Introduction

The main object of this paper is to establish the Yamada–Watanabe theory of uniqueness and existence of solutions of stochastic evolution equations in Banach spaces. The pioneering paper [YW] has initiated a comprehensive study of relations between essentially different types of uniqueness and existence (e.g. pathwise uniqueness, joint uniqueness in law, weak and strong existence) arising naturally in the study of SDEs (see e.g. [En], [J]) and the research in this direction is still active—even today, new surprising results are published (see e.g. [Ch]). Our intention is to give a presentation of these results for evolution equations in Banach spaces perturbed by a (generally) infinite-dimensional Wiener process.

Attacking this issue, we encounter two main obstacles which render the solution of this problem nontrivial. Firstly, unlike the finite-dimensional case, the continuity of trajectories of stochastic evolution equations is an open problem, and secondly, infinite-dimensional Wiener processes are not processes in a conventional sense: they are not Fréchet valued unless their covariance is of trace class. Being aware of these difficulties, no *prima facie* generalization of known proofs is possible and we must use different constructions which stem rather from the infinite-dimensional structure of the spaces we work in than from probabilistic reasons.

The paper also contains a comprehensive section of preliminary results on stochastic analysis in Banach spaces, namely a stochastic integral is constructed by a method alternative to the usual ones, Burkholder's inequality, Fubini's and Girsanov's theorems are proven, and theorems on equality of distributions of Bochner integrals, stochastic integrals and measurable selectors are given.

Concerning the principal content, we consider a stochastic semilinear equation (0.1) with an initial probability distribution; in other words, we are given purely deterministic quantities (transformations appearing in the equation and a measure on the state space), and before we can speak of any solution, we must specify what probability filtered space we work on and what Wiener process drives our equation. Then we can seek a stochastic process with the prescribed initial distribution solving the equation. We pose the following natural question: If there exists a probability space with a solving process, what conditions are sufficient to conclude that there exists a solving process on every probability space?

We will also be interested in the uniqueness point of view. By the Yamada–Watanabe theorem for SDEs, if an equation is pathwise unique (i.e. different paths of solutions have different initial values) then any two solutions living on possibly different probability spaces necessarily have the same probability distribution on the space of trajectories (uniqueness in law). Our second question is whether this is also true in the Banach space setting.

The third problem we treat is also inspired by the stochastic differential equations theory. Under suitable conditions there exists a deterministic function of two variables: the first corresponds to an initial value and the second to a path of a Wiener process. The function's value is a path of a solution with respect to the initial value and the Wiener process. We will present sufficient conditions for the existence of such a function in the case of stochastic evolution equations (Thm. 12.1, Thm. 13.2, Lemma E).

We also give an example of an equation which is *jointly unique in law* (Def. 1, Thm. 5) and another equation which is *jointly u-unique in law* (Def. 1, Thm. 3). The first example is based on Girsanov's theorem, therefore it concerns additive noise equations only, while the second one uses the measurable selectors approach and covers a fairly general class of multiplicative noise equations, namely all those with one-to-one diffusions.

Theorem 4 is a Banach space version of a remarkable theorem by A. S. Cherny [Ch] (who proved it for SDEs) which states that uniqueness in law is, in fact, equivalent to joint uniqueness in law provided we consider deterministic initial conditions.

An essential part of the proof of Theorems 3 and 4 is an alternative explicit form of the solution (Thm. 13), and therefore we decided to include the complete proof of stochastic Fubini's theorem (Prop. 6.1) together with its consequences (Thm. 12, Thm. 13) which were proven by the same method in a less general form and in the Hilbert space setting by A. Chojnowska-Michalik [ChM].

Next we prove three so called distribution-preserving theorems: for stochastic Bochner integrals (Thm. 8.3), for stochastic integrals (Thm. 8.6) and for measurable selectors (Prop. 8.8). The combination of the first two theorems will result in the fundamental solution-preserving theorem (Thm. 6) which gives a sufficient condition for a pair of a process and a Wiener process to be a solution in terms of their joint distribution on the space of trajectories.

Throughout this paper we work with Banach valued processes. Therefore, in the first part, we recall the construction of the stochastic integral in a separable 2-smooth Banach space—firstly because many proofs in this paper rely on it, and secondly because of its "directness"—the construction of the integral is free of any auxiliary embeddings even in the case of a cylindrical Wiener process with non-trace class covariance. We emphasize that, apart from the observation made in Step 1, the construction is more or less classical and we have just collected the "common knowledge" and sometimes gave shorter proofs in light of newer methods, many of them surveyed in [ChTV]. Our sources for stochastic integration and geometry of uniformly smooth Banach spaces were mainly works of A. L. Neidhardt [N] (construction of the integral), P. Assouad [A] (Burkholder inequality, geometry of 2-smooth Banach spaces), E. Dettweiler [D] (Burkholder inequality), W. Linde & A. Pietsch (characterization of integrands), G. Pisier [P] (geometry of 2-smooth Banach spaces), J. Hoffmann-Jørgensen (geometry of 2-smooth Banach spaces, characterization of integrands [ChTV]).

The cylindrical Wiener process is understood in the sense of M. Métivier & J. Pellaumail—we refer to their paper on cylindrical stochastic integration [MP] while the developments concerning uniqueness trace their origin back to stochastic differential equation results of T. Yamada and S. Watanabe [YW], H. J. Engelbert [En] and J. Jacod [J].

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Notation. 1. We will consider only *complete filtrations*, i.e. whenever $(\mathcal{F}_t)$ is a filtration on some probability space $(\Omega, \mathcal{F}, P)$ then $\mathcal{F}_0$ is supposed to contain all P-negligible sets in $\mathcal{F}$.

2. X stands for a separable 2-smooth Banach space (Def. 3.1), X^* for its topological dual space, $(x^*, x) \mapsto \langle x^*, x\rangle$ for the pairing between them and U will be a separable Hilbert space. If $h \in H$ and $x \in X$ then we will write $h \otimes_{L(U,X)} x = h \otimes x$ to denote the operator $U \to X : y \mapsto \langle y, h\rangle_U x$.

3. The *strong σ-algebra* on $L(U, X)$ is the smallest σ-algebra which renders the mappings $L(U, X) \to X : B \mapsto Bh$, $h \in U$, measurable. A mapping $g : (Y, \mathcal{Y}) \to L(U, X)$ is said to be *strongly measurable* provided that it is measurable with respect to the strong σ-algebra on $L(U, X)$, i.e. the mappings $Y \to X : y \mapsto g(h)$ are measurable for every $h \in U$.

4. X_1 is a separable Banach space such that X is continuously embedded in X_1, i.e. X is a subspace of X_1 and the identity mapping $i : X \to X_1$ is continuous.

5. Let $(\Omega, \mathcal{F}, P)$ be a probability space, $(Y, \mathcal{Y})$ a measurable space and $g : \Omega \to Y$ a measurable mapping. Then we define the image of P under g by $\mathfrak{Law}_P(g)(B) = P\{\omega : g(\omega) \in B\}$, $B \in \mathcal{Y}$.

6. In view of Definitions 1 and 7 we write briefly

$$\mathfrak{Law}_{P^1}(u^1, W^1) = \mathfrak{Law}_{P^2}(u^2, W^2), \quad \text{resp.} \quad \mathfrak{Law}_{P^1}(u^1) = \mathfrak{Law}_{P^2}(u^2)$$

instead of

$$\mathfrak{Law}_{P^1}(u^1(t_i), W^1(t_i, h_j) : i, j) = \mathfrak{Law}_{P^2}(u^2(t_i), W^2(t_i, h_j) : i, j),$$

resp.

$$\mathfrak{Law}_{P^1}(u^1(t_i) : i) = \mathfrak{Law}_{P^2}(u^2(t_i) : i)$$

for every partition $0 = t_0 < \cdots < t_n \le T$ and every $h_1, \ldots, h_m$ in U.

7. We will say that a process $(u_t : t \le T)$ on $(\Omega, \mathcal{F}, P, W)$ is $(u_0, W)_P$*-adapted* instead of saying that u is adapted to the P-augmentation of the filtration $\sigma(u_0, W(s, h) : s \le t,\ h \in U)$ in $\mathcal{F}$.

Main theorems. Consider the following stochastic evolution equation on $[0, T]$ in a separable 2-smooth Banach space X (Def. 3.1):

$$(0.1)\qquad u(t) = S_t u(0) + \int_0^t S_{t-s} f(s,u(s))\,ds + \int_0^t S_{t-s} g(s,u(s))\,dW_s, \quad 0 < t \le T,$$
$$\mathfrak{Law}\,(u(0)) = \mu,$$

where $S:(0,T]\to L(X_1,X)$ is a strongly measurable operator-valued function, Q a covariance operator on U (i.e. a symmetric nonnegative bounded linear operator), $f:[0,T]\times X\to X_1$ a measurable mapping, $g:[0,T]\times X\to L(U_0,X_1)$ a strongly measurable mapping (see Def. 2.1 for the definition of U_0) and μ a probability Borel measure on X.

We say that a 6-tuple $(\Omega,\mathcal{F},(\mathcal{F}_t),P,W,u)$ consisting of a filtered probability space, a Q-$(\mathcal{F}_t)$-Wiener process W on U (Def. 1.4) and a measurable X-valued process u on $[0,T]$ is a *solution* of (0.1) provided that

$$(0.2)\qquad P\Big[\int_0^t (\|S_{t-s}f(s,u(s))\|_X + \|S_{t-s}g(s,u(s))\|^2_{L_2(U_0,X)})\,ds < \infty\Big] = 1$$

for every $t\in(0,T]$, equation (0.1) is satisfied for every $t\in(0,T]$ and **there exists a sequence** $(x_n^*:n)$ **in** X^* **which separates points of** X **such that the real processes** $t\mapsto\langle x_n^*,u(t)\rangle$ **have continuous adapted modifications on** $[0,T]$ (hence u has a predictable modification by Corollary 11.2 and the integrals in (0.1) are well defined). The symbol $L_2(U_0,X)$ dentotes the space of radonifying operators (see Definition 2.3).

The last, sort of untypical, condition will be very important in what follows. We chose this formulation because such a sequence $(x_n^*:n)$ always exists if u is progressively measurable, $X=X_1$, (S_t) is a C_0-semigroup and the following integrability condition holds (see Thm. 13):

$$(0.3)\qquad P\Big[\int_0^T (\|f(s,u(s))\|_{X_1} + \|g(s,u(s))\|^2_{L(U_0,X_1)})\,ds < \infty\Big] = 1.$$

We will refer to (0.3) later on even in situations when X will not coincide with X_1.

Almost all results in this paper hold under fairly general conditions. Theorem 4 is exceptional in this sense—we do not know how to avoid additional assumptions. Namely we cannot take into account the uncountable number of conditions a process must satisfy to be a solution (as in (0.2) where the condition is to be satisfied for every $t\in[0,T]$). Therefore we introduce a single (but more restrictive) condition:

$$(0.4)\qquad P\Big[\int_0^T (M_1(f(s,u(s))) + M_2(g(s,u(s))))\,ds < \infty\Big] = 1,$$

where $M_1:X_1\to[0,\infty]$ is some measurable function with the following property: Whenever $y:[0,T]\to X$ is a measurable function such that

$$\int_0^T M_1(f(s,y_s))\,ds < \infty \quad \text{then} \quad \int_0^t \|S_{t-s}f(s,y_s)\|_X\,ds < \infty \quad \text{for every } t\in[0,T]$$

and $M_2:L(U_0,X_1)\to[0,\infty]$ is some strongly measurable function with the property:

Whenever $y : [0,T] \to X$ is a measurable function such that

$$\int_0^T M_2(g(s,y_s))\,ds < \infty \quad \text{then} \quad \int_0^t \|S_{t-s}g(s,y_s)\|^2_{L_2(U_0,X)}\,ds < \infty \quad \text{for every } t \in [0,T].$$

For instance, the choice of M_1 can be based on the inequality

$$\text{(a)} \qquad \|S_{t-s}f(s,y_s)\|_X \le \|S_{t-s}\|_{L(X_1,X)}\|f(s,y_s)\|_{X_1},$$

while the choice of M_2 arises typically in the cases when g takes values in $L_2(U_0,X_1)$, or when the covariance operator of the Wiener process is nuclear (i.e. $\operatorname{Tr} Q < \infty$) and g takes values in $L(U,X_1)$, or if (S_t) is p-summing for some $0 < p < \infty$ (see e.g. Chapter II.2.2 in [ChTV]). Then

$$\text{(b1)} \qquad \|S_{t-s}g(s,y_s)\|_{L_2(U_0,X)} \le \|S_{t-s}\|_{L(X_1,X)}\|g(s,y_s)\|_{L_2(U_0,X_1)}$$

by Definition 2.3, or

$$\text{(b2)} \qquad \|S_{t-s}g(s,y_s)\|_{L_2(U_0,X)} \le (\operatorname{Tr} Q)^{1/2}\|S_{t-s}\|_{L(X_1,X)}\|g(s,y_s)\|_{L(U,X_1)}$$

by Note 2.6, or

$$\text{(b3)} \qquad \|S_{t-s}g(s,y_s)\|_{L_2(U_0,X)} \le c_p\|S_{t-s}\|_{\varPi_p(X_1,X)}\|g(s,y_s)\|_{L(U_0,X_1)}$$

for some constant c_p by Proposition 2.4, where $\varPi_p(X_1,X)$ is the space of p-summing operators from X_1 to X (see e.g. [ChTV]).

Apparently, we can take $M_1 = 1$ if $\|f\|_{X_1}$ is bounded and $\|S\|_{L(X_1,X)} \in L^1(0,T)$, or

$$M_1 = \|\cdot\|_{X_1}^{\frac{1}{1-1/r}} \quad \text{if } \|S\|_{L(X_1,X)} \in L^r(0,T) \text{ for some } 1 < r \le \infty \text{ by (a).}$$

Analogously, we can consider $M_2 = 1$ if $\|g\|_{L_2(U_0,X_1)}$, resp. $\|g\|_{L(U,X_1)}$, resp. $\|g\|_{L(U_0,X_1)}$ is bounded and $\|S\|_{L(X_1,X)} \in L^2(0,T)$, resp. $\|S\|_{L(X_1,X)} \in L^2(0,T)$, resp. $\|S\|_{\varPi_p(X_1,X)} \in L^2(0,T)$, or

$$M_2 = \|\cdot\|_{L_2(U_0,X_1)}^{\frac{2}{1-1/q}}, \quad \text{resp.} \quad M_2 = \|\cdot\|_{L(U,X_1)}^{\frac{2}{1-1/q}}, \quad \text{resp.} \quad M_2 = \|\cdot\|_{L(U_0,X_1)}^{\frac{2}{1-1/q}}$$

if $\|S\|_{L(X_1,X)} \in L^{2q}(0,T)$, resp. $\|S\|_{L(X_1,X)} \in L^{2q}(0,T)$, resp. $\|S\|_{\varPi_p(X_1,X)} \in L^{2q}(0,T)$ for some $1 < q \le \infty$, by (b1), resp. (b2), resp. (b3).

We should mention that all results in this paper remain true if we consider only adapted solutions with norm continuous paths. Also, the integrability condition (0.2) is fairly general and can be replaced or complemented by e.g. (0.3) or (0.7), with slight modifications of the proofs. In these cases we usually cover smaller classes of solutions with better regularity of paths which, in the end, turns out to be important since the better regularity stays preserved.

In this paper we consider solutions on a bounded interval $[0,T]$ since this is sufficient for the questions of uniqueness even for solutions on $[0,\infty)$. But in Theorem 12.1, Theorem 13.2 and Lemma E we assert the existence of a functional R which assigns a trajectory of a solution on $[0,T]$ to the pair of the value of an initial condition and the trajectory of a Wiener process. This is the only part when the case $[0,\infty)$ is but slightly different so we point out the particular (notational) changes one must do in this situation:

- ▷ Each occurrence of $[0,T]$, resp. T has to be replaced by $[0,\infty)$, resp. ∞.
- ▷ The space $\mathfrak{C} = C([0,T],\mathbb{R}^{\mathbb{N}})$ has to be replaced by $C([0,\infty),\mathbb{R}^{\mathbb{N}})$, which, considered with the topology of locally uniform convergence, is again a Polish space whose Borel σ-algebra is generated by projections $(\pi_t : t < \infty)$.
- ▷ The function ϕ_t has to be replaced by $\phi_t : C([0,\infty),\mathbb{R}^{\mathbb{N}}) \to C([0,\infty),\mathbb{R}^{\mathbb{N}}) : \phi_t(f)(s) = f(t+s) - f(t)$.

With these changes the proofs go along the same lines.

Before we state the theorems we must give a few definitions.

Definition 1. We say that the equation (0.1) with the initial distribution μ

- ▷ is *pathwise unique* if whenever $(\Omega,\mathcal{F},(\mathcal{F}_t),P,W,u^1)$, $(\Omega,\mathcal{F},(\mathcal{F}_t),P,W,u^2)$ are solutions such that $P[u^1(0) = u^2(0)] = 1$ then $P[u^1(t) = u^2(t)] = 1$ for every $t \le T$.
- ▷ is *jointly unique in law* if whenever $(\Omega^1,\mathcal{F}^1,(\mathcal{F}^1_t),P^1,W^1,u^1)$, $(\Omega^2,\mathcal{F}^2,(\mathcal{F}^2_t),P^2,W^2,u^2)$ are solutions then $\mathfrak{Law}_{P^1}(u^1,W^1) = \mathfrak{Law}_{P^2}(u^2,W^2)$.
- ▷ is *jointly u-unique in law* for some solution $(\Omega,\mathcal{F},(\mathcal{F}_t),P,W,u)$ of (0.1) if whenever $(\Omega^1,\mathcal{F}^1,(\mathcal{F}^1_t),P^1,W^1,u^1)$ is another solution of (0.1) such that $\mathfrak{Law}_{P^1}(u^1)$ coincides with $\mathfrak{Law}_P(u)$ then $\mathfrak{Law}_{P^1}(u^1,W^1) = \mathfrak{Law}_P(u,W)$.
- ▷ is *unique in law* provided whenever $(\Omega^i,\mathcal{F}^i,(\mathcal{F}^i_t),P^i,W^i,u^i)$, $i = 1,2$, are solutions of (0.1) then $\mathfrak{Law}_{P^1}(u^1) = \mathfrak{Law}_{P^2}(u^2)$.
- ▷ has a *strong solution* if, for every probability filtered space $(\Omega,\mathcal{F},(\mathcal{F}_t),P,W)$ with a Q-$(\mathcal{F}_t)$-Wiener process W and an $\mathcal{F}_0$-measurable random variable u_0, there exists a process u such that $(\Omega,\mathcal{F},(\mathcal{F}),P,W,u)$ is a solution, $P[u(0) = u_0] = 1$ and u is $(u_0,W)_P$-adapted, i.e. u is adapted to the P-augmentation of the filtration $\sigma(u_0,W(s,h) : s \le t,\ h \in U)$ in $\mathcal{F}$.

Observe that joint uniqueness in law means uniqueness of the joint distribution measure on the space of functions, and the distribution of the initial condition u_0 in the definition of the strong solution is necessarily μ.

Another remark should be made on the notion of the *strong solution* which, in our definition, comprises more information, namely the adaptation of the solving process to the filtration generated by the initial condition and the driving Wiener process.

Now we can state the main results.

In Theorems 1 and 2, we give sufficient conditions for (0.1) to have a strong solution. Further, we show that pathwise uniqueness implies joint uniqueness in law, and we give a sufficient and necessary complementing condition for joint uniqueness in law to be equivalent to pathwise uniqueness.

Theorem 1. *Suppose that there exists a solution $(\Omega,\mathcal{F},(\mathcal{F}_t),P,W,u)$ such that the process u is $(u(0),W)_P$-adapted. Then*

- ▷ *Equation* (0.1) *has a strong solution.*
- ▷ *If joint uniqueness in law holds for* (0.1) *then so does pathwise uniqueness.*

Theorem 2. *Suppose that there exists a solution $(\Omega,\mathcal{F},(\mathcal{F}_t),P,W,u)$ and pathwise uniqueness holds for* (0.1). *Then*

▷ *Equation* (0.1) *has a strong solution.*

▷ *Joint uniqueness in law holds for* (0.1).

Unlike pathwise uniqueness, the notion of joint uniqueness in law, used in the preceding two theorems as a sufficient or a necessary condition for existence of strong solutions, has not been well investigated in the literature in connection with stochastic evolution equations, and so we are interested in examples of equations that have this property. In Theorem 3 we give a class of equations which are jointly u-unique in law. As a consequence, we find that equations unique in law with one-to-one diffusions are already jointly unique in law.

THEOREM 3. *Let* (S_t) *be a* C_0*-semigroup of bounded linear operators on* $L(X_1)$ *with* X_1 *reflexive. Let* $(\Omega, \mathcal{F}, (\mathcal{F}_t), P, W, u)$ *be a solution of* (0.1) *satisfying* (0.3). *Further suppose that* ([1])

$$dt \otimes P\{(s,\omega) : g(s, u(s,\omega)) \textit{ is not one-to-one in } L(U_0, X_1)\} = 0.$$

Then equation (0.1) *is jointly* u*-unique in law. In particular*, *if* (0.1) *is unique in law and* $g(s,x)$ *is one-to-one for every* $x \in X$ *and almost every* s *then* (0.1) *is jointly unique in law.*

Theorem 4 is an infinite-dimensional extension of a recent result on equivalence of uniqueness in law and joint uniqueness in law for stochastic differential equations in finite dimensions. It states that these two concepts of uniqueness coincide for stochastic equations in Banach spaces provided that the initial condition is deterministic.

THEOREM 4. *Let* (S_t) *be a* C_0*-semigroup of bounded linear operators on* $L(X_1)$ *with* X_1 *reflexive*, $x_0 \in X$ *and suppose that equation* (0.1) *with the initial condition* $\mu = \delta_{x_0}$ *is unique in law among the solutions satisfying* (0.3) *and* (0.4). *If* $S_t x_0 \in X$ *for* $t \in (0,T]$ *then equation* (0.1) *with the initial condition* δ_{x_0} *is jointly unique in law in the class of solutions satisfying* (0.3) *and* (0.4).

Theorem 5 brings an example of a particular equation which is jointly unique in law. Here the diffusion depends only on time and so we speak of a subclass of equations with additive noise. In fact, we can prove the joint uniqueness in law only in a smaller class of solutions determined by the condition (0.7); nonetheless, Theorems 1 and 2 hold in the class of solutions satisfying (0.7) as well.

THEOREM 5. *Let* $f : [0,T] \times X \to U_0$ *be measurable*, $g : [0,T] \to L(U_0, X_1)$ *strongly measurable*,

$$\int_0^t \|S_{t-s} g(s)\|^2_{L_2(U_0,X)}\, ds < \infty$$

for every $t \le T$ *and* μ *a Borel probability measure. Then joint uniqueness in law holds*

([1]) The set of one-to-one operators from $L(U_0, X_1)$ is strongly measurable. We remark that some authors use the word *injective* instead of *one-to-one* and we denote by dt the Lebesgue measure.

for the equation

$$(0.6)\qquad u(t) = S_t u(0) + \int_0^t S_{t-s} g(s) f(s, u(s))\, ds + \int_0^t S_{t-s} g(s)\, dW_s,$$
$$\mathfrak{Law}\,(u(0)) = \mu,$$

in the class of processes satisfying

$$(0.7)\qquad P\Big[\int_0^T \|f(s,u(s))\|_{U_0}^2 < \infty\Big] = 1.$$

Therefore, *if f is locally bounded and each solution has almost surely bounded* (*in particular*, *continuous*) *trajectories*, *then* (0.6) *is jointly unique in law.*

Theorem 6 is our basic tool throughout this work and we find it interesting in itself—therefore it appears in this section. It states that the property of a pair of a process and a Wiener process (u, W) defined on a certain stochastic base to be a solution of equation (0.1) is, in fact, a property of its joint law.

THEOREM 6. *Let $(\Omega^i, \mathcal{F}^i, (\mathcal{F}_t^i), P^i, W^i, u^i)$ be a probability filtered space*, *W^i a Q-$(\mathcal{F}_t^i)$-Wiener process and u^i a progressively measurable process*, $i = 1, 2$. *Suppose that $\mathfrak{Law}_{P^1}(u^1, W^1) = \mathfrak{Law}_{P^2}(u^2, W^2)$. If (u^1, W^1) satisfies* (0.1), (0.2) *for every $t \le T$ then so does (u^2, W^2).*

Results of similar nature. Apart from joint uniqueness in law and pathwise uniqueness we can also define, according to H. J. Engelbert, finer types of uniqueness so that the implications in Theorem 1 and 2 turn into equivalences. For instance, we know by Theorem 2 that pathwise uniqueness implies joint uniqueness in law. In this section we will produce an additional condition under which joint uniqueness in law becomes equivalent to pathwise uniqueness.

DEFINITION 7. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), P, W, u)$ be a solution of equation (0.1). We will say that (u, W)-*pathwise uniqueness* holds for (0.1) provided that whenever $(\Omega', \mathcal{F}', (\mathcal{F}_t'), P', W', u_1')$, $(\Omega', \mathcal{F}', (\mathcal{F}_t'), P', W', u_2')$ are solutions of (0.1) such that

▷ $\mathfrak{Law}_{P'}(u_1', W') = \mathfrak{Law}_{P'}(u_2', W') = \mathfrak{Law}_P(u, W)$,
▷ $P'[u_1'(0) = u_2'(0)] = 1$,

then $P'[u_1'(t) = u_2'(t)] = 1$ for every $t \le T$.

We also say that *u-pathwise uniqueness* holds for equation (0.1) provided that whenever $(\Omega', \mathcal{F}', (\mathcal{F}_t'), P', W', u_1')$, $(\Omega', \mathcal{F}', (\mathcal{F}_t'), P', W', u_2')$ are solutions of (0.1) such that

▷ $\mathfrak{Law}_{P'}(u_1') = \mathfrak{Law}_{P'}(u_2') = \mathfrak{Law}_P(u)$,
▷ $P'[u_1'(0) = u_2'(0)] = 1$,

then $P'[u_1'(t) = u_2'(t)] = 1$ for every $t \le T$.

The last property we need to define is *uniqueness in law* which is said to hold for (0.1) provided that $\mathfrak{Law}_{P^1}(u^1)$ coincides with $\mathfrak{Law}_{P^2}(u^2)$ for any two solutions $(\Omega^1, \mathcal{F}^1, (\mathcal{F}_t^1), P^1, W^1, u^1)$, $(\Omega^2, \mathcal{F}^2, (\mathcal{F}_t^2), P^2, W^2, u^2)$.

The following theorem gives conditions on (0.1) equivalent to pathwise uniqueness.

Theorem 8. *The following conditions on equation* (0.1) *are equivalent*:

(1) *Pathwise uniqueness holds and there exists a solution.*
(2) *Joint uniqueness in law holds and there exists a solution* $(\Omega, \mathcal{F}, (\mathcal{F}_t), P, W, u)$ *such that* (u, W)*-pathwise uniqueness holds.*
(3) *Joint uniqueness in law holds and there exists a solution* $(\Omega, \mathcal{F}, (\mathcal{F}_t), P, W, u)$ *such that* u *is* $(u(0), W)_P$*-adapted.*

In Theorem 9 we characterize the notion of (u, W)-pathwise uniqueness used in Theorem 8.

Theorem 9. *Suppose that* $(\Omega, \mathcal{F}, (\mathcal{F}_t), P, W, u)$ *is a solution of* (0.1). *Then the following conditions are equivalent*:

(1) (u, W)*-pathwise uniqueness holds.*
(2) *There exists a solution* $(\Omega', \mathcal{F}', (\mathcal{F}'_t), P', W', u')$ *such that* u' *is* $(u'(0), W')_{P'}$*-adapted and* $\mathfrak{Law}_{P'}(u', W') = \mathfrak{Law}_P(u, W)$.

