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Abstract

In this paper, the authors propose a new framework under which a theory of generalized Besov-type and Triebel–Lizorkin-type function spaces is developed. Many function spaces appearing in harmonic analysis fall under the scope of this new framework. The boundedness of the Hardy–Littlewood maximal operator or the related vector-valued maximal function on any of these function spaces is not required to construct these generalized scales of smoothness spaces. Instead, a key idea used is an application of the Peetre maximal function. This idea originates from recent findings in the abstract coorbit space theory obtained by Holger Rauhut and Tino Ullrich. In this new setting, the authors establish the boundedness of pseudo-differential operators based on atomic and molecular characterizations and also the boundedness of Fourier multipliers. Characterizations of these function spaces by means of differences and oscillations are also established. As further applications of this new framework, the authors reexamine and polish some existing results for many different scales of function spaces.

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1. Introduction

Different types of smoothness spaces play an important role in harmonic analysis, partial differential equations and approximation theory. For example, Sobolev spaces are widely used in the theory of elliptic partial differential equations. However, there are several partial differential equations for which the scale of Sobolev spaces is no longer sufficient. A proper generalization is given by the classical Besov and Triebel–Lizorkin function spaces. In recent years, it turned out to be necessary to generalize even further and replace the fundamental space $L^p(\mathbb{R}^n)$ by something more general, like a Lebesgue space with variable exponents ([11, 12]) or, more generally, an Orlicz space. Another direction is pursued via replacing $L^p(\mathbb{R}^n)$ by the Morrey space $\mathcal{M}_u^p(\mathbb{R}^n)$ [48, 52, 53] or generalizations thereof [43, 80, 82, 89, 95, 96, 97, 98, 100, 105]. Thus, the theory of function spaces has become more and more complicated. Moreover, results on atomic or molecular decompositions were often developed from scratch again and again for different scales.

A nice approach to unify the theory was proposed by Hedberg and Netrusov [24]. They developed an axiomatic approach to function spaces of Besov type and Triebel–Lizorkin type, in which the underlying function space is a quasi-normed space E of sequences of Lebesgue measurable functions on $\mathbb{R}^n$, satisfying some additional assumptions. The key property assumed in that approach is that the space E satisfies a vector-valued maximal inequality of Fefferman–Stein type, namely, for some $r \in (0,\infty)$ and $\lambda \in [0,\infty)$, there exists a positive constant C such that, for all $\{f_i\}_{i=0}^\infty \subset E$,

$$\|\{M_{r,\lambda} f_i\}_{i=0}^\infty\|_E \le C \|\{f_i\}_{i=0}^\infty\|_E$$

(see [24, Definition 1.1.1(b)]), where

$$M_{r,\lambda} f(x) := \sup_{R>0} \left\{ \frac{1}{R^n} \int_{|y|<R} |f(x+y)|^r (1+|y|)^{-r\lambda}\, dy \right\}^{1/r} \qquad \text{for all } x \in \mathbb{R}^n.$$

Related to [24], Ho [25] also developed a theory of function spaces on $\mathbb{R}^n$ under the additional assumption that the Hardy–Littlewood maximal operator M is bounded on the corresponding fundamental function space.

Another direction towards a unified treatment has been developed by Rauhut and Ullrich [68] based on the generalized abstract coorbit space theory. The coorbit space theory was originally developed by Feichtinger and Gröchenig [16, 21, 22] with the aim of providing a unified description of function spaces and their atomic decompositions. The classical theory uses locally compact groups together with integrable group representations as key ingredients. Based on the idea to measure smoothness via decay properties of an abstract wavelet transform one can in particular recover homogeneous Besov–Lizorkin–Triebel spaces as coorbits of Peetre spaces $\mathcal{P}^s_{p,q,a}(\mathbb{R}^n)$. The latter fact was observed recently by Ullrich [93]. In the next step Fornasier and Rauhut [17] observed that a locally compact group structure is not needed at all to develop a coorbit space theory. While the theory in [17] essentially applies only to coorbit spaces with respect to weighted Lebesgue spaces, Rauhut and Ullrich [68] extended this abstract theory to a wider variety of coorbit spaces. The main motivation was to cover inhomogeneous Besov–Lizorkin–Triebel spaces and generalizations thereof. Indeed, the Besov–Lizorkin–Triebel-type spaces appear as coorbits of Peetre type spaces $\mathcal{P}^w_{p,\mathcal{L},a}(\mathbb{R}^n)$ [68].

All the aforementioned theories are either not complete or in some situations too restrictive. Indeed, the boundedness of maximal operators of Hardy–Littlewood type or the related vector-valued maximal functions is always required and, moreover, the Plancherel–Pólya–Nikol'skiĭ inequality (see Lemma 1.1 below) and the Fefferman–Stein vector-valued inequality were key tools in developing a theory of function spaces of Besov and Triebel–Lizorkin type.

Despite the fact that the generalized coorbit space theory [68] so far only works for Banach spaces we mainly borrow techniques from there and combine them with recent ideas from the theory of Besov-type and Triebel–Lizorkin-type spaces (see [80, 82, 89, 97, 98, 99, 100, 105]) to build up our theory for quasi-normed spaces. With a view to applications also in microlocal analysis, we even introduce these spaces directly in weighted versions. The key idea, used in this new framework, is some delicate application of the sequence of the Peetre maximal functions

$$(\varphi_j^* f)_a(x) := \begin{cases} \sup\limits_{y\in\mathbb{R}^n} \dfrac{|\Phi * f(x+y)|}{(1+|y|)^a}, & j=0, \\ \sup\limits_{y\in\mathbb{R}^n} \dfrac{|\varphi_j * f(x+y)|}{(1+2^j|y|)^a}, & j\in\mathbb{N}, \end{cases} \tag{1.1}$$

for all $f\in\mathcal{S}'(\mathbb{R}^n)$, where Φ and φ are, respectively, as in (1.3) and (1.4) below, and $\varphi_j(\cdot) = 2^{jn}\varphi(2^j\cdot)$ for all $j\in\mathbb{N}$. Instead of the pure convolution $\varphi_j * f$ involved in the definitions of the classical Besov and Triebel–Lizorkin spaces, we make use of the Peetre maximal function $(\varphi_j^* f)_a$ already in the definitions of the spaces considered in the present paper. The second main feature is the fundamental space $\mathcal{L}(\mathbb{R}^n)$ involved in the definition (instead of $L^p(\mathbb{R}^n)$). This space is given in Section 2 via a list of fundamental assumptions ($\mathcal{L}1$) through ($\mathcal{L}6$). The key assumption is ($\mathcal{L}6$), which originates in [68] (see (2.2) below). The most important advantage of the Peetre maximal function in this framework is that $(\varphi_j^* f)_a$ can be pointwise controlled by a linear combination of some other Peetre maximal functions $(\psi_k^* f)_a$, whereas in the classical setting, $\varphi_j * f$ can only be dominated by a linear combination of the Hardy–Littlewood maximal function $M(|\psi_k * f|)$ of $\psi_k * f$ (see (1.5) below). This simple fact illustrates quite well that the boundedness of M on $\mathcal{L}(\mathbb{R}^n)$ is not required in the present setting. This represents the key advantage of our theory since, according to Example 1.2 and Section 11, we are now able to deal with a greater variety of spaces. However, we do not define abstract coorbit spaces here. Compared with the results in [68], the approach in the present paper has the following additional features:

- Extension of the decomposition results to quasi-normed spaces (Section 4);
- Sharpening the conditions on admissible atoms, molecules, and wavelets (Section 4);
- Intrinsic characterization for spaces on domains (Section 5);
- Boundedness of pseudo-differential operators (Section 6);
- Direct characterizations via differences and oscillations (Section 8).

Our general approach admits at least the treatment of the following list of function spaces as replacement for $L^p(\mathbb{R}^n)$ in the definition of generalized Besov–Lizorkin–Triebel-type spaces. For details we refer to Section 11.

Weighted Lebesgue spaces. Let ρ be a weight and $0 < p < \infty$. We let $L^p(\rho)$ denote the set of all Lebesgue measurable functions f for which the norm

$$\|f\|_{L^p(\rho)} := \left\{ \int_{\mathbb{R}^n} |f(x)|^p \rho(x)\, dx \right\}^{1/p}$$

is finite. Assume that $(1 + |\cdot|)^{-N_0} \in L^p(\rho)$ for some $N_0 \in (0,\infty)$ and the estimate

$$\|\chi_{Q_{jk}}\|_{L^p(\rho)} = \|\chi_{2^{-j}k+2^{-j}[0,1)^n}\|_{L^p(\rho)} \gtrsim 2^{-j\gamma}(1+|k|)^{-\delta}, \quad j \in \mathbb{Z}_+,\, k \in \mathbb{Z}^n, \tag{1.2}$$

holds for some $\gamma, \delta \in [0,\infty)$, where the implicit positive constant is independent of j and k. The space $L^p(\rho)$ is referred to as a *weighted Lebesgue space*. In harmonic analysis, a widely used condition for weights ρ is belonging to the Muckenhoupt class of weights, $A_p(\mathbb{R}^n)$ with $p \in [1,\infty]$ (see Example 1.2). However, some examples do not fall under the scope of the class $A_p(\mathbb{R}^n)$ in many branches of mathematics. We propose here a remedy to overcome this by considering (1.2). Observe that if $\rho \in A_p(\mathbb{R}^n)$ with $p \in [1,\infty]$, then (1.2) automatically holds for some $\gamma, \delta \in (0,\infty)$.

Morrey spaces. Let $\mathcal{L}(\mathbb{R}^n) := \mathcal{M}^p_u(\mathbb{R}^n)$, the *Morrey space*, with the norm defined by

$$\|f\|_{\mathcal{M}^p_u(\mathbb{R}^n)} := \sup_{x\in\mathbb{R}^n,\, r\in(0,\infty)} r^{n/p-n/u} \left[\int_{B(x,r)} |f(y)|^u\, dy \right]^{1/u},$$

with $0 < u \le p < \infty$.

Orlicz spaces. A *Young function* is a function $\Phi : [0,\infty) \to [0,\infty)$ which is convex and satisfies $\Phi(0) = 0$. Given a Young function Φ, the *mean Luxemburg norm* of f on a cube $Q \in \mathcal{Q}(\mathbb{R}^n)$ is defined by

$$\|f\|_{\Phi,Q} := \inf \left\{ \lambda > 0 : \frac{1}{|Q|} \int_Q \Phi\left(\frac{|f(x)|}{\lambda} \right) dx \le 1 \right\}.$$

If $\Phi(t) := t^p$ for all $t \in (0,\infty)$ with $p \in [1,\infty)$, then

$$\|f\|_{\Phi,Q} = \left[\frac{1}{|Q|} \int_Q |f(x)|^p\, dx \right]^{1/p},$$

that is, the mean Luxemburg norm coincides with the (normalized) L^p norm. The *Orlicz–Morrey space* $\mathcal{L}^{\Phi,\phi}(\mathbb{R}^n)$ consists of all locally integrable functions f on $\mathbb{R}^n$ for which the norm

$$\|f\|_{\mathcal{L}^{\Phi,\phi}(\mathbb{R}^n)} := \sup_{Q\in\mathcal{Q}(\mathbb{R}^n)} \phi(\ell(Q)) \|f\|_{\Phi,Q}$$

is finite.

Herz spaces. Let $p, q \in (0,\infty]$ and $\alpha \in \mathbb{R}$. We let $Q_0 := [-1,1]^n$ and

$$C_j := [-2^j, 2^j]^n \setminus [-2^{j-1}, 2^{j-1}]^n$$

for all $j \in \mathbb{N}$. The *inhomogeneous Herz space* $K^\alpha_{p,q}(\mathbb{R}^n)$ is defined to be the set of all measurable functions f for which the norm

$$\|f\|_{K^\alpha_{p,q}(\mathbb{R}^n)} := \|\chi_{Q_0} f\|_{L^p(\mathbb{R}^n)} + \left\{ \sum_{j=1}^\infty 2^{jq\alpha} \|\chi_{C_j} f\|^q_{L^p(\mathbb{R}^n)} \right\}^{1/q}$$

is finite, where we modify naturally the definition above when $q = \infty$.

Variable exponent Lebesgue spaces. Let $p(\cdot) : \mathbb{R}^n \to (0,\infty)$ be a measurable function such that $0 < \inf_{x\in\mathbb{R}^n} p(x) \le \sup_{x\in\mathbb{R}^n} p(x) < \infty$. The space $L^{p(\cdot)}(\mathbb{R}^n)$, the *Lebesgue space with variable exponent* $p(\cdot)$, is defined as the set of all measurable functions f for which the quantity $\int_{\mathbb{R}^n} |\varepsilon f(x)|^{p(x)}\,dx$ is finite for some $\varepsilon \in (0,\infty)$. We let

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} := \inf\left\{ \lambda > 0 : \int_{\mathbb{R}^n} \left[\frac{|f(x)|}{\lambda}\right]^{p(x)} dx \le 1 \right\}$$

for such a function f.

Amalgam spaces. Let p, $q \in (0,\infty]$ and $s \in \mathbb{R}$. Recall that $Q_{0z} := z + [0,1]^n$ for $z \in \mathbb{Z}^n$, the translate of the unit cube. For a locally Lebesgue integrable function f we define

$$\|f\|_{(L^p(\mathbb{R}^n),\ell^q(\langle z\rangle^s))} := \left\| \{(1+|z|)^s \|\chi_{Q_{0z}} f\|_{L^p(\mathbb{R}^n)}\}_{z\in\mathbb{Z}^n} \right\|_{\ell^q}.$$

Multiplier spaces. There is another variant of Morrey spaces. For $r \in [0, n/2)$, $\dot{X}_r(\mathbb{R}^n)$ is defined as the space of all functions $f \in L^2_{\rm loc}(\mathbb{R}^n)$ that satisfy

$$\|f\|_{\dot{X}_r(\mathbb{R}^n)} := \sup\{\|fg\|_{L^2(\mathbb{R}^n)} < \infty : \|g\|_{\dot{H}^r(\mathbb{R}^n)} \le 1\} < \infty,$$

where $\dot{H}^r(\mathbb{R}^n)$ stands for the completion of the space $\mathcal{D}(\mathbb{R}^n)$ with respect to the norm $\|u\|_{\dot{H}^r(\mathbb{R}^n)} := \|(-\Delta)^{r/2} u\|_{L^2(\mathbb{R}^n)}$ for $u \in \mathcal{D}(\mathbb{R}^n)$. Recall that $\mathcal{D}(\mathbb{R}^n)$ denotes the set of all $C^\infty(\mathbb{R}^n)$ functions on $\mathbb{R}^n$ with compact support, endowed with the inductive limit topology.

$\dot{\mathbf{B}}_\sigma$-spaces. The next example also falls under the scope of our generalized Triebel–Lizorkin type spaces. Let $\sigma \in [0,\infty)$, $p \in [1,\infty]$ and $\lambda \in [-n/p, 0]$. Then $\dot{B}_\sigma(L_{p,\lambda})(\mathbb{R}^n)$ is defined as the space of all $f \in L^p_{\rm loc}(\mathbb{R}^n)$ for which the norm

$$\|f\|_{\dot{B}_\sigma(L_{p,\lambda})(\mathbb{R}^n)} := \sup\left\{ \frac{1}{r^\sigma |Q|^{\lambda/n+1/p}} \|f\|_{L^p(Q)} : r \in (0,\infty),\, Q \subset Q(0,r) \right\}$$

is finite, where $Q(0,r) := \{x \in \mathbb{R}^n : |x| < r\}$ for $r \in (0,\infty)$.

Generalized Campanato spaces. We define

$$d_{p(\cdot)} := \min\{d \in \mathbb{Z}_+ : p_-(n+d+1) > n\}.$$

Then $L^q_{\rm comp}(\mathbb{R}^n)$ is defined to be the set of all $L^q(\mathbb{R}^n)$-functions with compact support. For a nonnegative integer d, let

$$L^{q,d}_{\rm comp}(\mathbb{R}^n) := \left\{ f \in L^q_{\rm comp}(\mathbb{R}^n) : \int_{\mathbb{R}^n} f(x) x^\alpha\,dx = 0,\ |\alpha| \le d \right\}.$$

Let us now describe the organization of the present paper. In Section 2, we describe the new setting we propose, which consists of a list of assumptions ($\mathcal{L}1$) through ($\mathcal{L}6$) on the fundamental space $\mathcal{L}(\mathbb{R}^n)$. Also several important consequences and further inequalities are provided.

In Section 3, starting from $\mathcal{L}(\mathbb{R}^n)$, we introduce two sorts of generalized Besov-type and Triebel–Lizorkin-type spaces, respectively (Definition 3.1). We justify these definitions by proving some properties, such as completeness (without assuming $\mathcal{L}(\mathbb{R}^n)$ is complete!), containing the Schwartz space $\mathcal{S}(\mathbb{R}^n)$, and embedding into the distributions $\mathcal{S}'(\mathbb{R}^n)$. An analogous statement holds with the classical 2-microlocal space $B^w_{1,1,a}(\mathbb{R}^n)$ as test functions and its dual, the space $B^{1/w}_{\infty,\infty,a}(\mathbb{R}^n)$, as distributions, which is an important observation for the wavelet characterization in Section 4. Therefore, the latter spaces, which have been studied intensively by Kempka [34, 35], appear naturally in our context.

In Section 4, we establish atomic and molecular decomposition characterizations (Theorem 4.5), which are further used in Section 6 to obtain the boundedness of some pseudo-differential operators from the Hörmander class $S^0_{1,\mu}(\mathbb{R}^n)$, with $\mu \in [0,1)$ (Theorems 6.6 and 6.11). In addition, characterizations using biorthogonal wavelet bases are given (see Theorem 4.12). Appropriate wavelets (analysis and synthesis) must be sufficiently smooth, fast decaying and provide enough vanishing moments. The precise conditions on these three issues are provided in Subsection 4.4 and allow for the selection of particular biorthogonal wavelet bases according to the well-known construction by Cohen, Daubechies and Feauveau [6]. Characterizations via orthogonal wavelets are contained in this setting.

Section 5 considers pointwise multipliers and the restriction of our function spaces to Lipschitz domains Ω and provides characterizations within the domain (avoiding extensions).

Section 6 considers Fourier multipliers and pseudo-differential operators, which shows that our new framework indeed works.

In Section 7, we obtain a sufficient condition for our function spaces to consist of continuous functions (Theorem 7.1). This is a preparatory step for Section 8, where we deal with differences and oscillations. Another issue of Section 7 is a further interesting application of the atomic decomposition result from Theorem 4.5. Under certain conditions on the scalar parameters involved (by still using a general fundamental space $\mathcal{L}(\mathbb{R}^n)$), our spaces degenerate to the well-known classical 2-microlocal Besov spaces $B^w_{\infty,\infty}(\mathbb{R})$.

In Section 8, we obtain a direct characterization in terms of differences and oscillations of these generalized Besov-type and Triebel–Lizorkin-type spaces (Theorems 8.2 and 8.6). Also, under some mild condition, $\mathcal{L}(\mathbb{R}^n)$ is shown to fall under our new framework (Theorem 9.6).

The Peetre maximal construction in the present paper makes it necessary to deal with an additional parameter $a \in (0,\infty)$ in the definition of function spaces. However, this new parameter a does not seem to play a significant role in a generic setting, although we do have an example showing that the space may depend upon a (see Example 3.4). We present several sufficient conditions in Section 9 which allow one to remove the parameter a (Assumption 8.1).

Homogeneous counterparts of the above are available and we describe them in Section 10. Finally, in Section 11 we present some well-known function spaces as examples of our abstract results and compare them with earlier contributions. We reexamine and polish some existing results for these known function spaces.

Notation. Next we clarify some conventions on the notation and review some basic definitions. In what follows, as usual, we use $\mathcal{S}(\mathbb{R}^n)$ to denote the classical topological vector space of all Schwartz functions on $\mathbb{R}^n$ and $\mathcal{S}'(\mathbb{R}^n)$ for its topological dual space endowed with weak-$*$ topology. For any $\varphi \in \mathcal{S}(\mathbb{R}^n)$, we use $\widehat{\varphi}$ to denote its *Fourier transform*, namely, for all $\xi \in \mathbb{R}^n$, $\widehat{\varphi}(\xi) := \int_{\mathbb{R}^n} e^{-i\xi x}\varphi(x)\,dx$. We denote *dyadic dilations* of a given function $\varphi \in \mathcal{S}(\mathbb{R}^n)$ by $\varphi_j(x) := 2^{jn}\varphi(2^j x)$ for all $j \in \mathbb{Z}$ and $x \in \mathbb{R}^n$. Throughout the paper we use a system (Φ, φ) of Schwartz functions satisfying

$$\operatorname{supp} \widehat{\Phi} \subset \{\xi \in \mathbb{R}^n : |\xi| \le 2\} \quad \text{and} \quad |\widehat{\Phi}(\xi)| \ge C > 0 \text{ if } |\xi| \le 5/3 \tag{1.3}$$

and

$$\operatorname{supp} \widehat{\varphi} \subset \{\xi \in \mathbb{R}^n : 1/2 \le |\xi| \le 2\} \quad \text{and} \quad |\widehat{\varphi}(\xi)| \ge C > 0 \text{ if } 3/5 \le |\xi| \le 5/3. \tag{1.4}$$

$L^1_{\rm loc}(\mathbb{R}^n)$ denotes the set of all locally integrable functions on $\mathbb{R}^n$; $L^\eta_{\rm loc}(\mathbb{R}^n)$ for any $\eta \in (0,\infty)$ is the set of all measurable functions on $\mathbb{R}^n$ such that $|f|^\eta \in L^1_{\rm loc}(\mathbb{R}^n)$; and $L^\infty_{\rm loc}(\mathbb{R}^n)$ is the set of all locally essentially bounded functions on $\mathbb{R}^n$. We also let M denote the *Hardy–Littlewood maximal operator* defined by setting, for all $f \in L^1_{\rm loc}(\mathbb{R}^n)$,

$$Mf(x) = M(f)(x) := \sup_{r>0} \frac{1}{r^n} \int_{|z-x|<r} |f(z)|\,dz \quad \text{for all } x \in \mathbb{R}^n. \tag{1.5}$$

One of the main tools in the classical theory of function spaces is the boundedness of M on a space of functions, say $L^p(\mathbb{R}^n)$ or its vector-valued extension $L^p(\ell^q)$, in connection with the Plancherel–Pólya–Nikol'skiĭ inequality connecting the Peetre maximal function and the Hardy–Littlewood maximal operator.

Lemma 1.1 ([90, p. 16]). *Let $\eta \in (0,1]$, $R \in (0,\infty)$ and $f \in \mathcal{S}'(\mathbb{R}^n)$ be such that* $\operatorname{supp} \widehat{f} \subset Q(0,R) := \{x \in \mathbb{R}^n : |x| < R\}$. *Then there exists a positive constant c_η such that, for all $x \in \mathbb{R}^n$,*

$$\sup_{y\in\mathbb{R}^n} \frac{|f(x-y)|}{(1+R|y|)^{n/\eta}} \le c_\eta [M(|f|^\eta)(x)]^{1/\eta}.$$

The following examples show situations when the boundedness of M can be achieved and when we cannot expect it.

Example 1.2. (i) Let $p \in (1,\infty)$. It is known that the Hardy–Littlewood operator M is not bounded on the weighted Lebesgue space $L^p(w)$ unless $w \in A_p(\mathbb{R}^n)$, where $A_p(\mathbb{R}^n)$ is the class of *Muckenhoupt weights* (see, for example, [19, 88] for their definitions and properties) such that

$$A_p(w) := \sup_{Q\in\mathcal{Q}} \left[\frac{1}{|Q|}\int_Q w(x)\,dx\right]\left[\frac{1}{|Q|}\int_Q [w(x)]^{-1/(p-1)}\,dx\right]^{p-1} < \infty.$$

Also observe that there exists a positive constant $C_{p,q}$ such that

$$\left\{\int_{\mathbb{R}^n}\left(\sum_{j=1}^{\infty}[Mf_j(x)]^q\right)^{q/p}w(x)\,dx\right\}^{1/p}\le C_{p,q}\left\{\int_{\mathbb{R}^n}\left[\sum_{j=1}^{\infty}|f_j(x)|^q\right]^{q/p}w(x)\,dx\right\}^{1/p}$$

holds for any $q\in(1,\infty]$ if and only if $w\in A_p(\mathbb{R}^n)$. There do exist doubling weights which do not belong to the Muckenhoupt class $A_\infty(\mathbb{R}^n)$ (see [14]).

(ii) There exists a function space such that even the operator $M_{r,\lambda}$ is difficult to control. For example, if $\mathcal{L}(\mathbb{R}^n):=L^{1+\chi_{\mathbb{R}^n_+}}(\mathbb{R}^n)$, which is the set of all measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{L^{1+\chi_{\mathbb{R}^n_+}}(\mathbb{R}^n)}:=\inf\left\{\lambda>0:\int_{\mathbb{R}^n_+}\left[\frac{|f(x)|}{\lambda}\right]^2dx+\int_{\mathbb{R}^n\setminus\mathbb{R}^n_+}\frac{|f(x)|}{\lambda}\,dx\le 1\right\}<\infty,$$

where $\mathbb{R}^n_+:=\{x=(x_1,\dots,x_n)\in\mathbb{R}^n:x_n\in(0,\infty)\}$, then it is well known that the maximal operator $M_{r,\lambda}$ is not bounded on $L^{1+\chi_{\mathbb{R}^n_+}}(\mathbb{R}^n)$ (see Lemma 11.11 below).

Throughout the paper, we denote by C a positive constant which is independent of the main parameters, but it may vary from line to line, while $C(\alpha,\beta,\dots)$ denotes a positive constant depending on the parameters α, β, $\dots$. The symbols $A\lesssim B$ and $A\lesssim_{\alpha,\beta,\dots}B$ mean, respectively, that $A\le CB$ and $A\le C(\alpha,\beta,\dots)B$. If $A\lesssim B$ and $B\lesssim A$, then we write $A\sim B$. If E is a subset of $\mathbb{R}^n$, we denote by χ_E its characteristic function. In what follows, for all a, $b\in\mathbb{R}$, let $a\vee b:=\max\{a,b\}$ and $a\wedge b:=\min\{a,b\}$. Also, we let $\mathbb{Z}_+:=\{0,1,2,\dots\}$. The notation $\lfloor x\rfloor$, for any $x\in\mathbb{R}$, means the maximal integer not larger than x. The following is our convention for dyadic cubes: For $j\in\mathbb{Z}$ and $k\in\mathbb{Z}^n$, denote by Q_{jk} the *dyadic cube* $2^{-j}([0,1)^n+k)$. Let $\mathcal{Q}(\mathbb{R}^n):=\{Q_{jk}:j\in\mathbb{Z},\ k\in\mathbb{Z}^n\}$,

$$\mathcal{Q}_j(\mathbb{R}^n):=\{Q\in\mathcal{Q}(\mathbb{R}^n):\ell(Q)=2^{-j}\}.$$

For any $Q\in\mathcal{Q}(\mathbb{R}^n)$, we let j_Q be $-\log_2\ell(Q)$, $\ell(Q)$ its *side length*, x_Q its *lower left corner* $2^{-j}k$ and c_Q its *center*. When the dyadic cube Q appears as an index, such as $\sum_{Q\in\mathcal{Q}(\mathbb{R}^n)}$ and $\{\cdot\}_{Q\in\mathcal{Q}(\mathbb{R}^n)}$, it is understood that Q runs over all dyadic cubes in $\mathbb{R}^n$. For any cube Q and $\kappa\in(0,\infty)$, we denote by κQ the cube with the same center as Q but κ times the side length of Q. Also, we write

$$\|\vec{\alpha}\|_1:=\sum_{j=1}^{n}\alpha^j \tag{1.6}$$

for a multiindex $\vec{\alpha}:=(\alpha^1,\dots,\alpha^n)\in\mathbb{Z}^n_+$. For $\sigma:=(\sigma_1,\dots,\sigma_n)\in\mathbb{Z}^n_+$, $\partial^\sigma:=(\partial/\partial x_1)^{\sigma_1}\cdots(\partial/\partial x_n)^{\sigma_n}$.

2. Fundamental settings and inequalities

2.1. Basic assumptions. First of all, we assume that $\mathcal{L}(\mathbb{R}^n)$ is a quasi-normed space of functions on $\mathbb{R}^n$. Following [3, p. 3], we denote by $M_0(\mathbb{R}^n)$ the topological vector space of all measurable complex-valued almost everywhere finite functions modulo null functions (i.e., any two functions coinciding almost everywhere are identified), topologized by

$$\rho_E(f) := \int_E \min\{1, |f(x)|\}\, dx,$$

where E is any subset of $\mathbb{R}^n$ with finite Lebesgue measure. It is easy to show that this topology is equivalent to the topology of convergence in measure on sets of finite measure, which makes $M_0(\mathbb{R}^n)$ a metrizable topological vector space (see [3, p. 30]).

First, we consider a mapping $\|\cdot\|_{\mathcal{L}(\mathbb{R}^n)} : M_0(\mathbb{R}^n) \to [0,\infty]$ satisfying the following fundamental conditions:

($\mathcal{L}1$) (Positivity) An element $f \in M_0(\mathbb{R}^n)$ satisfies $\|f\|_{\mathcal{L}(\mathbb{R}^n)} = 0$ if and only if $f = 0$.

($\mathcal{L}2$) (Homogeneity) Let $f \in M_0(\mathbb{R}^n)$ and $\alpha \in \mathbb{C}$. Then $\|\alpha f\|_{\mathcal{L}(\mathbb{R}^n)} = |\alpha|\, \|f\|_{\mathcal{L}(\mathbb{R}^n)}$.

($\mathcal{L}3$) (The θ-triangle inequality) The norm $\|\cdot\|_{\mathcal{L}(\mathbb{R}^n)}$ satisfies the θ-triangle inequality. That is, there exists a positive constant $\theta = \theta(\mathcal{L}(\mathbb{R}^n)) \in (0,1]$ such that

$$\|f+g\|^\theta_{\mathcal{L}(\mathbb{R}^n)} \le \|f\|^\theta_{\mathcal{L}(\mathbb{R}^n)} + \|g\|^\theta_{\mathcal{L}(\mathbb{R}^n)}$$

for all $f, g \in M_0(\mathbb{R}^n)$.

($\mathcal{L}4$) (The lattice property) If a pair $(f,g) \in M_0(\mathbb{R}^n) \times M_0(\mathbb{R}^n)$ satisfies $|g| \le |f|$, then $\|g\|_{\mathcal{L}(\mathbb{R}^n)} \le \|f\|_{\mathcal{L}(\mathbb{R}^n)}$.

($\mathcal{L}5$) (The Fatou property) Suppose that $\{f_j\}_{j=1}^\infty$ is a sequence of functions satisfying

$$\sup_{j\in\mathbb{N}} \|f_j\|_{\mathcal{L}(\mathbb{R}^n)} < \infty, \quad 0 \le f_1 \le f_2 \le \cdots .$$

Then the limit $f := \lim_{j\to\infty} f_j$ belongs to $\mathcal{L}(\mathbb{R}^n)$ and $\|f\|_{\mathcal{L}(\mathbb{R}^n)} \le \sup_{j\in\mathbb{N}} \|f_j\|_{\mathcal{L}(\mathbb{R}^n)}$.

Given a mapping $\|\cdot\|_{\mathcal{L}(\mathbb{R}^n)}$ satisfying ($\mathcal{L}1$) through ($\mathcal{L}5$), the space $\mathcal{L}(\mathbb{R}^n)$ is defined by

$$\mathcal{L}(\mathbb{R}^n) := \{f \in M_0(\mathbb{R}^n) : \|f\|_{\mathcal{L}(\mathbb{R}^n)} < \infty\}.$$

Let ρ be a weight. Note that $L^p(\rho)$ with $p \in (0,\infty)$ satisfies ($\mathcal{L}6$) below as long as ρ satisfies (1.2).

Remark 2.1. We point out that the assumptions ($\mathcal{L}1$), ($\mathcal{L}2$) and ($\mathcal{L}3$) can be replaced by the assumption that $\mathcal{L}(\mathbb{R}^n)$ is a quasi-normed linear space of functions. Indeed, if $(\mathcal{L}(\mathbb{R}^n), \|\cdot\|_{\mathcal{L}(\mathbb{R}^n)})$ is a quasi-normed linear space of function, then by the Aoki–Rolewicz theorem (see [2, 69]), there exists an equivalent quasi-norm $\|\!|\cdot|\!\|$ and $\widetilde{\theta} \in (0,1]$ such that, for all $f, g \in \mathcal{L}(\mathbb{R}^n)$,

$$\|\cdot\|_{\mathcal{L}(\mathbb{R}^n)} \sim \|\!|\cdot|\!\|, \qquad \|\!|f+g|\!\|^{\widetilde{\theta}} \le \|\!|f|\!\|^{\widetilde{\theta}} + \|\!|g|\!\|^{\widetilde{\theta}}. \tag{2.1}$$

Thus, $(\mathcal{L}(\mathbb{R}^n), \|\!|\cdot|\!\|)$ satisfies $(\mathcal{L}1)$, $(\mathcal{L}2)$ and $(\mathcal{L}3)$. Since all results are invariant with respect to taking equivalent quasi-norms, by (2.1), we know that all results are still true for the quasi-norm $\|\cdot\|_{\mathcal{L}(\mathbb{R}^n)}$.

Motivated by [68, 93], we also assume that $\mathcal{L}(\mathbb{R}^n)$ enjoys the following property:

$(\mathcal{L}6)$ (The non-degeneracy condition) The $(1+|\cdot|)^{-N_0}$ belongs to $\mathcal{L}(\mathbb{R}^n)$ for some $N_0 \in (0,\infty)$ and the estimate

$$\|\chi_{Q_{jk}}\|_{\mathcal{L}(\mathbb{R}^n)} = \|\chi_{2^{-j}k+2^{-j}[0,1)^n}\|_{\mathcal{L}(\mathbb{R}^n)} \gtrsim 2^{-j\gamma}(1+|k|)^{-\delta}, \qquad j \in \mathbb{Z}_+,\ k \in \mathbb{Z}^n, \tag{2.2}$$

holds for some $\gamma, \delta \in [0,\infty)$, where the implicit positive constant is independent of j and k.

We point out that $(\mathcal{L}6)$ is a key assumption, which makes our definitions of quasi-normed spaces a little different from that in [3]. This condition has been used by Rauhut and Ullrich [68, Definition 4.4] in order to define coorbits of Peetre type spaces in a reasonable way. Indeed, in [3], it is necessary to assume that $\chi_E \in \mathcal{L}(\mathbb{R}^n)$ if E is a measurable set of finite measure.

Moreover, from $(\mathcal{L}4)$ and $(\mathcal{L}5)$, we deduce the following Fatou property of $\mathcal{L}(\mathbb{R}^n)$.

Proposition 2.2. *If $\mathcal{L}(\mathbb{R}^n)$ satisfies $(\mathcal{L}4)$ and $(\mathcal{L}5)$, then, for all sequences $\{f_m\}_{m\in\mathbb{N}}$ of nonnegative functions of $\mathcal{L}(\mathbb{R}^n)$,*

$$\left\|\liminf_{m\to\infty} f_m\right\|_{\mathcal{L}(\mathbb{R}^n)} \le \liminf_{m\to\infty} \|f_m\|_{\mathcal{L}(\mathbb{R}^n)}.$$

Proof. Without loss of generality, we may assume that $\liminf_{m\to\infty} \|f_m\|_{\mathcal{L}(\mathbb{R}^n)} < \infty$. Recall that $\liminf_{m\to\infty} f_m = \sup_{m\in\mathbb{N}} \inf_{k\ge m}\{f_k\}$. For all $m \in \mathbb{N}$, let $g_m := \inf_{k\ge m}\{f_k\}$. Then $\{g_m\}_{m\in\mathbb{N}}$ is a sequence of nonnegative functions with $g_1 \le g_2 \le \cdots$. Moreover, by $(\mathcal{L}4)$, we conclude that

$$\sup_{m\in\mathbb{N}} \|g_m\|_{\mathcal{L}(\mathbb{R}^n)} \le \liminf_{m\to\infty} \|f_m\|_{\mathcal{L}(\mathbb{R}^n)} < \infty.$$

Then, from $(\mathcal{L}5)$, we further deduce that $\liminf_{m\to\infty} f_m = \sup_{m\in\mathbb{N}}\{g_m\} \in \mathcal{L}(\mathbb{R}^n)$ and

$$\left\|\liminf_{m\to\infty} f_m\right\|_{\mathcal{L}(\mathbb{R}^n)} \le \sup_{m\in\mathbb{N}} \|g_m\|_{\mathcal{L}(\mathbb{R}^n)} \le \liminf_{m\to\infty} \|f_m\|_{\mathcal{L}(\mathbb{R}^n)}. \quad \blacksquare$$

We also remark that the completeness of $\mathcal{L}(\mathbb{R}^n)$ is not necessary. It is of interest to have completeness automatically, as Proposition 3.16 below shows.

Let us additionally recall the following class $\mathcal{W}^{\alpha_3}_{\alpha_1,\alpha_2}$ of weights which was used recently in [68]. This class has been introduced for the definition of 2-microlocal Besov-Triebel–Lizorkin spaces; see [34, 35]. As in Example 1.2(ii), let

$$\mathbb{R}^{n+1}_+ := \{(x, x_{n+1}) : x \in \mathbb{R}^n,\ x_{n+1} \in (0,\infty)\}.$$

We also let $\mathbb{R}^{n+1}_{\mathbb{Z}_+} := \{(x,t) \in \mathbb{R}^{n+1}_+ : -\log_2 t \in \mathbb{Z}_+\}$.

Definition 2.3. Let $\alpha_1, \alpha_2, \alpha_3 \in [0,\infty)$. The class $\mathcal{W}^{\alpha_3}_{\alpha_1,\alpha_2}$ of weights is defined as the set of all measurable functions $w : \mathbb{R}^{n+1}_{\mathbb{Z}_+} \to (0,\infty)$ satisfying the following conditions:

(W1) There exists a positive constant C such that, for all $x \in \mathbb{R}^n$ and $j, \nu \in \mathbb{Z}_+$ with $j \geq \nu$,

$$C^{-1}2^{-(j-\nu)\alpha_1}w(x,2^{-\nu}) \leq w(x,2^{-j}) \leq C2^{-(\nu-j)\alpha_2}w(x,2^{-\nu}). \tag{2.3}$$

(W2) There exists a positive constant C such that, for all $x, y \in \mathbb{R}^n$ and $j \in \mathbb{Z}_+$,

$$w(x,2^{-j}) \leq Cw(y,2^{-j})(1+2^j|x-y|)^{\alpha_3}. \tag{2.4}$$

Given a weight w and $j \in \mathbb{Z}_+$, we often write

$$w_j(x) := w(x,2^{-j}) \quad (x \in \mathbb{R}^n,\ j \in \mathbb{Z}_+), \tag{2.5}$$

which is a convention used until the end of Section 9. With the convention (2.5), the conditions (2.3) and (2.4) read

$$C^{-1}2^{-(j-\nu)\alpha_1}w_\nu(x) \leq w_j(x) \leq C2^{-(\nu-j)\alpha_2}w_\nu(x)$$

and

$$w_j(x) \leq Cw_j(y)(1+2^j|x-y|)^{\alpha_3},$$

respectively. In what follows, for all $a \in \mathbb{R}$, $a_+ := \max(a,0)$.

Example 2.4. (i) The most familiar case, the classical Besov spaces $B^s_{p,q}(\mathbb{R}^n)$ and Triebel–Lizorkin spaces $F^s_{p,q}(\mathbb{R}^n)$, correspond to $w_j \equiv 2^{js}$ with $j \in \mathbb{Z}_+$ and $s \in \mathbb{R}$.

(ii) In general when $w_j(x)$ with $j \in \mathbb{Z}_+$ and $x \in \mathbb{R}^n$ is independent of x, then we see that $\alpha_3 = 0$. For example, when $w_j(x) \equiv 2^{js}$ for some $s \in \mathbb{R}$ and all $x \in \mathbb{R}^n$, then $w_j \in \mathcal{W}^0_{\max(0,-s),\max(0,s)}$.

(iii) Let $w \in \mathcal{W}^{\alpha_3}_{\alpha_1,\alpha_2}$ and $s \in \mathbb{R}$. Then the weight given by

$$\widetilde{w}_j(x) := 2^{js}w_j(x) \quad (x \in \mathbb{R}^n,\ j \in \mathbb{Z}_+)$$

belongs to the class $\mathcal{W}^{\alpha_3}_{(\alpha_1-s)_+,(\alpha_2+s)_+}$.

In the present paper, we consider six underlying function spaces, two of which are special cases of other four spaces. At first glance the definitions of $\ell^q(\mathcal{L}^w_\tau(\mathbb{R}^n,\mathbb{Z}_+))$ and $\ell^q(\mathcal{NL}^w_\tau(\mathbb{R}^n,\mathbb{Z}_+))$ might seem identical. However, in [82], we showed that they are different in general cases. In the present paper, we generalize this fact in Theorem 9.12.

Definition 2.5. Let $q \in (0,\infty]$ and $\tau \in [0,\infty)$. Suppose $w \in \mathcal{W}^{\alpha_3}_{\alpha_1,\alpha_2}$ with $\alpha_1, \alpha_2, \alpha_3 \in [0,\infty)$. Let w_j for $j \in \mathbb{Z}_+$ be as in (2.5).

(i) $\mathcal{L}^w(\ell^q(\mathbb{R}^n,\mathbb{Z}_+))$ is defined to be the set of all sequences $G := \{g_j\}_{j\in\mathbb{Z}_+}$ of measurable functions on $\mathbb{R}^n$ such that

$$\|G\|_{\mathcal{L}^w(\ell^q(\mathbb{R}^n,\mathbb{Z}_+))} := \left\|\left(\sum_{j=0}^\infty |w_jg_j|^q\right)^{1/q}\right\|_{\mathcal{L}(\mathbb{R}^n)} < \infty. \tag{2.6}$$

By analogy, the space $\mathcal{L}^w(\ell^q(\mathbb{R}^n,E))$ is defined for a subset $E \subset \mathbb{Z}$.

(ii) $\ell^q(\mathcal{L}^w(\mathbb{R}^n,\mathbb{Z}_+))$ is defined to be the set of all sequences $G := \{g_j\}_{j\in\mathbb{Z}_+}$ of measurable functions on $\mathbb{R}^n$ such that

$$\|G\|_{\ell^q(\mathcal{L}^w(\mathbb{R}^n,\mathbb{Z}_+))} := \left\{\sum_{j=0}^\infty \|w_jg_j\|^q_{\mathcal{L}(\mathbb{R}^n)}\right\}^{1/q} < \infty. \tag{2.7}$$

In analogy, the space $\ell^q(\mathcal{L}^w(\mathbb{R}^n,E))$ is defined for a subset $E \subset \mathbb{Z}$.

(iii) $\mathcal{L}^w_\tau(\ell^q(\mathbb{R}^n,\mathbb{Z}_+))$ is defined to be the set of all sequences $G := \{g_j\}_{j\in\mathbb{Z}_+}$ of measurable functions on $\mathbb{R}^n$ such that

$$\|G\|_{\mathcal{L}^w_\tau(\ell^q(\mathbb{R}^n,\mathbb{Z}_+))} := \sup_{P\in\mathcal{Q}(\mathbb{R}^n)} \frac{1}{|P|^\tau}\|\{\chi_P w_j g_j\}_{j=j_P\vee 0}^\infty\|_{\mathcal{L}^w(\ell^q(\mathbb{R}^n,\mathbb{Z}_+\cap[j_P,\infty)))} < \infty. \tag{2.8}$$

(iv) $\mathcal{EL}^w_\tau(\ell^q(\mathbb{R}^n,\mathbb{Z}_+))$ is defined to be the set of all sequences $G := \{g_j\}_{j\in\mathbb{Z}_+}$ of measurable functions on $\mathbb{R}^n$ such that

$$\|G\|_{\mathcal{EL}^w_\tau(\ell^q(\mathbb{R}^n,\mathbb{Z}_+))} := \sup_{P\in\mathcal{Q}(\mathbb{R}^n)} \frac{1}{|P|^\tau}\|\{\chi_P w_j g_j\}_{j=0}^\infty\|_{\mathcal{L}^w(\ell^q(\mathbb{R}^n,\mathbb{Z}_+))} < \infty. \tag{2.9}$$

(v) $\ell^q(\mathcal{L}^w_\tau(\mathbb{R}^n,\mathbb{Z}_+))$ is defined to be the set of all sequences $G := \{g_j\}_{j\in\mathbb{Z}_+}$ of measurable functions on $\mathbb{R}^n$ such that

$$\|G\|_{\ell^q(\mathcal{L}^w_\tau(\mathbb{R}^n,\mathbb{Z}_+))} := \sup_{P\in\mathcal{Q}(\mathbb{R}^n)} \frac{1}{|P|^\tau}\|\{\chi_P w_j g_j\}_{j=j_P\vee 0}^\infty\|_{\ell^q(\mathcal{L}^w(\mathbb{R}^n,\mathbb{Z}_+\cap[j_P,\infty)))} < \infty. \tag{2.10}$$

(vi) $\ell^q(\mathcal{NL}^w_\tau(\mathbb{R}^n,\mathbb{Z}_+))$ is defined to be the set of all sequences $G := \{g_j\}_{j\in\mathbb{Z}_+}$ of measurable functions on $\mathbb{R}^n$ such that

$$\|G\|_{\ell^q(\mathcal{NL}^w_\tau(\mathbb{R}^n,\mathbb{Z}_+))} := \left\{\sum_{j=0}^\infty \sup_{P\in\mathcal{Q}(\mathbb{R}^n)}\left[\frac{\|\chi_P w_j g_j\|_{\mathcal{L}(\mathbb{R}^n)}}{|P|^\tau}\right]^q\right\}^{1/q} < \infty. \tag{2.11}$$

When $q=\infty$, a natural modification is made in (2.6) through (2.11).

We also introduce the homogeneous counterparts of these spaces in Section 10. One of the reasons why we introduce $\mathcal{W}^{\alpha_3}_{\alpha_1,\alpha_2}$ is the necessity of describing the smoothness by using our new weighted function spaces more precisely than by using the classical Besov–Triebel–Lizorkin spaces. For example, in [103], Yoneda considered the following norm. In what follows, $\mathcal{P}(\mathbb{R}^n)$ denotes the set of all polynomials on $\mathbb{R}^n$.

EXAMPLE 2.6 ([103]). $\dot{B}^{-1,\sqrt{\cdot}}_{\infty\infty}(\mathbb{R}^n)$ denotes the set of all $f\in\mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$ for which the norm

$$\|f\|_{\dot{B}^{-1,\sqrt{\cdot}}_{\infty\infty}(\mathbb{R}^n)} := \sup_{j\in\mathbb{Z}} 2^{-j}\sqrt{|j|+1}\|\varphi_j * f\|_{L^\infty(\mathbb{R}^n)} < \infty.$$

If $\tau=0$, $a\in(0,\infty)$ and $w_j(x) := 2^{-j}\sqrt{|j|+1}$ for all $x\in\mathbb{R}^n$ and $j\in\mathbb{Z}$, then it can be shown that the space $\dot{B}^{-1,\sqrt{\cdot}}_{\infty\infty}(\mathbb{R}^n)$ and the space $\dot{B}^{w,0}_{L^\infty,\infty,a}(\mathbb{R}^n)$, introduced in Definition 10.3 below, coincide with equivalent norms. This can be proved by an argument similar to that used in the proof of [93, Theorem 2.9]; we omit the details. An inhomogeneous variant of this result is also true. Moreover, we refer to Subsection 11.9 for another example of non-trivial weights w. This is a special case of generalized smoothness. The weight w also plays a role of variable smoothness.

In the present paper, the spaces $\ell^q(\mathcal{L}^w_\tau(\mathbb{R}^n,\mathbb{Z}_+))$, $\ell^q(\mathcal{NL}^w_\tau(\mathbb{R}^n,\mathbb{Z}_+))$, $\mathcal{L}^w_\tau(\ell^q(\mathbb{R}^n,\mathbb{Z}_+))$ and $\mathcal{EL}^w_\tau(\ell^q(\mathbb{R}^n,\mathbb{Z}_+))$ play a central role, while $\ell^q(\mathcal{L}^w(\mathbb{R}^n,\mathbb{Z}_+))$ and $\mathcal{L}^w(\ell^q(\mathbb{R}^n,\mathbb{Z}_+))$ are auxiliary spaces.

By the monotonicity of ℓ^q, we immediately obtain the following useful conclusions. We omit the details.

Lemma 2.7. *Let* $0 < q_1 \le q_2 \le \infty$ *and* $\alpha_1, \alpha_2, \alpha_3, \tau \in [0, \infty)$ *and* $w \in \mathcal{W}^{\alpha_3}_{\alpha_1,\alpha_2}$. *Then*

$$\begin{aligned}
\ell^{q_1}(\mathcal{L}^w(\mathbb{R}^n, \mathbb{Z}_+)) &\hookrightarrow \ell^{q_2}(\mathcal{L}^w(\mathbb{R}^n, \mathbb{Z}_+)),\\
\mathcal{L}^w(\ell^{q_1}(\mathbb{R}^n, \mathbb{Z}_+)) &\hookrightarrow \mathcal{L}^w(\ell^{q_2}(\mathbb{R}^n, \mathbb{Z}_+)),\\
\ell^{q_1}(\mathcal{L}^w_\tau(\mathbb{R}^n, \mathbb{Z}_+)) &\hookrightarrow \ell^{q_2}(\mathcal{L}^w_\tau(\mathbb{R}^n, \mathbb{Z}_+)),\\
\ell^{q_1}(\mathcal{N}\mathcal{L}^w_\tau(\mathbb{R}^n, \mathbb{Z}_+)) &\hookrightarrow \ell^{q_2}(\mathcal{N}\mathcal{L}^w_\tau(\mathbb{R}^n, \mathbb{Z}_+)),\\
\mathcal{L}^w_\tau(\ell^{q_1}(\mathbb{R}^n, \mathbb{Z}_+)) &\hookrightarrow \mathcal{L}^w_\tau(\ell^{q_2}(\mathbb{R}^n, \mathbb{Z}_+)),\\
\mathcal{E}\mathcal{L}^w_\tau(\ell^{q_1}(\mathbb{R}^n, \mathbb{Z}_+)) &\hookrightarrow \mathcal{E}\mathcal{L}^w_\tau(\ell^{q_2}(\mathbb{R}^n, \mathbb{Z}_+))
\end{aligned}$$

in the sense of continuous embeddings.

2.2. Inequalities. Suppose that we are given a quasi-normed space $\mathcal{L}(\mathbb{R}^n)$ satisfying ($\mathcal{L}1$) through ($\mathcal{L}6$). The following lemma is immediately deduced from ($\mathcal{L}4$) and ($\mathcal{L}5$). We omit the details.

Lemma 2.8. *Let* $q \in (0, \infty]$ *and* w *be as in Definition* 2.5. *If* $\mathcal{L}(\mathbb{R}^n)$ *is a quasi-normed space, then*

(i) *the quasi-norms* $\|\cdot\|_{\ell^q(\mathcal{L}^w_0(\mathbb{R}^n,\mathbb{Z}_+))}$, $\|\cdot\|_{\ell^q(\mathcal{N}\mathcal{L}^w_0(\mathbb{R}^n,\mathbb{Z}_+))}$ *and* $\|\cdot\|_{\ell^q(\mathcal{L}^w(\mathbb{R}^n,\mathbb{Z}_+))}$ *are mutually equivalent;*

(ii) *the quasi-norms* $\|\cdot\|_{\mathcal{L}^w_0(\ell^q(\mathbb{R}^n,\mathbb{Z}_+))}$, $\|\cdot\|_{\mathcal{E}\mathcal{L}^w_0(\ell^q(\mathbb{R}^n,\mathbb{Z}_+))}$ *and* $\|\cdot\|_{\mathcal{L}^w(\ell^q(\mathbb{R}^n,\mathbb{Z}_+))}$ *are mutually equivalent.*

In view of Lemma 2.8, in what follows, we identify the spaces appearing, respectively, in (i) and (ii) of Lemma 2.8.

The fundamental estimates (2.13)–(2.16) below follow from the Hölder inequality and the conditions (W1) and (W2). However, we need to keep in mind that the condition (2.12) below is used throughout the present paper.

Lemma 2.9. *Let* $D_1, D_2, \alpha_1, \alpha_2, \alpha_3, \tau \in [0, \infty)$ *and* $q \in (0, \infty]$ *be fixed parameters satisfying*

$$D_1 \in (\alpha_1, \infty), \quad D_2 \in (n\tau + \alpha_2, \infty). \tag{2.12}$$

Suppose that $\{g_\nu\}_{\nu\in\mathbb{Z}_+}$ *is a given family of measurable functions on* $\mathbb{R}^n$ *and* $w \in \mathcal{W}^{\alpha_3}_{\alpha_1,\alpha_2}$. *For all* $j \in \mathbb{Z}_+$ *and* $x \in \mathbb{R}^n$, *let*

$$G_j(x) := \sum_{\nu=0}^{j} 2^{-(j-\nu)D_2} g_\nu(x) + \sum_{\nu=j+1}^{\infty} 2^{-(\nu-j)D_1} g_\nu(x).$$

If $\mathcal{L}(\mathbb{R}^n)$ *satisfies* ($\mathcal{L}1$) *through* ($\mathcal{L}4$), *then the following estimates, with the implicit positive constants independent of* $\{g_\nu\}_{\nu\in\mathbb{Z}_+}$, *hold:*

$$\|\{G_j\}_{j\in\mathbb{Z}_+}\|_{\ell^q(\mathcal{L}^w_\tau(\mathbb{R}^n,\mathbb{Z}_+))} \lesssim \|\{g_\nu\}_{\nu\in\mathbb{Z}_+}\|_{\ell^q(\mathcal{L}^w_\tau(\mathbb{R}^n,\mathbb{Z}_+))}, \tag{2.13}$$

$$\|\{G_j\}_{j\in\mathbb{Z}_+}\|_{\ell^q(\mathcal{N}\mathcal{L}^w_\tau(\mathbb{R}^n,\mathbb{Z}_+))} \lesssim \|\{g_\nu\}_{\nu\in\mathbb{Z}_+}\|_{\ell^q(\mathcal{N}\mathcal{L}^w_\tau(\mathbb{R}^n,\mathbb{Z}_+))}, \tag{2.14}$$

$$\|\{G_j\}_{j\in\mathbb{Z}_+}\|_{\mathcal{L}^w_\tau(\ell^q(\mathbb{R}^n,\mathbb{Z}_+))} \lesssim \|\{g_\nu\}_{\nu\in\mathbb{Z}_+}\|_{\mathcal{L}^w_\tau(\ell^q(\mathbb{R}^n,\mathbb{Z}_+))}, \tag{2.15}$$

$$\|\{G_j\}_{j\in\mathbb{Z}_+}\|_{\mathcal{E}\mathcal{L}^w_\tau(\ell^q(\mathbb{R}^n,\mathbb{Z}_+))} \lesssim \|\{g_\nu\}_{\nu\in\mathbb{Z}_+}\|_{\mathcal{E}\mathcal{L}^w_\tau(\ell^q(\mathbb{R}^n,\mathbb{Z}_+))}. \tag{2.16}$$

Proof. Let us prove (2.15). The other proofs are similar. Let us write

$$\mathrm{I}(P) := \frac{1}{|P|^\tau}\Big\|\chi_P\Big[\sum_{j=j_P\vee 0}^{\infty}\Big|\sum_{\nu=0}^{j} w_j 2^{(\nu-j)D_2} g_\nu\Big|^q\Big]^{1/q}\Big\|_{\mathcal{L}(\mathbb{R}^n)} + \frac{1}{|P|^\tau}\Big\|\chi_P\Big[\sum_{j=j_P\vee 0}^{\infty}\Big|\sum_{\nu=j+1}^{\infty} w_j 2^{(j-\nu)D_1} g_\nu\Big|^q\Big]^{1/q}\Big\|_{\mathcal{L}(\mathbb{R}^n)},$$

where P is a dyadic cube chosen arbitrarily. If $j,\nu\in\mathbb{Z}_+$ and $\nu\ge j$, then by (2.3), we know that, for all $x\in\mathbb{R}^n$,

$$w_j(x) \lesssim 2^{-\alpha_1(j-\nu)} w_\nu(x). \tag{2.17}$$

If $j,\nu\in\mathbb{Z}_+$ and $j\ge\nu$, then by (2.3), we see that, for all $x\in\mathbb{R}^n$,

$$w_j(x) \lesssim 2^{\alpha_2(j-\nu)} w_\nu(x). \tag{2.18}$$

If we combine (2.17) and (2.18), then we conclude that, for all $x\in\mathbb{R}^n$ and $j,\nu\in\mathbb{Z}_+$,

$$w_j(x) \lesssim \begin{cases} 2^{-\alpha_1(j-\nu)} w_\nu(x), & \nu\ge j,\\ 2^{\alpha_2(j-\nu)} w_\nu(x), & \nu\le j.\end{cases} \tag{2.19}$$

We need to show that

$$\mathrm{I}(P) \lesssim \|\{g_\nu\}_{\nu\in\mathbb{Z}_+}\|_{\mathcal{L}^w_\tau(\ell^q(\mathbb{R}^n,\mathbb{Z}_+))}$$

with the implicit constant independent of P and $\{g_\nu\}_{\nu\in\mathbb{Z}_+}$ in view of the definitions of $\{G_j\}_{j\in\mathbb{Z}_+}$ and $\|\{G_j\}_{j\in\mathbb{Z}_+}\|_{\mathcal{L}^w_\tau(\ell^q(\mathbb{R}^n,\mathbb{Z}_+))}$.

Suppose $q\in(0,1]$ for the moment. Then we deduce from (2.19) and ($\mathcal{L}4$) that

$$\mathrm{I}(P) \lesssim \frac{1}{|P|^\tau}\Big\|\chi_P\Big[\sum_{j=j_P\vee 0}^{\infty}\sum_{\nu=0}^{j} 2^{-(j-\nu)(D_2-\alpha_2)q}|w_\nu g_\nu|^q\Big]^{1/q}\Big\|_{\mathcal{L}(\mathbb{R}^n)} + \frac{1}{|P|^\tau}\Big\|\chi_P\Big[\sum_{j=j_P\vee 0}^{\infty}\sum_{\nu=j+1}^{\infty} 2^{-(\nu-j)(D_1-\alpha_1)q}|w_\nu g_\nu|^q\Big]^{1/q}\Big\|_{\mathcal{L}(\mathbb{R}^n)} \tag{2.20}$$

since, for all $r\in(0,1]$ and $\{a_j\}_j\subset\mathbb{C}$,

$$\Big(\sum_j |a_j|\Big)^r \le \sum_j |a_j|^r. \tag{2.21}$$

In (2.20), we change the order of the summations on the right-hand side to obtain

$$\mathrm{I}(P) \lesssim \frac{1}{|P|^\tau}\Big\|\chi_P\Big[\sum_{\nu=0}^{\infty}\sum_{j=\nu\vee j_P\vee 0}^{\infty} 2^{-(j-\nu)(D_2-\alpha_2)q}|w_\nu g_\nu|^q\Big]^{1/q}\Big\|_{\mathcal{L}(\mathbb{R}^n)} + \frac{1}{|P|^\tau}\Big\|\chi_P\Big[\sum_{\nu=j_P\vee 0}^{\infty}\sum_{j=j_P\vee 0}^{\nu} 2^{-(\nu-j)(D_1-\alpha_1)q}|w_\nu g_\nu|^q\Big]^{1/q}\Big\|_{\mathcal{L}(\mathbb{R}^n)}.$$

Now we decompose the summations with respect to ν according to $\nu\ge j_P\vee 0$ or $\nu< j_P\vee 0$. Since $D_2\in(\alpha_2+n\tau,\infty)$, we can choose $\epsilon\in(0,\infty)$ such that $D_2\in(\alpha_2+n\tau+\epsilon,\infty)$.

