0\le m\le l\le n} 2^{n\alpha-m\overline{\alpha}}(D_lE_n\widetilde{E}_mD_k)(x,y)\Big|\\ &\le \sum_{0\le m\le l\le n} 2^{n\alpha-m\overline{\alpha}}\Big|\int_X\int_X D_l(x,u)[(E_n\widetilde{E}_m)(u,z)-(E_n\widetilde{E}_m)(x,z)]D_k(z,y)\,d\mu(u)\,d\mu(z)\Big|\\ &\le C\sum_{0\le m\le l\le n} 2^{n\alpha-m\overline{\alpha}-(n-m)\delta\varepsilon''-m\varepsilon'}\\ &\quad\times\int_X\int_X |D_l(x,u)||D_k(z,y)|\frac{\varrho(u,x)^{(1-\delta)\varepsilon'}}{(2^{-m}+\varrho(u,z))^{d+\varepsilon'+(1-\delta)\varepsilon'}}\,d\mu(u)\,d\mu(z)\\ &\le C\frac{2^{-l(1-\delta)\varepsilon'}}{\varrho(x,y)^{d+(1-\delta)\varepsilon'}}\sum_{m=0}^{l}2^{m(\delta\varepsilon''-\overline{\alpha})}\sum_{n=m}^{l}2^{n(\alpha-\delta\varepsilon'')} \le C2^{l(\alpha-\overline{\alpha})}\frac{2^{-l(1-\delta)\varepsilon'}}{\varrho(x,y)^{d+(1-\delta)\varepsilon'}},\end{aligned}$$

where we choose $\delta \in (0,1)$ such that $\delta\varepsilon'' > \max(\alpha,\overline{\alpha})$. This is a desired estimate for Q_2.

For Q_3, we consider three cases. The first is $l = 0$. Then $l = n = m = 0$. Thus, by (3.24), we have a desired estimate. The second case is $l > 0$ and $\alpha - \overline{\alpha} > 0$. In this case, similarly to the estimate for Q_2, by (3.22) and (3.23), we have

$$|Q_3| \le \sum_{0\le m\le n\le l} 2^{n\alpha-m\overline{\alpha}}\Big|\int_X\int_X D_l(x,u)[(E_n\widetilde{E}_m)(u,z)-(E_n\widetilde{E}_m)(x,z)]D_k(z,y)\,d\mu(u)\,d\mu(z)\Big|$$
$$\le C\sum_{0\le m\le n\le l} 2^{n\alpha-m\overline{\alpha}-(n-m)\delta\varepsilon''-m\varepsilon'}$$
$$\times\int_X\int_X |D_l(x,u)||D_k(z,y)|\frac{\varrho(u,x)^{(1-\delta)\varepsilon'}}{(2^{-m}+\varrho(u,z))^{d+\varepsilon'+(1-\delta)\varepsilon'}}\,d\mu(u)\,d\mu(z)$$
$$\le C\frac{2^{-l(1-\delta)\varepsilon'}}{\varrho(x,y)^{d+(1-\delta)\varepsilon'}}\sum_{m=0}^{l}2^{m(\delta\varepsilon''-\overline{\alpha})}\sum_{n=m}^{l}2^{n(\alpha-\delta\varepsilon'')}\le C2^{l(\alpha-\overline{\alpha})}\frac{2^{-l(1-\delta)\varepsilon'}}{\varrho(x,y)^{d+(1-\delta)\varepsilon'}},$$

where we take $\delta\in(0,1)$ such that $\delta\varepsilon''>\max(\alpha,\overline{\alpha})$. This is a desired estimate. The third case is $l>0$ and $\alpha-\overline{\alpha}\le 0$. In this case, we take $\delta\in(0,1)$ such that $(1-\delta)\varepsilon'>\overline{\alpha}-\alpha$ and $\nu>0$ small enough such that $(1-\delta)\varepsilon'>\nu+\overline{\alpha}-\alpha$. By the above estimate, we have

$$|Q_3| \le \sum_{0\le m\le n\le l} 2^{n\alpha-m\overline{\alpha}}\Big|\int_X\int_X D_l(x,u)[(E_n\widetilde{E}_m)(u,z)-(E_n\widetilde{E}_m)(x,z)]D_k(z,y)\,d\mu(u)\,d\mu(z)\Big|$$
$$\le C\sum_{0\le m\le n\le l} 2^{n\alpha-m\overline{\alpha}-(n-m)\delta\varepsilon''-m\varepsilon'}$$
$$\times\int_X\int_X |D_l(x,u)||D_k(z,y)|\frac{\varrho(u,x)^{(1-\delta)\varepsilon'}}{(2^{-m}+\varrho(u,z))^{d+\varepsilon'+(1-\delta)\varepsilon'}}\,d\mu(u)\,d\mu(z)$$
$$\le C\frac{2^{-l(1-\delta)\varepsilon'}}{\varrho(x,y)^{d+(1-\delta)\varepsilon'-(\overline{\alpha}-\alpha)-\nu}}\sum_{m=0}^{l}2^{m(\delta\varepsilon''-\alpha+\nu)}\sum_{n=m}^{l}2^{n(\alpha-\delta\varepsilon'')}$$
$$\le C2^{l(\alpha-\overline{\alpha})}\frac{2^{-l((1-\delta)\varepsilon'-\nu-\overline{\alpha}+\alpha)}}{\varrho(x,y)^{d+(1-\delta)\varepsilon'-\nu-\overline{\alpha}+\alpha}},$$

which is also as desired.

Similarly to Case 1, the estimates for Q_4, Q_5 and Q_6 are, respectively, similar to those for Q_1, Q_2 and Q_3. We omit the details. This proves Case 2.

Case 3: $0\le k<l$ and $\varrho(x,y)>4A^2C2^{-k}$. In this case, the estimates for Q_1, Q_2 and Q_3 are completely similar to those in Case 2. Let us show how to estimate Q_4, Q_5 and Q_6. For Q_4, by (3.22), we have

$$|Q_4| = \Big|\sum_{0\le l\le n<m} 2^{n\alpha-m\overline{\alpha}}\int_X\int_X D_l(x,u)(E_n\widetilde{E}_m)(u,z)D_k(z,y)\,d\mu(u)\,d\mu(z)\Big|$$
$$\le C\sum_{0\le l\le n<m} 2^{n\alpha-m\overline{\alpha}-(m-n)\varepsilon''-n\varepsilon'}$$
$$\times\int_X\int_X |D_l(x,u)|\frac{1}{(2^{-n}+\varrho(u,z))^{d+\varepsilon'}}|D_k(z,y)|\,d\mu(u)\,d\mu(z)$$
$$\le C\varrho(x,y)^{-(d+\varepsilon')}\sum_{m=l}^{\infty}\sum_{n=l}^{\infty}2^{-m(\overline{\alpha}+\varepsilon'')}2^{n(\alpha-\varepsilon'+\varepsilon'')}\le C2^{l(\alpha-\overline{\alpha})}\frac{2^{-l\varepsilon'}}{\varrho(x,y)^{d+\varepsilon'}},$$

where in the second step to the last, we use the fact that $\varrho(u,z)\ge\varrho(x,y)/(2A^2)$ and we take $\varepsilon'>\varepsilon''>-\overline{\alpha}$. This is a desired estimate.

For Q_5, we always have $l > 0$. By (3.25) and (3.23), we have

$$
\begin{aligned}
|Q_5| &= \Big| \sum_{0 \le n < l \le m} 2^{n\alpha - m\overline{\alpha}} (D_l E_n \widetilde{E}_m D_k)(x,y) \Big| \\
&\le \sum_{0 \le n < l \le m} 2^{n\alpha - m\overline{\alpha}} \Big| \int_X \int_X D_l(x,u)[(E_n\widetilde{E}_m)(u,z) - (E_n\widetilde{E}_m)(x,z)] D_k(z,y)\, d\mu(u)\, d\mu(z) \Big| \\
&\le C \sum_{0 \le n < l \le m} 2^{n\alpha - m\overline{\alpha} - (m-n)\delta\varepsilon'' - n\varepsilon'} \\
&\quad \times \int_X \int_X |D_l(x,u)||D_k(z,y)| \frac{\varrho(u,x)^{(1-\delta)\varepsilon'}}{(2^{-n} + \varrho(u,z))^{d+\varepsilon'+(1-\delta)\varepsilon'}}\, d\mu(u)\, d\mu(z) \\
&\le C \frac{2^{-l(1-\delta)\varepsilon'}}{\varrho(x,y)^{d+(1-\delta)\varepsilon'}} \sum_{n=0}^{l} 2^{n(\delta\varepsilon''+\alpha)} \sum_{m=l}^{\infty} 2^{-m(\overline{\alpha}+\delta\varepsilon'')} \le C 2^{l(\alpha-\overline{\alpha})} \frac{2^{-l(1-\delta)\varepsilon'}}{\varrho(x,y)^{d+(1-\delta)\varepsilon'}},
\end{aligned}
$$

where we choose $\delta \in (0,1)$ such that $\delta\varepsilon'' > \max(-\alpha, -\overline{\alpha})$. This is a desired estimate for Q_5.

For Q_6, we consider two cases. The first is $\alpha - \overline{\alpha} > 0$. In this case, similarly to the estimate for Q_5, by (3.22) and (3.23), we have

$$
\begin{aligned}
|Q_6| &\le \sum_{0 \le n < m < l} 2^{n\alpha - m\overline{\alpha}} \Big| \int_X \int_X D_l(x,u)[(E_n\widetilde{E}_m)(u,z) - (E_n\widetilde{E}_m)(x,z)] D_k(z,y)\, d\mu(u)\, d\mu(z) \Big| \\
&\le C \sum_{0 \le n < m < l} 2^{n\alpha - m\overline{\alpha} - (m-n)\delta\varepsilon'' - n\varepsilon'} \\
&\quad \times \int_X \int_X |D_l(x,u)||D_k(z,y)| \frac{\varrho(u,x)^{(1-\delta)\varepsilon'}}{(2^{-n} + \varrho(u,z))^{d+\varepsilon'+(1-\delta)\varepsilon'}}\, d\mu(u)\, d\mu(z) \\
&\le C \frac{2^{-l(1-\delta)\varepsilon'}}{\varrho(x,y)^{d+(1-\delta)\varepsilon'}} \sum_{m=0}^{l} 2^{-m(\delta\varepsilon''+\overline{\alpha})} \sum_{n=0}^{m} 2^{n(\alpha+\delta\varepsilon'')} \le C 2^{l(\alpha-\overline{\alpha})} \frac{2^{-l(1-\delta)\varepsilon'}}{\varrho(x,y)^{d+(1-\delta)\varepsilon'}},
\end{aligned}
$$

where we take $\delta \in (0,1)$ such that $\delta\varepsilon'' > -\alpha$. This is a desired estimate. The second case is $\alpha - \overline{\alpha} \le 0$. In this case, we take $\delta \in (0,1)$ such that $(1-\delta)\varepsilon' > \overline{\alpha} - \alpha$ and $\nu > 0$ small enough such that $(1-\delta)\varepsilon' > \nu + \overline{\alpha} - \alpha$ and $(1-\delta)\varepsilon' - \nu - \overline{\alpha} + \alpha > s + \alpha - \overline{\alpha}$. By the above estimate, we have

$$
\begin{aligned}
|Q_6| &\le \sum_{0 \le n < m < l} 2^{n\alpha - m\overline{\alpha}} \Big| \int_X \int_X D_l(x,u)[(E_n\widetilde{E}_m)(u,z) - (E_n\widetilde{E}_m)(x,z)] D_k(z,y)\, d\mu(u)\, d\mu(z) \Big| \\
&\le C \sum_{0 \le n < m < l} 2^{n\alpha - m\overline{\alpha} - (m-n)\delta\varepsilon'' - n\varepsilon'} \\
&\quad \times \int_X \int_X |D_l(x,u)||D_k(z,y)| \frac{\varrho(u,x)^{(1-\delta)\varepsilon'}}{(2^{-n} + \varrho(u,z))^{d+\varepsilon'+(1-\delta)\varepsilon'}}\, d\mu(u)\, d\mu(z) \\
&\le C \frac{2^{-l(1-\delta)\varepsilon'}}{\varrho(x,y)^{d+(1-\delta)\varepsilon'-(\overline{\alpha}-\alpha)-\nu}} \sum_{n=0}^{l} 2^{n(\delta\varepsilon''+\overline{\alpha}+\nu)} \sum_{m=n+1}^{l} 2^{-m(\overline{\alpha}+\delta\varepsilon'')} \\
&\le C 2^{l(\alpha-\overline{\alpha})} \frac{2^{-l((1-\delta)\varepsilon'-\nu-\overline{\alpha}+\alpha)}}{\varrho(x,y)^{d+(1-\delta)\varepsilon'-\nu-\overline{\alpha}+\alpha}},
\end{aligned}
$$

which is also a desired estimate. This finishes the proof of Case 3.

Case 4: $0 \le k < l$ and $\varrho(x,y) \le 4A^2C2^{-k}$. Similarly to Case 1, we only estimate Q_1, Q_2 and Q_3. To do so, we choose $\eta_1 \in C^1(\mathbb{R})$, $\eta_1(x) = 1$ for $|x| \le 1$ and $\eta_1(x) = 0$ for $|x| \ge 2$ and we define $\eta_2(x) = 1 - \eta_1(x)$. By (3.25), we have

$$\begin{aligned}|Q_1| &= \Big|\sum_{0\le l\le m\le n} 2^{n\alpha - m\overline{\alpha}} \int_X\int_X D_l(x,u)(E_n\widetilde{E}_m)(u,z)D_k(z,y)\,d\mu(u)\,d\mu(z)\Big| \\ &\le \sum_{0\le l\le m\le n} 2^{n\alpha - m\overline{\alpha}} \Big|\int_X\int_X D_l(x,u)(E_n\widetilde{E}_m)(u,z)[D_k(z,y) - D_k(x,y)] \\ &\quad\times \eta_1\left(\frac{\varrho(z,x)}{2^{-l}}\right) d\mu(u)\,d\mu(z)\Big| \\ &\quad+ \sum_{0\le l\le m\le n} 2^{n\alpha - m\overline{\alpha}} \Big|\int_X\int_X D_l(x,u)[(E_n\widetilde{E}_m)(u,z) - (E_n\widetilde{E}_m)(x,z)] \\ &\quad\times [D_k(z,y) - D_k(x,y)]\eta_2\left(\frac{\varrho(z,x)}{2^{-l}}\right) d\mu(u)\,d\mu(z)\Big| \\ &= Q_1^1 + Q_1^2.\end{aligned}$$

For Q_1^1, we consider two cases. The first is $\alpha - \overline{\alpha} < 0$. In this case, by (3.22),

$$\begin{aligned}|Q_1^1| &\le C \sum_{0\le l\le m\le n} 2^{n\alpha - m\overline{\alpha} + k(d+\varepsilon) - l\varepsilon - (n-m)\varepsilon'' - m\varepsilon'} \\ &\quad\times \int_X\int_X |D_l(x,u)| \frac{1}{(2^{-m} + \varrho(u,z))^{d+\varepsilon'}} \left|\eta_1\left(\frac{\varrho(z,x)}{2^{-l}}\right)\right| d\mu(u)\,d\mu(z) \\ &\le C2^{(k-l)\varepsilon + kd} \sum_{n=l}^{\infty} 2^{n(\alpha - \varepsilon'')} \sum_{m=l}^{n} 2^{-m(\overline{\alpha} - \varepsilon'')} \\ &\le C2^{(k-l)\varepsilon + kd} \sum_{n=l}^{\infty} 2^{n(\alpha - \overline{\alpha})} \le C2^{l(\alpha - \overline{\alpha})} 2^{(k-l)\varepsilon + kd},\end{aligned}$$

where we take $\varepsilon'' > \alpha$. The second case is $\alpha - \overline{\alpha} \ge 0$. In this case, we take $\nu \in (0, \varepsilon')$ such that $\overline{\alpha} + \nu > \varepsilon''$. Since

$$\int_X (E_n\widetilde{E}_m)(u,z)\,d\mu(u) = 0,$$

we then have

$$\begin{aligned}|Q_1^1| &\le \sum_{0\le l\le m\le n} 2^{n\alpha - m\overline{\alpha}} \Big|\int_X\int_X [D_l(x,u) - D_l(x,z)](E_n\widetilde{E}_m)(u,z)[D_k(z,y) - D_k(x,y)] \\ &\quad\times \eta_1\left(\frac{\varrho(z,x)}{2^{-l}}\right) d\mu(u)\,d\mu(z)\Big| \\ &\le C \sum_{0\le l\le m\le n} 2^{n\alpha - m\overline{\alpha} + l(d+\nu) - (n-m)\varepsilon'' - m\varepsilon' + k(d+\varepsilon)} \\ &\quad\times \int_X\int_X \varrho(z,u)^{\nu} \frac{1}{(2^{-m} + \varrho(u,z))^{d+\varepsilon'}} \varrho(z,x)^{\varepsilon} \left|\eta_1\left(\frac{\varrho(z,x)}{2^{-l}}\right)\right| d\mu(u)\,d\mu(z) \\ &\le C2^{k\varepsilon - l\varepsilon + l\nu} 2^{kd} \sum_{n=l}^{\infty} 2^{n(\alpha - \varepsilon'')} \sum_{m=l}^{\infty} 2^{m(\varepsilon'' - \overline{\alpha} - \nu)} \le C2^{l(\alpha - \overline{\alpha})} 2^{(k-l)\varepsilon + kd},\end{aligned}$$

where we take $\varepsilon'' > \alpha$.

Now let us turn to estimating Q_1^2. We choose $\delta \in (0,1)$ and $\nu \in (0,\varepsilon)$ such that $\delta\varepsilon'' > \alpha$ and $\varepsilon - \varepsilon' < \nu < \min((1-\delta)\varepsilon', \varepsilon + \overline{\alpha} - \delta\varepsilon'')$. Thus, by (3.23), we have

$$|Q_1^2| \le C \sum_{0\le l\le m\le n} 2^{n\alpha - m\overline{\alpha} - (n-m)\delta\varepsilon'' - m\varepsilon' - k(d+\varepsilon)}$$
$$\times \int_X\int_X |D_l(x,u)| \frac{\varrho(x,u)^\nu}{(2^{-m} + \varrho(x,z))^{d+\varepsilon'+\nu}} \varrho(x,z)^\varepsilon \left|\eta_2\left(\frac{\varrho(z,x)}{2^{-l}}\right)\right| d\mu(u)\, d\mu(z)$$
$$\le C2^{-l\nu}2^{k(d+\varepsilon)} \sum_{n=l}^\infty 2^{n(\alpha-\delta\varepsilon'')} \sum_{m=l}^\infty 2^{m(\nu-\varepsilon+\delta\varepsilon''-\overline{\alpha})} \le C2^{l(\alpha-\overline{\alpha})}2^{(k-l)\varepsilon+kd}.$$

This finishes the estimate for Q_1.

For Q_2, we choose $\varepsilon_0 > s + \alpha - \overline{\alpha}$ and $\delta \in (0,1)$ such that $\delta\varepsilon'' > \alpha$ and

$$\varepsilon_0 < \min((1-\delta)\varepsilon' + \delta\varepsilon'' - \overline{\alpha}, \varepsilon'' + (1-\delta)\varepsilon').$$

By (3.25), (3.22) and (3.23), we have

$$|Q_2| = \left|\sum_{0\le m\le l\le n} 2^{n\alpha - m\overline{\alpha}} \int_X\int_X D_l(x,u)(E_n\widetilde{E}_m)(u,z)D_k(z,y)\, d\mu(u)\, d\mu(z)\right|$$
$$\le \sum_{0\le m\le l\le n} 2^{n\alpha - m\overline{\alpha}} \left|\int_X\int_X D_l(x,u)(E_n\widetilde{E}_m)(u,z)[D_k(z,y) - D_k(x,y)]\right.$$
$$\left.\times\, \eta_1\left(\frac{\varrho(z,x)}{2^{-l}}\right) d\mu(u)\, d\mu(z)\right|$$
$$+ \sum_{0\le m\le l\le n} 2^{n\alpha - m\overline{\alpha}} \left|\int_X\int_X D_l(x,u)[(E_n\widetilde{E}_m)(u,z) - (E_n\widetilde{E}_m)(x,z)]\right.$$
$$\left.\times\, [D_k(z,y) - D_k(x,y)]\eta_2\left(\frac{\varrho(z,x)}{2^{-l}}\right) d\mu(u)\, d\mu(z)\right|$$
$$\le C \sum_{0\le m\le l\le n} 2^{n\alpha - m\overline{\alpha} + k(d+\varepsilon) - l\varepsilon - (n-m)\varepsilon'' - m\varepsilon'}$$
$$\times \int_X\int_X |D_l(x,u)| \frac{1}{(2^{-m} + \varrho(u,z))^{d+\varepsilon'}} \left|\eta_1\left(\frac{\varrho(z,x)}{2^{-l}}\right)\right| d\mu(u)\, d\mu(z)$$
$$+ C \sum_{0\le m\le l\le n} 2^{n\alpha - m\overline{\alpha} - (n-m)\delta\varepsilon'' - m\varepsilon' - k(d+\varepsilon_0)} \int_X\int_X |D_l(x,u)|$$
$$\times \frac{\varrho(x,u)^{(1-\delta)\varepsilon'}}{(2^{-m} + \varrho(x,z))^{d+\varepsilon'+(1-\delta)\varepsilon'}} \varrho(x,z)^{\varepsilon_0} \left|\eta_2\left(\frac{\varrho(z,x)}{2^{-l}}\right)\right| d\mu(u)\, d\mu(z)$$
$$\le C2^{(k-l)\varepsilon+kd} \sum_{n=l}^\infty 2^{n(\alpha-\varepsilon'')} \sum_{m=0}^l 2^{-m(\overline{\alpha}-\varepsilon'')}$$
$$+ C2^{-l(1-\delta)\varepsilon'}2^{k(d+\varepsilon_0)} \sum_{n=l}^\infty 2^{n(\alpha-\delta\varepsilon'')} \sum_{m=0}^l 2^{m((1-\delta)\varepsilon' - \varepsilon_0 + \delta\varepsilon'' - \overline{\alpha})}$$
$$\le C2^{l(\alpha-\overline{\alpha})}2^{(k-l)\varepsilon+kd} + C2^{l(\alpha-\overline{\alpha})}2^{(k-l)\varepsilon_0+kd},$$

where we take $\varepsilon'' > \max(\alpha, \overline{\alpha})$. This is a desired estimate for Q_2 in this case.