In the following theorem we return to the notion of joint u-uniqueness in law which has already appeared in Theorem 3, and we clarify its position among the other types of pathwise uniqueness that we defined in this section.

Theorem 10. *Let* $(\Omega, \mathcal{F}, (\mathcal{F}_t), P, W, u)$ *be a solution of* (0.1). *Then the following conditions on* (0.1) *are equivalent*:

(1) u*-pathwise uniqueness holds.*
(2) *Joint* u*-uniqueness in law holds, and there exists a solution* $(\Omega', \mathcal{F}', (\mathcal{F}'_t), P', W', u')$ *such that* $\mathfrak{Law}_P(u) = \mathfrak{Law}_{P'}(u')$ *and* (u', W')*-pathwise uniqueness holds.*
(3) *Joint* u*-uniqueness in law holds, and there exists a solution* $(\Omega', \mathcal{F}', (\mathcal{F}'_t), P', W', u')$ *such that* $\mathfrak{Law}_{P'}(u')$ *coincides with* $\mathfrak{Law}_P(u)$ *and* u' *is* $(u'(0), W')_{P'}$*-adapted.*

We close this section by a straightforward comparison of joint uniqueness in law and uniqueness in law.

Theorem 11. *The following conditions are equivalent for* (0.1):

- ▷ *Joint uniqueness in law holds.*
- ▷ *Uniqueness in law holds and joint* u*-uniqueness in law holds for every solution* $(\Omega, \mathcal{F}, (\mathcal{F}_t), P, W, u)$.

Equivalent concepts of solutions. This section is devoted to various concepts of solutions to stochastic evolution equations in Banach spaces. Our definition of a solution to (0.1) is based on the variation-of-constants formula, where (S_t) is usually a C_0-semigroup of bounded linear operators on X; however, this is not the only definition used in the literature, and even in this paper, we will need to approach the solutions from different points of view.

We will be concerned with a more general problem than solution of a stochastic evolution equation. Namely, we will study three possible mathematically correct definitions of a formal stochastic differential

$$du = (Au + f)\,dt + g\,dW.$$

We assume that u, the drift f and the diffusion g are arbitrary progressively measurable processes, where no apriori mutual dependence between u, f and g is excluded. Hence we cover the problem of solutions of SPDE's.

The following two theorems state that different definitions of the above stochastic differential are equivalent. The only reason why we cannot compare all of them at once is that each definition demands different integrability assumptions. We remark that these results are essentially generalizations of the Chojnowska-Michalik theorem (see [ChM]) and the main tool in the proofs is the stochastic Fubini theorem (Proposition 6.1) that will also be proved in what follows.

THEOREM 12. *Let f be a progressively measurable process in X, g a progressively measurable process in $L_2(U_0, X)$, (S_t) a strongly continuous semigroup of linear operators on X generated by A, W a Q-Wiener process and u a progressively measurable X-valued process. Let also*

$$P\Big[\int_0^T (\|f(s)\| + \|g(s)\|^2_{L_2(U_0,X)})\,ds < \infty\Big] = 1.$$

Then

$$\text{(a)} \qquad P\Big[u(t) = S_t u(0) + \int_0^t S_{t-s} f(s)\,ds + \int_0^t S_{t-s} g(s)\,dW_s\Big] = 1$$

for every $t \le T$ if and only if u has a predictable modification with almost all trajectories in $L^1(0,T;X)$ such that

$$P\Big[\int_0^t u(s)\,ds \in D(A)\Big] = 1$$

and

$$\text{(b)} \qquad P\Big[u(t) = u(0) + A\int_0^t u(s)\,ds + \int_0^t f(s)\,ds + \int_0^t g(s)\,dW_s\Big] = 1$$

hold for every $t \le T$.

In that case

1. *The process $t \mapsto R_\lambda u(t)$ has a modification which is a norm continuous semimartingale for every λ from the resolvent set of A.*
2. *The process $t \mapsto \langle x^*, u(t)\rangle$ has a modification which is a continuous semimartingale for every $x^* \in D(A^*)$. In particular there exists a sequence $x^*_n \in D(A^*)$ which separates points of X such that $t \mapsto \langle x^*_n, u(t)\rangle$ is continuous for every $n \in \mathbb{N}$.*

THEOREM 13. *Let f be a progressively measurable process in X, g a progressively strongly measurable process in $L(U_0, X)$, (S_t) a strongly continuous semigroup of linear operators on X generated by A, W a Q-Wiener process and u a progressively measurable X-valued* *process. Let also*

$$P\Big[\int_0^T (\|f(s)\| + \|g(s)\|^2_{L(U_0,X)})\,ds < \infty\Big] = P\Big[\int_0^t \|S_{t-s}g(s)\|^2_{L_2(U_0,X)}\,ds < \infty\Big] = 1$$

for every $t \le T$. Then

$$P\Big[u(t) = S_t u(0) + \int_0^t S_{t-s}f(s)\,ds + \int_0^t S_{t-s}g(s)\,dW_s\Big] = 1 \tag{a}$$

holds for every $t \le T$ if and only if u satisfies

$$P\Big[\int_0^T\int_0^T |\langle x^*(t,s), u(s)\rangle|\,ds\,dt < \infty\Big] = 1 \tag{b}$$

for every measurable bounded function $x^* : [0,T]\times[0,T] \to X^*$ and

$$P\Big[\langle x^*, u_t\rangle = \langle x^*, u_0\rangle + \int_0^t \langle A^*x^*, u_s\rangle\,ds + \int_0^t \langle x^*, f_s\rangle\,ds + \int_0^t g_s^* x^*\,dW_s\Big] = 1 \tag{c}$$

for every $t \le T$, $x^* \in D(A^*)$.

Moreover, in that case, the conclusion (2) *of Theorem* 12 *holds and u has a predictable modification.*

Ideas of the proofs. Theorem 1 as well as Theorem 2 are consequences of the following general phenomenon. In both, and many other cases, there exists a time sequence of measurable functions $(R_t : t \le T)$ such that $P[u(t) = R_t(u(0), W)] = 1$ for every $t \le T$ (Lemma E). This means that the solution u depends only on the initial value and on the corresponding trajectory of the Wiener process. Moreover this dependence comes through the measurable transformations (R_t). Now it is enough to prove that whenever we take a filtered probability space $(\overline{\Omega}, \overline{\mathcal{F}}, (\overline{\mathcal{F}}_t), \overline{P}, \overline{W})$ with a Q-Wiener process $\overline{W}$ and an initial μ-distributed random variable $\overline{u}_0$ then the process $\overline{u}(t) = R_t(\overline{u}_0, \overline{W})$ completes the family $(\overline{\Omega}, \overline{\mathcal{F}}, (\overline{\mathcal{F}}_t), \overline{P}, \overline{W}, \overline{u})$ to be a strong solution starting from $\overline{u}_0$.

If we are in the situation of Theorem 1 then joint uniqueness in law implies pathwise uniqueness. Indeed, if $(\overline{\Omega}, \overline{\mathcal{F}}, (\overline{\mathcal{F}}_t), \overline{P}, \overline{W}, \overline{v})$ were another solution starting from $\overline{u}_0$ then, by joint uniqueness in law, we would have

$$\overline{P}[\overline{v}(t) = R_t(\overline{v}(0), \overline{W})] = \overline{P}[\overline{u}(t) = R_t(\overline{u}(0), \overline{W})] = 1$$

for every $t \le T$. But $\overline{P}[\overline{v}(0) = \overline{u}(0)] = 1$ and thus $\overline{P}[\overline{v}(t) = R_t(\overline{u}(0), \overline{W}) = \overline{u}(t)] = 1$ for every $t \le T$.

In the situation of Theorem 2 we must prove the uniqueness of the joint solution measure on the space of functions. Suppose that $(\Omega^i, \mathcal{F}^i, (\mathcal{F}^i_t), P^i, W^i, u^i)$, $i = 1,2$, are two solutions. Then, by pathwise uniqueness, we have $u^i(t) = R_t(u^i(0), W^i)$, $t \le T$, $i = 1,2$, and we see that to show the equality of the joint solution measures $\mathfrak{Law}_{P^i}(u^i, W^i)$, $i = 1,2$, it suffices to show the equality of $\mathfrak{Law}_{P^i}(u^i(0), W^i)$, $i = 1,2$, since (R_t) are measurable transformations. But $u^i(0)$ is $\mathcal{F}^i_0$-measurable and thus P^i-independent of W^i, hence $\mathfrak{Law}_{P^i}(u^i(0), W^i) = \mu \otimes \mathfrak{Law}_{P^i}(W^i)$. But $\mathfrak{Law}_{P^1}(W^1) = \mathfrak{Law}_{P^2}(W^2)$ because W^1 and W^2 have the same covariance Q.

In the proof of Theorems 3 and 4 we use a sort of "inversion formula" to express the intervening Wiener process in terms of the solution u whose distributions are supposed to coincide. We wish to write $W = \int g^{-1}(s)\,du$ for $u = \int g(s)\,dW$ but unfortunately, in our case we must proceed in steps using approximations given by measurable selectors (Theorem 8.8).

Theorem 5 is based on the Girsanov theorem (Proposition 7.1) and relies heavily (as do all results in this paper) on the fact that solutions of (0.1) are completely determined by the joint distribution $\mathfrak{Law}(u, W)$ of the solving process u and the Wiener process W. More precisely, if $(\Omega^1, \mathcal{F}^1, (\mathcal{F}_t^1), P^1, W^1, u^1)$ satisfies (0.1), (0.2) and u^1 is $(\mathcal{F}_t^1)$-progressively measurable (we do not assume any kind of path continuity of u^1) and $(\Omega^2, \mathcal{F}^2, (\mathcal{F}_t^2), P^2, W^2, u^2)$ is a filtered probability space with a Q-Wiener process W^2 and $(\mathcal{F}_t^2)$-progressively measurable process u^2 such that

$$\mathfrak{Law}_{P^1}(u^1(t_i), W^1(t_i, h_j) : i, j) = \mathfrak{Law}_{P^2}(u^2(t_i), W^2(t_i, h_j) : i, j)$$

for every partition $0 = t_0 < \cdots < t_n \le T$ and every finite number of vectors $h_1, \ldots, h_m$ in U then $(\Omega^2, \mathcal{F}^2, (\mathcal{F}_t^2), P^2, W^2, u^2)$ satisfies (0.1) and (0.2), which is just a summary of Theorem 6.

1. Cylindrical Wiener process

A classical stochastic process u in a separable Banach space X is a mapping from $[0,T]\times\Omega$ to X such that the restrictions $u_t : \Omega \to X$ are measurable for all $t \le T$. On the other hand, sometimes it is convenient to generalize this notion to a larger class of objects, the cylindrical processes. From the probabilistic point of view they are two-parameter real processes $(u(t, x^*) : t \le T,\ x^* \in X^*)$, where the first variable corresponds to time while the second to the elements of the topological dual space X^*. Moreover we want the linearity in the x^* variable. The motivation is the following. Suppose that we are given a classical process $(u(t) : t \le T)$ in X. Then $u(t, x^*) = \langle x^*, u(t)\rangle$ is a cylindrical process—it represents the decomposition of the classical process into coordinates. One of the reasons for introducing cylindrical processes is that we can define a Wiener process of a nonnuclear covariance and a stochastic integral with respect to it.

Definition 1.1. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(M(x^*) : x^* \in X^*)$ a family of real processes on $[0,T]$ such that $P[M_t(ax^* + y^*) = aM_t(x^*) + M_t(y^*)] = 1$ for every $t \in [0,T]$, $a \in \mathbb{R}$, $x^* \in X^*$, $y^* \in X^*$. Then M is called a *cylindrical process.*

Definition 1.2. We say that a cylindrical process $M = (M(x^*) : x^* \in X^*)$ on $[0,T]$ is *representable* provided there exists a stochastic process u in X defined on $[0,T]$ such that $P[\langle x^*, u_t\rangle = M_t(x^*)] = 1$ for all $t \le T$ and $x^* \in X^*$. Then we say that u is a *representation* of M; obviously, u is unique up to modification.

Definition 1.3. Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $(\mathcal{F}_t)$ and $\sigma \ge 0$. Then a continuous real $(\mathcal{F}_t)$-adapted process W on $[0,T]$ is called a *Wiener process with covariance* σ^2 provided

▷ $P[W_0 = 0] = 1$.
▷ $\sigma(W_t - W_s)$ is P-independent of $\mathcal{F}_s$ whenever $0 \le s < t \le T$.
▷ $\mathfrak{Law}_P(W_t - W_s) = \mathcal{N}(0, (t-s)\sigma^2)$ whenever $0 \le s < t \le T$.

In case $\sigma^2 = 1$ we say W is a *standard Wiener process.*

DEFINITION 1.4 Let U be a separable Hilbert space. A cylindrical process $W = (W(u) : u \in U)$ on $[0,T]$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ is called a *cylindrical Wiener process* provided that $(W_t(u) : t \le T)$ is an $(\mathcal{F}_t)$-Wiener process with covariance possibly depending on u (Def. 1.3) for every $u \in U$ and there exists a positive constant c such that $EW_T^2(u) \le c^2 \|u\|_U^2$, $u \in U$. The *covariance operator* of W is the unique operator $Q \in L(U)$ with $Q^* = Q$, $Q \ge 0$ such that

$$EW_t(x)W_s(y) = s\langle Qx, y\rangle_U = \langle W(x), W(y)\rangle(s)$$

for every $0 \le s < t \le T$, $x \in U$, $y \in U$, where $s \mapsto \langle W(x), W(y)\rangle(s)$ is the cross-variation process associated to $W(x)W(y)$.

Proof of the existence of Q. The mapping $(x,y) \mapsto EW_T(x)W_T(y)$ is a real bounded symmetric positive bilinear form on $U \times U$ so there exists a bounded symmetric positive operator Q on U satisfying $EW_T(x)W_T(y) = T\langle Qx, y\rangle_U$ for all $x, y \in U$:

$$\begin{aligned} EW_t(x)W_s(y) &= EE[W_t(x)/\mathcal{F}_s]W_s(y) = EW_s(x)W_s(y) \\ &= \frac{1}{4}\, E(W_s^2(x+y) - W_s^2(x-y)) = \frac{s}{4T}\, E(W_T^2(x+y) - W_T^2(x-y)) \\ &= s\langle Qx, y\rangle_U = \frac{1}{4}\,(\langle W(x+y)\rangle_s - \langle W(x-y)\rangle_s) = \langle W(x), W(y)\rangle_s. \ \blacksquare \end{aligned}$$

Now we will show that there exists a filtered probability space with a cylindrical Wiener process with given covariance operator Q:

CLAIM 1.5. *Let $Q \in L(U)$ with $Q^* = Q$, $Q \ge 0$. Then there exists a probability space $(\Omega, \mathcal{F}, P)$ with a complete filtration $(\mathcal{F}_t)$ and a cylindrical Wiener process $W = (W(u) : u \in U)$ on $[0,T]$ with covariance operator Q.*

Proof. Choose an orthonormal basis $(u_k : k)$ in U and a probability space $(\Omega, \mathcal{F}, P)$ with a complete filtration $(\mathcal{F}_t)$ in $\mathcal{F}$ which carries independent standard real Wiener processes $(\beta^k : k)$. Then define the real Wiener processes

$$W(u) = \sum_k \langle Q^{1/2}u, u_k\rangle_U \beta^k$$

for $u \in U$, where the sum converges uniformly in t in $L^2(\Omega)$. The family $(W(u) : u \in U)$ is a cylindrical Wiener process on U with covariance operator Q. $\blacksquare$

Now we will show the connection between cylindrical Wiener processes and classical Wiener processes. We know that the covariance operator of a Gaussian measure on a separable Hilbert space is necessarily nuclear, so every classical Wiener process is of nuclear covariance. The following theorem says that a cylindrical Wiener process is representable (Def. 1.2) if and only if the covariance operator is nuclear, and in that case the representation is a classical Wiener process.

Theorem 1.6. *Let W be a cylindrical Wiener process on $[0,T]$ with covariance operator Q. Then W has a continuous representation on $[0,T]$ if and only if Q is nuclear. In that case the representation is a U-valued Wiener process with covariance Q.*

Proof. Let $(u_k : k)$ be an orthonormal basis in U. Then the sum $\sum_k W(u_k)u_k$ converges in $L^2(\Omega, C([0,T],U))$ iff $\sum_k \|Q^{1/2}u_k\|^2 < \infty$ due to the Doob maximal inequality. So we only have to prove that the representation (when Q is nuclear) is a Wiener process. The fact that the increments are centered Gaussian follows immediately from the explicit formula $\sum_k W(u_k)u_k$ and the independence of the increments from Lévy's characterization theorem applied to the martingale $(W(u_k) : k \le N)$. ■

2. Radonifying mappings and the space U_0

The reason for the following definition is that we will work with Wiener processes with arbitrary covariance operators and as we will see later the only important information (regarding stochastic integration) lies in the reproducing kernel space U_0 which is continuously embedded in U.

Definition 2.1. Since $Q \in L(U)$ is a nonnegative operator we may define the square root $Q^{1/2} \in L(U)$ and $Q^{-1/2} : \mathrm{Rng}(Q^{1/2}) \to U$ defined as the inverse mapping of the one-to-one restriction $Q^{1/2}|_D : D \to \mathrm{Rng}(Q^{1/2})$, where $D = (\mathrm{Ker}\, Q^{1/2})^\perp$ is the orthogonal complement of $\mathrm{Ker}\, Q^{1/2}$ in U. We also define the separable Hilbert space $U_0 = \mathrm{Rng}(Q^{1/2}) \subseteq U$ with the inner product

$$\langle g_1, g_2\rangle_0 = \langle Q^{-1/2}g_1, Q^{-1/2}g_2\rangle_U, \quad g_1 \in U_0,\ g_2 \in U_0.$$

Now the mapping $Q^{1/2} : (D, \|\cdot\|_U) \to (U_0, \|\cdot\|_0)$ is an isometry.

Before we state some useful properties where the space U_0 intervenes we shall recall the definition of a *radonifying* operator. We know that the class of Hilbert–Schmidt operators is the state space for processes which are integrated with respect to a Wiener process in the Hilbertian case. If we want to pass to Banach spaces the class of radonifying operators appears. The following theorem is a synthesis of results due to Itô, Nisio, Fernique, Hoffmann-Jørgensen and Kwapień.

Theorem 2.2. *Let $(\eta_n : n \in \mathbb{N})$ be a sequence of real independent identically distributed centered Gaussian random variables, and $(x_n : n \in \mathbb{N})$ a sequence in a separable Banach space Y. Let $0 < p < \infty$ and define*

$$s_k = \sum_{n=1}^{k} \eta_n x_n, \quad k \in \mathbb{N}.$$

Then the following statements are equivalent:

- ▷ *The sequence $(s_k : k \in \mathbb{N})$ converges in L^p.*
- ▷ *The sequence $(s_k : k \in \mathbb{N})$ converges in norm almost surely.*

▷ *There exists a Borel probability measure ν on Y such that $\mathfrak{Law}(\langle x^*, s_k\rangle) \to \mathfrak{Law}_\nu(x^*)$ weakly in the space of measures for every $x^* \in Y^*$.*

In the third case, the measure ν is the distribution of the limit.

If, moreover, Y does not contain any subspace linearly homeomorphic to c_0 (e.g. a reflexive space) then the above conditions are also equivalent to:

▷ *The sequence $(s_k : k \in \mathbb{N})$ is bounded in L^p.*

Proof. See Chapter V in [ChTV]. ■

DEFINITION 2.3. Let U be a separable Hilbert space and (ξ_n) a sequence of independent standard Gaussian random variables defined on a probability space Ω. An operator $A \in L(U, X)$ is called *radonifying* provided that the series $\sum_n \xi_n A u_n$ converges in $L^2(\Omega, X)$ for some orthonormal basis (u_n) in U. We denote by $L_2(U, X)$ the space of radonifying operators and set

$$\|A\|^2_{L_2(U,X)} = E\Big\| \sum_n \xi_n A u_n \Big\|^2_X .$$

We also write $\|A\|_{L_2(U,X)} = \infty$ for $A \notin L_2(U, X)$.

It may seem that the definition of $L_2(U, X)$ and $\|A\|_{L_2(U,X)}$ depends on the choice of the orthonormal basis (u_n) and (ξ_n), but in fact it does not. Once $\sum_n \xi_n A u_n$ converges in $L^2(\Omega, X)$ it converges for all choices of orthonormal bases in U and for all choices of independent standard Gaussian random variables due to Theorem 2.2 because we already know that the probability distribution $\mathfrak{Law}(\sum_n \xi_n A u_n)$ is the Borel centered Gaussian measure on X with covariance $AA^* \in L(X^*, X)$, hence independent of (u_n) and (ξ_n), and

$$\|A\|^2_{L_2(U,X)} = \int_X \|x\|^2 \, d\mathcal{N}(0, AA^*).$$

Another consequence of Theorem 2.2 is that $A \in L_2(U, X)$ if and only if $\mathcal{N}(0, AA^*)$ exists as a Borel measure on X. For further details see [ChTV].

The following proposition is a handy tool for verification whether a composition of two operators is a radonifying operator provided either of them is. Indeed, we see that L_2 is an operator ideal. Moreover one can easily show, using Pietsch's factorization, that every p-summing operator is already radonifying (e.g. [LP]). Proposition 2.4 is due to W. Linde & A. Pietsch but our proof is based on Kahane's contraction principle. We note that $L_2(U, X)$ is an operator ideal even if X contains c_0 (e.g. [Ba]).

PROPOSITION 2.4. *Suppose that X does not contain any subspace linearly homeomorphic to c_0 and let $A \in L(U, X)$. Then the following conditions are equivalent.*

(1) *$A \in L_2(U, X)$.*

(2) *There exists $K \in [0, \infty)$ such that if $(\eta_n : n \in \mathbb{N})$ is a sequence of real standard Gaussian random variables then*

$$(*) \qquad E\Big\| \sum_{k=1}^n \eta_k A h_k \Big\|^2 \le K^2 \sup \Big\{ \sum_{k=1}^n \langle h, h_k\rangle^2 : \|h\| \le 1 \Big\}$$

for every $h_1, \ldots, h_n$ in U.

If these conditions hold, then $\|A\|_{L_2(U,X)}$ is the minimal K such that $(*)$ holds.

Proof. (2) implies (1) by Theorem 2.2. Suppose that (1) holds. Take arbitrary vectors $h_1, \ldots, h_n$ in U and an arbitrary orthonormal set $e_1, \ldots, e_n$ in U which contains $h_1, \ldots, h_n$ in its linear span. The left hand side of $(*)$ is now of the form

$$E\Big\|\sum_{k=1}^n\sum_{l=1}^n \eta_k f_{kl} A e_l\Big\|^2,$$

where $f_{kl} = \langle h_k, e_l\rangle$ is an $n\times n$-matrix. We can decompose (f_{kl}) into a matrix product $B_{n\times n}D_{n\times n}C_{n\times n}$ where B and C are unitary and D is diagonal. If we set $\xi = B^*\eta$, $\theta = C^*\xi$, $y_i = \sum_{l=1}^n c_{il}Ae_l$ then the left hand side of $(*)$ equals

$$E\Big\|\sum_{i=1}^n d_{ii}\xi_i y_i\Big\|^2 \le \max\{d_{ii}^2 : i\le n\}E\Big\|\sum_{i=1}^n \xi_i y_i\Big\|^2 = \|(f_{kl})\|^2 E\Big\|\sum_{l=1}^n \theta_l A e_l\Big\|^2$$

by the contraction principle (e.g. [ChTV, V.4, Proposition 4.1]) because ξ and θ are $\mathcal{N}(0, I_n)$-distributed. But

$$\|(f_{kl})\|^2 = \sup\Big\{\sum_{i=1}^n \langle h, h_i\rangle^2 : \|h\|\le 1\Big\}.$$

If we took $h_i = 0$ for $m < i \le n$ we would have

$$E\Big\|\sum_{k=1}^m \eta_k A h_k\Big\|^2 \le E\Big\|\sum_{l=1}^n \theta_l A e_l\Big\|^2 \sup\Big\{\sum_{k=1}^m \langle h, h_k\rangle^2 : \|h\|\le 1\Big\},$$

and consequently, letting $n\to\infty$,

$$E\Big\|\sum_{k=1}^m \eta_k A h_k\Big\|^2 \le \|A\|^2_{L_2(U,X)} \sup\Big\{\sum_{k=1}^m \langle h, h_k\rangle^2 : \|h\|\le 1\Big\}. \ \blacksquare$$

Now we are going to give a series of simple propositions leading to the fact that $(L_2(U,X), \|\ \|_{L_2(U,X)})$ is a separable Banach space which, in case X is a Hilbert space, coincides isometrically with the Hilbert–Schmidt operators. Moreover item (5) in the following proposition is the key to verifying whether an $L_2(U,X)$-valued mapping is Borel measurable or not.

Proposition 2.5. *Suppose that $A\in L(U,X)$ is radonifying. Then*

(1) $\|A\|_{L(U,X)} \le \|A\|_{L_2(U,X)}$.
(2) *$L_2(U,X)$ is a linear space and $\|\cdot\|_{L_2(U,X)}$ is a norm.*
(3) *$u\otimes x \in L_2(U,X)$ for all $u\in U$, $x\in X$, $\|u\otimes x\|_{L_2(U,X)} = \|u\|_U\|x\|_X$ and the finite-dimensional operators are dense in $L_2(U,X)$, so that $L_2(U,X)$ is separable.*
(4) *$\|\cdot\|_{L_2(U,X)}$ is complete.*
(5) *The Borel σ-algebra on the separable Banach space $L_2(U,X)$ is generated by the mappings $L_2(U,X)\to X : A\mapsto Au$, $u\in U$.*
(6) *$L_2(U,X)$ is a strongly measurable subset of $L(U,X)$.*

Proof. (1) The measure

$$\mu = \mathfrak{Law}\Big(\sum_n \xi_n A u_n\Big)$$

is Gaussian and centered on X and $\mu\{x : \langle x^*, x\rangle \in B\} = \mathcal{N}(0, \|A^*x^*\|_U^2)(B)$ for every $x^* \in X^*$, $B \in \mathbb{B}(\mathbb{R})$. Thus $\|A^*x^*\|_U \le \|A\|_{L_2(U,X)}$ for every $\|x^*\| \le 1$.

(2) and (3) are obvious.

(4) If (A_m) is $\|\ \|_{L_2(U,X)}$-Cauchy then it converges to $A \in L(U,X)$ in the uniform operator topology due to (1) and $\sum_n \xi_n A u_n$ coincides with the limit of $\sum_n \xi_n A_m u_n$ in $L^2(\Omega, X)$ due to Theorem 2.2.

(5) The mapping $A \mapsto Au$ is continuous for every $u \in U$ by (1), thus Borel measurable. On the other hand, if we denote by σ the σ-algebra generated by the mappings $A \mapsto Au$, $u \in U$, and we fix $B \in L_2(U,X)$ then the real mapping

$$A \mapsto E\Big\|\sum_k \xi_k(Au_k - Bu_k)\Big\|_X^2$$

is σ-measurable. Hence every ball in $L_2(U,X)$ belongs to σ, which ends the proof because $L_2(U,X)$ is separable.

(6) We have

$$L_2(U,X) = \bigcap_{k=1}^{\infty}\bigcup_{n=1}^{\infty}\bigcap_{m=n}^{\infty}\Big\{A \in L(U,X) : E\Big\|\sum_{j=n}^{m}\xi_j A u_j\Big\|^2 \le 1/k\Big\}$$

due to the completeness of $L^2(\Omega, X)$. ∎

Now we return to Definitions 2.1 and 2.3 and state a few simple observations which take the space U_0 into account.