From this, $D_1 \in (\alpha_1, \infty)$, the Hölder inequality, ($\mathcal{L}2$) and ($\mathcal{L}4$), it follows that

$$\begin{aligned}
\mathrm{I}(P) &\lesssim \|\{g_\nu\}_{\nu\in\mathbb{Z}_+}\|_{\mathcal{L}^w_\tau(\ell^q(\mathbb{R}^n,\mathbb{Z}_+))} \\
&\quad + \frac{1}{|P|^\tau}\Big\|\chi_P\Big[\sum_{\nu=0}^{j_P\vee 0}\sum_{j=j_P\vee 0}^{\infty} 2^{-(j-\nu)(D_2-\alpha_2)q}|w_\nu g_\nu|^q\Big]^{1/q}\Big\|_{\mathcal{L}(\mathbb{R}^n)} \\
&\lesssim \|\{g_\nu\}_{\nu\in\mathbb{Z}_+}\|_{\mathcal{L}^w_\tau(\ell^q(\mathbb{R}^n,\mathbb{Z}_+))} \\
&\quad + \frac{2^{-(j_P\vee 0)(D_2-\alpha_2-\epsilon)}}{|P|^\tau}\Big\|\chi_P\sum_{\nu=0}^{j_P\vee 0} 2^{\nu(D_2-\alpha_2-\epsilon)}|w_\nu g_\nu|\Big\|_{\mathcal{L}(\mathbb{R}^n)}.
\end{aligned} \tag{2.22}$$

We write $2^{j_P\vee 0-\nu}P$ for the $2^{j_P\vee 0-\nu}$-fold expansion of P as in our conventions at the end of Section 1. If we use the assumption ($\mathcal{L}3$), we see that

$$\begin{aligned}
\mathrm{I}(P) &\lesssim \|\{g_\nu\}_{\nu\in\mathbb{Z}_+}\|_{\mathcal{L}^w_\tau(\ell^q(\mathbb{R}^n,\mathbb{Z}_+))} \\
&\quad + \frac{2^{-(j_P\vee 0)(D_2-\alpha_2-\epsilon)}}{|P|^\tau}\Big\{\sum_{\nu=0}^{j_P\vee 0}\|2^{\nu(D_2-\alpha_2-\epsilon)}\chi_P w_\nu g_\nu\|^\theta_{\mathcal{L}(\mathbb{R}^n)}\Big\}^{1/\theta} \\
&\lesssim \|\{g_\nu\}_{\nu\in\mathbb{Z}_+}\|_{\mathcal{L}^w_\tau(\ell^q(\mathbb{R}^n,\mathbb{Z}_+))} + 2^{-(j_P\vee 0)(D_2-\alpha_2-\epsilon)} \\
&\quad \times\Big\{\sum_{\nu=0}^{j_P\vee 0}\Big[\frac{2^{\nu(D_2-\alpha_2-n\tau-\epsilon)+n\tau(j_P\vee 0)}}{|2^{(j_P\vee 0)-\nu}P|^\tau}\|\chi_{2^{(j_P\vee 0)-\nu}P}w_\nu g_\nu\|_{\mathcal{L}(\mathbb{R}^n)}\Big]^\theta\Big\}^{1/\theta} \\
&\lesssim \|\{g_\nu\}_{\nu\in\mathbb{Z}_+}\|_{\mathcal{L}^w_\tau(\ell^q(\mathbb{R}^n,\mathbb{Z}_+))}.
\end{aligned}$$

Since the dyadic cube P is arbitrary, by taking the supremum of all P, the proof of the case $q \in (0, 1]$ is complete.

When $q \in (1, \infty]$, choose $\kappa \in (0, \infty)$ such that $\kappa + \alpha_1 < D_1$ and $\kappa + n\tau + \alpha_2 < D_2$. Then, by the Hölder inequality, we are led to

$$\begin{aligned}
\mathrm{I}(P) \le \frac{1}{|P|^\tau}\Big\{&\Big\|\chi_P\Big[\sum_{j=j_P\vee 0}^{\infty}\sum_{\nu=0}^{j} 2^{-(j-\nu)(D_2-\kappa-\alpha_2)q}|w_\nu g_\nu|^q\Big]^{1/q}\Big\|_{\mathcal{L}(\mathbb{R}^n)} \\
&+\Big\|\chi_P\Big[\sum_{j=j_P\vee 0}^{\infty}\sum_{\nu=j+1}^{\infty} 2^{-(\nu-j)(D_2-\kappa-\alpha_2)q}|w_\nu g_\nu|^q\Big]^{1/q}\Big\|_{\mathcal{L}(\mathbb{R}^n)}\Big\},
\end{aligned}$$

where the only difference from (2.20) is that D_1 and D_2 are, respectively, replaced by $D_1 - \kappa$ and $D_2 - \kappa$. With this replacement, the same argument as above works. This finishes the proof of Lemma 2.9. ■

The following lemma is frequently used in the present paper; it appeared in [18, Lemmas B.1 and B.2], [20, p. 466], [24, Lemmas 1.2.8 and 1.2.9], [71, Lemma 1] or [93, Lemma A.3]. In the last reference the result is stated in terms of the continuous wavelet transform. Denote by ω_n the volume of the unit ball in $\mathbb{R}^n$ and by $C^L(\mathbb{R}^n)$ the space of all functions having continuous derivatives up to order L.

Lemma 2.10. *Let $j, \nu \in \mathbb{Z}_+$, $M, N \in (0, \infty)$, and $L \in \mathbb{N} \cup \{0\}$ satisfy $\nu \ge j$ and $N > M + L + n$. Suppose that $\phi_j \in C^L(\mathbb{R}^n)$ has the property that, for all $\|\vec{\alpha}\|_1 = L$,*

$$|\partial^{\vec{\alpha}}\phi_j(x)| \le A_{\vec{\alpha}}\frac{2^{j(n+L)}}{(1+2^j|x-x_j|)^M},$$

where $A_{\vec{\alpha}}$ *is a positive constant independent of* j, x *and* x_j. *Furthermore, suppose that* ϕ_ν *is another measurable function such that, for all* $\|\vec{\beta}\|_1 \le L-1$,

$$\int_{\mathbb{R}^n}\phi_\nu(y)y^{\vec{\beta}}\,dy = 0 \quad \textit{and, for all } x\in\mathbb{R}^n, \quad |\phi_\nu(x)| \le B\frac{2^{\nu n}}{(1+2^\nu|x-x_\nu|)^N},$$

where the former condition is supposed to be vacuous when $L=0$. *Then*

$$\left|\int_{\mathbb{R}^n}\phi_j(x)\phi_\nu(x)dx\right| \le \left(\sum_{\|\vec{\alpha}\|_1=L}\frac{A_{\vec{\alpha}}}{\vec{\alpha}!}\right)\frac{N-M-L}{N-M-L-n}B\omega_n\, 2^{jn-(\nu-j)L}(1+2^j|x_j-x_\nu|)^{-M}.$$

3. Besov-type and Triebel–Lizorkin-type spaces

3.1. Definitions. Through the spaces in Definition 2.5, we introduce the following Besov-type and Triebel–Lizorkin-type spaces on $\mathbb{R}^n$.

DEFINITION 3.1. Let $a \in (0,\infty)$, $\alpha_1, \alpha_2, \alpha_3, \tau \in [0,\infty)$, $q \in (0,\infty]$ and $w \in \mathcal{W}^{\alpha_3}_{\alpha_1,\alpha_2}$. Assume that Φ, $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfy, respectively, (1.3) and (1.4) and that $\mathcal{L}(\mathbb{R}^n)$ is a quasi-normed space satisfying ($\mathcal{L}1$) through ($\mathcal{L}4$). For any $f \in \mathcal{S}'(\mathbb{R}^n)$, let $\{(\varphi_j^* f)_a\}_{j\in\mathbb{Z}_+}$ be as in (1.1).

(i) The *inhomogeneous generalized Besov-type space* $B^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$ is defined to be the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{B^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} := \|\{(\varphi_j^* f)_a\}_{j\in\mathbb{Z}_+}\|_{\ell^q(\mathcal{L}^w_\tau(\mathbb{R}^n,\mathbb{Z}_+))} < \infty.$$

(ii) The *inhomogeneous generalized Besov–Morrey space* $\mathcal{N}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$ is defined to be the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{\mathcal{N}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} := \|\{(\varphi_j^* f)_a\}_{j\in\mathbb{Z}_+}\|_{\ell^q(\mathcal{N}\mathcal{L}^w_\tau(\mathbb{R}^n,\mathbb{Z}_+))} < \infty.$$

(iii) The *inhomogeneous generalized Triebel–Lizorkin-type space* $F^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$ is defined to be the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{F^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} := \|\{(\varphi_j^* f)_a\}_{j\in\mathbb{Z}_+}\|_{\mathcal{L}^w_\tau(\ell^q(\mathbb{R}^n,\mathbb{Z}_+))} < \infty.$$

(iv) The *inhomogeneous generalized Triebel–Lizorkin–Morrey space* $\mathcal{E}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$ is defined to be the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{\mathcal{E}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} := \|\{(\varphi_j^* f)_a\}_{j\in\mathbb{Z}_+}\|_{\mathcal{E}\mathcal{L}^w_\tau(\ell^q(\mathbb{R}^n,\mathbb{Z}_+))} < \infty.$$

The notation $A^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$ stands for either one of $B^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$, $\mathcal{N}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$, $F^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$ or $\mathcal{E}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$. When $\mathcal{L}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ and $w_j(x) := 2^{js}$ for $x \in \mathbb{R}^n$ and $j \in \mathbb{Z}_+$, we write

$$A^{s,\tau}_{p,q,a}(\mathbb{R}^n) := A^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n). \tag{3.1}$$

In what follows, if $\tau = 0$, we omit τ in the notation of the spaces introduced in Definition 3.1.

REMARK 3.2. Let us review what parameters the function spaces carry with.

(i) The function space $\mathcal{L}(\mathbb{R}^n)$ is equipped with $\theta, N_0, \gamma, \delta$ satisfying

$$\theta \in (0,1], \quad N_0 \in (0,\infty), \quad \gamma \in [0,\infty), \quad \delta \in [0,\infty). \tag{3.2}$$

(ii) The class $\mathcal{W}^{\alpha_3}_{\alpha_1,\alpha_2}$ of weights is equipped with $\alpha_1, \alpha_2, \alpha_3$ satisfying

$$\alpha_1, \alpha_2, \alpha_3 \in [0,\infty). \tag{3.3}$$

(iii) In general function spaces $A^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$, the indices τ, q and a satisfy

$$\tau \in [0,\infty), \qquad q \in (0,\infty], \qquad a \in (N_0+\alpha_3,\infty), \tag{3.4}$$

where in (3.28) below we need to assume $a \in (N_0+\alpha_3,\infty)$ in order to guarantee that $\mathcal{S}(\mathbb{R}^n)$ is contained in the function space.

In the following, we content ourselves with considering the case when $\mathcal{L}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ as an example, which still enables us to see why we introduce these function spaces in this way. Further examples are given in Section 11.

Example 3.3. Let $q \in (0,\infty]$, $s \in \mathbb{R}$ and $\tau \in [0,\infty)$. In [97, 98], the *Besov-type space* $B^{s,\tau}_{p,q}(\mathbb{R}^n)$ with $p \in (0,\infty]$ and the *Triebel–Lizorkin-type space* $F^{s,\tau}_{p,q}(\mathbb{R}^n)$ with $p \in (0,\infty)$ were, respectively, defined to be the sets of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{B^{s,\tau}_{p,q}(\mathbb{R}^n)} := \sup_{P\in\mathcal{Q}(\mathbb{R}^n)} \frac{1}{|P|^\tau}\left\{\sum_{j=j_P\vee 0}^{\infty}\left[\int_P |2^{js}\varphi_j * f(x)|^p\,dx\right]^{q/p}\right\}^{1/q} < \infty$$

and

$$\|f\|_{F^{s,\tau}_{p,q}(\mathbb{R}^n)} := \sup_{P\in\mathcal{Q}(\mathbb{R}^n)} \frac{1}{|P|^\tau}\left\{\int_P \left[\sum_{j=j_P\vee 0}^{\infty} |2^{js}\varphi_j * f(x)|^q\right]^{p/q} dx\right\}^{1/p} < \infty$$

with the usual modifications for $p=\infty$ or $q=\infty$. Here φ_0 is understood to be Φ. Then, we have shown in [45] that $B^{s,\tau}_{p,q,a}(\mathbb{R}^n)$ coincides with $B^{s,\tau}_{p,q}(\mathbb{R}^n)$ as long as $a \in (n/p,\infty)$. Likewise $F^{s,\tau}_{p,q,a}(\mathbb{R}^n)$ coincides with $F^{s,\tau}_{p,q}(\mathbb{R}^n)$ as long as $a \in (n/\min(p,q),\infty)$. Notice that $B^{s,0}_{p,q,a}(\mathbb{R}^n)$ and $F^{s,0}_{p,q,a}(\mathbb{R}^n)$ are isomorphic to $B^{s}_{p,q}(\mathbb{R}^n)$ and $F^{s}_{p,q}(\mathbb{R}^n)$ respectively by the Plancherel–Pólya–Nikol'skiĭ inequality (Lemma 1.1) and the Fefferman–Stein vector-valued inequality (see [15, 19, 20, 88]). This fact is generalized to our current setting. The atomic decomposition of these spaces can be found in [82, 104]. Needless to say, in this setting, $\mathcal{L}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ satisfies ($\mathcal{L}$1) through ($\mathcal{L}$6).

Observe that the function spaces $B^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$, $F^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$, $\mathcal{N}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$ and $\mathcal{E}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$ depend upon $a \in (0,\infty)$, as the following example shows.

Example 3.4. Let $m \in \mathbb{N}$, $b \in (0,\infty)$, $f_m(t) := [2\sin(2^{-2mb}t)/t]^m$ for all $t \in \mathbb{R}$, and $\mathcal{L}(\mathbb{R}) = L^p(\mathbb{R})$ with $p \in (0,\infty]$. If τ, a, q and w are as in Definition 3.1 with $w(x,1)$ independent of $x \in \mathbb{R}$, then $f_m \in B^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}) \cup F^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}) \cup \mathcal{N}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}) \cup \mathcal{E}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R})$ if and only if

$$p[\min(a,m)] > 1,$$

and, in this case, we have $f_m \in B^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}) \cap F^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}) \cap \mathcal{N}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}) \cap \mathcal{E}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R})$. To see this, notice that, for all $t \in \mathbb{R}$,

$$\widehat{\chi}_{[-2^{-mb},2^{-mb}]}(t) = \int_{-2^{-mb}}^{2^{-mb}} \cos(xt)\,dx = \frac{2\sin(2^{-mb}t)}{t},$$

which implies that

$$\widehat{f}_m := \overbrace{\chi_{[-2^{-mb},2^{-mb}]} * \cdots * \chi_{[-2^{-mb},2^{-mb}]}}^{m \text{ times}}$$

and that $\operatorname{supp} \widehat{f_m} \subset [-m2^{-mb}, m2^{-mb}]$. Choose $b \in (0,\infty)$ large enough that

$$[-m2^{-mb}, m2^{-mb}] \subset [-1/2, 1/2].$$

Let $\Phi, \varphi \in \mathcal{S}(\mathbb{R})$ satisfy (1.3) and (1.4), and assume additionally that

$$\chi_{B(0,1)} \le \widehat{\Phi} \le \chi_{B(0,2)} \quad \text{and} \quad \operatorname{supp} \widehat{\varphi} \subset \{\xi \in \mathbb{R} : 1/2 \le |\xi| \le 2\}.$$

Then, by the size of the frequency support, we see that $\Phi * f_m = f_m$ and $\varphi_j * f_m = 0$ for all $j \in \mathbb{N}$. Therefore, for all $x \in \mathbb{R}$,

$$(\Phi^* f_m)_a(x) = \sup_{z\in\mathbb{R}} \frac{|2\sin(2^{-mb}(x+z))|^m}{|x+z|^m(1+|z|)^a} \sim_m (1+|x|)^{\max(-a,-m)} \quad \text{and} \quad (\varphi_j^* f_m)_a(x) = 0,$$

which implies the claim. Here, "$\sim_m$" indicate that the implicit constants depend on m.

First, we wish to justify Definition 3.1. We show that the spaces $A^{s,\tau}_{p,q,a}(\mathbb{R}^n)$ are independent of the choices of Φ and φ by proving the following Theorem 3.5, which covers the local means as well. Notice that a special case $A^{s,\tau}_{p,q,a}(\mathbb{R}^n)$ of these results was dealt with in [99, 105].

THEOREM 3.5. *Let $a, \alpha_1, \alpha_2, \alpha_3, \tau, q$, w and $\mathcal{L}(\mathbb{R}^n)$ be as in Definition* 3.1. *Let $L \in \mathbb{Z}_+$ be such that*

$$L + 1 > \alpha_1 \vee (a + n\tau + \alpha_2). \tag{3.5}$$

Assume that $\Psi, \psi \in \mathcal{S}(\mathbb{R}^n)$ have the property that, for all $\alpha \in \mathbb{Z}_+^n$ with $\|\alpha\|_1 \le L$ and some $\varepsilon \in (0,\infty)$,

$$\widehat{\Psi}(\xi) \ne 0 \ \text{if } |\xi| < 2\varepsilon, \quad \partial^\alpha \widehat{\psi}(0) = 0, \quad \text{and} \quad \widehat{\psi}(\xi) \ne 0 \ \text{if } \varepsilon/2 < |\xi| < 2\varepsilon. \tag{3.6}$$

Let $\psi_j(\cdot) := 2^{jn}\psi(2^j\cdot)$ for all $j \in \mathbb{N}$ and $\{(\psi_j^* f)_a\}_{j\in\mathbb{Z}_+}$ be as in (1.1) *with Φ and φ replaced, respectively, by Ψ and ψ. Then*

$$\|f\|_{B^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} \sim \|\{(\psi_j^* f)_a\}_{j\in\mathbb{Z}_+}\|_{\ell^q(\mathcal{L}^w_\tau(\mathbb{R}^n,\mathbb{Z}_+))}, \tag{3.7}$$

$$\|f\|_{\mathcal{N}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} \sim \|\{(\psi_j^* f)_a\}_{j\in\mathbb{Z}_+}\|_{\ell^q(\mathcal{N}\mathcal{L}^w_\tau(\mathbb{R}^n,\mathbb{Z}_+))}, \tag{3.8}$$

$$\|f\|_{F^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} \sim \|\{(\psi_j^* f)_a\}_{j\in\mathbb{Z}_+}\|_{\mathcal{L}^w_\tau(\ell^q(\mathbb{R}^n,\mathbb{Z}_+))}, \tag{3.9}$$

$$\|f\|_{\mathcal{E}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} \sim \|\{(\psi_j^* f)_a\}_{j\in\mathbb{Z}_+}\|_{\mathcal{E}\mathcal{L}^w_\tau(\ell^q(\mathbb{R}^n,\mathbb{Z}_+))}, \tag{3.10}$$

with the implicit constants independent of f.

Proof. We only need to prove that, for all $f \in \mathcal{S}'(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$(\Psi^* f)_a(x) \lesssim (\Phi^* f)_a(x) + \sum_{\nu=1}^\infty 2^{-\nu(L+1-a)} \varphi_\nu^* f(x) \tag{3.11}$$

and

$$(\psi_j^* f)_a(x) \lesssim 2^{-j(L+1-a)}(\Phi^* f)_a(x) + \sum_{\nu=1}^\infty 2^{-|\nu-j|(L+1)+a[(j-\nu)\vee 0]}(\varphi_\nu^* f)_a(x). \tag{3.12}$$

Once we prove (3.11) and (3.12), we can apply Lemma 2.9 to deduce (3.7) through (3.10).

We now establish (3.12). The proof of (3.11) is easier and we omit the details. For a nonnegative integer L as in (3.5), by [72, Theorem 1.6], there exist $\Psi^\dagger, \psi^\dagger \in \mathcal{S}(\mathbb{R}^n)$ such that, for all β with $\|\beta\|_1 \le L$,

$$\int_{\mathbb{R}^n} \psi^\dagger(x) x^\beta \, dx = 0 \tag{3.13}$$

and

$$\Psi^\dagger * \Phi + \sum_{\nu=1}^{\infty} \psi_\nu^\dagger * \varphi_\nu = \delta_0 \tag{3.14}$$

in $\mathcal{S}'(\mathbb{R}^n)$, where $\psi_\nu^\dagger(\cdot) := 2^{\nu n}\psi^\dagger(2^\nu \cdot)$ for $\nu \in \mathbb{N}$ and δ_0 is the Dirac distribution at origin. We decompose ψ_j along (3.14) into

$$\psi_j = \psi_j * \Psi^\dagger * \Phi + \sum_{\nu=1}^{\infty} \psi_j * \psi_\nu^\dagger * \varphi_\nu. \tag{3.15}$$

From (3.6) and (3.13), together with Lemma 2.10, we infer that, for all $j \in \mathbb{Z}_+$ and $y \in \mathbb{R}^n$,

$$|\psi_j * \Psi^\dagger(y)| \lesssim \frac{2^{-j(L+1)}}{(1+|y|)^{n+1+a}} \quad \text{and} \quad |\psi_j * \psi_\nu^\dagger(y)| \lesssim \frac{2^{n(j\wedge\nu)-|j-\nu|(L+1)}}{(1+2^{j\wedge\nu}|y|)^{n+1+a}}. \tag{3.16}$$

By (3.16) and (3.15), we further see that, for all $j \in \mathbb{Z}_+$ and $x \in \mathbb{R}^n$,

$$\begin{aligned}
&\sup_{z\in\mathbb{R}^n} \frac{|\psi_j * f(x+z)|}{(1+2^j|z|)^a} \\
&\quad \lesssim 2^{-j(L+1-a)}(\Phi^* f)_a(x) + \sum_{\nu=1}^{\infty} 2^{-|j-\nu|(L+1)}(\varphi_\nu^* f)_a(x) \int_{\mathbb{R}^n} \frac{2^{n(j\wedge\nu)}(1+2^\nu|y|)^a}{(1+2^{j\wedge\nu}|y|)^{n+1+a}}\, dy \\
&\quad \lesssim 2^{-j(L+1-a)}(\Phi^* f)_a(x) + \sum_{\nu=1}^{\infty} 2^{-|j-\nu|(L+1)+a[(j-\nu)\vee 0]}(\varphi_\nu^* f)_a(x) \int_{\mathbb{R}^n} \frac{2^{n(j\wedge\nu)}\, dy}{(1+2^{j\wedge\nu}|y|)^{n+1}} \\
&\quad \sim 2^{-j(L+1-a)}(\Phi^* f)_a(x) + \sum_{\nu=1}^{\infty} 2^{-|j-\nu|(L+1)+a[(j-\nu)\vee 0]}(\varphi_\nu^* f)_a(x),
\end{aligned}$$

which completes the proof of (3.12) and hence of Theorem 3.5. ∎

Notice that the moment condition on Ψ in Theorem 3.5 is not necessary due to (3.6). Moreover, in view of the calculation presented in the proof of Theorem 3.5, we also have the following assertion.

Corollary 3.6. *Under the notation of Theorem 3.5, for some $N \in \mathbb{N}$ and all $x \in \mathbb{R}^n$, let*

$$\mathfrak{M}f(x, 2^{-j}) := \begin{cases} \sup\limits_{\psi} |\psi_j * f(x)|, & j \in \mathbb{N}, \\ \sup\limits_{\Psi} |\Psi * f(x)|, & j = 0, \end{cases}$$

where the supremum is taken over all ψ and Ψ in $\mathcal{S}(\mathbb{R}^n)$ satisfying

$$\sum_{\|\alpha\|_1 \le N} \sup_{x\in\mathbb{R}^n} (1+|x|)^N |\partial^\alpha \psi(x)| + \sum_{\|\alpha\|_1 \le N} \sup_{x\in\mathbb{R}^n} (1+|x|)^N |\partial^\alpha \Psi(x)| \le 1$$

as well as (3.6). *Then, if N is large enough, for all $f \in \mathcal{S}'(\mathbb{R}^n)$,*

$$\|f\|_{B^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} \sim \|\{\mathfrak{M}f(\cdot,2^{-j})\}_{j\in\mathbb{Z}_+}\|_{\ell^q(\mathcal{L}^w_\tau(\mathbb{R}^n,\mathbb{Z}_+))},$$
$$\|f\|_{\mathcal{N}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} \sim \|\{\mathfrak{M}f(\cdot,2^{-j})\}_{j\in\mathbb{Z}_+}\|_{\ell^q(\mathcal{N}\mathcal{L}^w_\tau(\mathbb{R}^n,\mathbb{Z}_+))},$$
$$\|f\|_{F^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} \sim \|\{\mathfrak{M}f(\cdot,2^{-j})\}_{j\in\mathbb{Z}_+}\|_{\mathcal{L}^w_\tau(\ell^q(\mathbb{R}^n,\mathbb{Z}_+))},$$
$$\|f\|_{\mathcal{E}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} \sim \|\{\mathfrak{M}f(\cdot,2^{-j})\}_{j\in\mathbb{Z}_+}\|_{\mathcal{E}\mathcal{L}^w_\tau(\ell^q(\mathbb{R}^n,\mathbb{Z}_+))},$$

with the implicit constants independent of f.

Another corollary is a characterization of these spaces via local means. Recall that $\Delta := \sum_{j=1}^n \partial^2/\partial x_j^2$ denotes the Laplacian.

Corollary 3.7. *Let $a, \alpha_1, \alpha_2, \alpha_3, \tau, q$, w and $\mathcal{L}(\mathbb{R}^n)$ be as in Definition* 3.1. *Assume that $\Psi \in C^\infty_{\rm c}(\mathbb{R}^n)$ satisfies $\chi_{B(0,1)} \le \Psi \le \chi_{B(0,2)}$. Assume, in addition, that $\psi = \Delta^{\ell_0+1}\Psi$ for some $\ell_0 \in \mathbb{Z}_+$ such that*

$$2\ell_0 + 1 > \alpha_1 \vee (a + n\tau + \alpha_2).$$

Let $\psi_j(\cdot) := 2^{jn}\psi(2^j\cdot)$ for all $j \in \mathbb{N}$ and $\{(\psi_j^* f)_a\}_{j\in\mathbb{Z}_+}$ be as in (1.1) *with Φ and φ replaced, respectively, by Ψ and ψ. Then, for all $f \in \mathcal{S}'(\mathbb{R}^n)$,*

$$\|f\|_{B^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} \sim \|\{(\psi_j^* f)_a\}_{j\in\mathbb{Z}_+}\|_{\ell^q(\mathcal{L}^w_\tau(\mathbb{R}^n,\mathbb{Z}_+))},$$
$$\|f\|_{\mathcal{N}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} \sim \|\{(\psi_j^* f)_a\}_{j\in\mathbb{Z}_+}\|_{\ell^q(\mathcal{N}\mathcal{L}^w_\tau(\mathbb{R}^n,\mathbb{Z}_+))},$$
$$\|f\|_{F^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} \sim \|\{(\psi_j^* f)_a\}_{j\in\mathbb{Z}_+}\|_{\mathcal{L}^w_\tau(\ell^q(\mathbb{R}^n,\mathbb{Z}_+))},$$
$$\|f\|_{\mathcal{E}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} \sim \|\{(\psi_j^* f)_a\}_{j\in\mathbb{Z}_+}\|_{\mathcal{E}\mathcal{L}^w_\tau(\ell^q(\mathbb{R}^n,\mathbb{Z}_+))},$$

with the implicit constants independent of f.

3.2. Fundamental properties. With the fundamental theorem on our function spaces stated and proven as above, we now take up some inclusion relations. The following lemma is immediately deduced from Lemma 2.7 and Definition 3.1.

Lemma 3.8. *Let $\alpha_1, \alpha_2, \alpha_3, \tau \in [0,\infty)$, $q, q_1, q_2 \in (0,\infty]$, $q_1 \le q_2$ and $w \in \mathcal{W}^{\alpha_3}_{\alpha_1,\alpha_2}$. Let $\mathcal{L}(\mathbb{R}^n)$ be a quasi-normed space satisfying* ($\mathcal{L}1$) *through* ($\mathcal{L}4$). *Then we have continuous embeddings*

$$B^{w,\tau}_{\mathcal{L},q_1,a}(\mathbb{R}^n) \hookrightarrow B^{w,\tau}_{\mathcal{L},q_2,a}(\mathbb{R}^n),$$
$$\mathcal{N}^{w,\tau}_{\mathcal{L},q_1,a}(\mathbb{R}^n) \hookrightarrow \mathcal{N}^{w,\tau}_{\mathcal{L},q_2,a}(\mathbb{R}^n),$$
$$F^{w,\tau}_{\mathcal{L},q_1,a}(\mathbb{R}^n) \hookrightarrow F^{w,\tau}_{\mathcal{L},q_2,a}(\mathbb{R}^n),$$
$$\mathcal{E}^{w,\tau}_{\mathcal{L},q_1,a}(\mathbb{R}^n) \hookrightarrow \mathcal{E}^{w,\tau}_{\mathcal{L},q_2,a}(\mathbb{R}^n),$$
$$B^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n),\ \mathcal{N}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n),\ F^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n),\ \mathcal{E}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n) \hookrightarrow \mathcal{N}^{w,\tau}_{\mathcal{L},\infty,a}(\mathbb{R}^n). \tag{3.17}$$

Remark 3.9. (i) It is well known that $F^s_{p,q}(\mathbb{R}^n) \hookrightarrow B^s_{p,\max(p,q)}(\mathbb{R}^n) \hookrightarrow B^s_{p,\infty}(\mathbb{R}^n)$ (see, for example, [90]). However, as an example in [73] shows, with $q \in (0,\infty]$ fixed, (3.17) is optimal in the sense that the continuous embedding $F^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n) \hookrightarrow \mathcal{N}^{w,\tau}_{\mathcal{L},r,a}(\mathbb{R}^n)$ holds for all admissible a, w, τ and $\mathcal{L}(\mathbb{R}^n)$ if and only if $r = \infty$.

(ii) From the definitions of the spaces $A^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$, we deduce that

$$A^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n) \hookrightarrow B^{w,\tau}_{\mathcal{L},\infty,a}(\mathbb{R}^n).$$

Indeed, for example, the proof of $\mathcal{E}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n) \hookrightarrow B^{w,\tau}_{\mathcal{L},\infty,a}(\mathbb{R}^n)$ is as follows:

$$\begin{aligned}\|f\|_{\mathcal{E}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} &= \sup_{P\in\mathcal{Q}(\mathbb{R}^n)} \frac{1}{|P|^\tau}\|\{\chi_{[j_P,\infty)}(j)\chi_P w_j(\varphi_j^* f)_a\}_{j=0}^\infty\|_{\mathcal{L}^w(\ell^q(\mathbb{R}^n,\mathbb{Z}_+))}\\ &\geq \sup_{P\in\mathcal{Q}(\mathbb{R}^n)} \sup_{j\geq j_P} \frac{1}{|P|^\tau}\|\chi_P w_j(\varphi_j^* f)_a\|_{\mathcal{L}(\mathbb{R}^n)} = \|f\|_{B^{w,\tau}_{\mathcal{L},\infty,a}(\mathbb{R}^n)}.\end{aligned}$$

Now we are going to discuss the lifting property, which also justifies our new framework of function spaces. Recall that, for all $f\in\mathcal{S}'(\mathbb{R}^n)$ and $\xi\in\mathbb{R}^n$, we let $((1-\Delta)^{s/2}f)\widehat{\ }(\xi) := (1+|\xi|^2)^{s/2}\widehat{f}(\xi)$ for all $\xi\in\mathbb{R}^n$.

Theorem 3.10. *Let a, α_1, α_2, α_3, τ, q, w and $\mathcal{L}(\mathbb{R}^n)$ be as in Definition 3.1 and $s\in\mathbb{R}$. For all $x\in\mathbb{R}^n$ and $j\in\mathbb{Z}_+$, let*

$$w^{(s)}(x,2^{-j}) := 2^{-js}w_j(x).$$

Then the lift operator $(1-\Delta)^{s/2}$ is bounded from $A^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$ to $A^{w^{(s)},\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$.

For the proof of Theorem 3.10, the following lemma is important. Once we prove this lemma, Theorem 3.10 is obtained by using (W1).

Lemma 3.11. *Let $a\in(0,\infty)$, $s\in\mathbb{R}$ and $\Phi,\varphi\in\mathcal{S}(\mathbb{R}^n)$ be such that*

$$\operatorname{supp}\widehat{\Phi}\subset\{\xi\in\mathbb{R}^n : |\xi|\leq 2\},\ \ \operatorname{supp}\widehat{\varphi}\subset\{\xi\in\mathbb{R}^n : 1/2\leq|\xi|\leq 2\} \text{ and } \widehat{\Phi}+\sum_{j=1}^\infty\widehat{\varphi_j}\equiv 1,$$

where $\varphi_j(\cdot) := 2^{jn}\varphi(2^j\cdot)$ for each $j\in\mathbb{N}$. Then there exists a positive constant C such that, for all $f\in\mathcal{S}'(\mathbb{R}^n)$ and $x\in\mathbb{R}^n$,

$$(\Phi^*((1-\Delta)^{s/2}f))_a(x) \leq C[(\Phi^* f)_a(x) + (\varphi_1^* f)_a(x)], \tag{3.18}$$

$$(\varphi_1^*((1-\Delta)^{s/2}f))_a(x) \leq C[(\Phi^* f)_a(x) + (\varphi_1^* f)_a(x) + (\varphi_2^* f)_a(x)], \tag{3.19}$$

$$(\varphi_j^*((1-\Delta)^{s/2}f))_a(x) \leq C2^{js}(\varphi_j^* f)_a(x) \quad \textit{for all } j\geq 2. \tag{3.20}$$

Proof. The proofs of (3.18) and (3.19) being simpler, let us prove (3.20). In view of the size of supports, we see that, for all $j\geq 2$ and $x\in\mathbb{R}^n$,

$$\begin{aligned}&(\varphi_j^*((1-\Delta)^{s/2}f))_a(x)\\ &\quad= \sup_{z\in\mathbb{R}^n}\frac{|\varphi_j * [(1-\Delta)^{s/2}f](x+z)|}{(1+2^j|z|)^a}\\ &\quad= \sup_{z\in\mathbb{R}^n}\frac{|(1-\Delta)^{s/2}(\varphi_{j-1}+\varphi_j+\varphi_{j+1}) * \varphi_j * f(x+z)|}{(1+2^j|z|)^a}\\ &\quad= \sup_{z\in\mathbb{R}^n}\frac{1}{(1+2^j|z|)^a}\left|\int_{\mathbb{R}^n}(1-\Delta)^{s/2}(\varphi_{j-1}+\varphi_j+\varphi_{j+1})(y)\varphi_j * f(x+z-y)\,dy\right|.\end{aligned}$$

Now let us show that, for all $j\geq 2$ and $y\in\mathbb{R}^n$,

$$|(1-\Delta)^{s/2}(\varphi_{j-1}+\varphi_j+\varphi_{j+1})(y)| \lesssim \frac{2^{j(s+n)}}{(1+2^j|y|)^{a+n+1}}. \tag{3.21}$$

Once we prove (3.21), by inserting it to the above equality we conclude the proof of (3.20).

To this end, we observe that, for all $j \geq 2$ and $y \in \mathbb{R}^n$,

$$(1-\Delta)^{s/2}\Big(\sum_{l=-1}^{1}\varphi_{j+l}\Big)(y) = \{(1+|\cdot|^2)^{s/2}[\widehat{\varphi}(2^{-j+1}\cdot)+\widehat{\varphi}(2^{-j}\cdot)+\widehat{\varphi}(2^{-j-1}\cdot)]\}^{\vee}(y)$$

Since, for all multiindices $\vec{\alpha}$, $j \geq 2$ and $\xi \in \mathbb{R}^n$, a pointwise estimate

$$\left|\partial^{\vec{\alpha}}((1+|\xi|^2)^{s/2}[\widehat{\varphi}(2^{-j+1}\xi)+\widehat{\varphi}(2^{-j}\xi)+\widehat{\varphi}(2^{-j-1}\xi)])\right| \lesssim 2^{(s-\|\vec{\alpha}\|_1)j}(1+2^{-j}|\xi|)^{-n-1}$$

holds, (3.21) follows from the definition of the Fourier transform. ■

The next Theorem 3.14 is mainly a consequence of the assumptions ($\mathcal{L}1$) through ($\mathcal{L}4$) and ($\mathcal{L}6$). To show it, we need to introduce a new class of weights, which are also used later.

Definition 3.12. Let $\alpha_1, \alpha_2, \alpha_3 \in [0,\infty)$. The class $\star$-$\mathcal{W}^{\alpha_3}_{\alpha_1,\alpha_2}$ of weights is defined as the set of all measurable functions $w : \mathbb{R}^n_{\mathbb{Z}_+} \to (0,\infty)$ satisfying (W1*) and (W2), where (W2) is defined as in Definition 2.3 and

(W1*) there exists a positive constant C such that, for all $x \in \mathbb{R}^n$ and $j, \nu \in \mathbb{Z}_+$ with $j \geq \nu$, $C^{-1}2^{(j-\nu)\alpha_1}w(x,2^{-\nu}) \leq w(x,2^{-j}) \leq C2^{-(\nu-j)\alpha_2}w(x,2^{-\nu})$.

It is easy to see that $\star$-$\mathcal{W}^{\alpha_3}_{\alpha_1,\alpha_2} \subsetneq \mathcal{W}^{\alpha_3}_{\alpha_1,\alpha_2}$.

Example 3.13. If $s \in [0,\infty)$ and $w_j(x) := 2^{js}$ for all $x \in \mathbb{R}^n$ and $j \in \mathbb{Z}_+$, then it is easy to see that $w \in \star$-$\mathcal{W}^0_{s,s}$.

Theorem 3.14. *Let $a, \alpha_1, \alpha_2, \alpha_3, \tau$ and q be as in Definition 3.1. If $w \in \star$-$\mathcal{W}^{\alpha_3}_{\alpha_1,\alpha_2}$ and $\mathcal{L}(\mathbb{R}^n)$ satisfies ($\mathcal{L}1$) through ($\mathcal{L}4$) and ($\mathcal{L}6$), then $A^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$ in the sense of continuous embedding.*

Proof. Let $\Phi, \varphi \in \mathcal{S}(\mathbb{R}^n)$ be as in Lemma 3.11. Then

$$\widehat{\Phi} + \sum_{j=1}^{\infty}\widehat{\varphi_j} \equiv 1. \tag{3.22}$$

We first assume that (W1*) holds with

$$\alpha_1 - N + n - \gamma + n\tau > 0 \quad \text{and} \quad N > \delta + n \tag{3.23}$$

for some $N \in (0,\infty)$.

For any $f \in A^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$, by the definition, we see that, for all $Q \in \mathcal{Q}(\mathbb{R}^n)$ with $j_Q \in \mathbb{N}$,

$$\frac{1}{|Q|^\tau}\|\chi_Q w(\cdot,2^{-j_Q})(\varphi^*_{j_Q}f)_a\|_{\mathcal{L}(\mathbb{R}^n)} \lesssim \|f\|_{A^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)}.$$

Consequently, from (W1*), we deduce that

$$\|\chi_Q w(\cdot,1)(\varphi^*_{j_Q}f)_a\|_{\mathcal{L}(\mathbb{R}^n)} \lesssim 2^{-j_Q(\alpha_1+n\tau)}\|f\|_{A^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)}. \tag{3.24}$$

Now let $\zeta \in \mathcal{S}(\mathbb{R}^n)$ and define

$$p(\zeta) := \sup_{x\in\mathbb{R}^n}(1+|x|)^{\alpha_3+N}\zeta(x).$$

Then from (3.24) and the partition $\{Q_{jk}\}_{k\in\mathbb{Z}^n}$ of $\mathbb{R}^n$, we infer that

$$\int_{\mathbb{R}^n} |\zeta(x)\varphi_j * f(x)|\,dx \lesssim p(\zeta) \sum_{k\in\mathbb{Z}^n} (1+|2^{-j}k|)^{-N-\alpha_3} \int_{Q_{jk}} |\varphi_j * f(x)|\,dx.$$

If we use the condition (W2) twice and the fact that $j \in [0,\infty)$, we obtain

$$\begin{aligned}\int_{\mathbb{R}^n} |\zeta(x)\varphi_j * f(x)|\,dx &\lesssim p(\zeta) \sum_{k\in\mathbb{Z}^n} (1+|2^{-j}k|)^{-N} \inf_{y\in Q_{jk}} w(y,1) \int_{Q_{jk}} |\varphi_j * f(x)|\,dx \\ &\lesssim p(\zeta) \sum_{k\in\mathbb{Z}^n} 2^{jN}(1+|k|)^{-N}|Q_{jk}| \inf_{y\in Q_{jk}} \{w(y,1)(\varphi_j^* f)_a(y)\}.\end{aligned}$$

Now we use (3.24) and the assumption ($\mathcal{L}6$) to conclude

$$\begin{aligned}\int_{\mathbb{R}^n} |\zeta(x)\varphi_j * f(x)|\,dx &\lesssim p(\zeta) \sum_{k\in\mathbb{Z}^n} 2^{j(N-n+\gamma)}(1+|k|)^{-N+\delta}\|\chi_{Q_{jk}} w(\cdot,1)(\varphi_j^* f)_a\|_{\mathcal{L}(\mathbb{R}^n)} \\ &\lesssim p(\zeta) \sum_{k\in\mathbb{Z}^n} 2^{-j(\alpha_1-N+n-\gamma+n\tau)}(1+|k|)^{-N+\delta}\|f\|_{A^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} \\ &\sim 2^{-j(\alpha_1-N+n-\gamma+n\tau)} p(\zeta)\|f\|_{A^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)}. \end{aligned} \tag{3.25}$$

By replacing φ_0 with Φ in the above argument, we see that

$$\int_{\mathbb{R}^n} |\zeta(x)\Phi * f(x)|\,dx \lesssim p(\zeta)\|f\|_{A^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)}. \tag{3.26}$$

Combining (3.22), (3.25) and (3.26), we then conclude that, for all $\zeta \in \mathcal{S}(\mathbb{R}^n)$,

$$|\langle f,\zeta\rangle| \le |\langle \Phi * f,\zeta\rangle| + \sum_{j=1}^{\infty} |\langle \varphi_j * f,\zeta\rangle| \lesssim p(\zeta)\|f\|_{A^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)}, \tag{3.27}$$

which implies that $f \in \mathcal{S}'(\mathbb{R}^n)$ and hence $A^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$ in the sense of continuous embedding.

We still need to remove the restriction (3.23). Indeed, for any $\alpha_1 \in [0,\infty)$ and $f \in A^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$, choose $s \in (-\infty,0)$ small enough that $\alpha_1 - s > \gamma+\delta-n\tau$. By Theorem 3.10, we have $(1-\Delta)^{s/2} f \in A^{w^{(s)},\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$. Then, defining a seminorm ρ by $\rho(\zeta) := p((1-\Delta)^{s/2}\zeta)$ for all $\zeta \in \mathcal{S}(\mathbb{R}^n)$, by (3.27), we have

$$\begin{aligned}|\langle f,\zeta\rangle| &= |\langle (1-\Delta)^{s/2} f, (1-\Delta)^{-s/2}\zeta\rangle| \\ &\lesssim \rho((1-\Delta)^{-s/2}\zeta)\|(1-\Delta)^{s/2} f\|_{A^{w^{(s)},\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} \lesssim p(\zeta)\|f\|_{A^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)}. \quad \blacksquare\end{aligned}$$

Remark 3.15. In the course of the proof of Theorem 3.14, the inequality

$$\int_{\kappa Q_{jk}} |\varphi_j * f(x)|\,dx \lesssim \kappa^M 2^{-j(\alpha_1+n+n\tau-\gamma)}(1+|k|)^{\delta}\|f\|_{A^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)}$$

is proved. Here $\kappa \ge 1$, M and the implicit constant are independent of j,k and κ.

It follows from Theorem 3.14 that we have the following conclusions, whose proof is similar to that of [90, pp. 48–49, Theorem 2.3.3]. For convenience, we give some details.

Proposition 3.16. *Let $a,\alpha_1,\alpha_2,\alpha_3,\tau$ and q be as in Definition 3.1. If $w \in \star\text{-}\mathcal{W}^{\alpha_3}_{\alpha_1,\alpha_2}$ and $\mathcal{L}(\mathbb{R}^n)$ satisfies ($\mathcal{L}1$) through ($\mathcal{L}6$), then the spaces $B^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$, $\mathcal{N}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$, $F^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$ and $\mathcal{E}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$ are complete.*

Proof. Due to similarity, we only give the proof for $F^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$. Let $\{f_l\}_{l\in\mathbb{N}}$ be a Cauchy sequence in $F^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$. Then from Theorem 3.14, it is also a Cauchy sequence in $\mathcal{S}'(\mathbb{R}^n)$. By the completeness of $\mathcal{S}'(\mathbb{R}^n)$, there exists an $f \in \mathcal{S}'(\mathbb{R}^n)$ such that, for all Schwartz functions φ, $\varphi * f_l \to \varphi * f$ pointwise as $l \to \infty$ and hence

$$\varphi * (f_l - f) = \lim_{m\to\infty} \varphi * (f_l - f_m)$$

pointwise. Therefore, for all $j \in \mathbb{Z}_+$ and $x \in \mathbb{R}^n$,

$$\sup_{z\in\mathbb{R}^n} \frac{|\varphi_j * (f_l - f)(x+z)|}{(1+2^j|z|)^a} \le \liminf_{m\to\infty} \sup_{z\in\mathbb{R}^n} \frac{|\varphi_j * (f_l - f_m)(x+z)|}{(1+2^j|z|)^a},$$

which, together with ($\mathcal{L}$4), the Fatou property of $\mathcal{L}(\mathbb{R}^n)$ in Proposition 2.2, and the Fatou property of ℓ^q, implies that

$$\limsup_{l\to\infty} \|f_l - f\|_{F^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} \le \limsup_{l\to\infty} \Big(\liminf_{m\to\infty} \|f_l - f_m\|_{F^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)}\Big) = 0.$$

Thus, $f = \lim_{m\to\infty} f_m$ in $F^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$, which shows that $F^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$ is complete. ∎

Assuming ($\mathcal{L}$6), we can prove that $\mathcal{S}(\mathbb{R}^n)$ is embedded into $A^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$.

Theorem 3.17. *Let $a, \alpha_1, \alpha_2, \alpha_3, \tau, q$ and w be as in Definition* 3.1. *Then if $\mathcal{L}(\mathbb{R}^n)$ satisfies* ($\mathcal{L}$1) *through* ($\mathcal{L}$6) *and*

$$a \in (N_0 + \alpha_3, \infty), \tag{3.28}$$

then $\mathcal{S}(\mathbb{R}^n) \hookrightarrow A^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$ in the sense of continuous embedding.

Proof. Let $f \in \mathcal{S}(\mathbb{R}^n)$. Then, for all $x \in \mathbb{R}^n$ and $j \in \mathbb{N}$, we have

$$\sup_{z\in\mathbb{R}^n} \frac{|\varphi_j * f(x+z)|}{(1+2^j|z|)^a} \lesssim \frac{1}{(1+|x|)^a} \sup_{y\in\mathbb{R}^n} (1+|y|)^{a+n+1}|f(y)|.$$

In view of (W2), ($\mathcal{L}$6) and (3.28), we have $(1+|\cdot|)^{-a}w(\cdot,1) \in \mathcal{L}(\mathbb{R}^n)$. Consequently,

$$\left\| w_j \sup_{z\in\mathbb{R}^n} \frac{|\varphi_j * f(\cdot+z)|}{(1+2^j|z|)^a} \right\|_{\mathcal{L}(\mathbb{R}^n)} \lesssim 2^{j\alpha_2} \sup_{y\in\mathbb{R}^n} (1+|y|)^{a+n+1}|f(y)|. \tag{3.29}$$

Let ϵ be a positive constant. Set $w^*_j(x) := 2^{-j(\alpha_2+n\tau+\epsilon)}w_j(x)$ for all $x \in \mathbb{R}^n$ and $j \in \mathbb{Z}_+$. The estimate (3.29) and its counterpart for $j = 0$ show that $\mathcal{S}(\mathbb{R}^n) \hookrightarrow A^{w^*,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$ and hence Theorem 3.10 shows that $\mathcal{S}(\mathbb{R}^n) \hookrightarrow A^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$. ∎

Motivated by Theorem 3.17, we postulate (3.28) on the parameter a here and below.

In analogy with Theorem 3.10, we have the following result on the boundedness of pseudo-differential operators of Hörmander–Mikhlin type.

Proposition 3.18. *Let $a, \alpha_1, \alpha_2, \alpha_3, \tau, q, w$ and $\mathcal{L}(\mathbb{R}^n)$ be as in Definition* 3.1. *Assume that $m \in C^\infty_{\rm c}(\mathbb{R}^n)$ has the property that, for all multiindices $\vec{\alpha}$,*

$$M_{\vec{\alpha}} := \sup_{\xi\in\mathbb{R}^n} (1+|\xi|)^{\|\vec{\alpha}\|_1}|\partial^{\vec{\alpha}}m(\xi)| < \infty.$$

Define $I_m f := (m\hat{f})^\vee$. Then the operator I_m is bounded on $A^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$ and there exists $K \in \mathbb{N}$ such that the operator norm is bounded by a positive constant multiple of $\sum_{\|\vec{\alpha}\|_1\le K} M_{\vec{\alpha}}$.

Proof. Going through an argument similar to the proof of Lemma 3.11, we are led to (3.21) with $s=0$ and $(1-\Delta)^{s/2}$ replaced by I_m. Except this change, the same argument works. We omit the details. ∎

In Chapter 5 below, we will give some further results on pseudo-differential operators. To conclude this section, we investigate an embedding of Sobolev type.

PROPOSITION 3.19. *Let $a, \alpha_1, \alpha_2, \alpha_3, \tau, q, w$ and $\mathcal{L}(\mathbb{R}^n)$ be as in Definition 3.1. Define*

$$w_j^*(x) := 2^{j(\tau-\gamma)}(1+|x|)^{\delta} w_j(x) \tag{3.30}$$

for all $x \in \mathbb{R}^n$ and $j \in \mathbb{Z}_+$. Then $A^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$ is embedded into $B^{w^*}_{\infty,\infty,a}(\mathbb{R}^n)$.

Observe that if $w \in \mathcal{W}^{\alpha_3}_{\alpha_1,\alpha_2}$, then $w^* \in \mathcal{W}^{\alpha_3+\delta}_{(\alpha_1+\gamma-\tau)_+,(\alpha_2+\tau-\gamma)_+}$ and hence

$$(w^*)^{-1} \in \mathcal{W}^{\alpha_3+\delta}_{(\alpha_2+\tau-\gamma)_+,(\alpha_1+\gamma-\tau)_+}.$$

Proof of Proposition 3.19. Let $P \in \mathcal{Q}_j(\mathbb{R}^n)$ be fixed for $j \in \mathbb{Z}_+$. Then, for all $x, z \in P$,

$$\frac{|\varphi_j * f(x+y)|}{(1+2^j|y|)^a} \lesssim \frac{|\varphi_j * f(z+(y+x-z))|}{(1+2^j|y+x-z|)^a},$$

where, when $j=0$, φ_0 is replaced by Φ. Consequently, by (W2), for all $x \in P$,

$$\begin{aligned}
w_j(x)(\varphi_j^* f)_a(x) &= \sup_{u\in P}\sup_{y\in\mathbb{R}^n} w_j(u)\frac{|\varphi_j * f(u+y)|}{(1+2^j|y|)^a} \\
&\lesssim \inf_{z\in P}\sup_{u\in P}\sup_{y\in\mathbb{R}^n} w_j(u)\frac{|\varphi_j * f(z+(y+u-z))|}{(1+2^j|y+u-z|)^a} \\
&\lesssim \inf_{z\in P}\sup_{u\in P}\sup_{w\in\mathbb{R}^n} w_j(u)\frac{|\varphi_j * f(z+w)|}{(1+2^j|w|)^a} \\
&\lesssim \inf_{z\in P}\sup_{y\in\mathbb{R}^n} w(z,2^{-j})\frac{|\varphi_j * f(z+y)|}{(1+2^j|y|)^a} \lesssim \inf_{z\in P} w(z,2^{-j})(\varphi_j^* f)_a(z).
\end{aligned}$$

Thus,

$$\sup_{x\in P} w_j(x)(\varphi_j^* f)_a(x) \lesssim \frac{1}{\|\chi_P\|_{\mathcal{L}(\mathbb{R}^n)}}\|\chi_P w_j \varphi_j^* * f\|_{\mathcal{L}(\mathbb{R}^n)} \le \frac{|P|^{\tau}}{\|\chi_P\|_{\mathcal{L}(\mathbb{R}^n)}}\|f\|_{A^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)},$$

which implies the desired result. ∎

It is also of essential importance to provide a duality result of the following type, when we consider the wavelet decomposition in Section 4.

In what follows, for $p, q \in (0,\infty]$, $w \in \mathcal{W}^{\alpha_3}_{\alpha_1,\alpha_2}$ with $\alpha_1, \alpha_2, \alpha_3 \in [0,\infty)$, w_j for $j \in \mathbb{Z}_+$ as in (2.5), the space $B^w_{p,q}(\mathbb{R}^n)$ is defined to be the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{B^w_{p,q}(\mathbb{R}^n)} := \|\{w_j\varphi_j * f\}_{j\in\mathbb{Z}_+}\|_{\ell^q(L^p(\mathbb{R}^n,\mathbb{Z}_+))} < \infty,$$

where $\Phi, \varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfy (1.3) and (1.4), $\varphi_0 := \Phi$ and $\varphi_j(\cdot) := 2^{jn}\varphi(2^j\cdot)$ for all $j \in \mathbb{N}$.

PROPOSITION 3.20. *Let $\alpha_1, \alpha_2, \alpha_3 \in [0,\infty)$ and $w \in \mathcal{W}^{\alpha_3}_{\alpha_1,\alpha_2}$. Assume, in addition, that there exist $\Phi, \varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfying (1.3) and (1.4) such that*

$$\Phi * \Phi + \sum_{j=1}^{\infty} \varphi_j * \varphi_j = \delta \quad \text{in } \mathcal{S}'(\mathbb{R}^n).$$

Any $g \in B^w_{\infty,\infty}(\mathbb{R}^n)$ defines a continuous functional, L_g, on $B^{w^{-1}}_{1,1}(\mathbb{R}^n)$ such that

$$L_g : f \in B^{w^{-1}}_{1,1}(\mathbb{R}^n) \mapsto \langle \Phi * g, \Phi * f \rangle + \sum_{j=1}^{\infty} \langle \varphi_j * g, \varphi_j * f \rangle \in \mathbb{C}.$$

Proof. The proof is straightforward. Indeed, for all $g \in B^w_{\infty,\infty}(\mathbb{R}^n)$ and $f \in B^{w^{-1}}_{1,1}(\mathbb{R}^n)$,

$$|\langle \Phi * g, \Phi * f \rangle| + \sum_{j=1}^{\infty} |\langle \varphi_j * g, \varphi_j * f \rangle| \leq \|g\|_{B^w_{\infty,\infty}(\mathbb{R}^n)} \|f\|_{B^{w^{-1}}_{1,1}(\mathbb{R}^n)}. \quad \blacksquare$$

We remark that the spaces $B^w_{p,q}(\mathbb{R}^n)$ were intensively studied by Kempka [34] and it was proved in [34, p. 134] that they are independent of the choices of Φ and φ.

4. Atomic decompositions and wavelets

Now we place ourselves once again in the setting of a quasi-normed space $\mathcal{L}(\mathbb{R}^n)$ satisfying only ($\mathcal{L}$1) through ($\mathcal{L}$6); recall that we do not need to use the Hardy–Littlewood maximal operator.

For a function F on $\mathbb{R}^{n+1}_{\mathbb{Z}_+} := \mathbb{R}^n \times \{2^{-j} : j \in \mathbb{Z}_+\}$, we define

$$\|F\|_{L^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^{n+1}_{\mathbb{Z}_+})} := \left\| \left\{ \sup_{y\in\mathbb{R}^n} \frac{|F(y,2^{-j})|}{(1+2^j|\cdot - y|)^a} \right\}_{j\in\mathbb{Z}_+} \right\|_{\ell^q(\mathcal{L}^w_\tau(\mathbb{R}^n,\mathbb{Z}_+))},$$

$$\|F\|_{\mathcal{N}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^{n+1}_{\mathbb{Z}_+})} := \left\| \left\{ \sup_{y\in\mathbb{R}^n} \frac{|F(y,2^{-j})|}{(1+2^j|\cdot - y|)^a} \right\}_{j\in\mathbb{Z}_+} \right\|_{\ell^q(\mathcal{N}\mathcal{L}^w_\tau(\mathbb{R}^n,\mathbb{Z}_+))},$$

$$\|F\|_{F^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^{n+1}_{\mathbb{Z}_+})} := \left\| \left\{ \sup_{y\in\mathbb{R}^n} \frac{|F(y,2^{-j})|}{(1+2^j|\cdot - y|)^a} \right\}_{j\in\mathbb{Z}_+} \right\|_{\mathcal{L}^w_\tau(\ell^q(\mathbb{R}^n,\mathbb{Z}_+))},$$

$$\|F\|_{\mathcal{E}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^{n+1}_{\mathbb{Z}_+})} := \left\| \left\{ \sup_{y\in\mathbb{R}^n} \frac{|F(y,2^{-j})|}{(1+2^j|\cdot - y|)^a} \right\}_{j\in\mathbb{Z}_+} \right\|_{\mathcal{E}\mathcal{L}^w_\tau(\ell^q(\mathbb{R}^n,\mathbb{Z}_+))}.$$

4.1. Atoms and molecules. Now we are going to consider atomic decompositions, where we use (1.6) to denote the length of multi-indices.

DEFINITION 4.1. Let $K \in \mathbb{Z}_+$ and $L \in \mathbb{Z}_+ \cup \{-1\}$.

(i) Let $Q \in \mathcal{Q}(\mathbb{R}^n)$. A (K,L)-*atom* (*for* $A^{s,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$) *supported near* Q is a $C^K(\mathbb{R}^n)$-function $\mathfrak{A}$ satisfying

$$\text{(support condition)} \qquad \operatorname{supp}(\mathfrak{A}) \subset 3Q,$$

$$\text{(size condition)} \qquad \|\partial^{\vec{\alpha}}\mathfrak{A}\|_{L^\infty(\mathbb{R}^n)} \le |Q|^{-\|\vec{\alpha}\|_1/n},$$

$$\text{(moment condition if } \ell(Q)<1) \qquad \int_{\mathbb{R}^n} x^{\vec{\beta}}\mathfrak{A}(x)\,dx = 0,$$

for all multiindices $\vec{\alpha}$ and $\vec{\beta}$ satisfying $\|\vec{\alpha}\|_1 \le K$ and $\|\vec{\beta}\|_1 \le L$. Here the moment condition with $L=-1$ is understood to be vacuous.

(ii) A set $\{\mathfrak{A}_{jk}\}_{j\in\mathbb{Z}_+,\,k\in\mathbb{Z}^n}$ of $C^K(\mathbb{R}^n)$-functions is called a *collection of* (K,L)-*atoms* (*for* $A^{s,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$) if each $\mathfrak{A}_{jk}$ is a (K,L)-atom supported near Q_{jk}.

DEFINITION 4.2. Let $K \in \mathbb{Z}_+$, $L \in \mathbb{Z}_+ \cup \{-1\}$ and $N \in \mathbb{R}$ satisfy

$$N > L + n.$$

(i) Let $Q \in \mathcal{Q}(\mathbb{R}^n)$. A (K,L)-*molecule* (*for* $A^{s,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$) *associated with a cube* Q is a $C^K(\mathbb{R}^n)$-function $\mathfrak{M}$ satisfying

$$\text{(decay condition)} \quad |\partial^{\vec{\alpha}}\mathfrak{M}(x)| \le (1+|x-c_Q|/\ell(Q))^{-N} \quad \text{for all } x \in \mathbb{R}^n,$$

$$\text{(moment condition if } \ell(Q)<1) \quad \int_{\mathbb{R}^n} y^{\vec{\beta}}\mathfrak{M}(y)\,dy = 0,$$

for all multiindices $\vec{\alpha}$ and $\vec{\beta}$ satisfying $\|\vec{\alpha}\|_1 \le K$ and $\|\vec{\beta}\|_1 \le L$. Here c_Q and $\ell(Q)$ denote, respectively, the center and the side length of Q, and the moment condition with $L=-1$ is understood to be vacuous.

(ii) A set $\{\mathfrak{M}_{jk}\}_{j\in\mathbb{Z}_+,\,k\in\mathbb{Z}^n}$ of $C^K(\mathbb{R}^n)$-functions is called a *collection of* (K,L)*-molecules* (*for* $A^{s,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$) if each $\mathfrak{M}_{jk}$ is a (K,L)-molecule associated with Q_{jk}.