Finally, we estimate Q_3. In this case, we choose $\delta \in (0,1)$ such that $(1-\delta)\varepsilon' > \overline{\alpha} - \alpha$ and $\varepsilon_0 > s + \alpha - \overline{\alpha}$ such that $\varepsilon_0 < \alpha - \overline{\alpha} + (1-\delta)\varepsilon'$. By (3.25), (3.22) and (3.23),

$$\begin{aligned}|Q_3| &= \Big|\sum_{0\le m\le n\le l} 2^{n\alpha-m\overline{\alpha}}\int_X\int_X D_l(x,u)(E_n\widetilde{E}_m)(u,z)D_k(z,y)\,d\mu(u)\,d\mu(z)\Big|\\
&\le \sum_{0\le m\le n\le l} 2^{n\alpha-m\overline{\alpha}}\Big|\int_X\int_X D_l(x,u)[(E_n\widetilde{E}_m)(u,z)-(E_n\widetilde{E}_m)(x,z)]\\
&\quad\times[D_k(z,y)-D_k(x,y)]\eta_1\Big(\frac{\varrho(z,x)}{2^{-l}}\Big)\,d\mu(u)\,d\mu(z)\Big|\\
&\quad+\sum_{0\le m\le n\le l} 2^{n\alpha-m\overline{\alpha}}\Big|\int_X\int_X D_l(x,u)[(E_n\widetilde{E}_m)(u,z)-(E_n\widetilde{E}_m)(x,z)]\\
&\quad\times[D_k(z,y)-D_k(x,y)]\eta_2\Big(\frac{\varrho(z,x)}{2^{-l}}\Big)\,d\mu(u)\,d\mu(z)\Big|\\
&\le C\sum_{0\le m\le n\le l} 2^{n\alpha-m\overline{\alpha}+k(d+\varepsilon)-l\varepsilon-(n-m)\delta\varepsilon''-m\varepsilon'}\int_X\int_X |D_l(x,u)|\\
&\quad\times\frac{\varrho(x,u)^{(1-\delta)\varepsilon'}}{(2^{-m}+\varrho(x,z))^{d+\varepsilon'+(1-\delta)\varepsilon'}}\varrho(x,z)^{\varepsilon}\Big|\eta_1\Big(\frac{\varrho(z,x)}{2^{-l}}\Big)\Big|\,d\mu(u)\,d\mu(z)\\
&\quad+C\sum_{0\le m\le n\le l} 2^{n\alpha-m\overline{\alpha}-(n-m)\delta\varepsilon''-m\varepsilon'-k(d+\varepsilon_0)}\int_X\int_X |D_l(x,u)|\\
&\quad\times\frac{\varrho(x,u)^{(1-\delta)\varepsilon'}}{(2^{-m}+\varrho(x,z))^{d+\varepsilon'+(1-\delta)\varepsilon'}}\varrho(x,z)^{\varepsilon_0}\Big|\eta_2\Big(\frac{\varrho(z,x)}{2^{-l}}\Big)\Big|\,d\mu(u)\,d\mu(z)\\
&\le C2^{(k-l)\varepsilon+kd-l(1-\delta)\varepsilon'}\sum_{m=0}^{l}2^{m((1-\delta)\varepsilon'+\delta\varepsilon''-\overline{\alpha})}\sum_{n=m}^{l}2^{n(\alpha-\delta\varepsilon'')}\\
&\quad+C2^{-l(1-\delta)\varepsilon'}2^{k(d+\varepsilon_0)}\sum_{m=0}^{l}2^{m((1-\delta)\varepsilon'+\delta\varepsilon''-\varepsilon_0-\overline{\alpha})}\sum_{n=m}^{l}2^{n(\alpha-\delta\varepsilon'')}\\
&\le C2^{l(\alpha-\overline{\alpha})}2^{(k-l)\varepsilon+kd}+C2^{l(\alpha-\overline{\alpha})}2^{(k-l)\varepsilon_0+kd},\end{aligned}$$

which is a desired estimate for Q_3.

Thus, (3.21) is true with $\varepsilon_0\in(s+\alpha-\overline{\alpha},\theta)$ and $\sigma\in(0,\theta)$.

This finishes the proof of (3.15) and (3.16) and the proof of Theorem 3.4.

We point out that Theorem 3.4 is also true when $\mu(X)=\infty$. Moreover, if $|s+\alpha|=|\overline{s}+\overline{\alpha}|<\theta$, it is also true for $\alpha=0$ or $\overline{\alpha}=0$.

Now let us give an application of the left inverses of fractional derivatives and Theorem 2.2. We establish Poincaré-type inequalities for functions in $F^s_{p2}(X)$ with $\mu(X)<\infty$, $1<p<\infty$ and with $s>0$ being small enough; see also [14] and [25, p. 39] for Poincaré inequalities for functions in Hajłasz–Sobolev spaces on metric spaces.

THEOREM 3.5. *Let $(X,\varrho,\mu)_{d,\theta}$ be a space of homogeneous type as in Definition* 0.1 *with $\mu(X)<\infty$. Let $1<p<\infty$. If $s>0$ is small enough, then there is a constant $C>0$ such that for all $f\in F^s_{p2}(X)$,*

$$\int_X\Big|f(x)-\frac{1}{\mu(X)}\int_X f(y)\,d\mu(y)\Big|^p\,d\mu(x)\le C\|I_{-s}f\|^p_{L^p(X)}\le C\|f\|^p_{F^s_{p2}(X)},\tag{3.27}$$

where C is independent of f, but it may depend on p, s and diam X.

Proof. Let $\{S_l\}_{l\in\mathbb{N}\cup\{0\}}$ be an approximation to the identity as in Definition 1.2 with $\varepsilon\in(0,\theta]$ and $s\in(-\varepsilon,\varepsilon)$. Let $E_l=S_l-S_{l-1}$ for $l\in\mathbb{N}$ and $E_0=S_0$. Let $f\in F^s_{p2}(X)$. Since $s>0$, by Proposition 1.2 and Lemma 1.10, we have $F^s_{pq}(X)\subset F^0_{p2}(X)=L^p(X)$. Moreover, by Remark 1.4, we can further suppose $f\in\mathcal{G}(\beta,\gamma)$ with $\theta/2\le\beta,\gamma\le\theta$. In fact, since $\mu(X)<\infty$, γ is not important. We then have the fractional derivative $I_{-s}f$ defined by

$$I_{-s}f=\sum_{l=0}^{\infty}2^{ls}E_l(f).$$

By Theorem 3.2, there is an $\alpha_1>0$ such that if $0<s<\alpha_1$, then $(I_{-s})_l^{-1}$ exists in $\mathcal{G}(\beta,\gamma)$. Thus, when $0<s<\alpha_1$, we have

$$f(x)=(I_{-s})_l^{-1}I_{-s}f(x)$$

for all $x\in X$. Moreover, by Theorem 3.2,

$$(I_{-s})_l^{-1}=\sum_{k=0}^{\infty}2^{-ks}\widetilde{E}_k,$$

where $\widetilde{E}_k$'s satisfy the same conditions as in Theorem 3.2. Let $g=I_{-s}f$. By Theorem 2.2 and Lemma 1.10, $g\in F^0_{p2}(X)=L^p(X)$, and

$$\|g\|_{L^p(X)}\le C\|f\|_{F^s_{p2}(X)},$$

where C is independent of f. We now write

$$\begin{aligned}
|f(x)-f(y)|&=\Big|\sum_{k=0}^{\infty}2^{-ks}[\widetilde{E}_k(g)(x)-\widetilde{E}_k(g)(y)]\Big|\\
&=\Big|\sum_{k=0}^{\infty}2^{-ks}\int_X[\widetilde{E}_k(x,z)-\widetilde{E}_k(y,z)]g(z)\,d\mu(z)\Big|\\
&\le\sum_{k=0}^{\infty}2^{-ks}\int_{\{z:\varrho(x,y)\le\frac{1}{2A}(2^{-k}+\varrho(x,z))\}}|\widetilde{E}_k(x,z)-\widetilde{E}_k(y,z)||g(z)|\,d\mu(z)\\
&\quad+\sum_{k=0}^{\infty}2^{-ks}\int_X|\widetilde{E}_k(x,z)||g(z)|\,d\mu(z)+\sum_{k=0}^{\infty}2^{-ks}\int_X|\widetilde{E}_k(y,z)||g(z)|\,d\mu(z)\\
&=R_1+R_2+R_3.
\end{aligned}$$

By Theorem 3.2(iii), we can choose some $\varepsilon'>s$ such that

$$\begin{aligned}
R_1&\le C\sum_{k=0}^{\infty}2^{-ks}\int_{\{z:\,\varrho(x,y)\le\frac{1}{2A}(2^{-k}+\varrho(x,z))\}}\left[\frac{\varrho(x,y)}{2^{-k}+\varrho(x,z)}\right]^{\varepsilon'}\\
&\quad\times\frac{2^{-k\varepsilon'}}{(2^{-k}+\varrho(x,z))^{d+\varepsilon'}}|g(z)|\,d\mu(z)\\
&\le C\sum_{k=0}^{\infty}2^{-ks}\int_X\frac{2^{-k\varepsilon'}}{(2^{-k}+\varrho(x,z))^{d+\varepsilon'}}|g(z)|\,d\mu(z)\\
&\le C\sum_{k=0}^{\infty}2^{-ks}M(g)(x)\le CM(g)(x),
\end{aligned}\tag{3.28}$$

where M is the Hardy–Littlewood maximal function of g, $C>0$ is independent of x, y and f, and in the second inequality to the last, we used the fact that $s>0$.

By Theorem 3.2(ii) and $s>0$, we have

$$R_2 \le C\sum_{k=0}^{\infty} 2^{-ks} \int_X \frac{2^{-k\varepsilon'}}{(2^{-k}+\varrho(x,z))^{d+\varepsilon'}} |g(z)|\,d\mu(z) \le C\sum_{k=0}^{\infty} 2^{-ks} M(g)(x) \le CM(g)(x), \tag{3.29}$$

where $C>0$ is independent of x, y and f.

Similarly, by Theorem 3.2(ii) and $s>0$, we have

$$R_3 \le C\sum_{k=0}^{\infty} 2^{-ks} \int_X \frac{2^{-k\varepsilon'}}{(2^{-k}+\varrho(y,z))^{d+\varepsilon'}} |g(z)|\,d\mu(z) \le C\sum_{k=0}^{\infty} 2^{-ks} M(g)(y) \le CM(g)(y), \tag{3.30}$$

where $C>0$ is independent of x, y and f.

By combining (3.28)–(3.30), we have

$$|f(x)-f(y)| \le C[M(g)(x)+M(g)(y)],$$

where $C>0$ is independent of x, y and f. From this, the $L^p(X)$-boundedness of the Hardy–Littlewood maximal function (see [4] and [25]) and Hölder's inequality, we deduce

$$\begin{aligned}\left|f(x)-\frac{1}{\mu(X)}\int_X f(y)\,d\mu(y)\right| &\le \frac{1}{\mu(X)}\int_X |f(x)-f(y)|\,d\mu(y)\\ &\le \frac{C}{\mu(X)}\int_X [M(g)(x)+M(g)(y)]\,d\mu(y)\\ &\le CM(g)(x)+\frac{C}{\mu(X)^{1/p}}\|g\|_{L^p(X)},\end{aligned}$$

where $C>0$ is independent of x, y, $\operatorname{diam} X$ and f. From this and the $L^p(X)$-boundedness of the Hardy–Littlewood maximal function, we finally conclude that when $0<s<\alpha_1$,

$$\begin{aligned}&\int_X \left|f(x)-\frac{1}{\mu(X)}\int_X f(y)\,d\mu(y)\right|^p d\mu(x)\\ &\qquad \le C\int_X M(g)(x)^p\,d\mu(x)+C\|g\|^p_{L^p(X)} \le C\|g\|^p_{L^p(X)} \le C\|f\|^p_{F^s_{p2}(X)},\end{aligned}$$

where $C>0$ is independent of f and it may depend on s, p and $\operatorname{diam} X$.

This finishes the proof of Theorem 3.5.

We mention here again that since $s>0$, $I_{-s}f$ is the discrete and inhomogeneous version of the fractional derivative of f introduced by Gatto, Segovia and Vági in [11]; see also [12].

We also remark that the difference between the Poincaré-type inequalities here and the Poincaré inequalities in [14] and [25] for functions in Hajłasz–Sobolev spaces on metric spaces is that we do not have the factor $(\operatorname{diam} X)^s$ on the right hand side of (3.27) and the positive constant C here also depends on this. We also note that even on $\mathbb{R}^n$, there are many domains such that the Poincaré inequality does not hold; see [25, p. 39].

4. Frame characterizations

In this section, we establish frame decomposition characterizations of $B^s_{pq}(X)$ and $F^s_{pq}(X)$ by using the discrete Calderón reproducing formulae established in [22]. These frame characterizations will play a key role in estimates of entropy numbers for compact embeddings between $B^s_{pq}(X)$ or $F^s_{pq}(X)$.

THEOREM 4.1. *Suppose that $\{S_k\}_{k=0}^{\infty}$ is an approximation to the identity as in Definition 1.2. Let $D_k = S_k - S_{k-1}$ for $k \in \mathbb{N}$ and $D_0 = S_0$. Then there exist families of linear operators $\widetilde{D}_k$ for $k \in \mathbb{N}$, functions $\widetilde{D}^{0,\nu}_\tau(x)$ for $\tau \in M_0$ and $\nu = 1, \ldots, N(0,\tau)$, and a fixed large $N \in \mathbb{N}$ satisfying the same conditions as in Lemma 1.7 such that for any fixed $y^{k,\nu}_\tau \in Q^{k,\nu}_\tau$ with $k \in \mathbb{N}$, $\tau \in M_k$ and $\nu \in \{1, \ldots, N(k,\tau)\}$ and all $f \in (\mathcal{G}(\beta_1,\gamma_1))'$ with $0 < \beta_1, \gamma_1 < \varepsilon$,*

$$f(x) = \sum_{\tau\in M_0} \sum_{\nu=1}^{N(0,\tau)} \mu(Q^{0,\nu}_\tau)\widetilde{D}^{0,\nu}_\tau(x) D^{0,\nu}_{\tau,1}(f) + \sum_{k=1}^{N} \sum_{\tau\in M_k} \sum_{\nu=1}^{N(k,\tau)} \mu(Q^{k,\nu}_\tau)\widetilde{D}_k(x, y^{k,\nu}_\tau) D^{k,\nu}_{\tau,1}(f) + \sum_{k=N+1}^{\infty} \sum_{\tau\in M_k} \sum_{\nu=1}^{N(k,\tau)} \mu(Q^{k,\nu}_\tau)\widetilde{D}_k(x, y^{k,\nu}_\tau) D_k(f)(y^{k,\nu}_\tau), \tag{4.1}$$

where the series converge in $(\mathcal{G}(\beta'_1,\gamma'_1))'$ with $\beta_1 < \beta'_1 < \varepsilon$ and $\gamma_1 < \gamma'_1 < \varepsilon$. Moreover,

(i) *if $f \in B^s_{pq}(X)$ with $-\varepsilon < s < \varepsilon$ and $1 \le p, q \le \infty$, then*

$$\|f\|_{B^s_{pq}(X)} \sim \Big\{ \sum_{k=0}^{N} \Big(\sum_{\tau\in M_k} \sum_{\nu=1}^{N(k,\tau)} [(\mu(Q^{k,\nu}_\tau))^{-s/d+1/p} |D^{k,\nu}_{\tau,1}(f)|]^p \Big)^{q/p} + \sum_{k=N+1}^{\infty} \Big(\sum_{\tau\in M_k} \sum_{\nu=1}^{N(k,\tau)} [(\mu(Q^{k,\nu}_\tau))^{-s/d+1/p} |D_k(f)(y^{k,\nu}_\tau)|]^p \Big)^{q/p} \Big\}^{1/q} \tag{4.2}$$

and the series in (4.1) also converge in the norm of $B^s_{pq}(X)$ if $1 \le p, q < \infty$;

(ii) *if $f \in F^s_{pq}(X)$ with $-\varepsilon < s < \varepsilon$, $1 < p < \infty$ and $1 < q \le \infty$, then*

$$\|f\|_{F^s_{pq}(X)} \sim \Big\| \Big\{ \sum_{k=0}^{N} \sum_{\tau\in M_k} \sum_{\nu=1}^{N(k,\tau)} [(\mu(Q^{k,\nu}_\tau))^{-s/d} |D^{k,\nu}_{\tau,1}(f)| \chi_{Q^{k,\nu}_\tau}(\cdot)]^q + \sum_{k=N+1}^{\infty} \sum_{\tau\in M_k} \sum_{\nu=1}^{N(k,\tau)} [(\mu(Q^{k,\nu}_\tau))^{-s/d} |D_k(f)(y^{k,\nu}_\tau)| \chi_{Q^{k,\nu}_\tau}(\cdot)]^q \Big\}^{1/q} \Big\|_{L^p(X)} \tag{4.3}$$

and the series in (4.1) also converge in the norm of $F^s_{pq}(X)$ if $1 < p, q < \infty$.

Proof. (4.1) is guaranteed by Lemma 1.7. We only need to show (4.2) and (4.3), and the convergence in the norms of $B^s_{pq}(X)$ or $F^s_{pq}(X)$ of the series in (4.1). Let us first show that the right hand sides of (4.2) and (4.3) are controlled, respectively, by the left hand sides of (4.2) and (4.3). For (4.2), we use Lemma 1.2. Let $f \in B^s_{pq}(X)$. Then there are linear operators $\widetilde{E}_l$'s with $l \in \mathbb{N} \cup \{0\}$ such that

$$f = \sum_{l=0}^{\infty} D_l \widetilde{E}_l(f), \tag{4.4}$$

where $\widetilde{E}_l$'s satisfy conditions (i) and (iii) of Remark 1.1 with ε replaced by any $\varepsilon' \in (0, \varepsilon)$, and the kernels of $\widetilde{E}_l$'s satisfy

$$\int_X \widetilde{E}_l(x,y)\, d\mu(y) = \int_X \widetilde{E}_l(x,y)\, d\mu(x) = \begin{cases} 1, & l = 0, 1, \dots, N, \\ 0, & l \geq N+1, \end{cases}$$

with $N \in \mathbb{N}$ as in the theorem. Let $1/p + 1/p' = 1$. For $k = 0, 1, \dots, N$, by (4.4), (2.12), (2.13), (2.15), Lemma 1.3 and Hölder's inequality, we have

$$\begin{aligned}
&\Big(\sum_{\tau \in M_k} \sum_{\nu=1}^{N(k,\tau)} [(\mu(Q^{k,\nu}_\tau))^{-s/d+1/p} |D^{k,\nu}_{\tau,1}(f)|]^p\Big)^{q/p} \qquad (4.5)\\
&= \Big(\sum_{\tau \in M_k} \sum_{\nu=1}^{N(k,\tau)} \Big[(\mu(Q^{k,\nu}_\tau))^{-s/d+1/p} \\
&\quad \times \Big(\sum_{l=0}^{\infty} \int_X \Big[\frac{1}{\mu(Q^{k,\nu}_\tau)} \int_{Q^{k,\nu}_\tau} |(D_k D_l)(z,y)|\, d\mu(z)\Big] |\widetilde{E}_l(f)(y)|\, d\mu(y)\Big)\Big]^p\Big)^{1/p} \\
&\leq \Big(\sum_{\tau \in M_k} \sum_{\nu=1}^{N(k,\tau)} (\mu(Q^{k,\nu}_\tau))^{-sp/d+1} \\
&\quad \times \Big\{\sum_{l=0}^{\infty} \Big(\int_X \Big[\frac{1}{\mu(Q^{k,\nu}_\tau)} \int_{Q^{k,\nu}_\tau} |(D_k D_l)(z,y)|\, d\mu(z)\Big] |\widetilde{E}_l(f)(y)|^p\, d\mu(y)\Big)^{1/p} \\
&\quad \times \Big(\frac{1}{\mu(Q^{k,\nu}_\tau)} \int_{Q^{k,\nu}_\tau} \int_X |(D_k D_l)(z,y)|\, d\mu(y)\, d\mu(z)\Big)^{1/p'}\Big\}^p\Big)^{1/p} \\
&\leq C\Big(\sum_{\tau \in M_k} \sum_{\nu=1}^{N(k,\tau)} 2^{ksp} \mu(Q^{k,\nu}_\tau) \Big[\sum_{l=0}^{\infty} 2^{-|k-l|\varepsilon/p'} \Big\{\int_X |\widetilde{E}_l(f)(y)|^p \\
&\quad \times \Big[\frac{1}{\mu(Q^{k,\nu}_\tau)} \int_{Q^{k,\nu}_\tau} |(D_k D_l)(z,y)|\, d\mu(z)\Big]\, d\mu(y)\Big\}^{1/p}\Big]^p\Big)^{1/p} \\
&\leq C \sum_{l=0}^{\infty} 2^{-|k-l|\varepsilon/p'} 2^{ks} \\
&\quad \times \Big\{\int_X |\widetilde{E}_l(f)(y)|^p \Big[\sum_{\tau \in M_k} \sum_{\nu=1}^{N(k,\tau)} \int_{Q^{k,\nu}_\tau} |(D_k D_l)(z,y)|\, d\mu(z)\Big]\, d\mu(y)\Big\}^{1/p} \\
&\leq C \sum_{l=0}^{\infty} 2^{-|k-l|\varepsilon} 2^{ks} \|\widetilde{E}_l(f)\|_{L^p(X)}.
\end{aligned}$$

By (4.4), (2.12), (2.13), (2.15), Lemma 1.3 and Hölder's inequality, for $k \geq N+1$,

$$(4.6)\qquad \Big(\sum_{\tau\in M_k}\sum_{\nu=1}^{N(k,\tau)}[(\mu(Q_\tau^{k,\nu}))^{-s/d+1/p}|D_k(f)(y_\tau^{k,\nu})|]^p\Big)^{1/p}$$
$$\leq C\Big(\sum_{\tau\in M_k}\sum_{\nu=1}^{N(k,\tau)}[\mu(Q_\tau^{k,\nu})]^{-sp/d+1}$$
$$\times\Big[\sum_{l=0}^{\infty}2^{-|k-l|\varepsilon/p'}\Big\{\int_X|(D_kD_l)(y_\tau^{k,\nu},y)||\widetilde{E}_l(f)(y)|^p\,d\mu(y)\Big\}^{1/p}\Big]^p\Big)^{1/p}$$
$$\leq C\sum_{l=0}^{\infty}2^{-|k-l|\varepsilon/p'}\Big\{\sum_{\tau\in M_k}\sum_{\nu=1}^{N(k,\tau)}[\mu(Q_\tau^{k,\nu})]^{-sp/d+1}$$
$$\times\int_X|(D_kD_l)(y_\tau^{k,\nu},y)||\widetilde{E}_l(f)(y)|^p\,d\mu(y)\Big\}^{1/p}$$
$$\leq C\sum_{l=0}^{\infty}2^{-|k-l|\varepsilon}2^{ks}\|\widetilde{E}_l(f)\|_{L^p(X)}.$$

From (4.5) and (4.6), by Hölder's inequality, we deduce that the right hand side of (4.3) is controlled by

$$(4.7)\qquad C\Big\{\sum_{k=0}^{\infty}\Big(\sum_{l=0}^{\infty}2^{-|k-l|\varepsilon}2^{ks}\|\widetilde{E}_l(f)\|_{L^p(X)}\Big)^q\Big\}^{1/q}$$
$$\leq C\Big\{\sum_{k=0}^{\infty}\Big(\sum_{l=0}^{k}2^{-(k-l)(\varepsilon-s)}2^{ls}\|\widetilde{E}_l(f)\|_{L^p(X)}\Big)^q\Big\}^{1/q}$$
$$+C\Big\{\sum_{k=0}^{\infty}\Big(\sum_{l=k+1}^{\infty}2^{-(l-k)(\varepsilon+s)}2^{ls}\|\widetilde{E}_l(f)\|_{L^p(X)}\Big)^q\Big\}^{1/q}$$
$$\leq C\Big\{\sum_{l=0}^{\infty}2^{lsp}\|\widetilde{E}_l(f)\|_{L^p(X)}^p\Big\}^{1/p}\leq C\|f\|_{B_{pq}^s(X)},$$

where C is independent of f and we have used Remark 2.1 of [20].