NOTE 2.6. *Let $A : U_0 \to X$ be a linear mapping and $B \in L(U,X)$. Then*

(1) $\|A\|_{L(U_0,X)} = \|AQ^{1/2}\|_{L(U,X)}$.
(2) $\|A\|_{L_2(U_0,X)} = \|AQ^{1/2}\|_{L_2(U,X)}$.
(3) $\|A^*x^*\|_{U_0} = \|(AQ^{1/2})^*x^*\|_U$ *for every* $x^* \in X^*$ *provided* $A \in L(U_0,X)$.
(4) $B|_{U_0} \in L_2(U_0,X)$ *and* $\|B\|_{L_2(U_0,X)} \le \|B\|_{L(U,X)}(\operatorname{Tr} Q)^{1/2}$ *provided* $Q \in L(U)$ *is nuclear.*
(5) $\overline{U_0}^U = D$.

Proof. (1)–(3) are direct consequences of the definitions and of the fact that $Q^{1/2}$ is an isometric isomorphism between D and U_0.

(4) Since $Q^{1/2}$ is Hilbert–Schmidt the series $\sum_k \xi_k Q^{1/2} u_n$ converges in $L^2(\Omega, U)$. Hence $\sum_k \xi_k B Q^{1/2} u_n$ converges in $L^2(\Omega, X)$ and the estimate follows immediately.

(5) $\overline{U_0}^U = \overline{\operatorname{Rng}(Q^{1/2})}^U = \overline{\operatorname{Rng}(Q^{1/2})^*}^U = (\operatorname{Ker} Q^{1/2})^\perp = D$. ∎

3. Stochastic integral

One of the reasons for including the construction of the stochastic integral was its straightforward Itô style—we have dropped the necessity of auxiliary spaces and embeddings (e.g. [DZ], [B1]–[B3] or [BG]), which, later on, will make all manipulations more transparent. None the less, we have arrived at the habitual stochastic integral whose properties we summarize at the end.

Let us consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P, W)$ with a Q-$(\mathcal{F}_t)$-Wiener process on U and X a 2-smooth Banach space (Def. 3.1).

Step 1. Since W is not necessarily a process in U we cannot define the X-valued random variable $A(W_t)$ for every $A \in L(U,X)$ but we can do so for A finite-dimensional (i.e. $Au = \sum_{k=1}^n \langle u, u_k\rangle x_k$ for some $u_k \in U$, $x_k \in X$) in the following way:

$$AW_t = \sum_{k=1}^n W_t(u_k)x_k.$$

We can easily see that $\langle x^*, AW_t\rangle = W_t(A^*x^*)$, hence the definition of AW_t is independent of the expansion of A, and AW_t coincides with the composition $A(W_t)$ provided W is a U-valued process (i.e. when the covariance Q is nuclear).

Step 2 (elementary integral). Now we are going to integrate simple finite-dimensional valued processes. Let ψ be a process with values in $L(U,X)$ such that $\psi(t) = \sum_{j=1}^m A_{ij} I_{F^{ij}}$ for $t_i < t \le t_{i+1}$, $i \le n$ for some partition $0 = t_1 < \cdots < t_{n+1} = T$, $(F^{ij} : j \le m)$ an $\mathcal{F}_{t_i}$-decomposition of Ω and A_{ij}, $i \le n$, $j \le m$, finite-dimensional operators in $L(U,X)$. Then the process

$$t \mapsto \int_0^t \psi\, dW = \sum_{i=1}^n \sum_{j=1}^m (A_{ij} W_{t\wedge t_{i+1}} - A_{ij} W_{t\wedge t_i}) I_{F^{ij}}$$

is a norm continuous L^2-martingale in X and

$$\left\langle x^*, \int_0^t \psi\, dW \right\rangle = \sum_{i=1}^n \sum_{j=1}^m \int_0^t I_{F^{ij}} I_{(t_i, t_{i+1}]}\, dW(A_{ij}x^*).$$

This definition is classical and one can define $\int_s^t \psi\, dW$, $0 \le s \le t$, in the same spirit as well.

Before we proceed to Step 3 we recall some properties and examples of 2-smooth Banach spaces.

Definition 3.1. A Banach space X is called 2-*smooth* provided there exists an equivalent norm $\|\ \|$ and a constant $c \ge 2$ such that $\|x+y\|^2 + \|x-y\|^2 \le 2\|x\|^2 + c\|y\|^2$ for any $x, y \in X$.

We note that there are other equivalent definitions, for instance in terms of asymptoticity of the modulus of smoothness of the norm due to P. Assouad, T. Figiel, J. Hoffmann-Jørgensen, G. Pisier (e.g. [P]) or in terms of martingale estimation [A], [P]. We chose the above definition because one can easily show by C^2-smoothness of $\|\cdot\|^\beta$, $\beta > 2$, that $L^p(\mu)$ spaces with arbitrary positive, not necessarily σ-finite measures μ are 2-smooth for every $2 \le p < \infty$. Obviously, by the parallelogram law, every Hilbert space is 2-smooth and closed subspaces and products of 2-smooth spaces are 2-smooth, hence the Sobolev spaces $W^{k,p}$ are 2-smooth for $2 \le p < \infty$, $k \ge 0$. Also, if A generates a holomorphic semigroup on a 2-smooth Banach space then the domains of the fractional powers $D(-A)^\alpha$, $\alpha \ge 0$, with the graph norm are 2-smooth since they are isometric isomorphs of X. Another observation is that a 2-smooth Banach space X is uniformly smooth, hence X is necessarily reflexive.

The following simple observation was made by P. Assouad [A] and it is the key to the forthcoming construction.

LEMMA 3.2. *Let X be a 2-smooth Banach space. Then there exists a constant C such that*

$$E\|M_n\|^2 \le C \sum_{k=1}^{n} E\|M_k - M_{k-1}\|^2, \qquad 1 \le n \le N,$$

for every L^2-martingale $(M_k, \mathcal{F}_k : k = 0, \ldots, N)$, $M_0 = 0$.

Proof. Let $\eta_2 \in L^2$, $\mathcal{A}$ a sub-σ-algebra and define $\eta_1 = E[\eta_2 \mid \mathcal{A}]$. Then

$$E\|2\eta_1 - \eta_2\|^2 + E\|\eta_2\|^2 \le 2E\|\eta_1\|^2 + cE\|\eta_2 - \eta_1\|^2$$

by 2-smoothness and

$$\|\eta_1\|^2 = \|E[2\eta_1 - \eta_2 \mid \mathcal{A}]\|^2 \le E[\|2\eta_1 - \eta_2\|^2 \mid \mathcal{A}].$$

Hence $E\|\eta_2\|^2 \le E\|\eta_1\|^2 + cE\|\eta_2 - \eta_1\|^2$ and the result follows by induction, applied step by step, on the martingale M. The constant c may change after returning to the original norm. ■

Step 3 (Burkholder inequality). There exist constants C_p, $0 < p < \infty$, such that the following estimate holds for every ψ of the form we have considered in Step 2:

$$E \sup\Big\{\Big\|\int_0^s \psi\, dW\Big\|^p : s \le t\Big\} \le C_p E\Big(\int_0^t \|\psi Q^{1/2}\|^2_{L_2(U,X)}\, ds\Big)^{p/2}. \tag{3.2}$$

The proof for p different from 2 will be postponed until (5.1). The left hand side of (3.2) is dominated by $4E\|\int_0^t \psi\, dw\|^2$ due to Doob's inequality and this can be further dominated by

$$4C \sum_{i=1}^{n} E\Big\|\int_{t_{i-1}}^{t_i} \psi\, dW\Big\|^2$$

by Lemma 3.2. To finish the proof we will refer to the following lemma.

LEMMA 3.3. *Let $p > 0$. Then there exists $c_p > 0$ such that*

$$\begin{aligned} E\Big\|\sum_{i=1}^{n}(A_i(W_t) - A_i(W_s))I_{F^i}\Big\|^p &= (t-s)^{p/2} \sum_{i=1}^{n} EI_{F_i} \int_X \|x\|^p\, d\mathcal{N}(0, A_i Q A_i^*) \\ &\le c_p (t-s)^{p/2} \sum_{i=1}^{n} EI_{F_i} \|A_i Q^{1/2}\|^p_{L_2(U,X)} \end{aligned}$$

for every $A_i \in L(U,X)$, $i \le n$, finite-dimensional, $s < t$ and $(F_i : i \le n)$ an $\mathcal{F}_s$-decomposition of Ω.

Proof. Since $(F^i : i \le n)$ is a decomposition of Ω we can interchange the sum and the norm in the left hand side term. Moreover $AW_t - AW_s$ is stochastically independent of $\mathcal{F}_s$ so we only have to show that

$$E\|AW_t - AW_s\|^p = (t-s)^{p/2} \int_X \|x\|^p\, d\mathcal{N}(0, AQA^*)$$

for $A \in L(U,X)$ finite-dimensional. But this is obvious since the distribution of $AW_t - AW_s$ is Gaussian centered on X with covariance $(t-s)AQA^*$. The second inequality follows from the fact that for any positive p,q there exists a positive constant a such that

$$\Big(\int_X \|x\|^p\,d\nu\Big)^{1/p} \le a\Big(\int_X \|x\|^q\,d\nu\Big)^{1/q}$$

for every Gaussian centered probability measure ν on X, which is a consequence of the Fernique theorem (e.g. [Ba]). ∎

Step 4 (stochastic L^2-integral). Having the Burkholder inequality for $p=2$ we can define the norm continuous X-valued L^2-martingale $t \mapsto \int_0^t \psi\,dW$ for a progressively measurable $L_2(U_0,X)$-valued random process ψ satisfying

$$E\int_0^T \|\psi(s)\|^2_{L_2(U_0,X)}\,ds < \infty$$

as a limit of integrals of simple processes with values in finite-dimensional operators of $L(U,X)$ in the space $L^2(\Omega, C([0,T],X))$ as we have done in Step 2. We recall Note 2.6(2) in view of the right hand side of (3.2), where

$$\|\psi Q^{1/2}\|^2_{L_2(U,X)} = \|\psi\|^2_{L_2(U_0,X)}$$

appears. The values of ψ outside of $U_0 = \operatorname{Rng} Q^{1/2}$ are not important in this estimation, thus we consider the more "appropriate" space $L_2(U_0,X)$. The only thing we now have to show is existence of simple processes ψ_n of the form considered in Step 2 which satisfy

$$\lim_{n\to\infty} E\int_0^T \|\psi_n(s) - \psi(s)\|^2_{L_2(U_0,X)}\,ds = 0.$$

To do this we are going to use the following classical lemma from [DZ, p. 16], which we present in the form adapted to our case:

LEMMA 3.4. *Let Y be a separable Banach space and Y_0 its countable dense subset. Then there exists a sequence of simple mappings $F_n : Y \to Y_0$ such that $\|F_n(y) - y\|_Y \searrow 0$ for every $y \in Y$. In particular we can take $Y = L_2(U_0,X)$ and Y_0 some dense countable subset of finite-dimensional operators in $L(U,X)$.*

Proof. Enumerate $Y_0 = \{z_1, z_2, \ldots\}$ and define

$$t_n(y) = \min\{i \le n : \|y - z_i\| = \min\{\|y - z_j\| : j \le n\}\}.$$

The functions $F_n(y) = z_{t_n(y)}$, $y \in Y$, $n \in \mathbb{N}$, clearly have the desired property. Regarding the particular case, we already know by Proposition 2.5(3) that the finite-dimensional operators of $L(U_0,X)$ are dense in $L_2(U_0,X)$ so we need only show that every $h_0 \otimes_{L(U_0,X)} x$, $h_0 \in U_0$, $x \in X$, can be approximated by some $h \otimes_{L(U,X)} x$, $h \in U$, $x \in X$. But

$$\langle Qh, h_0\rangle_{U_0} = \langle Q^{1/2}h, Q^{-1/2}h_0\rangle_U = \langle h, Q^{1/2}Q^{-1/2}h_0\rangle_U = \langle h, h_0\rangle_U$$

by definition of U_0 and selfadjointness of $Q^{1/2}$. Hence the restriction of $h \otimes_{L(U,X)} x \in L(U,X)$ to U_0 is $h \otimes_{L(U,X)} x|_{U_0} = Qh \otimes_{L(U_0,X)} x \in L(U_0,X)$ and

$$\begin{aligned}\|h \otimes_{L(U,X)} x - h_0 \otimes_{L(U_0,X)} x\|_{L_2(U_0,X)} \\ &= \|Qh \otimes_{L(U_0,X)} x - h_0 \otimes_{L(U_0,X)} x\|_{L_2(U_0,X)} \\ &= \|Qh - h_0\|_{U_0}\|x\|_X = \|Q^{1/2}h - Q^{-1/2}h_0\|_U\|x\|_X,\end{aligned}$$

where we have used Proposition 2.5(3). Now given $h_0 \in U_0$ we can always find $h \in U$ such that $\|Q^{1/2}h - Q^{-1/2}h_0\|_U$ is arbitrarily small because U_0 is dense in D and $Q^{-1/2}h_0 \in D$ by Note 2.6(5). ∎

Our process ψ is measurable from $([0,T]\times\Omega, \mathcal{P}_T)$ to $L_2(U_0,X)$, where $\mathcal{P}_T$ denotes the σ-algebra of progressively measurable sets. Hence, by the particular case of Lemma 3.4, the $F_n\psi$ are simple progressively measurable processes with values in finite-dimensional operators of $L(U,X)$ such that

$$E\int_0^T \|F_n\psi(s) - \psi(s)\|^2_{L_2(U_0,X)}\, ds \searrow 0$$

by the Lebesgue theorem. Since each $F_n\psi$ is of the form

$$\sum_{k=1}^m B_k I_{C_k},$$

where $(C_k : k \le m)$ is a $\mathcal{P}_T$-decomposition of $[0,T]\times\Omega$ and B_k is finite-dimensional in $L(U,X)$, we have to show that each I_{C_k} can be approximated by simple real processes in $L^2([0,T]\times\Omega)$; but this is a well known fact (e.g. [KS]).

Remark 3.5. Take a progressively measurable $L_2(U_0,X)$-valued process ψ such that

$$E\int_0^T \|\psi(s)\|^2_{L_2(U_0,X)}\, ds < \infty.$$

In view of Step 2 and ψ_n considered therein the processes

$$\eta_n(t) = \Big\langle x^*, \int_0^t \psi_n\, dW\Big\rangle^2 - \int_0^t \|(\psi_n(s)Q^{1/2})^*x^*\|^2_U\, ds$$

are real martingales for every $n \in \mathbb{N}$ and $x^* \in X^*$. Thus, if

$$\lim_{n\to\infty} E\int_0^T \|\psi_n(s) - \psi(s)\|^2_{L_2(U_0,X)}\, ds = 0$$

then $\eta_n(t)$ converges to

$$\Big\langle x^*, \int_0^t \psi\, dW\Big\rangle^2 - \int_0^t \|(\psi(s)Q^{1/2})^*x^*\|^2_U\, ds$$

in $L^1(\Omega)$ for every $t \le T$ because of Proposition 2.5(1), and consequently the process

$$t \mapsto \int_0^t \|\psi^*(s)x^*\|^2_{U_0}\, ds$$

is the quadratic variation process of $t \mapsto \langle x^*, \int_0^t \psi \, dW \rangle$ due to Note 2.6(3). Analogously one can show that if Y is another 2-smooth Banach space, and ϕ a progressively measurable $L_2(U_0, Y)$-valued process such that

$$E \int_0^T \|\phi(s)\|^2_{L_2(U_0,Y)} \, ds < \infty,$$

then

$$t \mapsto \int_0^t \langle \psi^*(s)x^*, \phi^*(s)y^* \rangle_{U_0} \, ds$$

is the cross-variation process associated to the real martingales $t \mapsto \langle x^*, \int_0^t \psi \, dW \rangle$ and $t \mapsto \langle y^*, \int_0^t \phi \, dW \rangle$ for $x^* \in X^*$, $y^* \in Y^*$.

Summary of Step 4. We have constructed a continuous X-valued L^2-martingale $t \mapsto \int_0^t \psi \, dW$ as a limit in $L^2(\Omega, C([0,T], X))$ for a progressively measurable process ψ with values in $L_2(U_0, X)$ and so the Burkholder inequality (3.2) holds with $p = 2$. The mapping $\psi \mapsto \int \psi \, dW$ is linear by construction.

Step 5 (general case). Now we will finish the construction of the stochastic integral by extending it to progressively measurable $L_2(U_0, X)$-valued processes ψ with P-almost all trajectories in $L^2([0,T], L_2(U_0, X))$ by the classical "localization" procedure. One defines the stopping times

$$t_n^* = \min\Big\{ t \le T : \int_0^t \|\psi(s)\|^2_{L_2(U_0,X)} \, ds \ge n \Big\}$$

with values in $[0,T]$ and then defines

$$\int_0^t \psi \, dW = \int_0^t I_{[0,t_n^*]} \psi \, dW \qquad \text{on } [0, t_n^*].$$

The process $t \mapsto \int_0^t \psi \, dW$ is a continuous local martingale in X. Yet, for the sake of correctness, we must first show:

LEMMA 3.6. *Let τ be a stopping time and ψ a progressively measurable process with values in $L_2(U_0, X)$ such that*

$$E \int_0^T \|\psi(s)\|^2_{L_2(U_0,X)} \, ds < \infty.$$

Then

$$\int_0^{t\wedge\tau} \psi \, dW = \int_0^t I_{[0,\tau]}(s)\psi(s) \, dW$$

for every $t \le T$.

Proof. Suppose that ψ is a bit more complicated than in Step 2, namely of the type $\sum_{k=1}^n f_k A_k$ where $(f_k : k \le n)$ are bounded real progressively measurable processes and

$(A_k : k \leq n)$ finite-dimensional operators from $L(U, X)$. Then, by a simple convergence argument and the second formula in Step 2, we have

$$\Big\langle x^*, \int_0^t \psi \, dW \Big\rangle = \sum_{k=1}^{n} \int_0^t g_k \, dW(A_k^* x^*)$$

for every $t \leq T$ and $x^* \in X^*$. Hence the claim holds for ψ of this type by the properties of real stochastic integrals. Now it suffices to take an approximating sequence ψ_n of simple processes from Step 2 such that

$$\lim_{n\to\infty} E \int_0^T \|\psi_n - \psi\|^2_{L_2(U_0,X)} \, ds = 0$$

and the proof is complete by using the Burkholder inequality (5.1) for $p = 2$. ∎

Summary of Step 5. We have extended the stochastic integral to progressively measurable processes ψ with values in $L_2(U_0, X)$ which satisfy

$$P\Big[\int_0^T \|\psi(s)\|^2_{L_2(U_0,X)} \, ds < \infty \Big] = 1.$$

The continuous X-valued process $t \mapsto \int_0^t \psi \, dW$ is a local martingale and

$$\int_0^{t\wedge\tau} \psi \, dW = \int_0^t I_{[0,\tau]}(s)\psi(s) \, dW \tag{3.6}$$

for every stopping time τ and time $t \leq T$. Moreover, if Y is another 2-smooth Banach space, ϕ a progressively measurable $L_2(U_0, Y)$-valued process such that

$$P\Big[\int_0^T \|\phi(s)\|^2_{L_2(U_0,Y)} \, ds < \infty \Big] = 1$$

then

$$t \mapsto \int_0^t \langle \psi^*(s)x^*, \phi^*(s)y^* \rangle_{U_0} \, ds$$

is the cross-variation process associated to the real local martingales $t \mapsto \langle x^*, \int_0^t \psi \, dW \rangle$ and $t \mapsto \langle y^*, \int_0^t \phi \, dW \rangle$ for $x^* \in X^*$, $y^* \in Y^*$.

Example 3.7. Let ψ be a progressively measurable $L_2(U_0, X)$-valued process with almost all trajectories in $L^2([0, T], L_2(U_0, X))$ and $h \in U$. Then

- ▷ $\phi(s, \omega) = h \otimes_{L(U,\mathbb{R})} 1$ is a constant process in $L_2(U_0, \mathbb{R})$.
- ▷ The restriction of ϕ to U_0 belongs to $L(U_0, \mathbb{R}) = U_0^*$ and thus can be identified with $Qh \in U_0$.
- ▷ $\phi^*(s, \omega) = Qh \in L(\mathbb{R}, U_0)$.
- ▷ $\int_0^t \phi(s) \, dW = \int_0^t Qh \, dW = W_t(h)$.
- ▷ $t \mapsto \int_0^t \langle x^*, \psi(s)Qh \rangle \, ds$ is the cross-variation process of $t \mapsto \langle x^*, \int_0^t \psi \, dW \rangle$ and $t \mapsto W_t(h)$, $x^* \in X^*$.

4. A convergence result

PROPOSITION 4.1. *Let ψ, ψ_n, $n \in \mathbb{N}$, be $L_2(U_0, X)$-valued progressively measurable processes with P-almost all trajectories in $L^2([0,T], L_2(U_0, X))$ such that*

$$\int_0^T \|\psi_n(s) - \psi(s)\|^2_{L_2(U_0,X)}\, ds$$

converges to 0 *in probability. Then*

$$\sup\Big\{\Big\|\int_0^s \psi_n\, dW - \int_0^s \psi\, dW\Big\| : s \le T\Big\} \to 0$$

in probability as well.

The proof is based on the following simple inequality: Define

$$A(t) = \sup\Big\{\Big\|\int_0^s \psi\, dW\Big\|^2 : s \le t\Big\}, \qquad B(t) = C_2 \int_0^t \|\psi(s)\|^2_{L_2(U_0,X)}\, ds.$$

Then A and B are continuous processes and

$$EA(\tau) \le EB(\tau) \tag{4.1}$$

for every stopping time $\tau \le T$. Now the claim follows from Lenglart's inequality (e.g. [KS, 1.4.15 and 1.4.17]).

Proof. Define t_n^* as in Step 5 of the previous section. Then $EA(\tau) = \lim EA(\tau \wedge t_n^*)$ as A is continuous and nondecreasing and

$$EA(\tau \wedge t_n^*) \le E \sup\Big\{\Big\|\int_0^s I_{[0,\tau\wedge t_n^*]}\psi\, dW\Big\|^2 : s \le T\Big\} \le C_2 E \int_0^{\tau\wedge t_n^*} \|\psi(s)\|^2_{L_2(U_0,X)}\, ds$$

by (5.1) for $p = 2$, which we may apply because

$$E\int_0^T \|I_{[0,\tau\wedge t_n^*]}\psi\|^2_{L_2(U_0,X)}\, ds < \infty. \ \blacksquare$$

REMARK 4.2. Let ψ be progressively measurable $L_2(U_0, X)$-valued processes with P-almost all trajectories in $L^2([0,T], L_2(U_0, X))$. Then, by Steps 4 and 5, there exists a sequence ψ_n, $n \in \mathbb{N}$, of simple processes that we have considered in Step 2 such that $\int_0^T \|\psi_n(s) - \psi(s)\|^2_{L_2(U_0,X)}\, ds$ converges to 0 in probability.

5. Burkholder inequality

There exist constants C_p, $0 < p < \infty$, such that

$$E\sup\Big\{\Big\|\int_0^s \psi\, dW\Big\|^p : s \le t\Big\} \le C_p E\Big(\int_0^t \|\psi\|^2_{L_2(U_0,X)}\, ds\Big)^{p/2} \tag{5.1}$$

for every progressively measurable $L_2(U_0, X)$-valued process ψ with trajectories in the space $L^2([0,T], L_2(U_0, X))$ for every $T > 0$.

We have already proven (5.1) in the case $p = 2$ in Steps 3 and 4 of the previous section. So, let $0 < p < \infty$, define processes

$$M(r) = \Big\| \int_0^r \psi \, dW \Big\|, \quad B(r) = \Big(\int_0^t \|\psi(s)\|^2_{L^2(U_0,X)} \, ds \Big)^{1/2}, \quad M^*(r) = \sup_{s \le r} M(s),$$

choose $\beta > 1$, $\delta > 0$, $\lambda > 0$, $t \ge 0$ and define stopping times $\tau_1 = \inf\{r : M(r) \ge \beta\lambda\}$, $\tau_2 = \inf\{r : M(r) \ge \lambda\}$, $\sigma = \inf\{r : B(r) \ge \delta\lambda\}$ and $\varrho_n = \inf\{r : M(r) \ge n\}$. The set

$$A_1 = [M^*(t \wedge \varrho_n) \ge \beta\lambda,\, B(t) < \delta\lambda]$$

is contained in the set

$$A_2 = \Big[\Big\| \int_0^{t\wedge\tau_1\wedge\sigma} \psi \, dW - \int_0^{t\wedge\tau_1\wedge\sigma} \psi \, dW \Big\| \ge \lambda(\beta - 1) \Big]$$

since $\tau_2 \le \tau_1 \le t \wedge \varrho_n \le t \le \sigma$, $M(\tau_1) = \lambda\beta$ and $M(\tau_2) = \lambda$ on A_1. Furthermore,

$$\begin{aligned}
E\Big\| \int_0^{t\wedge\tau_1\wedge\sigma\wedge\varrho_n} \psi \, dW - \int_0^{t\wedge\tau_2\wedge\sigma\wedge\varrho_n} \psi \, dW \Big\|^2 &\le C_2 E \int_0^t I_{(t\wedge\tau_2\wedge\sigma\wedge\varrho_n, t\wedge\tau_1\wedge\sigma\wedge\varrho_n]} \|\psi\|^2_{L^2} \, ds \\
&= C_2 E \Big\{ \int_0^t I_{(t\wedge\tau_2\wedge\sigma\wedge\varrho_n, t\wedge\tau_1\wedge\sigma\wedge\varrho_n]}(s) \|\psi(s)\|^2_{L^2(U_0,X)} \, ds \, I_{[M^*(t\wedge\varrho_n) \ge \lambda]} \Big\} \\
&\le C_2 E \Big\{ \int_0^{t\wedge\sigma} \|\psi(s)\|^2_{L^2(U_0,X)} \, ds \, I_{[M^*(t\wedge\varrho_n) \ge \lambda]} \Big\} \le C_2 \lambda^2 \delta^2 P[M^*(t \wedge \varrho_n) \ge \lambda]
\end{aligned}$$

by (3.2) applied for $p = 2$ and (3.6). Hence,

$$P(A_1) \le P(A_2) \le \frac{C_2 \delta^2}{(\beta - 1)^2} P[M^*(t \wedge \varrho_n) \ge \lambda]$$

and so

$$P[M^*(t \wedge \varrho_n) \ge \beta\lambda] \le P[B(t) \ge \delta\lambda] + \frac{C_2\delta^2}{(\beta-1)^2} P[M^*(t \wedge \varrho_n) \ge \lambda].$$

Integrating both sides with respect to $p\lambda^{p-1} \, d\lambda$ over $(0, \infty)$ we arrive at

$$\frac{1}{\beta^p} E(M^*(t \wedge \varrho_n))^p \le \frac{1}{\delta^p} E(B(t))^p + \frac{C_2\delta^2}{(\beta-1)^2} E(M^*(t \wedge \varrho_n))^p.$$

Now $M^*(t \wedge \varrho_n) \le n$ and if we choose $\delta < (\beta - 1) C_2^{-1/2} \beta^{-p/2}$ and define

$$C_p = \delta^{-p} \Big(\frac{1}{\beta^p} - \frac{C_2\delta^2}{(\beta-1)^2} \Big)^{-1}$$

then

$$E(M^*(t \wedge \varrho_n))^p \le C_p E(B(t))^p.$$

Letting n tend to infinity we get $E(M^*(t))^p \le C_p E(B(t))^p$. ∎

6. Fubini's theorem

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), P, W)$ be a probability filtered space with a Q-Wiener process W on U, set $\Omega_T = [0,T] \times \Omega$, and define $\mathcal{P}_T$ to be the σ-algebra of progressively measurable subsets of Ω_T and $ds \otimes P$ the product of the Lebesgue measure and P. We will write briefly $L^2(\Omega_T)$ for the space $L^2((\Omega_T, ds \otimes P), L_2(U_0, X))$.