DEFINITION 4.3. Let $\alpha_1,\alpha_2,\alpha_3,\tau\in[0,\infty)$, $a\in(N_0+\alpha_3,\infty)$ and $q\in(0,\infty]$, where N_0 is from ($\mathcal{L}6$). Suppose that $w\in\mathcal{W}^{\alpha_3}_{\alpha_1,\alpha_2}$. Let $\lambda:=\{\lambda_{jk}\}_{j\in\mathbb{Z}_+,\,k\in\mathbb{Z}^n}$ be a doubly indexed complex sequence. For $(x,2^{-j})\in\mathbb{R}^n_{\mathbb{Z}_+}$, let

$$\Lambda(x,2^{-j}) := \sum_{k\in\mathbb{Z}^n}\lambda_{jk}\chi_{Q_{jk}}(x).$$

We define the following inhomogeneous sequence spaces:

(i) $b^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$ is the set of all λ such that $\|\lambda\|_{b^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} := \|\Lambda\|_{L^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^{n+1}_{\mathbb{Z}_+})} < \infty$.

(ii) $n^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$ is the set of all λ such that $\|\lambda\|_{n^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} := \|\Lambda\|_{\mathcal{N}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^{n+1}_{\mathbb{Z}_+})} < \infty$.

(iii) $f^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$ is the set of all λ such that $\|\lambda\|_{f^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} := \|\Lambda\|_{F^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^{n+1}_{\mathbb{Z}_+})} < \infty$.

(iv) $e^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$ is the set of all λ such that $\|\lambda\|_{e^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} := \|\Lambda\|_{\mathcal{E}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^{n+1}_{\mathbb{Z}_+})} < \infty$.

When $\tau=0$, then τ is omitted from the above notation.

In the present paper we consider many types of atomic decompositions. To formulate them, we make the following definition.

DEFINITION 4.4. Let X be a function space embedded into $\mathcal{S}'(\mathbb{R}^n)$ and $\mathcal{X}$ a quasi-normed space of sequences. The pair $(X,\mathcal{X})$ is said to *admit atomic decompositions* if it satisfies the following two conditions:

(i) (Analysis condition) For any $f\in X$, there exist a collection of atoms, $\{\mathfrak{A}_{jk}\}_{j\in\mathbb{Z}_+,\,k\in\mathbb{Z}^n}$, and a complex sequence $\{\lambda_{jk}\}_{j\in\mathbb{Z}_+,\,k\in\mathbb{Z}^n}$ such that

$$f=\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^n}\lambda_{jk}\mathfrak{A}_{jk}$$

in $\mathcal{S}'(\mathbb{R}^n)$ and $\|\{\lambda_{jk}\}_{j\in\mathbb{Z}_+,\,k\in\mathbb{Z}^n}\|_{\mathcal{X}} \lesssim \|f\|_X$ with the implicit constant independent of f.

(ii) (Synthesis condition) Given a collection of atoms, $\{\mathfrak{A}_{jk}\}_{j\in\mathbb{Z}_+,\,k\in\mathbb{Z}^n}$, and a complex sequence $\{\lambda_{jk}\}_{j\in\mathbb{Z}_+,\,k\in\mathbb{Z}^n}$ satisfying $\|\{\lambda_{jk}\}_{j\in\mathbb{Z}_+,\,k\in\mathbb{Z}^n}\|_{\mathcal{X}}<\infty$, the series $f := \sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^n}\lambda_{jk}\mathfrak{A}_{jk}$ converges in $\mathcal{S}'(\mathbb{R}^n)$ and $\|f\|_X \lesssim \|\{\lambda_{jk}\}_{j\in\mathbb{Z}_+,\,k\in\mathbb{Z}^n}\|_{\mathcal{X}}$ with the implicit constant independent of $\{\lambda_{jk}\}_{j\in\mathbb{Z}_+,\,k\in\mathbb{Z}^n}$.

In analogy, a pair $(X,\mathcal{X})$ can be said to admit *molecular decompositions* or *wavelet decompositions*, where the definition of wavelets appears in Subsection 4.4 below.

In this section, we aim to prove the following conclusion.

Theorem 4.5. *Let $K \in \mathbb{Z}_+$, $L \in \mathbb{Z}_+$, $\alpha_1, \alpha_2, \alpha_3, \tau \in [0, \infty)$ and $q \in (0, \infty]$. Suppose that $w \in \mathcal{W}^{\alpha_3}_{\alpha_1,\alpha_2}$ and that* (3.28) *holds, namely, $a \in (N_0 + \alpha_3, \infty)$. Let δ be as in* ($\mathcal{L}6$). *Assume, in addition, that*

$$L > \alpha_3 + \delta + n - 1 + \gamma - n\tau + \alpha_1, \tag{4.1}$$

$$N > L + \alpha_3 + \delta + 2n, \tag{4.2}$$

$$K + 1 > \alpha_2 + n\tau,\ L + 1 > \alpha_1. \tag{4.3}$$

Then the pair $(A^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n), a^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n))$ admits atomic/molecular decompositions.

4.2. Proof of Theorem 4.5. The proof is made up of several lemmas. Our primary concern is the following question:

Do the series $\sum_{j=0}^{\infty} \sum_{k\in\mathbb{Z}^n} \lambda_{jk}\mathfrak{A}_{jk}$ and $\sum_{j=0}^{\infty} \sum_{k\in\mathbb{Z}^n} \lambda_{jk}\mathfrak{M}_{jk}$ converge in $\mathcal{S}'(\mathbb{R}^n)$?

Recall again that we are assuming only ($\mathcal{L}1$) through ($\mathcal{L}6$).

Lemma 4.6. *Let $\alpha_1, \alpha_2, \alpha_3 \in [0, \infty)$ and $w \in \mathcal{W}^{\alpha_3}_{\alpha_1,\alpha_2}$. Assume, in addition, that the parameters $K \in \mathbb{Z}_+$, $L \in \mathbb{Z}_+$ and $N \in (0, \infty)$ in Definition* 4.2 *satisfy* (4.1)–(4.3). *Assume that $\lambda := \{\lambda_{jk}\}_{j\in\mathbb{Z}_+,\, k\in\mathbb{Z}^n} \in b^{w,\tau}_{\mathcal{L},\infty,a}(\mathbb{R}^n)$ and $\{\mathfrak{M}_{jk}\}_{j\in\mathbb{Z}_+,\, k\in\mathbb{Z}^n}$ is a family of (K, L)-molecules. Then the series*

$$f = \sum_{j=0}^{\infty} \sum_{k\in\mathbb{Z}^n} \lambda_{jk}\mathfrak{M}_{jk} \tag{4.4}$$

converges in $\mathcal{S}'(\mathbb{R}^n)$.

Proof. Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Recall again that γ and δ are constants appearing in the assumption ($\mathcal{L}6$). By (4.1) and (4.2), we can choose $M \in (\alpha_3 + \delta + n, \infty)$ such that

$$L + 1 - \gamma - \alpha_1 - M + n\tau > 0 \quad \text{and} \quad N > L + M + n. \tag{4.5}$$

It follows from the definition of molecules and Lemma 2.10 that

$$\left|\int_{\mathbb{R}^n} \mathfrak{M}_{jk}(x)\varphi(x)\,dx\right| \lesssim 2^{-j(L+1)}(1 + 2^{-j}|k|)^{-M}.$$

By the assumption ($\mathcal{L}6$), we conclude that

$$\left|\int_{\mathbb{R}^n} \mathfrak{M}_{jk}(x)\varphi(x)\,dx\right| \lesssim 2^{-j(L+1-\gamma)}(1 + 2^{-j}|k|)^{-M}(1 + |k|)^{\delta}\|\chi_{Q_{jk}}\|_{\mathcal{L}(\mathbb{R}^n)}. \tag{4.6}$$

From the condition (W1), we deduce that, for all $j \in \mathbb{Z}_+$ and $x \in \mathbb{R}^n$, $2^{-j\alpha_1}w(x, 1) \lesssim w_j(x)$ and, from (W2), that, for all $x \in \mathbb{R}^n$, $w(0, 1) \lesssim w(x, 1)(1+|x|)^{\alpha_3}$. Combining these, we conclude that, for all $j \in \mathbb{Z}_+$ and $x \in \mathbb{R}^n$,

$$w(0, 1) \lesssim (1 + |x|)^{\alpha_3} 2^{j\alpha_1} w_j(x). \tag{4.7}$$

Consequently, we have

$$1 \lesssim (1 + |k|)^{\alpha_3} 2^{j\alpha_1} w_j(x) \tag{4.8}$$

for all $x \in Q_{jk}$ with $j \in \mathbb{Z}_+$ and $k \in \mathbb{Z}^n$. By (4.6) and (4.8), we further see that, for all $j \in \mathbb{Z}_+$ and $k \in \mathbb{Z}^n$,

$$\left|\lambda_{jk}\int_{\mathbb{R}^n}\mathfrak{M}_{jk}(x)\varphi(x)\,dx\right|\lesssim 2^{-j(L+1-\gamma-\alpha_1-M+n\tau)}(1+|k|)^{-M+\alpha_3+\delta}\|\lambda\|_{b^{w,\tau}_{\mathcal{L},\infty,a}(\mathbb{R}^n)}. \quad (4.9)$$

So by (4.5), this inequality can be summed over $j\in\mathbb{Z}_+$ and $k\in\mathbb{Z}^n$, which completes the proof. ∎

In view of Lemma 3.8, Lemma 4.6 is sufficient to ensure that, for any $f\in A^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$, the convergence in (4.4) takes place in $\mathcal{S}'(\mathbb{R}^n)$. Indeed, in view of Remark 3.9, without loss of generality, we may assume that $f\in B^{w,\tau}_{\mathcal{L},\infty,a}(\mathbb{R}^n)$. Then, by Lemma 4.6, the convergence in (4.4) takes place in $\mathcal{S}'(\mathbb{R}^n)$.

Next, we consider the synthesis part of Theorem 4.5.

LEMMA 4.7. *Let $s\in(0,\infty)$, $\alpha_1,\alpha_2,\alpha_3,\tau\in[0,\infty)$, $a\in(N_0+\alpha_3,\infty)$ and $q\in(0,\infty]$. Suppose that $w\in\mathcal{W}^{\alpha_3}_{\alpha_1,\alpha_2}$. Assume, in addition, that $K\in\mathbb{Z}_+$ and $L\in\mathbb{Z}_+$ satisfy* (4.1)–(4.3). *Let $\lambda:=\{\lambda_{jk}\}_{j\in\mathbb{Z}_+,\,k\in\mathbb{Z}^n}\in a^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$ and $\mathfrak{M}:=\{\mathfrak{M}_{jk}\}_{j\in\mathbb{Z}_+,\,k\in\mathbb{Z}^n}$ be a collection of (K,L)-molecules. Then the series*

$$f=\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^n}\lambda_{jk}\mathfrak{M}_{jk}$$

converges in $\mathcal{S}'(\mathbb{R}^n)$ and defines an element in $A^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$. Furthermore,

$$\|f\|_{A^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)}\lesssim\|\lambda\|_{a^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)},$$

with the implicit constant independent of f.

REMARK 4.8. One of the differences from the classical theory of molecules is that there is no need to distinguish Besov-type spaces and Triebel–Lizorkin-type spaces. Set $\sigma_p:=\max\{0,n/p-n\}$. For example, recall that in [92, Theorem 13.8] we need to assume

$$L\ge\max(-1,\lfloor\sigma_p-s\rfloor)\quad\text{or}\quad L\ge\max(-1,\lfloor\max(\sigma_p,\sigma_q)-s\rfloor)$$

according as we consider Besov spaces or Triebel–Lizorkin spaces. However, our approach does not require such a distinction. This seems due to the fact that we are using the Peetre maximal operator.

Proof of Lemma 4.7. The convergence of f in $\mathcal{S}'(\mathbb{R}^n)$ is a consequence of Lemma 4.6.

Let us prove $\|f\|_{A^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)}\lesssim\|\lambda\|_{a^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)}$. To this end, we fix $z\in\mathbb{R}^n$ and $j,l\in\mathbb{Z}_+$. Let us abbreviate $\sum_{k\in\mathbb{Z}^n}\lambda_{lk}\mathfrak{M}_{lk}$ to f_l. Then we have

$$\sup_{z\in\mathbb{R}^n}\frac{|\varphi_j*f_l(x+z)|}{(1+2^j|z|)^a}\lesssim\begin{cases}\displaystyle\sup_{z\in\mathbb{R}^n}\left\{\sum_{k\in\mathbb{Z}^n}\frac{2^{ln-(j-l)(L+1)}|\lambda_{lk}|}{(1+2^l|z|)^a(1+2^l|x+z-2^{-l}k|)^M}\right\}, & j\ge l,\\ \displaystyle\sup_{z\in\mathbb{R}^n}\left\{\sum_{k\in\mathbb{Z}^n}\frac{2^{jn-(l-j)(K+1)}|\lambda_{lk}|}{(1+2^j|z|)^a(1+2^j|x+z-2^{-l}k|)^M}\right\}, & j<l,\end{cases}$$

by Lemma 2.10, where M is as in (4.5). Consequently, as $1+2^j|z|\le1+2^{\max(j,l)}|z|$ for all $z\in\mathbb{R}^n$ and $j,l\in\mathbb{Z}_+$, we have

$$\sup_{z\in\mathbb{R}^n}\frac{|\varphi_j*f_k(x+z)|}{(1+2^j|z|)^a}$$
$$\lesssim\begin{cases}\sup_{z,w\in\mathbb{R}^n}\left\{\sum_{m\in\mathbb{Z}^n}\sum_{k\in\mathbb{Z}^n}\frac{2^{ln-(j-l)(L+1)}(1+2^l|w|)^{-a}|\lambda_{lm}|\chi_{Q_{lm}}(x+w)}{(1+2^l|x+z-2^{-l}k|)^M}\right\}, & j\geq l,\\ \sup_{z,w\in\mathbb{R}^n}\left\{\sum_{m\in\mathbb{Z}^n}\sum_{k\in\mathbb{Z}^n}\frac{2^{jn+(j-l)(K+1)}(1+2^l|w|)^{-a}|\lambda_{lm}|\chi_{Q_{lm}}(x+w)}{(1+2^j|x+z-2^{-l}k|)^M}\right\}, & j<l.\end{cases}$$

From

$$\sum_{k\in\mathbb{Z}^n}\frac{2^{ln}}{(1+2^l|x+z-2^{-l}k|)^M}+\sum_{k\in\mathbb{Z}^n}\frac{2^{jn}}{(1+2^j|x+z-2^{-l}k|)^M}\lesssim\int_{\mathbb{R}^n}\frac{1}{(1+|y|)^M}\,dy$$

and $M\in(\alpha_3+\delta+n,\infty)$, we conclude that

$$\sup_{z\in\mathbb{R}^n}\frac{|\varphi_j*f_l(x+z)|}{(1+2^j|z|)^a}\lesssim\begin{cases}2^{-(j-l)(L+1)}\sum_{m\in\mathbb{Z}^n}\left[\sup_{w\in\mathbb{R}^n}\frac{|\lambda_{lm}|\chi_{Q_{lm}}(x+w)}{(1+2^l|w|)^a}\right], & j\geq l,\\ 2^{(j-l)(K+1)}\sum_{m\in\mathbb{Z}^n}\left[\sup_{w\in\mathbb{R}^n}\frac{|\lambda_{lm}|\chi_{Q_{lm}}(x+w)}{(1+2^l|w|)^a}\right], & j<l.\end{cases}\tag{4.10}$$

If we now use (4.3) and Lemma 2.9, we obtain the desired result. ∎

With these preparations, let us prove Theorem 4.5. We investigate the case of $F^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$, the other cases being similar.

Proof of Theorem 4.5 (analysis part). Let $L\in\mathbb{Z}_+$ satisfying (4.1) be fixed. Let us choose $\Psi,\psi\in C^\infty_{\mathrm{c}}(\mathbb{R}^n)$ such that

$$\operatorname{supp}\Psi,\ \operatorname{supp}\psi\subset\{x=(x_1,\ldots,x_n):\max(|x_1|,\ldots,|x_n|)\leq 1\}\tag{4.11}$$

and

$$\int_{\mathbb{R}^n}\psi(x)x^{\vec{\beta}}\,dx=0\tag{4.12}$$

for all multiindices $\vec{\beta}$ with $\|\vec{\beta}\|_1\leq L$, and $\Psi*\Psi+\sum_{j=1}^\infty\psi_j*\psi_j=\delta_0$ in $\mathcal{S}'(\mathbb{R}^n)$, where $\psi_j:=2^{jn}\psi(2^j\cdot)$ for all $j\in\mathbb{N}$. Then, for all $f\in F^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$,

$$f=\Psi*\Psi*f+\sum_{j=1}^\infty\psi_j*\psi_j*f\tag{4.13}$$

in $\mathcal{S}'(\mathbb{R}^n)$. With this in mind, let us set, for all $j\in\mathbb{N}$ and $k\in\mathbb{Z}^n$,

$$\lambda_{0k}:=\int_{Q_{0k}}|\Psi*f(y)|\,dy,\quad \lambda_{jk}:=2^{jn}\int_{Q_{jk}}|\psi_j*f(y)|\,dy\tag{4.14}$$

and, for all $x\in\mathbb{R}^n$,

$$\mathfrak{A}_{0k}(x):=\frac{1}{\lambda_{0k}}\int_{Q_{0k}}\Psi(x-y)\Psi*f(y)\,dy,\ \mathfrak{A}_{jk}(x):=\frac{1}{\lambda_{jk}}\int_{Q_{jk}}\psi_j(x-y)\psi_j*f(y)\,dy.\tag{4.15}$$

In (4.15), if $\lambda_{jk}=0$ for some $j\in\mathbb{Z}_+$ and $k\in\mathbb{Z}^n$, we set $\mathfrak{A}_{jk}:=0$.

Observe that $f:=\sum_{j=0}^\infty\sum_{k\in\mathbb{Z}^n}\lambda_{jk}\mathfrak{A}_{jk}$ in $\mathcal{S}'(\mathbb{R}^n)$ by (4.13) and (4.15). Let us prove that $\mathfrak{A}_{jk}$, given by (4.15), is an atom supported near Q_{jk} modulo a multiplicative constant and that $\lambda := \{\lambda_{jk}\}_{j\in\mathbb{N},\, k\in\mathbb{Z}^n}$, given by (4.14), has the property that

$$\|\lambda\|_{f^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} \lesssim \|f\|_{F^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)}. \tag{4.16}$$

Observe that, when $x+z\in Q_{jk}$, by the Peetre inequality we have

$$\begin{aligned}\frac{2^{jn}}{(1+2^j|z|)^a}\int_{Q_{jk}}|\psi_j * f(y)|\,dy &= \frac{2^{jn}}{(1+2^j|z|)^a}\int_{x+z-Q_{jk}}|\psi_j * f(x+z-y)|\,dy\\ &\lesssim \int_{x+z-Q_{jk}}\frac{2^{jn}}{(1+2^j|z|)^a(1+2^j|y|)^a}|\psi_j * f(x+z-y)|\,dy\\ &\lesssim \int_{x+z-Q_{jk}}\frac{2^{jn}}{(1+2^j|z-y|)^a}|\psi_j * f(x+z-y)|\,dy\\ &\lesssim \sup_{w\in\mathbb{R}^n}\frac{|\psi_j * f(x-w)|}{(1+2^j|w|)^a}.\end{aligned}$$

Consequently,

$$\sup_{w\in Q_{jk}}\left\{\frac{2^{jn}}{(1+2^j|x-w|)^a}\int_{Q_{jk}}|\psi_j * f(y)|\,dy\right\} \lesssim \sup_{z\in\mathbb{R}^n}\frac{|\psi_j * f(x-z)|}{(1+2^j|z|)^a}. \tag{4.17}$$

Since $\{Q_{jk}\}_{k\in\mathbb{Z}^n}$ is a disjoint family for each fixed $j\in\mathbb{Z}_+$, (4.17) reads

$$\sup_{z\in\mathbb{R}^n}\frac{1}{(1+2^j|z|)^a}\Big|\sum_{k\in\mathbb{Z}^n}\lambda_{jk}\chi_{Q_{jk}}(x+z)\Big| \lesssim \sup_{z\in\mathbb{R}^n}\frac{|\psi_j * f(x-z)|}{(1+2^j|z|)^a}. \tag{4.18}$$

In particular, when $j=0$, we see that

$$\sup_{z\in\mathbb{R}^n}\frac{1}{(1+|z|)^a}\Big|\sum_{k\in\mathbb{Z}^n}\lambda_{0k}\chi_{Q_{0k}}(x+z)\Big| \lesssim \sup_{z\in\mathbb{R}^n}\frac{|\Psi * f(x-z)|}{(1+|z|)^a}. \tag{4.19}$$

Consequently, from (4.18) and (4.19), we deduce the estimate (4.16).

Meanwhile, via (4.11), a direct calculation of the size of supports yields

$$\operatorname{supp}(\mathfrak{A}_{jk}) \subset Q_{jk} + \operatorname{supp}(\psi_j) \subset 3Q_{jk} \tag{4.20}$$

and there exists a positive constant $C_{\vec{\alpha}}$ such that

$$|\partial^{\vec{\alpha}}\mathfrak{A}_{jk}(x)| = \frac{2^{j(\|\vec{\alpha}\|_1+n)}}{\lambda_{jk}}\left|\int_{Q_{jk}}\partial^{\vec{\alpha}}\psi(2^j(x-y))\psi_j * f(y)\,dy\right| \le C_{\vec{\alpha}}2^{j\|\vec{\alpha}\|_1} \tag{4.21}$$

for all multiindices $\vec{\alpha}$ as long as $\lambda_{jk}\neq 0$.

Keeping (4.20) and (4.21) in mind, let us show that each $\mathfrak{A}_{jk}$ is an atom modulo a positive multiplicative constant $\sum_{\|\vec{\alpha}\|_1\le K} C_{\vec{\alpha}}$. The support condition follows from (4.20). The size condition follows from (4.21). Finally, the moment condition follows from (4.12). ■

4.3. The regular case. Motivated by Remark 4.8, we now consider the regular case of Theorem 4.5, that is, the case $L=-1$. This is achieved by polishing a crude estimate (2.17). Our result is the following.

THEOREM 4.9. *Let $K\in\mathbb{N}\cup\{0\}$, $L=-1$, $\alpha_1,\alpha_2,\alpha_3,\tau\in[0,\infty)$ and $q\in(0,\infty]$. Suppose that $w\in\star\text{-}\mathcal{W}^{\alpha_3}_{\alpha_1,\alpha_2}$. Assume, in addition, that* (3.28) *and* (4.2) *hold, and that*

$$0 > \alpha_3+\delta+n+\gamma-n\tau-\alpha_1 \tag{4.22}$$

and

$$\alpha_1 > n\tau, \qquad K+1 > \alpha_2 + n\tau. \tag{4.23}$$

Then the pair $(A^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n), a^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n))$ *admits atomic/molecular decompositions.*

To prove Theorem 4.9 we need to modify Lemma 2.9.

LEMMA 4.10. *Let* D_1, D_2, $\alpha_1, \alpha_2, \alpha_3, \tau \in [0,\infty)$ *and* $q \in (0,\infty]$ *satisfy*

$$D_1 + \alpha_1 > 0 \quad \text{and} \quad D_2 - \alpha_2 > n\tau.$$

Let $\{g_\nu\}_{\nu\in\mathbb{Z}_+}$ *be a family of measurable functions on* $\mathbb{R}^n$ *and* $w \in \star\text{-}\mathcal{W}^{\alpha_3}_{\alpha_1,\alpha_2}$. *For all* $j \in \mathbb{Z}_+$ *and* $x \in \mathbb{R}^n$, *let*

$$G_j(x) := \sum_{\nu=j+1}^{\infty} 2^{-(\nu-j)D_1} g_\nu(x) + \sum_{\nu=0}^{j} 2^{-(j-\nu)D_2} g_\nu(x).$$

Then (2.13) *through* (2.16) *hold.*

Proof. The proof is based upon a modification of (2.19).

If, in Definition 3.12, we let $t := 2^{-\nu}$ and $s := 2^{-j}$ for $j, \nu \in \mathbb{Z}_+$ with $\nu \geq j$, we obtain

$$w_j(x) \lesssim 2^{\alpha_1(j-\nu)} w_\nu(x) \quad \text{for all } x \in \mathbb{R}^n. \tag{4.24}$$

If, in Definition 3.12, we let $t = 2^{-j}$ and $s = 2^{-\nu}$ for $j, \nu \in \mathbb{N}$ with $j \geq \nu$, we get

$$w_j(x) \lesssim 2^{\alpha_2(j-\nu)} w_\nu(x) \quad \text{for all } x \in \mathbb{R}^n. \tag{4.25}$$

Combining (4.24) and (4.25), we see that

$$w_j(x) \lesssim \begin{cases} 2^{\alpha_1(j-\nu)} w_\nu(x), & \nu \geq j, \\ 2^{\alpha_2(j-\nu)} w_\nu(x), & \nu \leq j, \end{cases} \tag{4.26}$$

for all $j, \nu \in \mathbb{Z}_+$. Let us write

$$\begin{aligned} \mathrm{I}(P) := {}& \frac{1}{|P|^\tau} \Big\| \chi_P \Big[\sum_{j=j_P\vee 0}^{\infty} \Big| \sum_{\nu=0}^{j} w_j 2^{(\nu-j)D_2} g_\nu \Big|^q \Big]^{1/q} \Big\|_{\mathcal{L}(\mathbb{R}^n)} \\ & + \frac{1}{|P|^\tau} \Big\| \chi_P \Big[\sum_{j=j_P\vee 0}^{\infty} \Big| \sum_{\nu=j+1}^{\infty} w_j 2^{(j-\nu)D_1} g_\nu \Big|^q \Big]^{1/q} \Big\|_{\mathcal{L}(\mathbb{R}^n)} \end{aligned}$$

for any dyadic cube P.

Let us suppose $q \in (0,1]$, since when $q \in (1,\infty]$, an argument similar to Lemma 2.9 works. Then we deduce, from (4.26) and ($\mathcal{L}4$), that

$$\begin{aligned} \mathrm{I}(P) \lesssim {}& \frac{1}{|P|^\tau} \Big\| \chi_P \Big[\sum_{j=j_P\vee 0}^{\infty} \sum_{\nu=0}^{j} 2^{-(j-\nu)(D_2-\alpha_2)q} |w_\nu g_\nu|^q \Big]^{1/q} \Big\|_{\mathcal{L}(\mathbb{R}^n)} \\ & + \frac{1}{|P|^\tau} \Big\| \chi_P \Big[\sum_{j=j_P\vee 0}^{\infty} \sum_{\nu=j+1}^{\infty} 2^{-(\nu-j)(D_1+\alpha_1)q} |w_\nu g_\nu|^q \Big]^{1/q} \Big\|_{\mathcal{L}(\mathbb{R}^n)} \end{aligned}$$

by (W1) and (2.21). We change the order of summations on the right-hand side of the above inequality to obtain

$$\mathrm{I}(P) \lesssim \frac{1}{|P|^\tau}\left\|\chi_P\Big[\sum_{\nu=0}^{\infty}\sum_{j=\nu\vee j_P\vee 0}^{\infty} 2^{-(j-\nu)(D_2-\alpha_2)q}|w_\nu g_\nu|^q\Big]^{1/q}\right\|_{\mathcal{L}(\mathbb{R}^n)}$$
$$+\frac{1}{|P|^\tau}\left\|\chi_P\Big[\sum_{\nu=j_P\vee 0}^{\infty}\sum_{j=j_P\vee 0}^{\nu} 2^{-(\nu-j)(D_1+\alpha_1)q}|w_\nu g_\nu|^q\Big]^{1/q}\right\|_{\mathcal{L}(\mathbb{R}^n)}.$$

Now we decompose the summand with respect to ν according to $j \ge j_P \vee 0$ or $j < j_P \vee 0$. Since $D_2 \in (\alpha_2 + n\tau, \infty)$, we can choose $\epsilon \in (0,\infty)$ such that $D_2 \in (\alpha_2 + n\tau + \epsilon, \infty)$. From this, $D_1 \in (-\alpha_1, \infty)$, the Hölder inequality, $(\mathcal{L}2)$ and $(\mathcal{L}4)$, it follows that

$$\mathrm{I}(P) \lesssim \|\{g_\nu\}_{\nu\in\mathbb{Z}_+}\|_{\mathcal{L}^w_\tau(\ell^q(\mathbb{R}^n,\mathbb{Z}_+))}$$
$$+\frac{1}{|P|^\tau}\left\|\chi_P\Big[\sum_{\nu=0}^{j_P\vee 0}\sum_{j=j_P\vee 0}^{\infty} 2^{-(j-\nu)(D_2-\alpha_2)q}|w_\nu g_\nu|^q\Big]^{1/q}\right\|_{\mathcal{L}(\mathbb{R}^n)}$$
$$\lesssim \|\{g_\nu\}_{\nu\in\mathbb{Z}_+}\|_{\mathcal{L}^w_\tau(\ell^q(\mathbb{R}^n,\mathbb{Z}_+))} + \frac{2^{-(j_P\vee 0)(D_2-\alpha_2-\epsilon)}}{|P|^\tau}\left\|\chi_P\sum_{\nu=0}^{j_P\vee 0}2^{\nu(D_2-\alpha_2-\epsilon)}|w_\nu g_\nu|\right\|_{\mathcal{L}(\mathbb{R}^n)},$$

which is just (2.22). Therefore, we can follow the same argument of the proof of Lemma 2.9. ■

Proof of Theorem 4.9. The proof is based upon reexamining that of Theorem 4.5. Recall that the latter proof consists of three parts: Lemma 4.6, Lemma 4.7 and the analysis condition. Let us start by modifying Lemma 4.6. By (4.22), we choose $M \in (\alpha_3+\delta+n, \infty)$ so that

$$-\gamma+\alpha_1-M+n\tau>0 \quad \text{and} \quad N > L+2n+\alpha_3+\delta. \tag{4.27}$$

Assuming that $w \in \star\text{-}\mathcal{W}^{\alpha_3}_{\alpha_1,\alpha_2}$, we see that α_1 in the proof of Theorem 4.5 and in the related statements can be replaced with $-\alpha_1$. More precisely, (4.7) changes to

$$w(0,1) \lesssim (1+|x|)^{\alpha_3}2^{-j\alpha_1}w_j(x) \qquad \text{for all } x\in\mathbb{R}^n \text{ and } j \in \mathbb{Z}_+.$$

Assuming $L=-1$, we can replace (4.9) with the following estimate: for all $j \in \mathbb{Z}_+$ and $k \in \mathbb{Z}^n$,

$$\left|\lambda_{jk}\int_{\mathbb{R}^n}\mathfrak{M}_{jk}(x)\varphi(x)\,dx\right| \lesssim 2^{-j(-\gamma+\alpha_1-M+n\tau)}(1+|k|)^{-M+\alpha_3+\delta}\|\lambda\|_{b^{w,\tau}_{\mathcal{L},\infty,a}(\mathbb{R}^n)}.$$

Since we are assuming (4.27), we have a counterpart for Lemma 4.6, that is, the series $f=\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^n}\lambda_{jk}\mathfrak{M}_{jk}$ converges in $\mathcal{S}'(\mathbb{R}^n)$.

Next, we reconsider Lemma 4.7. Its statement remains unchanged except that we substitute $L=-1$. Thus, the concluding estimate (4.10) changes to

$$\sup_{z\in\mathbb{R}^n}\frac{|\varphi_j * f_l(x+z)|}{(1+2^j|z|)^a} \lesssim \begin{cases} \displaystyle\sum_{m\in\mathbb{Z}^n}\left[\sup_{w\in\mathbb{R}^n}\frac{|\lambda_{lm}|\chi_{Q_{lm}}(x+w)}{(1+2^l|w|)^a}\right], & j \ge l,\\ \displaystyle 2^{(j-l)(K+1)}\sum_{m\in\mathbb{Z}^n}\left[\sup_{w\in\mathbb{R}^n}\frac{|\lambda_{lm}|\chi_{Q_{lm}}(x+w)}{(1+2^l|w|)^a}\right], & j<l.\end{cases}$$

Assuming (4.23), we can use Lemma 4.10 with $D_1=0$ and $D_2=K+1$.

Finally, the analysis part of the proof of Theorem 4.5 remains unchanged. Indeed, we did not use the condition for weights or the moment condition here. ■

4.4. Biorthogonal wavelet decompositions. We use biorthogonal wavelet bases on $\mathbb{R}$, namely, a system of scaling functions $(\psi^0, \widetilde{\psi}^0)$ and associated wavelets $(\psi^1, \widetilde{\psi}^1)$ satisfying

$$\langle \psi^0(\cdot - k), \widetilde{\psi}^0(\cdot - m)\rangle_{L^2(\mathbb{R})} = \delta_{k,m} \quad (k, m \in \mathbb{Z}),$$
$$\langle 2^{jn/2}\psi^1(2^j \cdot -k), 2^{\nu n/2}\widetilde{\psi}^1(2^\nu \cdot -m)\rangle_{L^2(\mathbb{R})} = \delta_{(j,k),(\nu,m)} \quad (j, k, \nu, m \in \mathbb{Z}),$$

where $\delta_{k,m} = 1$ if $k = m$ and $\delta_{k,m} = 0$ if $k \neq m$, $\delta_{(j,k),(\nu,m)}$ being defined similarly. Notice that, for all $f \in L^2(\mathbb{R}^n)$, we have

$$\begin{aligned} f &= \sum_{j,k\in\mathbb{Z}} 2^{jn}\langle f, \psi^1(2^j \cdot -k)\rangle_{L^2(\mathbb{R})}\widetilde{\psi}^1(2^j \cdot -k) \\ &= \sum_{j,k\in\mathbb{Z}} 2^{jn}\langle f, \widetilde{\psi}^1(2^j \cdot -k)\rangle_{L^2(\mathbb{R})}\psi^1(2^j \cdot -k) \\ &= \sum_{k\in\mathbb{Z}}\langle f, \psi^0(\cdot - k)\rangle_{L^2(\mathbb{R})}\widetilde{\psi}^0(\cdot - k) + \sum_{(j,k)\in\mathbb{Z}_+\times\mathbb{Z}} 2^{jn}\langle f, \psi^1(2^j \cdot -k)\rangle_{L^2(\mathbb{R})}\widetilde{\psi}^1(2^j \cdot -k) \\ &= \sum_{k\in\mathbb{Z}}\langle f, \widetilde{\psi}^0(\cdot - k)\rangle_{L^2(\mathbb{R})}\psi^0(\cdot - k) + \sum_{(j,k)\in\mathbb{Z}_+\times\mathbb{Z}} 2^{jn}\langle f, \widetilde{\psi}^1(2^j \cdot -k)\rangle_{L^2(\mathbb{R})}\psi^1(2^j \cdot -k) \end{aligned}$$

in $L^2(\mathbb{R})$. We construct a basis in $L^2(\mathbb{R}^n)$ by using the well-known tensor product procedure. Set $E := \{0,1\}^n \setminus \{(0,\dots,0)\}$. We need to consider the tensor products

$$\Psi^{\mathbf{c}} := \otimes_{j=1}^n \psi^{c_j} \quad \text{and} \quad \widetilde{\Psi}^{\mathbf{c}} := \otimes_{j=1}^n \widetilde{\psi}^{c_j}$$

for $\mathbf{c} := (c_1, \dots, c_n) \in \{0,1\}^n$. The following result is well known for orthonormal wavelets; see, for example, [6] and [94, Section 5.1]. However, it is straightforward to prove it for biorthogonal wavelets. Moreover, it can be arranged that the functions $\psi^0, \psi^1, \widetilde{\psi}^0, \widetilde{\psi}^1$ have compact supports.

As can be seen from the textbook [6], the existence of $\psi^0, \psi^1, \widetilde{\psi}^0, \widetilde{\psi}^1$ is guaranteed. Indeed, we just construct ψ^0, ψ^1 which are sufficiently smooth. Accordingly, we obtain $\widetilde{\psi}^0, \widetilde{\psi}^1$ which are almost as smooth as ψ^0, ψ^1. Finally, we obtain $\{\Psi^{\mathbf{c}}, \widetilde{\Psi}^{\mathbf{c}}\}_{\mathbf{c}\in E}$.

LEMMA 4.11. *Suppose that $\{\Psi^{\mathbf{c}}, \widetilde{\Psi}^{\mathbf{c}}\}_{\mathbf{c}\in E}$ is a biorthogonal system as above. Then for every $f \in L^2(\mathbb{R}^n)$,*

$$\begin{aligned} f = &\sum_{\mathbf{c}\in\{0,1\}^n}\sum_{k\in\mathbb{Z}^n}\langle f, \widetilde{\Psi}^{\mathbf{c}}(\cdot - k)\rangle_{L^2(\mathbb{R}^n)}\Psi^{\mathbf{c}}(\cdot - k) \\ &+ \sum_{\mathbf{c}\in E}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^n}\langle f, 2^{jn/2}\widetilde{\Psi}^{\mathbf{c}}(2^j \cdot -k)\rangle_{L^2(\mathbb{R}^n)}2^{jn/2}\Psi^{\mathbf{c}}(2^j \cdot -k) \end{aligned}$$

with convergence in $L^2(\mathbb{R}^n)$.

Notice that the above lemma covers the theory of wavelets (see, for example, [10, 26, 41, 94] for elementary facts) in that this reduces to a theory of wavelets when $\psi^0 = \widetilde{\psi}^0$ and $\psi^1 = \widetilde{\psi}^1$. In what follows we state conditions on the smoothness, the decay, and the number of vanishing moments for the wavelets $\psi^1, \widetilde{\psi}^1$ and the respective scaling functions $\psi^0, \widetilde{\psi}^0$ in order to make them suitable for our function spaces.

Recall first that $\alpha_1, \alpha_2, \alpha_3, \delta, \gamma, \tau$ are given in Definition 3.1. Suppose that the integers K, L, N satisfy

$$L > \alpha_3 + \delta + n - 1 + \gamma - n\tau + \alpha_1, \tag{4.28}$$

$$N > L + \alpha_3 + \delta + 2n, \tag{4.29}$$

$$K + 1 > \alpha_2 + n\tau, \qquad L + 1 > \alpha_1. \tag{4.30}$$

Assume that the $C^K(\mathbb{R})$-functions ψ^0, ψ^1 satisfy, for all $\alpha \in \mathbb{Z}_+$ with $\alpha \le K$,

$$|\partial^\alpha \psi^0(t)| + |\partial^\alpha \psi^1(t)| \lesssim (1 + |t|)^{-N}, \qquad t \in \mathbb{R}, \tag{4.31}$$

and

$$\int_{\mathbb{R}^n} t^\beta \psi^1(t)\, dt = 0 \tag{4.32}$$

for all $\beta \in \mathbb{Z}_+$ with $\beta \le L$. Similarly, the integers $\widetilde{K}, \widetilde{L}, \widetilde{N}$ are supposed to satisfy

$$\widetilde{L} > \alpha_3 + 2\delta + n - 1 + \gamma + \max(n/2, (\alpha_2 - \gamma)_+), \tag{4.33}$$

$$\widetilde{N} > \widetilde{L} + \alpha_3 + 2\delta + 2n, \tag{4.34}$$

$$\widetilde{K} + 1 > \alpha_1 + \gamma. \tag{4.35}$$

Let now the $C^{\widetilde{K}}(\mathbb{R})$-functions $\widetilde{\psi}^0$ and $\widetilde{\psi}^1$ satisfy, for all $\alpha \in \mathbb{Z}_+$ with $\alpha \le \widetilde{K}$,

$$|\partial^\alpha \widetilde{\psi}^0(t)| + |\partial^\alpha \widetilde{\psi}^1(t)| \lesssim (1 + |t|)^{-\widetilde{N}}, \qquad t \in \mathbb{R}, \tag{4.36}$$

and, for all $\beta \in \mathbb{Z}_+$ with $\beta \le \widetilde{L}$,

$$\int_{\mathbb{R}} t^\beta \widetilde{\psi}^1(t)\, dt = 0. \tag{4.37}$$

Assume, in addition, that

$$\widetilde{K} + 1 \ge \widetilde{L} > 2a + n\tau, \qquad \widetilde{N} > a + n. \tag{4.38}$$

Observe that (4.31) and (4.32) correspond to the decay condition and the moment condition of ψ^0 and ψ^1 in Definition 4.2, respectively. Let us now define the weight sequence

$$W_j(x) := [w_j^*(x)]^{-1} \wedge 2^{jn/2} \in \mathcal{W}^{\alpha_3+\delta}_{\max(n/2,(\alpha_2+\tau-\gamma)_+),(\alpha_1+\gamma-\tau)_+}, \tag{4.39}$$

where $x \in \mathbb{R}^n$, w_j^* is defined as in (3.30) and $j \in \mathbb{Z}_+$.

If $a \in (n + \alpha_3, \infty)$, using Proposition 9.5 below, which can be proved independently, together with the translation invariance of $L^\infty(\mathbb{R}^n)$ and $L^1(\mathbb{R}^n)$, we have

$$\|f\|_{B^\rho_{\infty,\infty,a}(\mathbb{R}^n)} \sim \sup_{j\in\mathbb{Z}_+} \|\rho_j(\varphi_j * f)\|_{L^\infty(\mathbb{R}^n)}, \ \|f\|_{B^\rho_{1,1,a}(\mathbb{R}^n)} \sim \sum_{j=0}^\infty \|\rho_j(\varphi_j * f)\|_{L^1(\mathbb{R}^n)} \tag{4.40}$$

for all $f \in \mathcal{S}'(\mathbb{R}^n)$ and $\rho \in \mathcal{W}^{\alpha_3}_{\alpha_1,\alpha_2}$. See also [45, Theorem 3.6] for a similar conclusion, where the case when ρ is independent of j is treated. Thus, if we assume that

$$a > n + \alpha_3 + \delta, \tag{4.41}$$

we see that

$$\|f\|_{B^{W^{-1}}_{\infty,\infty,a}(\mathbb{R}^n)} \sim \sup_{j\in\mathbb{Z}_+} \|W_j{}^{-1}(\varphi_j * f)\|_{L^\infty(\mathbb{R}^n)}, \; \|f\|_{B^{W}_{1,1,a}(\mathbb{R}^n)} \sim \sum_{j=0}^{\infty} \|W_j(\varphi_j * f)\|_{L^1(\mathbb{R}^n)}.$$

Observe that (4.33)–(4.35) guarantee that $(B^W_{1,1,a}(\mathbb{R}^n), b^W_{1,1,a}(\mathbb{R}^n))$ admits atomic/molecular characterizations; see Theorem 4.5 and the assumptions (4.28)–(4.30). Indeed, in $A^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$, we need to choose

$$A = B, \quad \mathcal{L}(\mathbb{R}^n) = L^1(\mathbb{R}^n), \quad q = 1, \quad w = W, \quad \tau = 0,$$

and hence, we have to replace $(\alpha_1, \alpha_2, \alpha_3)$ with

$$(\max(n/2, (\alpha_2 - \gamma)_+),\ \alpha_1 + \gamma, \alpha_3 + \delta)$$

and N_0 should be greater than n. Therefore, (4.28)–(4.30) become (4.33)–(4.35), respectively.

In view of Propositions 3.19 and 3.20, we define, for every $\mathbf{c} \in \{0,1\}^n$, a sequence $\{\lambda^{\mathbf{c}}_{j,k}\}_{j\in\mathbb{Z}_+, k\in\mathbb{Z}^n}$ by

$$\lambda^{\mathbf{c}}_{j,k} := \lambda^{\mathbf{c}}_{j,k}(f) := \langle f, 2^{jn/2}\widetilde{\Psi}^{\mathbf{c}}(2^j \cdot -k)\rangle, \quad j \in \mathbb{Z}_+,\ k \in \mathbb{Z}^n, \tag{4.42}$$

for a fixed $f \in B^{W^{-1}}_{\infty,\infty}(\mathbb{R}^n)$. In particular, when $\mathbf{c} = 0$, we let $\lambda^{\mathbf{c}}_{j,k} = 0$ whenever $j \in \mathbb{N}$.

It should be noticed that K and $\widetilde{K}$ may differ, as was the case in [68].

THEOREM 4.12. *Let $\alpha_1, \alpha_2, \alpha_3, \tau \in [0,\infty)$ and $q \in (0, \infty]$. Suppose that $\mathcal{L}(\mathbb{R}^n)$ satisfies $(\mathcal{L}1)$ through $(\mathcal{L}6)$, $w \in \mathcal{W}^{\alpha_3}_{\alpha_1,\alpha_2}$ and $a \in (N_0 + \alpha_3, \infty)$, where N_0 is as in $(\mathcal{L}6)$. Choose scaling functions $(\psi^0, \widetilde{\psi}^0) \in C^K(\mathbb{R}) \times C^{\widetilde{K}}(\mathbb{R})$ and associated wavelets $(\psi^1, \widetilde{\psi}^1) \in C^K(\mathbb{R}) \times C^{\widetilde{K}}(\mathbb{R})$ satisfying (4.31), (4.32), (4.36), (4.37), where $L, \widetilde{L}, N, \widetilde{N}, K, \widetilde{K} \in \mathbb{Z}_+$ are chosen according to (4.28), (4.29), (4.30), (4.33), (4.34), (4.35), (4.38) and (4.41). For every $f \in B^{W^{-1}}_{\infty,\infty}(\mathbb{R}^n)$ and every $\mathbf{c} \in \{0,1\}^n$, the sequences $\{\lambda^{\mathbf{c}}_{j,k}\}_{j\in\mathbb{Z}_+,k\in\mathbb{Z}^n}$ in (4.42) are well defined.*

(i) *The sequences $\{\lambda^{\mathbf{c}}_{j,k}\}_{j\in\mathbb{Z}_+,k\in\mathbb{Z}^n}$ belong to $a^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$ for all $\mathbf{c} \in \{0,1\}^n$ if and only if $f \in A^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$. Indeed, for all $f \in B^{W^{-1}}_{\infty,\infty}(\mathbb{R}^n)$,*

$$\sum_{\mathbf{c}\in\{0,1\}^n} \|\{\delta_{j,0}\langle f, \widetilde{\Psi}^{\mathbf{c}}(\cdot - k)\rangle\}_{j\in\mathbb{Z}_+,\, k\in\mathbb{Z}^n}\|_{a^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} + \sum_{\mathbf{c}\in E} \|\{\langle f, 2^{jn/2}\widetilde{\Psi}^{\mathbf{c}}(2^j \cdot -k)\rangle\}_{j\in\mathbb{Z}_+,\, k\in\mathbb{Z}^n}\|_{a^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} \sim \|f\|_{A^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)},$$

where "∞" is admitted on both sides.

(ii) *If $f \in A^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$, then*

$$f(\cdot) = \sum_{\mathbf{c}\in\{0,1\}^n} \sum_{k\in\mathbb{Z}^n} \lambda^{\mathbf{c}}_{0,k}\Psi^{\mathbf{c}}(\cdot - k) + \sum_{\mathbf{c}\in E}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^n} \lambda^{\mathbf{c}}_{j,k} 2^{jn/2}\Psi^{\mathbf{c}}(2^j \cdot -k) \tag{4.43}$$

in $\mathcal{S}'(\mathbb{R}^n)$. The equality (4.43) holds in $A^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$ if and only if the finite sequences are dense in $a^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$.

Proof. First, we show that if $f \in A^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$, then (4.43) holds in $\mathcal{S}'(\mathbb{R}^n)$. By (4.40) and (4.39), together with Proposition 3.19, the space $A^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$ can be embedded into

$B^{W^{-1}}_{\infty,\infty}(\mathbb{R}^n)$, which coincides with $B^{W^{-1}}_{\infty,\infty,a}(\mathbb{R}^n)$ when a satisfies (4.41). Fixing $\mathbf{c} \in \{0,1\}^n$ and letting $\{\lambda^{\mathbf{c}}_{j,k}\}_{j\in\mathbb{Z}_+,k\in\mathbb{Z}^n}$ be as in (4.42), we define

$$f^{\mathbf{c}}(\cdot) := \sum_{k\in\mathbb{Z}^n} \lambda^{\mathbf{c}}_{0,k}\Psi^{\mathbf{c}}(\cdot - k) + \sum_{j=1}^{\infty}\sum_{k\in\mathbb{Z}^n} \lambda^{\mathbf{c}}_{j,k} 2^{jn/2}\Psi^{\mathbf{c}}(2^j\cdot - k). \tag{4.44}$$

Noticing that $\Psi^{\mathbf{c}}(2^j\cdot - k)$ is a molecule modulo a multiplicative constant, by Lemma 4.7, we know that $f^{\mathbf{c}} \in A^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$ and

$$\begin{aligned}
\|f^{\mathbf{c}}\|_{A^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} &\lesssim \|\{\delta_{j,0}\lambda^{\mathbf{c}}_{0,k}\}_{j\in\mathbb{Z}_+,\,k\in\mathbb{Z}^n}\|_{a^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} + \|\{\lambda^{\mathbf{c}}_{j,k}\}_{j\in\mathbb{Z}_+,\,k\in\mathbb{Z}^n}\|_{a^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} \\
&\sim \|\{\delta_{j,0}\langle f^{\mathbf{c}}, \widetilde{\Psi}^{\mathbf{c}}(\cdot - k)\rangle\}_{j\in\mathbb{Z}_+,\,k\in\mathbb{Z}^n}\|_{a^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} \\
&\quad + \|\{\langle f^{\mathbf{c}}, 2^{jn/2}\widetilde{\Psi}^{\mathbf{c}}(2^j\cdot - k)\rangle\}_{j\in\mathbb{Z}_+,\,k\in\mathbb{Z}^n}\|_{a^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)}.
\end{aligned}$$

Then we further see that $f^{\mathbf{c}} \in B^{W^{-1}}_{\infty,\infty}(\mathbb{R}^n)$.

We now show that $f = \sum_{\mathbf{c}\in\{0,1\}^n} f^{\mathbf{c}}$. Indeed, for any

$$F \in B^{W}_{1,1}(\mathbb{R}^n)\ (\hookrightarrow B^{n/2}_{1,1}(\mathbb{R}^n) \hookrightarrow L^2(\mathbb{R}^n)), \tag{4.45}$$

if we let $\lambda^{\mathbf{c}}_{0,k}(F) = \langle F, \Psi^{\mathbf{c}}(\cdot - k)\rangle$ for all $k \in \mathbb{Z}^n$, and $\lambda^{\mathbf{c}}_{j,k}(F) = 2^{jn/2}\langle F, \Psi^{\mathbf{c}}(2^j\cdot - k)\rangle$ for all $j \in \mathbb{Z}_+$ and $k \in \mathbb{Z}^n$, then by Theorem 4.5, we conclude that

$$\sum_{\mathbf{c}\in E} \|\{\lambda^{\mathbf{c}}_{j,k}(F)\}_{j\in\mathbb{Z}_+,\,k\in\mathbb{Z}^n}\|_{b^{W}_{1,1}(\mathbb{R}^n)} \lesssim \|F\|_{B^{W}_{1,1}(\mathbb{R}^n)}. \tag{4.46}$$

From Lemma 4.11 and (4.45), we deduce that

$$F(\cdot) = \sum_{\mathbf{c}\in\{0,1\}^n}\sum_{k\in\mathbb{Z}^n} \lambda^{\mathbf{c}}_{0,k}(F)\widetilde{\Psi}^{\mathbf{c}}(\cdot - k) + \sum_{\mathbf{c}\in E}\sum_{j=1}^{\infty}\sum_{k\in\mathbb{Z}^n} \lambda^{\mathbf{c}}_{j,k}(F)2^{jn/2}\widetilde{\Psi}^{\mathbf{c}}(2^j\cdot - k) \tag{4.47}$$

in $L^2(\mathbb{R}^n)$; moreover, by (4.46), we also see that (4.47) holds in $B^{W}_{1,1}(\mathbb{R}^n)$.

Let $g := \sum_{\mathbf{c}\in\{0,1\}^n} f^{\mathbf{c}}$. Then $g \in B^{W^{-1}}_{\infty,\infty}(\mathbb{R}^n)$. By Proposition 3.20, together with (4.44) and (4.47), we see that $g(F) = f(F)$ for all $F \in B^{W}_{1,1}(\mathbb{R}^n)$, which gives $g = f$ immediately. Thus, (4.43) holds in $\mathcal{S}'(\mathbb{R}^n)$.

Thus, by Lemma 4.7 again, we obtain the "$\gtrsim$" relation in (i). Once we prove the "$\lesssim$" relation in (i), we immediately obtain the second conclusion in (ii), that is, (4.43) holds in $A^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$ if and only if the finite sequences are dense in $a^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$.

So it remains to prove the "$\lesssim$" relation in (i). Returning to the definition of the coupling $\langle f, 2^{jn/2}\widetilde{\Psi}^{\mathbf{c}}(2^j\cdot - k)\rangle$ (see Proposition 3.20), we have

$$\langle f, 2^{jn/2}\widetilde{\Psi}^{\mathbf{c}}(2^j\cdot - k)\rangle = 2^{jn/2}\langle \Phi * f, \Phi * \widetilde{\Psi}^{\mathbf{c}}(2^j\cdot - k)\rangle + \sum_{\ell=1}^{\infty} 2^{jn/2}\langle \varphi_\ell * f, \varphi_\ell * \widetilde{\Psi}^{\mathbf{c}}(2^j\cdot - k)\rangle.$$

In view of Lemma 2.10, we see that, for all $j, \ell \in \mathbb{Z}_+$, $k \in \mathbb{Z}^n$ and $x \in \mathbb{R}^n$,

$$|2^{jn}\varphi_\ell * \widetilde{\Psi}^{\mathbf{c}}(2^j x - k)| \lesssim 2^{\min(j,\ell)n - |\ell - j|\widetilde{L}}(1 + 2^{\min(j,\ell)}|x - 2^{-j}k|)^{-\widetilde{N}},$$

and hence, if $\widetilde{N} > a+n$ (see (4.38)), from $2^l \le 2^{\min(j,l)+|j-l|}$ we derive

$$
\begin{aligned}
&2^{jn}|\langle \varphi_\ell * f, \varphi_\ell * \widetilde{\Psi}^{\mathbf{c}}(2^j\cdot -k)\rangle| \\
&\qquad\lesssim 2^{\min(j,\ell)n-|\ell-j|\widetilde{L}} \int_{\mathbb{R}^n} \frac{|\varphi_\ell * f(x)|}{(1+2^{\min(j,\ell)}|x-2^{-j}k|)^{\widetilde{N}}}\,dx \\
&\qquad\lesssim 2^{\min(j,\ell)n-|\ell-j|\widetilde{L}} \sup_{y\in\mathbb{R}^n} \frac{|\varphi_\ell * f(y)|}{(1+2^\ell|y-2^{-j}k|)^a} \int_{\mathbb{R}^n} \frac{(1+2^\ell|x-2^{-j}k|)^a}{(1+2^{\min(j,\ell)}|x-2^{-j}k|)^{\widetilde{N}}}\,dx \\
&\qquad\lesssim 2^{-|\ell-j|(\widetilde{L}-a)} \sup_{y\in\mathbb{R}^n} \frac{|\varphi_\ell * f(y)|}{(1+2^\ell|y-2^{-j}k|)^a}
\end{aligned}
$$

with the implicit positive constant independent of j, ℓ, k and f. A similar estimate holds for $2^{jn/2}\langle \Phi * f, \Phi * \widetilde{\Psi}^{\mathbf{c}}(2^j\cdot -k)\rangle$. Consequently, as $(1+|y|)(1+|z|) \le (1+|y+z|)$ for all $y,z\in\mathbb{R}^n$, we see that, for all $x\in\mathbb{R}^n$,

$$
\begin{aligned}
&\sum_{k\in\mathbb{Z}^n}\sum_{\ell=1}^\infty 2^{jn}|\langle \varphi_\ell * f, \varphi_\ell * \widetilde{\Psi}^{\mathbf{c}}(2^j\cdot -k)\rangle|\chi_{Q_{jk}}(x) \\
&\qquad\lesssim \sum_{k\in\mathbb{Z}^n}\sum_{\ell=1}^\infty 2^{-|\ell-j|(\widetilde{L}-a)} \sup_{y\in\mathbb{R}^n} \frac{|\varphi_\ell * f(y)|}{(1+2^\ell|y-2^{-j}k|)^a}\chi_{Q_{jk}}(x) \\
&\qquad\lesssim \sum_{k\in\mathbb{Z}^n}\sum_{\ell=1}^\infty 2^{-|\ell-j|(\widetilde{L}-2a)} \sup_{y\in\mathbb{R}^n} \frac{|\varphi_\ell * f(y)|}{(1+2^\ell|y-x|)^a}\chi_{Q_{jk}}(x) \\
&\qquad\lesssim \sum_{\ell=1}^\infty 2^{-|\ell-j|(\widetilde{L}-2a)} \sup_{y\in\mathbb{R}^n} \frac{|\varphi_\ell * f(y)|}{(1+2^\ell|y-x|)^a},
\end{aligned}
$$

which, together with Lemma 2.9, implies the "$\lesssim$"-inequality in (i). ■

REMARK 4.13. (i) As in [68], biorthogonal systems in Theorem 4.12 can be replaced by frames.

(ii) Wavelet characterizations for some special cases of the function spaces in Theorem 4.12 are known; see, for example, [27, 29, 31, 94].

5. Pointwise multipliers and function spaces on domains

5.1. Pointwise multipliers. Let us recall that $\mathcal{B}^m(\mathbb{R}^n) := \bigcap_{\|\alpha\|_1 \le m} \{f \in C^m(\mathbb{R}^n) : \partial^\alpha f \in L^\infty(\mathbb{R}^n)\}$ for all $m \in \mathbb{Z}_+$. As an application of the atomic decomposition in the regular case, we can establish the following result.

THEOREM 5.1. *Let $\alpha_1, \alpha_2, \alpha_3, \tau \in [0, \infty)$ and $q \in (0, \infty]$. Suppose that $w \in \star\text{-}\mathcal{W}^{\alpha_3}_{\alpha_1,\alpha_2}$. Assume, in addition, that (3.28) holds. Then there exists $m_0 \in \mathbb{N}$ such that, for all $m \in \mathcal{B}^{m_0}(\mathbb{R}^n)$, the mapping $f \in \mathcal{S}(\mathbb{R}^n) \mapsto mf \in \mathcal{B}^{m_0}(\mathbb{R}^n)$ extends naturally to $A^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$ so that it has the property that*

$$\|mf\|_{B^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} \lesssim_m \|f\|_{B^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} \quad (f \in B^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)),$$
$$\|mf\|_{F^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} \lesssim_m \|f\|_{F^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} \quad (f \in F^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)),$$
$$\|mf\|_{\mathcal{N}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} \lesssim_m \|f\|_{\mathcal{N}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} \quad (f \in \mathcal{N}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)),$$
$$\|mf\|_{\mathcal{E}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} \lesssim_m \|f\|_{\mathcal{E}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} \quad (f \in \mathcal{E}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)).$$

Proof. Due to similarity, we only deal with the case of $B^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$.

Let α_1, α_2 and α_3 satisfy (4.22) and (4.23). We show the desired conclusion by induction. Let $m_0(w)$ be the smallest number such that $w^* \in \mathcal{W}^{\alpha_3}_{\alpha_1\alpha_2}$, where $w^*_\nu(x) := 2^{m_0(w)\nu} w_\nu(x)$ for all $\nu \in \mathbb{Z}_+$ and $x \in \mathbb{R}^n$. If $m_0(w)$ can be taken 0, then we use Theorem 4.12 to find that it suffices to define

$$(mf)(\cdot) := \sum_{\mathbf{c}\in\{0,1\}^n} \sum_{k\in\mathbb{Z}^n} \lambda^{\mathbf{c}}_{0,k} m(\cdot)\Psi^{\mathbf{c}}(\cdot - k) + \sum_{\mathbf{c}\in E} \sum_{j=0}^{\infty} \sum_{k\in\mathbb{Z}^n} \lambda^{\mathbf{c}}_{j,k} m(\cdot) 2^{jn/2} \Psi^{\mathbf{c}}(2^j \cdot - k),$$

which, together with Theorem 4.9 and the fact that $m(\cdot)2^{jn/2}\Psi^{\mathbf{c}}(2^j \cdot -k)$ is a molecule modulo a multiplicative constant, implies the desired conclusion in this case. Assume now that our theorem is true for the class of weights $m_0(w) \in \{0, 1, \dots, N\}$, where $N \in \mathbb{Z}_+$. For $m_0(w) = N+1$, let us write $f = (1-\Delta)^{-1} f - \sum_{j=1}^n {\partial_j}^2 (1-\Delta)^{-1} f$. Then we have

$$\begin{aligned} mf &= m(1-\Delta)^{-1} f - \sum_{j=1}^n m{\partial_j}^2(1-\Delta)^{-1} f \\ &= m(1-\Delta)^{-1} f - \sum_{j=1}^n \partial_j(m\partial_j(1-\Delta)^{-1} f) + \sum_{j=1}^n (\partial_j m)\partial_j((1-\Delta)^{-1} f). \end{aligned}$$

Notice that $(1-\Delta)^{-1} f$ and $\partial_j((1-\Delta)^{-1} f)$ belong to the space $B^{w^{**},\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$, where we write $w^{**}_\nu(x) := 2^\nu w_\nu(x)$ for all $\nu \in \mathbb{Z}_+$ and $x \in \mathbb{R}^n$. Notice that $m_0(w^{**}) = m_0(w) - 1$.