We now consider (4.3). First, by (2.12), (2.13) and (2.15), for $k=0,1,\dots,N$, $l\in\mathbb{N}\cup\{0\}$, $\tau\in M_k$ and $\nu=1,\dots,N(k,\tau)$, we have

$$(4.8)\qquad |D_{\tau,1}^{k,\nu}D_l\widetilde{E}_l(f)\chi_{Q_\tau^{k,\nu}}(x)|$$
$$=\Big|\int_X\Big[\frac{1}{\mu(Q_\tau^{k,\nu})}\int_{Q_\tau^{k,\nu}}(D_kD_l)(z,y)\,d\mu(z)\Big]\widetilde{E}_l(f)(y)\,d\mu(y)\Big|\chi_{Q_\tau^{k,\nu}}(x)$$
$$\leq C2^{-|k-l|\varepsilon}2^{(k\wedge l)d}\int_{\{y\in X:\,\varrho(x,y)\leq 2AC2^{-k\wedge l}\}}|\widetilde{E}_l(f)(y)|\,d\mu(y)\,\chi_{Q_\tau^{k,\nu}}(x)$$
$$\leq C2^{-|k-l|\varepsilon}M(\widetilde{E}_l(f))(x)\chi_{Q_\tau^{k,\nu}}(x),$$

where C is independent of k, l, ν, τ and x, and M is the Hardy–Littlewood maximal function. By (2.12), (2.13) and (2.15), for $k \geq N+1$, $l \in \mathbb{N} \cup \{0\}$, $\tau \in M_k$ and $\nu = 1, \dots, N(k,\tau)$, we have

$$\begin{aligned}(4.9)\quad & |D_k D_l \widetilde{E}_l(f)(y_\tau^{k,\nu})\chi_{Q_\tau^{k,\nu}}(x)| \\ &= \Big| \int_X (D_k D_l)(y_\tau^{k,\nu}, y)\widetilde{E}_l(f)(y)\, d\mu(y)\Big| \chi_{Q_\tau^{k,\nu}}(x) \\ &\leq C 2^{-|k-l|\varepsilon} 2^{(k\wedge l)d} \int_{\{y\in X:\, \varrho(y_\tau^{k,\nu}, y) \leq AC2^{-k\wedge l}\}} |\widetilde{E}_l(f)(y)|\, d\mu(y)\, \chi_{Q_\tau^{k,\nu}}(x) \\ &\leq C 2^{-|k-l|\varepsilon} 2^{(k\wedge l)d} \int_{\{y\in X:\, \varrho(x, y) \leq 2AC2^{-k\wedge l}\}} |\widetilde{E}_l(f)(y)|\, d\mu(y)\, \chi_{Q_\tau^{k,\nu}}(x) \\ &\leq C 2^{-|k-l|\varepsilon} M(\widetilde{E}_l(f))(x)\chi_{Q_\tau^{k,\nu}}(x),\end{aligned}$$

where C is independent of k, l, ν, τ and x. From (4.8) and (4.9), by Hölder's inequality and the Fefferman–Stein vector-valued inequality of [7], we deduce that the right hand side of (4.3) is controlled by

$$\begin{aligned}(4.10)\quad & C\Big\|\Big\{\sum_{k=0}^\infty \sum_{\tau\in M_k} \sum_{\nu=1}^{N(k,\tau)} 2^{ksq}\Big[\sum_{l=0}^\infty 2^{-|k-l|\varepsilon} M(\widetilde{E}_l(f))(\cdot)\Big]^q \chi_{Q_\tau^{k,\nu}}(\cdot)\Big\}^{1/q}\Big\|_{L^p(X)} \\ &= C\Big\|\Big\{\sum_{k=0}^\infty \Big[\sum_{l=0}^\infty 2^{-|k-l|\varepsilon} 2^{(k-l)s} 2^{ls} M(\widetilde{E}_l(f))(\cdot)\Big]^q\Big\}^{1/q}\Big\|_{L^p(X)} \\ &\leq C\Big\|\Big\{\sum_{l=0}^\infty 2^{lsq}[M(\widetilde{E}_l(f))(\cdot)]^q\Big\}^{1/q}\Big\|_{L^p(X)} \\ &\leq C\Big\|\Big\{\sum_{l=0}^\infty 2^{lsq}|\widetilde{E}_l(f)|^q\Big\}^{1/q}\Big\|_{L^p(X)} \leq C\|f\|_{F^s_{pq}(X)},\end{aligned}$$

where C is independent of f and we used Remark 2.2 of [20] again.

The reverse inequalities of (4.7) and (4.10) will be deduced from the proposition below.

Finally, let us show that the series in (4.1) also converge in the norm of $B^s_{pq}(X)$ or in the norm of $F^s_{pq}(X)$ to f when $f \in B^s_{pq}(X)$ and $1 \leq p, q < \infty$ or when $f \in F^s_{pq}(X)$ and $1 < p, q < \infty$. To do that, for $L \in \mathbb{N}$ and $L > N$, we define the partial sum, $S_L f$, of the series in (4.1) by

$$\begin{aligned}S_L f(x) = &\sum_{\tau\in M_0} \sum_{\nu=1}^{N(0,\tau)} \mu(Q_\tau^{0,\nu})\widetilde{D}_\tau^{0,\nu}(x) D_{\tau,1}^{0,\nu}(f) + \sum_{k=1}^N \sum_{\tau\in M_k} \sum_{\nu=1}^{N(k,\tau)} \mu(Q_\tau^{k,\nu})\widetilde{D}_k(x, y_\tau^{k,\nu}) D_{\tau,1}^{k,\nu}(f) \\ &+ \sum_{k=N+1}^L \sum_{\tau\in M_k} \sum_{\nu=1}^{N(k,\tau)} \mu(Q_\tau^{k,\nu})\widetilde{D}_k(x, y_\tau^{k,\nu}) D_k(f)(y_\tau^{k,\nu}).\end{aligned}$$

Since $f \in B^s_{pq}(X)$ or $f \in F^s_{pq}(X)$, the right hand sides of (4.2) and (4.3) are controlled, respectively, by $\|f\|_{B^s_{pq}(X)}$ and $\|f\|_{F^s_{pq}(X)}$. Thus, by Proposition 4.1 below, we know that as $L \to \infty$, $S_L f$ converges in the norm of $B^s_{pq}(X)$ to some $g \in B^s_{pq}(X)$ when $f \in B^s_{pq}(X)$, or $S_L f$ converges in the norm of $F^s_{pq}(X)$ to some $g \in F^s_{pq}(X)$ when $f \in F^s_{pq}(X)$. From this, we deduce that if $\max(-s, 0) < \beta < \varepsilon$ and $0 < \gamma < \varepsilon$, then $S_L f \to g$ in $(\mathcal{G}(\beta, \gamma))'$ as $L \to \infty$; see the proof of Proposition 4.1 below. By the assumption, we know that $S_L f \to g$ in $(\mathcal{G}(\beta_1', \gamma_1'))'$. Note that if $\beta_1 \geq \beta$ and $\gamma_1 \geq \gamma$, then $(\mathcal{G}(\beta, \gamma))' \subset (\mathcal{G}(\beta_1, \gamma_1))'$. From this, Lemma 1.7 and the above discussion, we deduce that $f = g$ in $(\mathcal{G}(\beta, \gamma))'$ for some β and γ satisfying $\max(-s, 0) < \beta < \varepsilon$ and $0 < \gamma < \varepsilon$. From this and the definitions of these spaces, we obtain $f = g$ also in the norm of $B^s_{pq}(X)$ when $f \in B^s_{pq}(X)$ or in the norm of $F^s_{pq}(X)$.

This finishes the proof of Theorem 4.1.

We remark that when $p = \infty$ or $q = \infty$, the series in (4.1) cannot converge in the norm of $B^s_{pq}(X)$ or $F^s_{pq}(X)$. This is well known when $X = \mathbb{R}^n$.

Now, we establish the reverse inequalities of (4.5) and (4.6). We will prove the following stronger proposition.

PROPOSITION 4.1. *With the notation of Theorem* 4.1, *let*

$$\{\lambda^{k,\nu}_\tau : k \in \mathbb{N} \cup \{0\},\ \tau \in M_k,\ \nu = 1, \ldots, N(k, \tau)\}$$

be a sequence of numbers.

(i) *If* $-\varepsilon < s < \varepsilon$, $1 \leq p,\ q \leq \infty$ *and*

$$\Big\{\sum_{k=0}^{\infty}\Big(\sum_{\tau\in M_k}\sum_{\nu=1}^{N(k,\tau)}[(\mu(Q^{k,\nu}_\tau))^{-s/d+1/p}|\lambda^{k,\nu}_\tau|]^p\Big)^{q/p}\Big\}^{1/q} < \infty, \tag{4.11}$$

then the series

$$\sum_{\tau\in M_0}\sum_{\nu=1}^{N(0,\tau)}\mu(Q^{0,\nu}_\tau)\widetilde{D}^{0,\nu}_\tau(x)\lambda^{0,\nu}_\tau + \sum_{k=1}^{\infty}\sum_{\tau\in M_k}\sum_{\nu=1}^{N(k,\tau)}\mu(Q^{k,\nu}_\tau)\widetilde{D}_k(x, y^{k,\nu}_\tau)\lambda^{k,\nu}_\tau \tag{4.12}$$

converge to some $f \in B^s_{pq}(X)$ *both in the norm of* $B^s_{pq}(X)$ *and in* $(\mathcal{G}(\beta, \gamma))'$ *with* $\max(-s, 0) < \beta < \varepsilon$ *and* $0 < \gamma < \varepsilon$ *when* $1 \leq p, q < \infty$ *and only in* $(\mathcal{G}(\beta, \gamma))'$ *when* $1 \leq p, q \leq \max(p, q) = \infty$. *Moreover,*

$$\|f\|_{B^s_{pq}(X)} \leq C\Big\{\sum_{k=0}^{\infty}\Big(\sum_{\tau\in M_k}\sum_{\nu=1}^{N(k,\tau)}[(\mu(Q^{k,\nu}_\tau))^{-s/d+1/p}|\lambda^{k,\nu}_\tau|]^p\Big)^{q/p}\Big\}^{1/q}, \tag{4.13}$$

where C *is independent of* f.

(ii) *If* $-\varepsilon < s < \varepsilon$, $1 < p < \infty$, $1 < q \leq \infty$ *and*

$$\Big\|\Big\{\sum_{k=0}^{\infty}\sum_{\tau\in M_k}\sum_{\nu=1}^{N(k,\tau)}[(\mu(Q^{k,\nu}_\tau))^{-s/d}|\lambda^{k,\nu}_\tau|\chi_{Q^{k,\nu}_\tau}(\cdot)]^q\Big\}^{1/q}\Big\|_{L^p(X)} < \infty, \tag{4.14}$$

then the series in (4.12) *converge to some* $f \in F^s_{pq}(X)$ *both in the norm of* $F^s_{pq}(X)$ *and in* $(\mathcal{G}(\beta, \gamma))'$ *with* $\max(-s, 0) < \beta < \varepsilon$ *and* $0 < \gamma < \varepsilon$ *when* $1 < p, q < \infty$, *and only in* $(\mathcal{G}(\beta, \gamma))'$ *when* $1 < p < \infty$ *and* $q = \infty$. *Moreover,*

$$\|f\|_{F^s_{pq}(X)} \leq C\Big\|\Big\{\sum_{k=0}^{\infty}\sum_{\tau\in M_k}\sum_{\nu=1}^{N(k,\tau)}[(\mu(Q^{k,\nu}_\tau))^{-s/d}|\lambda^{k,\nu}_\tau|\chi_{Q^{k,\nu}_\tau}(\cdot)]^q\Big\}^{1/q}\Big\|_{L^p(X)}, \tag{4.15}$$

where C *is independent of* f.

Proof. We first remark that if the series in (4.12) converge in the norm of $B^s_{pq}(X)$ when $1 \le p, q < \infty$ or in the norm of $F^s_{pq}(X)$ when $1 < p, q < \infty$, then by a duality argument, Lemma 1.8 and the facts that for $\max(-s, 0) < \beta < \varepsilon$ and $0 < \gamma < \varepsilon$,

$$\mathcal{G}(\beta, \gamma) \subset B^{-s}_{p'q'}(X) \cap F^{-s}_{p'q'}(X) \tag{4.16}$$

(see Remark 4.1 below), they also converge in $(\mathcal{G}(\beta, \gamma))'$; here and in what follows,

$$1/p + 1/p' = 1 = 1/q + 1/q'.$$

Thus, in these cases, we only need to show the former.

Let us first consider the convergence of the series (4.12) in the norm of $B^s_{pq}(X)$ when $1 \le p, q < \infty$. In these cases, when $p = 1$ or when $q = 1$, we need to use (1.7) of Remark 1.5. We first note that for all $k \in \mathbb{N} \cup \{0\}$ and all $\tau \in M_k$, $N(k, \tau)$ is a finite set; see the proof of Proposition 5.1. Now, if M_k is a finite set, by (4.16) or Remark 4.1, it is easy to see that

$$\sum_{\tau \in M_0} \sum_{\nu=1}^{N(0,\tau)} \mu(Q^{k,\nu}_\tau) \widetilde{D}^{0,\nu}_\tau(x) \lambda^{0,\nu}_\tau \quad \text{and} \quad \sum_{\tau \in M_k} \sum_{\nu=1}^{N(k,\tau)} \mu(Q^{k,\nu}_\tau) \widetilde{D}_k(x, y^{k,\nu}_\tau) \lambda^{k,\nu}_\tau$$

for $k \in \mathbb{N}$ are in $B^s_{pq}(X)$. We claim that this is also true if M_k is an infinite set. To show this, without loss of generality, we may assume that $M_k = \mathbb{N}$ and we only show this for $k \in \mathbb{N}$. The proof for $k = 0$ is just a literal repeat. Now, for any given $k, L \in \mathbb{N}$, we define

$$S^k_L = \sum_{\tau \in M_k,\, \tau \le L} \sum_{\nu=1}^{N(k,\tau)} \mu(Q^{k,\nu}_\tau) \widetilde{D}_k(x, y^{k,\nu}_\tau) \lambda^{k,\nu}_\tau.$$

We show that for any given $k \in \mathbb{N}$, $\{S^k_L\}_{L \in \mathbb{N}}$ is a Cauchy sequence in $B^s_{pq}(X)$, using Lemma 1.2 and a duality argument. Let $g \in B^{-s}_{p'q'}(X) \cap \mathcal{G}(\sigma, \sigma)$ for $0 < \sigma < \varepsilon$. We define the operator $\widetilde{D}^*_k$ by letting its kernel be $\widetilde{D}^*_k(x, y) = \widetilde{D}_k(y, x)$. By Hölder's inequality, for $L_1, L_2 \in \mathbb{N}$ with $L_1 < L_2$,

$$\begin{aligned} |\langle S^k_{L_2} - S^k_{L_1}, g\rangle| &= \Big| \sum_{\tau=L_1+1}^{L_2} \sum_{\nu=1}^{N(k,\tau)} \mu(Q^{k,\nu}_\tau) \lambda^{k,\nu}_\tau \widetilde{D}^*_k(g)(y^{k,\nu}_\tau) \Big| \\ &\le \Big(\sum_{\tau=L_1+1}^{L_2} \sum_{\nu=1}^{N(k,\tau)} [(\mu(Q^{k,\nu}_\tau))^{-s/d+1/p} |\lambda^{k,\nu}_\tau|]^p \Big)^{1/p} \\ &\quad \times \Big(\sum_{\tau=L_1+1}^{L_2} \sum_{\nu=1}^{N(k,\tau)} [(\mu(Q^{k,\nu}_\tau))^{s/d+1/p'} |\widetilde{D}^*_k(g)(y^{k,\nu}_\tau)|]^{p'} \Big)^{1/p'}. \end{aligned} \tag{4.17}$$

We now claim

$$\Big(\sum_{\tau=L_1+1}^{L_2} \sum_{\nu=1}^{N(k,\tau)} [(\mu(Q^{k,\nu}_\tau))^{s/d+1/p'} |\widetilde{D}^*_k(g)(y^{k,\nu}_\tau)|]^{p'} \Big)^{1/p'} \le C \|g\|_{B^{-s}_{p'q'}(X)}, \tag{4.18}$$

where C is independent of g, L_1 and L_2.

Since $y^{k,\nu}_\tau \in Q^{k,\nu}_\tau$, it is easy to see that for any $y \in Q^{k,\nu}_\tau$,

$$|\widetilde{D}_k(x, y^{k,\nu}_\tau)| \le C \frac{2^{-k\varepsilon'}}{(2^{-k} + \varrho(x, y))^{d+\varepsilon'}} \tag{4.19}$$

and

$$(4.20)\qquad |\widetilde{D}_k(x,y_\tau^{k,\nu})-\widetilde{D}_k(x',y_\tau^{k,\nu})|\le C\bigg(\frac{\varrho(x,x')}{2^{-k}+\varrho(x,y)}\bigg)^{\varepsilon'}\frac{2^{-k\varepsilon'}}{(2^{-k}+\varrho(x,y))^{d+\varepsilon'}}$$

for $\varrho(x,x')\le(2^{-k}+\varrho(x,y))/(2A)$, where C is independent of x, x', y, k, τ and ν. We now use (4.4) with g instead of f. By Lemma 1.7, $\widetilde{D}_\tau^{0,\nu}(x)$ for $\tau\in M_0$ and $\nu=1,\dots,N(0,\tau)$ also satisfies (4.19) and (4.20). Now, (4.19), (4.20) and a similar argument to (2.15) (see also (2.5) in [17] and (1.6) in [20]) show that for any $y\in Q_\tau^{0,\nu}$,

$$(4.21)\qquad \Big|\int_X \widetilde{D}_\tau^{0,\nu}(x)D_l(x,z)\,d\mu(x)\Big|\le C2^{-l\varepsilon'}\frac{1}{(1+\varrho(y,z))^{d+\varepsilon'}},$$

where $l\in\mathbb{N}\cup\{0\}$, $\tau\in M_0$, $\nu=1,\dots,N(0,\tau)$, and C is independent of z, y, l, τ and ν, and that for any $y\in Q_\tau^{k,\nu}$,

$$(4.22)\qquad |\widetilde{D}_k^*D_l(y_\tau^{k,\nu},z)|\le C2^{-|k-l|\varepsilon'}\frac{2^{-(k\wedge l)\varepsilon'}}{(2^{-(k\wedge l)}+\varrho(y,z))^{d+\varepsilon'}}$$

where $l\in\mathbb{N}\cup\{0\}$, $k\in\mathbb{N}$, $\tau\in M_k$, $\nu=1,\dots,N(k,\tau)$ and C is independent of z, y, k, l, τ and ν. Now, by using (4.22), (4.4) with g instead of f, and Hölder's inequality, we see that the left hand side of (4.18) is controlled by

$$C\Big(\sum_{\tau=L_1+1}^{L_2}\sum_{\nu=1}^{N(k,\tau)}[\mu(Q_\tau^{k,\nu})]^{sp'/d+1}$$
$$\times\Big[\sum_{l=0}^{\infty}2^{-|k-l|\varepsilon'/p}\Big\{\int_X|(\widetilde{D}_k^*D_l)(y_\tau^{k,\nu},z)||\widetilde{E}_l(g)(z)|^{p'}\,d\mu(z)\Big\}^{1/p'}\Big]^{p'}\Big)^{1/p'}$$
$$\le C\sum_{l=0}^{\infty}2^{-|k-l|\varepsilon'/p}2^{-ks}\Big(\int_X\Big[\sum_{\tau\in M_k}\sum_{\nu=1}^{N(k,\tau)}\mu(Q_\tau^{k,\nu})|(\widetilde{D}_k^*D_l)(y_\tau^{k,\nu},z)|\Big]|\widetilde{E}_l(g)(z)|^{p'}\,d\mu(z)\Big)^{1/p'}.$$