Proposition 6.1. *Let $(Y, \mathcal{Y}, \mu)$ be a finite measure space and $\psi : \Omega_T \times Y \to L_2(U_0, X)$ a $\mathcal{P}_T \otimes \mathcal{Y}$-measurable mapping such that*

$$\int_Y \|\psi(y)\|_{L^2(\Omega_T)}\, d\mu < \infty.$$

Then

(1) *the process $\int_Y \psi(y)\, d\mu$ indexed by $t \in [0,T]$ is progressively measurable and belongs to $L^2(\Omega_T)$.*

(2) *The process $\int_0^T \psi(y)\, dW$ indexed by $y \in Y$ has an $\mathcal{F}_T \otimes \mathcal{Y}$-measurable version $m : \Omega \times Y \to X$ such that*

$$P\Big[m(y) = \int_0^T \psi(y)\, dW\Big] = 1 \quad \textit{for } \mu\textit{-almost all } y \in Y.$$

(3) *We have*

$$P\Big[\int_Y m(y)\, d\mu = \int_0^T \Big(\int_Y \psi(y)\, d\mu\Big)\, dW\Big] = 1.$$

Proof. (1) follows from the following inequality: Let f be a nonnegative $\mathcal{P}_T \otimes \mathcal{Y}$-measurable function on $\Omega_T \times Y$. Then

$$\sqrt{\int_{\Omega_T} \Big(\int_Y f\, d\mu\Big)^2 ds \otimes P} \le \int_Y \|f(y)\|_{L^2(\Omega_T, \mathbb{R})}\, d\mu \tag{6.1}$$

because

$$\Big(\int_Y f(a,y)\, d\mu\Big)^2 = \int_{Y \times Y} f(a,y_1) f(a,y_2)\, d\mu\, d\mu$$

and we get (6.1) by the Schwarz inequality. Now suppose that m in (2) exists. Then by taking Ω instead of Ω_T in (6.1) we get

$$\sqrt{E\Big(\int_Y \|m(y)\|\, d\mu\Big)^2} \le \int_Y \sqrt{E\|m(y)\|^2}\, d\mu \le \sqrt{C_2} \int_Y \|\psi(y)\|_{L^2(\Omega_T)}\, d\mu \tag{6.2}$$

by the Burkholder inequality (5.1). Hence $\int_Y m(y)\, d\mu$ is defined P-almost everywhere. Now take ψ_n satisfying the assumption of the proposition such that the sequence of the integrals $\int_Y \|\psi_n(y) - \psi(y)\|_{L^2(\Omega_T)}\, d\mu$ converges to zero. Then there exists a subsequence $(n_k : k \in \mathbb{N})$ such that

(a) $\int_0^T \psi_{n_k}(y)\, dW \to \int_0^T \psi(y)\, dW$ in $L^2(\Omega, X)$ for μ-almost all $y \in Y$.

(b) $\int_0^T (\int_Y \psi_n(y)\, d\mu)\, dW \to \int_0^T (\int_Y \psi(y)\, d\mu)\, dW$ in $L^2(\Omega, X)$ for μ-almost all $y \in Y$.

(a) and (b) follow from the Burkholder inequality (5.1) and the estimate (6.1) because $\|\psi_{n_k}(y) - \psi(y)\|_{L^2(\Omega_T)} \to 0$ μ-almost everywhere. Let us introduce the set $\mathcal{D}$ of all $\mathcal{P}_T \otimes \mathcal{Y}$-measurable $L_2(U_0, X)$-valued processes ψ with

$$\int_Y \|\psi(y)\|_{L^2(\Omega_T)}\, d\mu < \infty$$

such that there exists an m satisfying (2) and (3). It is easy to see that $\mathcal{D}$ is a linear space and if we found $\psi_n \in \mathcal{D}$ such that

$$\int_Y \|\psi_n(y) - \psi(y)\|_{L^2(\Omega_T)}\, d\mu \to 0$$

then we would finish the proof. Indeed, take the corresponding functions m_n. Then the sequence $(m_n : n \in \mathbb{N})$ is Cauchy in $L^1(\Omega \times Y, X)$ due to (6.2) (apply the Jensen inequality) and ψ belongs to $\mathcal{D}$ due to (a) and (b). Now we will show how to construct the approximating sequence ψ_n. First consider mappings F_n on $L_2(U_0, X)$ as in Lemma 3.4. The simple functions $F_n\psi$ take values in finite-dimensional operators of $L(U, X)$. Moreover

$$\|F_n\psi(y) - \psi(y)\|_{L^2(\Omega_T)} \searrow 0$$

for μ-almost all $y \in Y$ by the Lebesgue theorem, hence

$$\int_Y \|F_n\psi(y) - \psi(y)\|_{L^2(\Omega_T)}\, d\mu \searrow 0,$$

and if $F_n\psi \in \mathcal{D}$, $n \in \mathbb{N}$, then $\psi \in D$. Now, to show that $F_n\psi \in \mathcal{D}$, we will take advantage of the fact that each $F_n\psi$ is bounded in $L_2(U_0, X)$ and

$$\int_Y \|\phi_n(y) - \phi(y)\|_{L^2(\Omega_T)}\, d\mu \to 0$$

if and only if

$$\int_{\Omega_T \times Y} \|\phi_n - \phi\|_{L_2(U_0,X)}\, ds\, dP\, d\mu \to 0$$

for ϕ_n uniformly bounded in $L_2(U_0, X)$. So as $F_n\psi$ is of the form

$$\sum_{k=1}^{m} I_{C_k} B_k,$$

where $(C_k : k \leq m)$ is a $\mathcal{P}_T \otimes \mathcal{Y}$-decomposition of $\Omega_T \times Y$ and B_k, $k \leq m$, are finite-dimensional operators in $L(U, X)$, we conclude that $F_n\psi \in \mathcal{D}$ provided $I_{C_k} B_k \in \mathcal{D}$ due to linearity of $\mathcal{D}$. Another reduction shows that this is true if

$$I_{C_k^1 \times C_k^2} B_k \in \mathcal{D}$$

for every $C_k^1 \in \mathcal{P}_T$, $C_k^2 \in \mathcal{Y}$ as I_{C_k} can be approximated by $I_{C_k^0}$ in $L^1(\Omega_T \times Y)$, where C_k^0 is a disjoint union of sets of the type $C_k^1 \times C_k^2$. Finally, as $I_{C_k^1}$ is a progressively measurable process, it can be approximated by simple uniformly bounded real processes in $L^1(\Omega_T)$, so we will finish the proof by showing that

$$I_{(s,t] \times C_s \times C_k^2} B_k \in \mathcal{D}$$

for $s < t$, $C_s \in \mathcal{F}_s$; but this is obvious. ∎

EXAMPLE 6.2. Let $g : [0,T] \times \Omega \to L_2(U_0, X)$ be such that $\langle x_n^*, g(h_k)\rangle : [0,T] \times \Omega \to \mathbb{R}$ is progressively measurable for $x_n^* \in X^*$, $h_k \in U_0$, $n \in \mathbb{N}$, $k \in \mathbb{N}$, where $(x_n^* : n)$ separates points in X and $(h_k : k \in \mathbb{N})$ is dense or orthonormal in U_0. Then g is progressively measurable by Proposition 2.5(5).

EXAMPLE 6.3. Let (S_t) be a continuous semigroup of linear operators on X generated by A, W a Q-Wiener process and $g : [0,T] \times \Omega \to L_2(U_0, X)$ a progressively measurable process such that

$$P\Big[\int_0^T \|g(s)\|^2_{L_2(U_0,X)}\, ds < \infty\Big] = 1.$$

Define

$$G(t) = \int_0^t g\, dW, \qquad C(t) = \int_0^t S_{t-s} g(s)\, dW.$$

Then

(1) C has an $(\mathcal{F}_t)$-predictable modification such that $C(\omega) \in L^2(0,T;X)$ P-almost surely.
(2) We have

$$P\Big[\int_0^t C(r)\, dr \in D(A)\Big] = P\Big[A\int_0^t C(r)\, dr = C(t) - G(t)\Big] = 1 \quad \text{for every } t \le T.$$

Proof. To simplify notation we will extend the operator-valued function S to negative times by $0 \in L(X)$.

First suppose that

$$E\int_0^T \|g(s)\|^2_{L_2(U_0,X)}\, ds < \infty.$$

Fix $t \in [0,T]$ and define

$$\psi_t(s,\omega,r) = S_{r-s} g(s,\omega) \quad \text{for } s \le t,\ r \le t.$$

Then, by Proposition 6.1, there exists a $\mathbb{B}([0,t]) \otimes \mathcal{F}_t$-measurable function $m_t : [0,t] \times \Omega \to X$ such that $P[m_t(r) = C(r)] = 1$ for almost all $r \in [0,t]$ and $m_t(\omega) \in L^2(0,t;X)$ for all $\omega \in \Omega$. Next take λ from the resolvent set of A and define

$$\psi(s,\omega,r) = S_{r-s} A R_\lambda g(s,\omega) \quad \text{for } s \le t,\ r \le t.$$

Then ψ satisfies the assumptions of Proposition 6.1 with $Y = [0,t]$ and we have

$$(s,\omega) \mapsto \int_Y \psi\, dr = S_{t-s} R_\lambda g(s,\omega) - R_\lambda g(s,\omega),$$

$P[m(r) = AR_\lambda m_t(r)] = 1$ for almost all $r \le t$ because

$$(*) \qquad AR_\lambda = \lambda R_\lambda - I_X$$

is bounded, and

$$AR_\lambda \int_0^t m_t(r)\, dr = \int_0^t m(r)\, dr = R_\lambda C(t) - R_\lambda G(t)$$

P-almost everywhere. But $\operatorname{Rng} R_\lambda = D(A)$ and, due to $(*)$,

$$P\Big[\int_0^t m_t(r)\,dr \in D(A)\Big] = 1.$$

Thus

$$P\Big[A\int_0^t m_T(r)\,dr = A\int_0^t m_t(r)\,dr = C(t) - G(t)\Big] = 1.$$

Moreover the process $t \mapsto \int_0^t m_T(r)\,dr$ is continuous and adapted as

$$P\Big[\int_0^t m_T(r)\,dr = \int_0^t m_t(r)\,dr\Big] = 1$$

for every $t \le T$ so

$$t \mapsto \begin{cases} A\int_0^t m_T(r)\,dr, & \int_0^t m_T(r)\,dr \in D(A), \\ 0, & \int_0^t m_T(r)\,dr \notin D(A), \end{cases}$$

is predictable because $D(A)$ is a Borel set in X and $A : D(A) \to X$ is Borel measurable. Consequently, C has a predictable modification. The general case follows directly from (1), (2) by localization of Step 5: Define

$$t_n^* = \inf\Big\{t \le T : \int_0^t \|g(s)\|^2_{L_2(U_0,X)}\,ds \ge n\Big\}, \qquad g_n(s,\omega) = g(s,\omega) I_{[0,t_n^*(\omega)]}(s).$$

Then the processes

$$C_n(t) = \int_0^t S_{t-s} g_n(s)\,dW_s, \quad G_n(t) = \int_0^t g_n\,dW$$

satisfy (1), (2). The set of (t,ω) where C_n is convergent is predictable and the limit is predictable as well. But this limit is a modification of C because $C_n(t)$ converges P-almost surely for every $t \le T$ as $P[t_n^* = T] \nearrow 1$ and $P[C_n(t) = C(t), t_n^* = T] = 1$ for every $t \le T$. ■

Example 6.4. Let (S_t) be a continuous semigroup of linear operators on a separable reflexive Banach space X generated by A, W a Q-Wiener process, $g : [0,T] \times \Omega \to L(U_0, X)$ a progressively measurable process with respect to the strong σ-algebra on $L(U_0, X)$. Further suppose that

$$P\Big[\int_0^T \|g(s)\|^2_{L(U_0,X)}\,ds < \infty\Big] = 1$$

and define

$$G_{x^*}(t) = \int_0^t g^* x^*\,dW, \quad C_{x^*}(t) = \int_0^t g^*(s) S^*_{t-s} x^*\,dW$$

for $x^* \in X^*$, $t \le T$. Then

(1) C_{x^*} has an $(\mathcal{F}_t)$-predictable modification such that $C_{x^*}(\omega) \in L^2(0,T)$ P-almost surely for every $x^* \in X^*$.

(2) C_{x^*} is a continuous process and

$$P\Big[\int_0^t C_{A^*x^*}(s)\,ds = C_{x^*}(t) - G_{x^*}(t)\Big] = 1 \quad \text{for every } t \le T$$

provided $x^* \in D(A^*)$.

Proof. C_{x^*} is a real adapted process which is continuous in probability by Proposition 4.1. Thus (1) is a consequence of [DZ, Proposition I.3.2]. Next suppose that

$$E\int_0^T \|g(s)\|^2_{L(U_0,X)}\,ds < \infty.$$

Then (2) follows immediately from Proposition 6.1, and the general case can be obtained in the same way as in Example 6.3 but this time with

$$t_n^* = \inf\Big\{t \le T : \int_0^t \|g(s)\|^2_{L(U_0,X)}\,ds \ge n\Big\}, \quad g_n(s,\omega) = g(s,\omega)I_{[0,t_n^*(\omega)]}(s). \ \blacksquare$$

Proof of Theorem 12. If we define $y_1(t) = S_t u(0)$ then

$$A\int_0^t y_1(s)\,ds = y_1(t) - u(0).$$

The process

$$y_2(t) = \int_0^t S_{t-s}f(s)\,ds$$

is obviously norm continuous, adapted and, by the classical Fubini theorem,

$$A\int_0^t y_2(s,\omega)\,ds = y_2(t,\omega) - \int_0^t f(s,\omega)\,ds$$

for every ω satisfying $\int_0^T \|f(s,\omega)\|\,ds < \infty$. Thus, by Example 6.3, the predictable process

$$y_3(t) = \int_0^t S_{t-s}g(s)\,dW$$

satisfies

$$A\int_0^t y_3(s)\,ds = y_3(t) - \int_0^t g(s)\,dW$$

almost everywhere, (a) implies (b) because $u(t) = y_1(t)+y_2(t)+y_3(t)$ almost everywhere, and (1), (2) obviously hold. On the other hand, if (b) holds, define $h(t) = y_1(t) + y_2(t) + y_3(t) - u(t)$. Then

$$h(t) = A\int_0^t h(s)\,ds$$

almost everywhere for every $t \le T$. Thus, computing $\int_0^r \langle x^*, S_{r-t}h(t)\rangle\,dt$, $x^* \in D(A^*)$, by the classical Fubini theorem, we get $\int_0^r h(s,\omega)\,ds = 0$ for every $r \le T$ and almost all ω. But $t \mapsto \langle x^*, h(t,\omega)\rangle$ is continuous for almost all ω for every choice of $x^* \in D(A^*)$, and thus $h(t,\omega)=0$ on $[0,T]$ for almost all ω because X is reflexive and $D(A^*)$ is norm dense in X^*. ■

Proof of Theorem 13. (a)⇒(b)&(c): In fact, we will prove a little more. (a) implies that

$$\langle x^*, u_t\rangle = \langle x^*, S_t u_0\rangle + \int_0^t \langle x^*, S_{t-s} f_s\rangle\, ds + \int_0^t g_s^* S_{t-s}^* x^*\, dW_s \tag{13.a}$$

for every $x^* \in X^*$, and this equality already implies (b) and (c). Thus we need not suppose that X is 2-smooth—X might be separable reflexive and (13.a) should hold. The process $t \mapsto \langle x^*, u(t)\rangle$ has a predictable modification for every $x^* \in X^*$ by Example 6.4, hence u has a predictable modification by Corollary 11.2. Now fix $x^* \in D(A^*)$ and define

$$y_3'(t) = \int_0^t g_s^* S_{t-s}^* A^* x^*\, dW_s.$$

Then, proceeding along the lines of Theorem 12, (c) follows from Example 6.4. To show (b) define the predictable processes

$$v_n(t) = \int_0^t S_{t-s} g_n(s)\, dW_s, \qquad n \in \mathbb{N}, \qquad v(t) = \int_0^t S_{t-s} g(s)\, dW_s$$

with g_n from Example 6.4. Then, by Proposition 6.1,

$$P\Big[\int_0^T\int_0^T |\langle x^*(t,s), v_n(s)\rangle|\, ds\, dt < \infty\Big] = 1.$$

But $P[v_n(t) I_{[t_n^*=T]} = v(t) I_{[t_n^*=T]}] = 1$ for every $t \le T$ and $P[t_n^* = T] \nearrow 1$, thus (b) holds as $t \mapsto S_t u(0) + \int_0^t S_{t-s} f(s)\, ds$ is norm continuous.

(b)&(c)⇒(a): Define

$$h(t) = u(t) - S_t u(0) - \int_0^t S_{t-s} f(s) - \int_0^t S_{t-s} g(s)\, dW_s.$$

Then, by the first part of the proof, h can be chosen predictable satisfying (b) and

$$P\Big[\langle x^*, h(t)\rangle = \int_0^t \langle A^* x^*, h(s)\rangle\, ds\Big] = 1$$

for every $t \le T$, $x^* \in D(A^*)$. Thus, computing $\int_0^r \langle S_{r-t}^* x^*, h(t)\rangle\, dt$ by Fubini's theorem, we conclude that $P[h(t) = 0] = 1$ for every $t \le T$. ■

7. The Girsanov theorem

PROPOSITION 7.1. *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), P, W)$ be a filtered probability space with a Q-Wiener process W on U, ϕ a progressively measurable process with values in U_0 satisfying*

$$E \exp\Big(\int_0^T \|\phi(s)\|_{U_0}^2\, ds\Big) < \infty$$

and ψ a progressively measurable process with values in $L_2(U_0,X)$ such that

$$P\Big[\int_0^T \|\psi(s)\|^2_{L_2(U_0,X)}\,ds\Big]=1.$$

Then

(1) *The process*

$$M_t=\exp\Big(-\int_0^t \phi\,dW-\frac{1}{2}\int_0^t \|\phi(s)\|^2_{U_0}\,ds\Big)$$

is a P-martingale on $[0,T]$.

(2) *The process*

$$\widetilde{W}_t(h)=W_t(h)+\int_0^t \langle\phi(s),h\rangle_U\,ds,\qquad t\le T,\ h\in U,$$

is a $\widetilde{P}$-Q-Wiener process on U with $\widetilde{P}(F)=\int_F M_T\,dP$.

(3) *We have*

$$\int_0^t \psi\,d\widetilde{W}=\int_0^t \psi\,dW+\int_0^t \psi(s)\phi(s)\,ds$$

almost everywhere for every $t\le T$.

(4) *If $(Y,\mathcal{Y})$ is a measurable space and $\xi:\Omega\to Y$ an $\mathcal{F}_0$-measurable random variable then $\mathfrak{Law}_{\widetilde{P}}(\xi)=\mathfrak{Law}_P(\xi)$.*

Proof. The proof goes along the lines of the proof of the classical Girsanov and Novikov theorem (e.g. [RY]). Hence (1) follows from the fact that the quadratic variation process of $t\mapsto\int_0^t\phi\,dW$ is $t\mapsto\int_0^t\|\phi(s)\|^2_{U_0}\,ds$ due to Remark 3.5. Here we use the isometric isomorphism between $L_2(U_0,\mathbb{R})$ and U_0. Using the same arguments as in Remark 3.5 we can show that

$$t\mapsto\int_0^t \langle\phi(s),Qh\rangle_{U_0}\,ds$$

is the cross-variation process associated to $t\mapsto\int_0^t\phi\,dW$ and $t\mapsto W_t(h)$. But $\langle\phi(s),Qh\rangle_{U_0}=\langle\phi(s),h\rangle_U$, which yields (2). If we take $\psi=I_{(s,t]}I_{F_s}h\otimes x$ for some $0\le s<t\le T$, $F_s\in\mathcal{F}_s$, $h\in U$, $x\in X$ then (3) obviously holds. Due to linearity, (3) holds for simple ψ that we have considered in Step 2 of the previous section as well. If ψ is general we can find a sequence $(\psi_n:n\in\mathbb{N})$ of simple processes of Step 2 such that

$$\int_0^T \|\psi_n(s)-\psi(s)\|^2_{L_2(U_0,X)}\,ds\to 0$$

in probability ($\widetilde{P}$ as well as P) by Remark 4.2, and the final equality is just the limiting argument of Proposition 4.1. Claim (4) is a consequence of (1). ∎

Remark. The measures $\widetilde{P}$, P are absolutely continuous with respect to each other so their null sets coincide, as do $\widetilde{P}$-convergence and P-convergence, and consequently the integrals in (3) do not depend on $\widetilde{P}$, resp. P due to Proposition 4.1.

8. Distribution of random integrals and measurable selectors

The goal of this section is to show a sufficient condition on a process u and a Q-Wiener process W to be a solution. More precisely we are going to prove that if the distributions of (u,W) and (v,B) coincide on the space of functions and (v,B) is a solution then so is (u,W).

In the last part we modify the selection theorem of [KRN] to open set mappings with the gain of distribution preserving selectors.

Distribution of random Bochner integrals

LEMMA 8.1. *Let $(Y,\mathcal{Y})$ be a measurable space, ξ^i a Y-valued random variable and $(f^i_j(t) : t \le T)$, $j \le N$, real bounded measurable processes on $(\Omega^i,\mathcal{F}^i,P^i)$, $i=1,2$, such that*

$$\mathfrak{Law}_{P^1}(f^1_j(r_l),\xi^1 : j\le N,\, l\le m) = \mathfrak{Law}_{P^2}(f^2_j(r_l),\xi^2 : j\le N,\, l\le m)$$

for every partition $0=r_0<\dots<r_m\le T$ in D^* for some $D^*\subseteq[0,T]$ of Lebesgue measure T. Then

$$\mathfrak{Law}_{P^1}\Big(\int_0^{t_k} f^1_j(s)\,ds,\xi^1 : k,j\Big) = \mathfrak{Law}_{P^2}\Big(\int_0^{t_k} f^2_j(s)\,ds,\xi^2 : k,j\Big)$$

for every partition $0=t_0<\dots<t_n\le T$.

Proof. First let ξ^i be real bounded, $N=1$ and $0<t\le T$ fixed. We are going to show that the Fourier transforms of the $\mathbb{R}^2$-valued random vectors $(\int_0^t f^i(s)\,ds,\xi^i)$ do not depend on i, thus they must coincide. Define

$$g^i(s)=\sqrt{-1}\Big(af^i(s)+\frac{b}{t}\,\xi^i\Big),\qquad a\in\mathbb{R},\ b\in\mathbb{R}.$$

Then

$$\int_{\Omega^i}\exp\Big(\int_0^t g^i(s)\,ds\Big)\,dP^i = 1+\sum_{k=1}^{\infty}\frac{1}{k!}\int_0^t\!\cdots\!\int_0^t\Big(\int_{\Omega^i} g^i(s_1)\cdots g^i(s_k)\,dP^i\Big)\,ds_1\cdots ds_k$$

by Fubini's theorem.

Now the general case can be proven by repeated application of the previous case: Fix a partition $0=t_0<\dots<t_n\le T$, $J\le N$, $K\le n$ and suppose that

$$\mathfrak{Law}_{P^i}\Big(\int_0^{t_k} f^i_{j_1}(s)\,ds, f^i_j(r_l),\xi^i : k\le K,\, j_1\le J,\, j\le N,\, l\le m\Big)$$

are equal for $i=1,2$ for every choice of $0=r_0<\dots<r_m\le T$ in D^*. Set

$$\eta^i=\Big(\int_0^{t_k} f^i_{j_1}(s)\,ds, f^1_j(r_l),\xi^1 : k\le K,\, j_1\le J,\, j\le N,\, l\le m\Big),\qquad i=1,2,$$

and fix a measurable set A in the state space of η^i. Then

$$\mathfrak{Law}_{P^1}(f^1_{j_0}(r^*_l), I_{[\eta^1\in A]} : l\le M) = \mathfrak{Law}_{P^2}(f^2_{j_0}(r^*_l), I_{[\eta^2\in A]} : l\le M)$$

for every partition $0 = r_0^* < \cdots < r_M^* \le T$ in D^* and $j_0 \le N$. Hence

$$\mathfrak{Law}_{P^1}\Big(\int_0^t f_{j_0}^1(s)\,ds, I_{[\eta^1\in A]}\Big) = \mathfrak{Law}_{P^2}\Big(\int_0^t f_{j_0}^2(s)\,ds, I_{[\eta^2\in A]}\Big)$$

for every $t \le T$, $j_0 \le N$ and A by the first part of the proof, which is, indeed, the induction step. ■

COROLLARY 8.2. *Suppose that $(f^i(t) : t \le T)$ is a $[0,\infty]$-valued measurable process on $(\Omega^i, \mathcal{F}^i, P^i)$, $i = 1,2$, such that*

$$\mathfrak{Law}_{P^1}(f^1(r_l) : l \le m) = \mathfrak{Law}_{P^2}(f^2(r_l) : l \le m)$$

for every partition $0 = r_0 < \cdots < r_m \le T$ in D^*. Then

$$P^1\Big[\int_0^T f^1(s)\,ds < \infty\Big] = P^2\Big[\int_0^T f^2(s)\,ds < \infty\Big].$$

Proof. The bounded measurable processes $t \mapsto f_n^i(t) = \max\{f^i(t), n\}$ satisfy the assumption of Lemma 8.1. Thus

$$P^1\Big[\int_0^T f_n^1(s)\,ds \le \Delta\Big] = P^2\Big[\int_0^T f_n^2(s)\,ds \le \Delta\Big]$$

for every $\Delta \in \mathbb{R}$ and we have

$$P^1\Big[\int_0^T f^1(s)\,ds \le \Delta\Big] = P^2\Big[\int_0^T f^2(s)\,ds \le \Delta\Big]$$

by Lévy's theorem. The claim now follows by letting Δ tend to infinity. ■

THEOREM 8.3. *Let $(Y, \mathcal{Y})$ be a measurable space, ξ^i a Y-valued random variable and $(f_j^i(t) : t \le T)$, $j \le N$, X-valued measurable processes on $(\Omega^i, \mathcal{F}^i, P^i)$, $i = 1,2$, satisfying*

$$P^1\Big[\int_0^T \|f_j^1(s)\|\,ds < \infty\Big] = P^2\Big[\int_0^T \|f_j^2(s)\|\,ds < \infty\Big] = 1, \qquad j \le N,$$

and

$$\mathfrak{Law}_{P^1}(f_j^1(r_l), \xi^1 : j \le N,\, l \le m) = \mathfrak{Law}_{P^2}(f_j^2(r_l), \xi^2 : j \le N,\, l \le m)$$

for every partition $0 = r_0 < \cdots < r_m \le T$ in D^* for some $D^* \subseteq [0,T]$ of Lebesgue measure T. Then

$$\mathfrak{Law}_{P^1}\Big(\int_0^{t_k} f_j^1(s)\,ds, \xi^1 : k, j\Big) = \mathfrak{Law}_{P^2}\Big(\int_0^{t_k} f_j^2(s)\,ds, \xi^2 : k, j\Big)$$

for every partition $0 = t_0 < \cdots < t_n \le T$.