Consequently, by the induction assumption, we have

$$\begin{aligned}\|m(1-\Delta)^{-1}f\|_{B^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} &\le \|m(1-\Delta)^{-1}f\|_{B^{w^{**},\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)}\\ &\lesssim_m \|(1-\Delta)^{-1}f\|_{B^{w^{**},\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} \lesssim_m \|f\|_{B^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)}.\end{aligned}$$

Analogously, by Proposition 3.18 and Theorem 3.10, we have

$$\begin{aligned}\|\partial_j(m\partial_j(1-\Delta)^{-1}f)\|_{B^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} &\lesssim \|m\partial_j(1-\Delta)^{-1}f\|_{B^{w^{**},\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)}\\ &\lesssim_m \|\partial_j(1-\Delta)^{-1}f\|_{B^{w^{**},\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} \lesssim_m \|f\|_{B^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)}\end{aligned}$$

and

$$\begin{aligned}\|(\partial_j m)\partial_j((1-\Delta)^{-1}f)\|_{B^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} &\le \|(\partial_j m)\partial_j((1-\Delta)^{-1}f)\|_{B^{w^{**},\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)}\\ &\lesssim_m \|\partial_j((1-\Delta)^{-1}f)\|_{B^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} \lesssim_m \|f\|_{B^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)},\end{aligned}$$

which completes the proof of Theorem 5.1. ■

5.2. Function spaces on domains. In what follows, let Ω be an open subset of $\mathbb{R}^n$, $\mathcal{D}(\Omega)$ denote the *space of all infinitely differentiable functions with compact support in* Ω endowed with the inductive topology, and $\mathcal{D}'(\Omega)$ its *topological dual* with the weak-$*$ topology which is called the *space of distributions on* Ω.

Now we aim at defining the spaces on Ω. Recall that a natural mapping

$$f \in \mathcal{S}'(\mathbb{R}^n) \mapsto f|\Omega \in \mathcal{D}'(\Omega)$$

is well defined.

DEFINITION 5.2. Let $s \in \mathbb{R}$, $a \in (0,\infty)$, $\alpha_1, \alpha_2, \alpha_3, \tau \in [0,\infty)$ and $q \in (0,\infty]$. Let $w \in \mathcal{W}^{\alpha_3}_{\alpha_1,\alpha_2}$.

(i) $B^{w,\tau}_{\mathcal{L},q,a}(\Omega)$ is defined to be the set of all $f \in \mathcal{D}'(\Omega)$ such that $f = g|\Omega$ for some $g \in B^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$, equipped with the norm

$$\|f\|_{B^{w,\tau}_{\mathcal{L},q,a}(\Omega)} := \inf\{\|g\|_{B^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} : g \in B^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n),\ f = g|\Omega\}.$$

(ii) $F^{w,\tau}_{\mathcal{L},q,a}(\Omega)$ is defined to be the set of all $f \in \mathcal{D}'(\Omega)$ such that $f = g|\Omega$ for some $g \in F^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$, equipped with the norm

$$\|f\|_{F^{w,\tau}_{\mathcal{L},q,a}(\Omega)} := \inf\{\|g\|_{F^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} : g \in F^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n),\ f = g|\Omega\}.$$

(iii) $\mathcal{N}^{w,\tau}_{\mathcal{L},q,a}(\Omega)$ is defined to be the set of all $f \in \mathcal{D}'(\Omega)$ such that $f = g|\Omega$ for some $g \in \mathcal{N}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$, equipped with the norm

$$\|f\|_{\mathcal{N}^{w,\tau}_{\mathcal{L},q,a}(\Omega)} := \inf\{\|g\|_{\mathcal{N}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} : g \in \mathcal{N}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n),\ f = g|\Omega\}.$$

(iv) $\mathcal{E}^{w,\tau}_{\mathcal{L},q,a}(\Omega)$ is defined to be the set of all $f \in \mathcal{D}'(\Omega)$ such that $f = g|\Omega$ for some $g \in \mathcal{E}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$, equipped with the norm

$$\|f\|_{\mathcal{E}^{w,\tau}_{\mathcal{L},q,a}(\Omega)} := \inf\{\|g\|_{\mathcal{E}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} : g \in \mathcal{E}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n),\ f = g|\Omega\}.$$

A routine argument shows that $B^{w,\tau}_{\mathcal{L},q,a}(\Omega)$, $F^{w,\tau}_{\mathcal{L},q,a}(\Omega)$, $\mathcal{E}^{w,\tau}_{\mathcal{L},q,a}(\Omega)$ and $\mathcal{N}^{w,\tau}_{\mathcal{L},q,a}(\Omega)$ are all quasi-Banach spaces.

Here we are interested in bounded Lipschitz domains. Let $\kappa : \mathbb{R}^{n-1} \to \mathbb{R}$ be a Lipschitz function. Then define

$$\Omega_{\kappa,+} := \{(x', x_n) \in \mathbb{R}^n : x_n > \kappa(x')\}$$
$$\Omega_{\kappa,-} := \{(x', x_n) \in \mathbb{R}^n : x_n < \kappa(x')\}.$$

Let $\sigma \in S_n$ be a permutation. Then define

$$\Omega_{\kappa,\pm;\sigma} := \{(x', x_n) \in \mathbb{R}^n : \sigma(x', x_n) \in \Omega_{\kappa,\pm}\}.$$

By a *Lipschitz domain*, we mean an open set of the form

$$\bigcup_{j=1}^{J} \sigma_j(\Omega_{f_j,+}) \cap \bigcup_{i=1}^{I} \tau_i(\Omega_{g_i,-}),$$

where the functions $f_1, \ldots, f_J$ and $g_1, \ldots, g_I$ are all Lipschitz functions and the mappings $\sigma_1, \ldots, \sigma_J$ and $\tau_1, \ldots, \tau_K$ belong to S_n. With Theorem 5.1, and a partition of unity, without loss of generality, we may assume that $\Omega := \Omega_{\kappa,\pm}$ for some Lipschitz function $\kappa : \mathbb{R}^n \to \mathbb{R}$. Furthermore, by symmetry, we only need to deal with the case when $\Omega := \Omega_{\kappa,+}$.

To specify, we let L be the *positive Lipschitz constant* of κ, the smallest number L such that for all $x', y' \in \mathbb{R}^{n-1}$, $|\kappa(x') - \kappa(y')| \le L|x' - y'|$. Also, we let K be the cone given by

$$K := \{(x', x_n) \in \mathbb{R}^n : L|x'| > -x_n\}.$$

We choose $\Psi \in \mathcal{D}(\mathbb{R}^n)$ so that $\operatorname{supp}\Psi \subset K$ and $\int_{\mathbb{R}^n} \Psi(x)\,dx \neq 0$. Let

$$\Phi(x) := \Psi(x) - \Psi_{-1}(x) = \Psi(x) - 2^{-n}\Psi(2^{-1}x)$$

for all $x \in \mathbb{R}^n$. Let $L \gg 1$ and choose $\eta, \psi \in C_{\mathrm{c}}^\infty(K)$ so that $\varphi := \eta - \eta_{-1}$ satisfies the moment condition of order L and $\psi * \Psi + \sum_{j=1}^\infty \varphi_j * \Phi_j = \delta$ in $\mathcal{S}'(\mathbb{R}^n)$. Define $\mathcal{M}^\Omega_{2^{-j},a} f(x)$, for all $j \in \mathbb{Z}_+$, $f \in \mathcal{D}'(\Omega)$ and $x \in \mathbb{R}^n$, by

$$\mathcal{M}^\Omega_{2^{-j},a} f(x) := \begin{cases} \sup\limits_{y\in\Omega} \dfrac{|\Psi * f(y)|}{(1+|x-y|)^a}, & j = 0, \\ \sup\limits_{y\in\Omega} \dfrac{|\Phi_j * f(y)|}{(1+2^j|x-y|)^a}, & j \in \mathbb{N} \end{cases}$$
$$= \begin{cases} \sup\limits_{y\in\Omega} \dfrac{|\langle f, \Psi(y-\cdot)\rangle|}{(1+|x-y|)^a}, & j = 0, \\ \sup\limits_{y\in\Omega} \dfrac{|\langle f, \Phi_j(y-\cdot)\rangle|}{(1+2^j|x-y|)^a}, & j \in \mathbb{N}. \end{cases}$$

Observe that this definition makes sense. More precisely, the pairings $\langle f, \Psi(y-\cdot)\rangle$ and $\langle f, \Phi_j(y-\cdot)\rangle$ are well defined, because $\Psi(y-\cdot)$ and $\Phi_j(y-\cdot)$ have compact support and, moreover, are supported on Ω, as the following calculation shows:

$$\operatorname{supp}(\Psi(y-\cdot)), \operatorname{supp}(\Phi_j(y-\cdot)) \subset y - K \subset \{y + z : |z_n| > K|z'|\} \subset \Omega.$$

Here we used the fact that $\Omega = \Omega_{\kappa,+}$ to obtain the last inclusion.

In what follows, the mapping $(x', x_n) \mapsto (x', 2\kappa(x') - x_n) =: (y', y_n)$ is said to *induce an isomorphism of $\mathcal{L}(\mathbb{R}^n)$ with equivalent norms* if $f \in \mathcal{L}(\mathbb{R}^n)$ if and only if $g_f(y', y_n) := f(x', 2\kappa(x') - x_n) \in \mathcal{L}(\mathbb{R}^n)$ and moreover $\|f\|_{\mathcal{L}(\mathbb{R}^n)} \sim \|g_f\|_{\mathcal{L}(\mathbb{R}^n)}$.

Now we aim to prove the following theorem.

THEOREM 5.3. *Let $\Omega := \Omega_{\kappa,+}$ be as above and assume that the reflection*

$$\iota : (x', x_n) \mapsto (x', 2\kappa(x') - x_n)$$

induces an isomorphism of $\mathcal{L}(\mathbb{R}^n)$ with equivalent norms. Then

(i) *$f \in B^{w,\tau}_{\mathcal{L},q,a}(\Omega)$ if and only if $f \in \mathcal{D}'(\Omega)$ and*

$$\|\{\chi_\Omega \mathcal{M}^\Omega_{2^{-j},a} f\}_{j\in\mathbb{Z}_+}\|_{\ell^q(\mathcal{L}^w_\tau(\mathbb{R}^n,\mathbb{Z}_+))} < \infty,$$

and there exists a positive constant C, independent of f, such that

$$C^{-1}\|f\|_{B^{w,\tau}_{\mathcal{L},q,a}(\Omega)} \le \|\{\chi_\Omega \mathcal{M}^\Omega_{2^{-j},a} f\}_{j\in\mathbb{Z}_+}\|_{\ell^q(\mathcal{L}^w_\tau(\mathbb{R}^n,\mathbb{Z}_+))} \le C\|f\|_{B^{w,\tau}_{\mathcal{L},q,a}(\Omega)}; \tag{5.1}$$

(ii) *$f \in F^{w,\tau}_{\mathcal{L},q,a}(\Omega)$ if and only if $f \in \mathcal{D}'(\Omega)$ and*

$$\|\{\chi_\Omega \mathcal{M}^\Omega_{2^{-j},a} f\}_{j\in\mathbb{Z}_+}\|_{\mathcal{L}^w_\tau(\ell^q(\mathbb{R}^n,\mathbb{Z}_+))} < \infty,$$

and there exists a positive constant C, independent of f, such that

$$C^{-1}\|f\|_{F^{w,\tau}_{\mathcal{L},q,a}(\Omega)} \le \|\{\chi_\Omega \mathcal{M}^\Omega_{2^{-j},a} f\}_{j\in\mathbb{Z}_+}\|_{\mathcal{L}^w_\tau(\ell^q(\mathbb{R}^n,\mathbb{Z}_+))} \le C^{-1}\|f\|_{F^{w,\tau}_{\mathcal{L},q,a}(\Omega)};$$

(iii) *$f \in \mathcal{N}^{w,\tau}_{\mathcal{L},q,a}(\Omega)$ if and only if $f \in \mathcal{D}'(\Omega)$ and*

$$\|\{\chi_\Omega \mathcal{M}^\Omega_{2^{-j},a} f\}_{j\in\mathbb{Z}_+}\|_{\ell^q(\mathcal{N}\mathcal{L}^w_\tau(\mathbb{R}^n,\mathbb{Z}_+))} < \infty,$$

and there exists a positive constant C, independent of f, such that

$$C^{-1}\|f\|_{\mathcal{N}^{w,\tau}_{\mathcal{L},q,a}(\Omega)} \le \|\{\chi_\Omega \mathcal{M}^\Omega_{2^{-j},a} f\}_{j\in\mathbb{Z}_+}\|_{\ell^q(\mathcal{N}\mathcal{L}^w_\tau(\mathbb{R}^n,\mathbb{Z}_+))} \le C\|f\|_{\mathcal{N}^{w,\tau}_{\mathcal{L},q,a}(\Omega)};$$

(iv) *$f \in \mathcal{E}^{w,\tau}_{\mathcal{L},q,a}(\Omega)$ if and only if $f \in \mathcal{D}'(\Omega)$ and*

$$\|\{\chi_\Omega \mathcal{M}^\Omega_{2^{-j},a} f\}_{j\in\mathbb{Z}_+}\|_{\mathcal{E}\mathcal{L}^w_\tau(\ell^q(\mathbb{R}^n,\mathbb{Z}_+))} < \infty,$$

and there exists a positive constant C, independent of f, such that

$$C^{-1}\|f\|_{\mathcal{E}^{w,\tau}_{\mathcal{L},q,a}(\Omega)} \le \|\{\chi_\Omega \mathcal{M}^\Omega_{2^{-j},a} f\}_{j\in\mathbb{Z}_+}\|_{\mathcal{E}\mathcal{L}^w_\tau(\ell^q(\mathbb{R}^n,\mathbb{Z}_+))} \le C\|f\|_{\mathcal{E}^{w,\tau}_{\mathcal{L},q,a}(\Omega)}.$$

Proof. By similarity, we only give the proof of (i). The second inequality of (5.1) follows from Corollary 3.6. Let us prove the first inequality of (5.1). Let $f \in B^{w,\tau}_{\mathcal{L},q,a}(\Omega)$. Choose $G \in B^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$ so that

$$G|\Omega = f, \quad \|f\|_{B^{w,\tau}_{\mathcal{L},q,a}(\Omega)} \le \|G\|_{B^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} \le 2\|f\|_{B^{w,\tau}_{\mathcal{L},q,a}(\Omega)}.$$

Define

$$F := \psi * \Psi * G + \sum_{j=1}^\infty \varphi_j * \Phi_j * G.$$

It is easy to see that $F|\Omega = f$ and $F \in \mathcal{S}'(\mathbb{R}^n)$, since $\psi * \Psi + \sum_{j=1}^\infty \varphi_j * \Phi_j = \delta$ in $\mathcal{S}'(\mathbb{R}^n)$. Then $\|f\|_{B^{w,\tau}_{\mathcal{L},q,a}(\Omega)} \lesssim \|F\|_{B^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)}$. To show the first inequality of (5.1), it suffices to show that

$$\|F\|_{B^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} \lesssim \|\{\chi_\Omega \mathcal{M}^\Omega_{2^{-j},a} f\}_{j\in\mathbb{Z}_+}\|_{\ell^q(\mathcal{L}^w_\tau(\mathbb{R}^n,\mathbb{Z}_+))}.$$

Since

$$\|F\|_{B^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} \lesssim \|\{\mathcal{M}^\Omega_{2^{-j},a} f\}_{j\in\mathbb{Z}_+}\|_{\ell^q(\mathcal{L}^w_\tau(\mathbb{R}^n,\mathbb{Z}_+))},$$

we only need to prove that

$$\|\{\mathcal{M}^{\Omega}_{2^{-j},a}f\}_{j\in\mathbb{Z}_+}\|_{\ell^q(\mathcal{L}^w_\tau(\mathbb{R}^n,\mathbb{Z}_+))}\lesssim\|\{\chi_\Omega\mathcal{M}^{\Omega}_{2^{-j},a}f\}_{j\in\mathbb{Z}_+}\|_{\ell^q(\mathcal{L}^w_\tau(\mathbb{R}^n,\mathbb{Z}_+))}.$$

To see this, noticing that if $(x',x_n)\in\Omega$ and $(y',y_n)\in\Omega$, since κ is a Lipschitz mapping, we conclude that

$$\begin{aligned}|x'-y'|^2+|y_n+x_n-2\kappa(x')|^2&\sim|x'-y'|^2+|y_n-\kappa(y')+x_n-\kappa(x')|^2\\&\sim|x'-y'|^2+|y_n-\kappa(y')-x_n+\kappa(x')|^2+|\kappa(y')-\kappa(x')|^2\\&\gtrsim|x'-y'|^2+|y_n-x_n|^2\sim|x-y|^2.\end{aligned}$$

From this, together with the isomorphism property with equivalent norms of the transform $(x',x_n)\in\mathbb{R}^n\setminus\Omega\mapsto(x',2\kappa(x')-x_n)\in\Omega$, we deduce that

$$\|\{\chi_{\mathbb{R}^n\setminus\Omega}\mathcal{M}^{\Omega}_{2^{-j},a}f\}_{j\in\mathbb{Z}_+}\|_{\ell^q(\mathcal{L}^w_\tau(\mathbb{R}^n,\mathbb{Z}_+))}\lesssim\|\{\chi_\Omega\mathcal{M}^{\Omega}_{2^{-j},a}f\}_{j\in\mathbb{Z}_+}\|_{\ell^q(\mathcal{L}^w_\tau(\mathbb{R}^n,\mathbb{Z}_+))},$$

which further implies the first inequality of (5.1). ■

To conclude Section 5, we present two examples concerning Theorem 5.3.

Example 5.4. It is necessary to assume that $(x',x_n)\mapsto(x',2\kappa(x')-x_n)$ induces an isomorphism of $\mathcal{L}(\mathbb{R}^n)$ with equivalent norms. Here is a counterexample which shows this.

Let $n=1$, $\mathcal{L}(\mathbb{R}):=L^1((1+t\chi_{(0,\infty)}(t))^{-N}\,dt)$ and $w_j(x):=1$ for all $x\in\mathbb{R}$ and $j\in\mathbb{Z}_+$. Consider the space $B^{0,0}_{\mathcal{L},\infty,2}((0,\infty))$, whose notation is based on the convention (3.1). A passage to the higher dimensional case is readily done. In this case the isomorphism is $t\in\mathbb{R}\mapsto -t\in\mathbb{R}$. Consider the corresponding maximal operators, for all $f\in\mathcal{D}'(0,\infty)$ and $t\in\mathbb{R}$,

$$\mathcal{M}^{(0,\infty)}_{1,2}f(t):=\sup_{s\in(0,\infty)}\frac{|\psi*f(s)|}{(1+|t-s|)^2}$$

and, for $j\in\mathbb{N}$,

$$\mathcal{M}^{(0,\infty)}_{2^{-j},2}f(t):=\sup_{s\in(0,\infty)}\frac{|\varphi_j*f(s)|}{(1+2^j|t-s|)^2},$$

where ψ and φ belong to $C^\infty_c((-2,-1))$ satisfying $\varphi=\Delta^L\psi$, and $\varphi_j(t)=2^j\varphi(2^jt)$ for all $t\in\mathbb{R}$ and $j\in\mathbb{N}$. Let $f_0\in C^\infty_c((2,5))$ be such that $\chi_{(3,4)}\le f_0\le\chi_{(2,5)}$. Set $f_a(t):=f_0(t-a)$ for all $t\in\mathbb{R}$ and some $a\gg1$. Then, for all $t\in\mathbb{R}$, we have

$$\mathcal{M}^{(0,\infty)}_{1,2}f_a(t)\sim\frac{1}{(1+|t-a|)^2}\quad\text{and}\quad\mathcal{M}^{(0,\infty)}_{2^{-j},2}f_a(t)\sim\frac{2^{-2jL}}{(1+|t-a|)^2}.$$

Consequently,

$$\|\{\chi_{(0,\infty)}\mathcal{M}^{(0,\infty)}_{2^{-j},2}f_a\}_{j\in\mathbb{Z}^+}\|_{\ell^\infty(\mathcal{L}(\mathbb{R},\mathbb{Z}_+))}\sim\int_0^\infty\frac{1}{(1+t)^N(1+|t-a|)^2}\,dt.$$

Let $\rho:\mathbb{R}\to\mathbb{R}$ be a smooth function satisfying $\chi_{(8/5,\infty)}\le\rho\le\chi_{(3/2,\infty)}$. If $f\in B^{0,0}_{\mathcal{L},\infty,2}(\mathbb{R}^n)$ is such that $f|(0,\infty)=f_a$, then $\|f\|_{B^{0,0}_{\mathcal{L},\infty,2}(\mathbb{R}^n)}=\|\rho f\|_{B^{0,0}_{\mathcal{L},\infty,2}(\mathbb{R}^n)}\lesssim\|f\|_{B^{0,0}_{\mathcal{L},\infty,2}(\mathbb{R}^n)}$ by Theorem 5.1. Consequently,

$$\|f_a\|_{B^{0,0}_{\mathcal{L},\infty,2}(\Omega)}\sim\|f_0\|_{B^{0,0}_{\mathcal{L},\infty,2}(\mathbb{R}^n)}\sim1/a.\tag{5.2}$$

Moreover,

$$
\begin{aligned}
&\|\{\chi_{(0,\infty)}\mathcal{M}^{(0,\infty)}_{2^{-j},2} f_a\}_{j\in\mathbb{Z}^+}\|_{\ell^\infty(\mathcal{L}(\mathbb{R},\mathbb{Z}_+))}\\
&\quad\sim \int_0^\infty \frac{1}{(1+t)^N(1+|t-a|)^2}\,dt \sim \left(\int_0^{a/2}+\int_{a/2}^\infty\right)\frac{1}{(1+t)^N(1+|t-a|)^2}\,dt\\
&\quad\lesssim \int_0^{a/2}\frac{1}{(1+t)^N(1+|a|)^2}\,dt+\int_{a/2}^\infty \frac{1}{(1+a)^N(1+|t-a|)^2}\,dt \lesssim \frac{1}{a^2}.
\end{aligned}
$$

In view of the above calculation and (5.2), the conclusion (5.1) of Theorem 5.3 fails unless we assume that $(x',x_{n+1})\mapsto(x',2\kappa(x')-x_{n+1})$ induces an isomorphism of $\mathcal{L}(\mathbb{R}^n)$.

Example 5.5. As examples satisfying the assumption of Theorem 5.3, we can list weak-L^p spaces, Orlicz spaces and Morrey spaces. For a detailed discussion of Orlicz spaces and Morrey spaces, see Section 10. Here we content ourselves with giving the definition of the norm and checking the assumption of Theorem 5.3 for Orlicz spaces and Morrey spaces.

(i) By a *Young function* we mean a convex homeomorphism $\Phi:[0,\infty)\to[0,\infty)$.

Given a Young function Φ, we define the *Orlicz space* $L^\Phi(\mathbb{R}^n)$ as the set of all measurable functions $f:\mathbb{R}^n\to\mathbb{C}$ such that

$$
\|f\|_{L^\Phi(\mathbb{R}^n)} := \inf\left\{\lambda\in(0,\infty) : \int_{\mathbb{R}^n}\Phi\left(\frac{|f(x)|}{\lambda}\right)dx\le 1\right\}<\infty.
$$

Indeed, to check the assumption of Theorem 5.3 for weak-L^p spaces and Orlicz spaces, we just have to bear in mind that the Jacobian of the involution ι is 1 and hence we can use the formula for the change of variables.

(ii) The *Morrey norm* $\|\cdot\|_{\mathcal{M}^p_u(\mathbb{R}^n)}$ with $0<u\le p\le\infty$ is given by

$$
\|f\|_{\mathcal{M}^p_u(\mathbb{R}^n)} := \sup_{x\in\mathbb{R}^n,\, r\in(0,\infty)} r^{n/p-n/u}\left[\int_{B(x,r)}|f(y)|^u\,dy\right]^{1/u},
$$

where $B(x,r)$ denotes the ball centered at x of radius $r\in(0,\infty)$ and f is a measurable function. Unlike the case of Orlicz spaces, for Morrey spaces, we need one more observation. Since $\iota\circ\iota=\mathrm{id}_{\mathbb{R}^n}$, we have only to prove that ι induces a bounded mapping on Morrey spaces. This can be shown as follows: Observe that $|x-y|<r$ implies $|\iota(x)-\iota(y)|<Dr$, since $\iota(x)=(x',2\kappa(x')-x_n)$ is a Lipschitz mapping with Lipschitz constant, say, D. Therefore, $\iota(B(x,r))\subset B(\iota(x),Dr)$. Hence

$$
\begin{aligned}
r^{n/p-n/u}\left[\int_{B(x,r)}|f(\iota(y))|^u\,dy\right]^{1/u} &= r^{n/p-n/u}\left[\int_{\iota(B(x,r))}|f(y)|^u\,dy\right]^{1/u}\\
&\le r^{n/p-n/u}\left[\int_{B(\iota(x),Dr)}|f(y)|^u\,dy\right]^{1/u}\\
&\le D^{n/u-n/p}\|f\|_{\mathcal{M}^p_u(\mathbb{R}^n)},
\end{aligned}
$$

which implies that ι induces a bounded mapping on the Morrey space $\mathcal{M}^p_u(\mathbb{R}^n)$ with norm less than or equal to $D^{n/u-n/p}$. As a result, we see that Morrey spaces satisfy the assumption of Theorem 5.3.

6. Boundedness of operators

Here, as announced in Section 1, we discuss the boundedness of pseudo-differential operators.

6.1. Boundedness of Fourier multipliers. We now refine Proposition 3.18. Throughout Section 6.1, we use a system (Φ, φ) of Schwartz functions satisfying (1.3) and (1.4).

For $\ell \in \mathbb{N}$ and $\alpha \in \mathbb{R}$, $m \in C^\ell(\mathbb{R}^n \setminus \{0\})$ is assumed to be such that, for all $\|\sigma\|_1 \le \ell$,

$$\sup_{R \in (1,\infty)} \left[R^{-n+2\alpha+2\|\sigma\|_1} \int_{R \le |\xi| < 2R} |\partial_\xi^\sigma m(\xi)|^2 \, d\xi \right] =: A_{\sigma,1} < \infty \tag{6.1}$$

and

$$\int_{|\xi|<1} |\partial_\xi^\sigma m(\xi)|^2 \, d\xi =: A_{\sigma,2} < \infty. \tag{6.2}$$

The *Fourier multiplier* T_m is defined by setting, for all $f \in \mathcal{S}(\mathbb{R}^n)$, $\widehat{T_m f} := m\widehat{f}$.

Lemma 6.1. *Let m be as in* (6.1) *and* (6.2) *and K its inverse Fourier transform. Then $K \in \mathcal{S}'(\mathbb{R}^n)$.*

Proof. Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Then

$$\langle K, \varphi \rangle = \int_{\mathbb{R}^n} m(\xi)\widehat{\varphi}(\xi) \, d\xi = \left(\int_{|\xi|\ge 1} + \int_{|\xi|<1} \right) m(\xi)\widehat{\varphi}(\xi) \, d\xi =: \mathrm{I}_1 + \mathrm{I}_2.$$

Let $M = n - \alpha + 1$. For I_1, by the Hölder inequality and (6.1), we see that

$$\begin{aligned}
|\mathrm{I}_1| &\lesssim \sum_{k=0}^\infty \int_{2^k \le |\xi| < 2^{k+1}} |m(\xi)| \, |\widehat{\varphi}(\xi)| \, d\xi \\
&\lesssim \sum_{k=0}^\infty \frac{\|(1+|\cdot|)^M \widehat{\varphi}\|_{L^\infty(\mathbb{R}^n)}}{(1+2^k)^M} \int_{2^k \le |\xi| < 2^{k+1}} |m(\xi)| \, d\xi \\
&\lesssim \sum_{k=0}^\infty \frac{2^{nk/2}\|(1+|\cdot|)^M \widehat{\varphi}\|_{L^\infty(\mathbb{R}^n)}}{(1+2^k)^M} \left[\int_{2^k \le |\xi| < 2^{k+1}} |m(\xi)|^2 \, d\xi \right]^{1/2} \\
&\lesssim \sum_{k=0}^\infty \frac{2^{k(n-\alpha)}\|(1+|\cdot|)^M \widehat{\varphi}\|_{L^\infty(\mathbb{R}^n)}}{(1+2^k)^M} \lesssim \|(1+|\cdot|)^M \widehat{\varphi}\|_{L^\infty(\mathbb{R}^n)}.
\end{aligned}$$

For I_2, by the Hölder inequality and (6.2), we conclude that

$$|\mathrm{I}_2| \lesssim \|\widehat{\varphi}\|_{L^\infty(\mathbb{R}^n)} \left[\int_{|\xi|<1} |m(\xi)|^2 \, d\xi \right]^{1/2} \lesssim \|\widehat{\varphi}\|_{L^\infty(\mathbb{R}^n)}.$$

This finishes the proof. ■

The next lemma concerns a piece of information adapted to our new setting.

LEMMA 6.2. *Let Ψ, ψ be Schwartz functions on $\mathbb{R}^n$ satisfying, respectively,* (1.3) *and* (1.4). *Assume, in addition, that m satisfies* (6.1) *and* (6.2). *If $a \in (0,\infty)$ and $\ell > a+n/2$, then there exists a positive constant C such that, for all $j \in \mathbb{Z}_+$,*

$$\int_{\mathbb{R}^n} (1+2^j|z|)^a |(K*\psi_j)(z)|\,dz \le C2^{-j\alpha},$$

where $\psi_0 = \Psi$ and $\psi_j(\cdot) = 2^{-jn}\psi(2^j\cdot)$.

Proof. The proof for $j \in \mathbb{N}$ is just [102, Lemma 3.2(i)] with $t = 2^{-j}$. So we still need to prove the case when $j = 0$. Its proof is simple but for convenience of the reader, we supply the details. When $j = 0$, choose μ such that $\mu > n/2$ and $a+\mu \le \ell$. From the Hölder inequality, the Plancherel theorem and (6.2), we deduce that

$$\begin{aligned}
&\left[\int_{\mathbb{R}^n} (1+|z|)^a |(K*\Psi)(z)|\,dz\right]^2 \\
&\qquad \lesssim \int_{\mathbb{R}^n} (1+|z|)^{-2\mu}\,dz \int_{\mathbb{R}^n} (1+|z|)^{2(a+\mu)} |(K*\Psi)(z)|^2\,dz \\
&\qquad \lesssim \int_{\mathbb{R}^n} (1+|z|)^{2\ell} |(K*\Psi)(z)|^2\,dz \\
&\qquad \lesssim \sum_{|\sigma|\le\ell} \int_{\mathbb{R}^n} |z^\sigma (K*\Psi)(z)|^2\,dz \lesssim \sum_{|\sigma|\le\ell} \int_{|\xi|<2} |\partial_\xi^\sigma[m(\xi)]|^2\,dz \lesssim 1,
\end{aligned}$$

which completes the proof. ■

Next we show that, in a suitable way, T_m can also be defined on the whole spaces $F^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$ and $B^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$. Let Φ, φ be Schwartz functions on $\mathbb{R}^n$ that satisfy, respectively, (1.3) and (1.4). Then there exist $\Phi^\dagger \in \mathcal{S}(\mathbb{R}^n)$, satisfying (1.3), and $\varphi^\dagger \in \mathcal{S}(\mathbb{R}^n)$, satisfying (1.4), such that

$$\Phi^\dagger * \Phi + \sum_{i=1}^\infty \varphi_i^\dagger * \varphi_i = \delta_0 \tag{6.3}$$

in $\mathcal{S}'(\mathbb{R}^n)$. For any $f \in F^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$ or $B^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$, we define a linear functional $T_m f$ on $\mathcal{S}(\mathbb{R}^n)$ by setting, for all $\phi \in \mathcal{S}(\mathbb{R}^n)$,

$$\langle T_m f, \phi\rangle := f*\Phi^\dagger*\Phi*\phi*K(0) + \sum_{i\in\mathbb{N}} f*\varphi_i^\dagger*\varphi_i*\phi*K(0) \tag{6.4}$$

as long as the right-hand side converges. In this sense, we say $T_m f \in \mathcal{S}'(\mathbb{R}^n)$. The following result shows that the right-hand side of (6.4) converges and $T_m f$ in (6.4) is well defined.

LEMMA 6.3. *Let $\ell \in (n/2,\infty)$, $\alpha \in \mathbb{R}$, $a \in (0,\infty)$, $\alpha_1,\alpha_2,\alpha_3,\tau \in [0,\infty)$, $q \in (0,\infty]$, $w \in \mathcal{W}^{\alpha_3}_{\alpha_1,\alpha_2}$ and $f \in F^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$ or $B^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$. Then the series in* (6.4) *is convergent and the definition of $T_m f$ is independent of the choice of $(\Phi^\dagger,\Phi,\varphi^\dagger,\varphi)$. Moreover, $T_m f \in \mathcal{S}'(\mathbb{R}^n)$.*

Proof. Due to similarity, we skip the proof for Besov spaces $B^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$. Assume first that $f \in F^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$. Let $(\Psi^\dagger,\Psi,\psi^\dagger,\psi)$ be another set of functions satisfying (6.3). Since

$\phi \in \mathcal{S}(\mathbb{R}^n)$, by the Calderón reproducing formula, we know that

$$\phi = \Psi^\dagger * \Psi * \phi + \sum_{j\in\mathbb{N}} \psi_j^\dagger * \psi_j * \phi$$

in $\mathcal{S}(\mathbb{R}^n)$. Thus,

$$\begin{aligned}
& f * \Phi^\dagger * \Phi * \phi * K(0) + \sum_{i\in\mathbb{N}} f * \varphi_i^\dagger * \varphi_i * \phi * K(0) \\
&\quad = f * \Phi^\dagger * \Phi * \Big(\Psi^\dagger * \Psi * \phi + \sum_{j\in\mathbb{N}} \psi_j^\dagger * \psi_j * \phi\Big) * K(0) \\
&\qquad + \sum_{i\in\mathbb{N}} f * \varphi_i^\dagger * \varphi_i * \Big(\Psi^\dagger * \Psi * \phi + \sum_{j\in\mathbb{N}} \psi_j^\dagger * \psi_j * \phi\Big) * K(0) \\
&\quad = f * \Phi^\dagger * \Phi * \Psi^\dagger * \Psi * \phi * K(0) + f * \Phi^\dagger * \Phi * \psi_1^\dagger * \psi_1 * \phi * K(0) \\
&\qquad + f * \varphi_1^\dagger * \varphi_1 * \Psi^\dagger * \Psi * \phi * K(0) + \sum_{i\in\mathbb{N}} \sum_{j=i-1}^{i+1} f * \varphi_i^\dagger * \varphi_i * \psi_j^\dagger * \psi_j * \phi * K(0),
\end{aligned}$$

where the last equality follows from the fact that $\varphi_i * \psi_j = 0$ if $|i-j| \geq 2$.

Notice that

$$\left| \int_{\mathbb{R}^n} f * \varphi_i(y-z)\varphi_i(-y)\, dy \right| \lesssim \sum_{k\in\mathbb{Z}^n} \frac{2^{in}}{(1+|k|)^M} \int_{Q_{ik}} |\varphi_i * f(y-z)|\, dy$$

for M sufficiently large. As $w \in \mathcal{W}_{\alpha_1,\alpha_2}^{\alpha_3}$, we see that

$$\int_{Q_{ik}} |\varphi_i * f(y-z)|\, dy \lesssim 2^{i(n-\alpha_1)}(1+2^i|z|)^{\alpha_3} 2^{-in\tau} \|f\|_{A^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)}.$$

Thus, by Lemma 6.2, we conclude that

$$\begin{aligned}
& \sum_{i\in\mathbb{N}} |f * \varphi_i * \varphi_i^\dagger * \psi_i * \psi_i^\dagger * \phi * K(0)| \\
&\quad = \sum_{i\in\mathbb{N}} \int_{\mathbb{R}^n} |f * \varphi_i * \varphi_i^\dagger(-z)\psi_i * \psi_i^\dagger * \phi * K(z)|\, dz \\
&\quad \lesssim \sum_{i\in\mathbb{N}} 2^{i(n-\alpha_1-n\tau)} \|f\|_{A^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} \int_{\mathbb{R}^n} \sum_{k\in\mathbb{Z}^n} \frac{2^{in}(1+2^i|z|)^{\alpha_3}}{(1+|k|)^M} |\psi_i * \psi_i * f(z)|\, dz \\
&\quad \lesssim \sum_{i\in\mathbb{N}} 2^{i(n-\alpha_1-n\tau)} 2^{in} \|f\|_{A^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} \int_{\mathbb{R}^n} (1+2^i|z|)^{\alpha_3} \int_{\mathbb{R}^n} \frac{2^{-iM}}{(1+|y-z|)^M} |\psi_i * f(y)|\, dy\, dz \\
&\quad \lesssim \sum_{i\in\mathbb{N}} 2^{i(2n-\alpha_1-n\tau+\alpha_3-M)} \|f\|_{A^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} \int_{\mathbb{R}^n} (1+2^i|y|)^{\alpha_3} |\psi_i * f(y)|\, dy \\
&\quad \lesssim \sum_{i\in\mathbb{N}} 2^{i(2n-\alpha_1-n\tau-M)} \|f\|_{A^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} \lesssim \|f\|_{A^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)},
\end{aligned}$$

where a is an arbitrary positive number.

By an argument similar to the above, we conclude that

$$\Big|f * \Phi^\dagger * \Phi * \Big(\Psi^\dagger * \Psi * \phi + \sum_{j\in\mathbb{N}} \psi_j^\dagger * \psi_j * \phi\Big) * K(0)\Big| \\ + \Big|\sum_{i\in\mathbb{N}} f * \varphi_i^\dagger * \varphi_i * \Big(\Psi^\dagger * \Psi * \phi + \sum_{j\in\mathbb{N}} \psi_j^\dagger * \psi_j * \phi\Big) * K(0)\Big| < \infty,$$

which, together with the Calderón reproducing formula, further implies that

$$\begin{aligned} f * \Phi^\dagger * \Phi * \phi * K(0) + \sum_{i\in\mathbb{N}} f * \varphi_i^\dagger * \varphi_i * \phi * K(0) &= f * \Phi^\dagger * \Phi * \Big(\Psi^\dagger * \Psi * \phi + \sum_{j\in\mathbb{N}} \psi_j^\dagger * \psi_j * \phi\Big) * K(0) \\ &\quad + \sum_{i\in\mathbb{N}} f * \varphi_i^\dagger * \varphi_i * \Big(\Psi^\dagger * \Psi * \phi + \sum_{j\in\mathbb{N}} \psi_j^\dagger * \psi_j * \phi\Big) * K(0) \\ &= f * \Psi^\dagger * \Psi * \Psi * K(0) + \sum_{i\in\mathbb{N}} f * \varphi_i^\dagger * \varphi_i * \Psi * K(0). \end{aligned}$$

Thus, $T_m f$ in (6.4) is independent of the choice of $(\Phi^\dagger, \Phi, \varphi^\dagger, \varphi)$. Moreover, the previous argument also implies that $T_m f \in \mathcal{S}'(\mathbb{R}^n)$. ∎

By Lemma 6.2, we immediately have the following conclusion; we omit the details.

LEMMA 6.4. *Let $\alpha \in \mathbb{R}$, $a \in (0,\infty)$, $\ell \in \mathbb{N}$, let $\Phi, \Psi \in \mathcal{S}(\mathbb{R}^n)$ satisfy* (1.3) *and let $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$ satisfy* (1.4). *Assume that m satisfies* (6.1) *and* (6.2) *and $f \in \mathcal{S}'(\mathbb{R}^n)$ is such that $T_m f \in \mathcal{S}'(\mathbb{R}^n)$. If $\ell > a + n/2$, then there exists a positive constant C such that, for all x, $y \in \mathbb{R}^n$ and $j \in \mathbb{Z}_+$,*

$$|(T_m f * \psi_j)(y)| \le C 2^{-j\alpha}(1 + 2^j|x-y|)^a (\varphi_j^* f)_a(x).$$

Now we are ready to prove the following conclusion.

THEOREM 6.5. *Let $\alpha \in \mathbb{R}$, $a \in (0,\infty)$, $\alpha_1, \alpha_2, \alpha_3, \tau \in [0,\infty)$, $q \in (0,\infty]$, $w \in \mathcal{W}_{\alpha_1,\alpha_2}^{\alpha_3}$ and $\widetilde{w}(x, 2^{-j}) = 2^{j\alpha} w(x, 2^{-j})$ for all $x \in \mathbb{R}^n$ and $j \in \mathbb{Z}_+$. Suppose that m satisfies* (6.1) *and* (6.2) *with $\ell \in \mathbb{N}$ and $\ell > a + n/2$. Then there exists a positive constant C_1 such that, for all $f \in F_{\mathcal{L},q,a}^{w,\tau}(\mathbb{R}^n)$,*

$$\|T_m f\|_{F_{\mathcal{L},q,a}^{\widetilde{w},\tau}(\mathbb{R}^n)} \le C_1 \|f\|_{F_{\mathcal{L},q,a}^{w,\tau}(\mathbb{R}^n)},$$

and a positive constant C_2 such that, for all $f \in B_{\mathcal{L},q,a}^{\widetilde{w},\tau}(\mathbb{R}^n)$,

$$\|T_m f\|_{B_{\mathcal{L},q,a}^{\widetilde{w},\tau}(\mathbb{R}^n)} \le C_2 \|f\|_{B_{\mathcal{L},q,a}^{w,\tau}(\mathbb{R}^n)}.$$

Similar assertions hold for $\mathcal{E}_{\mathcal{L},q,a}^{w,\tau}(\mathbb{R}^n)$ and $\mathcal{N}_{\mathcal{L},q,a}^{w,\tau}(\mathbb{R}^n)$.

Proof. By Lemma 6.4 we conclude that, if $\ell > a + n/2$, then for all $x \in \mathbb{R}^n$ and $j \in \mathbb{Z}_+$,

$$2^{j\alpha}(\psi_j^*(T_m f))_a(x) \lesssim (\varphi_j^* f)_a(x).$$

Then by the definitions of $F_{\mathcal{L},q,a}^{w,\tau}(\mathbb{R}^n)$ and $B_{\mathcal{L},q,a}^{w,\tau}(\mathbb{R}^n)$, we immediately deduce the desired conclusions. ∎

6.2. Boundedness of pseudo-differential operators. We consider the *class* $S^0_{1,\mu}(\mathbb{R}^n)$ with $\mu \in [0,1)$. Recall that a function $a \in C^\infty(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$ is said to belong to $S^m_{1,\mu}(\mathbb{R}^n)$ if

$$\sup_{x,\xi\in\mathbb{R}^n} (1+|\xi|)^{-m-\|\vec{\alpha}\|_1-\mu\|\vec{\beta}\|_1} |\partial_x^{\vec{\beta}} \partial_\xi^{\vec{\alpha}} a(x,\xi)| \lesssim_{\vec{\alpha},\vec{\beta}} 1$$

for all multiindices $\vec{\alpha}$ and $\vec{\beta}$. One defines, for all $x \in \mathbb{R}^n$,

$$a(X,D)(f)(x) := \int_{\mathbb{R}^n} a(x,\xi)\hat{f}(\xi)e^{ix\cdot\xi}\,d\xi,$$

first on $\mathcal{S}(\mathbb{R}^n)$, and then on $\mathcal{S}'(\mathbb{R}^n)$ via duality.

We aim to establish the following.

Theorem 6.6. *Let $w \in \mathcal{W}^{\alpha_3}_{\alpha_1,\alpha_2}$ with $\alpha_1,\alpha_2,\alpha_3 \in [0,\infty)$ and suppose a quasi-normed function space $\mathcal{L}(\mathbb{R}^n)$ satisfies* ($\mathcal{L}1$) *through* ($\mathcal{L}6$)*. Let $\mu \in [0,1)$, $\tau \in (0,\infty)$ and $q \in (0,\infty]$. Assume, in addition, that* (3.28) *holds, that is, $a \in (N_0+\alpha_3,\infty)$, where N_0 is as in* ($\mathcal{L}6$)*. Then all pseudo-differential operators with symbols in $S^0_{1,\mu}(\mathbb{R}^n)$ are bounded on $A^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$.*

With the following decomposition, we have only to consider the boundedness of $a(\cdot,\cdot) \in S^{-M_0}_{1,\mu}(\mathbb{R}^n)$ with an integer M_0 sufficiently large.

Lemma 6.7 ([88]). *Let $\mu \in [0,1)$, $a \in S^m_{1,\mu}(\mathbb{R}^n)$ and $N \in \mathbb{N}$. Then there exists a symbol $b \in S^m_{1,\mu}(\mathbb{R}^n)$ such that*

$$a(X,D) = (1+\Delta^{2N}) \circ b(X,D) \circ (1+\Delta^{2N})^{-1}.$$

Based upon Lemma 6.7, we plan to treat

$$\begin{aligned} A(X,D) &:= b(X,D) \circ (1+\Delta^{2N})^{-1} \in S^{-2N}_{1,\mu}(\mathbb{R}^n), \\ B(X,D) &:= \Delta^{2N} \circ b(X,D) \circ (1+\Delta^{2N})^{-1} \in S^0_{1,\mu}(\mathbb{R}^n). \end{aligned}$$

The following is one of the key observations in this subsection.

Lemma 6.8. *Let $\mu \in [0,1)$, w, q, τ, a and $\mathcal{L}$ be as in Theorem* 6.6*. Assume that $a \in S^0_{1,\mu}(\mathbb{R}^n)$ has the property that $a(\cdot,\xi) = 0$ if $|\xi| \ge 1/2$. Then $a(X,D)$ is bounded on $A^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$.*

Proof. We fix $\Phi \in \mathcal{S}(\mathbb{R}^n)$ so that $\hat{\Phi}(\xi) = 1$ whenever $|\xi| \le 1$ and $\hat{\Phi}(\xi) = 0$ whenever $|\xi| \ge 2$. Then, since $a(\cdot,\xi) = 0$ if $|\xi| \ge 1/2$, we know that, for all $f \in A^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$, $a(X,D)f = a(X,D)(\Phi * f)$. Hence, as the mapping $f \in A^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n) \mapsto \Phi * f \in A^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$ is continuous, without loss of generality we may assume that the frequency support of f is contained in $\{\xi \in \mathbb{R}^n : |\xi| \le 2\}$. Let $j \in \mathbb{Z}_+$ and $z \in \mathbb{R}^n$ be fixed. Then, for all $x \in \mathbb{R}^n$,

$$\begin{aligned} \varphi_j * [a(X,D)f](x) &= \int_{\mathbb{R}^n} \varphi_j(x-y)\left[\int_{\mathbb{R}^n} a(y,\xi)\widehat{f}(\xi)e^{i\xi y}\,d\xi\right]dy \\ &= \int_{\mathbb{R}^n}\left[\int_{\mathbb{R}^n} \varphi_j(x-y)a(y,\xi)e^{i\xi y}\,dy\right]\widehat{f}(\xi)\,d\xi \\ &= \int_{\mathbb{R}^n}\left[\int_{\mathbb{R}^n} \varphi_j(x-y)a(y,\cdot)e^{i\cdot y}\,dy\right]^{\wedge}(z)f(z)\,dz \end{aligned}$$

by the Fubini theorem. Notice that, again by the Fubini theorem,

$$\left[\int_{\mathbb{R}^n} \varphi_j(x-y)a(y,\cdot)e^{i\cdot y}\,dy\right]^\wedge(z) = \int_{\mathbb{R}^n} e^{-iz\xi}\left[\int_{\mathbb{R}^n} \varphi_j(x-y)a(y,\xi)e^{i\xi y}\,dy\right]d\xi$$
$$= \int_{\mathbb{R}^n} \varphi_j(x-y)\left[\int_{\mathbb{R}^n} a(y,\xi)e^{i\xi(y-z)}\,d\xi\right]dy.$$

Let us set $\tau_j := (4^{-j}\Delta)^{-L}\varphi_j$ with $L \in \mathbb{N}$ large enough, say

$$L = \lfloor a + n + \alpha_1 + \alpha_2 + 1 \rfloor.$$

Then $\tau_j \in \mathcal{S}(\mathbb{R}^n)$ and $\varphi_j(x) = 2^{-2jL}\Delta^L\tau_j(x)$ for all $j \in \mathbb{Z}_+$ and $x \in \mathbb{R}^n$. Consequently,

$$\left[\int_{\mathbb{R}^n} \varphi_j(x-y)a(y,\cdot)e^{i\cdot y}\,dy\right]^\wedge(z) = 2^{-2jL}\int_{\mathbb{R}^n} \tau_j(x-y)\Delta_y^L\left[\int_{\mathbb{R}^n} a(y,\xi)e^{i\xi(y-z)}\,d\xi\right]dy$$

by integration by parts.

Again by integration by parts, we conclude that

$$\Delta_y^L\left(\int_{\mathbb{R}^n} a(y,\xi)e^{i\xi(y-z)}\,d\xi\right)$$
$$= \sum_{\|\vec{\alpha}_1\|_1+\|\vec{\alpha}_2\|_1=2L} \int_{\mathbb{R}^n} [\xi^{\vec{\alpha}_2}\partial_y^{\vec{\alpha}_1}a(y,\xi)]e^{i\xi(y-z)}\,d\xi$$
$$= \frac{1}{(1+|y-z|^2)^L}\sum_{\|\vec{\alpha}_1\|_1+\|\vec{\alpha}_2\|_1=2L} \int_{\mathbb{R}^n} (1-\Delta_\xi)^L[\xi^{\vec{\alpha}_2}\partial_y^{\vec{\alpha}_1}a(y,\xi)]e^{i\xi(y-z)}\,d\xi.$$

Then, since $a \in S^0_{1,\mu}(\mathbb{R}^n)$ and $a(\cdot,\xi) = 0$ if $|\xi| \geq 1/2$, we see that, for all $\xi, y \in \mathbb{R}^n$,

$$|(1-\Delta_\xi)^L(\xi^{\vec{\alpha}_2}\partial_y^{\vec{\alpha}_1}a(y,\xi))| \lesssim \chi_{B(0,2)}(\xi), \tag{6.5}$$

and hence, for all $y, z \in \mathbb{R}^n$,

$$\left|\Delta_y^L\left(\int_{\mathbb{R}^n} a(y,\xi)e^{i\xi(y-z)}\,d\xi\right)\right| \lesssim \frac{1}{(1+|y-z|^2)^L}.$$

Consequently, for all $j \in \mathbb{Z}_+$ and $x, y, z \in \mathbb{R}^n$,

$$\left|\left[\int_{\mathbb{R}^n} \varphi_j(x-y)a(y,\cdot)e^{i\cdot y}\,dy\right]^\wedge(z)\right| \lesssim 2^{-2jL}\int_{\mathbb{R}^n} \frac{|\tau_j(x-y)|}{(1+|y-z|^2)^L}\,dy, \tag{6.6}$$

and hence

$$\frac{|\varphi_j * (a(X,D)f)(x+z)|}{(1+2^j|z|)^a} \lesssim \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{2^{-2jL}|\tau_j(x+z-y)|}{(1+2^j|z|)^a(1+|y-w|^2)^L}|f(w)|\,dy\,dw$$
$$\lesssim \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{2^{-2jL}|\tau_j(x+z-y)|}{(1+|z|)^a(1+|y-w|^2)^L}|f(w)|\,dy\,dw$$
$$\lesssim 2^{-2jL}\sup_{w\in\mathbb{R}^n} \frac{|f(x+w)|}{(1+|w|)^a}.$$

A similar argument also works for $\Phi * (a(X,D)f)$ (without using integration by parts) and we obtain

$$\frac{|\Phi * (a(X,D)f)(x+z)|}{(1+|z|)^a} \lesssim \sup_{w\in\mathbb{R}^n} \frac{|f(x+w)|}{(1+|w|)^a}.$$

With this pointwise estimate, the condition on L and the assumption that $\mu<1$, we obtain the desired result. ∎

If we reexamine the above calculation, we obtain the following:

LEMMA 6.9. *Assume that $\mu\in[0,1)$ and $a\in S^{-2M_0}_{1,\mu}(\mathbb{R}^n)$ satisfies $a(\cdot,\xi)=0$ if $2^{k-2}\le|\xi|\le 2^{k+2}$. Then $a(X,D)$ is bounded on $A^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$. Moreover, there exist a positive constant E and a positive constant $C(E)$, depending on E, such that the operator norm has the property that*

$$\|a(X,D)\|_{A^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)\to A^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)}\le C(E)2^{-Ek}$$

provided $M_0\in(1,\infty)$ is large enough.

Proof. Let us suppose that $M_0>2L+n$, where $L\in\mathbb{N}$ is chosen so that

$$L=\lfloor a+n+\alpha_1+\alpha_2+n\tau+1\rfloor. \tag{6.7}$$

Notice that this time $a(X,D)f=a(X,D)(\sum_{i=k-3}^{k+3}\varphi_i*f)$ for all $f\in A^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$. If we go through a similar argument as we did for (6.6) with the condition on L replaced by (6.7), we see that, for all $j\in\mathbb{Z}_+$ and $x,z\in\mathbb{R}^n$,

$$\left|\left[\int_{\mathbb{R}^n}\varphi_j(x-y)a(y,\cdot)e^{i\cdot y}\,dy\right]^\wedge(z)\right|\lesssim 2^{-2jL+k(4L-2M_0+n)}\int_{\mathbb{R}^n}\frac{|\tau_j(x-y)|}{(1+|y-z|^2)^L}\,dy. \tag{6.8}$$

Indeed, we just need to replace (6.5) in the proof of (6.6) by the following estimate, for all $k\in\mathbb{Z}_+$, $\xi,y\in\mathbb{R}^n$ and multi-indices α, β such that $\|\alpha\|_1+\|\beta\|_1=2L$:

$$|(1-\Delta_\xi)^L(\xi^\alpha\partial^\beta_y a(y,\xi))|\lesssim 2^{2k(2L-M_0)}\chi_{B(0,2^{k+2})\setminus B(0,2^{k-2})}(\xi).$$

By (6.8), we conclude that, for all $j\in\mathbb{Z}_+$ and $x,z\in\mathbb{R}^n$,

$$\begin{aligned}&\frac{|\varphi_j*(a(X,D)f)(x+z)|}{(1+2^j|z|)^a}\\ &\quad\lesssim\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{2^{-2jL+k(4L-2M_0+n)}|\tau_j(x+z-y)|}{(1+2^j|z|)^a(1+|y-w|^2)^L}\sum_{l=-3}^{3}|\varphi_{k+l}*f(w)|\,dy\,dw\\ &\quad\lesssim 2^{-2jL+k(4L-2M_0+a+n)}\sum_{l=-3}^{3}\sup_{w\in\mathbb{R}^n}\frac{|\varphi_{k+l}*f(x+w)|}{(1+2^{k+l}|w|)^a}.\end{aligned}$$

Consequently,

$$\frac{|\varphi_j(D)(a(X,D)f)(x+z)|}{(1+2^j|z|)^a}\lesssim 2^{-2jL+k(4L-2M_0+a+n)}\sup_{\substack{w\in\mathbb{R}^n\\ l\in[-3,3]\cap\mathbb{Z}}}\frac{|\varphi_{k+l}(D)f(x+w)|}{(1+2^{k+l}|w|)^a}. \tag{6.9}$$

Combining (6.9) and Lemma 2.9 completes the proof. ∎

In view of the atomic decomposition, we have the following conclusion.

LEMMA 6.10. *Let w be as in Theorem 6.6. Assume that $a\in S^0_{1,\mu}(\mathbb{R}^n)$ can be expressed as $a(X,D)=\Delta^{2M_0}\circ b(X,D)$ for some $b\in S^{-2M_0}_{1,\mu}(\mathbb{R}^n)$. Then $a(X,D)$ is bounded on $A^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$ as long as M_0 is large.*

Proof. For any $f\in A^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$, by Theorem 4.5, there exist a collection $\{\mathfrak{A}_{jk}\}_{j\in\mathbb{Z}_+,\,k\in\mathbb{Z}^n}$ of atoms and a complex sequence $\{\lambda_{jk}\}_{j\in\mathbb{Z}_+,\,k\in\mathbb{Z}^n}$ such that $f=\sum_{j=0}^\infty\sum_{k\in\mathbb{Z}^n}\lambda_{jk}\mathfrak{A}_{jk}$

in $\mathcal{S}'(\mathbb{R}^n)$ and $\|\{\lambda_{jk}\}_{j\in\mathbb{Z}_+,\,k\in\mathbb{Z}^n}\|_{a^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} \lesssim \|f\|_{A^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)}$. In the course of the proof of [75, Theorem 3.1], we have shown that the atoms $\{\mathfrak{A}_{jk}\}_{j\in\mathbb{Z}_+,\,k\in\mathbb{Z}^n}$ are transformed into molecules $\{a(X,D)\mathfrak{A}_{jk}\}_{j\in\mathbb{Z}_+,\,k\in\mathbb{Z}^n}$ satisfying the decay condition. However, if $a(X,D) = \Delta^{2M_0} \circ b(X,D)$, then atoms are transformed into molecules with moment condition of order $2M_0$. Therefore, via Theorem 4.5 letting $L = 2M_0$ completes the proof. ■

With Lemmas 6.8 through 6.10 in mind, we prove Theorem 6.6.

Proof of Theorem 6.6. We decompose $a(X,D)$ according to Lemma 6.7. We fix an integer M_0 large enough as in Lemmas 6.9 and 6.10. Write $A(X,D) := a(X,D) \circ (1+\Delta^{2M_0})^{-1}$ and $B(X,D) := \Delta^{2M_0} \circ a(X,D) \circ (1+\Delta^{2M_0})^{-1}$.

Let Φ and φ be as in (1.3) and (1.4) with $\widehat{\Phi}(\xi) + \sum_{j\in\mathbb{N}} \widehat{\varphi}(2^{-j}\xi) = 1$ for all $\xi \in \mathbb{R}^n$. Then by the Calderón reproducing formula, $f = \Phi * f + \sum_{j\in\mathbb{N}} \varphi_j * f$ in $\mathcal{S}'(\mathbb{R}^n)$ for all $f \in A^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$. Therefore,

$$\begin{aligned} a(X,D)f(x) &= \sum_{j=0}^{\infty} a(X,D)(\varphi_j * f)(x) \\ &= \sum_{j=0}^{\infty} \int_{\mathbb{R}^n} a(x,\xi)\widehat{\varphi}(2^{-j}\xi)\widehat{f}(\xi)e^{ix\xi}\,d\xi \\ &=: \sum_{j=0}^{\infty} \int_{\mathbb{R}^n} a_j(x,\xi)\widehat{f}(\xi)e^{ix\xi}\,d\xi =: \sum_{j=0}^{\infty} a_j(X,D)f(x) \end{aligned}$$

in $\mathcal{S}'(\mathbb{R}^n)$, where $a_j(x,\xi) := a(x,\xi)\widehat{\varphi}(2^{-j}\xi)$ for all $x,\xi \in \mathbb{R}^n$, and $a_j(X,D)$ is the related operator. It is easy to see that $a_j \in S^0_{1,\mu}(\mathbb{R}^n)$ with support in the annulus $2^{j-2} \le |\xi| \le 2^{j+2}$. Then by Lemmas 6.8 and 6.9, $A(X,D)$ is bounded on $A^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$. Moreover, Lemma 6.10 shows that $B(X,D)$ is bounded on $A^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$. Consequently, $a(X,D) = A(X,D) + B(X,D)$ is bounded on $A^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$. ■

Since molecules are mapped to molecules by pseudo-differential operators if we do not consider the moment condition, we have the following conclusion. We omit the details.