For any $l\in\mathbb{N}\cup\{0\}$ and any $z\in X$, by (4.21), we have

$$(4.23)\qquad \sum_{\tau\in M_0}\sum_{\nu=1}^{N(0,\tau)}\mu(Q_\tau^{0,\nu})\Big|\int_X\widetilde{D}_\tau^{0,\nu}(x)D_l(x,z)\,d\mu(x)\Big|$$
$$\le C\sum_{\tau\in M_0}\sum_{\nu=1}^{N(0,\tau)}\mu(Q_\tau^{0,\nu})2^{-l\varepsilon'}\frac{1}{(1+\varrho(y,z))^{d+\varepsilon'}}$$
$$\le C2^{-l\varepsilon'}\sum_{\tau\in M_0}\sum_{\nu=1}^{N(0,\tau)}\int_{Q_\tau^{0,\nu}}\frac{1}{(1+\varrho(x,z))^{d+\varepsilon'}}\,d\mu(x)\le C2^{-l\varepsilon'},$$

where y can be any point in $Q_\tau^{0,\nu}$ and C is independent of l, v, τ, z and y. For any $k\in\mathbb{N}$, $l\in\mathbb{N}\cup\{0\}$ and any $z\in X$, by (4.22), we have

$$(4.24)\qquad \sum_{\tau\in M_k}\sum_{\nu=1}^{N(k,\tau)}\mu(Q_\tau^{k,\nu})|(\widetilde{D}_k^*D_l)(y_\tau^{k,\nu},z)|$$
$$\le C\sum_{\tau\in M_k}\sum_{\nu=1}^{N(k,\tau)}\mu(Q_\tau^{k,\nu})2^{-|k-l|\varepsilon'}\frac{2^{-(k\wedge l)\varepsilon'}}{(2^{-(k\wedge l)}+\varrho(y,z))^{d+\varepsilon'}}$$
$$\le C2^{-|k-l|\varepsilon'}\sum_{\tau\in M_k}\sum_{\nu=1}^{N(k,\tau)}\int_{Q_\tau^{k,\nu}}\frac{2^{-(k\wedge l)\varepsilon'}}{(2^{-(k\wedge l)}+\varrho(x,z))^{d+\varepsilon'}}\,d\mu(x)\le C2^{-|k-l|\varepsilon'},$$

where y can be any point in $Q_\tau^{k,\nu}$ and C is independent of k, l, v, τ, z and y. Putting all these estimates together, we see that the left hand side of (4.18) is controlled by

$$C\sum_{l=0}^{\infty}2^{-|k-l|\varepsilon'}2^{(l-k)s}2^{-ls}\|\widetilde{E}_l(g)\|_{L^{p'}(X)}\le C\Big\{\sum_{l=0}^{\infty}2^{-lsq'}\|\widetilde{E}_l(g)\|^{q'}_{L^{p'}(X)}\Big\}^{1/q'}$$
$$\le C\|g\|_{B^{-s}_{p'q'}(X)},$$

where we have used some techniques similar to (4.7), and C is independent of g and $k\in\mathbb{N}\cup\{0\}$. Now by replacing (4.18) into (4.17) and by Lemma 1.8 and (1.7), we obtain

$$\|S^k_{L_2}-S^k_{L_1}\|_{B^s_{pq}(X)}\le C\Big(\sum_{\tau=L_1+1}^{L_2}\sum_{\nu=1}^{N(k,\tau)}[(\mu(Q_\tau^{k,\nu}))^{-s/d+1/p}|\lambda_\tau^{k,\nu}|]^p\Big)^{1/p}, \tag{4.25}$$

where C is independent of L_1 and L_2. Now, from this and (4.11), we deduce that $\{S^k_L\}_{L\in\mathbb{N}}$ is a Cauchy sequence. Thus, it converges in the norm of $B^s_{pq}(X)$ to

$$\sum_{\tau\in M_0}\sum_{\nu=1}^{N(0,\tau)}\mu(Q_\tau^{k,\nu})\widetilde{D}_\tau^{0,\nu}(x)\lambda_\tau^{0,\nu}$$

for $k=0$, and to

$$\sum_{\tau\in M_k}\sum_{\nu=1}^{N(k,\tau)}\mu(Q_\tau^{k,\nu})\widetilde{D}_k(x,y_\tau^{k,\nu})\lambda_\tau^{k,\nu}$$

for $k\in\mathbb{N}$. We still need to show that the first summation of the series in (4.12) also converges in the norm of $B^s_{pq}(X)$. To see this, for $L\in\mathbb{N}$, we define

$$S_L=\sum_{\tau\in M_0}\sum_{\nu=1}^{N(0,\tau)}\mu(Q_\tau^{k,\nu})\widetilde{D}_\tau^{0,\nu}(x)\lambda_\tau^{0,\nu}+\sum_{k=1}^{L}\sum_{\tau\in M_k}\sum_{\nu=1}^{N(k,\tau)}\mu(Q_\tau^{k,\nu})\widetilde{D}_k(x,y_\tau^{k,\nu})\lambda_\tau^{k,\nu}.$$

By a similar argument to the above, we can show that $\{S_L\}_{L\in\mathbb{N}}$ is also a Cauchy sequence in $B^s_{pq}(X)$. In fact, let $g\in B^{-s}_{p'q'}(X)\cap\mathcal{G}(\sigma,\sigma)$ for $0<\sigma<\varepsilon$ and $1/p+1/p=1=1/q+1/q'$. By Hölder's inequality, for $L_1,L_2\in\mathbb{N}$ and $L_1<L_2$,

$$\begin{aligned}|\langle S_{L_2}-S_{L_1},g\rangle|&=\Big|\sum_{k=L_1+1}^{L_2}\sum_{\tau\in M_k}\sum_{\nu=1}^{N(k,\tau)}\mu(Q_\tau^{k,\nu})\lambda_\tau^{k,\nu}\widetilde{D}_k^*(g)(y_\tau^{k,\nu})\Big|\\&\le\Big\{\sum_{k=L_1+1}^{L_2}\Big(\sum_{\tau\in M_k}\sum_{\nu=1}^{N(k,\tau)}[(\mu(Q_\tau^{k,\nu}))^{-s/d+1/p}|\lambda_\tau^{k,\nu}|]^p\Big)^{q/p}\Big\}^{1/q}\\&\quad\times\Big\{\sum_{k=L_1+1}^{L_2}\Big(\sum_{\tau\in M_k}\sum_{\nu=1}^{N(k,\tau)}[(\mu(Q_\tau^{k,\nu}))^{s/d+1/p'}|\widetilde{D}_k^*(g)(y_\tau^{k,\nu})|]^{p'}\Big)^{q'/p'}\Big\}^{1/q'}.\end{aligned} \tag{4.26}$$

We now claim that

$$\Big\{\sum_{k=L_1+1}^{L_2}\Big(\sum_{\tau\in M_k}\sum_{\nu=1}^{N(k,\tau)}[(\mu(Q_\tau^{k,\nu}))^{s/d+1/p'}|\widetilde{D}_k^*(g)(y_\tau^{k,\nu})|]^{p'}\Big)^{q'/p'}\Big\}^{1/q'}\le C\|g\|_{B^{-s}_{p'q'}(X)}, \tag{4.27}$$

where C is independent of g, L_1 and L_2.

By using (4.22), (4.4) with g instead of f, (4.24) and Hölder's inequality, we deduce that the left hand side of (4.27) is controlled by

$$C\Big\{\sum_{k=L_1+1}^{L_2}\Big(\sum_{\tau\in M_k}\sum_{\nu=1}^{N(k,\tau)}[\mu(Q_\tau^{k,\nu})]^{sp'/d+1}\Big[\sum_{l=0}^{\infty}2^{-|k-l|\varepsilon'/p} \times\Big\{\int_X|(\widetilde{D}_k^*D_l)(y_\tau^{k,\nu},z)||\widetilde{E}_l(g)(z)|\,d\mu(z)\Big\}^{1/p'}\Big]^{p'}\Big)^{q'/p'}\Big\}^{1/q'}$$

$$\le C\Big\{\sum_{k=L_1+1}^{L_2}\Big[\sum_{l=0}^{\infty}2^{-|k-l|\varepsilon'/p}2^{-ks}\Big(\int_X\Big[\sum_{\tau\in M_k}\sum_{\nu=1}^{N(k,\tau)}\mu(Q_\tau^{k,\nu})|(\widetilde{D}_k^*D_l)(y_\tau^{k,\nu},z)|\Big] \times|\widetilde{E}_l(g)(z)|^{p'}\,d\mu(z)\Big)^{1/p'}\Big]^{q'}\Big\}^{1/q'}$$

$$\le C\Big\{\sum_{k=0}^{\infty}\Big[\sum_{l=0}^{\infty}2^{-|k-l|\varepsilon'}2^{(l-k)s}2^{-ls}\|\widetilde{E}_l(g)\|_{L^{p'}(X)}\Big]^{q'}\Big\}^{1/q'}$$

$$\le C\Big\{\sum_{l=0}^{\infty}2^{-lsq'}\|\widetilde{E}_l(g)\|_{L^{p'}(X)}^{q'}\Big\}^{1/q'}\le C\|g\|_{B^{-s}_{p'q'}(X)},$$

where we have used some techniques similar to (4.7), and C is independent of g, L_1 and L_2.

Thus our claim (4.27) is true. By putting (4.27) into (4.26) and by Lemma 1.8 and (1.7), we obtain

$$\|S_{L_2}-S_{L_1}\|_{B^s_{pq}(X)}\le C\Big\{\sum_{k=L_1+1}^{L_2}\Big(\sum_{\tau\in M_k}\sum_{\nu=1}^{N(k,\tau)}[(\mu(Q_\tau^{k,\nu}))^{-s/d+1/p}|\lambda_\tau^{k,\nu}|]^p\Big)^{q/p}\Big\}^{1/q}, \tag{4.28}$$

where C is independent of L_1 and L_2. From this and (4.11), we deduce that $\{S_L\}_{L\in\mathbb{N}}$ is a Cauchy sequence in $B^s_{pq}(X)$. Thus, it converges in the norm of $B^s_{pq}(X)$ to some $f\in B^s_{pq}(X)$ when $1\le p,q<\infty$.

Now, consider the cases $1\le p,q\le\max(p,q)=\infty$. Since the cases $q=\infty$ and $1\le p<\infty$ can be dealt with similarly, we only consider the cases $p=\infty$ and $1\le q\le\infty$. In these cases, the right hand side of (4.25) may not converge to 0 as $L_1,L_2\to\infty$. This is also true for the right hand side of (4.28) when $q=\infty$. Thus, in these cases, the series in (4.12) may not converge in the norm of $B^s_{pq}(X)$. But, since $p'=1$, by (4.18), we see that the left hand side of (4.18) converges to 0 as $L_1,L_2\to\infty$. For $k\in\mathbb{N}$, let

$$S^k_\infty=\sum_{\tau\in M_k}\sum_{\nu=1}^{N(k,\tau)}\mu(Q_\tau^{k,\nu})\widetilde{D}_k(x,y_\tau^{k,\nu})\lambda_\tau^{k,\nu}.$$

Thus, for any given $g\in\mathcal{G}(\beta,\gamma)$ with $\max(-s,0)<\beta<\varepsilon$ and $0<\gamma<\varepsilon$, by Remark 4.1, (4.17) and (4.18), we see that as $L\to\infty$,

$$\langle S^k_\infty-S^k_L,g\rangle\to 0.$$

This just means that for any given $k\in\mathbb{N}$, S^k_L converges to S^k_∞ in $(\mathcal{G}(\beta,\gamma))'$. Similarly, let S_∞ be the series in (4.12). For $q=1$, by (4.26), Remark 4.1, (4.27) and the fact that as $L\to\infty$,

$$\sum_{k=L+1}^{\infty}\Big[\sup_{\tau\in M_k,\,\nu=1,\dots,N(k,\tau)}(\mu(Q_\tau^{k,\nu}))^{-s/d}|\lambda_\tau^{k,\nu}|\Big]\to 0$$

by (4.11), we find that for any given $g\in\mathcal{G}(\beta,\gamma)$ with $\max(-s,0)<\beta<\varepsilon$ and $0<\gamma<\varepsilon$,

$$\langle S_\infty - S_L, g\rangle\to 0$$

as $L\to\infty$. Thus, in this case, the series in (4.12) converge in $(\mathcal{G}(\beta,\gamma))'$. If $p=\infty$ and $1<q\le\infty$, for any given $g\in\mathcal{G}(\beta,\gamma)$ with $\max(-s,0)<\beta<\varepsilon$ and $0<\gamma<\varepsilon$, by (4.27) and Remark 4.1, we have

$$\Big\{\sum_{k=L+1}^{\infty}\Big(\sum_{\tau\in M_k}\sum_{\nu=1}^{N(k,\tau)}(\mu(Q_\tau^{k,\nu}))^{s/d+1}|\widetilde{D}_k^*(g)(y_\tau^{k,\nu})|\Big)^{q'}\Big\}^{1/q'}\to 0,$$

as $L\to\infty$. From this and (4.26), we deduce that $\langle S_\infty - S_L, g\rangle\to 0$ as $L\to\infty$. Thus, in these cases, the series in (4.12) also converge in $(\mathcal{G}(\beta,\gamma))'$.

To finish the proof of (i), we still need to estimate the norm of f. Let again $g\in B^{-s}_{p'q'}(X)\cap\mathcal{G}(\sigma,\sigma)$ for $0<\sigma<\varepsilon$. By Hölder's inequality,

$$\begin{aligned}
|\langle f,g\rangle| &= \Big|\sum_{\tau\in M_0}\sum_{\nu=1}^{N(0,\tau)}\mu(Q_\tau^{0,\nu})\lambda_\tau^{0,\nu}\int_X \widetilde{D}_\tau^{0,\nu}(x)g(x)\,d\mu(x)\\
&\quad+\sum_{k=1}^{\infty}\sum_{\tau\in M_k}\sum_{\nu=1}^{N(k,\tau)}\mu(Q_\tau^{k,\nu})\lambda_\tau^{k,\nu}\widetilde{D}_k^*(g)(y_\tau^{k,\nu})\Big|\\
&\le\Big\{\sum_{k=0}^{\infty}\Big(\sum_{\tau\in M_k}\sum_{\nu=1}^{N(k,\tau)}[(\mu(Q_\tau^{k,\nu}))^{-s/d+1/p}|\lambda_\tau^{k,\nu}|]^p\Big)^{q/p}\Big\}^{1/q}\\
&\quad\times\Big\{\Big(\sum_{\tau\in M_0}\sum_{\nu=1}^{N(0,\tau)}\Big[(\mu(Q_\tau^{0,\nu}))^{s/d+1/p'}\Big|\int_X\widetilde{D}_\tau^{0,\nu}(x)g(x)\,d\mu(x)\Big|\Big]^{p'}\Big)^{q'/p'}\\
&\quad+\sum_{k=1}^{\infty}\Big(\sum_{\tau\in M_k}\sum_{\nu=1}^{N(k,\tau)}[(\mu(Q_\tau^{k,\nu}))^{s/d+1/p'}|\widetilde{D}_k^*(g)(y_\tau^{k,\nu})|]^{p'}\Big)^{q'/p'}\Big\}^{1/q'}.
\end{aligned}$$

By Lemma 1.8 and (1.7), to obtain (4.13), we now only need to show

$$\begin{aligned}
&\Big\{\Big(\sum_{\tau\in M_0}\sum_{\nu=1}^{N(0,\tau)}\Big[(\mu(Q_\tau^{0,\nu}))^{s/d+1/p'}\Big|\int_X\widetilde{D}_\tau^{0,\nu}(x)g(x)\,d\mu(x)\Big|\Big]^{p'}\Big)^{q'/p'}\\
&+\sum_{k=N+1}^{\infty}\Big(\sum_{\tau\in M_k}\sum_{\nu=1}^{N(k,\tau)}[(\mu(Q_\tau^{k,\nu}))^{s/d+1/p'}|\widetilde{D}_k^*(g)(y_\tau^{k,\nu})|]^{p'}\Big)^{q'/p'}\Big\}^{1/q'}\le C\|g\|_{B^{-s}_{p'q'}(X)},
\end{aligned}\tag{4.29}$$

where C is independent of g.

To show this, by using (4.21), (4.22), (4.4) with g instead of f, Hölder's inequality, (4.23) and (4.24), we find that the left hand side of (4.29) is controlled by

$$
\begin{aligned}
&C\Big\{\Big(\sum_{\tau\in M_0}\sum_{\nu=1}^{N(0,\tau)}[\mu(Q_\tau^{0,\nu})]^{sp'/d+1}\\
&\qquad\times\Big[\sum_{l=0}^{\infty}2^{-l\varepsilon'/p}\Big\{\int_X\Big|\int_X\widetilde{D}_\tau^{0,\nu}(x)D_l(x,z)\,d\mu(x)\Big||\widetilde{E}_l(g)(z)|\,d\mu(z)\Big\}^{1/p'}\Big]^{p'}\Big)^{q'/p'}\\
&\qquad+\sum_{k=1}^{\infty}\Big(\sum_{\tau\in M_k}\sum_{\nu=1}^{N(k,\tau)}[\mu(Q_\tau^{k,\nu})]^{sp'/d+1}\\
&\qquad\times\Big[\sum_{l=0}^{\infty}2^{-|k-l|\varepsilon'/p}\Big\{\int_X|(\widetilde{D}_k^*D_l)(y_\tau^{k,\nu},z)||\widetilde{E}_l(g)(z)|\,d\mu(z)\Big\}^{1/p'}\Big]^{p'}\Big)^{q'/p'}\Big\}^{1/q'}\\
&\quad\le C\Big\{\Big[\sum_{l=0}^{\infty}2^{-l\varepsilon'/p}\Big\{\int_X\Big[\sum_{\tau\in M_0}\sum_{\nu=1}^{N(0,\tau)}\mu(Q_\tau^{0,\nu})\Big|\int_X\widetilde{D}_\tau^{0,\nu}(x)D_l(x,z)\,d\mu(x)\Big|\Big]\\
&\qquad\times|\widetilde{E}_l(g)(z)|^{p'}\,d\mu(z)\Big\}^{1/p'}\Big]^{q'}\\
&\qquad+\sum_{k=1}^{\infty}\Big[\sum_{l=0}^{\infty}2^{-|k-l|\varepsilon'/p}2^{-ks}\Big(\int_X\Big[\sum_{\tau\in M_k}\sum_{\nu=1}^{N(k,\tau)}\mu(Q_\tau^{k,\nu})|(\widetilde{D}_k^*D_l)(y_\tau^{k,\nu},z)|\Big]\\
&\qquad\times|\widetilde{E}_l(g)(z)|^{p'}\,d\mu(z)\Big)^{1/p'}\Big]^{q'}\Big\}^{1/q'}\\
&\quad\le C\Big\{\sum_{k=0}^{\infty}\Big[\sum_{l=0}^{\infty}2^{-|k-l|\varepsilon'}2^{(l-k)s}2^{-ls}\|\widetilde{E}_l(g)\|_{L^{p'}(X)}\Big]^{q'}\Big\}^{1/q'}\\
&\quad\le C\Big\{\sum_{l=0}^{\infty}2^{-lsq'}\|\widetilde{E}_l(g)\|_{L^{p'}(X)}^{q'}\Big\}^{1/q'}\le C\|g\|_{B_{p'q'}^{-s}(X)},
\end{aligned}
$$

where we have used some techniques similar to (4.7), and C is independent of g. Thus, (4.29) is true and the proof of (i) is finished.