Proof. First suppose that $X = \mathbb{R}$ and define the real functions

$$h_m(r) = \operatorname{sgn}(r)\max\{|r|, m\}, \qquad m \in \mathbb{N}.$$

Then, by Lemma 8.1, the measures

$$\mathfrak{Law}_{P^i}\Big(\int_0^{t_k} h_m(f_j^i(s))\,ds, \xi^i : k \le n,\, j \le N\Big)$$

are equal for $i = 1,2$ for every $m \in \mathbb{N}$ and the claim follows by letting m tend to infinity by Lebesgue's theorem. To prove the general case choose a sequence $(x_l^* : l \in \mathbb{N})$ in X^*

which separates points of X. Then, by an application of the previous case, we get the equality of the measures

$$\mathfrak{Law}_{P^i}\Big(\int_0^{t_k}\langle x_l^*, f_j^i(s)\rangle\,ds, \xi^i : k \le n,\ l \le L,\ j \le N\Big)$$

for $i = 1, 2$ for every $L \in \mathbb{N}$. But this is already equivalent to the conclusion of the theorem as $(x_l^* : l \in \mathbb{N})$ generates the Borel σ-algebra on X. ∎

REMARK 8.4. Notice that the probabilities appearing in condition (1) are, under the assumptions of the theorem, always equal by Corollary 8.2. Moreover, by obvious modification of the proof, the theorem holds true even if some of f^n's were $[0,\infty]$-valued.

Distribution of stochastic integrals

LEMMA 8.5. *Let $(Y, \mathcal{Y})$ be a measurable space, ξ^i a Y-valued random variable, B_m^i, $m \le M$, real $(\mathcal{F}_t^i)$-Wiener processes and $(g_j^i(t) : t \le T)$, $j \le N$, $(\mathcal{F}_t^i)$-progressively measurable bounded processes on $(\Omega^i, \mathcal{F}^i, (\mathcal{F}_t^i), P^i)$, $i = 1, 2$, such that*

$$\mathfrak{Law}_{P^1}(g_j^1(r_l), B_m^1(r_l), \xi^1 : j, l, m) = \mathfrak{Law}_{P^2}(g_j^2(r_l), B_m^2(r_l), \xi^2 : j, l, m)$$

for every partition $0 = r_0 < \cdots < r_m \le T$ in D^* for some $D^* \subseteq [0, T]$ of Lebesgue measure T. Then

$$\mathfrak{Law}_{P^1}\Big(\int_0^{t_k} g_j^1\,dB_m^1, \xi^1 : k, j, m\Big) = \mathfrak{Law}_{P^2}\Big(\int_0^{t_k} g_j^2\,dB_m^2, \xi^2 : k, j, m\Big)$$

for every partition $0 = t_0 < \cdots < t_n \le T$.

Proof. First suppose that all processes g_j^i are, in addition, continuous and $D^* = [0, T]$. Then

$$\mathfrak{Law}_{P^1}\Big(\int_0^{t_k} g_{jL}^1\,dB_m^1, \xi^1 : k, j, m\Big) = \mathfrak{Law}_{P^2}\Big(\int_0^{t_k} g_{jL}^2\,dB_m^2, \xi^2 : k, j, m\Big)$$

for approximations

$$g_{jL}^i(t) = \sum_{l=1}^{L} g_j^i(r_{l-1}) I_{(r_{l-1}, r_l]}(t)$$

with a subdivision $0 = r_0 < \cdots < r_L = T$ of $(t_k : k \le n)$. The claim now follows by letting the subdivisions' norm tend to zero. In the general case consider the continuous uniformly bounded processes

$$g_{jL}^i(t) = L \int_{\max\{t-1/L,0\}}^{t} g_j^i(s)\,ds = L\Big(\int_0^t g_j^i(s)\,ds - \int_0^{\max\{t-1/L,0\}} g_j^i(s)\,ds\Big).$$

We have

$$\mathfrak{Law}_{P^1}(g_{jL}^1(r_l), B_m^1(r_l), \xi^1 : j, l, m) = \mathfrak{Law}_{P^2}(g_{jL}^2(r_l), B_m^2(r_l), \xi^2 : j, l, m)$$

for every partition $0 = r_0 < \cdots < r_m \le T$ by Lemma 8.1. In fact, we should consider partitions in D^* but the intervening processes are continuous and D^* is dense in $[0, T]$.

Thus, by the previous case,

$$\mathfrak{Law}_{P^1}\Big(\int_0^{t_k} g^1_{jL}\,dB^1_m,\xi^1 : k,j,m\Big) = \mathfrak{Law}_{P^2}\Big(\int_0^{t_k} g^2_{jL}\,dB^2_m,\xi^2 : k,j,m\Big),$$

and the general claim follows by letting L tend to 0 as a sequence because $g^i_{jL} \to g^i_j$ $dt \otimes P^i$-almost everywhere on $[0,T]\times\Omega^1$. ∎

THEOREM 8.6. *Let $(Y,\mathcal{Y})$ be a measurable space, ξ^i a Y-valued random variable, W^i a Q-Wiener process on U and $(g^i_j(t) : t \le T)$, $j \le N$, $L_2(U_0,X)$-valued $(\mathcal{F}^i_t)$-progressively measurable processes on $(\Omega^i,\mathcal{F}^i,(\mathcal{F}^i_t),P^i)$, $i = 1,2$, satisfying*

$$P^1\Big[\int_0^T \|g^1_j(s)\|^2_{L_2(U_0,X)}\,ds < \infty\Big] = P^2\Big[\int_0^T \|g^2_j(s)\|^2_{L_2(U_0,X)}\,ds < \infty\Big] = 1$$

for every $j \le N$ and

$$\mathfrak{Law}_{P^1}(g^1_j(r_l),W^1_{r_l}(h_k),\xi^1 : j,l,k) = \mathfrak{Law}_{P^2}(g^2_j(r_l),W^2_{r_l}(h_k),\xi^2 : j,l,k)$$

for every $h_k \in U$, $k \le K$ and every partition $0 = r_0 < \dots < r_m \le T$ in D^* for some $D^* \subseteq [0,T]$ of Lebesgue measure T. Then

$$\mathfrak{Law}_{P^1}\Big(\int_0^{t_k} g^1_j\,dW^1,\xi^1 : k,j\Big) = \mathfrak{Law}_{P^2}\Big(\int_0^{t_k} g^2_j\,dW^2,\xi^2 : k,j\Big)$$

for every partition $0 = t_0 < \dots < t_n \le T$.

Proof. We start by taking the simple approximation $F_n : L_2(U_0,X) \to L(U,X)$, $n \in \mathbb{N}$, with values in the space of finite-dimensional operators of $L(U,X)$ as in Lemma 3.4 such that $\|F_nA - A\|_{L_2(U_0,X)} \searrow 0$ for every $A \in L_2(U_0,X)$. Thus each F_n is of the form

$$\sum_{k=1}^m I_{C_k}B_k,$$

where $(C_k : k \le m)$ is a measurable decomposition of $L_2(U_0,X)$ and $B_k \in L(U,X)$ are finite-dimensional. If we put $g^i_{jn} = F_ng^i_j$, we will have

$$(*)\qquad \lim_{n\to\infty}\int_0^T \|g^i_j(s) - g^i_{jn}(s)\|^2_{L_2(U_0,X)}\,ds = 0$$

P^i-almost everywhere, $i = 1,2$. Moreover

$$\mathfrak{Law}_{P^1}(g^1_{jn}(r_l),W^1_{r_l}(h_k),\xi^1 : j,l,k) = \mathfrak{Law}_{P^2}(g^2_{jn}(r_l),W^2_{r_l}(h_k),\xi^2 : j,l,k)$$

for every $h_k \in U$, $k \le K$ and every partition $0 = r_0 < \dots < r_m \le T$ in D^*. So, if we show that the measures

$$(**)\qquad \mathfrak{Law}_{P^i}\Big(\Big\langle x^*_l,\int_0^{t_k} g^i_{jm}\,dW^i\Big\rangle,\xi^i : l \le L,\ k \le n,\ j \le N\Big)$$

are equal for $i = 1,2$ for every $m \in \mathbb{N}$, $L \in \mathbb{N}$ and some sequence $(x^*_l : l \in \mathbb{N})$ which separates points of X (hence generates the Borel σ-algebra of X) we will prove the claim of the theorem using $(*)$ and Proposition 4.1 because $(**)$ implies

$$\mathfrak{Law}_{P^1}\Big(\int_0^{t_k} g^1_{jm}\,dW^1,\xi^1 : k,j\Big) = \mathfrak{Law}_{P^2}\Big(\int_0^{t_k} g^2_{jm}\,dW^2,\xi^2 : k,j\Big),\quad m \in \mathbb{N}.$$

But each g^i_{jn} is of the form

$$\sum_{k=1}^{m} I_{[g^i_j \in C_k]} B_k,$$

and recalling the proof of Lemma 3.6, we have

$$\Big\langle x^*, \int_0^t g^i_{jn}\, dW^i \Big\rangle = \sum_{k=1}^{m} \int_0^t I_{[g^i_j \in C_k]}\, dW^i(B^*_k x^*).$$

Thus $(**)$ follows from Lemma 8.5. ∎

Remark 8.7. Notice that the probabilities appearing in the assumptions of Theorem 8.6 are always equal by Corollary 8.2. Moreover, to verify condition (2), one should take advantage of Proposition 2.5(5).

Distribution of measurable selectors

Proposition 8.8. *Let $(\Omega^i, \mathcal{F}^i, (\mathcal{F}^i_t), P^i)$, $i = 1, 2$, be filtered probability spaces, $\mathbb{X}$ a Polish space, $(Y, \mathcal{Y})$ a measurable space, G a nonempty open set in $\mathbb{R}$, D^* a nonempty subset of $[0, T]$ and $\xi^i : \Omega^i \to Y$, $i = 1, 2$, measurable mappings. Let $H^i : [0, T] \times \Omega^i \times \mathbb{X} \to \mathbb{R}$, $i = 1, 2$, satisfy:*

1. *The mapping $[0, T] \times \Omega^i \to \mathbb{R} : (t, \omega) \mapsto H^i(t, \omega, y)$ is $(\mathcal{F}^i_t)$-progressively measurable for every $y \in \mathbb{X}$, $i = 1, 2$.*
2. *The mapping $\mathbb{X} \to \mathbb{R} : y \mapsto H^i(t, \omega, y)$ is continuous for every $(t, \omega) \in [0, T] \times \Omega^i$, $i = 1, 2$.*
3. *$\mathfrak{Law}_{P^1}(H^1(t_j, y_k), \xi^1 : j, k) = \mathfrak{Law}_{P^2}(H^2(t_j, y_k), \xi^2 : j, k)$ for every finite subset $\{t_0, \dots, t_n\}$ of D^* and for every $y_1, \dots, y_m$ in $\mathbb{X}$.*
4. *The set $\{y : H^i(t, \omega, y) \in G\}$ is nonempty for every $(t, \omega) \in [0, T] \times \Omega^i$, $i = 1, 2$.*

Then there exist $(\mathcal{F}^i_t)$-progressively measurable $\mathbb{X}$-valued processes s^i such that

$$\mathfrak{Law}_{P^1}(s^1(t_j), \xi^1 : j) = \mathfrak{Law}_{P^2}(s^2(t_j), \xi^2 : j)$$

for every $t_0, \dots, t_n$ in D^* and $H^i(t, \omega, s^i(t, \omega))$ belongs to $\overline{G}$.

Proof. Suppose that $d < 1$ is a complete metric on $\mathbb{X}$ and choose a countable dense subset $r_1, r_2, \dots$ of $\mathbb{X}$. Define $G^i(t, \omega) = \{y : H^i(t, \omega, y) \in G\}$ and construct a sequence of $(\mathcal{F}^i_t)$-progressively measurable $\mathbb{X}$-valued processes s^i_n in such a way that:

(a) $d(s^i_{n-1}(t, \omega), s^i_n(t, \omega)) < 2^{-n+1}$ for every $n > 0$ and $(t, \omega) \in [0, T] \times \Omega^i$, $i = 1, 2$.
(b) $d(s^i_n(t, \omega), G^i(t, \omega)) < 2^{-n}$ for every $n \ge 0$ and $(t, \omega) \in [0, T] \times \Omega^i$, $i = 1, 2$.
(c) $\mathfrak{Law}_{P^1}(s^1_n(t_j), H^1(t_j, y_k), \xi^1 : j, k) = \mathfrak{Law}_{P^2}(s^2_n(t_j), H^2(t_j, y_k), \xi^2 : j, k)$ for every $t_0, \dots, t_m$ in D^*, $y_1, \dots, y_M$ in $\mathbb{X}$ and $n \ge 0$.

First set $s^i_0(t, \omega)$ identically equal to r_0 and then, proceeding by induction, assuming that (a)–(c) hold for some s^i_{n-1}, $i = 1, 2$, $n \ge 1$, define

$$A^i_j = \{(t, \omega) : d(r_j, s^i_{n-1}(t, \omega)) < 2^{-n+1}\} \cap \{(t, \omega) : d(r_j, G^i(t, \omega)) < 2^{-n}\}$$

for $j \ge 0$ and $i = 1, 2$, which are $(\mathcal{F}^i_t)$-progressively measurable because

$$\{(t, \omega) : d(r_j, G^i(t, \omega)) < 2^{-n}\} = \bigcup_{l=1}^{\infty} \{(t, \omega) : H^i(t, \omega, r_l) \in G\,, d(r_j, r_l) < 2^{-n}\}.$$

Moreover

$$\bigcup_{j=0}^{\infty} A_j^i = [0,T] \times \Omega^i, \qquad i = 1,2,$$

by assumption (b). Thus defining

$$s_n^i = r_j \quad \text{on } A_j^i \setminus \bigcup_{l<j} A_l^i \quad \text{for } j \geq 0 \text{ and } i = 1,2$$

we complete the induction step and now it suffices to take s^i as the limit of s_n^i. ■

9. Proofs of Theorems 3 and 4

Proof of Theorem 3. We will start with the following lemma:

LEMMA 9.1. *Under the assumptions of Theorem* 3 *let* $(\Omega^1, \mathcal{F}^1, (\mathcal{F}_t^1), P^1, W^1, u^1)$ *be another solution of* (0.1) *such that* $\mathfrak{Law}_P(u) = \mathfrak{Law}_{P^1}(u^1)$. *Further suppose that* $\xi : \Omega \to Y$, $\xi^1 : \Omega^1 \to Y$ *are some random variables, where* $(Y, \mathcal{Y})$ *is some measurable space. Let also* x_i, *resp.* x_i^1, $i \leq n$, *be* $(\mathcal{F}_t)$, *resp.* $(\mathcal{F}_t^1)$*-progressively measurable processes in* X_1^* *such that*

$$P\Big[\int_0^T \|g^*(s,u(s))x_i(s)\|_{U_0}^2\,ds < \infty\Big] = 1 \quad \textit{for every } i \leq n \tag{1}$$

and

$$\mathfrak{Law}_P(u(t_j), x_i(s_j), \xi : i,j) = \mathfrak{Law}_{P^1}(u^1(t_j), x_i^1(s_j), \xi^1 : i,j) \tag{2}$$

for any finite sequences $t_0, \ldots, t_m$ *and* $s_0, \ldots, s_m$ *in* D^*, *where* D^* *is a subset of* $[0,T]$ *of Lebesgue measure* T. *Then*

$$\mathfrak{Law}_P\Big(u(t_j), \xi, \int_0^{t_j} g^*(s,u(s))x_i(s)\,dW_s : i,j\Big) = \mathfrak{Law}_{P^1}\Big(u^1(t_j), \xi^1, \int_0^{t_j} g^*(s,u^1(s))x_i^1(s)\,dW_s^1 : i,j\Big)$$

for every $0 = t_0 < \cdots < t_m \leq T$.

REMARK. Observe that (2) in Lemma 9.1 is equivalent to

$$\mathfrak{Law}_P(u(t_j), x_i(s_j), \xi : i,j) = \mathfrak{Law}_{P^1}(u^1(t_j), x_i^1(s_j), \xi^1 : i,j)$$

for every $t_0, \ldots, t_m$ in $[0,T]$ and $s_0, \ldots, s_m$ in D^* by the assumption (0.3). Indeed, (0.3) implies existence of a continuous modification of $t \mapsto \langle x^*, u_t \rangle$ for every $x^* \in D(A^*)$ (Theorem 13) and $D(A^*)$ separates points in X_1, so it generates the Borel σ-algebra in X! Consequently, given $0 = t_0 < \ldots < t_m \leq T$ we find some $t_j^k \in D^*$ such that $t_j^k \to t_j$ and extend the equality of the intervening laws to $t_0, \ldots, t_m$ by the above mentioned continuity.

Proof of Lemma 9.1. Note that (0.3) holds for u^1, as well as

$$P^1\Big[\int_0^T \|g^*(s,u^1(s))x_i^1(s)\|_{U_0}^2\,ds < \infty\Big] = 1$$

for every $i \le n$ by Corollary 8.2 (assuming (1) and (2) of Lemma 9.1). Hence we can apply Theorem 13 (together with the remark included in the proof) to obtain

$$\int_0^t g^*(s,u_s)R_\lambda^* y^*\,dW_s = \langle y^*, R_\lambda u_t\rangle - \langle y^*, R_\lambda u_0\rangle - \int_0^t \langle y^*, AR_\lambda u_s + R_\lambda f(s,u_s)\rangle\,ds$$

for every $y^* \in X_1^*$, where $R_\lambda = (\lambda - A)^{-1}$ for some λ in the resolvent set of the generator A of (S_t) in X_1. Similarly, we get an analogous equation for u^1. Now suppose that all x_i, x_i^1, $i \le n$, are bounded, norm continuous and $D^* = [0,T]$. Fix a partition $(t_j : j)$ of $[0,T]$ and define

$$x_{iLm}(t) = \sum_{l=0}^{L-1} F_m x_i(r_l) I_{(r_l,r_{l+1}]}(t), \qquad x_{iLm}^1(t) = \sum_{l=0}^{L-1} F_m x_i^1(r_l) I_{(r_l,r_{l+1}]}(t)$$

for some subdivision $0 = r_0 < \cdots < r_L = T$ of $(t_j : j)$ where F_m are the simple approximations of identity from Lemma 3.4 applied on the separable space X_1^*. Then the claim of the lemma is true for $R_\lambda^* x_{iLm}$ and $R_\lambda^* x_{iLm}^1$ by Theorem 8.3 because

$$\int_0^t g^*(s,u_s)R^* x_{iLm}\,dW_s = \sum_{l=0}^{L-1}\langle F_m x_i(r_l), Ru_{t\wedge r_{l+1}} - Ru_{t\wedge r_l}\rangle - \int_0^t \langle x_{iLm}, ARu_s + Rf(s,u_s)\rangle\,ds$$

and analogously for x_{iLm}^1. Letting $m \to \infty$ we get the claim for

$$x_{iL}(t) = \sum_{l=0}^{L-1} x_i(r_l) I_{(r_l,r_{l+1}]}(t), \qquad x_{iL}^1(t) = \sum_{l=0}^{L-1} x_i^1(r_l) I_{(r_l,r_{l+1}]}(t)$$

by Proposition 4.1. Letting the subdivision's norm tend to zero as L tends to infinity we deduce the claim of the lemma for $R_\lambda^* x_i$, $R_\lambda^* x_i^1$ by another application of Proposition 4.1. In the second step we will suppose that x_i, x_i^1 are bounded and D^* has Lebesgue measure T. Consider

$$y_{iL}(t) = L\int_{\max\{t-1/L,0\}}^t x_i(s)\,ds = L\Big(\int_0^t x_i(s)\,ds - \int_0^{\max\{t-1/L,0\}} x_i(s)\,ds\Big)$$

and analogously y_{iL}^1, where the integral is taken in the Banach space X_1^*. We know that

$$\mathfrak{Law}_P(u(t_j), y_{iL}(t_j) : i,j) = \mathfrak{Law}_{P^1}(u^1(t_j), y_{iL}^1(t_j) : i,j)$$

for every $0 = t_0 < \cdots < t_m \le T$ by Theorem 8.3. Thus, referring to the first part of the proof, we conclude that the claim of the lemma holds for $R_\lambda^* y_{iL}$, resp. $R_\lambda^* y_{iL}^1$, and so it holds for $R_\lambda^* x_i$, resp. $R_\lambda^* x_i^1$ by Proposition 4.1 as y_{iL}, resp. y_{iL}^1 converge in norm to x_i, resp. x_i^1 $dt\otimes P$, resp. $dt\otimes P^1$-almost everywhere (e.g. [DU]). On the other hand, $\lambda R_\lambda^* x^* \to x^*$ for every $x^* \in X_1^*$ as $\lambda \to \infty$ because X_1 is reflexive (so A^* is the generator of a C_0-semigroup on X_1^*). Hence the claim holds for x_i, x_i^1 as well by letting λ tend to infinity and by the uniform boundedness of x_i and x_i^1. Finally, to cover the general case, split the processes into

$$x_{im} = x_i I_{[\|x_i\| \le m]}, \qquad x_{im}^1 = x_i^1 I_{[\|x_i^1\| \le m]}.$$

The claim holds for x_{im}, x_{im}^1 by the previous part of the proof and

$$\int_0^T \|g^*(s,u(s))x_i - g^*(s,u(s))x_{im}\|_{U_0}^2\,ds = \int_0^T I_{[\|x_i\|>m]}\|g^*(s,u(s))x_i\|_{U_0}^2\,ds \xrightarrow{P\text{-a.e.}} 0.$$

Since the same reasoning holds in the analogous case the proof is complete. ∎

LEMMA 9.2. *Under the assumptions of Theorem* 3 *let* $(\Omega^1,\mathcal{F}^1,(\mathcal{F}^1_t),P^1,W^1,u^1)$ *be another solution of* (0.1) *such that* $\mathfrak{Law}_P(u)=\mathfrak{Law}_{P^1}(u^1)$. *Suppose that* $\xi:\Omega\to Y$, $\xi^1:\Omega^1\to Y$ *are random variables, where* $(Y,\mathcal{Y})$ *is a measurable space, such that*

$$\mathfrak{Law}_P(u(t_j),\xi:j)=\mathfrak{Law}_{P^1}(u^1(t_j),\xi^1:j)$$

for every $0=t_0<\cdots<t_m\le T$ *in* D^*, *where* D^* *is a subset of* $[0,T]$ *of Lebesgue measure* T. *Let* $p(t,x)\in L(U_0)$ *denote the orthogonal projection of* U_0 *onto the closed subspace* $(\operatorname{Ker} g(t,x))^\perp$. *Then* $p:[0,T]\times X\to L(U_0)$ *is strongly measurable and*

$$\mathfrak{Law}_P\Big(u(t_j),\xi,\int_0^{t_j}p(s,u(s))h_i\,dW_s:i,j\Big)=\mathfrak{Law}_{P^1}\Big(u^1(t_j),\xi^1,\int_0^{t_j}p(s,u^1(s))h_i\,dW^1_s:i,j\Big)$$

for every $0=t_0<\cdots<t_m\le T$ *and* $h_1,\dots,h_n$ *in* U_0.

Proof. We begin with the strong measurability of p. Fix $h\in U_0$ and let $a_0,a_1,a_2,\dots$ be a dense subset in U_0 with $a_0=0$. In this way we can define

$$\begin{aligned}\delta_\varepsilon(t,x)&=\inf\{\|h-a_i\|_{U_0}:\|g(t,x)a_i\|_{X_1}<\varepsilon,\ i=0,1,\dots\}\\&=\inf\{\|h-a\|_{U_0}:\|g(t,x)a\|_{X_1}<\varepsilon\}\end{aligned}$$

for $(t,x)\in[0,T]\times X$ and $q_{\varepsilon,m}(t,x)=a_i$, $(t,x)\in[0,T]\times X$, where i is the least index in $\{0,1,\dots\}$ such that

$$\|h-a_i\|_{U_0}<\delta_\varepsilon(t,x)+1/m$$

and $\|g(t,x)a_i\|_{X_1}<\varepsilon$. Both $\delta_\varepsilon(t,x)$ and $q_{\varepsilon,m}(t,x)$ are obviously measurable. But

$$\|q_{\varepsilon,m}-q_{\varepsilon,n}\|^2_{U_0}\le 2(\delta_\varepsilon+1/m)^2+2(\delta_\varepsilon+1/n)^2-4\delta_\varepsilon^2$$

by the parallelogram law. So $q_{\varepsilon,m}$ converges to some measurable q_ε in U_0 and we have $\|h-q_\varepsilon(t,x)\|_{U_0}=\delta_\varepsilon(t,x)$ and $\|g(t,x)(q_\varepsilon(t,x))\|_{X_1}\le\varepsilon$ for every $(t,x)\in[0,T]\times X$. Now let $0<\varepsilon_1<\varepsilon_2$. Using the parallelogram rule once again, we get

$$\|q_{\varepsilon_1}-q_{\varepsilon_2}\|^2_{U_0}\le 2\delta^2_{\varepsilon_1}+2\delta^2_{\varepsilon_2}-4\delta^2_{\varepsilon_2}.$$

But $\varepsilon\mapsto\delta_\varepsilon(t,x)$ is nonincreasing on $(0,\infty)$ and bounded by

$$\delta(t,x)=\inf\{\|h-a\|_{U_0}:\|g(t,x)a\|_{X_1}=0\}$$

for every $(t,x)\in[0,T]\times X$ so q_ε converges to some measurable $q:[0,T]\times X\to U_0$ as $\varepsilon\to0$ zero and we have $\|h-q(t,x)\|_{U_0}\le\delta(t,x)$ and $\|g(t,x)(q(t,x))\|_{X_1}=0$ for every $(t,x)\in[0,T]\times X$, so $\|h-q(t,x)\|_{U_0}=\delta(t,x)$. Hence $q(t,x)$ is the orthogonal projection of h onto $\operatorname{Ker} g(t,x)$, therefore $(t,x)\mapsto p(t,x)h=h-q(t,x)$ is measurable.

Now we can proceed to show the second assertion. Note that $\operatorname{Rng} g^*(t,x)$ is dense in $(\operatorname{Ker} g(t,x))^\perp=\operatorname{Rng} p(t,x)$ for every $(t,x)\in[0,T]\times X$, so define

$$H(t,\omega,x_1^*,\dots,x_n^*)=\sum_{i=1}^n\|g^*(t,u(t,\omega))x_i^*-p(t,u(t,\omega))h_i\|_{U_0}$$

on $[0,T]\times\Omega\times(X_1^*)^n$ and

$$H^1(t,\omega,x_1^*,\dots,x_n^*)=\sum_{i=1}^n\|g^*(t,u^1(t,\omega))x_i^*-p(t,u^1(t,\omega))h_i\|_{U_0}$$

on $[0,T]\times\Omega^1\times(X_1^*)^n$ for fixed $h_1,\dots,h_n$ in U_0. Now we can apply Proposition 8.8 to get progressively measurable processes x_{im}, x^1_{im}, $i\le n$, in X_1^* such that, for every $m\in\mathbb{N}$:

- ▷ $\mathfrak{Law}_P(u(t_j), \xi, x_{im}(s_j) : j, i) = \mathfrak{Law}_{P^1}(u^1(t_j), \xi^1, x^1_{im}(s_j) : j, i)$ for every partition $t_0, \ldots, t_N$ in D^* and every $s_0, \ldots, s_N$ in D^*.
- ▷ $\sum_{i=1}^n \|g^*(t, u(t,\omega))x_{im} - p(t, u(t,\omega))h_i\|_{U_0} \le 1/m$ everywhere on $[0,T] \times \Omega$.
- ▷ $\sum_{i=1}^n \|g^*(t, u^1(t,\omega))x^1_{im} - p(t, u^1(t,\omega))h_i\|_{U_0} \le 1/m$ everywhere on $[0,T] \times \Omega^1$.