THEOREM 6.11. *Under the condition of Theorem 4.9, pseudo-differential operators with symbols in $S^0_{1,1}(\mathbb{R}^n)$ are bounded on $A^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$.*

7. Embeddings

7.1. Embedding into $C(\mathbb{R}^n)$. Here we give a sufficient condition for our function spaces to be embedded into $C(\mathbb{R}^n)$. In what follows, $C(\mathbb{R}^n)$ denotes the set of all continuous functions on $\mathbb{R}^n$. Notice that we do not require that the functions of $C(\mathbb{R}^n)$ are bounded.

THEOREM 7.1. *Let $q \in (0,\infty]$, $a \in (0,\infty)$ and $\tau \in [0,\infty)$. Let $w \in \star\text{-}\mathcal{W}^{\alpha_3}_{\alpha_1,\alpha_2}$ with $\alpha_1,\alpha_2,\alpha_3 \in [0,\infty)$ and suppose a quasi-normed function space $\mathcal{L}(\mathbb{R}^n)$ satisfies $(\mathcal{L}1)$ through $(\mathcal{L}6)$ such that*

$$a+\gamma-\alpha_1-n\tau<0. \tag{7.1}$$

Then $A^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$ is embedded into $C(\mathbb{R}^n)$.

Proof. By Remark 3.9(ii), it suffices to consider $B^{w,\tau}_{\mathcal{L},\infty,a}(\mathbb{R}^n)$, into which $A^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$ is embedded. Also let us assume (3.22). Let us prove that $B^{w,\tau}_{\mathcal{L},\infty,a}(\mathbb{R}^n)$ is embedded into $C(\mathbb{R}^n)$. Fix $x \in \mathbb{R}^n$. From the definition of the Peetre maximal operator, we deduce that, for all $f \in B^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$, $j \in \mathbb{Z}_+$ and $y \in B(x,1)$,

$$\sup_{w\in B(x,1)}|\varphi_j*f(w)| \lesssim 2^{ja}\sup_{z\in\mathbb{R}^n}\frac{|\varphi_j*f(y+z)|}{(1+2^j|z|)^a}.$$

If we consider the $\mathcal{L}(\mathbb{R}^n)$-quasi-norm of both sides, then we obtain

$$\sup_{z\in B(x,1)}|\varphi_j*f(z)| \lesssim_x \frac{2^{ja}}{\|\chi_{B(x,2^{-j})}\|_{\mathcal{L}(\mathbb{R}^n)}}\|\chi_{B(x,2^{-j})}(\varphi_j^*f)_a\|_{\mathcal{L}(\mathbb{R}^n)}.$$

Notice that $w_j(x)=w(x,2^{-j})\ge 2^{j\alpha_1}w(x,1)$ for all $j\in\mathbb{Z}_+$ and $x\in\mathbb{R}^n$, and hence from (W2) and (7.1), it follows that

$$\sup_{z\in B(x,1)}|\varphi_j*f(z)| \lesssim_x 2^{j(a+\gamma-\alpha_1-n\tau)}\|f\|_{B^{w,\tau}_{\mathcal{L},\infty,a}(\mathbb{R}^n)}.$$

Since this implies that

$$f=\Phi*f+\sum_{j=1}^\infty\varphi_j*f$$

converges uniformly over any ball with radius 1, it follows that f is continuous. ■

7.2. Function spaces $A^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$ for τ large. The following theorem generalizes [101, Theorem 1] and explains what happens if τ is too large.

THEOREM 7.2. *Let $\omega \in \mathcal{W}^{\alpha_3}_{\alpha_1,\alpha_2}$ with $\alpha_1,\alpha_2,\alpha_3\in[0,\infty)$. Define a new index $\widetilde{\tau}$ by*

$$\widetilde{\tau}:=\limsup_{j\to\infty}\sup_{P\in\mathcal{Q}_j(\mathbb{R}^n)}\left[\frac{1}{nj}\log_2\frac{1}{\|\chi_P\|_{\mathcal{L}(\mathbb{R}^n)}}\right] \tag{7.2}$$

and a new weight $\widetilde{\omega}$ *by*

$$\widetilde{\omega}(x,2^{-j}) := 2^{jn(\tau-\widetilde{\tau})}\omega(x,2^{-j}), \quad x\in\mathbb{R}^n,\ j\in\mathbb{Z}_+.$$

Assume that

$$\tau > \widetilde{\tau} \geq 0. \tag{7.3}$$

Then

(i) $\widetilde{w}\in\mathcal{W}^{\alpha_3}_{(\alpha_1-n(\tau-\widetilde{\tau}))_+,(\alpha_2+n(\tau-\widetilde{\tau}))_+}$;

(ii) *for all* $q\in(0,\infty)$ *and* $a>\alpha_3+N_0$, $F^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$ *and* $B^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$ *coincide, respectively, with* $F^{\widetilde{w}}_{\infty,\infty,a}(\mathbb{R}^n)$ *and* $B^{\widetilde{w}}_{\infty,\infty,a}(\mathbb{R}^n)$ *with equivalent norms.*

Proof. We only prove $F^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$ coincides with $F^{\widetilde{w}}_{\infty,\infty,a}(\mathbb{R}^n)$. The assertion (i) can be proved as in Example 2.4(iii) and the proof for $B^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$ and $B^{\widetilde{w}}_{\infty,\infty,a}(\mathbb{R}^n)$ is similar.

By the atomic decomposition of $(F^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n), f^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n))$ and $(F^{\widetilde{w}}_{\infty,\infty,a}(\mathbb{R}^n), f^{\widetilde{w}}_{\infty,\infty,a}(\mathbb{R}^n))$, it suffices to show that $f^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n) = f^{\widetilde{w}}_{\infty,\infty,a}(\mathbb{R}^n)$ with norm equivalence. Recall that, for all $\lambda=\{\lambda_{jk}\}_{j\in\mathbb{Z}_+,k\in\mathbb{Z}^n}$,

$$\begin{aligned}
&\|\lambda\|_{f^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)}\\
&\quad= \sup_{P\in\mathcal{Q}(\mathbb{R}^n)}\frac{1}{|P|^\tau}\left\|\left[\sum_{j=j_P\vee 0}^{\infty}\left(\chi_P w_j\sup_{y\in\mathbb{R}^n}\frac{1}{(1+2^j|y|)^a}\sum_{k\in\mathbb{Z}^n}|\lambda_{jk}|\chi_{Q_{jk}}(\cdot+y)\right)^q\right]^{1/q}\right\|_{\mathcal{L}(\mathbb{R}^n)}
\end{aligned}$$

and

$$\begin{aligned}
\|\lambda\|_{f^{\widetilde{w}}_{\infty,\infty,a}(\mathbb{R}^n)} &= \sup_{x\in\mathbb{R}^n,\,j\in\mathbb{Z}_+}\widetilde{w}_j(x)\sup_{y\in\mathbb{R}^n}\frac{1}{(1+2^j|y|)^a}\sum_{k\in\mathbb{Z}^n}|\lambda_{jk}|\chi_{Q_{jk}}(x+y)\\
&= \sup_{(x,y)\in\mathbb{R}^n\times\mathbb{R}^n,\,j\in\mathbb{Z}_+}\widetilde{w}_j(x)\frac{1}{(1+2^j|y|)^a}\sum_{k\in\mathbb{Z}^n}|\lambda_{jk}|\chi_{Q_{jk}}(x+y).
\end{aligned}\tag{7.4}$$

By (7.4), there exist $j_0\in\mathbb{Z}_+$, $k_0\in\mathbb{Z}^n$ and $x_0,y_0\in\mathbb{R}^n$ such that

$$x_0+y_0\in Q_{j_0k_0} \quad\text{and}\quad \|\lambda\|_{f^{\widetilde{w}}_{\infty,\infty,a}(\mathbb{R}^n)}\sim\widetilde{w}_{j_0}(x_0)\frac{|\lambda_{j_0k_0}|}{(1+2^{j_0}|y_0|)^a}.$$

A geometric observation shows that there exists $P_0\in\mathcal{Q}(\mathbb{R}^n)$ whose side length is half that of $Q_{j_0k_0}$ and which satisfies $y_0+P_0\subset Q_{j_0k_0}$. Thus, for all $x\in P_0$, we have $|x-x_0|\lesssim 2^{-j_0}$ and hence

$$w_{j_0}(x_0)\leq w_{j_0}(x)(1+2^{j_0}|x-x_0|)^{\alpha_3}\lesssim w_{j_0}(x),$$

which, together with the assumption on τ, implies that

$$\begin{aligned}
&\|\lambda\|_{f^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)}\\
&\quad= \sup_{P\in\mathcal{Q}(\mathbb{R}^n)}\frac{1}{|P|^\tau}\left\|\left[\sum_{j=j_P\vee 0}^{\infty}\left(\chi_P w_j\sup_{y\in\mathbb{R}^n}\frac{1}{(1+2^j|y|)^a}\sum_{k\in\mathbb{Z}^n}|\lambda_{jk}|\chi_{Q_{jk}}(\cdot+y)\right)^q\right]^{1/q}\right\|_{\mathcal{L}(\mathbb{R}^n)}\\
&\quad\gtrsim \frac{1}{|P_0|^\tau}\left\|\chi_{P_0}w_{j_0}\frac{|\lambda_{j_0k_0}|}{(1+2^j|y_0|)^a}\chi_{Q_{jk}}(\cdot+y_0)\right\|_{\mathcal{L}(\mathbb{R}^n)}\\
&\quad\gtrsim \|\lambda\|_{f^{\widetilde{w}}_{\infty,\infty,a}(\mathbb{R}^n)}\frac{2^{-j_0n(\tau-\widetilde{\tau})}\|\chi_{P_0}\|_{\mathcal{L}(\mathbb{R}^n)}}{|P_0|^\tau}.
\end{aligned}$$

Consequently,

$$\|\lambda\|_{f^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} \gtrsim \|\lambda\|_{f^{\widetilde{w}}_{\infty,\infty,a}(\mathbb{R}^n)}. \tag{7.5}$$

To obtain the reverse inclusion, we calculate

$$\begin{aligned}
\|\lambda\|_{f^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} &= \sup_{P\in\mathcal{Q}(\mathbb{R}^n)} \frac{1}{|P|^\tau} \left\| \left(\sum_{j=j_P\vee 0}^{\infty} \left[\chi_P w_j \sup_{y\in\mathbb{R}^n} \frac{\sum_{k\in\mathbb{Z}^n} |\lambda_{jk}|\chi_{Q_{jk}}(\cdot+y)}{(1+2^j|y|)^a} \right]^q \right)^{1/q} \right\|_{\mathcal{L}(\mathbb{R}^n)} \\
&\le \|\lambda\|_{f^{\widetilde{w}}_{\infty,\infty,a}(\mathbb{R}^n)} \sup_{P\in\mathcal{Q}(\mathbb{R}^n)} \frac{1}{|P|^\tau} \left\| \chi_P \Big[\sum_{j=j_P\vee 0}^{\infty} \left(\frac{w_j}{\widetilde{w}_j}\right)^q \Big]^{1/q} \right\|_{\mathcal{L}(\mathbb{R}^n)}.
\end{aligned}$$

Using (7.3) and (7.4), we obtain

$$\begin{aligned}
\|\lambda\|_{f^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} &\le \|\lambda\|_{f^{\widetilde{w}}_{\infty,\infty,a}(\mathbb{R}^n)} \sup_{P\in\mathcal{Q}(\mathbb{R}^n)} \frac{1}{|P|^\tau} \left\| \chi_P \Big[\sum_{j=j_P\vee 0}^{\infty} 2^{-jnq(\tau-\widetilde{\tau})} \Big]^{1/q} \right\|_{\mathcal{L}(\mathbb{R}^n)} \\
&\sim \|\lambda\|_{f^{\widetilde{w}}_{\infty,\infty,a}(\mathbb{R}^n)} \sup_{P\in\mathcal{Q}(\mathbb{R}^n)} \frac{2^{-(j_P\vee 0)n(\tau-\widetilde{\tau})}}{|P|^\tau} \|\chi_P\|_{\mathcal{L}(\mathbb{R}^n)}.
\end{aligned}$$

Since $\widetilde{\tau}\in[0,\infty)$ and (7.2) holds, we see that

$$\begin{aligned}
\frac{2^{-(j_P\vee 0)n(\tau-\widetilde{\tau})}}{|P|^\tau} \|\chi_P\|_{\mathcal{L}(\mathbb{R}^n)} &\lesssim \frac{2^{-j_Pn(\tau-\widetilde{\tau})}}{|P|^\tau} \|\chi_P\|_{\mathcal{L}(\mathbb{R}^n)} \\
&\sim 2^{j_Pn\widetilde{\tau}} \|\chi_P\|_{\mathcal{L}(\mathbb{R}^n)} \sim |P|^{-\widetilde{\tau}} \|\chi_P\|_{\mathcal{L}(\mathbb{R}^n)} \lesssim 1.
\end{aligned}$$

Hence, we conclude that

$$\|\lambda\|_{f^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} \lesssim \|\lambda\|_{f^{\widetilde{w}}_{\infty,\infty,a}(\mathbb{R}^n)}. \tag{7.6}$$

Hence from (7.5) and (7.6), we deduce that $F^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$ and $F^{\widetilde{w}}_{\infty,\infty,a}(\mathbb{R}^n)$ coincide with equivalent norms. ■

8. Characterizations via differences and oscillations

In this section we are going to characterize our function spaces by means of differences and oscillations. To this end, we need some key constructions from Triebel [91].

For any $M \in \mathbb{N}$, Triebel [91, p. 173, Lemma 3.3.1] proved that there exist two smooth functions φ and ψ on $\mathbb{R}$ with $\operatorname{supp} \varphi \subset (0,1)$, $\operatorname{supp} \psi \subset (0,1)$, $\int_{\mathbb{R}} \varphi(\tau)\, d\tau = 1$ and $\varphi(t) - \frac{1}{2}\varphi(\frac{t}{2}) = \psi^{(M)}(t)$ for $t \in \mathbb{R}$. Let $\rho(x) := \prod_{\ell=1}^{n} \varphi(x_\ell)$ for all $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. For all $j \in \mathbb{Z}_+$ and $x \in \mathbb{R}^n$, let

$$T_j(x) := \sum_{m'=1}^{M} \sum_{m=1}^{M} \frac{(-1)^{M+m+m'+1}}{M!} \binom{M}{m'} \binom{M}{m} m^M (2^{-j} m m')^{-n} \rho\left(\frac{x}{2^{-j} m m'}\right),$$

where $\binom{M}{m}$ for $m \in \{1, \ldots, M\}$ denotes the binomial coefficient. For any $f \in \mathcal{S}'(\mathbb{R}^n)$, let

$$f^j := T_j * f \quad \text{for all } j \in \mathbb{Z}_+, \quad \text{and} \quad f^{-1} := 0. \tag{8.1}$$

From Theorem 3.5 and Triebel [91, pp. 174–175, Proposition 3.3.2], we immediately deduce the following useful conclusions, the details of whose proofs are omitted.

PROPOSITION 8.1. *Let $\alpha_1, \alpha_2, \alpha_3, \tau \in [0, \infty)$ and $q \in (0, \infty]$ and let $w \in \mathcal{W}^{\alpha_3}_{\alpha_1, \alpha_2}$. Choose $a \in (0, \infty)$ and $M \in \mathbb{N}$ such that*

$$M > \alpha_1 \vee (a + n\tau + \alpha_2). \tag{8.2}$$

For $j \in \mathbb{Z}_+$, $f \in \mathcal{S}'(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, let $F(x, 2^{-j}) := f^j(x) - f^{j-1}(x)$, where $\{f^j\}_{j=-1}^{\infty}$ is as in (8.1)*. Then:*

(i) *$f \in B^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$ if and only if $F \in L^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^{n+1}_{\mathbb{Z}_+})$ and $\|F\|_{L^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^{n+1}_{\mathbb{Z}_+})} < \infty$. Moreover, $\|f\|_{B^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} \sim \|F\|_{L^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^{n+1}_{\mathbb{Z}_+})}$ with the implicit constants independent of f.*

(ii) *$f \in \mathcal{N}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$ if and only if $F \in \mathcal{N}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^{n+1}_{\mathbb{Z}_+})$ and $\|F\|_{\mathcal{N}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^{n+1}_{\mathbb{Z}_+})} < \infty$. Moreover, $\|f\|_{\mathcal{N}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} \sim \|F\|_{\mathcal{N}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^{n+1}_{\mathbb{Z}_+})}$ with the implicit constants independent of f.*

(iii) *$f \in F^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$ if and only if $F \in P^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^{n+1}_{\mathbb{Z}_+})$ and $\|F\|_{P^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^{n+1}_{\mathbb{Z}_+})} < \infty$. Moreover, $\|f\|_{F^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} \sim \|F\|_{P^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^{n+1}_{\mathbb{Z}_+})}$ with the implicit constants independent of f.*

(iv) *$f \in \mathcal{E}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$ if and only if $F \in \mathcal{E}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^{n+1}_{\mathbb{Z}_+})$ and $\|F\|_{\mathcal{E}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^{n+1}_{\mathbb{Z}_+})} < \infty$. Moreover, $\|f\|_{\mathcal{E}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} \sim \|F\|_{\mathcal{E}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^{n+1}_{\mathbb{Z}_+})}$ with the implicit constants independent of f.*

8.1. Characterization by differences. In this section, we characterize our function spaces in terms of differences. For an arbitrary function f, we inductively define $\Delta_h^M f$

for $M \in \mathbb{N}$ and $h \in \mathbb{R}^n$ by

$$\Delta_h f := \Delta_h^1 f := f - f(\cdot - h) \quad \text{and} \quad \Delta_h^M f := \Delta_h(\Delta_h^{M-1} f),$$

and $\mathrm{J}^{(1)}_{a,w,\mathcal{L}}(f)$ and $\mathrm{J}^{(2)}_{a,w,\mathcal{L}}(f)$ with $a \in (0,\infty)$ and w_0 as in (2.5) by

$$\mathrm{J}^{(1)}_{a,w,\mathcal{L}}(f) := \sup_{P\in\mathcal{Q}(\mathbb{R}^n),\,|P|\geq 1} \frac{1}{|P|^\tau} \left\| \chi_P w_0 \sup_{y\in\mathbb{R}^n} \frac{|f(\cdot+y)|}{(1+|y|)^a} \right\|_{\mathcal{L}(\mathbb{R}^n)},$$

$$\mathrm{J}^{(2)}_{a,w,\mathcal{L}}(f) := \sup_{P\in\mathcal{Q}(\mathbb{R}^n)} \frac{1}{|P|^\tau} \left\| \chi_P w_0 \sup_{y\in\mathbb{R}^n} \frac{|f(\cdot+y)|}{(1+|y|)^a} \right\|_{\mathcal{L}(\mathbb{R}^n)}.$$

In what follows, we denote by $\fint_E f$ the *average of f over a measurable set E.*

THEOREM 8.2. *Let $a, \alpha_1, \alpha_2, \alpha_3, \tau \in [0,\infty)$, $u \in [1,\infty]$, $q \in (0,\infty]$ and $w \in \star\text{-}\mathcal{W}^{\alpha_3}_{\alpha_1,\alpha_2}$. If $M \in \mathbb{N}$, $\alpha_1 \in (a, M)$ and (8.2) holds, then there exists a positive constant $\widetilde{C} := C(M)$, depending on M, such that, for all $f \in \mathcal{S}'(\mathbb{R}^n) \cap L^1_{\mathrm{loc}}(\mathbb{R}^n)$, the following hold with the implicit constants independent of f:*

(i) $$\mathrm{I}_1 := \mathrm{J}^{(1)}_{a,w,\mathcal{L}}(f) + \left\| \left\{ \sup_{z\in\mathbb{R}^n} \left[\fint_{|h|\leq \widetilde{C}\,2^{-j}} \frac{|\Delta_h^M f(\cdot+z)|^u}{(1+2^j|z|)^{au}}\,dh \right]^{1/u} \right\}_{j\in\mathbb{Z}_+} \right\|_{\ell^q(\mathcal{L}^w_\tau(\mathbb{R}^n,\mathbb{Z}_+))} \sim \|f\|_{B^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)},$$

(ii) $$\mathrm{I}_2 := \mathrm{J}^{(1)}_{a,w,\mathcal{L}}(f) + \left\| \left\{ \sup_{z\in\mathbb{R}^n} \left[\fint_{|h|\leq \widetilde{C}\,2^{-j}} \frac{|\Delta_h^M f(\cdot+z)|^u}{(1+2^j|z|)^{au}}\,dh \right]^{1/u} \right\}_{j\in\mathbb{Z}_+} \right\|_{\mathcal{L}^w_\tau(\ell^q(\mathbb{R}^n,\mathbb{Z}_+))} \sim \|f\|_{F^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)},$$

(iii) $$\mathrm{I}_3 := \mathrm{J}^{(2)}_{a,w,\mathcal{L}}(f) + \left\| \left\{ \sup_{z\in\mathbb{R}^n} \left[\fint_{|h|\leq \widetilde{C}\,2^{-j}} \frac{|\Delta_h^M f(\cdot+z)|^u}{(1+2^j|z|)^{au}}\,dh \right]^{1/u} \right\}_{j\in\mathbb{Z}_+} \right\|_{\ell^q(\mathcal{N}\mathcal{L}^w_\tau(\mathbb{R}^n,\mathbb{Z}_+))} \sim \|f\|_{\mathcal{N}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)},$$

(iv) $$\mathrm{I}_4 := \mathrm{J}^{(2)}_{a,w,\mathcal{L}}(f) + \left\| \left\{ \sup_{z\in\mathbb{R}^n} \left[\fint_{|h|\leq \widetilde{C}\,2^{-j}} \frac{|\Delta_h^M f(\cdot+z)|^u}{(1+2^j|z|)^{au}}\,dh \right]^{1/u} \right\}_{j\in\mathbb{Z}_+} \right\|_{\mathcal{E}\mathcal{L}^w_\tau(\ell^q(\mathbb{R}^n,\mathbb{Z}_+))} \sim \|f\|_{\mathcal{E}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)}.$$

Proof. We only prove (i), since the proofs of the other items are similar. To this end, for any $f \in \mathcal{S}'(\mathbb{R}^n) \cap L^1_{\mathrm{loc}}(\mathbb{R}^n)$, since $\rho \in C^\infty_c(\mathbb{R}^n)$ (see [91, pp. 174–175, Proposition 3.3.2]), we conclude that, for all $j \in \mathbb{Z}_+$ and $x \in \mathbb{R}^n$,

$$f^j(x) := \sum_{m'=1}^{M} \sum_{m=1}^{M} \frac{(-1)^{M+m+m'-1}}{M!} \binom{M}{m'} \binom{M}{m} m^M \int_{\mathbb{R}^n} \rho(y) f(x - 2^{-j}mm'y)\,dy \tag{8.3}$$

and hence

$$f^j(x) - f^{j+1}(x) = \sum_{m'=1}^{M} \sum_{m=1}^{M} \frac{(-1)^{M+m+m'-1}}{M!} \binom{M}{m'} \binom{M}{m} m^M \int_{\mathbb{R}^n} \rho(y) f(x - 2^{-j}mm'y)\,dy$$

$$
\begin{aligned}
&-\sum_{m'=1}^{M}\sum_{m=1}^{M}\frac{(-1)^{M+m+m'-1}}{M!}\binom{M}{m'}\binom{M}{m}m^M\int_{\mathbb{R}^n}\rho(y)f(x-2^{-j-1}mm'y)\,dy\\
&=\sum_{m'=0}^{M}\sum_{m=1}^{M}\frac{(-1)^{M+m+m'-1}}{M!}\binom{M}{m'}\binom{M}{m}m^M\int_{\mathbb{R}^n}\rho(y)f(x-2^{-j}mm'y)\,dy\\
&\quad-\sum_{m'=0}^{M}\sum_{m=1}^{M}\frac{(-1)^{M+m+m'-1}}{M!}\binom{M}{m'}\binom{M}{m}m^M\int_{\mathbb{R}^n}\rho(y)f(x-2^{-j-1}mm'y)\,dy\\
&=\sum_{m=1}^{M}\frac{(-1)^{M+m-1}}{M!}\binom{M}{m}m^M\int_{\mathbb{R}^n}\rho(y)[\Delta^M_{2^{-j}my}f(x)-\Delta^M_{2^{-j-1}my}f(x)]\,dy.
\end{aligned}
$$

As a consequence, for all $x\in\mathbb{R}^n$ and $u\in[1,\infty]$,

$$
\sup_{z\in\mathbb{R}^n}\frac{|f^j(x+z)-f^{j+1}(x+z)|}{(1+2^j|z|)^a}\lesssim\sup_{z\in\mathbb{R}^n}\left[\fint_{|h|\le\widetilde{C}\,2^{-j}}\frac{|\Delta^M_h f(x+z)|^u}{(1+2^j|z|)^{au}}\,dh\right]^{1/u}. \tag{8.4}
$$

Moreover, as $T_0\in\mathcal{S}(\mathbb{R}^n)$ and $(1+|u|)^a\le(1+|u+y|)^a(1+|y|)^a$ for all $u,y\in\mathbb{R}^n$, we see that, for all $x\in\mathbb{R}^n$,

$$
\begin{aligned}
\sup_{y\in\mathbb{R}^n}\frac{|f^0(x+y)|}{(1+|y|)^a}&=\sup_{y\in\mathbb{R}^n}\frac{1}{(1+|y|)^a}\left|\int_{\mathbb{R}^n}T_0(u)f(x+y-u)\,du\right|\\
&\le\sup_{y\in\mathbb{R}^n}\int_{\mathbb{R}^n}\left|T_0(u+y)\right|\frac{(1+|u|)^a}{(1+|y|)^a}\frac{|f(x-u)|}{(1+|u|)^a}\,du\\
&\lesssim\sup_{u\in\mathbb{R}^n}\frac{|f(x+u)|}{(1+|u|)^a}.
\end{aligned}\tag{8.5}
$$

Combining (8.4) and (8.5) with Proposition 8.1 (here we need the assumption (8.2)), we conclude that

$$
\begin{aligned}
\mathrm{I}_1&\gtrsim\sup_{P\in\mathcal{Q}(\mathbb{R}^n),\,|P|\ge1}\frac{1}{|P|^\tau}\left\|\chi_P w_0\sup_{y\in\mathbb{R}^n}\frac{|(f^0-f^{-1})(\cdot+y)|}{(1+|y|)^a}\right\|_{\mathcal{L}(\mathbb{R}^n)}\\
&\quad+\left\|\left\{\sup_{z\in\mathbb{R}^n}\frac{|(f^j-f^{j-1})(\cdot+z)|}{(1+2^j|z|)^a}\right\}_{j\in\mathbb{Z}_+}\right\|_{\ell^q(\mathcal{L}^w_\tau(\mathbb{R}^n,\mathbb{Z}_+))}\sim\|f\|_{B^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)},
\end{aligned}
$$

as desired.

To show the reverse inequality, for any $f\in B^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)\cap L^1_{\mathrm{loc}}(\mathbb{R}^n)$, since $\{T_j\}_{j\in\mathbb{Z}_+}$ is an approximation to the identity (see [91, pp. 174–175, Proposition 3.3.2]), if we fix $|h|\le\widetilde{C}\,2^{-j}$ and $z\in\mathbb{R}^n$, then by [91, p. 195, (3.5.3/7)], we see that, for almost every $x\in\mathbb{R}^n$,

$$
\begin{aligned}
&\left[\fint_{|h|<2^{-j}}|\Delta^M_h f(x+z)|^u\,dy\right]^{1/u}\\
&\qquad\lesssim\sum_{l=1}^{\infty}\left\{|f_{j+l}(x+z)|+\left[\fint_{B(x+z,C2^{-j})}|f_{j+l}(y)|^u\,dh\right]^{1/u}\right\}\\
&\qquad\quad+\sup_{w\in B(x+z,C2^{-j})}\left|\int_{\mathbb{R}^n}D^\alpha T_0(y)f(w+2^{-j}y)\,dy\right|;
\end{aligned}
$$

here and in what follows, $f_j := f^j - f^{j-1}$ for all $j \in \mathbb{Z}_+$. Then

$$\begin{aligned}
&\left[\fint_{|h|<2^{-j}} \frac{|\Delta_h^M f(x+z)|^u}{(1+2^j|z|)^{au}}\,dh\right]^{1/u}\\
&\qquad\lesssim \sum_{l=1}^{\infty} \frac{|f_{j+l}(x+z)| + [\fint_{B(x+z,C2^{-j})} |f_{j+l}(y)|^u\,dy]^{1/u}}{(1+2^j|z|)^a}\\
&\qquad\quad + \sup_{w\in B(x+z,C2^{-j})} \frac{1}{(1+2^j|z|)^a}\left|\int_{\mathbb{R}^n} D^\alpha T_0(y) f(w+2^{-j}y)\,dy\right|. \qquad (8.6)
\end{aligned}$$

For the second term on the right-hand side of (8.6), we have

$$\begin{aligned}
&\sup_{z\in\mathbb{R}^n}\left[\sup_{w\in B(x+z,C2^{-j})} \frac{1}{(1+2^j|z|)^a}\left|\int_{\mathbb{R}^n} D^\alpha T_0(y) f(w+2^{-j}y)\,dy\right|\right]\\
&\qquad = \sup_{z\in\mathbb{R}^n}\left[\sup_{w\in B(0,C2^{-j})} \frac{|(D^\alpha T_0)_j * \widetilde{f}(x+z+w)|}{(1+2^j|z|)^a}\right]\\
&\qquad \le \sup_{z\in\mathbb{R}^n}\sup_{w\in B(0,C2^{-j})} \frac{|(D^\alpha T_0)_j * \widetilde{f}(x+z+w)|}{(1+2^j|z+w|)^a}\left[\frac{1+2^j(|z|+|w|)}{1+2^j|z|}\right]^a\\
&\qquad \lesssim \sup_{z\in\mathbb{R}^n} \frac{|(D^\alpha T_0)_j * \widetilde{f}(x+z)|}{(1+2^j|z|)^a},
\end{aligned}$$

where $\widetilde{f} := f(-\,\cdot)$. This observation, together with the fact that

$$\left\|\left\{\sup_{z\in\mathbb{R}^n} \frac{|(D^\alpha T_0)_j * \widetilde{f}(\cdot+z)|}{(1+2^j|z|)^a}\right\}_{j\in\mathbb{Z}_+}\right\|_{\ell^q(\mathcal{L}_\tau^w(\mathbb{R}^n,\mathbb{Z}_+))} \lesssim \|f\|_{B^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)},$$

implies that

$$\begin{aligned}
&\left\|\left\{\sup_{z\in\mathbb{R}^n}\sup_{w\in B(\cdot+z,C2^{-j})} \frac{1}{(1+2^j|z|)^a}\left|\int_{\mathbb{R}^n} D^\alpha T_0(y) f(w+2^{-j}y)\,dy\right|\right\}_{j\in\mathbb{Z}_+}\right\|_{\ell^q(\mathcal{L}_\tau^w(\mathbb{R}^n,\mathbb{Z}_+))}\\
&\qquad\qquad \lesssim \|f\|_{B^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)}.
\end{aligned}$$

For the first term on the right-hand side of (8.6), we see that, for all $x \in \mathbb{R}^n$,

$$\begin{aligned}
&\sup_{z\in\mathbb{R}^n}\left[\fint_{y\in B(x+z,2^{-j})} |f_{j+l}(y)|^u\,dy\right]^{1/u} \frac{1}{(1+2^j|z|)^a}\\
&\qquad \le \sup_{z\in\mathbb{R}^n}\left\{\sup_{y\in B(0,2^{-j})} \frac{|f_{j+l}(x+z+y)|}{(1+2^{j+l}|z+y|)^a}\left[\frac{1+2^{j+l}(|z|+|y|)}{1+2^j|z|}\right]^a\right\}\\
&\qquad \lesssim 2^{la}\sup_{z\in\mathbb{R}^n} \frac{|f_{j+l}(x+z)|}{(1+2^{j+l}|z|)^a}. \qquad (8.7)
\end{aligned}$$

Since $w \in \star\text{-}\mathcal{W}^{\alpha_3}_{\alpha_1,\alpha_2}$, we have $w_j(x) \lesssim 2^{-l\alpha_1} w_{j+l}(x)$ for all $x \in \mathbb{R}^n$ and $j, l \in \mathbb{Z}_+$, which, together with $\alpha_1 > a$ and (8.7), implies that

$$\left\|\left\{\sum_{l=1}^{\infty}\sup_{z\in\mathbb{R}^n}\left[\fint_{y\in B(\cdot+z,2^{-j})}|f_{j+l}(y)|^u\,dy\right]^{1/u}\frac{1}{(1+2^j|z|)^a}\right\}_{j\in\mathbb{Z}_+}\right\|_{\ell^q(\mathcal{L}_\tau^w(\mathbb{R}^n,\mathbb{Z}_+))}$$
$$\lesssim\left\|\left\{\sum_{l=1}^{\infty}2^{la}\sup_{z\in\mathbb{R}^n}\frac{|f_{j+l}(\cdot+z)|}{(1+2^{j+l}|z|)^a}\right\}_{j\in\mathbb{Z}_+}\right\|_{\ell^q(\mathcal{L}_\tau^w(\mathbb{R}^n,\mathbb{Z}_+))}$$
$$\lesssim\left\{\sum_{l=1}^{\infty}2^{la\widetilde{\theta}}\sup_{P\in\mathcal{Q}(\mathbb{R}^n)}\frac{1}{|P|^\tau}\left[\sum_{j=(0\vee j_P)}^{\infty}\left\|\chi_P w_j\sup_{z\in\mathbb{R}^n}\frac{|f_{j+l}(\cdot+z)|}{(1+2^{j+l}|z|)^a}\right\|_{\mathcal{L}(\mathbb{R}^n)}^q\right]^{\widetilde{\theta}/q}\right\}^{1/\widetilde{\theta}}$$
$$\lesssim\left\{\sum_{l=1}^{\infty}2^{-l(\alpha_1-a)\widetilde{\theta}}\sup_{P\in\mathcal{Q}(\mathbb{R}^n)}\frac{1}{|P|^\tau}\left[\sum_{j=(0\vee j_P)}^{\infty}\left\|\chi_P w_{j+l}\sup_{z\in\mathbb{R}^n}\frac{|f_{j+l}(\cdot+z)|}{(1+2^{j+l}|z|)^a}\right\|_{\mathcal{L}(\mathbb{R}^n)}^q\right]^{\widetilde{\theta}/q}\right\}^{1/\widetilde{\theta}}$$
$$\lesssim\sup_{P\in\mathcal{Q}(\mathbb{R}^n)}\frac{1}{|P|^\tau}\left\{\sum_{j=(0\vee j_P)}^{\infty}\left\|\chi_P w_j\sup_{z\in\mathbb{R}^n}\frac{|f_j(\cdot+z)|}{(1+2^j|z|)^a}\right\|_{\mathcal{L}(\mathbb{R}^n)}^q\right\}^{1/q}\sim\|f\|_{B^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)},$$

where we have chosen $\widetilde{\theta}\in(0,\min\{\theta,q\})$ and θ is as in ($\mathcal{L}3$).

Further, by (8.3), we see that, for all $x\in\mathbb{R}^n$,

$$f^0(x)=\sum_{m'=1}^{M}\sum_{m=1}^{M}\frac{(-1)^{M+m+m'-1}}{M!}\binom{M}{m'}\binom{M}{m}m^M\int_{\mathbb{R}^n}\rho(y)f(x-mm'y)\,dy$$

and

$$f(x)=\sum_{m=1}^{M}\frac{(-1)^{M+m+0-1}}{M!}\binom{M}{0}\binom{M}{m}m^M\int_{\mathbb{R}^n}\rho(y)f(x)\,dy,$$

which implies that, for all $x\in\mathbb{R}^n$,

$$|f(x)|=\left|\sum_{m=1}^{M}\frac{(-1)^{M+m-1}}{M!}\binom{M}{m}m^M\int_{\mathbb{R}^n}\rho(y)f(x)\,dy+f^0(x)-f^0(x)\right|$$
$$\lesssim\left|\sum_{m'=0}^{M}\sum_{m=1}^{M}\frac{(-1)^{M+m+m'-1}}{M!}\binom{M}{m'}\binom{M}{m}m^M\int_{\mathbb{R}^n}\rho(y)f(x-m'my)\,dy\right|+|f^0(x)|$$
$$\lesssim\left|\sum_{m=1}^{M}\frac{(-1)^{M+m-1}}{M!}\binom{M}{m}m^M\int_{\mathbb{R}^n}\rho(y)\Delta^M_{my}f(x)\,dy\right|+|f^0(x)|.$$

From this, we deduce that, for all $x\in\mathbb{R}^n$,

$$\sup_{y\in\mathbb{R}^n}\frac{|f(x+y)|}{(1+|y|)^a}\lesssim\sup_{y\in\mathbb{R}^n}\left[\fint_{|h|\lesssim1}\frac{|\Delta_h^M f(x+y)|^u}{(1+|y|)^{au}}\,dh\right]^{1/u}+\sup_{y\in\mathbb{R}^n}\frac{|f^0(x+y)|}{(1+|y|)^a},\tag{8.8}$$

which, together with the trivial inequality

$$\left\|\sup_{y\in\mathbb{R}^n}\frac{|f^0(\cdot+y)|}{(1+|y|)^a}\right\|_{\mathcal{L}(\mathbb{R}^n)}\lesssim\|f\|_{B^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)},$$

implies that

$$\mathrm{J}^{(1)}_{a,w,\mathcal{L}}(f) \lesssim \left\| \left\{ \sup_{z\in\mathbb{R}^n} \left[\oint_{|h|\le \widetilde{C}\, 2^{-j}} \frac{|\Delta_h^M f(\cdot+z)|^u}{(1+2^j|z|)^{au}}\,dh \right]^{1/u} \right\}_{j\in\mathbb{Z}_+} \right\|_{\ell^q(\mathcal{L}^w_\tau(\mathbb{R}^n,\mathbb{Z}_+))} + \|f\|_{B^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)}$$
$$\lesssim \|f\|_{B^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)}.$$

This finishes the proof of (i). ∎

If we further assume (7.1) holds, from Theorems 7.1 and 8.2 we immediately deduce the following conclusions. We omit the details.

COROLLARY 8.3. *Let $\alpha_1, \alpha_2, \alpha_3, \tau$, a, q and w be as in Theorem 8.2. Assume (7.1) and (8.2). Let $\{\mathrm{J}_j\}_{j=1}^4$ be as in Theorem 8.2. Then, with the implicit constants independent of f:*

(i) *$f \in B^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$ if and only if $f \in \mathcal{S}'(\mathbb{R}^n) \cap L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $\mathrm{J}_1 < \infty$; moreover, $\mathrm{J}_1 \sim \|f\|_{B^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)}$.*
(ii) *$f \in F^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$ if and only if $f \in \mathcal{S}'(\mathbb{R}^n) \cap L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $\mathrm{J}_2 < \infty$; moreover, $\mathrm{J}_2 \sim \|f\|_{F^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)}$.*
(iii) *$f \in \mathcal{N}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$ if and only if $f \in \mathcal{S}'(\mathbb{R}^n) \cap L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $\mathrm{J}_3 < \infty$; moreover, $\mathrm{J}_3 \sim \|f\|_{\mathcal{N}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)}$.*
(iv) *$f \in \mathcal{E}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$ if and only if $f \in \mathcal{S}'(\mathbb{R}^n) \cap L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $\mathrm{J}_4 < \infty$; moreover, $\mathrm{J}_4 \sim \|f\|_{\mathcal{E}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)}$.*

By the Peetre maximal function characterizations of the Besov space $B^s_{p,q}(\mathbb{R}^n)$ and the Triebel–Lizorkin space $F^s_{p,q}(\mathbb{R}^n)$ (see, for example, [93]), we know that, if $q \in (0,\infty]$, $\mathcal{L}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ and $w_j \equiv 2^{js}$ for some $s \in \mathbb{R}$ and all $j \in \mathbb{Z}_+$, then $B^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n) = B^s_{p,q}(\mathbb{R}^n)$ for all $p \in (0,\infty]$ and $a \in (n/p,\infty)$, and $F^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n) = F^s_{p,q}(\mathbb{R}^n)$ for all $p \in (0,\infty)$ and $a \in (n/\min\{p,q\},\infty)$. Then, applying Theorem 8.2, we have the corollary below. In what follows, for all measurable functions f, $a \in (0,\infty)$ and $x \in \mathbb{R}^n$, we define the *Peetre maximal function* of f as

$$f^*_a(x) := \sup_{z\in\mathbb{R}^n} \frac{|f(x+z)|}{(1+|z|)^a}.$$

COROLLARY 8.4. *Let $M \in \mathbb{N}$, $u \in [1,\infty]$ and $q \in (0,\infty]$.*

(i) *Let $p \in (0,\infty)$, $a \in (n/\min\{p,q\}, M/2)$ and $s \in (a, M-a)$. Then there exists a positive constant $\widetilde{C} := C(M)$, depending on M, such that $f \in F^s_{p,q}(\mathbb{R}^n)$ if and only if $f \in \mathcal{S}'(\mathbb{R}^n) \cap L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and*
$$\mathrm{J}_1 := \|f^*_a\|_{L^p(\mathbb{R}^n)} + \left\| \left\| \left\{ 2^{js} \sup_{z\in\mathbb{R}^n} \left[\oint_{|h|\le \widetilde{C}\, 2^{-j}} \frac{|\Delta_h^M f(\cdot+z)|^u}{(1+2^j|z|)^{au}}\,dh \right]^{1/u} \right\}_{j\in\mathbb{Z}_+} \right\|_{\ell^q(\mathbb{Z}_+)} \right\|_{L^p(\mathbb{R}^n)}$$
is finite. Moreover, J_1 is equivalent to $\|f\|_{F^s_{p,q}(\mathbb{R}^n)}$ with the equivalence constants independent of f.
(ii) *Let $p \in (0,\infty]$, $a \in (n/p, M/2)$ and $s \in (a, M-a)$. Then there exists a positive constant $\widetilde{C} := C(M)$, depending on M, such that $f \in B^s_{p,q}(\mathbb{R}^n)$ if and only if $f \in \mathcal{S}'(\mathbb{R}^n) \cap L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and*
$$\mathrm{J}_2 := \|f^*_a\|_{L^p(\mathbb{R}^n)} + \left\| \left\{ \left\| 2^{js} \sup_{z\in\mathbb{R}^n} \left[\oint_{|h|\le \widetilde{C}\, 2^{-j}} \frac{|\Delta_h^M f(\cdot+z)|^u}{(1+2^j|z|)^{au}}\,dh \right]^{1/u} \right\|_{L^p(\mathbb{R}^n)} \right\}_{j\in\mathbb{Z}_+} \right\|_{\ell^q(\mathbb{Z}_+)}$$

is finite. Moreover, J_2 *is equivalent to* $\|f\|_{B^s_{p,q}(\mathbb{R}^n)}$ *with the equivalence constants independent of* f.

Proof. Recall that by [85, Theorem 3.3.2] (see also [70, pp. 33–34]), $F^s_{p,q}(\mathbb{R}^n) \subset L^1_{\mathrm{loc}}(\mathbb{R}^n)$ if and only if either $p \in (0,1)$, $s \in [n(1/p-1),\infty)$ and $q \in (0,\infty]$, or $p \in [1,\infty)$, $s \in (0,\infty)$ and $q \in (0,\infty]$, or $p \in [1,\infty)$, $s = 0$ and $q \in (0,2]$; and $B^s_{p,q}(\mathbb{R}^n) \subset L^1_{\mathrm{loc}}(\mathbb{R}^n)$ if and only if either $p \in (0,\infty]$, $s \in (n\{\max(0,1/p-1)\},\infty)$ and $q \in (0,\infty]$, or $p \in (0,1]$, $s = n(1/p-1)$ and $q \in (0,1]$, or $p \in (1,\infty]$, $s = 0$ and $q \in (0,\min(p,2)]$. From this, the aforementioned Peetre maximal function characterizations of $B^s_{p,q}(\mathbb{R}^n)$ and $F^s_{p,q}(\mathbb{R}^n)$, and Theorem 8.2, we immediately deduce the conclusions of (i) and (ii). ■

We remark that the difference characterizations in Corollary 8.4 are a little different from the classical difference characterizations of Besov and Triebel–Lizorkin spaces in [91, Section 3.5.3]. Indeed, Corollary 8.4 can be seen as the Peetre maximal function version of [91, Theorem 3.5.3] in the case $u = \infty$. We also remark that the condition that $a \in (n/p, M)$ and $s \in (a,\infty)$ is necessary, since in the classical case, the condition $s \in (n/p,\infty)$ is necessary; see, for example, [5].

8.2. Characterization by oscillations. In this section, we characterize our function spaces in terms of oscillations.

Let $\mathbb{P}_M$ be the set of all polynomials of degree less than M. By convention $\mathbb{P}_{-1}$ stands for $\{0\}$. We define, for all $(x,t) \in \mathbb{R}^{n+1}_+$,

$$\mathrm{osc}^M_u f(x,t) := \inf_{P\in\mathbb{P}_M} \left[\frac{1}{|B(x,t)|}\int_{B(x,t)} |f(y)-P(y)|^u\,dy\right]^{1/u}.$$

We invoke the following estimates from [91].

LEMMA 8.5. *For any* $f \in \mathcal{S}'(\mathbb{R}^n)$, *let* $\{f^j\}_{j=-1}^\infty$ *be as in* (8.1). *Then there exists a positive constant* C *such that:*

(i) *for all* $j \in \mathbb{N}$ *and* $x \in \mathbb{R}^n$,

$$|f^j(x) - f^{j-1}(x)| \le C\,\mathrm{osc}^M_u f(x, 2^{-j}); \tag{8.9}$$

(ii) *for all* $j \in \mathbb{Z}_+$, $x \in \mathbb{R}^n$ *and* $y \in B(x,2^{-j})$,

$$\left|f^j(x) - \sum_{\|\alpha\|_1\le M-1} \frac{1}{\alpha!}D^\alpha f^j(x)(y-x)^\alpha\right| \le C2^{-jM} \sup_{z\in B(x,\,2^{-j})} \sum_{\|\alpha\|_1=M} |D^\alpha f^j(z)|. \tag{8.10}$$

Proof. Estimates (8.9) and (8.10) appear, respectively, in [91, p. 188] and [91, p. 182]. ■

THEOREM 8.6. *Let* $a,\alpha_1,\alpha_2,\alpha_3,\tau \in [0,\infty)$, $u \in [1,\infty]$, $q \in (0,\infty]$ *and* $w \in \star\text{-}\mathcal{W}^{\alpha_3}_{\alpha_1,\alpha_2}$. *If* $M \in \mathbb{N}$, $\alpha_1 \in (a,M)$ *and* (8.2) *holds, then, for all* $f \in \mathcal{S}'(\mathbb{R}^n) \cap L^1_{\mathrm{loc}}(\mathbb{R}^n)$, *the following hold with the implicit constants independent of* f:

(i)
$$\mathrm{H}_1 := \mathrm{J}^{(1)}_{a,w,\mathcal{L}}(f) + \left\|\left\{\sup_{z\in\mathbb{R}^n} \frac{\mathrm{osc}^M_u f(\cdot+z,2^{-j})}{(1+2^j|z|)^a}\right\}_{j\in\mathbb{Z}_+}\right\|_{\ell^q(\mathcal{L}^w_\tau(\mathbb{R}^n,\mathbb{Z}_+))} \sim \|f\|_{B^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)},$$

(ii) $$\mathrm{H}_2 := \mathrm{J}^{(1)}_{a,w,\mathcal{L}}(f) + \left\| \left\{ \sup_{z\in\mathbb{R}^n} \frac{\mathrm{osc}^M_u\, f(\cdot + z, 2^{-j})}{(1+2^j|z|)^a} \right\}_{j\in\mathbb{Z}_+} \right\|_{\mathcal{L}^w_\tau(\ell^q(\mathbb{R}^n,\mathbb{Z}_+))} \sim \|f\|_{F^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)},$$

(iii) $$\mathrm{H}_3 := \mathrm{J}^{(2)}_{a,w,\mathcal{L}}(f) + \left\| \left\{ \sup_{z\in\mathbb{R}^n} \frac{\mathrm{osc}^M_u\, f(\cdot + z, 2^{-j})}{(1+2^j|z|)^a} \right\}_{j\in\mathbb{Z}_+} \right\|_{\ell^q(\mathcal{N}\mathcal{L}^w_\tau(\mathbb{R}^n,\mathbb{Z}_+))} \sim \|f\|_{\mathcal{N}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)},$$

(iv) $$\mathrm{H}_4 := \mathrm{J}^{(2)}_{a,w,\mathcal{L}}(f) + \left\| \left\{ \sup_{z\in\mathbb{R}^n} \frac{\mathrm{osc}^M_u\, f(\cdot + z, 2^{-j})}{(1+2^j|z|)^a} \right\}_{j\in\mathbb{Z}_+} \right\|_{\mathcal{E}\mathcal{L}^w_\tau(\ell^q(\mathbb{R}^n,\mathbb{Z}_+))} \sim \|f\|_{\mathcal{E}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)}.$$

Proof. We only prove (ii) since the proofs of other items are similar.

By (8.5) and (8.9), we have

$$\begin{aligned}\mathrm{H}_2 \gtrsim & \sup_{P\in\mathcal{Q}(\mathbb{R}^n),\,|P|\ge 1} \frac{1}{|P|^\tau} \left\| \chi_P w_0 \sup_{y\in\mathbb{R}^n} \frac{|(f^0 - f^{-1})(\cdot+y)|}{(1+|y|)^a} \right\|_{\mathcal{L}(\mathbb{R}^n)} \\ &+ \left\| \left\{ \sup_{z\in\mathbb{R}^n} \frac{|(f^j - f^{j-1})(\cdot+z)|}{(1+2^j|z|)^a} \right\}_{j\in\mathbb{Z}_+} \right\|_{\mathcal{L}^w_\tau(\ell^q(\mathbb{R}^n,\mathbb{Z}_+))} \sim \|f\|_{F^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)}.\end{aligned}$$

For the reverse inequality, by (8.8) and Theorem 8.2(ii), we conclude that

$$\sup_{P\in\mathcal{Q}(\mathbb{R}^n),\,|P|\ge 1} \frac{1}{|P|^\tau} \left\| \chi_P w_0 \sup_{y\in\mathbb{R}^n} \frac{|f(\cdot+y)|}{(1+|y|)^a} \right\|_{\mathcal{L}(\mathbb{R}^n)} \lesssim \|f\|_{F^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)}.$$

Therefore, we only need to prove that

$$\left\| \left\{ \sup_{z\in\mathbb{R}^n} \frac{\mathrm{osc}^M_u\, f(\cdot + z, 2^{-j})}{(1+2^j|z|)^a} \right\}_{j\in\mathbb{Z}_+} \right\|_{\mathcal{L}^w_\tau(\ell^q(\mathbb{R}^n,\mathbb{Z}_+))} \lesssim \|f\|_{F^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)}.$$

We use the estimate [91, p. 188, (11)] with k_0 replaced by T_0: for all $x, z \in \mathbb{R}^n$,

$$\begin{aligned}&\mathrm{osc}^M_u\, f(x+z, 2^{-j}) \\ &\quad\lesssim \sum_{l=1}^\infty \fint_{y\in B(x+z,2^{-j})} |f_{j+l}(y)|\,dy + \sup_{w\in B(x+z,C2^{-j})} \left| \int_{\mathbb{R}^n} D^\alpha T_0(y) f(w+2^{-j}y)\,dy \right|,\end{aligned}$$

where C is a positive constant. Consequently, for all $x, z \in \mathbb{R}^n$,

$$\begin{aligned}\frac{\mathrm{osc}^M_u\, f(x+z, 2^{-j})}{(1+2^j|z|)^a} \lesssim & \sum_{l=1}^\infty \frac{\sup_{y\in B(x+z,2^{-j})} |f_{j+l}(y)|}{(1+2^j|z|)^a} \\ &+ \sup_{w\in B(x+z,C2^{-j})} \frac{1}{(1+2^j|z|)^a} \left| \int_{\mathbb{R}^n} D^\alpha T_0(y) f(w+2^{-j}y)\,dy \right|. \end{aligned} \tag{8.11}$$

Then by an argument similar to that used in the proof of Theorem 8.2, for the second term on the right-hand side of (8.11), we see that

$$\begin{aligned}&\left\| \left\{ \sup_{z\in\mathbb{R}^n} \sup_{w\in B(\cdot+z,C2^{-j})} \frac{1}{(1+2^j|z|)^a} \left| \int_{\mathbb{R}^n} D^\alpha T_0(y) f(w+2^{-j}y)\,dy \right| \right\}_{j\in\mathbb{Z}_+} \right\|_{\mathcal{L}^w_\tau(\ell^q(\mathbb{R}^n,\mathbb{Z}_+))} \\ &\qquad\lesssim \|f\|_{F^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)},\end{aligned}$$

It remains to consider the first term on the right-hand side of (8.11). Indeed, by $w \in \star$-$\mathcal{W}_{\alpha_1,\alpha_2}^{\alpha_3}$, we have $w_j(x) \lesssim 2^{-l\alpha_1} w_{j+l}(x)$ for all $j, l \in \mathbb{Z}_+$ and $x \in \mathbb{R}^n$, which, together with $\alpha_1 > a$ and (8.7), implies that

$$\begin{aligned}
&\left\|\left\{\sum_{l=1}^{\infty} \sup_{z\in\mathbb{R}^n} \frac{\sup_{y\in B(\cdot+z,2^{-j})} |f_{j+l}(y)|}{(1+2^j|z|)^a}\right\}_{j\in\mathbb{Z}_+}\right\|_{\mathcal{L}_\tau^w(\ell^q(\mathbb{R}^n,\mathbb{Z}_+))} \\
&\lesssim \left\|\left\{\sum_{l=1}^{\infty} 2^{la} \sup_{z\in\mathbb{R}^n} \frac{|f_{j+l}(\cdot+z)|}{(1+2^{j+l}|z|)^a}\right\}_{j\in\mathbb{Z}_+}\right\|_{\mathcal{L}_\tau^w(\ell^q(\mathbb{R}^n,\mathbb{Z}_+))} \\
&\lesssim \left\{\sum_{l=1}^{\infty} 2^{la\widetilde{\theta}} \sup_{P\in\mathcal{Q}(\mathbb{R}^n)} \frac{1}{|P|^\tau} \left\|\left(\sum_{j=(0\vee j_P)}^{\infty} \chi_P \left[w_j \sup_{z\in\mathbb{R}^n} \frac{|f_{j+l}(\cdot+z)|}{(1+2^{j+l}|z|)^a}\right]^q\right)^{1/q}\right\|_{\mathcal{L}(\mathbb{R}^n)}^{\widetilde{\theta}}\right\}^{1/\widetilde{\theta}} \\
&\lesssim \left\{\sum_{l=1}^{\infty} 2^{-l(\alpha_1-a)\widetilde{\theta}} \sup_{P\in\mathcal{Q}(\mathbb{R}^n)} \frac{1}{|P|^\tau} \left\|\left(\sum_{j=(0\vee j_P)}^{\infty} \chi_P \left[w_{j+l} \sup_{z\in\mathbb{R}^n} \frac{|f_{j+l}(\cdot+z)|}{(1+2^{j+l}|z|)^a}\right]^q\right)^{1/q}\right\|_{\mathcal{L}(\mathbb{R}^n)}^{\widetilde{\theta}}\right\}^{1/\widetilde{\theta}} \\
&\lesssim \sup_{P\in\mathcal{Q}(\mathbb{R}^n)} \frac{1}{|P|^\tau} \left\|\left\{\sum_{j=(0\vee j_P)}^{\infty} \chi_P \left[w_j \sup_{z\in\mathbb{R}^n} \frac{|f_j(\cdot+z)|}{(1+2^j|z|)^a}\right]^q\right\}^{1/q}\right\|_{\mathcal{L}(\mathbb{R}^n)} \sim \|f\|_{F_{\mathcal{L},q,a}^{w,\tau}(\mathbb{R}^n)},
\end{aligned}$$

where we have chosen $\widetilde{\theta} \in (0, \min\{\theta, q\})$. ■

If we further assume that (7.1) holds, then from Theorems 7.1 and 8.6, we immediately deduce the following conclusions. We omit the details.

Corollary 8.7. *Let $\alpha_1, \alpha_2, \alpha_3, \tau$, a, q and w be as in Theorem 8.6. Assume that (7.1) and (8.2) hold. Let $\{\mathrm{H}_j\}_{j=1}^4$ be as in Theorem 8.6. Then the following hold with the implicit constants independent of f:*

(i) *$f \in B_{\mathcal{L},q,a}^{w,\tau}(\mathbb{R}^n)$ if and only if $f \in \mathcal{S}'(\mathbb{R}^n) \cap L_{\mathrm{loc}}^1(\mathbb{R}^n)$ and $\mathrm{H}_1 < \infty$; moreover, $\mathrm{H}_1 \sim \|f\|_{B_{\mathcal{L},q,a}^{w,\tau}(\mathbb{R}^n)}$.*
(ii) *$f \in F_{\mathcal{L},q,a}^{w,\tau}(\mathbb{R}^n)$ if and only if $f \in \mathcal{S}'(\mathbb{R}^n) \cap L_{\mathrm{loc}}^1(\mathbb{R}^n)$ and $\mathrm{H}_2 < \infty$; moreover, $\mathrm{H}_2 \sim \|f\|_{F_{\mathcal{L},q,a}^{w,\tau}(\mathbb{R}^n)}$.*
(iii) *$f \in \mathcal{N}_{\mathcal{L},q,a}^{w,\tau}(\mathbb{R}^n)$ if and only if $f \in \mathcal{S}'(\mathbb{R}^n) \cap L_{\mathrm{loc}}^1(\mathbb{R}^n)$ and $\mathrm{H}_3 < \infty$; moreover, $\mathrm{H}_3 \sim \|f\|_{\mathcal{N}_{\mathcal{L},q,a}^{w,\tau}(\mathbb{R}^n)}$.*
(iv) *$f \in \mathcal{E}_{\mathcal{L},q,a}^{w,\tau}(\mathbb{R}^n)$ if and only if $f \in \mathcal{S}'(\mathbb{R}^n) \cap L_{\mathrm{loc}}^1(\mathbb{R}^n)$ and $\mathrm{H}_4 < \infty$; moreover, $\mathrm{H}_4 \sim \|f\|_{\mathcal{E}_{\mathcal{L},q,a}^{w,\tau}(\mathbb{R}^n)}$.*

Again, applying the Peetre maximal function characterizations of the spaces $B_{p,q}^s(\mathbb{R}^n)$ and $F_{p,q}^s(\mathbb{R}^n)$ (see, for example, [93]), and Theorem 8.6, we have the following corollary. Its proof is similar to that of Corollary 8.4. We omit the details.

Corollary 8.8. *Let $M \in \mathbb{N}$, $u \in [1, \infty]$ and $q \in (0, \infty]$.*

(i) *Let $p \in (0, \infty)$, $a \in (n/\min\{p, q\}, M)$ and $s \in (a, M - a)$. Then $f \in F_{p,q}^s(\mathbb{R}^n)$ if and only if $f \in \mathcal{S}'(\mathbb{R}^n) \cap L_{\mathrm{loc}}^1(\mathbb{R}^n)$ and*

$$\mathrm{K}_1 := \|f_a^*\|_{L^p(\mathbb{R}^n)} + \left\|\left\|\left\{2^{js} \sup_{z\in\mathbb{R}^n} \frac{\mathrm{osc}_u^M f(\cdot+z, 2^{-j})}{(1+2^j|z|)^a}\right\}_{j\in\mathbb{Z}_+}\right\|_{\ell^q(\mathbb{Z}_+)}\right\|_{L^p(\mathbb{R}^n)} < \infty.$$

Moreover, K_1 *is equivalent to* $\|f\|_{F^s_{p,q}(\mathbb{R}^n)}$ *with the equivalence constants independent of* f*.*

(ii) *Let* $p \in (0, \infty]$*,* $a \in (n/p, M)$ *and* $s \in (a, M - a)$*. Then* $f \in B^s_{p,q}(\mathbb{R}^n)$ *if and only if* $f \in \mathcal{S}'(\mathbb{R}^n) \cap L^1_{\mathrm{loc}}(\mathbb{R}^n)$ *and*

$$\mathrm{K}_2 := \|f^*_a\|_{L^p(\mathbb{R}^n)} + \left\| \left\{ \left\| 2^{js} \sup_{z \in \mathbb{R}^n} \frac{\operatorname{osc}^M_u f(\cdot + z, 2^{-j})}{(1 + 2^j|z|)^a} \right\|_{L^p(\mathbb{R}^n)} \right\}_{j \in \mathbb{Z}_+} \right\|_{\ell^q(\mathbb{Z}_+)} < \infty.$$

Moreover, K_2 *is equivalent to* $\|f\|_{B^s_{p,q}(\mathbb{R}^n)}$ *with the equivalence constants independent of* f*.*

Again, Corollary 8.8 can be seen as the Peetre maximal function version of [91, Theorem 3.5.1] in the case $u \in [1, \infty]$.