Now let us prove (ii). We first remark that, in a similar way, we can show that the series in (4.12) converge in the norm of $F_{pq}^s(X)$ to some $f\in F_{pq}^s(X)$ when $1<p,q<\infty$. We omit the details. Now we establish (4.15) for $1<p,q<\infty$. For any $g\in F_{p'q'}^{-s}(X)\cap\mathcal{G}(\sigma,\sigma)$, by (4.4) with g instead of f, we have

$$
\begin{aligned}
\text{(4.30)}\qquad \langle f,g\rangle&=\sum_{\tau\in M_0}\sum_{\nu=1}^{N(0,\tau)}\lambda_\tau^{0,\nu}\mu(Q_\tau^{0,\nu})\int_X\widetilde{D}_\tau^{0,\nu}(x)g(x)\,d\mu(x)\\
&\quad+\sum_{k=1}^{\infty}\sum_{\tau\in M_k}\sum_{\nu=1}^{N(k,\tau)}\lambda_\tau^{k,\nu}\mu(Q_\tau^{k,\nu})\widetilde{D}_k^*(g)(y_\tau^{k,\nu})\\
&=\sum_{l=0}^{\infty}\sum_{\tau\in M_0}\sum_{\nu=1}^{N(0,\tau)}\lambda_\tau^{0,\nu}\mu(Q_\tau^{0,\nu})\int_X\Big[\int_X\widetilde{D}_\tau^{0,\nu}(x)D_l(x,z)\,d\mu(x)\Big]\widetilde{E}_l(g)(z)\,d\mu(z)\\
&\quad+\sum_{l=0}^{\infty}\sum_{k=1}^{\infty}\sum_{\tau\in M_k}\sum_{\nu=1}^{N(k,\tau)}\lambda_\tau^{k,\nu}\mu(Q_\tau^{k,\nu})\widetilde{D}_k^*D_l\widetilde{E}_l(g)(y_\tau^{k,\nu})
\end{aligned}
$$

$$= \sum_{l=0}^{\infty} \int_X \Big\{ \sum_{\tau\in M_0} \sum_{\nu=1}^{N(0,\tau)} \lambda_\tau^{0,\nu} \mu(Q_\tau^{0,\nu}) \Big[\int_X \widetilde{D}_\tau^{0,\nu}(x) D_l(x,z)\, d\mu(x) \Big] \Big\} \widetilde{E}_l(g)(z)\, d\mu(z)$$
$$+ \sum_{l=0}^{\infty} \sum_{k=1}^{\infty} \int_X \widetilde{E}_l(g)(z) \Big[\sum_{\tau\in M_k} \sum_{\nu=1}^{N(k,\tau)} \lambda_\tau^{k,\nu} \mu(Q_\tau^{k,\nu}) (\widetilde{D}_k^* D_l)(y_\tau^{k,\nu}, z) \Big]\, d\mu(z).$$

By (4.21) and Lemma 1.9, we have

$$\sum_{\tau\in M_0} \sum_{\nu=1}^{N(0,\tau)} \Big| \lambda_\tau^{0,\nu} \mu(Q_\tau^{0,\nu}) \Big[\int_X \widetilde{D}_\tau^{0,\nu}(x) D_l(x,z)\, d\mu(x) \Big] \Big| \le C 2^{-l\varepsilon'} M\Big(\sum_{\tau\in M_0} \sum_{\nu=1}^{N(0,\tau)} |\lambda_\tau^{0,\nu}| \chi_{Q_\tau^{0,\nu}} \Big)(z), \tag{4.31}$$

where M is the Hardy–Littlewood maximal function and C is independent of l, τ, ν and z. By (4.22) and Lemma 1.9, for $k \in \mathbb{N}$,

$$\sum_{\tau\in M_k} \sum_{\nu=1}^{N(k,\tau)} |\lambda_\tau^{k,\nu} \mu(Q_\tau^{k,\nu}) (\widetilde{D}_k^* D_l)(y_\tau^{k,\nu}, z)| \le C 2^{-|k-l|\varepsilon'} 2^{(k\wedge l)d} 2^{[k-(k\wedge l)]d} M\Big(\sum_{\tau\in M_k} \sum_{\nu=1}^{N(k,\tau)} |\lambda_\tau^{k,\nu}| \chi_{Q_\tau^{k,\nu}} \Big)(z), \tag{4.32}$$

where C is independent of k, l, τ, ν and z. Thus, by combining (4.31) and (4.32) with (4.30), we have

$$\begin{aligned}
|\langle f, g\rangle| &\le C \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} 2^{-|k-l|\varepsilon'} \int_X \Big[M\Big(\sum_{\tau\in M_k} \sum_{\nu=1}^{N(k,\tau)} |\lambda_\tau^{k,\nu}| \chi_{Q_\tau^{k,\nu}} \Big)(z) \Big] |\widetilde{E}_l(g)(z)|\, d\mu(z) \\
&\le C \int_X \Big\{ \sum_{l=0}^{\infty} 2^{-lsq'} |\widetilde{E}_l(g)(z)|^{q'} \Big\}^{1/q'} \\
&\quad \times \Big\{ \sum_{l=0}^{\infty} \Big[\sum_{k=0}^{\infty} 2^{-|k-l|\varepsilon'} 2^{(l-k)s} 2^{ks} M\Big(\sum_{\tau\in M_k} \sum_{\nu=1}^{N(k,\tau)} |\lambda_\tau^{k,\nu}| \chi_{Q_\tau^{k,\nu}} \Big)(z) \Big]^q \Big\}^{1/q} d\mu(z) \\
&\le C \int_X \Big\{ \sum_{l=0}^{\infty} 2^{-lsq'} |\widetilde{E}_l(g)(z)|^{q'} \Big\}^{1/q'} \\
&\quad \times \Big\{ \sum_{k=0}^{\infty} 2^{ksq} \Big[M\Big(\sum_{\tau\in M_k} \sum_{\nu=1}^{N(k,\tau)} |\lambda_\tau^{k,\nu}| \chi_{Q_\tau^{k,\nu}} \Big)(z) \Big]^q \Big\}^{1/q} d\mu(z) \\
&\le C \Big\| \Big\{ \sum_{k=0}^{\infty} \Big[M\Big(\sum_{\tau\in M_k} \sum_{\nu=1}^{N(k,\tau)} (\mu(Q_\tau^{k,\nu}))^{-s/d} |\lambda_\tau^{k,\nu}| \chi_{Q_\tau^{k,\nu}} \Big)(\cdot) \Big]^q \Big\}^{1/q} \Big\|_{L^p(X)} \\
&\quad \times \Big\| \Big\{ \sum_{l=0}^{\infty} 2^{-lsq'} |\widetilde{E}_l(g)(\cdot)|^{q'} \Big\}^{1/q'} \Big\|_{L^{p'}(X)} \\
&\le C \Big\| \Big\{ \sum_{k=0}^{\infty} \Big[M\Big(\sum_{\tau\in M_k} \sum_{\nu=1}^{N(k,\tau)} (\mu(Q_\tau^{k,\nu}))^{-s/d} |\lambda_\tau^{k,\nu}| \chi_{Q_\tau^{k,\nu}} \Big)(\cdot) \Big]^q \Big\}^{1/q} \Big\|_{L^p(X)} \|g\|_{F^{-s'}_{p'q'}(X)},
\end{aligned}$$

where we have used the Fefferman–Stein vector-valued inequality in [7] and some techniques similar to (4.10). From this, by Lemma 1.8, it is easy to deduce (4.15) when $1 < p, q < \infty$.

We still need to show (ii) for $1 < p < \infty$ and $q = \infty$. Since in these cases, we do not have dual spaces, we have to directly use the definitions by combining some estimates.

Let us first show the series in (4.12) converge in $(\mathcal{G}(\beta,\gamma))'$ with $\max(-s,0) < \beta < \varepsilon$ and $0 < \gamma < \varepsilon$ in these cases. We only show this for the sum on k in (4.12); the proof for the sum on τ is similar. Let $g \in \mathcal{G}(x_0, 1, \beta, \gamma) \equiv \mathcal{G}(\beta,\gamma)$ with $x_0 \in X$. By a similar proof to (4.22), for any $z \in Q_\tau^{k,\nu}$,

$$\Big|\int_X \widetilde{D}_k(x, y_\tau^{k,\nu}) g(x)\, d\mu(x)\Big| \le C 2^{-k\beta} \frac{1}{(1+\varrho(x_0,z))^{d+\beta}},$$

where C is independent of k, τ, ν and z. From this and the arbitrariness of z, we deduce that for any $L \in \mathbb{N}$,

$$\begin{aligned}
\Big|\sum_{k>L}\sum_{\tau\in M_k}\sum_{\nu=1}^{N(k,\tau)} &\mu(Q_\tau^{k,\nu})\lambda_\tau^{k,\nu}\int_X D_k(x, y_\tau^{k,\nu}) g(x)\, d\mu(x)\Big| \\
&\le C\sum_{k>L} 2^{-k\beta}\sum_{\tau\in M_k}\sum_{\nu=1}^{N(k,\tau)} \mu(Q_\tau^{k,\nu})|\lambda_\tau^{k,\nu}| \frac{1}{(1+\varrho(x_0,z))^{d+\beta}} \\
&\le C\sum_{k>L} 2^{-k\beta}\sum_{\tau\in M_k}\sum_{\nu=1}^{N(k,\tau)} \mu(Q_\tau^{k,\nu})|\lambda_\tau^{k,\nu}| \int_{Q_\tau^{k,\nu}} \frac{1}{(1+\varrho(x_0,y))^{d+\beta}}\, d\mu(y) \\
&\le C\sum_{k>L} 2^{-k(\beta+s)}\int_X \Big\{\sup_{k\in\mathbb{N}\cup\{0\},\, \tau\in M_k,\, \nu=1,\dots,N(k,\tau)} [\mu(Q_\tau^{k,\nu})]^{-s/d}|\lambda_\tau^{k,\nu}|\chi_{Q_\tau^{k,\nu}}(y)\Big\} \\
&\quad\times \frac{1}{(1+\varrho(x_0,y))^{d+\beta}}\, d\mu(y) \\
&\le C\Big\|\sup_{k\in\mathbb{N}\cup\{0\},\, \tau\in M_k,\, \nu=1,\dots,N(k,\tau)} [\mu(Q_\tau^{k,\nu})]^{-s/d}|\lambda_\tau^{k,\nu}|\chi_{Q_\tau^{k,\nu}}(\cdot)\Big\|_{L^p(X)} \\
&\quad\times \sum_{k>L} 2^{-k(\beta+s)}\Big\{\int_X \frac{1}{(1+\varrho(x_0,y))^{(d+\beta)p'}}\, d\mu(y)\Big\}^{1/p'} \\
&\le C\sum_{k>L} 2^{-k(\beta+s)} \to 0,
\end{aligned}$$

as $L \to \infty$, since $\beta > -s$, where C is independent of k and L. This shows that the series in (4.12) converge in $(\mathcal{G}(\beta,\gamma))'$ in these cases.

Finally, let us establish (4.15) in these cases. Let $\{D_l\}_{l\in\mathbb{N}\cup\{0\}}$ be as in Theorem 4.1 and let f be the series in (4.12). For $l \in \mathbb{N}\cup\{0\}$, by what we have just proved,

$$\begin{aligned}
D_l(f)(x) = &\sum_{\tau\in M_0}\sum_{\nu=1}^{N(0,\tau)} \mu(Q_\tau^{0,\nu})(D_l\widetilde{D}_\tau^{0,\nu})(x)\lambda_\tau^{0,\nu} \\
&+ \sum_{k=1}^{\infty}\sum_{\tau\in M_k}\sum_{\nu=1}^{N(k,\tau)} \mu(Q_\tau^{k,\nu})(D_l\widetilde{D}_k)(x, y_\tau^{k,\nu})\lambda_\tau^{k,\nu}
\end{aligned}$$

in $(\mathcal{G}(\beta,\gamma))'$ with $\max(-s,0) < \beta < \varepsilon$ and $0 < \gamma < \varepsilon$.

By (4.21), for any $l \in \mathbb{N} \cup \{0\}$, any $y \in Q_\tau^{0,\nu}$ and any $\varepsilon' \in (0, \varepsilon)$, we have

$$
\begin{aligned}
(4.33)\quad \Big| \sum_{\tau \in M_0} \sum_{\nu=1}^{N(0,\tau)} \mu(Q_\tau^{0,\nu})(D_l \widetilde{D}_\tau^{0,\nu})(x)\lambda_\tau^{0,\nu}\Big| &\le C \sum_{\tau \in M_0} \sum_{\nu=1}^{N(0,\tau)} 2^{-l\varepsilon'} \frac{|\lambda_\tau^{0,\nu}|}{(1+\varrho(x,y))^{d+\varepsilon'}} \\
&\le C \sum_{\tau \in M_0} \sum_{\nu=1}^{N(0,\tau)} |\lambda_\tau^{0,\nu}| 2^{-l\varepsilon'} \int_{Q_\tau^{0,\nu}} \frac{1}{(1+\varrho(x,z))^{d+\varepsilon'}}\, d\mu(z) \\
&\le C 2^{-l\varepsilon'} \int_X \{ \sup_{\tau \in M_0,\, \nu = 1, \dots, N(k,\tau)} [\mu(Q_\tau^{0,\nu})]^{-s/d} |\lambda_\tau^{0,\nu}| \chi_{Q_\tau^{0,\nu}}(z)\} \frac{1}{(1+\varrho(x,z))^{d+\varepsilon'}}\, d\mu(z),
\end{aligned}
$$

where C is independent of x.

Similarly, by (4.22), for any $l \in \mathbb{N} \cup \{0\}$, any $y \in Q_\tau^{0,\nu}$ and any $\varepsilon' \in (0, \varepsilon)$, we obtain

$$
\begin{aligned}
(4.34)\quad &\Big| \sum_{k=1}^{\infty} \sum_{\tau \in M_k} \sum_{\nu=1}^{N(k,\tau)} \mu(Q_\tau^{k,\nu})(D_l \widetilde{D}_k)(x, y_\tau^{k,\nu})\lambda_\tau^{k,\nu}\Big| \\
&\quad \le C \sum_{k=1}^{\infty} \sum_{\tau \in M_k} \sum_{\nu=1}^{N(k,\tau)} \mu(Q_\tau^{k,\nu}) |\lambda_\tau^{k,\nu}| 2^{-|k-l|\varepsilon'} \frac{2^{-(k\wedge l)\varepsilon'}}{(2^{-(k\wedge l)} + \varrho(x,y))^{d+\varepsilon'}} \\
&\quad \le C \sum_{k=1}^{\infty} \sum_{\tau \in M_k} \sum_{\nu=1}^{N(k,\tau)} |\lambda_\tau^{k,\nu}| 2^{-|k-l|\varepsilon'} \int_{Q_\tau^{k,\nu}} \frac{2^{-(k\wedge l)\varepsilon'}}{(2^{-(k\wedge l)} + \varrho(x,z))^{d+\varepsilon'}}\, d\mu(z) \\
&\quad \le C \int_X \{ \sup_{k \in \mathbb{N},\, \tau \in M_k,\, \nu = 1, \dots, N(k,\tau)} [\mu(Q_\tau^{k,\nu})]^{-s/d} |\lambda_\tau^{k,\nu}| \chi_{Q_\tau^{k,\nu}}(z)\} \\
&\quad\quad \times \sum_{k=1}^{\infty} 2^{-|k-l|\varepsilon'} 2^{-ks} \frac{2^{-(k\wedge l)\varepsilon'}}{(2^{-(k\wedge l)} + \varrho(x,z))^{d+\varepsilon'}}\, d\mu(z),
\end{aligned}
$$

where C is independent of l and x.

By combining (4.33) and (4.34) and by Hölder's inequality, for all $l \in \mathbb{N} \cup \{0\}$ and all $x \in X$,

$$
\begin{aligned}
2^{ls} |D_l(f)(x)| &\le C \int_X \{ \sup_{k \in \mathbb{N},\, \tau \in M_k,\, \nu = 1, \dots, N(k,\tau)} [\mu(Q_\tau^{k,\nu})]^{-s/d} |\lambda_\tau^{k,\nu}| \chi_{Q_\tau^{k,\nu}}(z)\} \\
&\quad \times \sum_{k=0}^{\infty} 2^{-|k-l|\varepsilon'} 2^{-ks} \frac{2^{-(k\wedge l)\varepsilon'}}{(2^{-(k\wedge l)} + \varrho(x,z))^{d+\varepsilon'}}\, d\mu(z) \\
&\le C \Big[\int_X \{ \sup_{k \in \mathbb{N},\, \tau \in M_k,\, \nu = 1, \dots, N(k,\tau)} [\mu(Q_\tau^{k,\nu})]^{-s/d} |\lambda_\tau^{k,\nu}| \chi_{Q_\tau^{k,\nu}}(z)\}^p \\
&\quad \times \sum_{k=0}^{\infty} 2^{-|k-l|\varepsilon'} 2^{-ks} \frac{2^{-(k\wedge l)\varepsilon'}}{(2^{-(k\wedge l)} + \varrho(x,z))^{d+\varepsilon'}}\, d\mu(z) \Big]^{1/p},
\end{aligned}
$$

where C is independent of l and x.

From this, it is easy to deduce

$$\|f\|_{F^s_{p\infty}(X)} = \|\sup_{l\in\mathbb{N}\cup\{0\}} 2^{ls}|D_l(f)|\|_{L^p(X)}$$
$$\le C\|\sup_{k\in\mathbb{N},\,\tau\in M_k,\,\nu=1,\dots,N(k,\tau)} [\mu(Q^{k,\nu}_\tau)]^{-s/d}|\lambda^{k,\nu}_\tau|\chi_{Q^{k,\nu}_\tau}(\cdot)\|_{L^p(X)},$$

where C is independent of f.

This finishes the proof of Proposition 4.1.

We remark that Theorem 4.1 and Proposition 4.1 are true for both $\mu(X) < \infty$ and $\mu(X) = \infty$.

Remark 4.1. Let $s \in (-\varepsilon, \varepsilon)$, $\max(s, 0) < \beta < \varepsilon$ and $0 < \gamma$. Then

$$\mathcal{G}(\beta,\gamma) \subset B^s_{pq}(X) \quad \text{for } 1 \le p, q \le \infty,$$
$$\mathcal{G}(\beta,\gamma) \subset F^s_{pq}(X) \quad \text{for } 1 < p < \infty \text{ and } 1 < q \le \infty.$$

This is true for both $\mu(X) < \infty$ and $\mu(X) = \infty$ and it can be easily seen from the proof of Theorem 2.2 in [20]. See also Remark 2.1 in [20] and the remark in [23, p. 100]. In both remarks, it is also required that $\max(-s, 0) < \gamma < \varepsilon$, which is in fact not necessary.

Remark 4.2. We point out that the methods applied for the cases $F^s_{p\infty}(X)$ also work for all other cases.

5. Embeddings

In this section, we first estimate the entropy numbers of compact embeddings between $B^s_{pq}(X)$ or $F^s_{pq}(X)$ spaces when $\mu(X) < \infty$ by using the frame characterizations of these spaces, namely, Theorem 4.1 and Proposition 4.1. Some limiting embeddings between these spaces are also obtained. We remark that the atomic decompositions of these spaces are not enough to obtain these estimates.

Let us now recall the definition of the entropy numbers; see [6] and [33]. In the following, if B is a quasi-Banach space, then $\mathcal{U}_B = \{b \in B : \|b\|_B \le 1\}$ stands for the unit ball in B.

Definition 5.1. Let A and B be quasi-Banach spaces and T be a linear continuous operator from A to B. Then for all $k \in \mathbb{N}$, the kth *entropy number*, $e_k(T)$, of T is defined by

$$e_k(T) = \inf\Big\{\varepsilon > 0 : T(\mathcal{U}_A) \subset \bigcup_{j=1}^{2^{k-1}} (b_j + \varepsilon\mathcal{U}_B) \text{ for some } b_1, \dots, b_{2^{k-1}} \in B\Big\}.$$

By using some ideas from the proof of Proposition 20.5 in [33], we can now establish upper estimates for the entropy numbers of compact embeddings between $B^s_{pq}(X)$ and $F^s_{pq}(X)$ spaces when $\mu(X) < \infty$. We point out that our results for $B^s_{pq}(X)$ when X is a d-set (see [33]) and $0 < s < 1$ are included in Proposition 20.5 in [33]. The other cases, even when X is a d-set, are new. Since there is no quarkonial decomposition on spaces of homogeneous type, which plays a key role in [33], the new idea here is to use the frame decompositions for $B^s_{pq}(X)$ and $F^s_{pq}(X)$, discussed in Theorem 4.1 and Proposition 4.1.

Proposition 5.1. *Let* $(X, \varrho, \mu)_{d,\theta}$ *be a space of homogeneous type with* $\mu(X) < \infty$*. Let* $B^s_{pq}(X)$ *for* $1 \le p, q \le \infty$ *and* $F^s_{pq}(X)$ *for* $1 < p < \infty$ *and* $1 < q \le \infty$ *be the spaces in Definition* 1.3 *with* $|s| < \theta$*. Let* $-\theta < s_2 < s_1 < \theta$*.*

(i) *If* $1 \le p_1, p_2 \le \infty$, $1 \le q_1, q_2 \le \infty$ *and*

$$\delta_+ = s_1 - s_2 - d\left(\frac{1}{p_1} - \frac{1}{p_2}\right)_+ > 0,$$

where $x_+ = \max(x, 0)$*, then the embedding of* $B^{s_1}_{p_1q_1}(X)$ *into* $B^{s_2}_{p_2q_2}(X)$ *is compact and there is a constant* $C > 0$ *such that*

$$e_k(\mathrm{id} : B^{s_1}_{p_1q_1}(X) \to B^{s_2}_{p_2q_2}(X)) \le Ck^{-(s_1-s_2)/d} \quad \text{for all } k \in \mathbb{N}.$$

(ii) *If* $1 < p_1, p_2 < \infty$, $1 < q_1, q_2 \le \infty$ *and* $\delta_+ > 0$*, then the embedding of* $F^{s_1}_{p_1q_1}(X)$ *into* $F^{s_2}_{p_2q_2}(X)$ *is compact and there is a constant* $C > 0$ *such that*

$$e_k(\mathrm{id} : F^{s_1}_{p_1q_1}(X) \to F^{s_2}_{p_2q_2}(X)) \le Ck^{-(s_1-s_2)/d} \quad \text{for all } k \in \mathbb{N}.$$

Proof. By Proposition 13.6 in [33], it is easy to see that for $s \in (-\theta, \theta)$, $1 < p < \infty$ and $1 < q \le \infty$, we have the following continuous embedding:

$$B^s_{pu}(X) \subset F^s_{pq}(X) \subset B^s_{pv}(X), \tag{5.1}$$

where $u = \min(p, q)$ and $v = \max(p, q)$. By (5.1) it is easy to see that it is sufficient to prove (i). We consider two cases.

Case 1: $p_2 \ge p_1$. In this case, we have

$$\delta = \delta_+ = s_1 - s_2 - d\left(\frac{1}{p_1} - \frac{1}{p_2}\right) > 0.$$

We will use Theorem 4.1 and Proposition 4.1. In the following part of this section, for M_k of Lemma 1.3, we will also write M_k for the set $\{1, \dots, M_k\}$. We first claim that if $\mu(X) < \infty$ and we take $\delta = 1/2$ in Lemma 1.3, then in Lemma 1.3 we have M_k satisfying $M_k \sim 2^{kd}$. In fact, by Lemma 1.3(i), (iv), we have

$$\mu(X) = \mu\Big(\bigcup_{\tau \in M_k} Q^k_\tau\Big) = \sum_{\tau \in M_k} \mu(Q^k_\tau) \le C2^{-kd} M_k.$$

Thus, $M_k \ge C2^{kd}$. By Lemma 1.3(i), (v), we then have

$$\mu(X) = \mu\Big(\bigcup_{\tau \in M_k} Q^k_\tau\Big) = \sum_{\tau \in M_k} \mu(Q^k_\tau) \ge \sum_{\tau \in M_k} \mu(B(z^k_\tau, a_0 2^{-k})) \ge C2^{-kd} M_k.$$

From this, we see that $M_k \le C2^{kd}$. Thus our claim holds. In a similar way, we can show $N(k, \tau) \sim 2^{jd}$ for any $k \in \mathbb{N} \cup \{0\}$ and $\tau \in M_k$. Thus, for any fixed $j \in \mathbb{N}$,

$$\sum_{\tau \in M_k} N(k, \tau) \sim 2^{kd}.$$

In the rest of this proof, we denote $\sum_{\tau \in M_k} N(k, \tau)$ by $\widetilde{M}_k$ for $k \in \mathbb{N} \cup \{0\}$ and we use the same notation of Theorem 4.1.