But now we are exactly in the situation of Lemma 9.1 and the claim of Lemma 9.2 is proven if we let m tend to infinity, by Proposition 4.1. ■

Proof of Theorem 3. First we remark that the set of one-to-one operators from $L(U_0, X_1)$ is strongly measurable. Indeed, both U_0 and X_1 are separable reflexive, hence a linear bounded operator $B \in L(U_0, X_1)$ is one-to-one if and only if the range of its adjoint operator B^* is norm dense in U_0, i.e. $\inf\{\|B^* x^*_k - h_j\| : k \in \mathbb{N}\} = 0$ for every $j \in \mathbb{N}$, where $(x^*_k : k)$, resp. $(h_j : j)$ are norm dense countable subsets of X^*_1, resp. U_0. As a consequence, if $(\Omega^1, \mathcal{F}^1, (\mathcal{F}^1_t), P^1, W^1, u^1)$ is another solution of (0.1) such that $\mathfrak{Law}_P(u) = \mathfrak{Law}_{P^1}(u^1)$ then

$$dt \otimes P^1\{(s,\omega) : g(s, u^1(s,\omega)) \text{ is not one-to-one in } L(U_0, X_1)\} = 0.$$

From Lemma 9.2 we deduce that, given $h \in U$, $p(s, u(s,\omega))Qh = Qh$ for $dt\otimes P$-almost all $(s,\omega) \in [0,T]\times\Omega$ as well as $p(s, u^1(s,\omega))Qh = Qh$ for $dt\otimes P^1$-almost all $(s,\omega) \in [0,T]\times\Omega^1$ and thus

$$P\Big[\int_0^t p(s, u(s))Qh\, dW = \int_0^t Qh\, dW = W_t(h)\Big] = 1$$

for every $t \in [0,T]$ by Example 3.7, and an analogous equality holds for (u^1, W^1). Hence Lemma 9.2 yields the assertion of Theorem 3. ■

Proof of Theorem 4. We will start with an auxiliary lemma referring once again to the notation of Section 11:

LEMMA 9.3. *Let X_1 be reflexive, $x_0 \in X$, and let (S_t) be a C_0-semigroup of bounded linear operators on $L(X_1)$. Suppose that equation* (0.1) *with $\mu = \delta_{x_0}$ has the uniqueness in law property for solutions satisfying* (0.3) *and* (0.4). *Let $S_t x_0 \in X$ for every $t \in (0,T]$. Then the σ-algebra $\sigma(u_t : t \le T)$ is independent of $\mathcal{G}_0$ for every solution $(\Omega, \mathcal{G}, (\mathcal{G}_t), P, u, W)$ of* (0.1), (0.3), (0.4) *starting from $x_0 \in X$.*

Proof. In compliance with the notation of Section 11 (preceding the proof of Theorems 1 and 2) we fix an orthonormal basis $(h^*_k : k \in \mathbb{N})$ in U and write W_{dec} for the continuous $\mathbb{R}^{\mathbb{N}}$-valued process $(W(h^*_k) : k \in \mathbb{N})$. We also fix a sequence $(x^*_k \in X^*_1 : k \in \mathbb{N})$ in $D(A^*)$ which separates points in X_1 (hence in X). Denote by $e : X \to \mathbb{R}^{\mathbb{N}}$ the continuous embedding $x \mapsto (\langle x^*_k, x\rangle : k \in \mathbb{N})$ and consider the extended (Borel measurable) inverse $e^{-1} : \mathbb{R}^{\mathbb{N}} \to X$, where $e^{-1}(y) = 0$ for $y \notin \mathrm{Rng}(e)$ (see Lemma 11.1). The remark included in the proof of Theorem 13 ensures a continuous modification of the $\mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$-valued process $t \mapsto (eu_t, W_{\mathrm{dec}}(t))$, hence there exists a mapping (kernel) $k : \Omega \times \mathbb{B}(\mathfrak{C}\times\mathfrak{C}) \to [0,1]$ such that

(a) $\Omega \to [0,1] : \omega \mapsto k(\omega, V)$ is $\mathcal{G}_0$-measurable for every $V \in \mathbb{B}(\mathfrak{C}\times\mathfrak{C})$.
(b) $\mathbb{B}(\mathfrak{C}\times\mathfrak{C}) \to [0,1] : V \mapsto k(\omega, V)$ is a probability measure for every $\omega \in \Omega$.
(c) $\int_{G_0} k_\omega(V)\, dP = \int_{G_0} I_V(eu, W_{\mathrm{dec}})\, dP$ for every $G_0 \in \mathcal{G}_0$ and $V \in \mathbb{B}(\mathfrak{C}\times\mathfrak{C})$.

To simplify the notation we will write

$$\pi^i : \mathfrak{C} \times \mathfrak{C} \to C([0,T],\mathbb{R}^{\mathbb{N}}) : (g_1, g_2) \mapsto g_i, \quad i = 1, 2.$$

The existence of k is guaranteed e.g. by [Ed, Corollary 3.3] as $C([0,T],\mathbb{R}^{\mathbb{N}})$ is a Polish space. Equality (c) can now be rewritten as

$$(*) \qquad \int_{G_0} k(\omega, [(\pi^1, \pi^2) \in V])\, dP(\omega) = \int_{G_0} I_V(eu, W_{\mathrm{dec}})\, dP.$$

The space $\mathfrak{C}\times\mathfrak{C}$ endowed with the filtration $(\mathbb{B}_t \otimes \mathbb{B}_t)_{t\in[0,T]}$ in $\mathbb{B}\otimes\mathbb{B}$ is a filtered measurable space and π^1, π^2 are adapted continuous $\mathbb{R}^{\mathbb{N}}$-valued processes. Firstly we will show that

$$B_t(h) = \begin{cases} \sum_k \langle h_k^*, h\rangle_U \pi_k^2(t) & \text{on } V_h^t, \\ 0 & \text{off } V_h^t, \end{cases} \qquad h \in U,\ t \le T,$$

where

$$V_h = \Big\{(t, g_1, g_2) \in [0,T] \times \mathfrak{C} \times \mathfrak{C} : \sum_k \langle h_k^*, h\rangle_U \pi_k^2(t, g_1, g_2) \text{ is convergent}\Big\}$$

$$= \bigcap_{j=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} \Big\{(t, g_1, g_2) \in [0,T] \times \mathfrak{C} \times \mathfrak{C} : \Big|\sum_k \langle h_k^*, h\rangle_U \pi_k^2(t, g_1, g_2)\Big| \le 1/j\Big\}$$

defines a cylindrical Q-$(\mathbb{B}_t \otimes \mathbb{B}_t)$-Wiener process on $\mathfrak{C} \times \mathfrak{C}$ under the probability measure k_ω for P-almost all $\omega \in \Omega$. To this end fix $N \in \mathbb{N}$, $0 \le s < t \le T$, $A \in \mathbb{B}_s \otimes \mathbb{B}_s$, $B \in \mathbb{B}(\mathbb{R}^N)$ and set $Y^N = (\pi_1^2, \ldots, \pi_N^2)$. Then $(*)$ yields

$$k_\omega(A \cap [Y_t^N - Y_s^N \in B]) = k_\omega(A)\mathcal{N}(0, (t-s)[\langle Q^{1/2}h_i^*, Q^{1/2}h_j^*\rangle]_{i,j})(B)$$

for P-almost all ω. But since $\mathbb{B}_s \otimes \mathbb{B}_s$ and $\mathbb{B}(\mathbb{R}^N)$ are countably generated, there exists a set, say $G_1 \in \mathcal{G}_0$, such that Y^N is an $\mathbb{R}^N$-valued k_ω-$(\mathbb{B}_t \otimes \mathbb{B}_t)$-Wiener process for every $N \in \mathbb{N}$ and $\omega \in G_1$. As a consequence, $\sum_k \langle h_k^*, h\rangle_U \pi_k^2(t)$ is a sum of sign-invariant random variables, hence it converges in measure (under k_ω) if and only if it converges k_ω-almost surely (for $\omega \in G_1$). So, given $h \in U$, we get $k_\omega[V_h^t] = 1$ for every $t \le T$ and every $\omega \in G_1$. Passing to the limit in $(*)$ we get

$$(**) \qquad \int_{\Omega} k(\omega, [(\pi^1, B_{t_1}h_1, \ldots, B_{t_m}h_m) \in V])\, dP(\omega) = \int_{\Omega} I_V(eu, W_{t_1}h_1, \ldots, W_{t_m}h_m)\, dP$$

for every $t_1, \ldots, t_m$ in $[0,T]$, $h_1, \ldots, h_m$ in U and $V \in \mathbb{B} \otimes \mathbb{B}(\mathbb{R}^m)$.

Now fix $t \in (0,T]$. We are going to show that

$$\text{(aa)} \qquad \begin{aligned} &k_\omega\Big[\int_0^T (\|f(r, e^{-1}\pi_r^1)\|_{X_1} + \|g(r, e^{-1}\pi_r^1)\|^2_{L(U_0,X_1)})\, dr < \infty\Big] = 1, \\ &k_\omega\Big[\int_0^T (M_1(f(r, e^{-1}\pi_r^1)) + M_2(g(r, e^{-1}\pi_r^1)))\, dr < \infty\Big] = 1, \end{aligned}$$

and second that

$$\text{(bb)} \qquad k_\omega\Big[e^{-1}\pi_t^1 = S_t e^{-1}\pi_0^1 + \int_0^t S_{t-r} f(r, e^{-1}\pi_r^1)\, dr + \int_0^t S_{t-r} g(r, e^{-1}\pi_r^1)\, dB_r\Big] = 1,$$

$$\text{(cc)} \qquad k_\omega[e^{-1}\pi_0^1 = x_0] = 1$$

for every ω from some $G_2 \in \mathcal{G}_0$, $G_2 \subseteq G_1$, $P(G_2) = 1$. The fact that (aa) and (cc) hold for P-almost all ω follows immediately from $(*)$ and Proposition 9.4. To show that (bb) holds define

$$O = \{(r, y) \in [0, t] \times \mathbb{R}^{\mathbb{N}} : S_{t-r} g(r, e^{-1} y) \in L_2(U_0, X)\}$$

and

$$\psi : [0, t] \times \mathbb{R}^{\mathbb{N}} : (r, y) \mapsto S_{t-r} g(r, e^{-1} y) I_O(r, y).$$

Choosing $\varphi_1(r, y) = I_O(r, y)$ in Proposition 9.4 we get

$$\text{(dd)} \qquad k_\omega \Big[\int_0^t I_O(r, \pi_r^1)\, dr = t \Big] = 1,$$

$$\text{(ee)} \qquad k_\omega [\pi_t^1 \in \operatorname{Rng} e] = 1$$

for P-almost all $\omega \in \Omega$ by $(*)$. We can introduce, as in the remark following Lemma 3.4, simple approximations $F_n : L_2(U_0, X) \to L(U, X)$ such that each F_n takes only finitely many values, and moreover all of them are finite-dimensional operators in $L(U, X)$, and $\|F_n(B) - B\|_{L_2(U_0,X)} \searrow 0$ for every $B \in L_2(U_0, X)$. Consider an equidistant partition $0 = t_0 < \cdots < t_k = t$ and define measurable mappings from $\mathfrak{C}$ to $L_2(U_0, X)$ as in Proposition 9.4:

$$Z_i(y) = \begin{cases} (m/t) \int_{t_{i-1}}^{t_i} \psi(r, y_r)\, dr & \text{if } \int_0^t \|\psi(r, y_r)\|^2_{L_2(U_0,X)}\, dr < \infty, \\ 0 & \text{if } \int_0^t \|\psi(r, y_r)\|^2_{L_2(U_0,X)}\, dr = \infty, \end{cases}$$

and

$$Z_t(y) = \begin{cases} 0 & \text{for } 0 \le t \le t_1, \\ Z_i(y) & \text{for } t_i < t \le t_{i+1},\ i = 1, \dots, m-1. \end{cases}$$

Note that Z (depending on m) is a predictable process with values in $L_2(U_0, X)$, and so is $(Z_t(\pi^1))$. Now the composition process $t \mapsto F_n \circ Z_t(y)$ is piecewise constant and takes only finitely many values, all of them finite-dimensional operators in $L(U, X)$. It is easy to verify that

$$(3^*) \qquad \int_{G_1} k\Big(\omega, \Big[\Big(\pi^1, \int_0^t F_n Z_r(\pi^1)\, dB_r\Big) \in V\Big]\Big)\, dP = \int_\Omega I_V\Big(eu, \int_0^t F_n Z_r(eu)\, dW_r\Big)\, dP$$

for every $V \in \mathbb{B} \otimes \mathbb{B}(X)$ because both stochastic integrals are elementary and are defined exclusively by Borel compositions, and so (3^*) holds by $(**)$ in a straightforward manner. Indeed, the left hand side integral in (3^*) is considered only on G_1 as we know that $P(G_1) = 1$ (so G_1 can be exchanged with Ω) and B is a Q-Wiener process under k_ω only for $\omega \in G_1$. Now, letting $n \to \infty$, we obtain

$$(4^*) \qquad \int_{G_2} \Big(\int_{\mathfrak{C}} \varphi\Big(\pi^1, \int_0^t Z_r(\pi^1)\, dB_r\Big)\, dk_\omega \Big)\, dP = \int_\Omega \varphi\Big(eu, \int_0^t Z_r(eu)\, dW_r\Big)\, dP$$

for every bounded continuous function $\varphi : \mathfrak{C} \times X \to \mathbb{R}$ since

$$\int_0^t F_n Z_r(\pi^1)\, dB_r \to \int_0^t Z_r(\pi^1)\, dB_r$$

in measure (under k_ω) for every $\omega \in G_2$ by (aa), where $\int_0^t Z_r(\pi^1)\, dB_r$ depends on ω and is defined only for $\omega \in G_1$. In the last step we recall the dependence of Z on m and let

$m \to \infty$ to get

$$(5^*) \qquad \int_{G_2} k\Big(\omega, \Big[\Big(\pi^1, \int_0^t \psi(r, \pi^1_r)\, dB_r\Big) \in V\Big]\Big)\, dP = \int_{\Omega} I_V\Big(eu, \int_0^t \psi(r, eu_r)\, dW_r\Big)\, dP$$

by the same argument as in the previous step. Now we apply (dd) to get

$$\int_{G_2} k\Big(\omega, \Big[\Big(\pi^1, \int_0^t S_{t-r} g(r, e^{-1}\pi^1_r)\, dB_r\Big) \in V\Big]\Big)\, dP = \int_{\Omega} I_V\Big(eu, \int_0^t S_{t-r} g(r, u_r)\, dW_r\Big)\, dP,$$

which, together with Proposition 9.4, implies (bb).

Now we can find a set $G_3 \subseteq G_1$, $G_3 \in \mathcal{G}_0$, such that (aa), (bb), (cc) and (ee) hold for every $\omega \in G_3$ and every $t \in J$, where J is some countable dense subset of $[0,T]$. We claim that the process $t \mapsto e^{-1}\pi^1_t$ is a solution under k_ω of (0.1), (0.3) and (0.4) with the initial condition x_0. To see this, choose $\omega \in G_3$ and define

$$v_t = S_t x_0 + \int_0^t S_{t-s} f(s, e^{-1}\pi^1_s)\, ds + \int_0^t S_{t-s} g(s, e^{-1}\pi^1_s)\, dB_s$$

under k_ω. As remarked in the proof of Theorem 13, the process $t \mapsto v_t$ is an X-valued process predictable in X_1 with respect to the k_ω-augmentation of $(\mathbb{B}_t \otimes \mathbb{B}_t)$ in $\mathbb{B} \otimes \mathbb{B}$, hence predictable in X (all summands belong to X) and $t \mapsto e(v_t)$ has a continuous modification. But $k_\omega[v_t = e^{-1}\pi^1_t] = 1$ for every $t \in J$ by (aa). This implies that $k_\omega[e(v_t) = \pi^1_t] = 1$ for every $t \in J$ by (ee). In consequence, $k_\omega[e(v_t) = \pi^1_t] = 1$ for every $t \in [0,T]$ by continuity of both ev and π^1. Therefore $k_\omega[v_t = e^{-1}\pi^1_t] = 1$ for every $t \in [0,T]$ as v takes values in X, whence the claim. In particular,

$$P[u_{t_i} \in B_i,\, i \le n] = k_\omega[e^{-1}\pi_{t_i} \in B_i,\, i \le n] = k_\omega[\pi_{t_i} \in e[B_i],\, i \le n]$$

for every $\omega \in G_3$, $t_1, \ldots, t_n$ in $[0,T]$ and $B_1, \ldots, B_n$ in $\mathbb{B}(X)$, where the first equality holds by uniqueness in law. Given $G \in \mathcal{G}_0$ with $G \subseteq G_3$ we get

$$P(G) P[u_{t_i} \in B_i,\, i \le n] = \int_G k_\omega[\pi_{t_i} \in e[B_i],\, i \le n]\, dP = P[G \cap [u_{t_i} \in B_i,\, i \le n]]$$

by $(*)$ so the independence of $\sigma(u_t : 0 \le t \le T)$ on $\mathcal{G}_0$ has just been shown. ∎

PROPOSITION 9.4. *Let $\varphi_1 : [0,T] \times \mathbb{R}^{\mathbb{N}} \to [0,\infty]$ and $\varphi_2 : [0,T] \times \mathbb{R}^{\mathbb{N}} \to Y$ be measurable functions, where Y is some separable Banach space. Then, for some $0 \le a < b \le T$, the mappings*

$$\mathfrak{C} \to [0,\infty] : y \mapsto \int_a^b \varphi_1(r, y_r)\, dr,$$

$$\mathfrak{C} \to Y : y \mapsto \begin{cases} \int_a^b \varphi_2(r, y_r)\, dr & \textit{if } \int_0^T \|\varphi_2(r, y_r)\|_Y\, dr < \infty, \\ 0 & \textit{if } \int_0^T \|\varphi_2(r, y_r)\|_Y\, dr = \infty, \end{cases}$$

are Borel measurable.

Proof of Theorem 4. Consider $q : [0,T] \times X \to L(U_0)$, where $q(t,x)$ is the orthogonal projection in U_0 onto the closed subspace $\operatorname{Ker} g(t,x)$. Then q is strongly measurable by Lemma 9.2. Next consider a solution $(\Omega, \mathcal{F}, (\mathcal{F}_t), P, W, u)$ of (0.1), (0.3), (0.4) starting from x_0 and extend the stochastic base (by product extension of probability spaces) so that there exist Q-$(\mathcal{F}_t)$-Wiener processes B and C such that B, C and (u, W) are independent. Further define

$$w_t(h) = \int_0^t p(s,u_s)Qh\,dW_s + \int_0^t q(s,u_s)Qh\,dB_s,$$

$$v_t(h) = \int_0^t p(s,u_s)Qh\,dC_s + \int_0^t q(s,u_s)Qh\,dW_s$$

for $h \in U$. Then

$$\begin{aligned}\langle w(h_1), w(h_2)\rangle_t &= \langle v(h_1), v(h_2)\rangle_t \\ &= \int_0^t \left(\langle p(s,u_s)Qh_1, p(s,u_s)Qh_2\rangle_{U_0} + \langle q(s,u_s)Qh_1, q(s,u_s)Qh_2\rangle_{U_0}\right) ds \\ &= t\langle Qh_1, Qh_2\rangle_{U_0} = t\langle Q^{1/2}h_1, Q^{1/2}h_2\rangle_U\end{aligned}$$

by independence of W, B and C. Thus w and v are Q-$(\mathcal{F}_t)$-Wiener processes. Moreover

$$\langle w(h_1), v(h_2)\rangle_t = \int_0^t \langle p(s,u_s)Qh_1, q(s,u_s)Qh_2\rangle_{U_0}\,ds = 0,$$

which, by Lévy's theorem, implies that

- ▷ $a_t - a_s$ is independent of $\mathcal{F}_s$,
- ▷ $a_t^1 - a_s^1$ is independent of $a_t^2 - a_s^2$,

for every $0 \le s < t \le T$, where $a = (a^1, a^2)$, $a^1 = (w(h_1), \dots, w(h_m))$ and $a^2 = (v(\widetilde{h}_1), \dots, v(\widetilde{h}_n))$ with $h_1, \dots, h_m, \widetilde{h}_1, \dots, \widetilde{h}_n$ belonging to U. This means that the σ-algebra $\sigma(w_t(h) - w_s(h))$ is independent of

$$\mathcal{G}_s = \mathcal{F}_s \vee \sigma(v_r(\widetilde{h}) - v_s(\widetilde{h}) : r \ge s,\ \widetilde{h} \in U) = \mathcal{F}_s \vee \sigma(v_r(\widetilde{h}) : r \ge 0,\ \widetilde{h} \in U).$$

Hence w is a Q-$(\mathcal{G}_t)$-Wiener process. Now, if ψ is an $(\mathcal{F}_t)$-simple process (as we have assumed in Step 1 of the construction of the stochastic integral) then one verifies that

$$\text{(a)} \qquad \int_0^t \psi_s\,dw_s = \int_0^t \psi_s p(s,u_s)\,dW_s + \int_0^t \psi_s q(s,u_s)\,dB_s,$$

$$\text{(b)} \qquad \int_0^t \psi_s\,dv_s = \int_0^t \psi_s p(s,u_s)\,dC_s + \int_0^t \psi_s q(s,u_s)\,dW_s,$$

directly by definition. A density argument implies that the above equalities hold for every $(\mathcal{F}_t)$-progressively measurable process ψ satisfying

$$P\Big[\int_0^t \|\psi_s\|^2_{L_2(U_0,X)}\,ds < \infty\Big] = 1$$

by the construction of the stochastic integral and by the ideal property of $L_2(U_0, X)$ as shown in Proposition 2.4, in particular

$$\|\psi_s p(s,u_s)\|_{L_2(U_0,X)} \le \|\psi_s\|_{L_2(U_0,X)}\|p(s,u_s)\|_{L(U_0)} \le \|\psi_s\|_{L_2(U_0,X)}$$

and analogously for $q(s,u_s)$. In consequence, taking $\psi_s = S_{t-s}g(s,u_s)$, we get

$$\int_0^t S_{t-s}g(s,u_s)\,dw_s = \int_0^t S_{t-s}g(s,u_s)p(s,u_s)\,dW_s = \int_0^t S_{t-s}g(s,u_s)\,dW_s$$

by (a), and since the stochastic integral

$$\int_0^t S_{t-s}g(s,u_s)\,dw_s$$

is the same under the filtration $(\mathcal{F}_t)$ and $(\mathcal{G}_t)$ we conclude that $(\Omega,\mathcal{F},(\mathcal{G}_t),P,w,u)$ is a solution of (0.1), (0.3) and (0.4) starting from x_0, hence u is independent of $\mathcal{G}_0$ by Lemma 9.3. In particular, $\sigma(u_t : 0\le t\le T)$ is independent of $\sigma(v_t(h) : t\ge 0, h\in U)$, and this is the crucial point of the proof. Indeed, consider another solution $(\Omega^1,\mathcal{F}^1,(\mathcal{F}^1_t),P^1,u^1,W^1)$ of (0.1), (0.3) and (0.4) such that $\mathfrak{Law}_P(u) = \mathfrak{Law}_{P^1}(u^1)$ on the extended space (as at the beginning of this proof) so that it supports two Q-$(\mathcal{F}^1_t)$-Wiener processes B^1 and C^1 such that B^1, C^1 and (u^1,W^1) are independent. We define u^1 and v^1 in the same way as we defined u and v to infer that $\sigma(u^1_t : 0\le t\le T)$ is independent of $\sigma(v^1_t(h) : t\ge 0, h\in U)$. But this implies that $\mathfrak{Law}_P(u,v) = \mathfrak{Law}_{P^1}(u^1,v^1)$, and consequently

$$\text{(c)}\qquad \mathfrak{Law}_P\Big(u_{t_j}, \int_0^{t_j} q(s,u_s)h_i\,dv_s : i,j\Big) = \mathfrak{Law}_{P^1}\Big(u^1_{t_j}, \int_0^{t_j} q(s,u^1_s)h_i\,dv^1_s : i,j\Big)$$

for every $0=t_0<\cdots<t_m\le T$ and $h_1,\dots,h_n$ in U_0 by Theorem 8.6. But

$$\int_0^t q(s,u_s)h\,dv_s = \int_0^t q(s,u_s)h\,dW_s \quad\text{and}\quad \int_0^t q(s,u^1_s)h\,dv^1_s = \int_0^t q(s,u^1_s)h\,dW^1_s$$

by (b), hence, incorporating this fact to (c) and applying Lemma 9.2 we obtain

$$\begin{aligned}\mathfrak{Law}_P\Big(u_{t_j}, &\int_0^{t_j} q(s,u_s)h_i\,dW_s, \int_0^{t_j} p(s,u_s)h_i\,dW_s : i,j\Big)\\ &= \mathfrak{Law}_{P^1}\Big(u^1_{t_j}, \int_0^{t_j} q(s,u^1_s)h_i\,dW^1_s, \int_0^{t_j} p(s,u^1_s)h_i\,dW^1_s : i,j\Big)\end{aligned}$$

for every $0=t_0<\cdots<t_m\le T$ and $h_1,\dots,h_n$ in U_0. This, in particular, means that

$$\mathfrak{Law}_P(u_{t_j}, W_{t_j}(\widetilde{h}_i) : i,j) = \mathfrak{Law}_{P^1}(u^1_{t_j}, W^1_{t_j}(\widetilde{h}_i) : i,j)$$

for every $0=t_0<\cdots<t_m\le T$ and $\widetilde{h}_1,\dots,\widetilde{h}_n$ in U because $\int_0^t Q\widetilde{h}\,dW = W_t(\widetilde{h})$ and $\int_0^t Q\widetilde{h}\,dW^1 = W^1_t(\widetilde{h})$ for every $\widetilde{h}\in U$ by Example 3.7. ■

10. Proofs of Theorems 5 and 6

Before we give the details we recall a version of Lévy's theorem:

PROPOSITION 10.1. *Consider a filtered probability space $(\Omega^i,\mathcal{F}^i,(\mathcal{F}^i_t),P^i)$ with a d-dimensional continuous local $(\mathcal{F}^i_t)$-martingale M^i, $M^i_0=0$, defined on $[0,T]$, and an $(\mathcal{F}_0)$-measurable random variable ξ^i with values in a measurable space $(Y,\mathcal{Y})$, $i=1,2$. Suppose that the cross-variation $(d\times d)$-matrix*

$$V_{jk}(t) = (\langle M^i_j, M^i_k\rangle_t)$$

is deterministic, independent of i *and* $\mathfrak{Law}_{P^1}(\xi^1) = \mathfrak{Law}_{P^2}(\xi^2)$. *Then*

$$\mathfrak{Law}_{P^1}(M_j^1(t_k), \xi^1 : j \le d,\ k \le n) = \mathfrak{Law}_{P^2}(M_j^2(t_k), \xi^2 : j \le d,\ k \le n)$$

for every partition $0 = t_0 < \cdots < t_n = T$.