9. Isomorphisms between spaces

In this section, under some additional assumptions on $\mathcal{L}(\mathbb{R}^n)$, we establish some isomorphisms between $A^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$ spaces. First, in Subsection 9.1, we prove that if the parameter a is sufficiently large, then $A^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$ coincides with $A^{w,\tau}_{\mathcal{L},q}(\mathbb{R}^n)$, which is independent of a. In Subsection 9.2, we give some further assumptions on $\mathcal{L}(\mathbb{R}^n)$ which ensure that $\mathcal{L}(\mathbb{R}^n)$ coincides with $\mathcal{E}^{0,0}_{\mathcal{L},2,a}(\mathbb{R}^n)$. Finally, in Subsection 9.3, under some additional assumptions on $\mathcal{L}(\mathbb{R}^n)$, we prove that $\mathcal{E}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$ and $F^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$ coincide.

9.1. The role of the new parameter a. The new parameter a, which we added, seems not to play any significant role. We now consider some conditions which permit removing a from the definition of $A^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$.

Here we consider the following conditions.

ASSUMPTION 9.1. Let $\eta_{j,R}(x) := 2^{jn}(1+2^j|x|)^{-R}$ for $j \in \mathbb{Z}_+$, $R \gg 1$ and $x \in \mathbb{R}^n$.

($\mathcal{L}7$) There exist $R \gg 1$, $r \in (0,\infty)$ and a positive constant $C(R,r)$, depending on R and r, such that, for all $f \in \mathcal{L}(\mathbb{R}^n)$ and $j \in \mathbb{Z}_+$,

$$\|w_j(\eta_{j,R} * |f|^r)^{1/r}\|_{\mathcal{L}(\mathbb{R}^n)} \le C(R,r)\|w_j f\|_{\mathcal{L}(\mathbb{R}^n)}.$$

($\mathcal{L}7^\star$) There exist $r \in (0,\infty)$ and a positive constant $C(r)$, depending on r, such that, for all $f \in \mathcal{L}(\mathbb{R}^n)$ and $j \in \mathbb{Z}_+$,

$$\|w_j M(|f|^r)^{1/r}\|_{\mathcal{L}(\mathbb{R}^n)} \le C(r)\|w_j f\|_{\mathcal{L}(\mathbb{R}^n)}.$$

($\mathcal{L}8$) Let $q \in (0,\infty]$. There exist $R \gg 1$, $r \in (0,\infty)$ and a positive constant $C(R,r,q)$, depending on R, r and q, such that, for all $\{f_j\}_{j\in\mathbb{N}} \subset \mathcal{L}(\mathbb{R}^n)$,

$$\|\{w_j(\eta_{j,R} * |f_j|^r)^{1/r}\}_{j\in\mathbb{Z}_+}\|_{\mathcal{L}(\ell^q(\mathbb{R}^n,\mathbb{Z}_+))} \le C(R,r,q)\|\{w_j f_j\}_{j\in\mathbb{Z}_+}\|_{\mathcal{L}(\ell^q(\mathbb{R}^n,\mathbb{Z}_+))}.$$

($\mathcal{L}8^\star$) Let $q \in (0,\infty]$. There exist $r \in (0,\infty)$ and a positive constant $C(r,q)$, depending on r and q, such that, for all $\{f_j\}_{j\in\mathbb{N}} \subset \mathcal{L}(\mathbb{R}^n)$,

$$\|\{w_j[M(|f_j|^r)]^{1/r}\}_{j\in\mathbb{Z}_+}\|_{\mathcal{L}(\ell^q(\mathbb{R}^n,\mathbb{Z}_+))} \le C(r,q)\|\{w_j f_j\}_{j\in\mathbb{Z}_+}\|_{\mathcal{L}(\ell^q(\mathbb{R}^n,\mathbb{Z}_+))}.$$

We now claim that in most cases the parameter a is only auxiliary, by proving the following theorem.

THEOREM 9.2. *Let $\alpha_1, \alpha_2, \alpha_3, \tau \in [0,\infty)$, $a \in (N_0+\alpha_3,\infty)$ and $q \in (0,\infty]$, where N_0 is as in* ($\mathcal{L}6$)*. Let $w \in \mathcal{W}^{\alpha_3}_{\alpha_1,\alpha_2}$, $\tau \in [0,\infty)$ and $q \in (0,\infty]$. Assume that $\Phi, \varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfy, respectively,* (1.3) *and* (1.4)*.*

(i) *Assume that* ($\mathcal{L}7$) *holds and, in addition,* $a \gg 1$. *Then*

$$\|f\|_{B^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} \sim \left\| \left\{ \left[\int_{\mathbb{R}^n} \frac{2^{jn}|\varphi_j * f(y)|^r}{(1+2^j|\cdot - y|)^{ar}}\, dy \right]^{1/r} \right\}_{j\in\mathbb{Z}_+} \right\|_{\ell^q(\mathcal{L}^w_\tau(\mathbb{R}^n,\mathbb{Z}_+))}$$
$$\sim \|\{\varphi_j * f\}_{j\in\mathbb{Z}_+}\|_{\ell^q(\mathcal{L}^w_\tau(\mathbb{R}^n,\mathbb{Z}_+))}$$

and

$$\|f\|_{\mathcal{N}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} \sim \left\| \left\{ \left[\int_{\mathbb{R}^n} \frac{2^{jn}|\varphi_j * f(y)|^r}{(1+2^j|\cdot - y|)^{ar}}\, dy \right]^{1/r} \right\}_{j\in\mathbb{Z}_+} \right\|_{\ell^q(\mathcal{N}\mathcal{L}^w_\tau(\mathbb{R}^n,\mathbb{Z}_+))}$$
$$\sim \|\{\varphi_j * f\}_{j\in\mathbb{Z}_+}\|_{\ell^q(\mathcal{N}\mathcal{L}^w_\tau(\mathbb{R}^n,\mathbb{Z}_+))}$$

with the implicit constants independent of f. *In particular, if* ($\mathcal{L}7^\star$) *holds, then the above equivalences hold.*

(ii) *Assume that* ($\mathcal{L}8$) *holds and, in addition,* $a \gg 1$. *Then*

$$\|f\|_{F^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} \sim \left\| \left\{ \left[\int_{\mathbb{R}^n} \frac{2^{jn}|\varphi_j * f(y)|^r}{(1+2^j|\cdot - y|)^{ar}}\, dy \right]^{1/r} \right\}_{j\in\mathbb{Z}_+} \right\|_{\mathcal{L}^w_\tau(\ell^q(\mathbb{R}^n,\mathbb{Z}_+))}$$
$$\sim \|\{\varphi_j * f\}_{j\in\mathbb{Z}_+}\|_{\mathcal{L}^w_\tau(\ell^q(\mathbb{R}^n,\mathbb{Z}_+))} \tag{9.1}$$

and

$$\|f\|_{\mathcal{E}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} \sim \left\| \left\{ \left[\int_{\mathbb{R}^n} \frac{2^{jn}|\varphi_j * f(y)|^r}{(1+2^j|\cdot - y|)^{ar}}\, dy \right]^{1/r} \right\}_{j\in\mathbb{Z}_+} \right\|_{\mathcal{E}\mathcal{L}^w_\tau(\ell^q(\mathbb{R}^n,\mathbb{Z}_+))}$$
$$\sim \|\{\varphi_j * f\}_{j\in\mathbb{Z}_+}\|_{\mathcal{E}\mathcal{L}^w_\tau(\ell^q(\mathbb{R}^n,\mathbb{Z}_+))}$$

with the implicit constants independent of f. *In particular, if* ($\mathcal{L}8^\star$) *holds, then the above equivalences hold.*

Motivated by Theorem 9.2, let us define

$$\|f\|_{B^{w,\tau}_{\mathcal{L},q}(\mathbb{R}^n)} := \|\{\varphi_j * f\}_{j\in\mathbb{Z}_+}\|_{\ell^q(\mathcal{L}^w_\tau(\mathbb{R}^n,\mathbb{Z}_+))},$$
$$\|f\|_{\mathcal{N}^{w,\tau}_{\mathcal{L},q}(\mathbb{R}^n)} := \|\{\varphi_j * f\}_{j\in\mathbb{Z}_+}\|_{\ell^q(\mathcal{N}\mathcal{L}^w_\tau(\mathbb{R}^n,\mathbb{Z}_+))},$$
$$\|f\|_{F^{w,\tau}_{\mathcal{L},q}(\mathbb{R}^n)} := \|\{\varphi_j * f\}_{j\in\mathbb{Z}_+}\|_{\mathcal{L}^w_\tau(\ell^q(\mathbb{R}^n,\mathbb{Z}_+))},$$
$$\|f\|_{\mathcal{E}^{w,\tau}_{\mathcal{L},q}(\mathbb{R}^n)} := \|\{\varphi_j * f\}_{j\in\mathbb{Z}_+}\|_{\mathcal{E}\mathcal{L}^w_\tau(\ell^q(\mathbb{R}^n,\mathbb{Z}_+))},$$

for all $f \in \mathcal{S}'(\mathbb{R}^n)$ as long as the assumptions of Theorem 9.2 are satisfied.

LEMMA 9.3. *Let* $\alpha_1, \alpha_2, \alpha_3, \tau \in [0,\infty)$, $a \in (N_0 + \alpha_3, \infty)$, $q \in (0,\infty]$ *and* $\varepsilon \in (0,\infty)$. *Assume that* Φ, $\varphi \in \mathcal{S}(\mathbb{R}^n)$ *satisfy, respectively,* (1.3) *and* (1.4). *Then:*

(i) *For all* $f \in \mathcal{S}'(\mathbb{R}^n)$,

$$\|f\|_{B^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} \gtrsim \left\| \left\{ \left[\int_{\mathbb{R}^n} \frac{2^{jn}|\varphi_j * f(y)|^r}{(1+2^j|\cdot - y|)^{ar+n+\varepsilon}}\, dy \right]^{1/r} \right\}_{j\in\mathbb{Z}_+} \right\|_{\ell^q(\mathcal{L}^w_\tau(\mathbb{R}^n,\mathbb{Z}_+))}, \tag{9.2}$$

$$\|f\|_{B^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} \lesssim \left\| \left\{ \left[\int_{\mathbb{R}^n} \frac{2^{jn}|\varphi_j * f(y)|^r}{(1+2^j|\cdot - y|)^{ar}}\, dy \right]^{1/r} \right\}_{j\in\mathbb{Z}_+} \right\|_{\ell^q(\mathcal{L}^w_\tau(\mathbb{R}^n,\mathbb{Z}_+))}, \tag{9.3}$$

$$\|f\|_{\mathcal{N}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} \gtrsim \left\| \left\{ \left[\int_{\mathbb{R}^n} \frac{2^{jn}|\varphi_j * f(y)|^r}{(1+2^j|\cdot - y|)^{ar+n+\varepsilon}} \, dy \right]^{1/r} \right\}_{j\in\mathbb{Z}_+} \right\|_{\ell^q(\mathcal{N}\mathcal{L}^w_\tau(\mathbb{R}^n,\mathbb{Z}_+))}, \tag{9.4}$$

$$\|f\|_{\mathcal{N}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} \lesssim \left\| \left\{ \left[\int_{\mathbb{R}^n} \frac{2^{jn}|\varphi_j * f(y)|^r}{(1+2^j|\cdot - y|)^{ar}} \, dy \right]^{1/r} \right\}_{j\in\mathbb{Z}_+} \right\|_{\ell^q(\mathcal{N}\mathcal{L}^w_\tau(\mathbb{R}^n,\mathbb{Z}_+))}, \tag{9.5}$$

where φ_0 is understood to be Φ and the implicit constants are independent of f.

(ii) *For all $f \in \mathcal{S}'(\mathbb{R}^n)$,*

$$\|f\|_{F^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} \gtrsim \left\| \left\{ \left[\int_{\mathbb{R}^n} \frac{2^{jn}|\varphi_j * f(y)|^r}{(1+2^j|\cdot - y|)^{ar+n+\varepsilon}} \, dy \right]^{1/r} \right\}_{j\in\mathbb{Z}_+} \right\|_{\mathcal{L}^w_\tau(\ell^q(\mathbb{R}^n,\mathbb{Z}_+))}, \tag{9.6}$$

$$\|f\|_{F^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} \lesssim \left\| \left\{ \left[\int_{\mathbb{R}^n} \frac{2^{jn}|\varphi_j * f(y)|^r}{(1+2^j|\cdot - y|)^{ar}} \, dy \right]^{1/r} \right\}_{j\in\mathbb{Z}_+} \right\|_{\mathcal{L}^w_\tau(\ell^q(\mathbb{R}^n,\mathbb{Z}_+))}, \tag{9.7}$$

$$\|f\|_{\mathcal{E}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} \gtrsim \left\| \left\{ \left[\int_{\mathbb{R}^n} \frac{2^{jn}|\varphi_j * f(y)|^r}{(1+2^j|\cdot - y|)^{ar+n+\varepsilon}} \, dy \right]^{1/r} \right\}_{j\in\mathbb{Z}_+} \right\|_{\mathcal{E}\mathcal{L}^w_\tau(\ell^q(\mathbb{R}^n,\mathbb{Z}_+))}, \tag{9.8}$$

$$\|f\|_{\mathcal{E}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} \lesssim \left\| \left\{ \left[\int_{\mathbb{R}^n} \frac{2^{jn}|\varphi_j * f(y)|^r}{(1+2^j|\cdot - y|)^{ar}} \, dy \right]^{1/r} \right\}_{j\in\mathbb{Z}_+} \right\|_{\mathcal{E}\mathcal{L}^w_\tau(\ell^q(\mathbb{R}^n,\mathbb{Z}_+))}, \tag{9.9}$$

where φ_0 is understood to be Φ and the implicit constants are independent of f.

Proof. Estimates (9.2), (9.4), (9.6) and (9.8) are immediate from the definitions, while (9.3), (9.5), (9.7) and (9.9) depend on the following estimate: By [93, (2.29)], we see that, for all $t \in [1,2]$, $N \gg 1$, $r \in (0,\infty)$, $\ell \in \mathbb{N}$ and $x \in \mathbb{R}^n$,

$$[(\phi^*_{2^{-\ell}t} f)_a(x)]^r \lesssim \sum_{k=0}^{\infty} 2^{-kNr} 2^{(k+\ell)n} \int_{\mathbb{R}^n} \frac{|((\phi_{k+\ell})_t * f)(y)|^r}{(1+2^\ell|x-y|)^{ar}} \, dy.$$

In particular, when $l = 0$, for all $x \in \mathbb{R}^n$, we have

$$(\phi^*_t f)_a(x) \lesssim \left[\sum_{k=0}^{\infty} 2^{-kNr} 2^{kn} \int_{\mathbb{R}^n} \frac{|((\phi_k)_t * f)(y)|^r}{(1+|x-y|)^{ar}} \, dy \right]^{1/r}. \tag{9.10}$$

If we combine Lemma 2.9 and (9.10), we obtain the desired result. ∎

The key to the proof of Theorem 9.2 is the following dilation estimate. The next lemma translates the assumptions ($\mathcal{L}7$) and ($\mathcal{L}8$) into our function spaces.

LEMMA 9.4. *Let $\{F_j\}_{j\in\mathbb{Z}_+}$ be a sequence of positive measurable functions on $\mathbb{R}^n$.*

(i) *If ($\mathcal{L}7$) holds, then*

$$\|\{(\eta_{j,2R} * [F_j]^r)^{1/r}\}_{j\in\mathbb{Z}_+}\|_{\ell^q(\mathcal{L}^w_\tau(\mathbb{R}^n,\mathbb{Z}_+))} \lesssim \|\{F_j\}_{j\in\mathbb{Z}_+}\|_{\ell^q(\mathcal{L}^w_\tau(\mathbb{R}^n,\mathbb{Z}_+))},$$
$$\|\{(\eta_{j,2R} * [F_j]^r)^{1/r}\}_{j\in\mathbb{Z}_+}\|_{\ell^q(\mathcal{N}\mathcal{L}^w_\tau(\mathbb{R}^n,\mathbb{Z}_+))} \lesssim \|\{F_j\}_{j\in\mathbb{Z}_+}\|_{\ell^q(\mathcal{N}\mathcal{L}^w_\tau(\mathbb{R}^n,\mathbb{Z}_+))},$$

with the implicit constants independent of $\{F_j\}_{j\in\mathbb{Z}_+}$.

(ii) *If ($\mathcal{L}8$) holds, then*

$$\|\{(\eta_{j,2R} * [F_j]^r)^{1/r}\}_{j\in\mathbb{Z}_+}\|_{\mathcal{L}^w_\tau(\ell^q(\mathbb{R}^n,\mathbb{Z}_+))} \lesssim \|\{F_j\}_{j\in\mathbb{Z}_+}\|_{\mathcal{L}^w_\tau(\ell^q(\mathbb{R}^n,\mathbb{Z}_+))}, \tag{9.11}$$
$$\|\{(\eta_{j,2R} * [F_j]^r])^{1/r}\}_{j\in\mathbb{Z}_+}\|_{\mathcal{E}\mathcal{L}^w_\tau(\ell^q(\mathbb{R}^n,\mathbb{Z}_+))} \lesssim \|\{F_j\}_{j\in\mathbb{Z}_+}\|_{\mathcal{E}\mathcal{L}^w_\tau(\ell^q(\mathbb{R}^n,\mathbb{Z}_+))},$$

with the implicit constants independent of $\{F_j\}_{j\in\mathbb{Z}_+}$.

Proof. Due to similarity, we only prove (9.11).

For all sequences $F = \{F_j\}_{j\in\mathbb{Z}_+}$ of positive measurable functions on $\mathbb{R}^n$, define

$$\|F\| := \|\{F_j\}_{j\in\mathbb{Z}_+}\|_{\mathcal{L}_\tau^w(\ell^q(\mathbb{R}^n,\mathbb{Z}_+))}.$$

Then $\|\cdot\|$ is still a quasi-norm. By the Aoki–Rolewicz theorem (see [2, 69]), there exists a quasi-norm $|\!|\!|\cdot|\!|\!|$ and $\widetilde{\theta} \in (0,1]$ such that, for all sequences F and G, $\|F\| \sim |\!|\!|F|\!|\!|$ and

$$|\!|\!|F+G|\!|\!|^{\widetilde{\theta}} \le |\!|\!|F|\!|\!|^{\widetilde{\theta}} + |\!|\!|G|\!|\!|^{\widetilde{\theta}}.$$

Therefore,

$$\begin{aligned}
&\Big\|\Big\{\Big[\sum_{l=0}^\infty \eta_{k,2R}*(G_{k,l})^r\Big]^{1/r}\Big\}_{k\in\mathbb{Z}_+}\Big\|^{\widetilde{\theta}}_{\mathcal{L}_\tau^w(\ell^q(\mathbb{R}^n,\mathbb{Z}_+))}\\
&\qquad\sim \Big|\!\Big|\!\Big|\Big\{\Big[\sum_{l=0}^\infty \eta_{k,2R}*(G_{k,l})^r\Big]^{1/r}\Big\}_{k\in\mathbb{Z}_+}\Big|\!\Big|\!\Big|^{\widetilde{\theta}}\\
&\qquad\lesssim \sum_{l=0}^\infty |\!|\!|\{[\eta_{k,2R}*(G_{k,l})^r]^{1/r}\}_{k\in\mathbb{Z}_+}|\!|\!|^{\widetilde{\theta}}\\
&\qquad\sim \sum_{l=0}^\infty \|\{[\eta_{k,2R}*(G_{k,l})^r]^{1/r}\}_{k\in\mathbb{Z}_+}\|^{\widetilde{\theta}}_{\mathcal{L}_\tau^w(\ell^q(\mathbb{R}^n,\mathbb{Z}_+))}
\end{aligned}\tag{9.12}$$

for all sequences $\{G_{k,l}\}_{k,l\in\mathbb{Z}_+}$ of positive measurable functions.

We fix a dyadic cube P. Our goal is to prove

$$\mathrm{I} := \Big\|\Big(\sum_{k=j_P\vee 0}^\infty \chi_P(w_k)^q[\eta_{k,2R}*(F_k)^r]^{q/r}\Big)^{1/q}\Big\|_{\mathcal{L}(\mathbb{R}^n)} \lesssim |P|^\tau \|\{F_j\}_{j\in\mathbb{Z}_+}\|_{\mathcal{L}_\tau^w(\ell^q(\mathbb{R}^n,\mathbb{Z}_+))} \tag{9.13}$$

with the implicit constant independent of $\{F_j\}_{j\in\mathbb{Z}_+}$ and P.

By using (9.12), we conclude that

$$\mathrm{I} \lesssim \Big\{\sum_{m\in\mathbb{Z}^n}\Big[\Big\|\Big(\sum_{k=j_P\vee 0}^\infty \chi_P(w_k)^q[\eta_{k,2R}*(\chi_{\ell(P)m+P}F_k)^r]^{q/r}\Big)^{1/q}\Big\|_{\mathcal{L}(\mathbb{R}^n)}\Big]^{\min(\theta,q,r)}\Big\}^{\frac{1}{\min(\theta,q,r)}}.$$

A geometric observation shows that

$$\tfrac{1}{2}|m|\ell(P) \le |x-y| \le 2n|m|\ell(P)$$

whenever $x\in P$ and $y \in \ell(P)m+P$ with $|m|\ge 2$. Hence, for all $m\in\mathbb{Z}^n$ and $x\in\mathbb{R}^n$,

$$\begin{aligned}
\eta_{k,2R}*(\chi_{\ell(P)m+P}F_k)^r(x) &= \int_{\ell(P)m+P} 2^{kn}(1+2^k|x-y|)^{-R}(1+2^k|x-y|)^{-R}[F_k(y)]^r\,dy\\
&\lesssim \frac{1}{|m|^R}\int_{\ell(P)m+P} 2^{kn}[1+2^k|m|\ell(P)]^{-R}[F_k(y)]^r\,dy\\
&\lesssim \frac{1}{|m|^R}\eta_{j_P,R}*[\chi_{\ell(P)m+P}(F_k)^r](x).
\end{aligned}$$

From this and ($\mathcal{L}$8), we further conclude that

$$\begin{aligned}
\mathrm{I} &\lesssim \Big\{\sum_{m\in\mathbb{Z}^n}\Big[\Big\|\Big(\sum_{k=j_P\vee 0}^\infty [\eta_{k,2R}*(\chi_{\ell(P)m+P}F_k)^r]^{q/r}\Big)^{1/q}\Big\|_{\mathcal{L}(\mathbb{R}^n)}\Big]^{\min(\theta,q,r)}\Big\}^{\frac{1}{\min(\theta,q,r)}}\\
&\lesssim |P|^\tau\|\{F_j\}_{j\in\mathbb{Z}_+}\|_{\mathcal{L}_\tau^w(\ell^q(\mathbb{R}^n,\mathbb{Z}_+))}. \qquad \blacksquare
\end{aligned}$$

Proof of Theorem 9.2. Due to similarity, we only prove the estimates for $F^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$.

By Lemma 9.3, we have

$$\|f\|_{F^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} \lesssim \left\| \left\{ \left[\int_{\mathbb{R}^n} \frac{2^{jn}|\varphi_j * f(y)|^r}{(1+2^j|\cdot - y|)^{ar}}\, dy \right]^{1/r} \right\}_{j\in\mathbb{Z}_+} \right\|_{\mathcal{L}^w_\tau(\ell^q(\mathbb{R}^n,\mathbb{Z}_+))}. \tag{9.14}$$

Observe that the right-hand side of (9.14) is just

$$\|\{(\eta_{j,ar} * [|\varphi_j * f(\cdot)|^r])^{1/r}\}_{j\in\mathbb{Z}_+}\|_{\mathcal{L}^w_\tau(\ell^q(\mathbb{R}^n,\mathbb{Z}_+))}.$$

By Lemma 9.4,

$$\left\| \left\{ \left[\int_{\mathbb{R}^n} \frac{2^{jn}|\varphi_j * f(y)|^r\, dy}{(1+2^j|\cdot - y|)^{ar}} \right]^{1/r} \right\}_{j\in\mathbb{Z}_+} \right\|_{\mathcal{L}^w_\tau(\ell^q(\mathbb{R}^n,\mathbb{Z}_+))} \lesssim \|f\|_{F^{w,\tau}_{\mathcal{L},q}(\mathbb{R}^n)}. \tag{9.15}$$

Also, it follows trivially from the definition of $F^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$ that

$$\|f\|_{F^{w,\tau}_{\mathcal{L},q}(\mathbb{R}^n)} \le \|f\|_{F^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)}. \tag{9.16}$$

Combining (9.14)–(9.16), we obtain (9.1). ∎

Proposition 9.5. *Let $q \in [1,\infty]$. Assume that $\theta = 1$ in the assumption ($\mathcal{L}3$) and, additionally, there exist some $M \in (0,\infty)$ and a positive constant $C(M)$, depending on M, such that, for all $f \in \mathcal{L}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,*

$$\|f(\cdot - x)\|_{\mathcal{L}(\mathbb{R}^n)} \le C(M)(1+|x|)^M \|f\|_{\mathcal{L}(\mathbb{R}^n)}. \tag{9.17}$$

Then, whenever $a \gg 1$,

$$\|f\|_{B^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} \sim \left\| \left\{ \left[\int_{\mathbb{R}^n} \frac{2^{jn}|\varphi_j * f(y)|^r}{(1+2^j|\cdot - y|)^{ar}}\, dy \right]^{1/r} \right\}_{j\in\mathbb{Z}_+} \right\|_{\ell^q(\mathcal{L}^w_\tau(\mathbb{R}^n,\mathbb{Z}_+))} \sim \|f\|_{B^{w,\tau}_{\mathcal{L},q}(\mathbb{R}^n)},$$

$$\|f\|_{\mathcal{N}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} \sim \left\| \left\{ \left[\int_{\mathbb{R}^n} \frac{2^{jn}|\varphi_j * f(y)|^r}{(1+2^j|\cdot - y|)^{ar}}\, dy \right]^{1/r} \right\}_{j\in\mathbb{Z}_+} \right\|_{\ell^q(\mathcal{N}\mathcal{L}^w_\tau(\mathbb{R}^n,\mathbb{Z}_+))} \sim \|f\|_{\mathcal{N}^{w,\tau}_{\mathcal{L},q}(\mathbb{R}^n)},$$

with the implicit constants independent of f.

It is not clear whether the counterpart of Proposition 9.5 for $\mathcal{E}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$ and $F^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$ is true or not.

Proof of Proposition 9.5. We concentrate on the B-scale, the proof for the $\mathcal{N}$-scale being similar. By Theorem 9.2, we see that

$$\|f\|_{B^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} \lesssim \sup_{P\in\mathcal{Q}(\mathbb{R}^n)} \frac{1}{|P|^\tau} \left\{ \sum_{k=j_P\vee 0}^\infty \left\| \chi_P \left[w_k \left(\int_{\mathbb{R}^n} \frac{2^{kn}|\varphi_k * f(y)|^r}{(1+2^k|\cdot - y|)^{ar+n+\varepsilon}}\, dy \right)^{1/r} \right] \right\|_{\mathcal{L}(\mathbb{R}^n)}^q \right\}^{1/q}.$$

Now that $\theta = 1$, we can use the triangle inequality to obtain

$$\|f\|_{B^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} \lesssim \sup_{P\in\mathcal{Q}(\mathbb{R}^n)} \frac{1}{|P|^\tau} \left\{ \sum_{k=j_P\vee 0}^\infty \|\chi_P w_k [\varphi_k * f]\|_{\mathcal{L}(\mathbb{R}^n)}^q \right\}^{1/q}$$

whenever $a \gg 1$. The reverse inequality being trivial, we obtain the desired estimates. ∎

To conclude this section, with Theorems 4.12 and 9.2 proved, we have already obtained the biorthogonal wavelet decompositions of Morrey spaces; see also Section 11.2 below.

9.2. Identification of the space $\mathcal{L}(\mathbb{R}^n)$. The following lemma is a natural extension with $|\cdot|$ in the definition of $\|f\|_{\mathcal{L}(\mathbb{R}^n)}$ replaced by $\ell^2(\mathbb{Z})$. In this subsection, we *always assume* that $\theta = 1$ in ($\mathcal{L}3$) and that, for any set E of finite measure, there exists a positive constant $C(E)$, depending on E, such that, for all $f \in \mathcal{L}(\mathbb{R}^n)$,

$$\int_E |f(x)|\,dx \le C(E)\|f\|_{\mathcal{L}(\mathbb{R}^n)}. \tag{9.18}$$

In this case $\mathcal{L}(\mathbb{R}^n)$ is a Banach space of functions and the *dual space* $\mathcal{L}'(\mathbb{R}^n)$ can be defined.

THEOREM 9.6. *Let $\mathcal{L}$ be as above, let $\psi, \varphi \in \mathcal{S}(\mathbb{R}^n)$ be even and satisfy, respectively,* (1.3) *and* (1.4), *and let $N \in \mathbb{N}$. Suppose that $a \in (N, \infty)$ and*

$$(1+|\cdot|)^{-N} \in \mathcal{L}(\mathbb{R}^n) \cap \mathcal{L}'(\mathbb{R}^n). \tag{9.19}$$

Assume, in addition, that there exists a positive constant C such that, for any finite sequence $\{\varepsilon_k\}_{k=1}^{k_0} \subset \{-1, 1\}$, $f \in \mathcal{L}(\mathbb{R}^n)$ and $g \in \mathcal{L}'(\mathbb{R}^n)$,

$$\begin{cases} \left\|\psi * f + \sum_{k=1}^{k_0} \varepsilon_k \varphi_k * f\right\|_{\mathcal{L}(\mathbb{R}^n)} \le C\|f\|_{\mathcal{L}(\mathbb{R}^n)}, \\ \left\|\psi * g + \sum_{k=1}^{k_0} \varepsilon_k \varphi_k * g\right\|_{\mathcal{L}'(\mathbb{R}^n)} \le C\|g\|_{\mathcal{L}'(\mathbb{R}^n)}. \end{cases} \tag{9.20}$$

Then $\mathcal{L}(\mathbb{R}^n)$ and $\mathcal{L}'(\mathbb{R}^n)$ are embedded into $\mathcal{S}'(\mathbb{R}^n)$, and $\mathcal{L}(\mathbb{R}^n)$ and $\mathcal{E}^{0,0}_{\mathcal{L},2,a}(\mathbb{R}^n)$ coincide.

Proof. The fact that $\mathcal{L}(\mathbb{R}^n)$ and $\mathcal{L}'(\mathbb{R}^n)$ are embedded into $\mathcal{S}'(\mathbb{R}^n)$ is a simple consequence of (9.18) and (9.19). By using the Rademacher sequence $\{r_j\}_{j=1}^\infty$, we obtain

$$\begin{aligned} \left\|\Big(\sum_{j=1}^\infty |\varphi_j * f|^2\Big)^{1/2}\right\|_{\mathcal{L}(\mathbb{R}^n)} &= \lim_{k_0\to\infty} \left\|\Big(\sum_{j=1}^{k_0} |\varphi_j * f|^2\Big)^{1/2}\right\|_{\mathcal{L}(\mathbb{R}^n)} \\ &\lesssim \lim_{k_0\to\infty} \left\|\sum_{j=1}^{k_0} \int_0^1 |r_j(t)\varphi_j * f|\,dt\right\|_{\mathcal{L}(\mathbb{R}^n)}, \end{aligned}$$

which, together with the assumption $a > N$, Theorem 9.2 and (9.20), implies that

$$\|f\|_{\mathcal{E}^{0,0}_{\mathcal{L},2,a}(\mathbb{R}^n)} \sim \left\|\Big(|\psi * f|^2 + \sum_{j=1}^\infty |\varphi_j * f|^2\Big)^{1/2}\right\|_{\mathcal{L}(\mathbb{R}^n)} \lesssim \|f\|_{\mathcal{L}(\mathbb{R}^n)}.$$

If we fix $g \in C_c^\infty(\mathbb{R}^n)$, we see that

$$\int_{\mathbb{R}^n} f(x)g(x)\,dx = \int_{\mathbb{R}^n} \psi * f(x)\psi * g(x)\,dx + \sum_{j=1}^\infty \int_{\mathbb{R}^n} \varphi_j * f(x)\varphi_j * g(x)\,dx.$$

From Theorem 9.2, the Hölder inequality and the duality, we deduce that

$$\|f\|_{\mathcal{L}(\mathbb{R}^n)} \lesssim \sup\{\|f\|_{\mathcal{E}^{0,0}_{\mathcal{L},2,a}(\mathbb{R}^n)}\|g\|_{\mathcal{E}^{0,0}_{\mathcal{L}',2,a}(\mathbb{R}^n)} : g \in C_c^\infty(\mathbb{R}^n),\ \|g\|_{\mathcal{L}'(\mathbb{R}^n)} = 1\}.$$

Since we have proved that $\mathcal{L}'(\mathbb{R}^n)$ is embedded into $\mathcal{E}^{0,0}_{\mathcal{L}',2,a}(\mathbb{R}^n)$, by the second estimate of (9.20), we conclude that

$$\|f\|_{\mathcal{L}(\mathbb{R}^n)} \lesssim \|f\|_{\mathcal{E}^{0,0}_{\mathcal{L},2,a}(\mathbb{R}^n)}.$$

The reverse inequality was already proved before. ∎

Let $\mathcal{L}(\mathbb{R}^n)$ be a Banach space of functions and define

$$\mathcal{L}^p(\mathbb{R}^n) := \{f : \mathbb{R}^n \to \mathbb{C} : f \text{ is measurable and } |f|^p \in \mathcal{L}(\mathbb{R}^n)\}$$

for $p \in (0,\infty)$, and $\|f\|_{\mathcal{L}^p(\mathbb{R}^n)} := \|\,|f|^p\,\|_{\mathcal{L}(\mathbb{R}^n)}^{1/p}$ for all $f \in \mathcal{L}^p(\mathbb{R}^n)$. A criterion for (9.20) to hold is given in the book [9]. Here we invoke the following fact.

PROPOSITION 9.7. *Let $\mathcal{L}(\mathbb{R}^n)$ be a Banach space of functions such that $\mathcal{L}^p(\mathbb{R}^n)$ is a Banach space of functions and the maximal operator M is bounded on $(\mathcal{L}^p(\mathbb{R}^n))'$ for some $p \in (1,\infty)$.*

Assume, in addition, that $\mathcal{Z}$ is a set of pairs (f,g) of positive measurable functions such that, for all $p_0 \in (1,\infty)$ and $w \in A_{p_0}(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} [f(x)]^{p_0} w(x)\,dx \lesssim_{A_{p_0}(w)} \int_{\mathbb{R}^n} [g(x)]^{p_0} w(x)\,dx \tag{9.21}$$

with the implicit constant depending on the weight constant $A_{p_0}(w)$ of w, but not on (f,g). Then $\|f\|_{\mathcal{L}(\mathbb{R}^n)} \lesssim \|g\|_{\mathcal{L}(\mathbb{R}^n)}$ for all $(f,g) \in \mathcal{Z}$, with the implicit constant independent of (f,g).

A direct consequence of this proposition is a criterion for (9.20) to hold.

THEOREM 9.8. *Let $\mathcal{L}(\mathbb{R}^n)$ be a Banach space of functions such that $\mathcal{L}^p(\mathbb{R}^n)$ and $(\mathcal{L}')^p(\mathbb{R}^n)$ are Banach spaces of functions and the maximal operator M is bounded on $(\mathcal{L}^p(\mathbb{R}^n))'$ and $((\mathcal{L}')^p(\mathbb{R}^n))'$ for some $p \in (1,\infty)$. Then (9.20) holds. In particular, if $a > N$ and $(1+|\cdot|)^{-N} \in \mathcal{L}(\mathbb{R}^n) \cap \mathcal{L}'(\mathbb{R}^n)$, then $\mathcal{L}(\mathbb{R}^n)$ and $\mathcal{L}'(\mathbb{R}^n)$ are embedded into $\mathcal{S}'(\mathbb{R}^n)$, and $\mathcal{L}(\mathbb{R}^n)$ and $\mathcal{E}^{0,0}_{\mathcal{L},2,a}(\mathbb{R}^n)$ coincide.*

Proof. We have only to check (9.20). Let

$$\mathcal{Z} = \Big\{\Big(\psi * f + \sum_{k=1}^{N} \varepsilon_k \varphi_k * f, f\Big) : f \in \mathcal{L}(\mathbb{R}^n),\ N \in \mathbb{N},\ \{\varepsilon_k\}_{k\in\mathbb{N}} \subset \{-1,1\}\Big\}.$$

Then (9.21) holds according to the well-known Calderón–Zygmund theory (see [13, Chapter 7], for example). Thus, (9.20) holds. ∎

9.3. F-spaces and $\mathcal{E}$-spaces. As we have seen in [82], when $\mathcal{L}(\mathbb{R}^n)$ is the Morrey space $\mathcal{M}^p_q(\mathbb{R}^n)$, we have $\mathcal{E}^{s,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n) = F^{s,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$ with norm equivalence. The same happens under some mild assumptions (9.22) and (9.24) below. Recall that $\mathcal{L}(\mathbb{R}^n)$ carries the parameter N_0 from ($\mathcal{L}6$).

THEOREM 9.9. *Let $a \in (N_0 + \alpha_3, \infty)$, $q \in (0,\infty]$ and $s \in \mathbb{R}$. Assume that $\mathcal{L}(\mathbb{R}^n)$ satisfies the assumption ($\mathcal{L}8$) and there exist positive constants C and τ_0 such that, for all $P \in \mathcal{Q}(\mathbb{R}^n)$,*

$$C^{-1}\|\chi_P\|_{\mathcal{L}(\mathbb{R}^n)} \le |P|^{\tau_0} \le C\|\chi_P\|_{\mathcal{L}(\mathbb{R}^n)}. \tag{9.22}$$

Then for all $\tau \in [0,\tau_0)$, $\mathcal{E}^{s,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n) = F^{s,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$ with equivalent norms.

Proof. By the definition of $\|\cdot\|_{\mathcal{E}^{s,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)}$ and $\|\cdot\|_{F^{s,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)}$, we need only show that

$$F^{s,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n) \hookrightarrow \mathcal{E}^{s,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n). \tag{9.23}$$

In view of the atomic decomposition theorem (Theorem 4.5), instead of proving (9.23) directly, we can reduce the matter to the level of sequence spaces. So we have only to prove

$$f^{s,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n) \hookrightarrow e^{s,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n).$$

First, by ($\mathcal{L}8$),

$$\begin{aligned}\|\lambda\|_{e^{s,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} &:= \sup_{P\in\mathcal{Q}(\mathbb{R}^n)} \frac{1}{|P|^\tau}\left\|\left[\sum_{j=0}^\infty\left(\chi_P 2^{js}\sup_{y\in\mathbb{R}^n}\frac{\sum_{k\in\mathbb{Z}^n}|\lambda_{jk}|\chi_{Q_{jk}}(\cdot+y)}{(1+2^j|y|)^a}\right)^q\right]^{1/q}\right\|_{\mathcal{L}(\mathbb{R}^n)}\\ &\sim \sup_{P\in\mathcal{Q}(\mathbb{R}^n)} \frac{1}{|P|^\tau}\left\|\left[\sum_{j=0}^\infty\left(\chi_P 2^{js}\sum_{k\in\mathbb{Z}^n}|\lambda_{jk}|\chi_{Q_{jk}}\right)^q\right]^{1/q}\right\|_{\mathcal{L}(\mathbb{R}^n)}.\end{aligned}$$

Similarly, by ($\mathcal{L}8$),

$$\begin{aligned}\|\lambda\|_{f^{s,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} &:= \sup_{P\in\mathcal{Q}(\mathbb{R}^n)} \frac{1}{|P|^\tau}\left\|\left[\sum_{j=j_P\vee 0}^\infty\left(\chi_P 2^{js}\sup_{y\in\mathbb{R}^n}\frac{\sum_{k\in\mathbb{Z}^n}|\lambda_{jk}|\chi_{Q_{jk}}(\cdot+y)}{(1+2^j|y|)^a}\right)^q\right]^{1/q}\right\|_{\mathcal{L}(\mathbb{R}^n)}\\ &\sim \sup_{P\in\mathcal{Q}(\mathbb{R}^n)} \frac{1}{|P|^\tau}\left\|\left[\sum_{j=j_P\vee 0}^\infty\left(\chi_P 2^{js}\sum_{k\in\mathbb{Z}^n}|\lambda_{jk}|\chi_{Q_{jk}}\right)^q\right]^{1/q}\right\|_{\mathcal{L}(\mathbb{R}^n)}.\end{aligned}$$

Thus, it suffices to show that, for all dyadic cubes P with $j_P\ge 1$,

$$\mathrm{I} := \frac{1}{|P|^\tau}\left\|\left[\sum_{j=0}^{j_P-1}\left(\chi_P 2^{js}\sum_{k\in\mathbb{Z}^n}|\lambda_{jk}|\chi_{Q_{jk}}\right)^q\right]^{1/q}\right\|_{\mathcal{L}(\mathbb{R}^n)} \lesssim \|\lambda\|_{f^{s,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)}.$$

For all $j\in\{0,\ldots,j_P-1\}$, there exists a unique $k\in\mathbb{Z}^n$ such that $P\cap Q_{jk}\neq\emptyset$. Set $\lambda_j:=\lambda_{jk}$ and $Q_j:=Q_{jk}$; then for all $j\in\{0,\ldots,j_P-1\}$, by (9.22), we have

$$\begin{aligned}\frac{2^{js}|\lambda_j|}{|Q_j|^{\tau-\tau_0}} &\sim \frac{\|2^{js}|\lambda_j|\chi_{Q_j}\|_{\mathcal{L}(\mathbb{R}^n)}}{|Q_j|^\tau}\\ &\lesssim \frac{1}{|Q_j|^\tau}\left\|\left[\sum_{i=j}^\infty\left(\chi_{Q_j}2^{is}\sum_{k\in\mathbb{Z}^n}|\lambda_{ik}|\chi_{Q_{ik}}\right)^q\right]^{1/q}\right\|_{\mathcal{L}(\mathbb{R}^n)} \lesssim \|\lambda\|_{f^{s,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)},\end{aligned}$$

which implies that

$$\begin{aligned}\mathrm{I} &= \frac{1}{|P|^\tau}\left\|\left[\sum_{j=0}^{j_P-1}\left(\chi_P 2^{js}\sum_{k\in\mathbb{Z}^n}|\lambda_{jk}|\chi_{Q_{jk}}\right)^q\right]^{1/q}\right\|_{\mathcal{L}(\mathbb{R}^n)} \lesssim \frac{\|\chi_P\|_{\mathcal{L}(\mathbb{R}^n)}}{|P|^\tau}\left(\sum_{j=0}^{j_P-1}2^{jsq}|\lambda_j|^q\right)^{1/q}\\ &\lesssim \|\lambda\|_{f^{s,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)}|P|^{\tau_0-\tau}\left[\sum_{j=0}^{j_P-1}|Q_j|^{q(\tau-\tau_0)}\right]^{1/q} \lesssim \|\lambda\|_{f^{s,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)}. \quad\blacksquare\end{aligned}$$

The following is a variant of Theorem 9.9.

THEOREM 9.10. *Let $\tau\in[0,\infty)$ and $q\in(0,\infty]$. Assume that there exist a positive constant A and a positive constant $C(A)$, depending on A, such that, for all $P\in\mathcal{Q}(\mathbb{R}^n)$ and $k\in\mathbb{Z}_+$,*

$$\|\chi_P w_{j_P-k}\|_{\mathcal{L}(\mathbb{R}^n)} \le C(A)2^{-Ak}\|\chi_{2^kP}w_{j_P-k}\|_{\mathcal{L}(\mathbb{R}^n)} \tag{9.24}$$

and assume that ($\mathcal{L}8$) holds. Then $\mathcal{E}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n) = F^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$ with equivalent norms for all $\tau\in[0,A)$.

Proof. By the definition, we have only to show that $F^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n) \hookrightarrow \mathcal{E}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$. By Theorem 4.5, we know that this reduces to investigating the corresponding sequence spaces. First, by ($\mathcal{L}8$),

$$\begin{aligned}\|\lambda\|_{e^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} &:= \sup_{P\in\mathcal{Q}(\mathbb{R}^n)} \frac{1}{|P|^\tau}\left\|\left[\sum_{j=0}^{\infty}\left(\chi_P w_j \sup_{y\in\mathbb{R}^n}\frac{\sum_{k\in\mathbb{Z}^n}|\lambda_{jk}|\chi_{Q_{jk}}(\cdot+y)}{(1+2^j|y|)^a}\right)^q\right]^{1/q}\right\|_{\mathcal{L}(\mathbb{R}^n)}\\ &\sim \sup_{P\in\mathcal{Q}(\mathbb{R}^n)} \frac{1}{|P|^\tau}\left\|\left[\sum_{j=0}^{\infty}\left(\chi_P w_j \sum_{k\in\mathbb{Z}^n}|\lambda_{jk}|\chi_{Q_{jk}}\right)^q\right]^{1/q}\right\|_{\mathcal{L}(\mathbb{R}^n)}.\end{aligned}$$

Similarly, by ($\mathcal{L}8$), we also conclude that

$$\begin{aligned}\|\lambda\|_{F^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} &:= \sup_{P\in\mathcal{Q}(\mathbb{R}^n)} \frac{1}{|P|^\tau}\left\|\left[\sum_{j=j_P\vee 0}^{\infty}\left(\chi_P w_j \sup_{y\in\mathbb{R}^n}\frac{\sum_{k\in\mathbb{Z}^n}|\lambda_{jk}|\chi_{Q_{jk}}(\cdot+y)}{(1+2^j|y|)^a}\right)^q\right]^{1/q}\right\|_{\mathcal{L}(\mathbb{R}^n)}\\ &\sim \sup_{P\in\mathcal{Q}(\mathbb{R}^n)} \frac{1}{|P|^\tau}\left\|\left[\sum_{j=j_P\vee 0}^{\infty}\left(\chi_P w_j \sum_{k\in\mathbb{Z}^n}|\lambda_{jk}|\chi_{Q_{jk}}\right)^q\right]^{1/q}\right\|_{\mathcal{L}(\mathbb{R}^n)}.\end{aligned}$$

Thus, it suffices to show that, for all dyadic cubes P with $j_P \ge 1$,

$$\mathrm{I} := \frac{1}{|P|^\tau}\left\|\left[\sum_{j=0}^{j_P-1}\left(\chi_P w_j \sum_{k\in\mathbb{Z}^n}|\lambda_{jk}|\chi_{Q_{jk}}\right)^q\right]^{1/q}\right\|_{\mathcal{L}(\mathbb{R}^n)} \lesssim \|\lambda\|_{f^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)}.$$

For all $j \in \{0,\ldots,j_P-1\}$, there exists a unique $k \in \mathbb{Z}^n$ such that $P\cap Q_{jk} \ne \emptyset$. Set $\lambda_j := \lambda_{jk}$ and $Q_j := Q_{jk}$; then for all $j \in \{0,\ldots,j_P-1\}$,

$$\frac{1}{|Q_j|^\tau}\|w_j\lambda_j\chi_{Q_j}\|_{\mathcal{L}(\mathbb{R}^n)} \le \frac{1}{|Q_j|^\tau}\left\|\left[\sum_{i=j}^{\infty}\left(\chi_{Q_j} w_i \sum_{k\in\mathbb{Z}^n}|\lambda_{ik}|\chi_{Q_{ik}}\right)^q\right]^{1/q}\right\|_{\mathcal{L}(\mathbb{R}^n)} \le \|\lambda\|_{f^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)}.$$

Assume $q \in [1,\infty]$ for the moment. Then by the assumption $q \in [1,\infty]$ and the triangle inequality for $\|\cdot\|^\theta_{\mathcal{L}(\mathbb{R}^n)}$, we see that

$$\begin{aligned}\mathrm{I} &= \frac{1}{|P|^\tau}\left\|\left[\sum_{j=0}^{j_P-1}\left(\chi_P w_j \sum_{k\in\mathbb{Z}^n}|\lambda_{jk}|\chi_{Q_{jk}}\right)^q\right]^{1/q}\right\|_{\mathcal{L}(\mathbb{R}^n)}\\ &\le \frac{1}{|P|^\tau}\left\|\sum_{j=0}^{j_P-1}\chi_P w_j|\lambda_j|\chi_{Q_j}\right\|_{\mathcal{L}(\mathbb{R}^n)} \le \frac{1}{|P|^\tau}\left[\sum_{j=0}^{j_P-1}\|\chi_P w_j\lambda_j\chi_{Q_j}\|^\theta_{\mathcal{L}(\mathbb{R}^n)}\right]^{1/\theta}.\end{aligned}$$

If we use the assumption (9.24), we see that

$$\mathrm{I} \le \|\lambda\|_{f^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)}\frac{1}{|P|^\tau}\left[\sum_{j=0}^{j_P-1}2^{-jA\theta}|Q_j|^{\tau\theta}\right]^{1/\theta} \lesssim \|\lambda\|_{f^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)}.$$

For $q \in (0,1)$, since $\mathcal{L}^{1/q}(\mathbb{R}^n)$ is still a quasi-normed space of functions, by the Aoki–Rolewicz theorem (see [2, 69]), there exist an equivalent quasi-norm $|\!|\!|\cdot|\!|\!|$ and $\widetilde{\theta} \in (0,1]$ such that, for all $f,g \in \mathcal{L}^{1/q}(\mathbb{R}^n)$,

$$\|f\|_{\mathcal{L}^{1/q}(\mathbb{R}^n)} \sim |\!|\!|f|\!|\!|, \qquad |\!|\!|f+g|\!|\!|^{\widetilde{\theta}} \le |\!|\!|f|\!|\!|^{\widetilde{\theta}} + |\!|\!|g|\!|\!|^{\widetilde{\theta}}.$$

It follows that

$$\begin{aligned}
\mathrm{I}^{\widetilde{\theta}} &\lesssim \frac{1}{|P|^{\tau\widetilde{\theta}}} \sum_{j=0}^{j_P-1} \|\!|\chi_P w_j \lambda_j \chi_{Q_j}\|\!|^{\widetilde{\theta}} \sim \frac{1}{|P|^{\tau\widetilde{\theta}}} \sum_{j=0}^{j_P-1} \|\chi_P w_j \lambda_j \chi_{Q_j}\|_{\mathcal{L}(\mathbb{R}^n)}^{\widetilde{\theta}} \\
&\lesssim \frac{1}{|P|^{\tau\widetilde{\theta}}} \sum_{j=0}^{j_P-1} 2^{-jA\widetilde{\theta}} \|w_j \lambda_j \chi_{Q_j}\|_{\mathcal{L}(\mathbb{R}^n)}^{\widetilde{\theta}} \\
&\lesssim \|\lambda\|_{f^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)}^{\widetilde{\theta}} \frac{1}{|P|^{\tau\widetilde{\theta}}} \sum_{j=0}^{j_P-1} 2^{-jA\widetilde{\theta}} |Q_j|^{\tau\widetilde{\theta}} \lesssim \|\lambda\|_{f^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)}^{\widetilde{\theta}}. \qquad \blacksquare
\end{aligned}$$

Remark 9.11. In many examples (see Section 11), it is not hard to show that (9.22) holds.

The following theorem generalizes [82, Theorem 1.1].

Theorem 9.12. *Let $\omega \in \mathcal{W}^{\alpha_3}_{\alpha_1,\alpha_2}$ with $\alpha_1, \alpha_2, \alpha_3 \in [0,\infty)$.*

(i) *Assume $\tau \in (0,\infty)$, $q \in (0,\infty)$ and ($\mathcal{L}7$) holds. If $a \gg 1$, then $\mathcal{N}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$ is a proper subspace of $B^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$.*
(ii) *If $a \in (0,\infty)$ and $\tau \in [0,\infty)$, then $\mathcal{N}^{w,\tau}_{\mathcal{L},\infty,a}(\mathbb{R}^n) = B^{w,\tau}_{\mathcal{L},\infty,a}(\mathbb{R}^n)$ with equivalent norms.*

Proof. Since (ii) is immediate from the definition, we only prove (i). By ($\mathcal{L}7$) and Theorems 4.5 and 9.2, we see that

$$\begin{aligned}
&\|\lambda\|_{b^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} \\
&\quad = \sup_{P\in\mathcal{Q}(\mathbb{R}^n)} \frac{1}{|P|^{\tau}} \left\{ \sum_{j=j_P\vee 0}^{\infty} \left\| \chi_P w_j \sup_{y\in\mathbb{R}^n} \frac{1}{(1+2^j|y|)^a} \sum_{k\in\mathbb{Z}^n} |\lambda_{jk}| \chi_{Q_{jk}}(\cdot+y) \right\|_{\mathcal{L}(\mathbb{R}^n)}^q \right\}^{1/q} \\
&\quad \sim \sup_{P\in\mathcal{Q}(\mathbb{R}^n)} \frac{1}{|P|^{\tau}} \left\{ \sum_{j=j_P\vee 0}^{\infty} \left\| \chi_P w_j \sum_{k\in\mathbb{Z}^n} |\lambda_{jk}| \chi_{Q_{jk}} \right\|_{\mathcal{L}(\mathbb{R}^n)}^q \right\}^{1/q}
\end{aligned}$$

and

$$\begin{aligned}
&\|\lambda\|_{n^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} \\
&\quad = \left\{ \sum_{j=0}^{\infty} \sup_{P\in\mathcal{Q}(\mathbb{R}^n)} \frac{1}{|P|^{\tau q}} \left\| \chi_P w_j \sup_{y\in\mathbb{R}^n} \frac{1}{(1+2^j|y|)^a} \sum_{k\in\mathbb{Z}^n} |\lambda_{jk}| \chi_{Q_{jk}}(\cdot+y) \right\|_{\mathcal{L}(\mathbb{R}^n)}^q \right\}^{1/q} \\
&\quad \sim \left\{ \sum_{j=0}^{\infty} \sup_{P\in\mathcal{Q}(\mathbb{R}^n)} \frac{1}{|P|^{\tau q}} \left\| \chi_P w_j \sum_{k\in\mathbb{Z}^n} |\lambda_{jk}| \chi_{Q_{jk}} \right\|_{\mathcal{L}(\mathbb{R}^n)}^q \right\}^{1/q}.
\end{aligned}$$

We abbreviate

$$Q_{j(1,\ldots,1)} := \overbrace{[2^{-j}, 2^{1-j}) \times \cdots \times [2^{-j}, 2^{1-j})}^{n \text{ times}}$$

to R_j for all $j \in \mathbb{Z}$ and set

$$\lambda_Q := \begin{cases} \|w_j \chi_{R_j}\|_{\mathcal{L}(\mathbb{R}^n)}^{-1} |R_j|^{\tau}, & Q = R_j \text{ for some } j \in \mathbb{Z}, \\ 0, & Q \neq R_j \text{ for any } j \in \mathbb{Z}. \end{cases}$$

Then we have

$$\|\lambda\|_{b^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} \sim \sup_{P\in\mathcal{Q}(\mathbb{R}^n)} \frac{1}{|P|^\tau}\left\{\sum_{j=j_P\vee 0}^{\infty} \|\chi_{P\cap R_j} w_j \lambda_{R_j}\|^q_{\mathcal{L}(\mathbb{R}^n)}\right\}^{1/q}.$$

In order that the inner summand is not zero, there are there possibilities: (a) P contains $\{R_k, R_{k+1}, \ldots\}$; (b) P agrees with R_k for some $k \in \mathbb{Z}$; (c) P is a proper subset of R_k for some $k \in \mathbb{Z}$. Case (c) dose not yield the supremum, while case (a) can be covered by (b). Hence

$$\begin{aligned}\|\lambda\|_{b^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} &\sim \sup_{k\in\mathbb{Z}} \frac{1}{|R_k|^\tau}\left\{\sum_{j=k\vee 0}^{\infty} \|\chi_{R_k\cap R_j} w_j \lambda_{R_j}\|^q_{\mathcal{L}(\mathbb{R}^n)}\right\}^{1/q} \\ &\sim \sup_{k\in\mathbb{Z}} \frac{1}{|R_k|^\tau} \|\chi_{R_k} w_k \lambda_{R_k}\|_{\mathcal{L}(\mathbb{R}^n)} \sim 1. \end{aligned} \tag{9.25}$$

On the other hand, keeping in mind that q is finite, we have

$$\|\lambda\|_{n^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} \geq \left\{\sum_{j=0}^{\infty} \sup_{k\in\mathbb{Z}} \frac{1}{|R_k|^{\tau q}} \|\chi_{R_k\cap R_j} w_j \lambda_{R_j}\|^q_{\mathcal{L}(\mathbb{R}^n)}\right\}^{1/q} = \infty.$$

This, together with Theorem 4.1, the atomic decompositions of $(B^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n), b^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n))$ and $(\mathcal{N}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n), n^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n))$, and (9.25), completes the proof. ■

10. Homogeneous spaces

What we have done so far can be extended to homogeneous cases. Here we give definitions and state theorems but the proofs are omitted.

Following Triebel [90], we let

$$\mathcal{S}_\infty(\mathbb{R}^n) := \left\{ \varphi \in \mathcal{S}(\mathbb{R}^n) : \int_{\mathbb{R}^n} \varphi(x) x^\gamma \, dx = 0 \text{ for all multi-indices } \gamma \in \mathbb{Z}_+^n \right\}$$

and consider $\mathcal{S}_\infty(\mathbb{R}^n)$ as a subspace of $\mathcal{S}(\mathbb{R}^n)$, including the topology. Write $\mathcal{S}'_\infty(\mathbb{R}^n)$ to denote the topological dual of $\mathcal{S}_\infty(\mathbb{R}^n)$, that is, the set of all continuous linear functionals on $\mathcal{S}_\infty(\mathbb{R}^n)$. We endow $\mathcal{S}'_\infty(\mathbb{R}^n)$ with the weak-$*$ topology. Let $\mathcal{P}(\mathbb{R}^n)$ be the set of all polynomials on $\mathbb{R}^n$. It is well known that $\mathcal{S}'_\infty(\mathbb{R}^n) = \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$ as topological spaces (see, for example, [105, Proposition 8.1]).

To develop a theory of homogeneous spaces, we need to modify the class of weights. Let $\mathbb{R}^{n+1}_{\mathbb{Z}} := \{(x,t) \in \mathbb{R}^{n+1}_+ : \log_2 t \in \mathbb{Z}\}$.

Definition 10.1. Let $\alpha_1, \alpha_2, \alpha_3 \in [0,\infty)$. We define the class $\dot{\mathcal{W}}^{\alpha_3}_{\alpha_1,\alpha_2}$ of weights as the set of all measurable functions $w : \mathbb{R}^{n+1}_{\mathbb{Z}} \to (0,\infty)$ satisfying the following conditions:

(H-W1) There exists a positive constant C such that, for all $x \in \mathbb{R}^n$ and $j, \nu \in \mathbb{Z}$ with $j \geq \nu$,

$$C^{-1} 2^{-(j-\nu)\alpha_1} w(x, 2^{-\nu}) \leq w(x, 2^{-j}) \leq C 2^{-(\nu-j)\alpha_2} w(x, 2^{-\nu}).$$

(H-W2) There exists a positive constant C such that, for all $x, y \in \mathbb{R}^n$ and $j \in \mathbb{Z}$,

$$w_j(x) \leq C w(y, 2^{-j})(1 + 2^j|x-y|)^{\alpha_3}.$$

The class $\star$-$\dot{\mathcal{W}}^{\alpha_3}_{\alpha_1,\alpha_2}$ is defined by making modifications similar to Definition 3.12.

As we did for the inhomogeneous case, we write $w_j(x) := w(x, 2^{-j})$ for $x \in \mathbb{R}^n$ and $j \in \mathbb{Z}$.