Now suppose $f \in B^{s_1}_{p_1 q_1}(X)$. By Theorem 4.1, we have

$$f(x) = \sum_{\tau\in M_0}\sum_{\nu=1}^{N(0,\tau)} \mu(Q^{0,\nu}_\tau)\widetilde{D}^{0,\nu}_\tau(x)D^{0,\nu}_{\tau,1}(f) + \sum_{k=1}^{N}\sum_{\tau\in M_k}\sum_{\nu=1}^{N(k,\tau)} \mu(Q^{k,\nu}_\tau)\widetilde{D}_k(x,y^{k,\nu}_\tau)D^{k,\nu}_{\tau,1}(f) + \sum_{k=N+1}^{\infty}\sum_{\tau\in M_k}\sum_{\nu=1}^{N(k,\tau)} \mu(Q^{k,\nu}_\tau)\widetilde{D}_k(x,y^{k,\nu}_\tau)D_k(f)(y^{k,\nu}_\tau). \tag{5.2}$$

Moreover,

$$\|f\|_{B^{s_1}_{p_1q_1}(X)} \sim \Big\{ \sum_{k=0}^{N}\Big(\sum_{\tau\in M_k}\sum_{\nu=1}^{N(k,\tau)} [(\mu(Q^{k,\nu}_\tau))^{-s_1/d+1/p_1}|D^{k,\nu}_{\tau,1}(f)|]^{p_1}\Big)^{q_1/p_1} + \sum_{k=N+1}^{\infty}\Big(\sum_{\tau\in M_k}\sum_{\nu=1}^{N(k,\tau)} [(\mu(Q^{k,\nu}_\tau))^{-s_1/d+1/p_1}|D_k(f)(y^{k,\nu}_\tau)|]^{p_1}\Big)^{q_1/p_1}\Big\}^{1/q_1}. \tag{5.3}$$

Let

$$\eta^{k,\nu}_\tau = \begin{cases} 2^{-k\delta}2^{k(s_1-d/p_1)}D^{k,\nu}_{\tau,1}(f), & k=0,1,\dots,N,\ \tau\in M_k,\ \nu=1,\dots,N(k,\tau),\\ 2^{-k\delta}2^{k(s_1-d/p_1)}D_k(f)(y^{k,\nu}_\tau), & k=N+1,\dots,\ \tau\in M_k,\ \nu=1,\dots,N(k,\tau).\end{cases}$$

We now define the (nonlinear) operator S from $B^{s_1}_{p_1q_1}(X)$ to $l_{q_1}(2^{\nu\delta}l^{\widetilde{M}_\nu}_{p_1})$ by letting

$$Sf = \eta = \{\eta^{k,\nu}_\tau : k\in\mathbb{N}\cup\{0\},\ \tau\in M_k \text{ and } \nu=1,\dots,N(k,\tau)\} \tag{5.4}$$

for $f\in B^{s_1}_{p_1q_1}(X)$ having the above decomposition (5.2). Here by $l_{q_1}(2^{\nu\delta}l^{\widetilde{M}_\nu}_{p_1})$ we mean the linear space of all complex sequences

$$\lambda = \{\lambda^{k,\nu}_\tau : k\in\mathbb{N}\cup\{0\},\ \tau=1,\dots,M_k \text{ and } \nu=1,\dots,N(k,\tau)\}$$

endowed with the norm

$$\|\lambda\|_{l_{q_1}(2^{\nu\delta}l^{\widetilde{M}_\nu}_{p_1})} = \Big\{\sum_{k=0}^{\infty}\Big[\sum_{\tau\in M_k}\sum_{\nu=1}^{N(k,\tau)} 2^{k\delta p_1}|\lambda^{k,\nu}_\tau|^{p_1}\Big]^{q_1/p_1}\Big\}^{1/q_1};$$

see [33, p. 38]. By (5.2) and (5.3), S is bounded from $B^{s_1}_{p_1q_1}(X)$ to $l_{q_1}(2^{\nu\delta}l^{\widetilde{M}_\nu}_{p_1})$. That is, there is a constant $C>0$ such that for all $f\in B^{s_1}_{p_1q_1}(X)$, we have

$$\|Sf\|_{l_{q_1}(2^{\nu\delta}l^{\widetilde{M}_\nu}_{p_1})} \le C\|f\|_{B^{s_1}_{p_1q_1}(X)}.$$

Now we define another linear operator T from $l_{q_2}(l^{\widetilde{M}_\nu}_{p_2})$ to $B^{s_2}_{p_2q_2}(X)$ by letting

$$T\kappa = \sum_{\tau\in M_0}\sum_{\nu=1}^{N(0,\tau)} \kappa^{0,\nu}_\tau 2^{-k(s_2-d/p_2)}\mu(Q^{0,\nu}_\tau)\widetilde{D}^{0,\nu}_\tau(x) + \sum_{k=1}^{\infty}\sum_{\tau\in M_k}\sum_{\nu=1}^{N(k,\tau)} \kappa^{k,\nu}_\tau 2^{-k(s_2-d/p_2)}\mu(Q^{k,\nu}_\tau)\widetilde{D}_k(x,y^{k,\nu}_\tau) \tag{5.5}$$

for

$$\kappa = \{\kappa_\tau^{k,\nu} : k \in \mathbb{N} \cup \{0\},\ \tau = 1, \ldots, M_k \text{ and } \nu = 1, \ldots, N(k,\tau)\} \in l_{q_2}(l_{p_2}^{\widetilde{M}_\nu}).$$

By Proposition 4.1, T is also bounded from $l_{q_2}(l_{p_2}^{\widetilde{M}_\nu})$ to $B^{s_2}_{p_2q_2}(X)$. That is, there is a constant $C > 0$ such that for all $\kappa \in l_{q_2}(l_{p_2}^{\widetilde{M}_\nu})$,

$$\|T\kappa\|_{B^{s_2}_{p_2q_2}(X)} \le C\|\kappa\|_{l_{q_2}(l_{p_2}^{\widetilde{M}_\nu})}.$$

Let $\mathrm{id} : l_{q_1}(2^{\nu\delta} l_{p_1}^{\widetilde{M}_\nu}) \to l_{q_2}(l_{p_2}^{\widetilde{M}_\nu})$. Then, by Theorem 9.2 in [33], id is compact. By our above definitions of S and T, that is, (5.4) and (5.5), and (5.2), it is easy to see that

$$\mathrm{id}(B^{s_1}_{p_1q_1}(X) \to B^{s_2}_{p_2q_2}(X)) = T \circ \mathrm{id} \circ S.$$

Thus, $\mathrm{id} : B^{s_1}_{p_1q_1}(X) \to B^{s_2}_{p_2q_2}(X)$ is compact. Moreover, by Theorem 9.2 with $u_1 = u_2 = \infty$ and by Proposition 5.4(ii) in [33], we have

$$e_k(\mathrm{id} : B^{s_1}_{p_1q_1}(X) \to B^{s_2}_{p_2q_2}(X)) \le Ce_k(\mathrm{id} : l_{q_1}(2^{\nu\delta} l_{p_1}^{\widetilde{M}_\nu}) \to l_{q_2}(l_{p_2}^{\widetilde{M}_\nu})) \le Ck^{-(s_1-s_2)/d}.$$

This finishes the proof of Case 1.

Case 2: $p_2 < p_1$. In this case, we first show that

$$B^{s_2}_{p_1q_2}(X) \subset B^{s_2}_{p_2q_2}(X). \tag{5.6}$$

To show (5.6), let $\{S_k\}_{k=0}^\infty$ be an approximation to the identity as in Definition 1.2. Let $E_k = S_k - S_{k-1}$ for $k \in \mathbb{N}$ and $E_0 = S_0$. Since $\mu(X) < \infty$, by Hölder's inequality, we have

$$\|E_k(f)\|_{L^{p_2}(X)} \le \|E_k(f)\|_{L^{p_1}(X)} \mu(X)^{1/p_2 - 1/p_1}.$$

From this and Definition 1.3, we have (5.6). Now our result in this case can be deduced from (5.6) and Case 1 applied to $p_1 = p_2$.

This finishes the proof of Proposition 5.1.

Now we are going to use Proposition 5.1, Theorem 1.1 and Lemma 1.10 to establish lower estimates for those entropy numbers in Proposition 5.1; see also Theorem 20.6 and Theorem 23.2 in [33]. We also remark that if X is a d-set and $0 < s < 1$, our results on $B^s_{pq}(X)$ are included in Theorem 20.6 in [33] and the other cases are new.

THEOREM 5.1. *Let $(X, \varrho, \mu)_{d,\theta}$ be a space of homogeneous type with $\mu(X) < \infty$. Let $B^s_{pq}(X)$ for $1 \le p, q \le \infty$ and $F^s_{pq}(X)$ for $1 < p < \infty$ and $1 < q \le \infty$ be the spaces in Definition 1.3 with $|s| < \theta$. Let $-\theta < s_2 < s_1 < \theta$.*

(i) *If $1 \le p_1, p_2 \le \infty$, $1 \le q_1, q_2 \le \infty$ and*

$$\delta_+ = s_1 - s_2 - d\left(\frac{1}{p_1} - \frac{1}{p_2}\right)_+ > 0,$$

then the embedding of $B^{s_1}_{p_1q_1}(X)$ into $B^{s_2}_{p_2q_2}(X)$ is compact and

$$e_k(\mathrm{id} : B^{s_1}_{p_1q_1}(X) \to B^{s_2}_{p_2q_2}(X)) \sim k^{-(s_1-s_2)/d} \quad \text{for all } k \in \mathbb{N}.$$

(ii) *If $1 < p_1, p_2 < \infty$, $1 < q_1, q_2 \le \infty$ and $\delta_+ > 0$, then the embedding of $F^{s_1}_{p_1q_1}(X)$ into $F^{s_2}_{p_2q_2}(X)$ is compact and*

$$e_k(\mathrm{id} : F^{s_1}_{p_1q_1}(X) \to F^{s_2}_{p_2q_2}(X)) \sim k^{-(s_1-s_2)/d} \quad \text{for all } k \in \mathbb{N}.$$

Proof. Similarly to the proof of Proposition 5.1, we only need to show (i). The estimate of e_k from above by $Ck^{-(s_1-s_2)/d}$ is covered by Proposition 5.1. To establish the estimate from below, we use some ideas of the proofs of Theorems 20.6 and 23.2 in [33]. We have to show that there is a constant $C > 0$ such that

$$e_k(\mathrm{id} : B^{s_1}_{p_1q_1}(X) \to B^{s_2}_{p_2q_2}(X))k^{(s_1-s_2)/d} \geq C$$

for all $k \in \mathbb{N}$. Assume that there is no such $C > 0$. Then we find a sequence $k_j \to \infty$ such that

$$e_{k_j}(\mathrm{id} : B^{s_1}_{p_1q_1}(X) \to B^{s_2}_{p_2q_2}(X))k_j^{(s_1-s_2)/d} \to 0 \tag{5.7}$$

as $j \to \infty$. We can always find $\theta > s_3 > s_1$ and $-\theta < s_4 < s_2$ such that by Proposition 5.1, for $k \in \mathbb{N}$,

$$e_k(\mathrm{id} : B^{s_3}_{22}(X) \to B^{s_1}_{p_1q_1}(X)) \leq Ck^{-(s_3-s_1)/d}, \tag{5.8}$$

$$e_k(\mathrm{id} : B^{s_2}_{P_2q_2}(X) \to B^{s_4}_{22}(X)) \leq Ck^{-(s_4-s_2)/d}. \tag{5.9}$$

By (5.7)–(5.9) and the multiplication property of entropy numbers (see (5.8) in [33] or [6]),

$$e_{3k_j}(\mathrm{id} : B^{s_3}_{22}(X) \to B^{s_4}_{22}(X))k_j^{(s_3-s_4)/d} \to 0 \tag{5.10}$$

as $j \to \infty$. We may assume $s_4 < 0 < s_3$. By Lemma 1.10, we have

$$L^2(X) = F^0_{22}(X) = B^0_{22}(X).$$

Taking $\sigma \in (0, 1)$ such that $(1 - \sigma)s_3 + \sigma s_4 = 0$, by Definition 1.3, we obtain

$$\|f\|_{L^2(X)} \leq C\|f\|^{1-\theta}_{B^{s_3}_{22}(X)}\|f\|^{\theta}_{B^{s_4}_{22}(X)}. \tag{5.11}$$

By the interpolation property for entropy numbers in [6, p. 13], we deduce from (5.10) and (5.11) that

$$\begin{aligned} clk_j^{s_3/d}e_{3k_j}&(\mathrm{id} : B^{s_3}_{22}(X) \to L^2(X)) \\ &\leq C[e_{3k_j}(\mathrm{id} : B^{s_3}_{22}(X) \to B^{s_4}_{22}(X))k_j^{(s_3-s_4)/d}]^{\theta} \to 0 \end{aligned} \tag{5.12}$$

as $j \to \infty$. We will show this is impossible. Choose two C^∞ nonnegative functions, φ and ψ, on $\mathbb{R}$ with supports in $(-a_0, a_0)$, where a_0 is as in Lemma 1.3. Then choose $C_{j,\tau}$ such that

$$C_{j,\tau}2^{jd}\int_X \varphi(2^j\varrho(x, z^j_\tau))\psi(2^j\varrho(x, z^j_\tau))\, d\mu(x) = 1 \tag{5.13}$$

for $j \in \mathbb{N} \cup \{0\}$ and $\tau \in M_j$, where $M_j \sim 2^{jd}$; see the proof of Proposition 5.1. Here, as in the proof of Proposition 5.1, we identify M_k of Lemma 1.3 with the set $\{1, \ldots, M_k\}$ for $k \in \mathbb{N} \cup \{0\}$. Moreover, we can suppose $\varphi(x) \geq C$ and $\psi(x) \geq C$ when $x \in (-a_0/2, a_0/2)$. Then, by (0.1), we may assume that there are constants $0 < C_1 \leq C_2 < \infty$ such that $C_1 \leq C_{j,\tau} \leq C_2$ for all $j \in \mathbb{N} \cup \{0\}$ and $\tau \in M_j$. We now define a linear operator A from $2^{j(s_3-d/2)}l_2^{M_j}$ to $B^{s_3}_{22}(X)$ by letting

$$A\{a_\tau : \tau = 1, \ldots, M_j\} = \sum_{\tau=1}^{M_j} a_\tau\varphi(2^j\varrho(x, z^j_\tau))$$

and a linear operator B from $L^2(X)$ into $2^{-jd/2}l_2^{M_j}$ by letting

$$Bf = \Big\{C_{j,\tau}2^{jd}\int_X f(x)\psi(2^j\varrho(x,z_\tau^j))\,d\mu(x) : \tau = 1,\dots,M_j\Big\}.$$

Noting that $2^{jd/2}\varphi(2^j\varrho(x,z_\tau^j))$ is an ε-block for Q_τ^j, multiplied with an unimportant normalizing constant, by Theorem 1.1, we have

$$\|A\{a_\tau : \tau = 1,\dots,M_j\}\|_{B_{22}^{s_3}(X)} \le C2^{j(s_3-d/2)}\|\{a_\tau : \tau = 1,\dots,M_j\}\|_{l_2^{M_j}},$$

where C is independent of j. Now, let

$$b_\tau^j = C_{j,\tau}2^{jd}\int_X f(x)\psi(2^j\varrho(x,z_\tau^j))\,d\mu(x).$$

By Lemma 1.3(v), if $\tau_1 \ne \tau_2$, then

$$\operatorname{supp}\psi(2^j\varrho(\cdot,z_{\tau_1}^j)) \cap \operatorname{supp}\psi(2^j\varrho(\cdot,z_{\tau_2}^j)) = \emptyset.$$

By this fact and Hölder's inequality, we have

$$|b_\tau^j|^2 \le C2^{2jd}\int_{\{x:\,\varrho(x,z_{\tau_2}^j)\le a_0 2^{-j}\}} |f(x)|^2\,d\mu(x)2^{-jd} \le C2^{jd}\int_{Q_\tau^j}|f(x)|^2\,d\mu(x)$$

and

$$\|Bf\|_{l_2^{M_j}} = \Big(\sum_{\tau=1}^{M_j}|b_\tau^j|^2\Big)^{1/2} \le C2^{jd/2}\|f\|_{L^2(X)},$$

where Q_τ^j is as in Lemma 1.3 and C is independent of j. Thus, A and B are bounded linear operators with operator norms independent of j. Moreover, if we let id^j be the embedding from $2^{j(s_3-d/2)}l_2^{M_j}$ to $2^{-jd/2}l_2^{M_j}$ and id be the embedding from $B_{22}^{s_3}(X)$ to $L^2(X)$, then, by (5.13), we have $\mathrm{id}^j = B\circ \mathrm{id}\circ A$ and consequently, by Proposition 6.4 in [33], we have

$$e_k(\mathrm{id}^j) \le Ce_k(\mathrm{id}) \quad \text{for all } k\in\mathbb{N}, \tag{5.14}$$

where C is independent of j and k. By Proposition 5.2 with $k = 2M_j \sim 2^{jd}$ in [33], we obtain

$$e_{C2^{jd}}(\mathrm{id}^j) = 2^{-j(s_3-d/2)}2^{-jd/2}e_{c2^{jd}}(\mathrm{id} : l_2^{M_j} \to l_2^{M_j}) \ge C'2^{-js_3}, \tag{5.15}$$

where $C > 0$ and $C' > 0$ are independent of j. By (5.15) and (5.14), it is easy to deduce that there is a constant $C > 0$ such that for all $k\in\mathbb{N}$,

$$e_k(\mathrm{id} : B_{22}^{s_3}(X) \to L^2(X)) \ge Ck^{-s_3/d},$$

which implies that (5.12) is impossible.

This finishes the proof of Theorem 5.1.

Now, let us consider some limiting embeddings between these spaces which correspond to the case $\delta_+ = 0$ of Theorem 5.1. We first have the following theorem; see [17] for its homogeneous version. The main idea of the proof is also similar to that in [17]. For completeness, we give the details. Moreover, we correct a mistake in the proof in [17].

THEOREM 5.2. *Let $(X, \varrho, \mu)_{d,\theta}$ be a space of homogeneous type. Let $B^s_{pq}(X)$ for $1 \le p, q \le \infty$ and $F^s_{pq}(X)$ for $1 < p < \infty$ and $1 < q \le \infty$ be the spaces as in Definition* 1.3 *with $|s| < \theta$. Let $-\theta < s_2 < s_1 < \theta$. Then*

(i) $B^{s_1}_{p_1 q}(X) \subset B^{s_2}_{p_2 q}(X)$ *for* $1 \le q \le \infty$, $1 \le p_1, p_2 \le \infty$ *and* $-\theta < s_1 - d/p_1 = s_2 - d/p_2 < \theta$;

(ii) $F^{s_1}_{p_1 q_1}(X) \subset F^{s_2}_{p_2 q_2}(X)$ *for* $1 < p_1, p_2 < \infty$, $1 < q_1, q_2 \le \infty$ *and* $-\theta < s_1 - d/p_1 = s_2 - d/p_2 < \theta$.

Proof. We use the inhomogeneous Calderón reproducing formulae of [18]. Suppose $\{S_k\}_{k=0}^\infty$ is an approximation to the identity with $\varepsilon \in (0, \theta]$. Let $E_k = S_k - S_{k-1}$ for $k \in \mathbb{N}$ and $E_0 = S_0$. Then by Lemma 1.2, for all $f \in (\mathcal{G}(\beta, \gamma))'$ with $0 < \beta, \gamma < \varepsilon$, there is a sequence of linear operators $\{\widetilde{E}_k\}_{k=0}^\infty$ and an $N \in \mathbb{N}$ such that

$$f = \sum_{k=0}^{\infty} \widetilde{E}_k E_k(f) \tag{5.16}$$

in $(\mathcal{G}(\beta, \gamma))'$ with $\beta' > \beta$ and $\gamma' > \gamma$, where the kernel, $\widetilde{E}_k(x, y)$, of $\widetilde{E}_k$ satisfies

$$\int_X \widetilde{E}_k(x, y)\, d\mu(y) = \int_X \widetilde{E}_k(x, y)\, d\mu(x) = \begin{cases} 1 & \text{if } k = 0, 1, \ldots, N, \\ 0 & \text{if } k = N + 1, \ldots, \end{cases}$$

and (i) and (ii) of Remark 1.1 with ε replaced by any $\varepsilon' \in (0, \varepsilon)$. We take $\varepsilon' \in (0, \varepsilon)$ such that $-\varepsilon' < s_2 < s_1 < \varepsilon'$ and $-\varepsilon' < s_1 - d/p_1 = s_2 - d/p_2 < \varepsilon'$. By a similar proof to (2.15) (see also (2.5) in [17] and (1.6) in [20]), we can show that for all $k, j \in \mathbb{N} \cup \{0\}$,

$$|(E_k \widetilde{E}_j)(x, y)| \le C 2^{-|k-j|\varepsilon'} \frac{2^{-(k \wedge j)\varepsilon'}}{(2^{-(k \wedge j)} + \varrho(x, y))^{d+\varepsilon'}}, \tag{5.17}$$

where C is independent of k, j, x and y. Noting that $p_2/p_1 > 1$, by (5.16), Hölder's inequality, Young's inequality and (5.17), for any $k \in \mathbb{N} \cup \{0\}$,

$$\begin{aligned}
\|E_k(f)\|_{L^{p_2}(X)} &= \Big\| \sum_{j=0}^{\infty} E_k \widetilde{E}_j E_j(f) \Big\|_{L^{p_2}(X)} \\
&\le \sum_{j=0}^{\infty} \Big\{ \int_X \Big[\int_X |(E_k \widetilde{E}_j)(x, y)| |E_j(f)(y)|^{p_1}\, d\mu(y) \Big]^{p_2/p_1} \\
&\quad \times \Big[\int_X |(E_k \widetilde{E}_j)(x, y)|\, d\mu(y) \Big]^{p_2/p_1'} d\mu(x) \Big\}^{1/p_2} \\
&\le C \sum_{j=0}^{\infty} 2^{-|k-j|\varepsilon'/p_1'} \Big\{ \int_X \Big[\int_X |(E_k \widetilde{E}_j)(x, y)| |E_j(f)(y)|^{p_1}\, d\mu(y) \Big]^{p_2/p_1} d\mu(x) \Big\}^{1/p_2} \\
&\le C \sum_{j=0}^{\infty} 2^{-|k-j|\varepsilon'/p_1'} \Big\{ \int_X |E_j(f)(y)|^{p_1} \Big[\int_X |(E_k \widetilde{E}_j)(x, y)|^{p_2/p_1}\, d\mu(x) \Big]^{p_1/p_2} d\mu(y) \Big\}^{1/p_1} \\
&\le C \sum_{j=0}^{\infty} 2^{-|k-j|\varepsilon'} 2^{-(k \wedge j) d(1/p_2 - 1/p_1)} \|E_j(f)\|_{L^{p_1}(X)},
\end{aligned}$$

where $1/p_1 + 1/p_1' = 1$.