Proof. The classical Lévy characterization theorem implies that $\sigma(M_t^i - M_s^i)$ is P^i-independent of $\mathcal{F}_s^i$ for every $0 \le s < t \le T$, $i = 1, 2$, and $\mathfrak{Law}_{P^i}(M_t^i - M_s^i) = \mathcal{N}(0, V_t - V_s)$. In particular we have equality of the marginal measures

$$\mathfrak{Law}_{P^1}(M_j^1(t_k) : j, k) = \mathfrak{Law}_{P^2}(M_j^2(t_k) : j, k)$$

and P^i-independence implies

$$\mathfrak{Law}_{P^i}(M_j^i(t_k), \xi^i : j, k) = \mathfrak{Law}_{P^i}(M_j^i(t_k) : j, k) \otimes \mathfrak{Law}_{P^i}(\xi^i). \ \blacksquare$$

Proof of Theorem 5. Let $(\Omega^i, \mathcal{F}^i, (\mathcal{F}_t^i), P^i, W^i, u^i)$ be a filtered probability space with a Q-Wiener process W^i and u^i, $i = 1, 2$, a progressively measurable process satisfying (0.6), (0.7). It will be convenient to extend the operator-valued function S to nonpositive times by $0 \in L(X_1, X)$ to obtain a strongly measurable family of operators in $L(X_1, X)$ on the real line. So, if we define the stopping times

$$\tau_k^i = T \wedge \min\Big\{t \le T : \int_0^t \|f(s, u^i(s))\|_{U_0}^2\, ds \ge k\Big\}, \quad i = 1, 2,$$

then

$$M^i(t) = \exp\bigg(-\int_0^{t\wedge\tau_k^i} f(s, u^i(s))\, dW^i - \frac{1}{2}\int_0^{t\wedge\tau_k^i} \|f(s, u^i(s))\|_{U_0}^2\, ds\bigg), \quad i = 1, 2,$$

is a P^i-martingale by Proposition 7.1. Following Proposition 7.1 we define a new measure

$$\widetilde{P}^i(F) = \int_F M_T^i\, dP^i$$

on $\mathcal{F}^i$ and a family of nonrandom strongly measurable $L(U_0, X)$-valued processes

$$\psi^t(s) = S_{t-s} g(s), \quad t \in [0, T],$$

such that $\psi^t(s) \in L_2(U_0, X)$ for almost all $s \le T$. Indeed, $L_2(U_0, X)$ is a strongly measurable subset of $L(U_0, X)$ by Proposition 2.5(6) so we can define the stochastic integral of ψ^t in an unambiguous way as $\int \phi^t\, dW$ with any $L_2(U_0, X)$-valued progressively measurable process ϕ^t which coincides with ψ^t $dt \otimes P$-almost everywhere. Then

$$\widetilde{W}_t^i(h) = W_t^i(h) + \int_0^t \langle f(s, u^i(s)), h\rangle_U I_{[0,\tau_k^i]}(s)\, ds, \quad t \le T,\ h \in U,$$

is a Q-$(\mathcal{F}_t^i)$-Wiener process on U with respect to $\widetilde{P}^i$, and

$$\int_0^r \psi^t\, d\widetilde{W}^i = \int_0^r \psi^t\, dW^i + \int_0^r \psi^t(s) f(s, u^i(s)) I_{[0,\tau_k^i]}(s)\, ds \tag{10.1}$$

almost everywhere for every $r \le T$ and $t \le T$ by Proposition 7.1. Now the mutual cross-variation processes of the real local $(\mathcal{F}_t^i, \widetilde{P}^i)$-martingales $\widetilde{W}^i(h_1)$, $\widetilde{W}^i(h_2)$, $\langle x_1^*, \int \psi^t\, d\widetilde{W}^i\rangle$ and $\langle x_2^*, \int \psi^t\, d\widetilde{W}^i\rangle$ are nonrandom and independent of i for every choice of $h_1 \in U$, $h_2 \in U$, $x_1^* \in X$, $x_2^* \in X$ by Summary of Step 5 and Example 3.7. Hence the probability measures

$$\mathfrak{Law}_{\widetilde{P}^i}\Big(\Big\langle x_j^*, \int_0^{t_n} \psi^{t_n}\, d\widetilde{W}^i\Big\rangle, \widetilde{W}^i_{t_n} h_m, u^i(0) : j, n, m\Big)$$

are equal for $i = 1, 2$ by Proposition 10.1 for every partition $0 = t_0 < \cdots < t_N \le T$, $x_1^*, \ldots, x_J^*$ in X^* and $h_1, \ldots, h_M$ in U. Consequently, the measures

$$\mathfrak{Law}_{\widetilde{P}^i}\Big(\int_0^{t_n} \psi^{t_n}\, d\widetilde{W}^i, \widetilde{W}^i_{t_n} h_m, u^i(0) : n, m\Big) \tag{10.2}$$

coincide for $i = 1, 2$ since $u^1(0)$, $u^2(0)$ have the same law. The process

$$z^i(t) = \begin{cases} S_t u^i(0) + \int_0^t \psi^t\, d\widetilde{W}^i, & 0 < t \le T, \\ u^i(0), & t = 0, \end{cases}$$

has a predictable modification

$$u^i(t) - \int_0^t S_{t-s} g(s) f(s, u^i(s)) I_{(\tau_k^i, T]}(s)\, ds \qquad \text{for } i = 1, 2$$

by (10.1), and due to (10.2), we get

$$\mathfrak{Law}_{\widetilde{P}^1}(z^1(t_n), \widetilde{W}^1_{t_n} h_m : n, m) = \mathfrak{Law}_{\widetilde{P}^2}(z^2(t_n), \widetilde{W}^2_{t_n} h_m : n, m)$$

for every partition $0 = t_0 < \cdots < t_N \le T$ and $h_1, \ldots, h_M$ in U. Consider the auxiliary process

$$e^i(t) = \int_0^t \|f(s, z^i(s))\|^2_{U_0}\, ds, \quad t \le T,\ i = 1, 2.$$

Then, by Remark 8.4,

$$\mathfrak{Law}_{\widetilde{P}^1}(z^1(t_n), \widetilde{W}^1_{t_n} h_m, e^1(t_n) : n, m) = \mathfrak{Law}_{\widetilde{P}^2}(z^2(t_n), \widetilde{W}^2_{t_n} h_m, e^2(t_n) : n, m),$$
$$\mathfrak{Law}_{\widetilde{P}^1}(z^1(t_n), \widetilde{W}^1_{t_n} h_m, \tau_k^1 : n, m) = \mathfrak{Law}_{\widetilde{P}^2}(z^2(t_n), \widetilde{W}^2_{t_n} h_m, \tau_k^2 : n, m),$$

since the process z^i coincides with u^i $dt \otimes \widetilde{P}^i$-almost everywhere on $[0, \tau_k^i]$ by definition of z^i and thus

$$[\tau_k^i \le \Delta] = [\inf\{\max\{e^i(q) - k, 0\} : q \in \mathbb{Q} \cap [0, \Delta]\} = 0]$$

modulo a $\widetilde{P}^i$-negligible set for every $0 \le \Delta < T$, which is already sufficient for the equality of the measures above.

Now, as already observed,

$$l^i(t) = f(t, z^i(t)) I_{[0,\tau_k^i]}(t) = f(t, u^i(t)) I_{[0,\tau_k^i]}(t)$$

$dt \otimes \widetilde{P}^i$-almost everywhere on $[0, T] \times \Omega^i$, so the measures

$$\mathfrak{Law}_{\widetilde{P}^i}\Big(z^i(t_n), \widetilde{W}^i_{t_n} h_m, l^i(t_n), \int_0^{t_n} l^i(s)\, ds : n, m\Big)$$

are equal for $i = 1, 2$ for every partition $0 = t_0 < \cdots < t_N \le T$ and $h_1, \ldots, h_M$ in U by Theorem 8.3. Consequently,

$$\mathfrak{Law}_{\widetilde{P}^1}(z^1(t_n), W^1_{t_n} h_m, l^1(t_n) : n, m) = \mathfrak{Law}_{\widetilde{P}^2}(z^2(t_n), W^2_{t_n} h_m, l^2(t_n) : n, m)$$

since

$$W_t^i h = \widetilde{W}_t^i h - \Big\langle \int_0^t l^i(s)\, ds, h \Big\rangle_U$$

for every $t \le T$ and $h \in U$ by the definition of $\widetilde{W}^i$. Finally,

$$\mathfrak{Law}_{\widetilde{P}^i} \Big(z^i(t_n), W_{t_n}^i h_m, \int_0^T l^i\, dW^i, \int_0^T \|l^i(s)\|_{U_0}^2\, ds : n, m \Big)$$

are equal for $i = 1, 2$ by Theorems 8.3 and 8.6, and since

$$M^i(T) = \exp\Big(- \int_0^T l^i\, dW^i - \frac{1}{2} \int_0^T \|l^i(s)\|_{U_0}^2\, ds \Big)$$

we conclude that $\mathfrak{Law}_{\widetilde{P}^i}(z^i(t_n), W_{t_n}^i h_m, M^i(T) : n, m)$ are equal for $i = 1, 2$. But this implies

$$\mathfrak{Law}_{P^1}(z^1(t_n), W_{t_n}^1 h_m : n, m) = \mathfrak{Law}_{P^2}(z^2(t_n), W_{t_n}^2 h_m : n, m)$$

by the definition of the measure $\widetilde{P}^i$. Now, if we observe that $z^i(t) = z_k^i(t) \to u^i(t)$ P^i-almost everywhere, the claim follows by letting k tend to infinity. ■

Proof of Theorem 6. Fix $t \in (0, T]$, set $S_0 = 0 \in L(X_1, X)$ and define the processes

$$\eta_1^i(s) = S_{t-s} f(s, u^i(s)), \quad \theta_2^i(s) = S_{t-s} g(s, u^i(s)), \quad s \le t,\ i = 1, 2.$$

Then η_1^i is measurable with values in X and θ_2^i is $(\mathcal{F}_t^i)$-progressively measurable with values in $L(U_0, X)$ such that $\eta_2^i \in L_2(U_0, X)\, ds \otimes P^i$-almost everywhere. Since $L_2(U_0, X)$ is a strongly measurable subset of $L(U_0, X)$ by Proposition 2.5(6), we define the $(\mathcal{F}_t^i)$-progressively measurable $L_2(U_0, X)$-valued processes

$$\eta_2^i = \theta_2^i I_{[\theta_2^i \in L_2(U_0, X)]}, \quad i = 1, 2,$$

which satisfy

$$P^i[\eta_2^i(s) = S_{t-s} g(s, u^i(s))] = 1, \quad i = 1, 2,$$

for almost all $s \le T$. Hence the measures

$$\mathfrak{Law}_{P^i}(u^i(t), u^i(0), \eta_1^i(r_l), \eta_2^i(r_l), W_{r_l}^i h_k : l, k)$$

are equal for $i = 1, 2$ for every $h_1, \ldots, h_m$ in U and every partition $0 = r_0 < \cdots < r_L \le T$ in some set $D^* \subseteq [0, T]$ of Lebesgue measure T. Thus, by Corollary 8.2, (0.2) holds for u^2. Moreover we conclude that

$$\mathfrak{Law}_{P^1} \Big(u^1(t), S_t u^1(0), \int_0^t \eta_1^1(s)\, ds, \int_0^t \eta_2^1\, dW^1 \Big) = \mathfrak{Law}_{P^2} \Big(u^2(t), S_t u^2(0), \int_0^t \eta_1^2(s)\, ds, \int_0^t \eta_2^2\, dW^2 \Big)$$

by Theorems 8.3 and 8.6, so

$$\begin{aligned} 1 &= P^1 \Big[u^1(t) = S_t u^1(0) + \int_0^t S_{t-s} f(s, u^1(s))\, ds + \int_0^t S_{t-s} g(s, u^1(s))\, dW^1 \Big] \\ &= P^2 \Big[u^2(t) = S_t u^2(0) + \int_0^t S_{t-s} f(s, u^2(s))\, ds + \int_0^t S_{t-s} g(s, u^2(s))\, dW^2 \Big]. \end{aligned}$$ ■

11. Preliminaries to the proofs of Theorems 1 and 2

Before we proceed to the proofs of Theorems 1 and 2 we make a simple but important observation on a bi-Borel embedding of a separable reflexive Banach space into $\mathbb{R}^{\mathbb{N}}$ (Lemma 11.1). More precisely, we re-establish the results of Yamada and Watanabe for mild solutions (0.1) in 2-smooth Banach spaces. The proof in the finite-dimensional case relies on the disintegration of the joint solution measure on the Polish state space of continuous functions which is associated to the trajectories of the solutions. But, in contrast to the finite-dimensional case, we do not know in general whether the solutions are norm continuous. So it is not possible to consider $C([0,T],X)$ as the state space for the trajectories although we need some Polish space for the disintegration theory. On the other hand we know that, in many cases (e.g. Theorem 13), there exists a sequence $(x_n^* : n \in \mathbb{N})$ in X^* which separates points of X such that $t \mapsto \langle x_n^*, u(t)\rangle$ is a continuous process. So one way out is to consider the solutions with the above, rather weak continuity property, and $C([0,T],\mathbb{R}^{\mathbb{N}})$ as the state space for the natural coordinate decomposition of the trajectory. We are also going to use the following notation:

- $\triangleright$ $\mathfrak{C} = C([0,T],\mathbb{R}^{\mathbb{N}})$, $\mathbb{B} = \mathbb{B}(C([0,T],\mathbb{R}^{\mathbb{N}}))$.
- $\triangleright$ $\pi_t : C([0,T],\mathbb{R}^{\mathbb{N}}) \to \mathbb{R}^{\mathbb{N}} : f \mapsto f(t)$, $t \le T$.
- $\triangleright$ $\varphi_t : C([0,T],\mathbb{R}^{\mathbb{N}}) \to C([0,T],\mathbb{R}^{\mathbb{N}}) : \varphi_t(f)(s) = f(t \wedge s)$, $t \le T$.
- $\triangleright$ $\phi_t : C([0,T],\mathbb{R}^{\mathbb{N}}) \to C([0,T],\mathbb{R}^{\mathbb{N}}) : \phi_t(f)(s) = f((t+s) \wedge T) - f(t)$, $t \le T$.
- $\triangleright$ $\mathbb{B}_t = \mathbb{B}(C([0,T],\mathbb{R}^{\mathbb{N}})) = \sigma(\pi_s : s \le t)$, $t \le T$.

One can easily verify that the Borel σ-algebra $\mathbb{B}(C([0,T],\mathbb{R}^{\mathbb{N}}))$ coincides with $\sigma(\pi_s : s \le T)$ so the mappings $\varphi_t : (\mathfrak{C},\mathbb{B}_t) \to (\mathfrak{C},\mathbb{B})$, $\phi_t : (\mathfrak{C},\mathbb{B}) \to (\mathfrak{C},\mathbb{B})$ are measurable and $\mathbb{B}_t(C([0,T],\mathbb{R}^{\mathbb{N}})) = \sigma(\varphi_t)$, $t \le T$. Next we define

- $\triangleright$ $\Omega^* = X \times \mathfrak{C} \times \mathfrak{C} \times \mathfrak{C}$,
- $\triangleright$ $\mathcal{F}^* = \mathbb{B}(X) \otimes \mathbb{B} \otimes \mathbb{B} \otimes \mathbb{B}$,
- $\triangleright$ $\mathcal{F}_t^* = \mathbb{B}(X) \otimes \mathbb{B}_t \otimes \mathbb{B}_t \otimes \mathbb{B}_t$,

and if ν is a measure on $(\Omega^*,\mathcal{F}^*)$ we will write $\mathcal{F}_t^\nu = \mathcal{F}_t^* \vee \sigma\{N \in \mathcal{F}^* : \nu(n) = 0\}$ for the ν-augmentation of $\mathcal{F}_t^*$ in $\mathcal{F}^*$. *We fix an orthonormal basis* $(h_k^* : k \in \mathbb{N})$ *in* U *throughout this section* and if W is a Q-Wiener process on U then we denote by

- $\triangleright$ $t \mapsto W_{\mathrm{dec}}(t) = (W_t(h_k^*) : k \in \mathbb{N})$ the continuous process in $\mathbb{R}^{\mathbb{N}}$ and
- $\triangleright$ $\mathcal{W} = \mathfrak{Law}(W_{\mathrm{dec}})$ its distribution on $(\mathfrak{C},\mathbb{B})$ which depends only on Q.

Finally, let μ be a probability Borel measure on X. Then we denote by

- $\triangleright$ $\mathcal{G}_t^{\mu\otimes\mathcal{W}} = \mathbb{B}(X) \otimes \mathbb{B}_t \vee \sigma\{N \in \mathbb{B}(X) \otimes \mathbb{B} : \mu \otimes \mathcal{W}(N) = 0\}$ the augmentation of $\mathbb{B}(X) \otimes \mathbb{B}_t$ in $\mathbb{B}(X) \otimes \mathbb{B}$ with respect to $\mu \otimes \mathcal{W}$,

and, to shorten the notation, we will write

- $\triangleright$ $x : \Omega^* \to X : (a,b,c^1,c^2) \mapsto a$.
- $\triangleright$ $w : \Omega^* \to \mathfrak{C} : (a,b,c^1,c^2) \mapsto b$.
- $\triangleright$ $y^i : \Omega^* \to \mathfrak{C} : (a,b,c^1,c^2) \mapsto c^i$, $i = 1,2$.

A decomposition result is discussed next:

LEMMA 11.1. *Let X be a separable reflexive Banach space (e.g. 2-smooth) and $(x_n^*:n\in\mathbb{N})$ a sequence in X^* which separates points of X. Then the image* $\operatorname{Rng} e$ *of the one-to-one mapping $e : X \to \mathbb{R}^{\mathbb{N}} : x \mapsto (\langle x_n^*, x\rangle : n \in \mathbb{N})$ is a Borel subset in $\mathbb{R}^{\mathbb{N}}$ and the extended inverse $e^{-1} : \mathbb{R}^{\mathbb{N}} \to X$, $e^{-1}(y) = 0$ for $y \notin \operatorname{Rng} e$, is Borel measurable.*

Proof. The system $\mathcal{S} = \{B \subseteq X : e[B]$ is Borel measurable in $\mathbb{R}^{\mathbb{N}}\}$ contains closed balls and X as these are weakly σ-compact and e is continuous with respect to the weak topology in X. Hence $\mathcal{S}$ is a σ-algebra, whence the claim follows. ∎

COROLLARY 11.2. *Let $(Y, \mathcal{Y})$ be a measurable space, $(\Omega^i, \mathcal{F}^i, (\mathcal{F}_t^i), P^i, \xi^i, \eta^i)$, $i = 1, 2$, two filtered probability spaces, where ξ^1 is an $(\mathcal{F}_t^1)$-predictable X-valued process, $\xi^2 = (\xi_n^2 : n \in \mathbb{N})$ a family of real $(\mathcal{F}_t^2)$-predictable processes and η^i, $i = 1, 2$, some Y-valued random variables. Suppose that*

$$\mathfrak{Law}_{P^1}(\langle x_m^*, \xi^1(t_n)\rangle, \eta^1 : m, n) = \mathfrak{Law}_{P^2}(\xi_m^2(t_n), \eta^2 : m, n)$$

for every partition $0 = t_0 < \cdots < t_N \le T$, every $x_1^*, \ldots, x_M^*$, $M \in \mathbb{N}$ and $N \in \mathbb{N}$, where $(x_n^* : n \in \mathbb{N})$ is some sequence in X^* which separates points of X. Then there exists a predictable X-valued process ξ on $(\Omega^2, \mathcal{F}^2, (\mathcal{F}_t^2), P^2)$ such that

$$\mathfrak{Law}_{P^1}(\xi^1(t_n), \eta^1 : n) = \mathfrak{Law}_{P^2}(\xi(t_n), \eta^2 : n)$$

for every partition $0 = t_0 < \cdots < t_N \le T$ and $t \mapsto \langle x_n^*, \xi(t)\rangle$ is a modification of $t \mapsto \xi_n^2(t)$, $n \in \mathbb{N}$.

Proof. Consider the mapping e associated to $(x_n^* : n \in \mathbb{N})$ and define $\xi = e^{-1}(\xi^2)$. Then ξ is a predictable process in X because ξ^2 is predictable in $\mathbb{R}^{\mathbb{N}}$ and e^{-1} is measurable. Moreover

$$P^2[e(\xi(t)) = \xi^2(t)] = P^2[e(e^{-1}\xi^2(t)) = \xi^2(t)] = P^1[e(e^{-1}\xi^1(t)) = \xi^1(t)] = 1$$

for every $t \le T$ and the equality of the laws follows from the fact that $(x_n^* : n \in \mathbb{N})$ generates the Borel σ-algebra on X. ∎

12. Proof of Theorem 2

In fact, we are going to prove a more general statement. Theorem 2 is its immediate consequence.

THEOREM 12.1. *Let μ be a Borel probability measure on X, $(\Omega, \mathcal{F}, P, W, u)$ a solution of* (0.1) *satisfying* (0.2), *and let $(x_n^* : n \in \mathbb{N})$ be a sequence in X^* which separates points of X such that the processes $t \mapsto \langle x_n^*, u(t)\rangle$ have continuous adapted modifications. Then there exists a probability measure P^* on $(\Omega^*, \mathcal{F}^*)$, a Q-$(\mathcal{F}_t^{P^*})$-Wiener process W^* and $(\mathcal{F}_t^{P^*})$-predictable X-valued processes Z^1, Z^2 such that $(\Omega^*, \mathcal{F}^*, (\mathcal{F}_t^{P^*}), P^*, W^*, Z^i)$ satisfies* (0.1), (0.2) *and the processes $t \mapsto \langle x_n^*, Z^i(t)\rangle$ have continuous adapted modifications, $i = 1, 2$. Moreover*

(1) $P^*[Z^1(0) = Z^2(0)] = 1$.
(2) $\mathfrak{Law}_{P^*}(Z^1, W^*) = \mathfrak{Law}_{P^*}(Z^2, W^*) = \mathfrak{Law}_P(u, W)$.

(3) *If we knew that* $P^*[Z^1(t)=Z^2(t)]=1$ *for* t *from some dense subset of* $[0,T]$ *then there would exist a mapping* $\widetilde{R}: X\times\mathfrak{C}\to\mathfrak{C}$ *measurable in the sense*

$$\widetilde{R}:(X\times\mathfrak{C},\mathcal{G}_t^{\mu\otimes\mathcal{W}})\to(\mathfrak{C},\mathbb{B}_t),\quad t\le T,$$

such that whenever $(\Omega^1,\mathcal{F}^1,(\mathcal{F}^1_t),P^1,W^1,u^1_0)$ *is a filtered probability space with a* Q*-*$(\mathcal{F}^1_t)$*-Wiener process* W^1 *and an* $\mathcal{F}^1_0$*-measurable random variable* u^1_0 *with distribution* $\mathfrak{Law}_{P^1}(u^1_0)=\mu$, *if we define the predictable process* $u^1(t)$ *to be* $e^{-1}\pi_t\widetilde{R}(u^1_0,W^1_{\mathrm{dec}})$, *then the family* $(\Omega^1,\mathcal{F}^1,(\mathcal{F}^1_t),P^1,W^1,u^1)$ *is a strong solution of* (0.1) *satisfying* (0.2) *with the initial condition* u^1_0 *and* $\mathfrak{Law}_{P^1}(u^1,W^1)=\mathfrak{Law}_P(u,W)$.

We will start the proof of Theorem 12.1 by recalling one of the versions of the classical theorem on disintegration of measures. See Corollary 3.3 in [Ed] for the proof.

PROPOSITION 12.2. *Let* Z *be a Polish space,* $(H,\mathcal{H})$ *a measurable space and* q *a probability measure on* $\mathcal{H}\otimes\mathbb{B}(Z)$. *Then there exists a kernel* $q: H\times\mathbb{B}(Z)\to[0,1]$ *such that*

(1) *The mapping* $H\to[0,1]: h\mapsto q(h,F)$ *is* $\mathcal{H}$*-measurable for every* $F\in\mathbb{B}(Z)$.
(2) *The mapping* $\mathbb{B}(Z)\to[0,1]: F\mapsto g(h,F)$ *is a probability measure.*
(3) $q(B\times F)=\int_B q(h,F)\,dq_1(h)$ *for every* $B\in\mathcal{H}$, $F\in\mathbb{B}(Z)$, *where* $q_1(B)=q(B\times Z)$ *for* $B\in\mathcal{H}$ *is the marginal measure of* q.

Proof of Theorem 12.1. Let e be the mapping associated to $(x^*_n: n\in\mathbb{N})$ by Lemma 11.1. Then the modification of the random vector $(u(0),W_{\mathrm{dec}},eu-eu(0))$ takes values in $X\times\mathfrak{C}\times\mathfrak{C}$ by the assumption of the theorem. Next consider the measure

$$\mathfrak{Law}_P(u(0),W_{\mathrm{dec}},eu-eu(0))\quad\text{ on }\mathbb{B}(X)\otimes\mathbb{B}\otimes\mathbb{B}$$

and denote by q^t its restriction to $\mathbb{B}(X)\otimes\mathbb{B}_t\otimes\mathbb{B}$, $t\le T$. By Proposition 12.2, there exist the corresponding kernels $q^t: X\times\mathfrak{C}\times\mathbb{B}\to[0,1]$, $t\le T$, where we take $(H,\mathcal{H})=(X\times\mathfrak{C},(X)\otimes\mathbb{B}_t)$, $t\le T$, and $Z=\mathfrak{C}$. The measure P^* is then defined by

$$P^*(B)=\int\limits_{X\times\mathfrak{C}}\Big(\int\limits_{\mathfrak{C}\times\mathfrak{C}} q^T_{(x,w)}\otimes q^T_{(x,w)}(B^{(x,w)})\Big)\,d\mu\otimes\mathcal{W}(x,w),\quad B\in\mathcal{F}^*,\tag{12.1}$$

where $B^{(x,w)}=\{(y^1,y^2):(x,w,y^1,y^2)\in B\}$ is the cut-set in $(x,w)\in X\times\mathfrak{C}$, the solutions Z^i are defined as

$$Z^i(t,x,w,y^1,y^2)=x+e^{-1}y^i(t)\tag{12.2}$$

and the Q-Wiener process as

$$W^*_{x,w,y^1,y^2}(h)=\sum_{k=1}^{\infty}\langle h,h^*_k\rangle_U w_k,\quad h\in U,\tag{12.3}$$

where the sum is taken in $L^2((\Omega^*,\mathcal{F}^*P^*),C[0,T])$ and $w=(w_k:k\in\mathbb{N})\in\mathfrak{C}$.

Proof of the first part of Theorem 12.1 (in a sequence of lemmas)

LEMMA A. *Let* $t\le T$. *Then* $\sigma(\phi_t(W_{\mathrm{dec}}))$ *is* P*-independent of* $\mathcal{F}_t$. *In particular*

(1) $\sigma(u(0),\varphi_t(W_{\mathrm{dec}}),\varphi_t(eu-eu(0)))$ *is* P*-independent of* $\sigma(\phi_t(W_{\mathrm{dec}}))$.
(2) $\mathbb{B}_t$ *is* $\mathcal{W}$*-independent of* $\sigma(\phi_t)$ *on* $(\mathfrak{C},\mathbb{B})$.

(3) $\mathfrak{Law}_P(u(0), W_{\mathrm{dec}}) = \mu \otimes \mathcal{W}$.

(4) *$\mu \otimes \mathcal{W}$ is the q_1^t-marginal of q^t, $t \le T$, from Proposition* 12.2.

