0. Introduction

The aim of the work presented here is to connect two fields of functional analysis, on one hand the theory of sequence spaces and on the other hand the nonlinear theory of algebras of generalized functions, with the emphasis on the description of the latter. Associative differential algebras of generalized functions, containing the (embedded) delta distribution, with the ordinary product of continuous functions do not exist, as was proved by Schwartz [73]. But with the ordinary multiplication of smooth functions, such algebras do exist. One of the first and today most widely studied and used constructions has been introduced by Colombeau [8]. Nowadays, the theory of these so-called Colombeau type algebras is well-established and it is affirmed through many applications especially in nonlinear problems with strong singularities. Here we refer to the books [5, 8, 9, 59, 60, 63] and to the numerous papers given in the references, while we apologize for all undue omissions. We also want to point out the progress made in the direction of PDE and differential geometry with applications in general relativity done by the DIANA group [24–27, 33, 34, 37, 38, 43, 45–47].

On the other hand, sequence spaces of various type are a basic notion in investigations of various branches of functional analysis [48, 49, 50, 51, 52, 53]. In this paper we show that Colombeau type algebras can be reconsidered as a class of sequence space algebras. We hope that our investigations in the field of generalized function algebras can serve as a motivation for those who are more interested in the functional analysis of sequence spaces.

At the time when we started our work, the results of [24, 25, 26, 27] related to the topology, and in general to functional analysis in the framework of Colombeau type generalized function algebras, were not known. Even now (five years later) they are not known properly. We would like to point out that this work significantly extends the well known theory relating to sharp topology. We will not give details about this work but advice the reader to consult the cited papers.

The present paper extends our previous publications [13–15], where we elaborated separately on the general construction, on the issue of embeddings of distributions, ultradistributions and generalized hyperfunctions, and on functoriality and the different notions of association which we cast into a unified scheme, with new examples and developments relating to Maddox' sequence spaces, and sheaf theory.

Colombeau constructed his well-known algebras by algebraic methods. No topology appeared in his construction. As we already mentioned, the different topologies and convergence structures defined on $\mathcal{G}$ appeared afterwards. Our first task in this paper is to give a purely topological description of Colombeau type spaces. Let us mention that these types of sequence spaces appear frequently in describing the structure of (expanded) periodic distributions, ultradistributions and hyperfunctions. Our formulation of Colombeau-like algebras should convince by its conceptual simplicity: In fact, all these classes of algebras are simply determined by a (locally convex) space E, and a sequence of weights $r : \mathbb{N} \to \mathbb{R}_+$ (or sequence of sequences) which serves to construct an ultrametric on the sequence space $E^{\mathbb{N}}$. As a first, motivating example, note that $r = 1/\log$ just gives Colombeau's algebra: Indeed, the ring of Colombeau generalized numbers is $\overline{\mathbb{C}} \equiv \{x \in \mathbb{C}^{\mathbb{N}} : \limsup |x_n|^{1/\log n} < \infty\}/\{x \in \mathbb{C}^{\mathbb{N}} : \limsup |x_n|^{1/\log n} = 0\}$ and idem for the space $\mathcal{G}(\Omega)$ (see Subsection 1.1.2 for details).

The sequence $r = (r_n)_n$ is assumed to be decreasing to zero. This implies that sequence spaces under consideration ($\subset E^{\mathbb{N}}$) contain as a subspace $E \sim \operatorname{diag} E^{\mathbb{N}}$ and that they induce the discrete topology on E. This is well-known for the sharp topology for Colombeau type algebras. But our analysis implies that if one has a Colombeau type algebra containing the Dirac delta distribution δ as an embedded Colombeau generalized function, then the topology induced on the basic space must be discrete. This result is analogous to Schwartz' "impossibility result" concerning the product of distributions (cf. Remark 43 and Subsection 3.1). It shows, through topology, the importance and the validity of the Colombeau idea for the construction of Colombeau type algebras.

An important and in a sense a leading motivation for the analysis of the class of sequence spaces is the fact that distribution, ultradistribution and hyperfunction type spaces can be embedded in corresponding sequence spaces of this class. An important part of the paper is devoted to embeddings since this justifies the joint interest for sequence spaces and for generalized function algebras. The embeddings of Schwartz' spaces into the Colombeau algebra $\mathcal{G}$ are very well known, but for ultradistribution and hyperfunction type spaces new results are given. The problem of multiplication of regular enough functions (smooth, ultradifferentiable or quasianalytic), embedded into corresponding algebras, is also analysed.

To complete the analysis of the relation between this approach and previous results, we introduce in Section 4 an important generalization which is to consider sequences of sequences of weights. This way, we can describe other Colombeau type algebras, not based on polynomial scales, as for example asymptotic algebras [16] and Egorov type algebras.

This justifies turning then, in Section 5, to nowadays classical questions like functorial aspects of Colombeau type algebras [70, 71], in order to apply the following scheme in standard applications: if a classical differential problem for regular data has a unique solution such that the map associating the solution to the initial data satisfies convenient growth conditions (with respect to the chosen scale of weights), then this same problem can be transferred to corresponding sequence spaces, where it also allows for a unique solution. That way, differential problems with singular data can be solved *ad hoc* in such spaces.