Definition 10.2. Let $q \in (0,\infty]$ and $\tau \in [0,\infty)$. Suppose, in addition, that $w \in \dot{\mathcal{W}}^{\alpha_3}_{\alpha_1,\alpha_2}$ with $\alpha_1, \alpha_2, \alpha_3 \in [0,\infty)$.

(i) $\ell^q(\mathcal{L}^w_\tau(\mathbb{R}^n, \mathbb{Z}))$ is defined to be the space of all sequences $G := \{g_j\}_{j\in\mathbb{Z}}$ of measurable functions on $\mathbb{R}^n$ such that

$$\|G\|_{\ell^q(\mathcal{L}^w_\tau(\mathbb{R}^n,\mathbb{Z}))} := \sup_{P \in \mathcal{Q}(\mathbb{R}^n)} \frac{1}{|P|^\tau} \|\{\chi_P w_j g_j\}_{j=j_P}^\infty\|_{\mathcal{L}^w(\ell^q(\mathbb{R}^n,\mathbb{Z}))} < \infty. \tag{10.1}$$

(ii) $\ell^q(\mathcal{NL}^w_\tau(\mathbb{R}^n,\mathbb{Z}))$ is defined to be the space of all sequences $G := \{g_j\}_{j\in\mathbb{Z}}$ of measurable functions on $\mathbb{R}^n$ such that

$$\|G\|_{\ell^q(\mathcal{NL}^w_\tau(\mathbb{R}^n,\mathbb{Z}))} := \left\{ \sum_{j=-\infty}^{\infty} \sup_{P\in\mathcal{Q}(\mathbb{R}^n)} \left(\frac{\|\chi_P w_j g_j\|_{\mathcal{L}(\mathbb{R}^n)}}{|P|^\tau} \right)^q \right\}^{1/q} < \infty. \tag{10.2}$$

(iii) $\mathcal{L}^w_\tau(\ell^q(\mathbb{R}^n,\mathbb{Z}))$ is defined to be the space of all sequences $G := \{g_j\}_{j\in\mathbb{Z}}$ of measurable functions on $\mathbb{R}^n$ such that

$$\|G\|_{\mathcal{L}^w_\tau(\ell^q(\mathbb{R}^n,\mathbb{Z}))} := \sup_{P\in\mathcal{Q}(\mathbb{R}^n)} \frac{1}{|P|^\tau} \|\{\chi_P w_j g_j\}_{j=j_P}^{\infty}\|_{\ell^q(\mathcal{L}^w(\mathbb{R}^n,\mathbb{Z}))} < \infty. \tag{10.3}$$

(iv) $\mathcal{EL}^w_\tau(\ell^q(\mathbb{R}^n,\mathbb{Z}))$ is defined to be the space of all sequences $G := \{g_j\}_{j\in\mathbb{Z}}$ of measurable functions on $\mathbb{R}^n$ such that

$$\|G\|_{\mathcal{EL}^w_\tau(\ell^q(\mathbb{R}^n,\mathbb{Z}))} := \sup_{P\in\mathcal{Q}(\mathbb{R}^n)} \frac{1}{|P|^\tau} \|\{\chi_P w_j g_j\}_{j=-\infty}^{\infty}\|_{\ell^q(\mathcal{L}^w(\mathbb{R}^n,\mathbb{Z}))} < \infty. \tag{10.4}$$

When $q = \infty$, a natural modification is made in (10.1) through (10.4), and τ is omitted in the notation when $\tau = 0$.

10.1. Homogeneous Besov-type and Triebel–Lizorkin-type spaces. Based upon the inhomogeneous case, we present the following definitions.

Definition 10.3. Let $a \in (0,\infty)$, $\alpha_1, \alpha_2, \alpha_3, \tau \in [0,\infty)$, $q \in (0,\infty]$ and $w \in \dot{\mathcal{W}}^{\alpha_3}_{\alpha_1,\alpha_2}$. Assume also that $\mathcal{L}(\mathbb{R}^n)$ is a quasi-normed space satisfying $(\mathcal{L}1)$ through $(\mathcal{L}4)$ and that $\varphi \in \mathcal{S}_\infty(\mathbb{R}^n)$ satisfies (1.4). For all $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$, $x \in \mathbb{R}^n$ and $j \in \mathbb{Z}$, let

$$(\varphi_j^* f)_a(x) := \sup_{y\in\mathbb{R}^n} \frac{|\varphi_j * f(x+y)|}{(1+2^j|y|)^a}. \tag{10.5}$$

(i) The *homogeneous generalized Besov-type space* $\dot{B}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$ is defined to be the space of all $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$ such that

$$\|f\|_{\dot{B}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} := \|\{(\varphi_j^* f)_a\}_{j\in\mathbb{Z}}\|_{\ell^q(\mathcal{L}^w_\tau(\mathbb{R}^n,\mathbb{Z}))} < \infty.$$

(ii) The *homogeneous generalized Besov-Morrey space* $\dot{\mathcal{N}}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$ is defined to be the space of all $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$ such that

$$\|f\|_{\dot{\mathcal{N}}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} := \|\{(\varphi_j^* f)_a\}_{j\in\mathbb{Z}}\|_{\ell^q(\mathcal{NL}^w_\tau(\mathbb{R}^n,\mathbb{Z}))} < \infty.$$

(iii) The *homogeneous generalized Triebel–Lizorkin-type space* $\dot{F}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$ is defined to be the space of all $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$ such that

$$\|f\|_{\dot{F}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} := \|\{(\varphi_j^* f)_a\}_{j\in\mathbb{Z}}\|_{\mathcal{L}^w_\tau(\ell^q(\mathbb{R}^n,\mathbb{Z}))} < \infty.$$

(iv) The *homogeneous generalized Triebel–Lizorkin-Morrey space* $\dot{\mathcal{E}}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$ is defined to be the space of all $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$ such that

$$\|f\|_{\dot{\mathcal{E}}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} := \|\{(\varphi_j^* f)_a\}_{j\in\mathbb{Z}}\|_{\mathcal{EL}^w_\tau(\ell^q(\mathbb{R}^n,\mathbb{Z}))} < \infty.$$

(v) Denote by $\dot{\mathcal{A}}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$ any of the above spaces.

Example 10.4. One of the advantages of introducing the class $\dot{\mathcal{W}}^{\alpha_3}_{\alpha_1,\alpha_2}$ is that intersections of these function spaces still fall under this scope. Indeed, let $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2,$

$\beta_3, \tau \in [0,\infty)$, $q, q_1, q_2 \in (0, \infty]$, $w \in \dot{\mathcal{W}}^{\alpha_3}_{\alpha_1,\alpha_2}$ and $w' \in \dot{\mathcal{W}}^{\beta_3}_{\beta_1,\beta_2}$. Then it is easy to see $\dot{\mathcal{A}}^{w,\tau}_{\mathcal{L},q_1,a}(\mathbb{R}^n) \cap \dot{\mathcal{A}}^{w',\tau}_{\mathcal{L},q_1,a}(\mathbb{R}^n) = \dot{\mathcal{A}}^{w+w',\tau}_{\mathcal{L},q_1,a}(\mathbb{R}^n)$.

The following lemma is immediate from the definitions (cf. Lemma 3.8).

LEMMA 10.5. *Let $\alpha_1, \alpha_2, \alpha_3, \tau \in [0,\infty)$, $q, q_1, q_2 \in (0, \infty]$ and $w \in \dot{\mathcal{W}}^{\alpha_3}_{\alpha_1,\alpha_2}$. Then*

$$\begin{aligned}
\dot{B}^{w,\tau}_{\mathcal{L},q_1,a}(\mathbb{R}^n) &\hookrightarrow \dot{B}^{w,\tau}_{\mathcal{L},q_2,a}(\mathbb{R}^n),\\
\dot{\mathcal{N}}^{w,\tau}_{\mathcal{L},q_1,a}(\mathbb{R}^n) &\hookrightarrow \dot{\mathcal{N}}^{w,\tau}_{\mathcal{L},q_2,a}(\mathbb{R}^n),\\
\dot{F}^{w,\tau}_{\mathcal{L},q_1,a}(\mathbb{R}^n) &\hookrightarrow \dot{F}^{w,\tau}_{\mathcal{L},q_2,a}(\mathbb{R}^n),\\
\dot{\mathcal{E}}^{w,\tau}_{\mathcal{L},q_1,a}(\mathbb{R}^n) &\hookrightarrow \dot{\mathcal{E}}^{w,\tau}_{\mathcal{L},q_2,a}(\mathbb{R}^n),\\
\dot{F}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n) &\hookrightarrow \dot{\mathcal{N}}^{w,\tau}_{\mathcal{L},\infty,a}(\mathbb{R}^n)
\end{aligned}$$

in the sense of continuous embeddings.

The next theorem is a homogeneous counterpart of Theorem 3.14.

THEOREM 10.6. *Let $\alpha_1, \alpha_2, \alpha_3, \tau \in [0,\infty)$, $q \in (0, \infty]$ and $w \in \dot{\mathcal{W}}^{\alpha_3}_{\alpha_1,\alpha_2}$. Then $\dot{B}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$ and $\dot{F}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$ are continuously embedded into $\mathcal{S}'_\infty(\mathbb{R}^n)$.*

Proof. In view of Lemma 10.5, we have only to prove that

$$\dot{B}^{w,\tau}_{\mathcal{L},\infty,a}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'_\infty(\mathbb{R}^n).$$

Suppose that Φ satisfies (1.3) and $\widehat{\Phi}$ equals 1 in a neighborhood of the origin. We write $\varphi(\cdot) := \Phi(\cdot) - 2^{-n}\Phi(2^{-1}\cdot)$ and define $L_1(f) := f - \Phi * f$ for all $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$. Then by Theorem 3.14, we have $L_1(\dot{B}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)) \hookrightarrow \mathcal{S}'(\mathbb{R}^n) \hookrightarrow \mathcal{S}'_\infty(\mathbb{R}^n)$. Therefore, we need to prove that

$$L_2(f) := \sum_{j=-\infty}^{0} \varphi_j * f$$

converges in $\mathcal{S}'_\infty(\mathbb{R}^n)$ and that L_2 is a continuous operator from $\dot{B}^{w,\tau}_{\mathcal{L},\infty,a}(\mathbb{R}^n)$ to $\mathcal{S}'_\infty(\mathbb{R}^n)$.

Notice that, for all $\alpha \in \mathbb{Z}^n_+$, $j \in \mathbb{Z}$ and $x \in \mathbb{R}^n$,

$$|\partial^\alpha(\varphi_j * f)(x)| \lesssim 2^{j\|\alpha\|_1}((\partial^\alpha\varphi)^*_j f)_a(x).$$

Consequently, for any $\kappa \in \mathcal{S}_\infty(\mathbb{R}^n)$, we have, for $\alpha \in \mathbb{Z}^n_+$,

$$\begin{aligned}
\left|\int_{\mathbb{R}^n} \kappa(x)\partial^\alpha(\varphi_j * f)(x)\,dx\right| &\le \int_{\mathbb{R}^n} |\kappa(x)\partial^\alpha(\varphi_j * f)(x)|\,dx\\
&\le 2^{j\|\alpha\|_1}\int_{\mathbb{R}^n} |\kappa(x)|((\partial^\alpha\varphi)^*_j f)_a(x)\,dx.
\end{aligned}$$

Now we use the condition (H-W2) to conclude that, for $\alpha \in \mathbb{Z}^n_+$,

$$\begin{aligned}
\left|\int_{\mathbb{R}^n} \kappa(x)\partial^\alpha(\varphi_j * f)(x)\,dx\right| &\le 2^{j(\|\alpha\|_1-\alpha_1)}\int_{\mathbb{R}^n} \frac{|\kappa(x)|}{w(x,1)} w_j(x)((\partial^\alpha\varphi)^*_j f)_a(x)\,dx\\
&\le 2^{j(\|\alpha\|_1-\alpha_1)}\int_{\mathbb{R}^n} \frac{1}{(1+|x|)^M} w_j(x)((\partial^\alpha\varphi)^*_j f)_a(x)\,dx
\end{aligned}$$

$$= 2^{j(\|\alpha\|_1-\alpha_1)} \sum_{k\in\mathbb{Z}^n} \int_{Q_{jk}} \frac{1}{(1+|x|)^M} w_j(x)((\partial^\alpha\varphi)_j^* f)_a(x)\,dx$$
$$\lesssim 2^{j(\|\alpha\|_1-\alpha_1-M)} \sum_{k\in\mathbb{Z}^n} (|k|+1)^{-M} \int_{Q_{jk}} w_j(x)((\partial^\alpha\varphi)_j^* f)_a(x)\,dx.$$

By ($\mathcal{L}6$) and (H-W2), together with Theorem 10.7 below, we further see that, for $\alpha \in \mathbb{Z}_+^n$,

$$\left| \int_{\mathbb{R}^n} \kappa(x)\partial^\alpha(\varphi_j * f)(x)\,dx \right| \lesssim 2^{j(\|\alpha\|_1-\alpha_1-\delta_0)} \sum_{k\in\mathbb{Z}^n} (|k|+1)^{-M+\delta_0} \|w_j((\partial^\alpha\varphi)_j^* f)_a\|_{\mathcal{L}(\mathbb{R}^n)}$$
$$\lesssim 2^{j(\|\alpha\|_1-\alpha_1-\delta_0)} \|f\|_{\dot{B}^{w,\tau}_{\mathcal{L},\infty,a}(\mathbb{R}^n)}.$$

Therefore, the summation defining $L_2(f)$ converges in $\mathcal{S}'_\infty(\mathbb{R}^n)$. ∎

We remark that these homogeneous spaces have many properties similar to those of their inhomogeneous counterparts. However, similar to the classical homogeneous Besov spaces and Triebel–Lizorkin spaces (see [90, p. 238]), some of the most striking features of the spaces $B^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$, $F^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$, $\mathcal{N}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$ and $\mathcal{E}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$ have no counterparts, such as the boundedness of pointwise multipliers in Section 5.

10.2. Characterizations. We have the following counterparts of Theorem 3.5.

THEOREM 10.7. *Let $a, \alpha_1, \alpha_2, \alpha_3, \tau, q$, w and $\mathcal{L}(\mathbb{R}^n)$ be as in Definition* 10.3. *Assume that $\psi \in \mathcal{S}_\infty(\mathbb{R}^n)$ has the property that*

$$\widehat{\psi}(\xi) \neq 0 \qquad \text{if } \varepsilon/2 < |\xi| < 2\varepsilon$$

for some $\varepsilon \in (0,\infty)$. Let $\psi_j(\cdot) := 2^{jn}\psi(2^j\cdot)$ for all $j \in \mathbb{Z}$ and $\{(\psi_j^* f)_a\}_{j\in\mathbb{Z}}$ be as in (10.5) *with φ replaced by ψ. Then*

$$\|f\|_{\dot{B}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} \sim \|\{(\psi_j^* f)_a\}_{j\in\mathbb{Z}}\|_{\ell^q(\mathcal{L}^w_\tau(\mathbb{R}^n,\mathbb{Z}))},$$
$$\|f\|_{\dot{\mathcal{N}}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} \sim \|\{(\psi_j^* f)_a\}_{j\in\mathbb{Z}}\|_{\ell^q(\mathcal{N}\mathcal{L}^w_\tau(\mathbb{R}^n,\mathbb{Z}))},$$
$$\|f\|_{\dot{F}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} \sim \|\{(\psi_j^* f)_a\}_{j\in\mathbb{Z}}\|_{\mathcal{L}^w_\tau(\ell^q(\mathbb{R}^n,\mathbb{Z}))},$$
$$\|f\|_{\dot{\mathcal{E}}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} \sim \|\{(\psi_j^* f)_a\}_{j\in\mathbb{Z}}\|_{\mathcal{E}\mathcal{L}^w_\tau(\ell^q(\mathbb{R}^n,\mathbb{Z}))},$$

with the implicit constants independent of f.

We also characterize these function spaces in terms of local means (see Corollary 3.6).

COROLLARY 10.8. *Under the notation of Theorem* 10.7, *let*

$$\mathfrak{M}f(x,2^{-j}) := \sup_{\psi} |\psi_j * f(x)|$$

for all $(x,2^{-j}) \in \mathbb{R}^{n+1}_{\mathbb{Z}}$ and $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$, where the supremum is taken over all ψ in $\mathcal{S}_\infty(\mathbb{R}^n)$ such that

$$\sum_{|\alpha|\le N} \sup_{x\in\mathbb{R}^n} (1+|x|)^N |\partial^\alpha\psi(x)| \le 1$$

and, for some $\varepsilon \in (0,\infty)$,

$$\int_{\mathbb{R}^n} \xi^\alpha \widehat{\psi}(\xi)\,d\xi = 0, \quad \widehat{\psi}(\xi) \neq 0 \qquad \text{if } \varepsilon/2 < |\xi| < 2\varepsilon.$$

If N is large enough, then for all $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$,

$$\|f\|_{\dot{B}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} \sim \|\{\mathfrak{M}f(\cdot, 2^{-j})\}_{j\in\mathbb{Z}}\|_{\ell^q(\mathcal{L}^w_\tau(\mathbb{R}^n,\mathbb{Z}))},$$

$$\|f\|_{\dot{\mathcal{N}}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} \sim \|\{\mathfrak{M}f(\cdot, 2^{-j})\}_{j\in\mathbb{Z}}\|_{\ell^q(\mathcal{N}\mathcal{L}^w_\tau(\mathbb{R}^n,\mathbb{Z}))},$$

$$\|f\|_{\dot{F}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} \sim \|\{\mathfrak{M}f(\cdot, 2^{-j})\}_{j\in\mathbb{Z}}\|_{\mathcal{L}^w_\tau(\ell^q(\mathbb{R}^n,\mathbb{Z}))},$$

$$\|f\|_{\dot{\mathcal{E}}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} \sim \|\{\mathfrak{M}f(\cdot, 2^{-j})\}_{j\in\mathbb{Z}}\|_{\mathcal{E}\mathcal{L}^w_\tau(\ell^q(\mathbb{R}^n,\mathbb{Z}))},$$

with the implicit constants independent of f.

10.3. Atomic decompositions. Now we place ourselves once again in the setting of a quasi-normed space $\mathcal{L}(\mathbb{R}^n)$ satisfying only ($\mathcal{L}1$) through ($\mathcal{L}6$). Now we are going to consider the atomic decompositions of the spaces in Definition 10.3.

DEFINITION 10.9 (cf. Definition 4.1). Let $K \in \mathbb{Z}_+$ and $L \in \mathbb{Z}_+ \cup \{-1\}$.

(i) Let $Q \in \mathcal{Q}(\mathbb{R}^n)$. A *$(K, L)$-atom (for $\dot{A}^{s,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$)* supported near a cube Q is a $C^K(\mathbb{R}^n)$-function a satisfying

$$\text{(the support condition)} \quad \operatorname{supp}(a) \subset 3Q,$$

$$\text{(the size condition)} \quad \|\partial^\alpha a\|_{L^\infty(\mathbb{R}^n)} \le |Q|^{-\|\alpha\|_1/n},$$

$$\text{(the moment condition)} \quad \int_{\mathbb{R}^n} x^\beta a(x)\, dx = 0,$$

for all multiindices α and β satisfying $\|\alpha\|_1 \le K$ and $\|\beta\|_1 \le L$. Here the moment condition with $L = -1$ is understood to be vacuous.

(ii) A set $\{a_{jk}\}_{j\in\mathbb{Z},\, k\in\mathbb{Z}^n}$ of $C^K(\mathbb{R}^n)$-functions is called a *collection of (K, L)-atoms (for $\dot{A}^{s,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$)* if each a_{jk} is a (K, L)-atom supported near Q_{jk}.

DEFINITION 10.10 (cf. Definition 4.2). Let $K \in \mathbb{Z}_+$, $L \in \mathbb{Z}_+ \cup \{-1\}$ and $N \gg 1$.

(i) Let $Q \in \mathcal{Q}(\mathbb{R}^n)$. A *$(K, L)$-molecule (for $\dot{A}^{s,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$)* supported near a cube Q is a $C^K(\mathbb{R}^n)$-function $\mathfrak{M}$ satisfying

$$\text{(the decay condition)} \quad |\partial^\alpha \mathfrak{M}(x)| \le (1 + |x - c_Q|/\ell(Q))^{-N} \text{ for all } x \in \mathbb{R}^n,$$

$$\text{(the moment condition)} \quad \int_{\mathbb{R}^n} x^\beta \mathfrak{M}(x)\, dx = 0,$$

for all multiindices α and β satisfying $\|\alpha\|_1 \le K$ and $\|\beta\|_1 \le L$. Here c_Q and $\ell(Q)$ denote, respectively, the center and the side length of Q, and the moment condition with $L = -1$ is understood to be vacuous.

(ii) A collection $\{\mathfrak{M}_{jk}\}_{j\in\mathbb{Z},\, k\in\mathbb{Z}^n}$ of $C^K(\mathbb{R}^n)$-functions is called a *collection of (K, L)-molecules (for $\dot{A}^{s,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$)* if each $\mathfrak{M}_{jk}$ is a (K, L)-molecule supported near Q_{jk}.

For a function F on $\mathbb{R}^{n+1}_{\mathbb{Z}}$, we define

$$\|F\|_{L^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^{n+1}_{\mathbb{Z}})} := \left\| \left\{ \sup_{y\in\mathbb{R}^n} \frac{|F(y,2^{-j})|}{(1+2^j|\cdot-y|)^a} \right\}_{j\in\mathbb{Z}} \right\|_{\ell^q(\mathcal{L}^w_\tau(\mathbb{R}^n,\mathbb{Z}))},$$

$$\|F\|_{\mathcal{N}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^{n+1}_{\mathbb{Z}})} := \left\| \left\{ \sup_{y\in\mathbb{R}^n} \frac{|F(y,2^{-j})|}{(1+2^j|\cdot-y|)^a} \right\}_{j\in\mathbb{Z}} \right\|_{\ell^q(\mathcal{N}\mathcal{L}^w_\tau(\mathbb{R}^n,\mathbb{Z}))},$$

$$\|F\|_{F^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^{n+1}_{\mathbb{Z}})} := \left\| \left\{ \sup_{y\in\mathbb{R}^n} \frac{|F(y,2^{-j})|}{(1+2^j|\cdot-y|)^a} \right\}_{j\in\mathbb{Z}} \right\|_{\mathcal{L}^w_\tau(\ell^q(\mathbb{R}^n,\mathbb{Z}))},$$

$$\|F\|_{\mathcal{E}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^{n+1}_{\mathbb{Z}})} := \left\| \left\{ \sup_{y\in\mathbb{R}^n} \frac{|F(y,2^{-j})|}{(1+2^j|\cdot-y|)^a} \right\}_{j\in\mathbb{Z}} \right\|_{\mathcal{E}\mathcal{L}^w_\tau(\ell^q(\mathbb{R}^n,\mathbb{Z}))}.$$

Definition 10.11 (cf. Definition 4.3). Let $\alpha_1, \alpha_2, \alpha_3, \tau \in [0,\infty)$ and $q \in (0,\infty]$. Suppose that $w \in \dot{\mathcal{W}}^{\alpha_3}_{\alpha_1,\alpha_2}$. Assume that $\Phi, \varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfy, respectively, (1.3) and (1.4). Define $\Lambda : \mathbb{R}^{n+1}_{\mathbb{Z}} \to \mathbb{C}$ by setting, for all $(x, 2^{-j}) \in \mathbb{R}^{n+1}_{\mathbb{Z}}$,

$$\Lambda(x,2^{-j}) := \sum_{m\in\mathbb{Z}^n} \lambda_{jm}\chi_{Q_{jm}}(x),$$

when $\lambda := \{\lambda_{jm}\}_{j\in\mathbb{Z},\, m\in\mathbb{Z}^n}$, a doubly-indexed complex sequence, is given.

(i) The homogeneous sequence space $\dot{b}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$ is defined to be the space of all λ such that $\|\lambda\|_{\dot{b}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} := \|\Lambda\|_{\dot{L}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^{n+1}_{\mathbb{Z}})} < \infty$.
(ii) The homogeneous sequence space $\dot{n}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$ is defined to be the space of all λ such that $\|\lambda\|_{\dot{n}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} := \|\Lambda\|_{\dot{\mathcal{N}}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^{n+1}_{\mathbb{Z}})} < \infty$.
(iii) The homogeneous sequence space $\dot{f}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$ is defined to be the space of all λ such that $\|\lambda\|_{\dot{f}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} := \|\Lambda\|_{\dot{F}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^{n+1}_{\mathbb{Z}})} < \infty$.
(iv) The homogeneous sequence space $\dot{e}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$ is defined to be the space of all λ such that $\|\lambda\|_{\dot{e}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} := \|\Lambda\|_{\dot{\mathcal{E}}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^{n+1}_{\mathbb{Z}})} < \infty$.

As we did for inhomogeneous spaces, we present the following definition.

Definition 10.12 (cf. Definition 4.4). Let X be a function space continuously embedded into $\mathcal{S}'_\infty(\mathbb{R}^n)$ and $\mathcal{X}$ a quasi-normed space of sequences. The pair $(X, \mathcal{X})$ is said to *admit atomic decompositions* if it satisfies the following two conditions:

(i) For any $f \in X$, there exist a collection of atoms, $\{a_{jk}\}_{j\in\mathbb{Z},\, k\in\mathbb{Z}^n}$, and a sequence $\{\lambda_{jk}\}_{j\in\mathbb{Z},\, k\in\mathbb{Z}^n}$ such that $f = \sum_{j=-\infty}^{\infty}\sum_{k\in\mathbb{Z}^n}\lambda_{jk}a_{jk}$ in $\mathcal{S}'_\infty(\mathbb{R}^n)$ and $\|\{\lambda_{jk}\}_{j\in\mathbb{Z},\, k\in\mathbb{Z}^n}\|_{\mathcal{X}} \lesssim \|f\|_X$ with the implicit constant independent of f.
(ii) Suppose that $\{a_{jk}\}_{j\in\mathbb{Z},\, k\in\mathbb{Z}^n}$ is a collection of atoms, and $\{\lambda_{jk}\}_{j\in\mathbb{Z},\, k\in\mathbb{Z}^n}$ a sequence such that $\|\{\lambda_{jk}\}_{j\in\mathbb{Z},\, k\in\mathbb{Z}^n}\|_{\mathcal{X}} < \infty$. Then the series $f := \sum_{j=-\infty}^{\infty}\sum_{k\in\mathbb{Z}^n}\lambda_{jk}a_{jk}$ converges in $\mathcal{S}'_\infty(\mathbb{R}^n)$ and $\|f\|_X \lesssim \|\{\lambda_{jk}\}_{j\in\mathbb{Z},\, k\in\mathbb{Z}^n}\|_{\mathcal{X}}$ with the implicit constant independent of $\{\lambda_{jk}\}_{j\in\mathbb{Z},\, k\in\mathbb{Z}^n}$.

Analogously one defines the notion of a pair $(X, \mathcal{X})$ *admitting molecular decompositions.*

Theorem 10.13. *Let $\alpha_1, \alpha_2, \alpha_3, \tau \in [0,\infty)$ and $q \in (0,\infty]$. Suppose that $w \in \dot{\mathcal{W}}^{\alpha_3}_{\alpha_1,\alpha_2}$ and* (3.28) *and* (4.1)–(4.3) *hold. Then the pair $(\dot{A}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n), \dot{a}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n))$ admits atomic decompositions.*

In principle, the proof of Theorem 10.13 is analogous to that of Theorem 4.5: we just need to modify the related proofs. In particular, we have to prove the following counterpart of Lemma 4.7.

LEMMA 10.14. *Let* $\alpha_1, \alpha_2, \alpha_3 \in [0,\infty)$ *and* $w \in \dot{\mathcal{W}}^{\alpha_3}_{\alpha_1,\alpha_2}$. *Assume that* $K \in \mathbb{Z}_+$ *and* $L \in \mathbb{Z}_+$ *satisfy* (4.1)–(4.3). *Let* $\lambda := \{\lambda_{jk}\}_{j\in\mathbb{Z},\, k\in\mathbb{Z}^n} \in \dot{b}^{w,\tau}_{\mathcal{L},\infty,a}(\mathbb{R}^n)$ *and* $\{\mathfrak{M}_{jk}\}_{j\in\mathbb{Z},\, k\in\mathbb{Z}^n}$ *be a family of molecules. Then* $f = \sum_{j=-\infty}^{\infty}\sum_{k\in\mathbb{Z}^n}\lambda_{jk}\mathfrak{M}_{jk}$ *converges in* $\mathcal{S}'_\infty(\mathbb{R}^n)$.

Proof. Let $\varphi \in \mathcal{S}_\infty(\mathbb{R}^n)$. Lemma 4.7 shows $f_+ := \sum_{j=1}^{\infty}\sum_{k\in\mathbb{Z}^n}\lambda_{jk}\mathfrak{M}_{jk}$ converges in $\mathcal{S}'_\infty(\mathbb{R}^n)$. So we need to prove $f_- := \sum_{j=-\infty}^{0}\sum_{k\in\mathbb{Z}^n}\lambda_{jk}\mathfrak{M}_{jk}$ converges in $\mathcal{S}'_\infty(\mathbb{R}^n)$.

Let $M \gg 1$. From Lemma 2.10, the definition of molecules and the fact that $\varphi \in \mathcal{S}_\infty(\mathbb{R}^n)$, it follows that, for all $j \le 0$ and $k \in \mathbb{Z}^n$,

$$\left|\int_{\mathbb{R}^n}\mathfrak{M}_{jk}(x)\varphi(x)\,dx\right| \lesssim 2^{j(M+1)}(1+2^{-j}|k|)^{-N}.$$

By ($\mathcal{L}6$), we conclude that

$$\left|\int_{\mathbb{R}^n}\mathfrak{M}_{jk}(x)\varphi(x)\,dx\right| \lesssim 2^{j(M+1-\gamma)}(1+2^{-j}|k|)^{-N}(1+|k|)^{\delta_0}\|\chi_{Q_{jk}}\|_{\mathcal{L}(\mathbb{R}^n)}.$$

Consequently,

$$\left|\lambda_{jk}\int_{\mathbb{R}^n}\mathfrak{M}_{jk}(x)\varphi(x)\,dx\right| \lesssim 2^{j(M+1-\gamma-\alpha_1)}(1+|k|)^{-N+\alpha_3+\delta_0}\|\lambda\|_{b^{w,\tau}_{\mathcal{L},\infty,a}(\mathbb{R}^n)}.$$

By the assumption, this inequality is summable over $j \le 0$ and $k \in \mathbb{Z}^n$, which completes the proof. ■

The homogeneous version of Theorem 4.9, which is the regular case of decompositions, is given below; its proof is similar to that of Theorem 4.9. We omit the details.

THEOREM 10.15. *Let* $K \in \mathbb{Z}_+$, $L = -1$, $\alpha_1, \alpha_2, \alpha_3, \tau \in [0,\infty)$ *and* $q \in (0,\infty]$. *Suppose that* $w \in \star\text{-}\dot{\mathcal{W}}^{\alpha_3}_{\alpha_1,\alpha_2}$. *Assume, in addition, that* (3.28), (4.2), (4.22) *and* (4.23) *hold, namely,* $a \in (N_0+\alpha_3,\infty)$. *Then* $(\dot{A}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n), \dot{a}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n))$ *admits atomic/molecular decompositions.*

10.4. Boundedness of operators. We first focus on the counterpart of Theorem 6.5. To this end, for $\ell \in \mathbb{N}$ and $\alpha \in \mathbb{R}$, let $m \in C^\ell(\mathbb{R}^n\backslash\{0\})$ be such that, for all $\|\sigma\|_1 \le \ell$,

$$\sup_{R\in(0,\infty)}\left[R^{-n+2\alpha+2\|\sigma\|_1}\int_{R\le|\xi|<2R}|\partial_\xi^\sigma m(\xi)|^2\,d\xi\right] \le A_\sigma < \infty. \tag{10.6}$$

The *Fourier multiplier* T_m is defined by setting, for all $f \in \mathcal{S}_\infty(\mathbb{R}^n)$, $\widehat{T_m f} := m\widehat{f}$.

We remark that when $\alpha = 0$, the condition (10.6) is just the classical Hörmander condition (see, for example, [88, p. 263]). A typical example satisfying (10.6) with $\alpha = 0$ is the kernel of the Riesz transform R_j given by $\widehat{R_j f}(\xi) := -i\frac{\xi_i}{|\xi|}\widehat{f}(\xi)$ for all $\xi \in \mathbb{R}^n \setminus \{0\}$ and $j \in \{1,\dots,n\}$. When $\alpha \ne 0$, a typical example satisfying (10.6) for any $\ell \in \mathbb{N}$ is given by $m(\xi) := |\xi|^{-\alpha}$ for $\xi \in \mathbb{R}^n \setminus \{0\}$; another example is the symbol of the differential operator ∂^σ of order $\alpha := \sigma_1 + \cdots + \sigma_n$ with $\sigma := (\sigma_1,\dots,\sigma_n) \in \mathbb{Z}^n_+$.

It was proved in [102] that the Fourier multiplier T_m is bounded on some Besov-type and Triebel–Lizorkin-type spaces for suitable indices.

Let m be as in (10.6) and K its inverse Fourier transform. To obtain the boundedness of T_m on $\dot{B}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$ and $\dot{F}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$, we need [102, Lemma 3.1]:

Lemma 10.16. $K \in \mathcal{S}'_\infty(\mathbb{R}^n)$.

The next lemma is [4, Lemma 4.1]; see also [102, Lemma 3.2].

Lemma 10.17. *Let ψ be a Schwartz function on $\mathbb{R}^n$ satisfying* (1.4). *Assume that m satisfies* (10.6). *If $a \in (0,\infty)$ and $\ell > a + n/2$, then there exists a positive constant C such that, for all $j \in \mathbb{Z}$,*

$$\int_{\mathbb{R}^n} (1+2^j|z|)^a |(K * \psi_j)(z)|\, dz \le C 2^{-j\alpha}.$$

Next we show that, in a suitable way, T_m can also be defined on the whole spaces $\dot{F}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$ and $\dot{B}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$. Let φ be a Schwartz function on $\mathbb{R}^n$ satisfy (1.4). Then there exists $\varphi^\dagger \in \mathcal{S}(\mathbb{R}^n)$ satisfying (1.4) such that

$$\sum_{i\in\mathbb{Z}} \varphi_i^\dagger * \varphi_i = \delta_0 \tag{10.7}$$

in $\mathcal{S}'_\infty(\mathbb{R}^n)$. For any $f \in \dot{F}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$ or $\dot{B}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$, we define a linear functional $T_m f$ on $\mathcal{S}_\infty(\mathbb{R}^n)$ by setting, for all $\phi \in \mathcal{S}_\infty(\mathbb{R}^n)$,

$$\langle T_m f, \phi\rangle := \sum_{i\in\mathbb{Z}} f * \varphi_i^\dagger * \varphi_i * \phi * K(0) \tag{10.8}$$

as long as the right-hand side converges. In this sense, we say $T_m f \in \mathcal{S}'_\infty(\mathbb{R}^n)$. The following result shows that $T_m f$ in (10.8) is well defined.

Lemma 10.18. *Let $\ell \in (n/2,\infty)$, $\alpha \in \mathbb{R}$, $a \in (0,\infty)$, $\alpha_1, \alpha_2, \alpha_3, \tau \in [0,\infty)$, $q \in (0,\infty]$, $w \in \dot{\mathcal{W}}^{\alpha_3}_{\alpha_1,\alpha_2}$ and $f \in \dot{F}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$ or $\dot{B}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$. Then the series in* (10.8) *is convergent and the sum on the right-hand side of* (10.8) *is independent of the choice of the pair $(\varphi^\dagger, \varphi)$. Moreover, $T_m f \in \mathcal{S}'_\infty(\mathbb{R}^n)$.*

Proof. By similarity, we only consider $f \in \dot{F}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$. Let $(\psi^\dagger, \psi)$ be another pair satisfying (10.7). Since $\phi \in \mathcal{S}_\infty(\mathbb{R}^n)$, by the Calderón reproducing formula we have

$$\phi = \sum_{j\in\mathbb{Z}} \psi_j^\dagger * \psi_j * \phi$$

in $\mathcal{S}_\infty(\mathbb{R}^n)$. Thus,

$$\begin{aligned}\sum_{i\in\mathbb{Z}} f * \varphi_i^\dagger * \varphi_i * \phi * K(0) &= \sum_{i\in\mathbb{Z}} f * \varphi_i^\dagger * \varphi_i * \Big(\sum_{j\in\mathbb{Z}} \psi_j^\dagger * \psi_j * \phi\Big) * K(0)\\ &= \sum_{i\in\mathbb{Z}} \sum_{j=i-1}^{i+1} f * \varphi_i^\dagger * \varphi_i * \psi_j^\dagger * \psi_j * \phi * K(0),\end{aligned}$$

where the last equality follows from the fact that $\varphi_i * \psi_j = 0$ if $|i-j| \ge 2$.

Similar to the argument in Lemma 6.3, we see that

$$\sum_{i=0}^{\infty} |f * \varphi_i * \varphi_i^\dagger * \psi_i * \psi_i^\dagger * \phi * K(0)| \lesssim \|f\|_{\dot{F}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)},$$

where a is an arbitrary positive number. When $i<0$, notice that, for all $z\in\mathbb{R}^n$,

$$\begin{aligned}
&\int_{\mathbb{R}^n}|\varphi_i*f(y-z)|\,|\varphi_i(-y)|\,dy\\
&\qquad\lesssim\sum_{k\in\mathbb{Z}^n}\frac{2^{in}}{(1+2^i|2^{-i}k|)^a}\int_{Q_{ik}}|\varphi_i*f(y-z)|\,dy\\
&\qquad\lesssim\sum_{k\in\mathbb{Z}^n}\frac{2^{in-i\alpha_1}(1+2^i|z|)^{\alpha_3}}{(1+2^i|2^{-i}k|)^{a-\alpha_3}}\inf_{y\in Q_{ik}}\omega(y-z,2^{-i})\int_{Q_{ik}}|\varphi_i*f(y-z)|\,dy\\
&\qquad\lesssim\sum_{k\in\mathbb{Z}^n}\frac{2^{-i\alpha_1}(1+2^i|z|)^{\alpha_3}}{(1+2^i|2^{-i}k|)^{a-\alpha_3}}\inf_{y\in Q_{ik}}\{\omega(y-z,2^{-i})|\varphi_i^*f(y-z)|\}\\
&\qquad\lesssim 2^{in-i\alpha_1}(1+2^i|z|)^{\alpha_3}2^{-in\tau}\|f\|_{\dot A^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)},
\end{aligned}$$

which, together with the fact that, for M sufficiently large and all $y,z\in\mathbb{R}^n$,

$$|\psi_i*\phi(y-z)|\lesssim 2^{iM}\frac{2^{in}}{(1+2^i|y-z|)^{n+M}},$$

and Lemma 10.17, further implies that

$$\begin{aligned}
&\sum_{i<0}|f*\varphi_i*\varphi_i^\dagger*\psi_i*\psi_i^\dagger*\phi*K(0)|\\
&\qquad=\sum_{i<0}\int_{\mathbb{R}^n}|f*\varphi_i*\varphi_i^\dagger(-z)\psi_i*\psi_i^\dagger*\phi*K(z)|\,dz\\
&\qquad\lesssim\sum_{i<0}2^{in-i\alpha_1}2^{-in\tau}\|f\|_{\dot F^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)}\int_{\mathbb{R}^n}(1+2^i|z|)^{\alpha_3}|\psi_i*\psi_i^\dagger*\phi*K(z)|\,dz\\
&\qquad\lesssim\sum_{i<0}2^{in-i\alpha_1}2^{iM}2^{-in\tau}\|f\|_{\dot F^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{2^{in}(1+2^i|z|)^{\alpha_3}}{(1+2^i|y-z|)^{n+M}}|\psi_i^\dagger*K(y)|\,dy\,dz\\
&\qquad\lesssim\sum_{i<0}2^{2in+iM-i\alpha_1}\|f\|_{\dot F^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)}\lesssim\|f\|_{\dot F^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)},
\end{aligned}$$

where we have chosen $M>\alpha_1-2n$.

Similar to the previous arguments, we see that

$$\Big|\sum_{i\in\mathbb{Z}}\sum_{j=i-1}^{i+1}f*\varphi_i^\dagger*\varphi_i*\psi_j^\dagger*\psi_j*\phi*K(0)\Big|\lesssim\|f\|_{\dot F^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)}.$$

Thus, T_mf in (10.8) is independent of the choice of $(\varphi^\dagger,\varphi)$. Moreover, the previous argument also implies that $T_mf\in\mathcal{S}'_\infty(\mathbb{R}^n)$, which completes the proof. ∎

Next, Lemma 10.17 immediately yields the following result; we omit the details.

LEMMA 10.19. *Let $\alpha\in\mathbb{R}$, $a\in(0,\infty)$, $\ell\in\mathbb{N}$ and φ, $\psi\in\mathcal{S}_\infty(\mathbb{R}^n)$ satisfy* (1.4). *Assume that m satisfies* (10.6) *and $f\in\mathcal{S}'_\infty(\mathbb{R}^n)$ is such that $T_mf\in\mathcal{S}'_\infty(\mathbb{R}^n)$. If $\ell>a+n/2$, then there exists a positive constant C such that, for all $x,y\in\mathbb{R}^n$ and $j\in\mathbb{Z}$,*

$$|(T_mf*\psi_j)(y)|\le C2^{-j\alpha}(1+2^j|x-y|)^a(\varphi_j^*f)_a(x).$$

THEOREM 10.20. *Let $\alpha \in \mathbb{R}$, $a \in (0,\infty)$, $\alpha_1, \alpha_2, \alpha_3, \tau \in [0,\infty)$, $q \in (0, \infty]$, $w \in \dot{\mathcal{W}}^{\alpha_3}_{\alpha_1,\alpha_2}$ and $\widetilde{w}(x,2^{-j}) = 2^{j\alpha} w(x,2^{-j})$ for all $x \in \mathbb{R}^n$ and $j \in \mathbb{Z}$. Suppose that m satisfies* (10.6) *with $\ell \in \mathbb{N}$ and $\ell > a + n/2$. Then there exists a positive constant C_1 such that, for all $f \in \dot{F}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$, $\|T_m f\|_{\dot{F}^{\widetilde{w},\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} \le C_1 \|f\|_{\dot{F}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)}$ and a positive constant C_2 such that, for all $f \in \dot{B}^{\widetilde{w},\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$, $\|T_m f\|_{\dot{B}^{\widetilde{w},\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)} \le C_2 \|f\|_{\dot{B}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)}$. Similar assertions hold for $\dot{\mathcal{E}}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$ and $\dot{\mathcal{N}}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$.*

Proof. By Lemma 10.19, we see that, if $\ell > a + n/2$, then for all $j \in \mathbb{Z}$ and $x \in \mathbb{R}^n$,

$$2^{j\alpha}(\psi_j^*(T_m f))_a(x) \lesssim (\varphi_j^* f)_a(x).$$

Then the definitions of $\dot{F}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$ and $\dot{B}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$ immediately yield the desired conclusions. ■

The following analogue to Theorem 3.10 can be proven similarly. We omit the details.

THEOREM 10.21. *Let $s \in [0,\infty)$, $a > \alpha_3 + N_0$, $\alpha_1, \alpha_2, \alpha_3, \tau \in [0,\infty)$, $q \in (0,\infty]$ and $w \in \dot{\mathcal{W}}^{\alpha_3}_{\alpha_1,\alpha_2}$. Set $w^*(x,2^{-j}) := 2^{js} w_j(x)$ for all $x \in \mathbb{R}^n$ and $j \in \mathbb{Z}$. Then the lift operator $(-\Delta)^{s/2}$ is bounded from $\dot{A}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$ to $\dot{A}^{w^*,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$.*

We consider the class $\dot{S}^0_{1,\mu}(\mathbb{R}^n)$ with $\mu \in [0,1)$. Recall that a $C^\infty(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$-function a is said to belong to the class $\dot{S}^m_{1,\mu}(\mathbb{R}^n)$ if

$$\sup_{x,\xi \in \mathbb{R}^n} |\xi|^{-m-\|\vec{\alpha}\|_1 - \mu\|\vec{\beta}\|_1} |\partial_x^{\vec{\beta}} \partial_\xi^{\vec{\alpha}} a(x,\xi)| \lesssim_{\vec{\alpha},\vec{\beta}} 1$$

for all multiindices $\vec{\alpha}$ and $\vec{\beta}$. One defines

$$a(X,D)(f)(x) := \int_{\mathbb{R}^n} a(x,\xi) \hat{f}(\xi) e^{ix\cdot\xi}\, d\xi$$

for all $f \in \mathcal{S}_\infty(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$. Theorem 6.6 has the following counterpart, whose proof is similar and omitted.

THEOREM 10.22. *Let $w \in \dot{\mathcal{W}}^{\alpha_3}_{\alpha_1,\alpha_2}$ with $\alpha_1, \alpha_2, \alpha_3 \in [0,\infty)$ and let a quasi-normed function space $\mathcal{L}(\mathbb{R}^n)$ satisfy ($\mathcal{L}$1) through ($\mathcal{L}$6). Let $\mu \in [0,1)$, $\tau \in (0,\infty)$ and $q \in (0,\infty]$. Assume, in addition, that* (3.28) *holds, that is, $a \in (N_0 + \alpha_3, \infty)$, where N_0 is as in ($\mathcal{L}$6). Then pseudo-differential operators with symbol in $\dot{S}^0_{1,\mu}(\mathbb{R}^n)$ are bounded on $\dot{A}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$.*

10.5. Function spaces $\dot{A}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$ for τ large. Now we have the following counterpart for Theorem 7.2.

THEOREM 10.23. *Let $\omega \in \dot{\mathcal{W}}^{\alpha_3}_{\alpha_1,\alpha_2}$ with $\alpha_1, \alpha_2, \alpha_3 \ge 0$. Define a new index $\widetilde{\tau}$ by*

$$\widetilde{\tau} := \limsup_{j\to\infty} \sup_{P \in \mathcal{Q}_j(\mathbb{R}^n)} \left[\frac{1}{nj} \log_2 \frac{1}{\|\chi_P\|_{\mathcal{L}(\mathbb{R}^n)}}\right]$$

and a new weight $\widetilde{\omega}$ by

$$\widetilde{\omega}(x,2^{-j}) := 2^{jn(\tau-\widetilde{\tau})} \omega(x,2^{-j}), \quad x \in \mathbb{R}^n,\ j \in \mathbb{Z}.$$

Assume that $\tau > \widetilde{\tau} \ge 0$. Then

(i) $\widetilde{w} \in \dot{\mathcal{W}}^{\alpha_3}_{(\alpha_1 - n(\tau-\widetilde{\tau}))_+,(\alpha_2 + n(\tau-\widetilde{\tau}))_+}$;

(ii) *for all* $q\in(0,\infty)$ *and* $a>\alpha_3+N_0$, $\dot F^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$ *and* $\dot B^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)$ *coincide, respectively, with* $\dot F^{\widetilde{w}}_{\infty,\infty,a}(\mathbb{R}^n)$ *and* $\dot B^{\widetilde{w}}_{\infty,\infty,a}(\mathbb{R}^n)$ *with equivalent norms.*

10.6. Characterizations via differences and oscillations. We can extend Theorems 8.2 and 8.6 to homogeneous spaces as follows; the proofs are omitted.

THEOREM 10.24. *Let* $a,\alpha_1,\alpha_2,\alpha_3,\tau\in[0,\infty)$, $u\in[1,\infty]$, $q\in(0,\infty]$ *and* $w\in\star\text{-}\mathcal{W}^{\alpha_3}_{\alpha_1,\alpha_2}$. *If* $M\in\mathbb{N}$, $\alpha_1\in(a,M)$ *and* (8.2) *holds, then there exists a positive constant* $\widetilde{C}:=C(M)$ *such that, for all* $f\in\mathcal{S}'_\infty(\mathbb{R}^n)\cap L^1_{\rm loc}(\mathbb{R}^n)$, *the following hold with the implicit constants independent of* f*:*

(i)
$$\left\|\left\{\sup_{z\in\mathbb{R}^n}\left[\fint_{|h|\le\widetilde{C}\,2^{-j}}\frac{|\Delta_h^M f(\cdot+z)|^u}{(1+2^j|z|)^{au}}\,dh\right]^{1/u}\right\}_{j\in\mathbb{Z}}\right\|_{\ell^q(\mathcal{L}^w_\tau(\mathbb{R}^n,\mathbb{Z}))}\sim\|f\|_{\dot B^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)},$$

(ii)
$$\left\|\left\{\sup_{z\in\mathbb{R}^n}\left[\fint_{|h|\le\widetilde{C}\,2^{-j}}\frac{|\Delta_h^M f(\cdot+z)|^u}{(1+2^j|z|)^{au}}\,dh\right]^{1/u}\right\}_{j\in\mathbb{Z}}\right\|_{\mathcal{L}^w_\tau(\ell^q(\mathbb{R}^n,\mathbb{Z}))}\sim\|f\|_{\dot F^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)},$$

(iii)
$$\left\|\left\{\sup_{z\in\mathbb{R}^n}\left[\fint_{|h|\le\widetilde{C}\,2^{-j}}\frac{|\Delta_h^M f(\cdot+z)|^u}{(1+2^j|z|)^{au}}\,dh\right]^{1/u}\right\}_{j\in\mathbb{Z}}\right\|_{\ell^q(\mathcal{N}\mathcal{L}^w_\tau(\mathbb{R}^n,\mathbb{Z}))}\sim\|f\|_{\dot{\mathcal{N}}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)},$$

(iv)
$$\left\|\left\{\sup_{z\in\mathbb{R}^n}\left[\fint_{|h|\le\widetilde{C}\,2^{-j}}\frac{|\Delta_h^M f(\cdot+z)|^u}{(1+2^j|z|)^{au}}\,dh\right]^{1/u}\right\}_{j\in\mathbb{Z}}\right\|_{\mathcal{E}\mathcal{L}^w_\tau(\ell^q(\mathbb{R}^n,\mathbb{Z}))}\sim\|f\|_{\dot{\mathcal{E}}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)}.$$

THEOREM 10.25. *Let* $a,\alpha_1,\alpha_2,\alpha_3,\tau\in[0,\infty)$, $u\in[1,\infty]$, $q\in(0,\infty]$ *and* $w\in\star\text{-}\dot{\mathcal{W}}^{\alpha_3}_{\alpha_1,\alpha_2}$. *If* $M\in\mathbb{N}$, $\alpha_1\in(a,M)$ *and* (8.2) *holds, then, for all* $f\in\mathcal{S}'_\infty(\mathbb{R}^n)\cap L^1_{\rm loc}(\mathbb{R}^n)$, *the following hold with the implicit constants independent of* f*:*

(i)
$$\left\|\left\{\sup_{z\in\mathbb{R}^n}\frac{\operatorname{osc}^M_u f(\cdot+z,2^{-j})}{(1+2^j|z|)^a}\right\}_{j\in\mathbb{Z}}\right\|_{\ell^q(\mathcal{L}^w_\tau(\mathbb{R}^n,\mathbb{Z}))}\sim\|f\|_{\dot B^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)},$$

(ii)
$$\left\|\left\{\sup_{z\in\mathbb{R}^n}\frac{\operatorname{osc}^M_u f(\cdot+z,2^{-j})}{(1+2^j|z|)^a}\right\}_{j\in\mathbb{Z}}\right\|_{\mathcal{L}^w_\tau(\ell^q(\mathbb{R}^n,\mathbb{Z}))}\sim\|f\|_{\dot F^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)},$$

(iii)
$$\left\|\left\{\sup_{z\in\mathbb{R}^n}\frac{\operatorname{osc}^M_u f(\cdot+z,2^{-j})}{(1+2^j|z|)^a}\right\}_{j\in\mathbb{Z}}\right\|_{\ell^q(\mathcal{N}\mathcal{L}^w_\tau(\mathbb{R}^n,\mathbb{Z}))}\sim\|f\|_{\dot{\mathcal{N}}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)},$$

(iv)
$$\left\|\left\{\sup_{z\in\mathbb{R}^n}\frac{\operatorname{osc}^M_u f(\cdot+z,2^{-j})}{(1+2^j|z|)^a}\right\}_{j\in\mathbb{Z}}\right\|_{\mathcal{E}\mathcal{L}^w_\tau(\ell^q(\mathbb{R}^n,\mathbb{Z}))}\sim\|f\|_{\dot{\mathcal{E}}^{w,\tau}_{\mathcal{L},q,a}(\mathbb{R}^n)}.$$

Next, we transplant Theorems 9.6 and 9.8 to the homogeneous case. Again, since the proofs are similar to the respective inhomogeneous cases, we omit the details.

THEOREM 10.26. *Suppose that* $a>N$ *and that* (9.19) *is satisfied:*

$$(1+|\cdot|)^{-N}\in\mathcal{L}(\mathbb{R}^n)\cap\mathcal{L}'(\mathbb{R}^n).$$

Assume, in addition, that there exists a positive constant C *such that, for any finite sequence* $\{\varepsilon_k\}_{k=-k_0}^{k_0}$ *taking values* $\{-1,1\}$,

$$\left\|\sum_{k=-k_0}^{k_0}\varepsilon_k\varphi_k*f\right\|_{\mathcal{L}(\mathbb{R}^n)}\le C\|f\|_{\mathcal{L}(\mathbb{R}^n)},\quad\left\|\sum_{k=-k_0}^{k_0}\varepsilon_k\varphi_k*g\right\|_{\mathcal{L}'(\mathbb{R}^n)}\le C\|g\|_{\mathcal{L}'(\mathbb{R}^n)}\tag{10.9}$$

for all $f \in \mathcal{L}(\mathbb{R}^n)$ *and* $g \in \mathcal{L}'(\mathbb{R}^n)$. *Then* $\mathcal{L}(\mathbb{R}^n)$ *and* $\mathcal{L}'(\mathbb{R}^n)$ *are embedded into* $\mathcal{S}'_\infty(\mathbb{R}^n)$, *and* $\mathcal{L}(\mathbb{R}^n)$ *and* $\dot{\mathcal{E}}^{0,0}_{\mathcal{L},2,a}(\mathbb{R}^n)$ *coincide.*

THEOREM 10.27. *Let* $\mathcal{L}(\mathbb{R}^n)$ *be a Banach space of functions such that* $\mathcal{L}^p(\mathbb{R}^n)$ *and* $(\mathcal{L}')^p(\mathbb{R}^n)$ *are Banach spaces of functions and the maximal operator* M *is bounded on* $(\mathcal{L}^p(\mathbb{R}^n))'$ *and on* $((\mathcal{L}')^p(\mathbb{R}^n))'$ *for some* $p \in (1,\infty)$. *Then* (10.9) *holds. In particular, if* $a > N$ *and* $(1+|\cdot|)^{-N} \in \mathcal{L}(\mathbb{R}^n) \cap \mathcal{L}'(\mathbb{R}^n)$, *then* $\mathcal{L}(\mathbb{R}^n)$ *and* $\mathcal{L}'(\mathbb{R}^n)$ *are embedded into* $\mathcal{S}'_\infty(\mathbb{R}^n)$, *and* $\mathcal{L}(\mathbb{R}^n)$ *and* $\dot{\mathcal{E}}^{0,0}_{\mathcal{L},2,a}(\mathbb{R}^n)$ *coincide.*

As a corollary, $\mathcal{L}(\mathbb{R}^n)$ enjoys the following characterization.

COROLLARY 10.28. *Let* $\mathcal{L}(\mathbb{R}^n)$ *be a Banach space of functions such that* $\mathcal{L}^p(\mathbb{R}^n)$ *and* $(\mathcal{L}')^p(\mathbb{R}^n)$ *are Banach spaces of functions and the maximal operator* M *is bounded on* $(\mathcal{L}^p(\mathbb{R}^n))'$ *and on* $((\mathcal{L}')^p(\mathbb{R}^n))'$ *for some* $p \in (1,\infty)$. *If* $a > N$ *and* $(1+|\cdot|)^{-N} \in \mathcal{L}(\mathbb{R}^n) \cap \mathcal{L}'(\mathbb{R}^n)$, *then*

$$\begin{aligned}\|f\|_{\mathcal{L}(\mathbb{R}^n)} &\sim \left\| \left\{ \sup_{z\in\mathbb{R}^n} \left[⨏_{|h|\le \widetilde{C}\,2^{-j}} \frac{|\Delta_h^M f(\cdot+z)|^u}{(1+2^j|z|)^{au}}\,dh \right]^{1/u} \right\}_{j\in\mathbb{Z}} \right\|_{\mathcal{E}\mathcal{L}^1_0(\ell^2(\mathbb{R}^n,\mathbb{Z}))} \\ &\sim \left\| \left\{ \sup_{z\in\mathbb{R}^n} \frac{\operatorname{osc}_u^M f(\cdot+z,2^{-j})}{(1+2^j|z|)^a} \right\}_{j\in\mathbb{Z}} \right\|_{\mathcal{E}\mathcal{L}^1_0(\ell^2(\mathbb{R}^n,\mathbb{Z}))}\end{aligned}$$

with the implicit constants independent of $f \in \mathcal{L}(\mathbb{R}^n)$.

11. Applications and examples

Now we present some examples for $\mathcal{L}(\mathbb{R}^n)$ and survey what has been obtained recently.

11.1. Weighted Lebesgue spaces. Let ρ be a weight and $p \in (0,\infty)$. We let $L^p(\rho)$ denote the set of all Lebesgue measurable functions f for which the norm

$$\|f\|_{L^p(\rho)} := \left[\int_{\mathbb{R}^n} |f(x)|^p \rho(x)\,dx\right]^{1/p}$$

is finite. Assume that $(1+|\cdot|)^{-N_0} \in L^p(\rho)$ for some $N_0 \in (0,\infty)$ and the estimate

$$\|\chi_{Q_{jk}}\|_{L^p(\rho)} = \|\chi_{2^{-j}k+2^{-j}[0,1)^n}\|_{L^p(\rho)} \gtrsim 2^{-j\gamma}(1+|k|)^{-\delta}, \quad j \in \mathbb{Z}_+,\ k \in \mathbb{Z}^n \tag{11.1}$$

holds for some $\gamma, \delta \in [0,\infty)$, where the implicit constant is independent of j and k. The space $L^p(\rho)$ is referred to as the *weighted Lebesgue space.*

In this example, N_0 and γ, δ are included in (11.1). The assumption (3.2) actually reads

$$\mathcal{L}(\mathbb{R}^n) := L^p(\rho), \quad \theta := \min\{1,p\},$$

and $\mathcal{L}(\mathbb{R}^n)$ satisfies ($\mathcal{L}1$) through ($\mathcal{L}6$). Notice that if ρ satisfies

$$\rho(x+y) \lesssim (1+|y|)^M \rho(y) \quad \text{for all } x, y \in \mathbb{R}^n,$$

then ρ satisfies (9.17), and if $\rho \in A_\infty(\mathbb{R}^n) = \bigcup_{1\le u<\infty} A_u(\mathbb{R}^n)$, then ρ satisfies ($\mathcal{L}8$). Moreover (3.3) actually reads

$$w_j(x) := 1 \quad \text{for all } x \in \mathbb{R}^n \text{ and } j \in \mathbb{Z}_+, \qquad \alpha_1 = \alpha_2 = \alpha_3 = 0.$$

Hence, (3.4) is replaced by $\tau \in [0,\infty)$, $q \in (0,\infty]$, $a > N_0$.