Proof. The process $B = (W(h_1^*), \ldots, W(h_N^*))$ is an N-dimensional $(\mathcal{F}_t)$-Wiener process with covariance

$$(\langle Q^{1/2} h_i^*, Q^{1/2} h_j^* \rangle_U)_{ij}$$

so, by Lévy's characterization theorem, the σ-algebra $\sigma(B(r) - B(t) : r \in [t, T])$ is P-independent of $\mathcal{F}_t$ for every $N \in \mathbb{N}$, hence so is $\sigma(W_{\mathrm{dec}}(r) - W_{\mathrm{dec}}(t) : r \in [t, T])$. The claim (1) holds because

$$\sigma(u(0), \varphi_t(W_{\mathrm{dec}}), \varphi_t(eu - eu(0))) \subseteq \mathcal{F}_t,$$

(2) because $\mathbb{B}_t = \sigma(\varphi_t)$, and (3) and (4) are obvious. ∎

Lemma B. *Let $t \le T$, $A \in \mathbb{B}$, $F \in \mathcal{F}_t^{P^*}$ and $B = X \times \phi_t^{-1}[A] \times \mathfrak{C} \times \mathfrak{C}$. Then $P^*(B \cap F) = P^*(B) P^*(F)$.*

Proof. First we prove that

$$\mu \otimes \mathcal{W}\{(x, w) \in X \times \mathfrak{C} : q^t(x, w, F_t) = q^T(x, w, F_t)\} = 1 \tag{B.1}$$

for every $F_t \in \mathbb{B}_t$. To do so fix $G_1 \in \mathcal{B}X$ and define

$$\mathcal{D} = \Big\{ G \in \mathbb{B} : \int\limits_{G_1 \times G} q^t(x, w, F_t)\, d\mu \otimes \mathcal{W}(x, w) = q^T(G_1 \times G \times F_t) \Big\}.$$

It is easy to see that $\mathcal{D}$ is a Dynkin class and if we show that it contains all sets of the type $\varphi_t^{-1}[G_2] \cap \phi_t^{-1}[G_3]$ for all $G_2 \in \mathbb{B}$, $G_3 \in \mathbb{B}$ we will know that $\mathcal{D}$ is all of $\mathbb{B}$. Thus

$$\begin{aligned}
&\int\limits_{G_1 \times (\varphi_t^{-1}[G_2] \cap \phi_t^{-1}[G_3])} q^t(x, w, F_t)\, d\mu \otimes \mathcal{W}(x, w) \\
&\qquad = \int\limits_{\mathfrak{C}} \underbrace{\Big(\int\limits_X I_{G_1}(x) q^t(x, w, F_t)\, d\mu(x) \Big) I_{G_2}(\varphi_t(w))}_{\mathbb{B}_t\text{-measurable}} \ \underbrace{I_{G_3}(\phi_t(w))}_{\sigma(\phi_t)\text{-measurable}} \ d\mathcal{W}(w)
\end{aligned}$$

by Fubini's theorem. But $\mathbb{B}_t$ is $\mathcal{W}$-independent of $\sigma(\phi_t)$ by Lemma A(2) so the above equals

$$\begin{aligned}
&\Big(\int\limits_{G_1 \times \varphi_t^{-1}[G_2]} q^t(x, w, F_t)\, d\mu \otimes \mathcal{W}(x, w) \Big) \mathcal{W}(\phi_t^{-1}[G_3]) \\
&\qquad = \mathfrak{Law}_P(u(0), W_{\mathrm{dec}}, eu - eu(0))(G_1 \times \varphi_t^{-1}[G_2] \times F_t)\, \mathfrak{Law}_P(W_{\mathrm{dec}})(\phi_t^{-1}[G_3])
\end{aligned}$$

and since $F_t = \varphi_t^{-1}[V_0]$ for some $V_0 \in \mathbb{B}$, this is

$$\begin{aligned}
&P[(u(0), \varphi_t(W_{\mathrm{dec}}), \varphi_t(eu - eu(0))) \in G_1 \times G_2 \times V_0] P[\phi_t(W_{\mathrm{dec}}) \in G_3] \\
&\qquad = \mathfrak{Law}_P(u(0), W_{\mathrm{dec}}, eu - eu(0))(G_1 \times (\varphi_t^{-1}[G_2] \cap \phi_t^{-1}[G_3]) \times F_t).
\end{aligned}$$

Finally we infer that $\int_V q^t(x, w, F_t)\, d\mu \otimes \mathcal{W}(x, w) = \int_V q^T(x, w, F_t)\, d\mu \otimes \mathcal{W}(x, w)$ for every $V \in \mathbb{B}(X) \otimes \mathbb{B}$, proving (B.1).

Now we turn to the proof of Lemma B: Fix $U_1 \in \mathbb{B}(X)$, $U_2 \in \mathbb{B}_t$, $U_3 \in \mathbb{B}_t$, $U_4 \in \mathbb{B}_t$ and define $C = U_1 \times U_2 \times U_3 \times U_4$. Then

$$\begin{aligned} P^*(C \cap B) &= \int_{X\times\mathfrak{C}} I_{U_1}(x) I_{U_2}(w) I_A(\phi_t(w)) q^T(x,w,U_3) q^T(x,w,U_4)\, d\mu \otimes \mathcal{W}(x,w) \\ &= \int_{X\times\mathfrak{C}} I_{U_1}(x) I_{U_2}(w) I_A(\phi_t(w)) q^t(x,w,U_3) q^t(x,w,U_4)\, d\mu \otimes \mathcal{W}(x,w) \end{aligned}$$

by (B.1). In view of Fubini's theorem this equals

$$\begin{aligned} &\int_{\mathfrak{C}} \underbrace{\Big(\int_X I_{U_1}(x) q^t(x,w,U_3) q^t(x,w,U_4)\, d\mu(x)\Big) I_{U_2}(w)}_{\mathbb{B}_t\text{-measurable}} \quad \underbrace{I_A(\phi_t w)}_{\sigma(\phi_t)\text{-measurable}} \quad d\mathcal{W}(w) \\ &\qquad = \underbrace{\int_{X\times\mathfrak{C}} I_{U_1}(x) I_{U_2}(w) q^1(x,w,U_3) q^t(x,w,U_4)\, d\mu \otimes d\mathcal{W}(x,w)}_{P^*(C)}\, P^*(B) \end{aligned}$$

due to Lemma A(2). So we have proven that $P^*(F \cap B) = P^*(F)P^*(B)$ for every $F \in \mathbb{B}(X) \otimes \mathbb{B}_t \otimes \mathbb{B}_t \otimes \mathbb{B}_t = \mathcal{F}_t^*$; but the same is obviously true for every $F \in \mathcal{F}_t^{P^*}$. ■

Lemma C. *The processes Z^i, $i = 1, 2$, defined in* (12.2) *are $(\mathcal{F}_t^{P^*})$-predictable, W^* defined in* (12.3) *is a Q-$(\mathcal{F}_t^{P^*})$-Wiener process, and* (1) *and* (2) *of Theorem* 12.1 *hold. In particular, $(\Omega^*, \mathcal{F}^*, (\mathcal{F}_t^{P^*}), P^*, W^*, Z^i)$ satisfies* (0.1), (0.2) *for $i = 1, 2$ by Theorem* 6.

Proof. First note that, by the definition of the measure P^*,

$$\mathfrak{Law}_{P^*}(x, w, y^i) = \mathfrak{Law}_P(u(0), W_{\mathrm{dec}}, eu - eu(0)). \tag{12.4}$$

So $\mathfrak{Law}_{P^*}(W^*_{\mathrm{dec}}, ex + y^i) = \mathfrak{Law}_P(W_{\mathrm{dec}}, eu)$, $i = 1, 2$, and, by Corollary 11.2,

$$P^*[y^i(t) \in \operatorname{Rng} e] = 1, \quad t \le T, \tag{12.5}$$

Z^i, $i = 1, 2$, are predictable in X and

$$\mathfrak{Law}_{P^*}(Z^i(t_j), W^*_{t_j}(h_k^*) : j \le n, k \le K) = \mathfrak{Law}_P(u(t_j), W_{t_j}(h_k^*) : j \le n, k \le K), \tag{12.6}$$

$i = 1, 2$, for every partition $0 = t_0 < \cdots < t_n \le T$ and every $K \in \mathbb{N}$. Denoting by $B = (W(h_1^*), \ldots, W(h_N^*))$ the continuous N-dimensional $(\mathcal{F}_t^*)$-adapted process we see that $\mathfrak{Law}_{P^*}(B_t - B_s)$ is the N-dimensional centered Gaussian measure with covariance

$$(\langle Q^{1/2} h_i^*, Q^{1/2} h_j^* \rangle_U)_{ij}$$

and, by Lemma B, $\sigma(B_t - B_s)$ is $\mathcal{F}_t^{P^*}$-independent. Thus we conclude that B is an N-dimensional $(\mathcal{F}_t^{P^*})$-Wiener process for every $N \in \mathbb{N}$, and consequently, as

$$\begin{aligned} E_{P^*} \Big\| \sum_{k=m}^n \langle h, h_k^* \rangle_U W^*(h_k^*) \Big\|^2_{C([0,T])} &\le 4T \sum_{k=m}^n \sum_{l=m}^n \langle h, h_k^* \rangle_U \langle Q^{1/2} h_k^*, Q^{1/2} h_l^* \rangle_U \langle h, h_l^* \rangle_U \\ &= 4T \Big\| Q^{1/2} \Big(\sum_{k=m}^n \langle h, h_k^* \rangle_U h_k^* \Big) \Big\|_U^2 \end{aligned}$$

by the Doob inequality, the series (12.3) is convergent and defines a Q-$(\mathcal{F}_t^{P^*})$-Wiener process while (2) in Theorem 12.1 follows from (12.6) by the linearity of W^* and by the fact that the linear span of $(h_k^* : k \in \mathbb{K})$ is dense in U. By definition of P^*, we also have $P^*[y^i(0) = 0] = 1$, $i = 1, 2$, which yields (1) of Theorem 12.1.

Proof of the second part of Theorem 12.1 (in a sequence of lemmas). Suppose that we know that $P^*[Z^1(t) = Z^2(t)] = 1$ for t from some dense subset of $[0,T]$. Then, by the definition (12.2) of Z^i and by (12.5),

$$1 = P^*[y^1 = y^2] = P^*[X \times \mathfrak{C} \times D] = \int\limits_{X\times\mathfrak{C}} q^T_{(x,w)} \otimes q^T_{(x,w)}(D)\, d\mu \otimes \mathcal{W}(x,w),$$

where $D = \{(y^1,y^2) \in \mathfrak{C} \times \mathfrak{C} : y^1 = y^2\}$ is the diagonal of $\mathfrak{C} \times \mathfrak{C}$. So the set

$$M = \{(x,w) : q^T_{(x,w)} \otimes q^T_{(x,w)}(D) = 1\} \in \mathbb{B}(X) \otimes \mathbb{B}$$

is of $\mu\otimes\mathcal{W}$-measure 1 and, by Fubini's theorem, $q^T_{(x,w)}$ must be a Dirac measure for every $(x,w) \in M$. Denoting by $k(x,w) \in \mathfrak{C}$, $(x,w) \in M$, the corresponding mass point we have

$$[k \in B] = [q^T(B) = 1] = [q^t(B) = 1]$$

modulo a $\mu \otimes \mathcal{W}$-zero set for every $F_t \in \mathbb{B}_t$ by (B.1). Hence k, extended by 0 off M, is $(X \times \mathfrak{C}, \mathcal{G}^{\mu\otimes\mathcal{W}}_t) \to (\mathfrak{C}, \mathbb{B}_t)$ measurable for every $t \le T$. The proof of the second part of Theorem 12.1 will now follow from Lemmas D and E below.

LEMMA D. *Write* $t \mapsto \widetilde{R}_t(x,w) = ex + k_t(x,w) \in \mathbb{R}^{\mathbb{N}}$. *Then*

(D.1) $\widetilde{R} : (X \times \mathfrak{C}, \mathcal{G}^{\mu\otimes\mathcal{W}}_t) \to (\mathfrak{C}, \mathbb{B}_t)$ *is measurable for every* $t \le T$,

(D.2) $\mathfrak{Law}_P(eu, W_{\mathrm{dec}}) = \mathfrak{Law}_{\mu\otimes\mathcal{W}}(\widetilde{R}, \widetilde{w})$ *on* $\mathbb{B} \otimes \mathbb{B}$,

(D.3) $\mu \otimes \mathcal{W}\{(x,w) : \pi_0 \widetilde{R}(x,w) = ex\} = 1$,

where $\widetilde{w} : (X,\mathfrak{C}) \to \mathfrak{C} : (x,w) \mapsto w$.

Proof. (D.1) is obviously true so we will show (D.2) and (D.3):

$$\begin{aligned} P^*(B) &= \int\limits_M \delta_{k(x,w)} \otimes \delta_{k(x,w)}(B^{(x,w)})\, d\mu \otimes \mathcal{W}(x,w) \\ &= \mu \otimes \mathcal{W}\{(x,w) : (x,w,k(x,w),k(x,w)) \in B\} \end{aligned}$$

by the definition (12.1) of P^* for every $B \in \mathcal{F}^*$. So (D.2) follows from (12.4). For (D.3), observe that

$$1 = P[\pi_0(eu - eu(0)) = 0] = P^*\{(x,w,y^1,y^2) : y^1(0) = 0\},$$

so the claim follows from the first part of the proof. ■

LEMMA E. *Let* μ *be a Borel probability measure on* X, $(\Omega, \mathcal{F}, P, W, u)$ *a solution of* (0.1) *satisfying* (0.2), *and let* $(x^*_n : n \in \mathbb{N})$ *be a sequence in* X^* *which separates points of* X *such that the processes* $t \mapsto \langle x^*_n, u(t)\rangle$ *have continuous adapted modifications. Let also* $\widetilde{R} : X \times \mathfrak{C} \to \mathfrak{C}$ *be a function satisfying* (D.1)–(D.3) *of Lemma* D. *Then, whenever* $(\Omega^1, \mathcal{F}^1, (\mathcal{F}^1_t), W^1, u^1_0)$ *is a filtered probability space with a* Q-$(\mathcal{F}^1_t)$-*Wiener process* W^1 *and an* $\mathcal{F}^1_0$-*measurable random variable* u^1_0 *with distribution* $\mathfrak{Law}_{P^1}(u^1_0) = \mu$, *the family* $(\Omega^1, \mathcal{F}^1, (\mathcal{F}^1_t), W^1, u^1)$, *where* $u^1(t) = e^{-1}\pi_t \widetilde{R}(u^1_0, W^1_{\mathrm{dec}})$ *is a predictable process, is a strong solution of* (0.1) *satisfying* (0.2) *with the initial condition* u^1_0 *and* $\mathfrak{Law}_{P^1}(u^1, W^1) = \mathfrak{Law}_P(u, W)$.

Proof. First of all, note that $\mathfrak{Law}_{P^1}(u^1_0, W^1_{\mathrm{dec}}) = \mu\otimes\mathcal{W}$ by Lemma A, so if we denote by $\mathcal{H}_t$ the P^1-augmentation of $\sigma(u^1_0, \pi_s W^1_{\mathrm{dec}} : s \le t)$ then $\mathcal{H}_t \subseteq \mathcal{F}^1_t$, the mapping $(u^1_0, W^1_{\mathrm{dec}}) : (\Omega^1, \mathcal{H}_t) \to (X \times \mathfrak{C}, \mathcal{G}^{\mu\otimes\mathcal{W}}_t)$ is clearly measurable and u^1 is $(\mathcal{H}_t)$-predictable. Moreover

$$\mathfrak{Law}_{P^1}(u^1(t_j), W^1_{\mathrm{dec}} : j) = \mathfrak{Law}_{P^1}(e^{-1}\pi_{t_j}\widetilde{R}(u^1_0, W^1_{\mathrm{dec}}), W^1_{\mathrm{dec}} : j)$$
$$= \mathfrak{Law}_{\mu\otimes\mathcal{W}}(e^{-1}\pi_{t_j}\widetilde{R}, \widetilde{w} : j) = \mathfrak{Law}_P(e^{-1}\pi_{t_j}eu, W_{\mathrm{dec}}) = \mathfrak{Law}_P(u(t_j), W_{\mathrm{dec}}),$$

by the assumption (D.2), and consequently

$$\mathfrak{Law}_{P^1}(u^1(t_j), W^1_{t_j}(h^*_k) : j \le n, k \le K) = \mathfrak{Law}_P(u(t_j), W_{t_j}(h^*_k) : j \le n, k \le K)$$

for every partition $0 = t_0 < \cdots < t_n \le T$ and every $K \in \mathbb{N}$. But since $(h^*_k : k \in \mathbb{N})$ spans densely in U we get the above equality with arbitrary h_k's due to linearity of W, W^1. Hence (u^1, W^1) satisfies (0.1), (0.2). We have

$$\mathfrak{Law}_P(eu(t)) = \mathfrak{Law}_{\mu\otimes\mathcal{W}}(\pi_t\widetilde{R}), \quad t \le T,$$

by (D.2), so using Corollary 11.2, we see that $\mu\otimes\mathcal{W}[\pi_t\widetilde{R} \in \operatorname{Rng} e] = 1$ and

$$P^1[u^1(0) = u^1_0] = P^1[e^{-1}\pi_0\widetilde{R}(u^1_0, W^1_{\mathrm{dec}}) = u^1_0] = \mu\otimes\mathcal{W}\{(x,w) : e^{-1}\pi_0\widetilde{R}(x,w) = x\}$$
$$= \mu\otimes\mathcal{W}\{(x,w) : \pi_0\widetilde{R}(x,w) = ex\} = 1$$

by (D.3). ∎

13. Proof of Theorem 1

The idea is, again, to construct a function $\widetilde{R}$ satisfying the assumptions of Lemma E but firstly we will recall a fairly well known fact on representations of "suitably" measurable functions. The proof can be found in [DM, 12-I-18].

Proposition 13.1. *Let $(\Omega, \mathcal{F}, \nu)$ be a measure space, $(Y, \mathcal{Y})$ a measurable space, $\mathcal{A}$ a sub-σ-algebra of $\mathcal{F}$, Z a Polish space, and $f : \Omega \to Y$ an arbitrary function. Denote by $\sigma(f)$ the σ-algebra generated by f and $\mathcal{A}^\nu = \mathcal{A} \vee \{N \in \mathcal{F} : \nu(N) = 0\}$ the ν-augmentation of $\mathcal{A}$ in $\mathcal{F}$.*

(1) *If $g : \Omega \to Z$ is a $\sigma(f)$-measurable mapping then there exists a measurable mapping $h : (Y, \mathcal{Y}) \to Z$ such that $g = hf$.*
(2) *If $g : \Omega \to Z$ is $\mathcal{A}^\nu$-measurable then there exists an $\mathcal{A}$-measurable function h such that $g = h$ ν-almost everywhere.*

Theorem 13.2. *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), P, W, u)$ be a strong solution of* (0.1), (0.2) *such that $x^*_n \in X^*$, $n = 1, 2, \ldots$, separate points of X and the processes $t \mapsto \langle x^*_n, u(t)\rangle$, $n \in \mathbb{N}$, have continuous modifications. Then there exists a function $\widetilde{R}$ satisfying* (D.1)–(D.3) *of Lemma* D, *and consequently the conclusions of Lemma* E *hold.*

Proof. Denote by $\mathcal{H}_t$ the P-augmentation of $\sigma(u(0), \pi_s W_{\mathrm{dec}} : s \le t)$. Then the mapping $eu : (\Omega, \mathcal{H}_t) \to (\mathfrak{C}, \mathbb{B}_t)$ is measurable for every $t \le T$ by assumption. Consequently, there exists a measurable mapping $\widetilde{R} : (X \times \mathfrak{C}, \mathbb{B}(X) \otimes \mathbb{B}) \to (\mathfrak{C}, \mathbb{B})$ as well as mappings $r_t : (X \times \mathfrak{C}, \mathbb{B}(X) \otimes \mathbb{B}_t) \to \mathbb{R}^{\mathbb{N}}$ such that

$$P[\widetilde{R}(u(0), W_{\mathrm{dec}}) = eu] = 1$$

and $P[r_t(u(0), W_{\mathrm{dec}}) = \pi_t eu] = 1$, $t \le T$, by Proposition 13.1. But then

$$\mu\otimes\mathcal{W}[r_t = \pi_t\widetilde{R}] = P[r_t(u(0), W_{\mathrm{dec}}) = \pi_t\widetilde{R}(u(0), W_{\mathrm{dec}})] = 1, \quad t \le T,$$

so $\widetilde{R} : (X \times \mathfrak{C}, \mathcal{G}^{\mu\otimes\mathcal{W}}_t) \to (\mathfrak{C}, \mathbb{B}_t)$ for every $t \le T$ by Lemma A. ∎

Proof of Theorem 1. The strong existence follows immediately from Lemma E so suppose that joint uniqueness in law holds for (0.1) and that we have a solution v of (0.1) on some filtered probability space $(\Omega', \mathcal{F}', (\mathcal{F}'_t), P', W')$. Then

$$P'[v(t) = e^{-1}\pi_t \widetilde{R}(v(0), W'_{\mathrm{dec}})] = P[u(t) = e^{-1}\pi_t \widetilde{R}(u(0), W_{\mathrm{dec}})] = 1, \quad t \le T,$$

by the joint uniqueness in law and Theorem 13.2. Hence we see that (0.1) is pathwise unique. ∎

14. Proofs of Theorems 8, 9 and 10

We use the notation of Section 11.

Proof of Theorem 8. (1) implies (2) and (3) by Theorem 2, while (3) implies (1) by Theorem 1. Suppose that (2) holds. Then the assumptions of Theorem 12.1 are satisfied for the original solution $(\Omega, \mathcal{F}, (\mathcal{F}_t), P, W, u)$ so if $(\Omega', \mathcal{F}', (\mathcal{F}'_t), P', W', u')$ is a solution then so is $(\Omega', \mathcal{F}', (\mathcal{F}'_t), P', W', v')$, where $v'(t) = e^{-1}\pi_t \widetilde{R}(u'(0), W'_{\mathrm{dec}})$. Consequently,

$$P'[u'(t) = e^{-1}\pi_t \widetilde{R}(u'(0), W'_{\mathrm{dec}})] = P'[v'(t) = e^{-1}\pi_t \widetilde{R}(v'(0), W'_{\mathrm{dec}})]$$

for every $t \le T$ as (u', W') and (v', W') have the same law by joint uniqueness in law. The latter probability is 1 because $P'[u'(0) = v'(0)] = 1$ and the proof is complete. This yields pathwise uniqueness for (0.1). ∎

Proof of Theorem 9. (1) implies (2) by Theorem 12.1. Now suppose that (2) holds. Then (u, W)-pathwise uniqueness holds if and only if (u', W')-pathwise uniqueness holds, so let $(\Omega^1, \mathcal{F}^1, (\mathcal{F}^1_t), P^1, W^1, u^1)$ be a solution of (0.1) such that $\mathfrak{Law}_{P^1}(u^1, W^1) = \mathfrak{Law}_{P'}(u', W')$. Then $(\Omega^1, \mathcal{F}^1, (\mathcal{F}^1_t), P^1, W^1, v^1)$ is a solution of (0.1) and $\mathfrak{Law}_{P^1}(v^1, W^1) = \mathfrak{Law}_{P^1}(u^1, W^1)$ with

$$v^1(t) = e^{-1}\pi_t \widetilde{R}(u^1(0), W^1_{\mathrm{dec}})$$

by Theorem 13.2. Moreover $P^1[u^1(0) = v^1(0)] = 1$ and thus

$$P^1[u^1(t) = e^{-1}\pi_t \widetilde{R}(u^1(0), W^1_{\mathrm{dec}})] = P^1[v^1(t) = e^{-1}\pi_t \widetilde{R}(v^1(0), W^1_{\mathrm{dec}})] = 1$$

for every $t \le T$, proving (1). ∎

Proof of Theorem 10. Suppose that (1) holds. Then the assumptions of Theorem 12.1 are satisfied for the original solution $(\Omega, \mathcal{F}, (\mathcal{F}_t), P, W, u)$, so consider another solution $(\Omega', \mathcal{F}', (\mathcal{F}'_t), P', W', v')$ such that $\mathfrak{Law}_P(u) = \mathfrak{Law}_{P'}(v')$. Then, by Theorem 12.1, $(\Omega', \mathcal{F}', (\mathcal{F}'_t), P', W', u')$ is again a solution of (0.1) with $\mathfrak{Law}_P(u, W) = \mathfrak{Law}_{P'}(u', W')$, where

$$u'(t) = e^{-1}\pi_t \widetilde{R}(v'(0), W'), \quad t \le T,$$

is $(u'(0), W')_{P'}$-adapted. But $P'[u'(t) = v'(t)] = 1$, $t \le T$, by u-pathwise uniqueness as $P'[u'(0) = v'(0)] = 1$. This implies joint u-uniqueness in law for (0.1), and (2) and (3) hold.

If (2) holds then the assumptions of Theorem 12.1 are satisfied for the original solution $(\Omega', \mathcal{F}', (\mathcal{F}'_t), P', W', u')$ and we will show that it has the desired properties. Indeed, $(\Omega', \mathcal{F}', (\mathcal{F}'_t), P', W', v')$ is a $(u'(0), W')_{P'}$-adapted solution, where $v'(t) = e^{-1}\pi_t \widetilde{R}(u'(0), W')$, $t \le T$. Moreover, by (u', W')-pathwise uniqueness, $P'[u'(t) = v'(t)]$ for every $t \le T$.

To show that (3) implies (1) consider a solution $(\Omega^1,\mathcal{F}^1,(\mathcal{F}^1_t),P^1,W^1,u^1)$ such that $\mathfrak{Law}_{P'}(u')=\mathfrak{Law}_{P^1}(u^1)$. Then, by Theorem 13.2, $(\Omega^1,\mathcal{F}^1,(\mathcal{F}^1_t),P^1,W^1,v^1)$ is also a solution with $\mathfrak{Law}_{P'}(u')=\mathfrak{Law}_{P^1}(v^1)$, where

$$v^1(t)=e^{-1}\pi_t\widetilde{R}(u^1(0),W^1),\qquad t\le T.$$

But we know that

$$P^1[u^1(t)=e^{-1}\pi_t\widetilde{R}(u^1(0),W^1_{\mathrm{dec}})]=P^1[v^1(t)=e^{-1}\pi_t\widetilde{R}(v^1(0),W^1_{\mathrm{dec}})]=1$$

for every $t\le T$ by joint u-uniqueness in law and the fact that $P^1[u^1(0)=v^1(0)]=1$. This implies u-pathwise uniqueness for (0.1). ■

Notation used

$\mathbb{B}(X)$	Borel σ-algebra of X
X^*	dual space to X
L^p	Lebesgue space of p-integrable functions
$E[f/\mathcal{A}]$	conditional expectation
$\sigma\mathcal{S}$	σ-hull over $\mathcal{S}$
$\sigma(f_\alpha:\alpha\in A)$	σ-hull over $(f_\alpha:\alpha\in A)$
$\mathfrak{Law}_\mu(f)$	distribution of f with respect to μ
$\langle M\rangle$, resp. $\langle M,N\rangle$	variation, resp. cross-variation process [RY, Section IV.1]
U_0	Definition 2.1
$\mathcal{N}(x,Q)$	Gaussian probability with mean x and covariance Q
$L(U,X)$	linear bounded operators from U to X
$L_2(U,X)$	radonifying operators from U to X, Definition 2.3
$C([0,T],Z)$	continuous functions from $[0,T]$ to Z
$\mathcal{P}_T$	σ-algebra of progressively measurable sets
e, e^{-1}	11.1
$\mathfrak{C}$	Section 11
$\mathbb{B}$, $\mathbb{B}_t$	Section 11
π_t	Section 11
φ_t	Section 11
ϕ_t	Section 11
$(\Omega^*,\mathcal{F}^*,(\mathcal{F}^*_t))$	Section 11
$\mathcal{F}^\nu_t$	Section 11
W_{dec}	Section 11
$\mathcal{W}$	Section 11
$\mathcal{G}^{\mu\otimes\mathcal{W}}_t$	Section 11

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