Finally, it occurs frequently that exact solutions are not required, and in spaces of generalized functions the notion of weak solutions has often been used, in the sense of different types of association. These concepts can be nicely described in our sequential approach, which is done in Section 6. Indeed, we give a generalized and unified scheme of a large number of tools of this kind, which can be found in various places in the existing literature.

1. The basic construction

Let us now present the construction in detail for the simplest possible case. The situation here is included in the more general constructions of the next section, but the underlying principle and the proofs will be more evident here. This is also the setting pertaining to the definition of rings of generalized constants.

We follow the convention that $0 \in \mathbb{N}$, $\mathbb{R}_+ = [0, \infty)$ and denote by $\mathbb{N}^*$, $\mathbb{R}^*_{(+)}$, $\mathbb{C}^*$ the respective sets without 0.

1.1. Seminormed algebras and rings of generalized numbers. Consider a sequence $r \in \mathbb{R}_+^{\mathbb{N}}$ decreasing to zero, and a seminormed algebra (E, p) over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, such that $\exists C > 0, \forall a, b \in E : p(ab) \leq Cp(a)p(b)$.

1.1.1. *Ultranorms and associated ultrametric sequence spaces.* Now define (1) for $f \in E^{\mathbb{N}}$,

$$\|f\|_{p,r} := \limsup_{n\to\infty} p(f_n)^{r_n}.$$

This is well defined for any $f \in E^{\mathbb{N}}$, with values in $\overline{\mathbb{R}}_+ \equiv \mathbb{R}_+ \cup \{\infty\}$. In the particular case $(E, p) = (\mathbb{K}, |\cdot|)$, we will sometimes write $|\cdot|_r$ for $\|\cdot\|_{|\cdot|,r}$.

LEMMA 1. *For any $f, g \in E^{\mathbb{N}}$ and $\lambda \in \mathbb{K}^*$, $\|\lambda f\|_{p,r} = \|f\|_{p,r}$ and*

$$\|f + g\|_{p,r} \leq \max(\|f\|_{p,r}, \|g\|_{p,r}), \|f \cdot g\|_{p,r} \leq \|f\|_{p,r}\|g\|_{p,r}. \tag{1.1}$$

If there is $M > 0$ such that $M \geq p(f_n) \geq 1/M$ for n large enough, in particular if f is a constant sequence (of elements with nonzero seminorm), then $\|f\|_{p,r} = 1$.

We will sometimes summarize these properties by referring to $\|\cdot\|_{p,r}$ as an ultra(pseudo)(semi)norm (which is not a seminorm, by lack of $\mathbb{C}$-homogeneity).

The last statement also implies that if a sequence $(f^m)_{m\in\mathbb{N}}$ of elements $f^m \in E^{\mathbb{N}}$ converges (componentwise) to $f \in E^{\mathbb{N}}$, then $\|f^m - f\|_{p,r}$ does *not* in general converge to 0, even if $f^m \to f$ uniformly in E. For example, if f, f^m are elements of E, embedded as constant sequences in $E^{\mathbb{N}}$, such that $p(f - f^m) \neq 0$, then $\|f^m - f\|_{p,r} = 1$ for all m.

Proof. The property $\lim r_n = 0$ entails $\forall M > 0$: $\lim M^{r_n} = 1$ and thus the last statement. With $p(\lambda f_n) \leq |\lambda| p(f_n)$, this gives $\|\lambda f\|_{p,r} = \|f\|_{p,r}$. Together with $p(f_n g_n) \leq Cp(f_n)p(g_n)$, we obtain the inequality for the product. Finally, using $p(f_n + g_n) \leq p(f_n) + p(g_n) \leq 2\max\{p(f_n), p(g_n)\}$ this also gives the ultrametric triangular inequality. ■

PROPOSITION-DEFINITION 2. *With the above definitions, consider the sets*

$$\mathcal{F}_{p,r} = \{f \in E^{\mathbb{N}} \mid \|f\|_{p,r} < \infty\} \quad \textit{and} \quad \mathcal{K}_{p,r} = \{f \in E^{\mathbb{N}} \mid \|f\|_{p,r} = 0\}.$$

(1) For $r_n = 0$, we use in this formula the (unusual) convention $0^0 = 0$.

(i) *$\mathcal{F}_{p,r}$ is a subalgebra of $E^{\mathbb{N}}$, and $\mathcal{K}_{p,r}$ is an ideal of $\mathcal{F}_{p,r}$; thus*

$$\mathcal{G}_{p,r} = \mathcal{F}_{p,r}/\mathcal{K}_{p,r}$$

is an algebra. Instead of $\mathcal{F}_{p,r}$, $\mathcal{K}_{p,r}$ and $\mathcal{G}_{p,r}$, we also use the notations $\mathcal{F}_r(E,p)$, $\mathcal{K}_r(E,p)$ and especially $\mathcal{G}_r(E,p)$.