11.2. Morrey spaces

Morrey spaces. To begin, we consider the case when $\mathcal{L}(\mathbb{R}^n) := \mathcal{M}^p_u(\mathbb{R}^n)$, the Morrey space. Recall that the definition was given in Example 5.5. Besov–Morrey spaces and Triebel–Lizorkin–Morrey spaces are function spaces whose norms are obtained by replacing L^p-norms with Morrey norms. More precisely, the *Besov–Morrey norm* $\|\cdot\|_{\mathcal{N}^s_{pqr}(\mathbb{R}^n)}$ is given by

$$\|f\|_{\mathcal{N}^s_{pqr}(\mathbb{R}^n)} := \|\Phi * f\|_{\mathcal{M}^p_q(\mathbb{R}^n)} + \left[\sum_{j=1}^{\infty} 2^{jsr}\|\varphi_j * f\|^r_{\mathcal{M}^p_q(\mathbb{R}^n)}\right]^{1/r}$$

and the *Triebel–Lizorkin–Morrey norm* $\|\cdot\|_{\mathcal{E}^s_{pqr}(\mathbb{R}^n)}$ is given by

$$\|f\|_{\mathcal{E}^s_{pqr}(\mathbb{R}^n)} := \|\Phi * f\|_{\mathcal{M}^p_q(\mathbb{R}^n)} + \left\| \left(\sum_{j=1}^{\infty} 2^{jsr} |\varphi_j * f|^r \right)^{1/r} \right\|_{\mathcal{M}^p_q(\mathbb{R}^n)}$$

for $0 < q \le p < \infty$, $r \in (0, \infty]$ and $s \in \mathbb{R}$, where Φ and φ are, respectively, as in (1.3) and (1.4), and $\varphi_j(\cdot) = 2^{jn}\varphi(2^j \cdot)$ for all $j \in \mathbb{N}$. The spaces $\mathcal{N}^s_{pqr}(\mathbb{R}^n)$ and $\mathcal{E}^s_{pqr}(\mathbb{R}^n)$ are the sets of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that the norms $\|f\|_{\mathcal{N}^s_{pqr}(\mathbb{R}^n)}$ and $\|f\|_{\mathcal{E}^s_{pqr}(\mathbb{R}^n)}$ are finite, respectively. Let $\mathcal{A}^s_{pqr}(\mathbb{R}^n)$ denote either $\mathcal{N}^s_{pqr}(\mathbb{R}^n)$ or $\mathcal{E}^s_{pqr}(\mathbb{R}^n)$. Write

$$B^{w,\tau}_{p,u,q,a}(\mathbb{R}^n) := B^{w,\tau}_{\mathcal{M}^p_u,q,a}(\mathbb{R}^n) \quad \text{and} \quad F^{w,\tau}_{p,u,q,a}(\mathbb{R}^n) := F^{w,\tau}_{\mathcal{M}^p_u,q,a}(\mathbb{R}^n).$$

If we let $w_j(x) := 2^{js}$ ($x \in \mathbb{R}^n$, $j \in \mathbb{Z}_+$) with $s \in \mathbb{R}$, then it is easy to show that $\mathcal{N}^s_{p,u,q,a}(\mathbb{R}^n) := \mathcal{N}^{s,0}_{p,u,q,a}(\mathbb{R}^n)$ coincides with $\mathcal{N}^s_{puq}(\mathbb{R}^n)$ when $a > n/\min(1,u)$, and that $F^s_{p,u,q,a}(\mathbb{R}^n) := F^{s,0}_{p,u,q,a}(\mathbb{R}^n)$ coincides with $\mathcal{E}^s_{puq}(\mathbb{R}^n)$ when $a > n/\min(1,u,q)$. Indeed, this is just a matter of applying the Plancherel–Pólya–Nikol'skiĭ inequality (Lemma 1.1) and the maximal inequalities obtained in [80, 89]. These function spaces are dealt with in [80, 89].

Observe that ($\mathcal{L}1$) through ($\mathcal{L}6$) hold in this case.

There exists another point of view on these function spaces. Recall that the function space $A^{s,\tau}_{p,q}(\mathbb{R}^n)$, defined by (3.1), originated from [97, 98, 99]. The following is known, which is extended in our Theorem 9.12.

Proposition 11.1 ([104, Theorem 1.1]). *Let $s \in \mathbb{R}$.*

(i) *If $0<p<u<\infty$ and $q \in (0,\infty)$, then $\mathcal{N}^s_{upq}(\mathbb{R}^n)$ is a proper subset of $B^{s,1/p-1/u}_{p,q}(\mathbb{R}^n)$.*

(ii) *If $0 < p < u < \infty$ and $q = \infty$, then $\mathcal{N}^s_{upq}(\mathbb{R}^n) = B^{s,1/p-1/u}_{p,q}(\mathbb{R}^n)$ with equivalent norms.*

(iii) *If $0 < p \le u < \infty$ and $q \in (0,\infty]$, then $\mathcal{E}^s_{upq}(\mathbb{R}^n) = F^{s,1/p-1/u}_{p,q}(\mathbb{R}^n)$ with equivalent norms.*

An analogue for homogeneous spaces is also true.

Other related spaces are inhomogeneous *Hardy–Morrey spaces* $h\mathcal{M}^p_q(\mathbb{R}^n)$, whose norm is given by

$$\|f\|_{h\mathcal{M}^p_q(\mathbb{R}^n)} := \left\| \sup_{0<t\le 1} |t^{-n}\Phi(t^{-1}\cdot) * f| \right\|_{\mathcal{M}^p_q(\mathbb{R}^n)}$$

for all $f \in \mathcal{S}'(\mathbb{R}^n)$ and $0 < q \le p < \infty$, where Φ is as in (1.3).

Now in this example (3.2) actually reads

$$\mathcal{L}(\mathbb{R}^n) := \mathcal{M}^p_q(\mathbb{R}^n), \quad \theta := \min\{1,q\}, \quad N_0 := n/p + 1, \quad \gamma := n/p, \quad \delta := 0,$$

and $\mathcal{L}(\mathbb{R}^n)$ satisfies ($\mathcal{L}1$) through ($\mathcal{L}6$) and ($\mathcal{L}8$) (see [79, 89]). Moreover (3.3) reads

$$w_j(x) := 1 \quad \text{for all } x \in \mathbb{R}^n \text{ and } j \in \mathbb{Z}_+, \quad \alpha_1 = \alpha_2 = \alpha_3 = 0.$$

Hence, (3.4) is replaced by

$$\tau \in [0,\infty), \qquad q \in (0,\infty], \qquad a > n/p + 1.$$

We refer to [32, 33, 43, 74, 75, 80, 83] for more details and applications of Hardy–Morrey spaces, Besov–Morrey spaces and Triebel–Lizorkin–Morrey spaces. Indeed, in [43, 74, 80], Besov–Morrey spaces and their applications are investigated; Triebel–Lizorkin–Morrey spaces are dealt with in [74, 75, 80]; Hardy–Morrey spaces are defined and considered in [32, 33, 75, 83] and Hardy–Morrey spaces are applied to PDE in [33]. We also refer to [30] for more related results about Besov–Morrey spaces and Triebel–Lizorkin–Morrey spaces, where weighted settings are covered.

Generalized Morrey spaces. We can also consider generalized Morrey spaces. Let $p \in (0,\infty)$ and $\phi : (0,\infty) \to (0,\infty)$ be a suitable function. For a function f locally in $L^p(\mathbb{R}^n)$, we set

$$\|f\|_{\mathcal{M}_{\phi,p}(\mathbb{R}^n)} := \sup_{Q \in \mathcal{Q}(\mathbb{R}^n)} \phi(\ell(Q)) \left[\frac{1}{|Q|}\int_Q |f(x)|^p\,dx\right]^{1/p},$$

where $\ell(Q)$ denotes the side length of the cube Q. The *generalized Morrey space* $\mathcal{M}_{\phi,p}(\mathbb{R}^n)$ is defined to be the space of all functions f locally in $L^p(\mathbb{R}^n)$ such that $\|f\|_{\mathcal{M}_{\phi,p}(\mathbb{R}^n)} < \infty$. Let $\mathcal{L}(\mathbb{R}^n) := \mathcal{M}_{\phi,p}(\mathbb{R}^n)$. Observe that ($\mathcal{L}1$) through ($\mathcal{L}6$) are true under a suitable condition on ϕ. At least ($\mathcal{L}1$) through ($\mathcal{L}5$) hold without assuming any condition on ϕ. Morrey–Campanato spaces with growth function ϕ were first introduced by Spanne [86, 87] and Peetre [67], which treat singular integrals of convolution type. In 1991, Mizuhara [54] studied the boundedness of the Hardy–Littlewood maximal operator on Morrey spaces with growth function ϕ. Later in 1994, Nakai [56] considered the boundedness of singular integrals (with non-convolution kernel), and fractional integral operators on Morrey spaces with growth function ϕ. In [58], Nakai defined the space $\mathcal{M}_{\phi,p}(\mathbb{R}^n)$. Later, this type of function space was used in [44, 56, 76]. We refer to [60] for more details. In [57, p. 445], Nakai proved the following (see [78, (10.6)] as well).

Proposition 11.2. *Let $p \in (0,\infty)$ and $\phi : (0,\infty) \to (0,\infty)$ be an arbitrary function. Then there exists a function $\phi^* : (0,\infty) \to (0,\infty)$ such that*

$$\phi^*(t) \text{ is nondecreasing and } [\phi^*(t)]^p t^{-n} \text{ is nonincreasing,} \tag{11.2}$$

and $\mathcal{M}_{\phi,p}(\mathbb{R}^n)$ and $\mathcal{M}_{\phi^*,p}(\mathbb{R}^n)$ coincide.

We rephrase ($\mathcal{L}8$) by using (11.2) as follows.

Proposition 11.3 ([73, Theorem 2.5]). *Suppose that $\phi : (0,\infty) \to (0,\infty)$ is an increasing function. Assume that $\phi : (0,\infty) \to (0,\infty)$ satisfies*

$$\int_r^\infty \phi(t)\,\frac{dt}{t} \sim \phi(r) \tag{11.3}$$

for all $r \in (0,\infty)$. Then, for all $u \in (1,\infty]$ and all sequences $\{f_j\}_{j=1}^\infty$ of measurable functions,

$$\Big\|\Big(\sum_{j=1}^\infty [Mf_j]^u\Big)^{1/u}\Big\|_{\mathcal{M}_{\phi,p}(\mathbb{R}^n)} \sim \Big\|\Big(\sum_{j=1}^\infty |f_j|^u\Big)^{1/u}\Big\|_{\mathcal{M}_{\phi,p}(\mathbb{R}^n)}$$

with the implicit constants independent of $\{f_j\}_{j=1}^\infty$.

Remark 11.4. In [73], it was actually assumed that

$$\int_r^\infty \phi(t)\,\frac{dt}{t} \lesssim \phi(r) \quad \text{for all } r \in (0,\infty). \tag{11.4}$$

However, under the assumption (11.2), the conditions (11.3) and (11.4) are mutually equivalent.

Now in this example, (3.2) reads

$$\mathcal{L}(\mathbb{R}^n) := \mathcal{M}_{\phi,p}(\mathbb{R}^n), \quad \theta := 1, \quad N_0 := n/p + 1, \quad \gamma := n/p, \quad \delta := 0$$

and $\mathcal{L}(\mathbb{R}^n)$ satisfies ($\mathcal{L}8$) by Proposition 11.3 and also ($\mathcal{L}1$) through ($\mathcal{L}6$). Moreover (3.3) reads

$$w_j(x) := 1 \quad \text{for all } x \in \mathbb{R}^n \text{ and } j \in \mathbb{Z}_+, \quad \alpha_1 = \alpha_2 = \alpha_3 = 0.$$

Hence, (3.4) is replaced by

$$\tau \in [0, \infty), \quad q \in (0, \infty], \quad a > n/p + 1.$$

11.3. Orlicz spaces. Recall the definition of Orlicz spaces given in Example 5.5. The proof of the following estimate can be found in [8].

LEMMA 11.5. *If a Young function* Φ *satisfies*

$$\text{(doubling condition)} \sup_{t>0} \frac{\Phi(2t)}{\Phi(t)} < \infty, \quad (\nabla_2\text{-condition}) \inf_{t>0} \frac{\Phi(2t)}{\Phi(t)} > 2,$$

then for all $u \in (1, \infty]$ *and all sequences* $\{f_j\}_{j=1}^\infty$ *of measurable functions,*

$$\Big\| \Big(\sum_{j=1}^\infty [M f_j]^u \Big)^{1/u} \Big\|_{L^\Phi(\mathbb{R}^n)} \sim \Big\| \Big(\sum_{j=1}^\infty |f_j|^u \Big)^{1/u} \Big\|_{L^\Phi(\mathbb{R}^n)} \tag{11.5}$$

with the implicit constants independent of $\{f_j\}_{j=1}^\infty$.

Thus, by Lemma 11.5, $L^\Phi(\mathbb{R}^n)$ satisfies ($\mathcal{L}8$). In this example $\mathcal{L}(\mathbb{R}^n) := L^\Phi(\mathbb{R}^n)$ also satisfies ($\mathcal{L}1$) through ($\mathcal{L}6$) with (3.2), and (3.3) reading

$$\mathcal{L}(\mathbb{R}^n) := L^\Phi(\mathbb{R}^n), \quad \theta := 1, \quad N_0 := n + 1, \quad \gamma := n, \quad \delta := 0.$$

Indeed, since Φ is a Young function, we have

$$\int_{\mathbb{R}^n} \Phi(2^{jn} \chi_{Q_{j0}}(x))\, dx = 2^{-jn} \Phi(2^{jn}) \geq 1.$$

Consequently, $\|\chi_{Q_{j0}}\|_{L^\Phi(\mathbb{R}^n)} \geq 2^{-jn}$. Moreover as before,

$$w_j(x) := 1 \text{ for all } x \in \mathbb{R}^n \text{ and } j \in \mathbb{Z}_+, \quad \alpha_1 = \alpha_2 = \alpha_3 = 0.$$

Hence (3.4) now reads

$$\tau \in [0, \infty), \quad q \in (0, \infty], \quad a > n + 1.$$

This example can be generalized somewhat. Given a Young function Φ, define the *mean Luxemburg norm* of f on a cube $Q \in \mathcal{Q}(\mathbb{R}^n)$ by

$$\|f\|_{\Phi,Q} := \inf\Big\{ \lambda > 0 : \frac{1}{|Q|} \int_Q \Phi\Big(\frac{|f(x)|}{\lambda} \Big)\, dx \leq 1 \Big\}.$$

When $\Phi(t) := t^p$ for all $t \in (0, \infty)$ with $p \in [1, \infty)$, we have

$$\|f\|_{\Phi,Q} = \Big[\frac{1}{|Q|} \int_Q |f(x)|^p\, dx \Big]^{1/p},$$

that is, the mean Luxemburg norm coincides with the (normalized) L^p norm. The *Orlicz–Morrey space* $\mathcal{L}^{\Phi,\phi}(\mathbb{R}^n)$ consists of all locally integrable functions f on $\mathbb{R}^n$ for which the norm

$$\|f\|_{\mathcal{L}^{\Phi,\phi}(\mathbb{R}^n)} := \sup_{Q\in\mathcal{Q}(\mathbb{R}^n)} \phi(\ell(Q))\|f\|_{\Phi,Q}$$

is finite. As stated in [77, Section 1], we can assume without loss of generality that the real functions $t \mapsto \phi(t)$ and $t \mapsto t^n\phi(t)^{-1}$ are both increasing on $(0,\infty)$.

Using [77, Proposition 2.17], we extend [37, 38] and [77, Proposition 2.17] to the vector-valued version. In the next proposition, we shall establish that ($\mathcal{L}8$) holds provided that

$$\int_1^t \Phi(t/s)\,ds \le \Phi(Ct) \quad (t\in(0,\infty))$$

for some positive constant C and for all $t\in(1,\infty)$.

PROPOSITION 11.6. *Let $q\in(0,\infty]$. Let Φ be a normalized Young function. Then the following are equivalent:*

(i) *The maximal operator M is locally bounded in the norm determined by Φ, that is, there exists a positive constant C such that, for all cubes $Q\in\mathcal{Q}(\mathbb{R}^n)$,*

$$\|M(g\chi_Q)\|_{\Phi,Q} \le C\|g\|_{\Phi,Q}.$$

(ii) *The function space $\mathcal{L}(\mathbb{R}^n) := \mathcal{L}^{\Phi,\phi}(\mathbb{R}^n)$ satisfies ($\mathcal{L}8$) with some $0<r<q$ and $w\equiv 1$. Namely, there exist $R\gg 1$ and $r\in(0,\infty)$ such that*

$$\|\{(\eta_{j,R}*|f_j|^r)^{1/r}\}_{j\in\mathbb{Z}_+}\|_{\mathcal{L}^{\Phi,\phi}(\ell^q(\mathbb{R}^n,\mathbb{Z}_+))} \lesssim \|\{f_j\}_{j\in\mathbb{Z}_+}\|_{\mathcal{L}^{\Phi,\phi}(\ell^q(\mathbb{R}^n,\mathbb{Z}_+))}$$

for all $\{f_j\}_{j\in\mathbb{N}}\subset\mathcal{L}^{\Phi,\phi}(\mathbb{R}^n)$, where the implicit constant is independent of $\{f_j\}_{j\in\mathbb{N}}$.

(iii) *For some positive constant C and all $t\in(1,\infty)$,*

$$\int_1^t \frac{t}{s}\Phi'(s)\,ds \le \Phi(Ct).$$

(iv) *For some positive constant C and all $t\in(1,\infty)$,*

$$\int_1^t \Phi(t/s)\,ds \le \Phi(Ct).$$

Therefore, a theory of Besov–Orlicz spaces and Triebel–Lizorkin–Orlicz spaces similar to the theory of Besov–Morrey spaces and Triebel–Lizorkin–Morrey spaces as in [32, 33, 43, 74, 75, 80, 83] can be developed as before.

Proof of Proposition 11.6. The proof is based upon a minor modification of known results. However, not having found the proof in the literature, we outline it here. In [77, Proposition 2.17] we have shown that (i), (iii) and (iv) are mutually equivalent. It is clear that (ii) implies (i). Therefore, we need to prove that (iv) implies (ii). In [77, Claim 5.1] we have also shown that the space $\mathcal{L}^{\Phi,\phi}(\mathbb{R}^n)$ remains the same if we change the value $\Phi(t)$ for $t\le 1$. Therefore, we can and do assume

$$\int_0^t \frac{t}{s}\Phi'(s)\,ds \le \Phi(Ct)$$

for all $t \in (0,\infty)$. Consequently,

$$\begin{aligned}
&\int_{\mathbb{R}^n} \Phi\Big(\Big\{\sum_{j=1}^{\infty}[M(|f_j|^r)(x)]^{q/r}\Big\}^{1/q}\Big)\,dx\\
&\qquad = \int_0^\infty \Phi'(t)\Big|\Big\{x\in\mathbb{R}^n : \Big(\sum_{j=1}^{\infty}[M(|f_j|^r)(x)]^{q/r}\Big)^{1/q} > t\Big\}\Big|\,dt\\
&\qquad \lesssim \int_{\mathbb{R}^n}\int_0^\infty \frac{\Phi'(t)}{t}\chi_{\{x\in\mathbb{R}^n\,:\,[\sum_{j=1}^\infty |f_j(x)|^q]^{1/q}>t/2\}}(x)\Big[\sum_{j=1}^{\infty}|f_j(x)|^q\Big]^{1/q}\,dt\\
&\qquad \lesssim \int_{\mathbb{R}^n}\Phi\Big(C_0\Big[\sum_{j=1}^{\infty}|f_j(x)|^q\Big]^{1/q}\Big)\,dx
\end{aligned}$$

for some positive constant C_0. This implies that whenever

$$\Big\|\Big[\sum_{j=1}^{\infty}|f_j|^q\Big]^{1/q}\Big\|_{L^{\Phi,\phi}(\mathbb{R}^n)} \le \frac{1}{C_0},$$

we have

$$\int_{\mathbb{R}^n}\Phi\Big(\Big\{\sum_{j=1}^{\infty}[M(|f_j|^r)(x)]^{q/r}\Big\}^{1/q}\Big)\,dx \le 1.$$

From the definition of the Orlicz norm $\|\cdot\|_{L^{\Phi,\phi}(\mathbb{R}^n)}$, we have (11.5). Once we obtain (11.5), we can go through the same argument as in [79, Theorem 2.4]. We omit the details. ∎

In this example, if we assume the conditions of Proposition 11.6, then ($\mathcal{L}1$) through ($\mathcal{L}6$) hold with (3.2) and (3.3) reading

$$\mathcal{L}(\mathbb{R}^n) := L^{\Phi,\phi}(\mathbb{R}^n), \quad \theta := 1, \quad N_0 := n+1, \quad \gamma := n, \quad \delta := 0.$$

Indeed, since Φ is a Young function, again we have

$$2^{jn}\int_{\mathbb{R}^n}\Phi(\chi_{Q_{j0}}(x)/\lambda)\,dx = \Phi(\lambda^{-1})$$

for $\lambda > 0$. Consequently, $\|\chi_{Q_{j0}}\|_{\Phi,Q_{j0}} = 1/\Phi^{-1}(1)$ and hence

$$\phi(2^{-j})\|\chi_{Q_{j0}}\|_{\Phi,Q_{j0}} = \phi(2^{-j}) = \phi(2^{-j})2^{jn}2^{-jn} \ge \phi(1)2^{-jn}.$$

Here we invoked the assumption that $\phi(t)t^{-n}$ is a decreasing function.

Since $\mathcal{L}^{\Phi,\phi}(\mathbb{R}^n)$ satisfies ($\mathcal{L}8$), we obtain $M\chi_{[-1,1]^n} \in \mathcal{L}^{\Phi,\phi}(\mathbb{R}^n)$, showing that $N_0 := n$ will do in this setting.

Moreover, as before,

$$w_j(x) := 1 \quad \text{for all } x\in\mathbb{R}^n \text{ and } j\in\mathbb{Z}_+, \quad \alpha_1=\alpha_2=\alpha_3=0.$$

Hence (3.4) now reads

$$\tau\in[0,\infty), \quad q\in(0,\infty], \quad a>n+1.$$

Finally, we remark that Orlicz spaces are examples to which the results in Subsection 9.2 apply.

11.4. Herz spaces. Let $p,q \in (0,\infty]$ and $\alpha \in \mathbb{R}$. We let $Q_0 := [-1,1]^n$ and $C_j := [-2^j,2^j]^n \setminus [-2^{j-1},2^{j-1}]^n$ for all $j \in \mathbb{N}$. Define the *inhomogeneous Herz space* $K^\alpha_{p,q}(\mathbb{R}^n)$ to be the set of all measurable functions f for which the norm

$$\|f\|_{K^\alpha_{p,q}(\mathbb{R}^n)} := \|\chi_{Q_0} f\|_{L^p(\mathbb{R}^n)} + \Big(\sum_{j=1}^\infty 2^{jq\alpha} \|\chi_{C_j} f\|^q_{L^p(\mathbb{R}^n)}\Big)^{1/q}$$

is finite, where we modify naturally the definition above when $p = \infty$ or $q = \infty$.

The following is shown by Izuki [28], which is ($\mathcal{L}8$) of this case. A complete theory of Herz-type spaces was given in [46].

Proposition 11.7. *Let $p \in (1,\infty)$, $q,u \in (0,\infty]$ and $\alpha \in (-1/p, 1/p')$. Then, for all sequences $\{f_j\}_{j=1}^\infty$ of measurable functions,*

$$\Big\|\Big(\sum_{j=1}^\infty [Mf_j]^u\Big)^{1/u}\Big\|_{K^\alpha_{p,q}(\mathbb{R}^n)} \sim \Big\|\Big(\sum_{j=1}^\infty |f_j|^u\Big)^{1/u}\Big\|_{K^\alpha_{p,q}(\mathbb{R}^n)}$$

with the implicit constants independent of $\{f_j\}_{j=1}^\infty$.

In this example ($\mathcal{L}1$) through ($\mathcal{L}6$) hold with (3.2)–(3.4) reading

$$\mathcal{L}(\mathbb{R}^n) := K^\alpha_{p,q}(\mathbb{R}^n), \quad \theta := \min(1,p,q), \quad N_0 := n/q + 1 + \max(\alpha,0), \quad \gamma := n/p + \alpha,$$
$$\delta := 0, \quad w_j(x) := 1 \quad \text{for all } x \in \mathbb{R}^n \text{ and } j \in \mathbb{Z}_+, \quad \alpha_1 = \alpha_2 = \alpha_3 = 0,$$
$$\tau \in [0,\infty), \quad q \in (0,\infty], \quad a \in (n/q+1,\infty).$$

By Proposition 11.7, we know that ($\mathcal{L}8$) holds as well.

Therefore, again a theory of Besov–Herz spaces and Triebel–Lizorkin–Herz spaces similar to the theory of Besov–Morrey spaces and Triebel–Lizorkin–Morrey spaces as in [32, 33, 43, 74, 75, 80, 83] can be developed as before. A homogeneous counterpart of the above is available. Define the *homogeneous Herz space* $\dot{K}^\alpha_{p,q}(\mathbb{R}^n)$ to be the set of all measurable functions f for which the norm

$$\|f\|_{\dot{K}^\alpha_{p,q}(\mathbb{R}^n)} := \Big[\sum_{j=-\infty}^\infty \|2^{jq\alpha}\chi_{C_j} f\|^q_{L^p(\mathbb{R}^n)}\Big]^{1/q}$$

is finite, where we modify naturally the definition above when $q = \infty$.

An analogous result is available but we do not go into details.

11.5. Variable exponent Lebesgue spaces. Starting from the recent work by Diening [11], there exist a series of results of the theory of variable exponent function spaces. Let $p(\cdot) : \mathbb{R}^n \to (0,\infty)$ be a measurable function such that $0 < \inf_{x\in\mathbb{R}^n} p(x) \le \sup_{x\in\mathbb{R}^n} p(x) < \infty$. Then $L^{p(\cdot)}(\mathbb{R}^n)$, the Lebesgue space with variable exponent $p(\cdot)$, is defined as the set of all measurable functions f for which the quantity $\int_{\mathbb{R}^n} |\varepsilon f(x)|^{p(x)}\,dx$ is finite for some $\varepsilon \in (0,\infty)$. We let

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} := \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n} \left[\frac{|f(x)|}{\lambda}\right]^{p(x)} dx \le 1\right\}$$

for such a function f. As a special case of the theory of Nakano and Luxemburg [47, 62, 63], we see $(L^{p(\cdot)}(\mathbb{R}^n), \|\cdot\|_{L^{p(\cdot)}(\mathbb{R}^n)})$ is a quasi-normed space. It is customary to let $p_+ := \sup_{x\in\mathbb{R}^n} p(x)$ and $p_- := \inf_{x\in\mathbb{R}^n} p(x)$.

The following was shown in [7] and hence we have ($\mathcal{L}8$) for $L^{p(\cdot)}(\mathbb{R}^n)$.

Proposition 11.8. *Suppose that* $p(\cdot) : \mathbb{R}^n \to (0,\infty)$ *is a function satisfying*

$$1 < p_- := \inf_{x\in\mathbb{R}^n} p(x) \le p_+ := \sup_{x\in\mathbb{R}^n} p(x) < \infty, \tag{11.6}$$

$$\text{(log-Hölder continuity)} \quad |p(x)-p(y)| \lesssim \frac{1}{\log(1/|x-y|)} \quad \textit{for all } |x-y| \le 1/2, \tag{11.7}$$

$$\text{(decay condition)} \quad |p(x)-p(y)| \lesssim \frac{1}{\log(e+|x|)} \quad \textit{for all } |y| \ge |x|. \tag{11.8}$$

Let $u \in (1,\infty]$. *Then, for all sequences* $\{f_j\}_{j=1}^\infty$ *of measurable functions,*

$$\left\| \left(\sum_{j=1}^\infty [Mf_j]^u \right)^{1/u} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \sim \left\| \left(\sum_{j=1}^\infty |f_j|^u \right)^{1/u} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}$$

with the implicit constants independent of $\{f_j\}_{j=1}^\infty$.

In this example ($\mathcal{L}1$) through ($\mathcal{L}6$) hold with the parameters in (3.2)–(3.4) satisfying

$$\mathcal{L}(\mathbb{R}^n) := L^{p(\cdot)}(\mathbb{R}^n), \quad \theta := \min(1,p_-), \quad N_0 := n/p_- + 1, \quad \gamma := n/p_-, \quad \delta := 0,$$
$$w_j(x) := 1 \quad \text{for all } x \in \mathbb{R}^n \text{ and } j \in \mathbb{Z}_+, \quad \alpha_1 = \alpha_2 = \alpha_3 = 0,$$
$$\tau \in [0,\infty), \quad q \in (0,\infty], \quad a > n/p_- + 1.$$

Also, by Proposition 11.8, we have ($\mathcal{L}8$) as well. For simplicity, let us write $A^{s,\tau}_{p(\cdot),q}(\mathbb{R}^n)$ instead of $A^{s,\tau}_{L^{p(\cdot)}(\mathbb{R}^n),q,a}(\mathbb{R}^n)$.

The function space $A^{s,0}_{p(\cdot),q}(\mathbb{R}^n)$ is well investigated and we have the following proposition, for example.

Proposition 11.9 ([61]). *Let* $f \in \mathcal{S}'(\mathbb{R}^n)$ *and* $p(\cdot)$ *satisfy* (11.6)–(11.8). *Then the following are equivalent:*

(i) *f belongs to the local Hardy space $h^{p(\cdot)}(\mathbb{R}^n)$ with variable exponent $p(\cdot)$, that is,*

$$\|f\|_{h^{p(\cdot)}(\mathbb{R}^n)} := \left\| \sup_{0<t\le 1} |t^{-n}\Phi(t^{-1}\cdot) * f| \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} < \infty;$$

(ii) *f satisfies*

$$\|f\|_{F^0_{p(\cdot),2}(\mathbb{R}^n)} := \|\Phi * f\|_{L^{p(\cdot)}(\mathbb{R}^n)} + \left\| \left(\sum_{j=1}^\infty |\varphi_j * f|^2 \right)^{1/2} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} < \infty.$$

Lemma 1.1, Theorem 9.2, and Propositions 11.8 and 11.9 yield

Proposition 11.10. *The function space $h^{p(\cdot)}(\mathbb{R}^n)$ coincides with $F^{0,0}_{p(\cdot),2,a}(\mathbb{R}^n)$ whenever $a \gg 1$.*

Recall that Besov/Triebel–Lizorkin spaces with variable exponent date back to the works by Almeida and Hästö [1] and Diening, Hästö and Roudenko [12]. Xu investigated the fundamental properties of $A^s_{p(\cdot),q}(\mathbb{R}^n)$ [95, 96]. Among other things he obtained atomic decomposition results. Just as for $A^s_{p(\cdot),q}(\mathbb{R}^n)$, in [64], Noi and Sawano have investigated the complex interpolation of $F^{s_0}_{p_0(\cdot),q_0}(\mathbb{R}^n)$ and $F^{s_1}_{p_1(\cdot),q_1}(\mathbb{R}^n)$.

Finally, as announced in Section 1, we show the unboundedness of the Hardy–Littlewood maximal operator and the maximal operator $M_{r,\lambda}$.

LEMMA 11.11. *The maximal operator $M_{r,\lambda}$ is not bounded on $L^{1+\chi_{\mathbb{R}^n_+}}(\mathbb{R}^n)$, for all $r \in (0,\infty)$ and $\lambda \in (0,\infty)$. In particular, the Hardy–Littlewood maximal operator M is not bounded on $L^{1+\chi_{\mathbb{R}^n_+}}(\mathbb{R}^n)$.*

Proof. Let $r, \lambda \in (0,\infty)$. Consider $f_r(x) := \chi_{[-r,0]}(x_n)\chi_{[-1,1]^{n-1}}(x_1,\dots,x_{n-1})$ for all $x = (x_1,\dots,x_n)$ in $\mathbb{R}^n$. Then, for all x in the support of f_r, we have

$$M_{r,\lambda} f_r(x) \sim M f_r(x) \sim \chi_{[-r,r]}(x_n)\chi_{[-1,1]^{n-1}}(x_1,\dots,x_{n-1}).$$

Hence $\|M_{r,\lambda} f\|_{L^{1+\chi_{\mathbb{R}^n_+}}(\mathbb{R}^n)} \gtrsim r^{-1/2}$, while $\|f\|_{L^{1+\chi_{\mathbb{R}^n_+}}(\mathbb{R}^n)} \sim r^{-1}$, showing the unboundedness. ∎

Lebesgue spaces with variable exponent date back to the works by Orlicz and Nakano [66, 62, 63], where the case $p_+ < \infty$ is considered. When $p_+ \le \infty$, Sharapudinov considered $L^{p(\cdot)}([0,1])$ [84] and then Kováčik and Rákosník extended the theory to domains [40].

11.6. Amalgam spaces. Let p, $q \in (0,\infty]$ and $s \in \mathbb{R}$. Let $Q_{0z} := z + [0,1]^n$ for $z \in \mathbb{Z}^n$ be the translation of the unit cube. For a Lebesgue locally integrable function f we define

$$\|f\|_{(L^p(\mathbb{R}^n),\ell^q(\langle z\rangle^s))} := \|\{(1+|z|)^s \|\chi_{Q_{0z}} f\|_{L^p(\mathbb{R}^n)}\}_{z\in\mathbb{Z}^n}\|_{\ell^q}.$$

In this example ($\mathcal{L}1$) through ($\mathcal{L}6$) hold with (3.2)–(3.4) reading

$$\mathcal{L}(\mathbb{R}^n) := (L^p(\mathbb{R}^n), \ell^q(\langle z\rangle^s)), \quad \theta := \min(1,p,q), \quad N_0 := n+1+s, \quad \gamma := n/p,$$
$$\delta := \max(-s,0), \quad w_j(x) := 1 \quad \text{for all } x \in \mathbb{R}^n \text{ and } j \in \mathbb{Z}_+, \quad \alpha_1 = \alpha_2 = \alpha_3 = 0.$$
$$\tau \in [0,\infty), \quad q \in (0,\infty], \quad a > n+1+s.$$

The following is shown essentially in [36] (actually, in [36] the boundedness of singular integral operators was established). Using the technique employed in [19, p. 498], we get

PROPOSITION 11.12. *Let $q, u \in (1,\infty]$, $p \in (1,\infty)$ and $s \in \mathbb{R}$. Then, for all sequences $\{f_j\}_{j=1}^\infty$ of measurable functions,*

$$\Big\|\Big(\sum_{j=1}^\infty [M f_j]^u\Big)^{1/u}\Big\|_{(L^p(\mathbb{R}^n),\ell^q(\langle z\rangle^s))} \sim \Big\|\Big(\sum_{j=1}^\infty |f_j|^u\Big)^{1/u}\Big\|_{(L^p(\mathbb{R}^n),\ell^q(\langle z\rangle^s))}$$

with the implicit constants independent of $\{f_j\}_{j=1}^\infty$.

Therefore, ($\mathcal{L}6$) holds and the results above apply to these *amalgam spaces*. Note that amalgam spaces can be used to describe the range of the Fourier transform; see [81] for details.

11.7. Multiplier spaces. There is another variant of Morrey spaces:

DEFINITION 11.13. For $r \in [0,n/2)$, $\dot{X}_r(\mathbb{R}^n)$ is defined as the space of all functions $f \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$ that satisfy

$$\|f\|_{\dot{X}_r(\mathbb{R}^n)} := \sup\{\|fg\|_{L^2(\mathbb{R}^n)} < \infty : \|g\|_{\dot{H}^r(\mathbb{R}^n)} \le 1\} < \infty,$$

where $\dot{H}^r(\mathbb{R}^n)$ stands for the completion of $\mathcal{D}(\mathbb{R}^n)$ with respect to the norm $\|u\|_{\dot{H}^r(\mathbb{R}^n)} := \|(-\Delta)^{r/2}u\|_{L^2(\mathbb{R}^n)}$.

We refer to [51] for the field of multiplier spaces. Here and below we place ourselves in the setting of $\mathbb{R}^n$ with $n \geq 3$.

We will characterize the above norm in terms of $\dot{H}^r(\mathbb{R}^n)$-capacity and wavelets. Here we recall the definition of capacity (see [50, 51]). Denote by $\mathcal{K}$ the set of all compact sets in $\mathbb{R}^n$.

Definition 11.14 ([51]). Let $r \in [0, n/2)$ and $e \in \mathcal{K}$. Then $\operatorname{cap}(e, \dot{H}^r(\mathbb{R}^n))$ stands for the $\dot{H}^r$-*capacity*, defined by

$$\operatorname{cap}(e, \dot{H}^r(\mathbb{R}^n)) := \inf\{\|u\|^2_{\dot{H}^r(\mathbb{R}^n)} : u \in \mathcal{D}(\mathbb{R}^n),\ u \geq 1 \text{ on } e\}.$$

Set $1/u := 1/2 - r/n$, that is, $u = 2n/(n-2r)$. Notice that by the Sobolev embedding theorem, we have

$$|e|^{1/u} = \|\chi_e\|_{L^u(\mathbb{R}^n)} \leq \|u\|_{L^u(\mathbb{R}^n)} \lesssim \|u\|_{\dot{H}^r(\mathbb{R}^n)}$$

for all $u \in \mathcal{D}(\mathbb{R}^n)$. Consequently,

$$\operatorname{cap}(e, \dot{H}^r(\mathbb{R}^n)) \geq |e|^{(n-2r)/n}. \tag{11.9}$$

Let us now formulate our main result. We choose a system $\{\psi_{\varepsilon,jk}\}_{\varepsilon \in \{1,\dots,2^n-1\},\, j \in \mathbb{Z},\, k \in \mathbb{Z}^n}$ so that it forms a complete orthonormal basis of $L^2(\mathbb{R}^n)$ and

$$\psi_{\varepsilon,jk}(x) = \psi_\varepsilon(2^j x - k) \quad \text{for all } j \in \mathbb{Z},\ k \in \mathbb{Z}^n \text{ and } x \in \mathbb{R}^n.$$

Proposition 11.15 ([23, 51]). *Let $r \in [0, n/2)$ and $f \in L^2_{\mathrm{loc}}(\mathbb{R}^n) \cap \mathcal{S}'(\mathbb{R}^n)$. Then the following are equivalent:*

(i) $f \in \dot{X}_r(\mathbb{R}^n)$.

(ii) *f can be expanded as follows:*

$$f = \sum_{\varepsilon=1}^{2^n-1} \sum_{(j,k)\in\mathbb{Z}\times\mathbb{Z}^n} \lambda_{\varepsilon,jk}\psi_{\varepsilon,jk} \quad \textit{in the topology of } \mathcal{S}'(\mathbb{R}^n),$$

where

$$\sum_{\varepsilon=1}^{2^n-1} \sum_{(j,k)\in\mathbb{Z}\times\mathbb{Z}^n} |\lambda_{\varepsilon,jk}|^2 \int_e |\psi_{\varepsilon,jk}(x)|^2 [M\chi_e(x)]^{4/5}\, dx \leq (C_1)^2 \operatorname{cap}(e, \dot{H}^r(\mathbb{R}^n))$$

for $e \in \mathcal{K}$.

(iii) *If $n \geq 3$ then f can be expanded as follows:*

$$f = \sum_{\varepsilon=1}^{2^n-1} \sum_{(j,k)\in\mathbb{Z}\times\mathbb{Z}^n} \lambda_{\varepsilon,jk}\psi_{\varepsilon,jk} \quad \textit{in the topology of } \mathcal{S}'(\mathbb{R}^n),$$

where

$$\sum_{\varepsilon=1}^{2^n-1} \sum_{(j,k)\in\mathbb{Z}\times\mathbb{Z}^n} |\lambda_{\varepsilon,jk}|^2 \int_{\mathbb{R}^n} |\psi_{\varepsilon,jk}(x)|^2\, dx \leq (C_2)^2 \operatorname{cap}(e, \dot{H}^r(\mathbb{R}^n))$$

for $e \in \mathcal{K}$.

Furthermore, the smallest values of C_1 and C_2 are both equivalent to $\|f\|_{\dot{X}_r(\mathbb{R}^n)}$.

To show that this function space falls under the scope of our theory, set

$$\|F\|^{(1)}_{\dot{X}_r(\mathbb{R}^n)} := \sup_{e\in\mathcal{K}}\left\{\frac{1}{\operatorname{cap}(e,\dot{H}^r(\mathbb{R}^n))}\int_{\mathbb{R}^n}|F(x)|^2\,dx\right\}^{1/2},$$

$$\|F\|^{(2)}_{\dot{X}_r(\mathbb{R}^n)} := \sup_{e\in\mathcal{K}}\left\{\frac{1}{\operatorname{cap}(e,\dot{H}^r(\mathbb{R}^n))}\int_{e}|F(x)|^2[M\chi_e(x)]^{4/5}\,dx\right\}^{1/2}.$$

Then $\dot{X}^{(i)}_r(\mathbb{R}^n)$, $i\in\{1,2\}$, denotes the set of all measurable function $F:\mathbb{R}^n\to\mathbb{C}$ for which $\|F\|^{(i)}_{\dot{X}_r(\mathbb{R}^n)}<\infty$.

The following lemma, which can be used to check ($\mathcal{L}6$), is known.

Lemma 11.16 ([23, Lemma 2.1]). *Let e be a compact set and $\kappa\in(0,\infty)$. Define $E_\kappa=\{x\in\mathbb{R}^n: M\chi_e(x)>\kappa\}$. Then*

$$\operatorname{cap}(\overline{E_\kappa},\dot{H}^r(\mathbb{R}^n))\lesssim\kappa^{-2}\operatorname{cap}(e,\dot{H}^r(\mathbb{R}^n))$$

By (11.9) and Lemma 11.16, ($\mathcal{L}1$) through ($\mathcal{L}6$) hold with (3.2) reading

$$\mathcal{L}(\mathbb{R}^n):=\dot{X}^{(i)}_r(\mathbb{R}^n)\quad\text{for } i\in\{1,2\},\quad \theta:=1,\quad N_0:=n+1,\quad \gamma:=2,\quad \delta:=0.$$

In this case the condition (3.3) on w is trivial:

$$w_j(x):=1\quad\text{for all } j\in\mathbb{Z}_+ \text{ and } x\in\mathbb{R}^n,\qquad \alpha_1=\alpha_2=\alpha_3=0.$$

Consequently, (3.4) reads

$$\tau\in[0,\infty),\quad q\in(0,\infty],\quad a>n+1.$$

In view of Proposition 11.15 we make the following definition.

Definition 11.17. For any given sequence $\lambda:=\{\lambda_{jk}\}_{j\in\mathbb{Z}_+,k\in\mathbb{Z}^n}$, let

$$\|\lambda\|^{(1)}_{\dot{x}_r(\mathbb{R}^n)}:=\|\lambda\|_{\dot{b}^{0,0}_{\dot{X}^{(1)}_r(\mathbb{R}^n),2}},\qquad \|\lambda\|^{(2)}_{\dot{x}_r(\mathbb{R}^n)}:=\|\lambda\|_{\dot{b}^{0,0}_{\dot{X}^{(2)}_r(\mathbb{R}^n),2}}.$$

Then $\dot{x}^{(i)}_r(\mathbb{R}^n)$ for $i\in\{1,2\}$ is the set of all sequences $\lambda:=\{\lambda_{jk}\}_{j\in\mathbb{Z}_+,\,k\in\mathbb{Z}^n}$ for which $\|\lambda\|^{(i)}_{\dot{x}_r(\mathbb{R}^n)}$ is finite.

In [23], essentially, we have shown the following conclusions.

Proposition 11.18. *Let $r\in(0,n/2)$.*

(i) *If $n\geq 3$, then $(\dot{X}_r(\mathbb{R}^n),\dot{x}^{(1)}_r(\mathbb{R}^n))$ admits atomic/molecular decompositions.*
(ii) *If $n\geq 1$, then $(\dot{X}_r(\mathbb{R}^n),\dot{x}^{(2)}_r(\mathbb{R}^n))$ admits atomic/molecular decompositions.*

Thanks to Proposition 9.5, this can be improved as follows.

Proposition 11.19. *Let $r\in(0,n/2)$ and $n\geq 1$. Then $(\dot{X}_r(\mathbb{R}^n),\dot{x}^{(1)}_r(\mathbb{R}^n))$ admits atomic/molecular decompositions.*

11.8. $\dot{B}_\sigma(\mathbb{R}^n)$ spaces. The next example also falls under the scope of our generalized Triebel–Lizorkin type spaces.

Definition 11.20. Let $\sigma \in [0,\infty)$, $p \in [1,\infty]$ and $\lambda \in [-n/p, 0]$. Then $\dot{B}_\sigma(L_{p,\lambda})(\mathbb{R}^n)$ is defined as the space of all $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ for which the norm

$$\|f\|_{\dot{B}_\sigma(L_{p,\lambda})(\mathbb{R}^n)} := \sup\left\{\frac{1}{r^\sigma |Q|^{\lambda/n+1/p}}\|f\|_{L^p(Q)} : r \in (0,\infty),\, Q \subset Q(0,r)\right\}$$

is finite.

In this example ($\mathcal{L}1$) through ($\mathcal{L}6$) hold with (3.2) and (3.3) reading

$$\mathcal{L}(\mathbb{R}^n) := \dot{B}_\sigma(L_{p,\lambda})(\mathbb{R}^n), \quad \theta := 1, \quad N_0 := -\lambda + 1, \quad \gamma := -\lambda, \quad \delta := 0,$$
$$w_j(x) := 1 \text{ for all } j \in \mathbb{Z}_+ \text{ and } x \in \mathbb{R}^n, \quad \alpha_1 = \alpha_2 = \alpha_3 = 0.$$

Hence (3.4) now reads

$$\tau \in [0,\infty), \quad q \in (0,\infty], \quad a > -\lambda + 1.$$

We remark that $\dot{B}^\sigma(\mathbb{R}^n)$-spaces have been introduced recently to unify λ-central Morrey spaces, λ-central mean oscillation spaces and usual Morrey–Campanato spaces [49]. Recall that in Lemma 1.1 we have defined $Q(0,r)$. We refer to [39] for further generalizations.

Definition 11.21 ([42]). Let $p \in (1,\infty)$, $\sigma \in (0,\infty)$, $\lambda \in [-n/p, -\sigma)$ and let φ satisfy (1.3) and (1.4). Given $f \in \mathcal{S}'(\mathbb{R}^n)$, set

$$\|f\|_{\dot{B}_\sigma(L^D_{p,\lambda})(\mathbb{R}^n)} := \sup_{\substack{r\in(0,\infty)\\ Q\in\mathcal{Q}(\mathbb{R}^n),\, Q\subset Q(0,r)}} \frac{1}{r^\sigma|Q|^{\lambda/n+1/p}}\left\|\left(\sum_{j=j_Q}^\infty |\varphi_j * f|^2\right)^{1/2}\right\|_{L^p(Q)}.$$

Then $\dot{B}_\sigma(L^D_{p,\lambda})(\mathbb{R}^n)$ denotes the space of all $f \in \mathcal{S}'(\mathbb{R}^n)$ for which $\|f\|_{\dot{B}_\sigma(L^D_{p,\lambda})(\mathbb{R}^n)}$ is finite.

Lemma 11.22 ([42]). *Let $p \in (1,\infty)$, $u \in (1,\infty]$, $\sigma \in [0,\infty)$ and $\lambda \in (-\infty,0)$. Assume, in addition, that $\sigma + \lambda < 0$. Then*

$$\left\|\left(\sum_{j=1}^\infty [Mf_j]^u\right)^{1/u}\right\|_{\dot{B}_\sigma(L_{p,\lambda})(\mathbb{R}^n)} \sim \left\|\left(\sum_{j=1}^\infty |f_j|^u\right)^{1/u}\right\|_{\dot{B}_\sigma(L_{p,\lambda})(\mathbb{R}^n)}$$

with the implicit constants independent of $\{f_j\}_{j=1}^\infty \subset \dot{B}_\sigma(L_{p,\lambda})(\mathbb{R}^n)$.

Proposition 11.23 ([42]). *Let $p \in (1,\infty)$, $\sigma \in (0,\infty)$ and $\lambda \in [-n/p,-\sigma)$. Then*

$$\dot{B}_\sigma(L^D_{p,\lambda})(\mathbb{R}^n) \quad \text{and} \quad \dot{B}_\sigma(L_{p,\lambda})(\mathbb{R}^n)$$

coincide. More precisely, the following hold:

(i) $\dot{B}_\sigma(L_{p,\lambda})(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$ *in the sense of continuous embedding.*
(ii) $\dot{B}_\sigma(L^D_{p,\lambda})(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n) \cap L^p_{\mathrm{loc}}(\mathbb{R}^n)$ *in the sense of continuous embedding.*
(iii) $f \in \dot{B}_\sigma(L_{p,\lambda})(\mathbb{R}^n)$ *if and only if* $f \in \dot{B}_\sigma(L^D_{p,\lambda})(\mathbb{R}^n)$ *and the norms are mutually equivalent.*
(iv) *Different choices of φ yield equivalent* $\|\cdot\|_{\dot{B}_\sigma(L^D_{p,\lambda})(\mathbb{R}^n)}$ *norms.*

The atomic decomposition of $\dot{B}_\sigma(\mathbb{R}^n)$ is as follows. First we introduce the sequence space.

DEFINITION 11.24. Let $\sigma \in [0,\infty)$, $p \in [1,\infty]$ and $\lambda \in [-n/p, 0]$. Then $\dot{b}_\sigma(L^D_{p,\lambda})(\mathbb{R}^n)$ is defined to be the space of all $\lambda := \{\lambda_{jk}\}_{j\in\mathbb{Z}_+,\, k\in\mathbb{Z}^n}$ such that

$$\|\lambda\|_{\dot{b}_\sigma(L^D_{p,\lambda})(\mathbb{R}^n)} := \sup_{\substack{r\in(0,\infty) \\ Q\in\mathcal{Q}(\mathbb{R}^n),\ Q\subset Q(0,r)}} \frac{1}{r^\sigma |Q|^{\lambda/n+1/p}} \Big\| \sum_{j=j_Q}^{\infty} \lambda_{jk}\chi_{Q_{jk}} \Big\|_{L^p(Q)} < \infty.$$

In view of Theorem 6.6, we have the following direct corollary of Theorem 4.5.

THEOREM 11.25. *The pair* $(\dot{B}_\sigma(L^D_{p,\lambda})(\mathbb{R}^n), \dot{b}_\sigma(L^D_{p,\lambda})(\mathbb{R}^n))$ *admits atomic/molecular decompositions.*

11.9. Generalized Campanato spaces. Returning to the variable exponent setting described in Section 11.5, we define

$$d_{p(\cdot)} := \min\{d \in \mathbb{Z}_+ : p_-(n+d+1) > n\}.$$

Let $L^q_{\mathrm{comp}}(\mathbb{R}^n)$ be the set of all $L^q(\mathbb{R}^n)$-functions with compact support. For a nonnegative integer d, let

$$L^{q,d}_{\mathrm{comp}}(\mathbb{R}^n) := \left\{ f \in L^q_{\mathrm{comp}}(\mathbb{R}^n) : \int_{\mathbb{R}^n} f(x)x^\alpha\,dx = 0,\ \|\alpha\|_1 \le d \right\}.$$

Likewise if Q is a cube, we write

$$L^{q,d}(Q) := \left\{ f \in L^q(Q) : \int_Q f(x)x^\alpha\,dx = 0,\ \|\alpha\|_1 \le d \right\},$$

where $L^q(Q)$ is the closed subspace of functions in $L^q(\mathbb{R}^n)$ having support in Q.

Recall that $\mathcal{P}_d(\mathbb{R}^n)$ is the set of all polynomials having degree at most d. For a locally integrable function f, a cube Q and a nonnegative integer d, there exists a unique polynomial $P \in \mathcal{P}_d(\mathbb{R}^n)$ such that, for all $q \in \mathcal{P}_d(\mathbb{R}^n)$,

$$\int_Q [f(x) - P(x)]q(x)\,dx = 0.$$

Denote this unique polynomial P by $P^d_Q f$. It follows immediately from the definition that $P^d_Q g = g$ if $g \in \mathcal{P}_d(\mathbb{R}^n)$.

We postulate the following conditions on $\phi : \mathbb{R}^{n+1}_+ \to (0,\infty)$:

(A1) (Doubling condition) There exist positive constants M_1 and M_2 such that

$$M_1 \le \frac{\phi(x,2r)}{\phi(x,r)} \le M_2 \quad (x \in \mathbb{R}^n,\ r \in (0,\infty)).$$

(A2) (Compatibility condition) There exist positive constants M_3 and M_4 such that

$$M_3 \le \frac{\phi(x,r)}{\phi(y,r)} \le M_4 \quad (x, y \in \mathbb{R}^n,\ r \in (0,\infty),\ |x-y| \le r).$$

(A3) (∇_2-condition) There exists a positive constant M_5 such that

$$\int_0^r \frac{\phi(x,t)}{t}\,dt \le M_5\phi(x,r) \quad (x \in \mathbb{R}^n,\ r \in (0,\infty)).$$

(A4) (Δ_2-condition) There exists a positive constant M_6 such that

$$\int_r^\infty \frac{\phi(x,t)}{t^{d+2}}\,dt \le M_6\frac{\phi(x,r)}{r^{d+1}} \quad \text{for some integer } d \in [0,\infty).$$

(A5) (Uniform condition) $\sup_{x\in\mathbb{R}^n} \phi(x,1) < \infty$.

Here the constants $M_1, \dots, M_6$ need to be specified for later considerations.

Notice that the Morrey–Campanato space with variable growth function $\phi(x,r)$ was first introduced by Nakai [55, 59] by using an idea from [65]. In [56], Nakai established the boundedness of the Hardy–Littlewood maximal operator, singular integral operators (of Calderón–Zygmund type), and fractional integral operators on Morrey spaces with variable growth function $\phi(x,r)$.

Recently, Nakai and Sawano considered a more general version in [61].

Let us say that $\phi : \mathcal{Q}(\mathbb{R}^n) \to (0,\infty)$ is a *nice function* if there exists $b \in (0,1)$ such that, for all cubes $Q \in \mathcal{Q}(\mathbb{R}^n)$,

$$\frac{1}{\phi(Q)}\left[\frac{1}{|Q|}\int_Q |f(x) - P_Q^d f(x)|^q\,dx\right]^{1/q} > b$$

for some $f \in \mathcal{L}_{q,\phi,d}(\mathbb{R}^n)$ with norm 1. In [61, Lemma 6.1], we showed that ϕ can be assumed to be nice. Actually, there exists a nice function $\phi^\dagger$ such that $\mathcal{L}_{q,\phi,d}(\mathbb{R}^n)$ and $\mathcal{L}_{q,\phi^\dagger,d}(\mathbb{R}^n)$ coincide as sets and the norms are equivalent [61, Lemma 6.1].

Definition 11.26 ([61]). Let $\phi : \mathbb{R}^{n+1}_+ \to (0,\infty)$ be a function, which is not necessarily nice, and $f \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$. Define, when $q \in (1,\infty)$,

$$\|f\|_{\mathcal{L}_{q,\phi,d}(\mathbb{R}^n)} := \sup_{(x,t)\in\mathbb{R}^{n+1}_+} \frac{1}{\phi(x,t)}\left\{\frac{1}{|Q(x,t)|}\int_{Q(x,t)} |f(y) - P^d_{Q(x,t)} f(y)|^q\,dy\right\}^{1/q},$$

and

$$\|f\|_{\mathcal{L}_{\infty,\phi,d}(\mathbb{R}^n)} := \sup_{(x,t)\in\mathbb{R}^{n+1}_+} \frac{1}{\phi(x,t)}\|f - P^d_{Q(x,t)} f\|_{L^\infty(Q(x,t))}.$$

The *Campanato space* $\mathcal{L}_{q,\phi,d}(\mathbb{R}^n)$ is defined to be the set of all f such that $\|f\|_{\mathcal{L}_{q,\phi,d}(\mathbb{R}^n)}$ is finite.

Definition 11.27 ([61]). Let $q \in [1,\infty]$, suppose φ satisfies (1.4) and let $\phi : \mathbb{R}^{n+1}_+ \to (0,\infty)$ be a function. A distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ is said to belong to $\mathcal{L}^D_{q,\phi}(\mathbb{R}^n)$ if

$$\|f\|_{\mathcal{L}^D_{q,\phi}(\mathbb{R}^n)} := \sup_{(x,t)\in\mathbb{R}^{n+1}_{\mathbb{Z}}} \frac{1}{\phi(x,t)}\left\{\frac{1}{|Q(x,t)|}\int_{Q(x,t)} |\varphi_{(\log_2 t^{-1})} * f(y)|^q\,dy\right\}^{1/q} < \infty.$$

Proposition 11.28 ([61]). *Assume* (A1) *through* (A5). *Then*

(i) *The spaces $\mathcal{L}^D_{q,\phi}(\mathbb{R}^n)$ and $\mathcal{L}_{q,\phi,d}(\mathbb{R}^n)$ coincide. More precisely, the following hold:*
  - (a) *Let $f \in \mathcal{L}^D_{q,\phi}(\mathbb{R}^n)$. Then there exists $P \in \mathcal{P}(\mathbb{R}^n)$ such that $f - P \in \mathcal{L}_{q,\phi,d}(\mathbb{R}^n)$. In this case, $\|f-P\|_{\mathcal{L}_{q,\phi,d}(\mathbb{R}^n)} \lesssim \|f\|_{\mathcal{L}^D_{q,\phi}(\mathbb{R}^n)}$ with the implicit constant independent of f.*
  - (b) *If $f \in \mathcal{L}_{q,\phi,d}(\mathbb{R}^n)$, then $f \in \mathcal{L}^D_{q,\phi}(\mathbb{R}^n)$ and $\|f\|_{\mathcal{L}^D_{q,\phi}(\mathbb{R}^n)} \lesssim \|f\|_{\mathcal{L}_{q,\phi,d}(\mathbb{R}^n)}$ with the implicit constant independent of f. In particular, the definition of $\mathcal{L}^D_{q,\phi}(\mathbb{R}^n)$ is independent of the admissible choices of φ: Any $\varphi \in \mathcal{S}(\mathbb{R}^n)$ does the job as long as $\chi_{Q(0,1)} \le \widehat{\varphi} \le \chi_{Q(0,2)}$.*

(ii) *The function space $\mathcal{L}^D_{q,\phi}(\mathbb{R}^n)$ is independent of q.*

In view of Definition 11.27, if we assume that ϕ satisfies (A1) through (A5), then we have the following proposition.

Proposition 11.29. *Let φ satisfy* (1.4). *If $\phi : \mathcal{Q}(\mathbb{R}^n) \to (0,\infty)$ satisfies* (A1) *through* (A5), *then*

$$\|f\|_{\mathcal{L}^D_{\infty,\phi}(\mathbb{R}^n)} \sim \sup_{(x,t)\in\mathbb{R}^{n+1}_{\mathbb{Z}}} \frac{1}{\phi(Q(x,t))} \sup_{y\in Q(x,t)} \left\{ \sup_{z\in\mathbb{R}^n} \frac{|\varphi_{(\log_2 t^{-1})} * f(y+z)|}{(1+t^{-1}|z|)^a} \right\}$$

whenever $a \gg 1$, with the implicit constants independent of f.

To prove Proposition 11.29, we just need to check (9.17) by using (A1) and (A2). We omit the details.

Definition 11.30. Define

$$\|\lambda\|_{l^D_{\infty,\phi}(\mathbb{R}^n)} := \sup_{(x,t)\in\mathbb{R}^{n+1}_{\mathbb{Z}}} \frac{1}{\phi(Q(x,t))} \sup_{y\in Q(x,t)} \left\{ \sup_{z\in\mathbb{R}^n} \frac{1}{(1+t^{-1}|z|)^a} \sum_{k\in\mathbb{Z}^n} |\lambda_{(\log_2 t^{-1})k}| \chi_{Q_{(\log_2 t^{-1})k}} \right\}.$$

Now in this example ($\mathcal{L}1$) through ($\mathcal{L}6$) hold with the parameters in (3.2) and (3.3) satisfying

$$\mathcal{L}(\mathbb{R}^n) := L^\infty(\mathbb{R}^n), \quad \theta := 1, \quad N_0 := 0, \quad \gamma := 0, \quad \delta := 0$$

and $w(x,t) := 1/\phi(Q(x,t))$ for all $x \in \mathbb{R}^n$ and $t \in (0,\infty)$, $\alpha_1 = \log_2 {M_1}^{-1}$, $\alpha_2 = \log_2 M_2$, $\alpha_3 = \log_2 (M_2/M_1)$, respectively. Furthermore, unlike the preceding examples, we choose

$$\tau = 0, \quad q = \infty, \quad a > N_0 + \log_2 (M_2/M_1).$$

Therefore, $\mathcal{L}_{q,\phi,d}(\mathbb{R}^n)$ and $\mathcal{L}^D_{q,\phi}(\mathbb{R}^n)$ fall under the scope of our theory.

Theorem 11.31. *Under the conditions* (A1) *through* (A5), *the pair* $(\mathcal{L}^D_{\infty,\phi}(\mathbb{R}^n), l^D_{\infty,\phi}(\mathbb{R}^n))$ *admits atomic/molecular decompositions.*

Theorem 11.31 is just a consequence of Theorem 4.5. We omit the details.

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