From this and $s_1 - d/p_1 = s_2 - d/p_2$, it follows that

$$\begin{aligned}\|f\|_{B^{s_2}_{p_2 q}(X)} &= \Big\{\sum_{k=0}^{\infty} 2^{ks_2 q}\|E_k(f)\|^q_{L^{p_2}(X)}\Big\}^{1/q} \\ &\le C\Big\{\sum_{k=0}^{\infty} 2^{ks_2 q}\Big[\sum_{j=0}^{\infty} 2^{-|k-j|\varepsilon'} 2^{-(k\wedge j)d(1/p_2-1/p_1)}\|E_j(f)\|_{L^{p_1}(X)}\Big]^q\Big\}^{1/q} \\ &\le C\Big\{\sum_{k=0}^{\infty}\Big[\sum_{j=0}^{k} 2^{(k-j)(s_2-\varepsilon')} 2^{js_1}\|E_j(f)\|_{L^{p_1}(X)}\Big]^q\Big\}^{1/q} \\ &\quad + C\Big\{\sum_{k=0}^{\infty}\Big[\sum_{j=k+1}^{\infty} 2^{-(j-k)(s_1+\varepsilon')} 2^{js_1}\|E_j(f)\|_{L^{p_1}(X)}\Big]^q\Big\}^{1/q} \\ &\le C\Big\{\sum_{j=0}^{\infty} 2^{js_1 q}\|E_j(f)\|^q_{L^{p_1}(X)}\Big\}^{1/q} \le C\|f\|_{B^{s_1}_{p_1 q}(X)},\end{aligned}$$

where we used the Young inequality for number sequences, and C is independent of f. This proves (i).

To prove (ii), by homogeneity, without loss of generality, we may suppose $\|f\|_{F^{s_1}_{p_1 q_1}(X)} = 1$. From this, (5.16), Hölder's inequality and (5.17), we deduce that for any $k \in \mathbb{N}\cup\{0\}$ and any $x \in X$,

$$\begin{aligned}|E_k(f)(x)| &= \Big|\sum_{j=0}^{\infty} E_k\widetilde{E}_j E_j(f)(x)\Big| \le C\sum_{j=0}^{\infty} 2^{-|k-j|\varepsilon'} 2^{(k\wedge j)d/p_1}\|E_j(f)\|_{L^{p_1}(X)} \\ &\le C\sum_{j=0}^{\infty} 2^{-|k-j|\varepsilon'} 2^{(k\wedge j)d/p_1} 2^{-js_1}.\end{aligned}$$

Thus, for any fixed $N \in \mathbb{N}\cup\{0\}$, we have

$$\begin{aligned}&\Big\{\sum_{k=0}^{N} 2^{ks_2 q_2}|E_k(f)(x)|^{q_2}\Big\}^{1/q_2} \\ &\le C\Big\{\sum_{k=0}^{N} 2^{ks_2 q_2}\Big|\sum_{j=0}^{\infty} 2^{-|k-j|\varepsilon'} 2^{(k\wedge j)d/p_1} 2^{-js_1}\Big|^{q_2}\Big\}^{1/q_2} \\ &\le C\Big\{\sum_{k=0}^{N} 2^{ks_2 q_2}\Big|\sum_{j=0}^{k} 2^{-(k-j)\varepsilon'} 2^{jd/p_1} 2^{-js_1}\Big|^{q_2}\Big\}^{1/q_2} \\ &\quad + C\Big\{\sum_{k=0}^{N} 2^{ks_2 q_2}\Big|\sum_{j=k+1}^{\infty} 2^{-(j-k)\varepsilon'} 2^{kd/p_1} 2^{-js_1}\Big|^{q_2}\Big\}^{1/q_2} \\ &= C\Big\{\sum_{k=0}^{N} 2^{kdq_2/p_2}\Big[\sum_{j=0}^{k} 2^{-(k-j)(\varepsilon'-s_1+d/p_1)}\Big]^{q_2}\Big\}^{1/q_2} \\ &\quad + C\Big\{\sum_{k=0}^{N} 2^{kdq_2/p_2}\Big[\sum_{j=k+1}^{\infty} 2^{-(j-k)(\varepsilon'+s_1)}\Big]^{q_2}\Big\}^{1/q_2} \\ &\le C\Big\{\sum_{k=0}^{N} 2^{kdq_2/p_2}\Big\}^{1/q_2} = C_0 2^{Nd/p_2},\end{aligned} \tag{5.18}$$

since $s_1 > -\varepsilon'$ and $\varepsilon' > s_1 - d/p_1$, where C_0 is independent of N and we have used the fact that $s_1 - d/p_1 = s_2 - d/p_2$.

On the other hand, for any $N \in \mathbb{N} \cup \{0\}$,

$$\begin{aligned}(5.19)\qquad &\Big\{\sum_{k=N+1}^{\infty} 2^{ks_2q_2}|E_k(f)(x)|^{q_2}\Big\}^{1/q_2}\\ &\le \Big\{\sum_{k=N+1}^{\infty} 2^{k(s_2-s_1)q_2}2^{ks_1q_2}|E_k(f)(x)|^{q_2}\Big\}^{1/q_2}\\ &\le \Big\{\sum_{k=N+1}^{\infty} 2^{k(s_2-s_1)q_2}\Big\}^{1/q_2}\Big\{\sum_{k=0}^{\infty} 2^{ks_1q_1}|E_k(f)(x)|^{q_1}\Big\}^{1/q_1}\\ &\le C_0 2^{N(s_2-s_1)}\Big\{\sum_{k=0}^{\infty} 2^{ks_1q_1}|E_k(f)(x)|^{q_1}\Big\}^{1/q_1},\end{aligned}$$

since $s_2 < s_1$, where C_0 is independent of N. In particular,

$$\begin{aligned}(5.20)\qquad &\Big\{\sum_{k=0}^{\infty} 2^{ks_2q_2}|E_k(f)(x)|^{q_2}\Big\}^{1/q_2}\\ &\le \Big\{\sum_{k=0}^{\infty} 2^{k(s_2-s_1)q_2}\Big\}^{1/q_2}\Big\{\sum_{k=0}^{\infty} 2^{ks_1q_1}|E_k(f)(x)|^{q_1}\Big\}^{1/q_1}\\ &\le C_0\Big\{\sum_{k=0}^{\infty} 2^{ks_1q_1}|E_k(f)(x)|^{q_1}\Big\}^{1/q_1}.\end{aligned}$$

Thus, noting that $p_2 > p_1$, by (5.18)–(5.20), we have

$$\begin{aligned}\|f\|^{p_2}_{F^{s_2}_{p_2q_2}(X)} &= p_2\int_0^{\infty} t^{p_2-1}\mu\Big(\Big\{x\in X:\Big[\sum_{k=0}^{\infty} 2^{ks_2q_2}|E_k(f)(x)|^{q_2}\Big]^{1/q_2} > t\Big\}\Big)\,dt\\ &= p_2\int_0^{2C_0} t^{p_2-1}\mu\Big(\Big\{x\in X:\Big[\sum_{k=0}^{\infty} 2^{ks_2q_2}|E_k(f)(x)|^{q_2}\Big]^{1/q_2} > t\Big\}\Big)\,dt\\ &\quad+\sum_{N=0}^{\infty}\int_{2C_02^{Nd/p_2}}^{2C_02^{(N+1)d/p_2}} t^{p_2-1}\mu\Big(\Big\{x\in X:\Big[\sum_{k=0}^{N} 2^{ks_2q_2}|E_k(f)(x)|^{q_2}\Big]^{1/q_2}\\ &\quad+\Big[\sum_{k=N+1}^{\infty} 2^{ks_2q_2}|E_k(f)(x)|^{q_2}\Big]^{1/q_2} > t\Big\}\Big)\,dt\\ &\le p_2(2C_0)^{p_2-p_1}\int_0^{2C_0} t^{p_1-1}\mu\Big(\Big\{x\in X:\Big[\sum_{k=0}^{\infty} 2^{ks_1q_1}|E_k(f)(x)|^{q_1}\Big]^{1/q_1} > t/C_0\Big\}\Big)\,dt\\ &\quad+\sum_{N=0}^{\infty}\int_{2C_02^{Nd/p_2}}^{2C_02^{(N+1)d/p_2}} t^{p_2-1}\mu\Big(\Big\{x\in X:\Big[\sum_{k=N+1}^{\infty} 2^{ks_2q_2}|E_k(f)(x)|^{q_2}\Big]^{1/q_2} > t/2\Big\}\Big)\,dt\\ &\le C + C\sum_{N=0}^{\infty}\int_{2C_02^{Nd/p_2}}^{2C_02^{(N+1)d/p_2}} t^{p_2-1}\end{aligned}$$

$$\times\,\mu\Big(\Big\{x\in X:\Big[\sum_{k=N+1}^{\infty}2^{ks_1q_1}|E_k(f)(x)|^{q_1}\Big]^{1/q_1}>t2^{N(s_1-s_2)}/(2C_0)\Big\}\Big)\,dt$$

$$\le C+C\sum_{N=0}^{\infty}\int\limits_{2^{Nd/p_1}}^{2^{(N+1)d/p_1}}t^{p_1-1}\mu\Big(\Big\{x\in X:\Big[\sum_{k=N+1}^{\infty}2^{ks_1q_1}|E_k(f)(x)|^{q_1}\Big]^{1/q_1}>t\Big\}\Big)\,dt\le C.$$

This proves (ii) and finishes the proof of Theorem 5.2.

We remark that Theorem 5.2 is true even when $\mu(X)=\infty$. However, the embeddings in Theorem 5.2 cannot be compact even when X is a compact space of homogeneous type. For example, when X is a d-set, one can find a proof of this fact in [33, pp. 169–170].

Now we consider some limiting compact embeddings for spaces of homogeneous type. First, we need to estimate some embedding constants. Let $\max(1,d)<p\le q<\infty$. Then by Theorem 5.2 and Proposition 1.2, $B^{d/p}_{pp}(X)\subset L^q(X)$ and $F^{d/p}_{p2}(X)\subset L^q(X)$. Let $\mathrm{id}_{p,q}$ be one of these embedding operators. Our theorem below corresponds to Theorem 2.7.2 in [6], but our proof is essentially different. The key for the proof in [6] is Nikol'skiĭ's well known inequality for L^q functions with Fourier transforms having compact supports; see [6] and [31]. Since there is no theory of the Fourier transform on spaces of homogeneous type, we use approximations to the identity and the inhomogeneous Calderón reproducing formulae of [18]. The main ideas of our proof are similar to that of Theorem 5.2.

Theorem 5.3. *Let* $\max(1,d)<p<\infty$. *Then there is a constant* $C>0$ *depending on* p *such that*

$$\|\mathrm{id}_{p,q}\|\le Cq^{1-1/p}\quad\textit{for every } q \textit{ with } p\le q<\infty. \tag{5.21}$$

Proof. We use the notation of the proof of Theorem 5.2. Let $\{E_k\}_{k=0}^{\infty}$ and $\{\widetilde{E}_k\}_{k=0}^{\infty}$ be as in that proof. By (5.17), Hölder's inequality and Young's inequality, we have

$$\|E_k\widetilde{E}_jE_j(f)\|_{L^q(X)}=\Big\{\int\limits_X\Big|\int\limits_X(E_k\widetilde{E}_j)(x,y)E_j(f)(y)\,d\mu(y)\Big|^q\,d\mu(x)\Big\}^{1/q}$$
$$\le\Big\{\int\limits_X\Big[\int\limits_X|(E_k\widetilde{E}_j)(x,y)||E_j(f)(y)|^p\,d\mu(y)\Big]^{q/p}\Big[\int\limits_X|(E_k\widetilde{E}_j)(x,y)|\,d\mu(y)\Big]^{q/p'}\,d\mu(x)\Big\}^{1/q}$$
$$\le C2^{-|k-j|\varepsilon'/p'}\Big\{\int\limits_X\Big[\int\limits_X|(E_k\widetilde{E}_j)(x,y)||E_j(f)(y)|^p\,d\mu(y)\Big]^{q/p}\,d\mu(x)\Big\}^{1/q}$$
$$\le C2^{-|k-j|\varepsilon'/p'}\Big\{\int\limits_X|E_j(f)(y)|^p\Big[\int\limits_X|(E_k\widetilde{E}_j)(x,y)|^{q/p}\,d\mu(x)\Big]^{p/q}\,d\mu(y)\Big\}^{1/p}$$
$$\le C2^{-|k-j|\varepsilon'}2^{-(k\wedge j)d(1/q-1/p)}\|E_j(f)\|_{L^p(X)},$$

where $1/p_1+1/p_1'=1$, $\varepsilon'>0$ is as in (5.17) and C is independent of q.

From this, (5.16) and Hölder's inequality, we deduce that

$$\|f\|_{L^q(X)}\le\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\|E_k\widetilde{E}_jE_j(f)\|_{L^q(X)} \tag{5.22}$$
$$=\sum_{j=0}^{\infty}\sum_{k=0}^{j}\|E_k\widetilde{E}_jE_j(f)\|_{L^q(X)}+\sum_{j=0}^{\infty}\sum_{k=j+1}^{\infty}\cdots$$

$$\le C\sum_{j=0}^{\infty}\sum_{k=0}^{j}2^{-(j-k)\varepsilon'}2^{-kd(1/q-1/p)}\|E_j(f)\|_{L^p(X)}$$
$$+C\sum_{j=0}^{\infty}\sum_{k=j+1}^{\infty}2^{-(k-j)\varepsilon'}2^{-jd(1/q-1/p)}\|E_j(f)\|_{L^p(X)}$$
$$\le C\sum_{j=0}^{\infty}2^{-jd(1/q-1/p)}\|E_j(f)\|_{L^p(X)}$$
$$\le C\Big\{\sum_{j=0}^{\infty}2^{-jdp'/q}\Big\}^{1/p'}\Big\{\sum_{j=0}^{\infty}2^{jd}\|E_j(f)\|^p_{L^p(X)}\Big\}^{1/p}\le Cq^{1-1/p}\|f\|_{B^{d/p}_{pp}(X)},$$

where C is independent of q. This shows (5.21) in the case of $B^{d/p}_{pp}(X)$.

Now let us prove (5.21) for $F^{d/p}_{p2}(X)$. If $d\ge 2$, by (5.1) and (5.21) in the case of $B^{d/p}_{pp}(X)$, we deduce (5.21) for $F^{d/p}_{p2}(X)$ and $d<p<\infty$. If $0<d<2$, by (5.1) and (5.32) in the case of $B^{d/p}_{pp}(X)$, we deduce (5.32) for $F^{d/p}_{p2}(X)$ and $2\le p<\infty$. We still need to show (5.32) in the case of $F^{d/p}_{p2}(X)$ for $0<d<2$, $\max(1,d)<p<2$ and $p\le q<\infty$. We only show this for $2\le q<\infty$. The extension to the case $p\le q<2$ is obvious. We need to establish an inequality similar to (5.22) with $B^{d/p}_{pp}(X)$ replaced by $F^{d/p}_{p2}(X)$. By homogeneity, we may suppose $\|f\|_{F^{d/p}_{p2}(X)}=1$. From (5.16), Hölder's inequality and (5.17), we deduce that for any $k\in\mathbb{N}\cup\{0\}$ and any $x\in X$,

$$|E_k(f)(x)|=\Big|\sum_{j=0}^{\infty}E_k\widetilde{E}_jE_j(f)(x)\Big|\le C\sum_{j=0}^{\infty}2^{-|k-j|\varepsilon'}2^{(k\wedge j)d/p}\|E_j(f)\|_{L^p(X)}$$
$$\le C\sum_{j=0}^{\infty}2^{-|k-j|\varepsilon'}2^{(k\wedge j)d/p}2^{-jd/p}, \tag{5.23}$$

where $\varepsilon'\in(0,\varepsilon)$, C is independent of q, and C depends on ε'.

Thus, for any fixed $N\in\mathbb{N}\cup\{0\}$, by (5.16), (5.23) and Hölder's inequality, we have

$$\sum_{k=0}^{N}|E_k(f)(x)|\le C\sum_{k=0}^{N}\sum_{j=0}^{\infty}|E_k\widetilde{E}_jE_j(f)(x)|$$
$$\le C\sum_{k=0}^{N}\sum_{j=0}^{k}2^{-(k-j)\varepsilon'}+C\sum_{k=0}^{N}\sum_{j=k+1}^{\infty}2^{-(j-k)\varepsilon'}2^{kd/p}2^{-jd/p}=C_1N, \tag{5.24}$$

where C_1 is independent of N and q. We also have

$$\sum_{k=N+1}^{\infty}|E_k(f)(x)|\le C\sum_{k=N+1}^{\infty}2^{-kd/p}\Big\{\sum_{k=0}^{\infty}2^{2kd/p}|E_k(f)(x)|^2\Big\}^{1/2}$$
$$\le C_12^{-Nd/p}\Big\{\sum_{k=0}^{\infty}2^{2kd/p}|E_k(f)(x)|^2\Big\}^{1/2}, \tag{5.25}$$

and in particular,

$$\sum_{k=0}^{\infty}|E_k(f)(x)|\le C_1\Big\{\sum_{k=0}^{\infty}2^{2kd/p}|E_k(f)(x)|^2\Big\}^{1/2}, \tag{5.26}$$

where C_1 is independent of N and q.

By (5.24)–(5.26), we have

$$\begin{aligned}
\|f\|^q_{L^q(X)} &= q\int_0^\infty t^{q-1}\mu(\{x\in X: |f(x)|>t\})\,dt\\
&\le q\int_0^{2C_1} t^{q-1}\mu(\{x\in X: |f(x)|>t\})\,dt + \sum_{N=1}^\infty q\int_{2C_1N}^{2C_1(N+1)}\cdots\\
&\le q(2C_1)^{q-p}\int_0^{2C_1} t^{p-1}\mu\Big(\Big\{x\in X: C_1\Big[\sum_{k=0}^\infty 2^{2kd/p}|E_k(f)(x)|^2\Big]^{1/2}>t\Big\}\Big)\,dt\\
&\quad+\sum_{N=1}^\infty q\int_{2C_1N}^{2C_1(N+1)} t^{q-1}\mu\Big(\Big\{x\in X: \sum_{k=0}^N|E_k(f)(x)|+\sum_{k=N+1}^\infty|E_k(f)(x)|>t\Big\}\Big)\,dt\\
&\le C(2C_1)^{q-p}q+\sum_{N=1}^\infty q\int_{2C_1N}^{2C_1(N+1)} t^{q-1}\mu\Big(\Big\{x\in X: \sum_{k=N+1}^\infty|E_k(f)(x)|>t/2\Big\}\Big)\,dt\\
&\le C(2C_1)^{q-p}q+\sum_{N=1}^\infty q\int_{2C_1N}^{2C_1(N+1)} t^{q-1}\\
&\quad\times\mu\Big(\Big\{x\in X: C_12^{-Nd/p}\Big[\sum_{k=0}^\infty 2^{2kd/p}|E_k(f)(x)|^2\Big]^{1/2}>t/2\Big\}\Big)\,dt\\
&\le C(2C_1)^{q-p}q+C\sum_{N=1}^\infty q\frac{(2C_1)^qN^{q-p}}{2^{Nd}}\int_{N2^{Nd/p}}^{(N+1)2^{Nd/p}} t^{p-1}\\
&\quad\times\mu\Big(\Big\{x\in X: \Big[\sum_{k=0}^\infty 2^{2kd/p}|E_k(f)(x)|^2\Big]^{1/2}>t\Big\}\Big)\,dt,
\end{aligned}$$

where C and C_1 are independent of q. Moreover, it is easy to see that there is a constant $C_{p,d}$ independent of q and N such that

$$N\le C_{p,d}(q-p)^{1-1/p}2^{Nd/(q-p)}.$$

Hence,

$$\|f\|^q_{L^q(X)}\le C(2C_1)^{q-p}q+Cq(2C_1)^qC_{p,d}^{q-p}(q-p)^{(1-1/p)(q-p)}.$$

Noting that $q^{1/q}\le C$ and $(q-p)^{-p(1-1/p)/q}\le C_p$, where C and C_p are independent of q, we have

$$\|f\|_{L^q(X)}\le Cq^{1-1/p}$$

and (5.32) holds for $2\le q<\infty$.

This finishes the proof of Theorem 5.3.

Based on this theorem, we can now consider some limiting compact embeddings. Let us first recall the definition of the spaces $L^p(\log L)_a(X)$; see [6], [33], [24] and [1].

DEFINITION 5.2. Let $(X,\varrho,\mu)_{d,\theta}$ be a space of homogeneous type as in Definition 0.1 with $\mu(X)<\infty$.

(i) Let $0 < p < \infty$ and $a \in \mathbb{R}$. Then $L^p(\log L)_a(X)$ is the set of all μ-measurable complex-valued functions f such that

$$\int_X |f(x)|^p \log^{ap}(2 + |f(x)|)\, d\mu(x) < \infty.$$

(ii) Let $a < 0$. Then $L^\infty(\log L)_a(X)$ is the set of all μ-measurable complex-valued functions f for which there exists a constant $\lambda > 0$ such that

$$\int_X \exp\{[\lambda|f(x)|]^{-1/a}\}\, d\mu(x) < \infty.$$

This is just a special case of Definition 6.11 in [1, p. 252]; see also [24] for another equivalent definition. By introducing some equivalent norms in these spaces, one can show that if $1 < p < \infty$ and $a \in \mathbb{R}$, or $p = \infty$ and $a < 0$, or $p = 1$ and $a \geq 0$, then the spaces $L^p(\log L)_a(X)$ can be regarded as Banach spaces; see Theorem 8.3 in [1] and [6, pp. 66–67]. One can also find some basic properties of these spaces in [6]. For example, by the above definition, one can easily show the following proposition; see Proposition 1 in [6, p. 67].

PROPOSITION 5.2. *Let $(X, \varrho, \mu)_{d,\theta}$ be a space of homogeneous type with $\mu(X) < \infty$.*

(i) *Let $0 < \sigma < p < \infty$ and $-\infty < a_2 < a_1 < \infty$. Then*

$$L^{p+\sigma}(X) \subset L^p(\log L)_{a_1}(X) \subset L^p(\log L)_{a_2}(X) \subset L^{p-\sigma}(X),$$
$$L^p(\log L)_\sigma(X) \subset L^p(X) \subset L^p(\log L)_{-\sigma}(X).$$

(ii) *Let $-\infty < b_1 < b_2 < 0$. Then*

$$L^\infty(X) \subset L^\infty(\log L)_{b_2}(X) \subset L^\infty(\log L)_{b_1}(X).$$

Moreover, one can show the following proposition by repeating the proof of Theorem 1 in Section 2.6.2 of [6]. We omit the details.

PROPOSITION 5.3. *Suppose that $0 < p \leq \infty$, $a < 0$ and $\mu(X) < \infty$. Then $L^p(\log L)_a(X)$ is the set of all measurable functions $f : X \to \mathbb{C}$ such that*

$$\left\{\int_0^\varepsilon [\sigma^{-a}\|f\|_{L^{p^\sigma}(X)}]^p \frac{d\sigma}{\sigma}\right\}^{1/p} < \infty \tag{5.27}$$

(*with the usual modification if $p = \infty$*) *for $\varepsilon > 0$, and* (5.27) *defines an equivalent quasi-norm on $L^p(\log L)_a(X)$. Furthermore,* (5.27) *can be replaced by the equivalent quasi-norm*

$$\left\{\sum_{j=J}^\infty 2^{jap}\|f\|^p_{L^{p^{\sigma_j}}(X)}\right\}^{1/p} < \infty \tag{5.28}$$

(*with the usual modification if $p = \infty$*) *for $J \in \mathbb{N}$. Here $1/p^\sigma = 1/p + \sigma/d$ and $\sigma_j = 2^{-j}$.*

Now we can establish the following limiting compact embeddings; see Theorem 2.7.3 in [6].

THEOREM 5.4. *Let $(X, \varrho, \mu)_{d,\theta}$ be a space of homogeneous type with $\mu(X) < \infty$. Let $\max(1, d) < p < \infty$ and $a < 0$.*

(i) *The embedding*

$$\mathrm{id} : B^{d/p}_{pp}(X) \to L^\infty(\log L)_a(X)$$

exists if and only if $a \le 1/p - 1$*, and it is compact if and only if* $a < 1/p - 1$.

(ii) *The embedding*

$$\mathrm{id} : F^{d/p}_{p2}(X) \to L^\infty(\log L)_a(X)$$

exists if and only if $a \le 1/p - 1$*, and it is compact if and only if* $a < 1/p - 1$.

Proof. The main idea of the proof is similar to that of Theorem 2.7.3 in [6]. We only show case (i). Obviously, (5.28) in Proposition 5.3 is equivalent to

$$\|f\|_{L^\infty(\log L)_a(X)} \sim \sup_{j\in\mathbb{N}} j^a \|f\|_{L^j(X)} \tag{5.29}$$

for $a < 0$. By (5.29), Theorem 5.3 and its proof, it is easy to see that

$$\mathrm{id} : B^{d/p}_{pp}(X) \to L^\infty(\log L)_a(X) \tag{5.30}$$

exists and is continuous if $a \le 1/p - 1$. On the other hand, if X is a bounded domain in $\mathbb{R}^n$ with C^∞ boundary, then Theorem 2.7.2 in [6] shows that the embedding in (5.30) does not exist if $a > 1/p - 1$ and is not compact if $a = 1/p - 1$. We now show that if $a < 1/p - 1$, then id in (5.30) is compact. In fact, by Theorem 5.1, if $\max(1, d) < q < \infty$, then

$$\mathrm{id} : F^{d/q}_{q2}(X) \to L^q(X)$$

is compact. By Theorem 5.2, we have

$$B^{d/p}_{pp}(X) = F^{d/p}_{pp}(X) \subset F^{d/q}_{q2}(X)$$

if $\max(1, d) < p \le q < \infty$. Thus,

$$\mathrm{id} : B^{d/p}_{pp}(X) \to L^q(X) \tag{5.31}$$

is compact. Then by (5.31) and (5.29), we can easily show that (5.30) is compact if $a < 1/p - 1$.

This finishes the proof of Theorem 5.4.

We now turn to estimating the entropy numbers for the compact embeddings in Theorem 5.4. We will consider more general cases. We first claim that if $1 < p < \infty$, $s > 0$ and $1 \le p^s \le \infty$, then the embedding

$$\mathrm{id} : B^s_{p^s 1}(X) \to L^p(X) \tag{5.32}$$

is continuous. In fact, by (5.1), Theorem 5.2 and Lemma 1.10, we have

$$B^s_{p^s 1}(X) \subset F^s_{p^s 2}(X) \subset F^0_{p2}(X) = L^p(X).$$

Noting that (5.1) is true even when $\mu(X) = \infty$, we know that (5.32) is true for both $\mu(X) = \infty$ and $\mu(X) < \infty$. But, even when $\mu(X) < \infty$, the embedding in (5.32) cannot be compact; see (21.2) in [33]. However, if we replace $L^p(X)$ by $L^p(\log L)_a(X)$ with some $a < 0$, we get a compact embedding. In fact, we will give a similar result to Theorem 21.7 in [33], which is more general than this claim. We need the fact that if $\mu(X) < \infty$, $0 < \sigma < s$, $p^s \ge 1$ and $1 \le q \le \infty$, then

$$(5.33)\quad e_k(\mathrm{id}: B^s_{p^s q}(X) \to L^{p^\sigma}(X)) \le C\delta^{-1-2(1/p^s-1/p^\sigma)}k^{-\delta/d+1/p^\sigma-1/p^s}$$
$$\le C\sigma^{-1-2(s-\sigma)/d}k^{-s/d} \le C'\sigma^{-1-2s/d}k^{-s/d}$$

for all $k \in \mathbb{N}$, where $\delta = s - d/p^s + d/p^\sigma = \sigma$ and positive constants C and C' are independent of σ. (5.33) can be proved similarly to (i) of Proposition 5.1 by replacing Theorem 9.2 in [33] by Corollary 9.4 in [33]; see also (21.14) in [33].

THEOREM 5.5. *Let $(X, \varrho, \mu)_{d,\theta}$ be a space of homogeneous type with $\mu(X) < \infty$. Let $1 < p \le \infty$, $s > 0$, $p^s \ge 1$, $1 \le q \le \infty$ and $a < -1 - 2s/d$. Then the embedding of $B^s_{p^s q}(X)$ into $L^p(\log L)_a(X)$ is compact and*

$$e_k(\mathrm{id}: B^s_{p^s q}(X) \to L^p(\log L)_a(X)) \sim k^{-s/d} \quad \textit{for all } k \in \mathbb{N}.$$

The proof is a literal repeat of Theorem 21.7 in [33] by replacing (21.14) and (21.4) in [33], respectively, by (5.33) and

$$(5.34)\quad e_k(\mathrm{id}: B^s_{p^s q}(X) \to L^{p^\sigma}(X)) \sim k^{-s/d}$$

for all $k \in \mathbb{N}$, where $0 < \sigma < s$, $p^s \ge 1$ and $1 \le q \le \infty$. (5.34) is a simple corollary of Theorem 5.1.

Similarly to Corollary 21.10 in [33], by Theorem 5.5, Proposition 5.1 and Proposition 5.3, we can also deduce the following corollary; see [33, pp. 178–179] for the details.

COROLLARY 5.1. *Let $(X, \varrho, \mu)_{d,\theta}$ be a space of homogeneous type with $\mu(X) < \infty$. Let $1 < p \le \infty$, $s > 0$, $p^s \ge 1$ and $-(d+2s)/d \le a < 0$. Then the embedding of $B^s_{p^s 1}(X)$ into $L^p(\log L)_a(X)$ is compact and for any $\varepsilon > 0$, there is a constant $C_\varepsilon > 0$ such that*

$$e_k(\mathrm{id}: B^s_{p^s 1}(X) \to L^p(\log L)_a(X)) \le C_\varepsilon k^{\frac{s}{d+2s}a+\varepsilon} \quad \textit{for all } k \in \mathbb{N}.$$

6. Relations with Sobolev spaces on metric spaces

Now let ϱ in Definition 0.1 be a metric. In this case, we can choose $\theta = 1$ in (0.2). Then $(X, \varrho, \mu)_{d,1}$ is an Ahlfors d-regular metric measure space if we further assume the Borel measure μ to be a Borel regular measure; see [25, p. 62]. But, for the rest of this section, it is enough to assume that μ is just a finite positive Borel measure. In this section, for such an Ahlfors d-regular metric measure space, we discuss the relationship between the spaces $W^{1,p}(X, \varrho, \mu)$ for $1 < p \le \infty$ defined by Hajłasz in [14] and the spaces $B^s_{pq}(X)$ and $F^s_{pq}(X)$. Let us first recall the definition of $W^{1,p}(X, \varrho, \mu)$; see [14], [16], [15] and [25].

DEFINITION 6.1. Let (X, ϱ, μ) be a metric space (X, ϱ) with a finite positive Borel measure μ and $\mu(X) < \infty$. Let $1 < p \le \infty$. The Sobolev space $W^{1,p}(X, \varrho, \mu)$ is defined by

$$\begin{aligned} W^{1,p}(X, \varrho, \mu) = \{u \in L^p(X) : {} & \text{there is a set } E \subset X,\ \mu(E) = 0, \\ & \text{and a function } g \ge 0,\ g \in L^p(X) \text{ such that} \\ & |u(x) - u(y)| \le \varrho(x, y)(g(x) + g(y)) \text{ for all } x, y \in X \setminus E\}, \end{aligned}$$

where g is called a *generalized gradient* of u. Moreover, we define

$$\|u\|_{W^{1,p}(X,\varrho,\mu)} = \|u\|_{L^p(X)} + \inf_g \|g\|_{L^p(X)},$$

where the infimum is taken over all generalized gradients of u.

The theorem below clears up the relationship between $W^{1,p}(X,\varrho,\mu)$ and the spaces $B^s_{pq}(X)$ and $F^s_{pq}(X)$.

THEOREM 6.1. *Let* $(X,\varrho,\mu)_{d,1}$ *be an Ahlfors d-regular metric measure space with* $\mu(X) < \infty$. *Then*

(i) $W^{1,p}(X,\varrho,\mu)\subset B^s_{pq}(X)$ *for* $1\le q\le\infty$, $1<p\le\infty$ *and* $-1<s<1$;
(ii) $W^{1,p}(X,\varrho,\mu)\subset F^s_{pq}(X)$ *for* $1<q\le\infty$, $1<p<\infty$ *and* $-1<s<1$.

Proof. Let $\{S_k\}_{k=0}^\infty$ be an approximation to the identity as in Definition 1.2 (or Remark 1.1) with $\varepsilon=1$. Let $E_k=S_k-S_{k-1}$ for $k\in\mathbb{N}$ and $E_0=S_0$. Let $f\in W^{1,p}(X,\varrho,\mu)$ and g be any generalized gradient of f. We first establish the estimates

$$|E_0(f)(x)|\le C\mu(x)^{1-1/p}\|f\|_{L^p(X)}, \tag{6.1}$$

$$|E_j(f)(x)|\le C2^{-j}M(g)(x)\quad \text{for } j\in\mathbb{N}, \tag{6.2}$$

where M is the Hardy–Littlewood maximal function on X and C in both (6.1) and (6.2) is independent of x, j, f and g.

For (6.1), by Hölder's inequality and (i) in Definition 1.2, we have

$$|E_0(f)(x)|=\Big|\int_X E_0(x,y)f(y)\,d\mu(y)\Big|\le C\int_X|f(y)|\,d\mu(y)\le C\mu(X)^{1-1/p}\|f\|_{L^p(X)}.$$

For (6.2), since $j\in\mathbb{N}$ and $\int_X E_j(x,y)\,d\mu(y)=0$, we have

$$\begin{aligned}|E_j(f)(x)|&=\Big|\int_X E_j(x,y)f(y)\,d\mu(y)\Big|=\Big|\int_X E_j(x,y)[f(y)-f(x)]\,d\mu(y)\Big|\\&\le C2^{-j}\int_X|E_j(x,y)|(g(x)+g(y))\,d\mu(y)\\&\le C2^{-j}\{g(x)+M(g)(x)\}\le C2^{-j}M(g)(x).\end{aligned}$$

Now let $f\in W^{1,p}(X,\varrho,\mu)$ and g be any generalized gradient of f. Then, by (6.1) and (6.2), we have

$$\begin{aligned}\|f\|_{F^s_{pq}(X)}&=\Big\|\Big\{\sum_{j=0}^\infty 2^{jsq}|E_j(f)(x)|^q\Big\}^{1/q}\Big\|_{L^p(X)}\\&\le\|E_0(f)\|_{L^p(X)}+\Big\|\Big\{\sum_{j=1}^\infty 2^{jsq}|E_j(f)(x)|^q\Big\}^{1/q}\Big\|_{L^p(X)}\\&\le C\|f\|_{L^p(X)}+C\Big\|\Big\{\sum_{j=1}^\infty 2^{j(s-1)q}\Big\}^{1/q}M(g)\Big\|_{L^p(X)}\le C[\|f\|_{L^p(X)}+\|g\|_{L^p(X)}],\end{aligned}$$

since $s<1$ and M is bounded on $L^p(X)$ for $1<p\le\infty$; see Theorem 2.2 in [25] and Theorem 14.13 in [15]. By taking the infimum over g, we obtain

$$\|f\|_{F^s_{pq}(X)}\le C\|f\|_{W^{1,p}(X,\varrho,\mu)},$$

where C is independent of f. This proves (ii). The proof of (i) is similar.

This finishes the proof of Theorem 6.1.

7. Quadratic forms

In this section, we give some applications of the estimates of entropy numbers obtained in Section 5 to the spectral theory of positive-definite self-adjoint operators relative to some quadratic forms. The main ideas come from [33]. See [36] and [35] for more applications.

Let us recall some basic facts of [6] and [33]. Let B be a (complex) quasi-Banach space and T be a compact operator on B. Edmunds and Triebel [6] have shown that the spectrum of T, apart from the point 0, consists only of eigenvalues of finite algebraic multiplicity; see also [37] for the case of Banach spaces. Let λ be an eigenvalue of T and I be the identity operator on B. The algebraic multiplicity of λ is defined to be the dimension of the space $\bigcup_{k=1}^{\infty} \ker(T-\lambda I)^k$; see [6]. Let $\{\mu_k(T)\}_{k\in\mathbb{N}}$ be the sequence of all nonzero eigenvalues of T, repeated according to algebraic multiplicity and ordered so that

$$|\mu_1(T)| \ge |\mu_2(T)| \ge \ldots \to 0. \tag{7.1}$$

If T has only $m \in \mathbb{N}$ different eigenvalues and M is the sum of their algebraic multiplicities, then let $\mu_n(T) = 0$ for all $n > M$.

The following inequality, called Carl's inequality, connects spectral properties of compact operators with their geometry described in terms of entropy numbers.

Lemma 7.1. *Let B be a* (*complex*) *quasi-Banach space and T be a compact operator on B. Let $\{\mu_k(T)\}_{k\in\mathbb{N}}$ be the sequence of all nonzero eigenvalues of T, repeated according to algebraic multiplicity and ordered as in* (7.1). *Then for all $k \in \mathbb{N}$,*

$$|\mu_k(T)| \le \sqrt{2}\, e_k(T). \tag{7.2}$$

If B is a (complex) Banach space, (7.2) was obtained by Carl [2]. Lemma 7.1 was proved by Edmunds and Triebel [6]. This inequality plays a key role in applications of estimates of entropy numbers to estimates of the eigenvalues for differential operators; see [33], [6], [36] and [35].

We also need to use approximation numbers; see [33, pp. 191–192] and [6] for some basic properties of approximation numbers.

Definition 7.1. Let A and B be complex quasi-Banach spaces and let T be a bounded operator from A into B. Then

$$a_k(T) = \inf\{\|T-S\| : S \in L(A,B),\ \operatorname{rank} S < k\}, \quad k \in \mathbb{N},$$

is the kth *approximation number* of T, where $\operatorname{rank} S$ is the dimension of the range of S.

Let $(X,\varrho,\mu)_{d,\theta}$ be a homogeneous type space as in Definition 0.1. For $|s| < \theta$, we let

$$H^s(X) = B^s_{22}(X) = F^s_{22}(X).$$

Then $H^s(X)$ is a Hilbert space with scalar product

$$(f,g)_{H^s(X)} = \tfrac{1}{4}\{\|f+g\|^2_{H^s(X)} - \|f-g\|^2_{H^s(X)} + i\|f+ig\|^2_{H^s(X)} - i\|f-ig\|^2_{H^s(X)}\}$$

for all $f,g \in H^s(X)$; see [32, p. 95] or [33, p. 193]. If $0 < s < \theta$, by Proposition 1.2 and Lemma 1.10, we have $F^s_{22}(X) \subset F^0_{22}(X) \subset L^2(X)$ and there is a constant $C > 0$ such that

$$\|f\|_{L^2(X)} \le C\|f\|_{H^s(X)} \tag{7.3}$$

for all $f \in H^s(X)$. Thus, according to §24.2 in [33],

$$a(f,g) = (f,g)_{H^s(X)}, \quad D = H^s(X),$$

is a closed quadratic form in the Hilbert space $L^2(X)$. Let A_s be the related self-adjoint operator according to (24.9) in [33], namely,

$$a(f,g) = (A_s^{1/2} f, A_s^{1/2} g)_{L^2(X)}$$

for all $f, g \in \mathrm{dom}(A_s^{1/2}) = H^s(X)$. By (7.3), this operator is positive-definite and we have

$$\|A_s^{1/2} f\|_{L^2(X)} = \|f\|_{H^s(X)}, \quad \mathrm{dom}(A_s^{1/2}) = H^s(X).$$

The following theorem is a version of Theorem 25.2 of [33] in spaces of homogeneous type.

THEOREM 7.1. *Let $(X, \varrho, \mu)_{d,\theta}$ be a homogeneous type space with $\mu(X) < \infty$. Let $\theta > s > 0$ and let A_s be the operator as above, in particular,*

$$(f,g)_{H^s(X)} = (A_s f, g)_{L^2(X)}, \quad f \in \mathrm{dom}(A_s),\ g \in H^s(X).$$

Then A_s is a positive-definite self-adjoint operator in $L^2(X)$ with pure point spectrum, and there are two numbers $0 < C_1 \le C_2 < \infty$ with

$$C_1 k^{2s/d} \le \mu_k \le C_2 k^{2s/d}, \quad k \in \mathbb{N}, \tag{7.4}$$

where μ_k's are the eigenvalues of A_s ordered by (7.1).

Proof. We follow the proof of Theorem 25.2 in [33]. The eigenvalues of the nonnegative compact self-adjoint operator $A_s^{-1/2}$ in $L^2(X)$ are $\nu_k = \mu_k^{-1/2}$. Furthermore, $A_s^{-1/2}$ is an isomorphism from $L^2(X)$ onto $H^s(X)$. Thus, since

$$A_s^{-1/2}(L_2(X) \to L_2(X)) = \mathrm{id}(H^s(X) \to L^2(X)) \circ A_s^{-1/2}(L^2(X) \to H^s(X)), \tag{7.5}$$

(7.3), Theorem 5.1 and Lemma 7.1 imply that

$$\nu_k \le C e_k(\mathrm{id} : H^s(X) \to L^2(X)) \le C k^{-s/d}, \quad k \in \mathbb{N}.$$

This proves the left hand side of (7.4).

We now prove the converse assertion. We use the construction in the proof of Theorem 5.1; see also Step 2 of the proof of Theorem 20.6 in [33]. Let the notation be as in the proof of Theorem 5.1. In particular, we define the linear operator A from $2^{j(s-d/2)} l_2^{M_j}$ to $H^s(X)$ and the linear operator B from $L^2(X)$ into $2^{-jd/2} l_2^{M_j}$ as in that proof. Note that $2^{jd/2}\varphi(2^j \varrho(x, z_\tau^j))$ is an ε-block for Q_τ^j, multiplied with an unimportant normalizing constant. By Theorem 1.1, we now have

$$\|A\{a_\tau : \tau = 1, \ldots, M_j\}\|_{H^s(X)} \le C 2^{j(s-d/2)} \|\{a_\tau : \tau = 1, \ldots, M_j\}\|_{l_2^{M_j}},$$

where C is independent of j. Now, by the proof of Theorem 5.1, we also have

$$\|Bf\|_{l_2^{M_j}} \le C 2^{jd/2} \|f\|_{L^2(X)},$$

where C is independent of j. Thus, A and B are bounded linear operators with operator norms independent of j. Moreover, if we let id^j be the embedding from $2^{j(s-d/2)} l_2^{M_j}$ into

$2^{-jd/2}l_2^{M_j}$ and id be the embedding from $H^s(X)$ to $L^2(X)$, then, by (5.13), we have $\mathrm{id}^j = B \circ \mathrm{id} \circ A$ and consequently, by the multiplication properties of the approximation numbers which may be found e. g. in [33], (24.13), we have

$$a_k(\mathrm{id}^j) \le C a_k(\mathrm{id}) \tag{7.6}$$

for all $k \in \mathbb{N}$, where C is independent of j and k. It is easy to see that id^j has the same approximation numbers as the embedding from $2^{js}l_2^{M_j}$ to $l_2^{M_j}$. By (7.5), (7.6) and Proposition 24.5(iii) of [33], we obtain

$$a_k(\mathrm{id}^j : 2^{js}l_2^{M_j} \to l_2^{M_j}) \le c\nu_k, \quad k \in \mathbb{N}.$$

Hence by Proposition 24.5(ii) of [33] with $k = M_j - 1 \sim 2^{jd}$, we have

$$2^{-js} \le C'\nu_{C2^{jd}}, \quad j \in \mathbb{N}.$$

This proves the right hand inequality of (7.4) and finishes the proof of Theorem 7.1.

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