(ii) *The function*

$$d_{p,r} : \mathcal{F}_{p,r} \times \mathcal{F}_{p,r} \to \mathbb{R}_+, \qquad (f,g) \mapsto \|f-g\|_{p,r},$$

is an ultrapseudometric on $\mathcal{F}_{p,r}$, inducing on $\mathcal{F}_{p,r}$ the structure of a topological ring such that the intersection of neighborhoods of zero equals $\mathcal{K}_{p,r}$. Multiplication by scalars $\lambda \in \mathbb{K}$ is not *continuous, because $\|\lambda f\|_{p,r} = \|f\|_{p,r}$ does not go to zero as $\lambda \to 0$ in $\mathbb{K}$. Thus, $\mathcal{F}_{p,r}$ is not a topological $\mathbb{K}$-algebra, but it* is *a topological algebra over the ring $\mathcal{F}_{|\cdot|,r} \subset \mathbb{K}^{\mathbb{N}}$ endowed with the topology given by $|\cdot|_r = \|\cdot\|_{|\cdot|,r}$.*

(iii) *$\mathcal{G}_{p,r} = \mathcal{F}_{p,r}/\mathcal{K}_{p,r}$ is a Hausdorff topological ring and topological algebra over the generalized numbers (2) $\mathbb{C}_r = \mathcal{G}_{|\cdot|,r}$, the quotient topology being the same as the topology induced by the ultrametric*

$$\widetilde{d}_{p,r} : \mathcal{G}_{p,r} \times \mathcal{G}_{p,r} \to \mathbb{R}_+, \qquad ([f],[g]) \mapsto d_{p,r}(f,g),$$

where $[f],[g] \in \mathcal{G}_{p,r}$ are the classes of $f,g \in \mathcal{F}_{p,r}$.

Proof. (i) This is an immediate consequence of the preceding lemma.

(ii) Well-definedness (values $< \infty$), reflexivity and symmetry of $d_{p,r}(\cdot,\cdot)$ are obvious. The ultrametric property $\forall f,g,h \in \mathcal{F}_{p,r} : d(f,g) = \max(d(f,h), d(h,g))$ follows by applying the lemma to $x = f-h$, $y = h-g$ in place of f,g. Continuity of addition and multiplication is also a consequence of (1.1). Thus, $d_{p,r}$ makes $\mathcal{F}_{p,r}$ a topological ring.

(iii) Let us first show that $\widetilde{d}_{p,r}$ is well defined, i.e. $d_{p,r}(f+j,g) = d_{p,r}(f,g)$ for $j \in \mathcal{K}_{p,r}$. This is equivalent to $\|x+j\| = \|x\|$ with $x = f-g$, which is again an immediate consequence of (1.1) and the definition of $\mathcal{K}_{p,r}$. Thus, $\widetilde{d}_{p,r}$ does not depend on the choice of representatives.

To show that the quotient topology is the same as the one induced by the ultrametric $\widetilde{d}_{p,r}$, it is sufficient to consider the base of neighborhoods of 0. The assertion follows from the fact that $\widetilde{d}_{p,r}(0,F) = 0 \Leftrightarrow F \in \mathcal{K}_{p,r}$. Since $\widetilde{d}_{p,r}(0, F\cdot G) = \widetilde{d}_{p,r}(0,F)\widetilde{d}_{p,r}(0,G)$, $\mathcal{G}_{p,r}$ is a topological ring, and as a metric space, it is Hausdorff. ■

Summarizing, such Colombeau type spaces are nothing else than the usual construction of associated Hausdorff spaces for the topological subspaces of $E^{\mathbb{N}}$ on which the ultrapseudometric $d_{p,r}$ is defined. This will remain true for the more involved constructions given in the following subsections.

It is also immediate to see that in the definition of the space $\mathcal{F}_{p,r}$ (resp. $\mathcal{K}_{p,r}$), one could "simplify" lim sup to sup (resp. lim). This is usually done in the theory of sequence spaces (see Subsection 1.1.5). We prefer, however, to insist on the ultrametric structure, and therefore express both spaces using always the same ultra-seminorm $\|\cdot\|_{p,r}$.

(2) See also the next subsection 1.1.2.

REMARK 3 (on notation). The notations $\mathcal{F}_{p,r}$, $\mathcal{K}_{p,r}$, $\mathcal{G}_{p,r}$ introduced in our previous papers [13–15] are handy in proofs; however, the notation $\mathcal{G}_r(E,p)$ reflects better the functorial character of the construction (see also Section 5).

1.1.2. *Colombeau generalized numbers.* The setting considered here is used to define rings of generalized numbers. For this, E is the underlying field $\mathbb{R}$ or $\mathbb{C}$, and $p = |\cdot|$ the absolute value. The resulting factor algebra $\mathcal{G}_{|\cdot|,r}$, with topology given by $|\cdot|_r = \|\!|\cdot|\!\|_{|\cdot|,r}$, will be denoted by $\mathbb{R}_r$ or $\mathbb{C}_r$. As already explained in the introduction, for $r = 1/\log$, we get the ring $\overline{\mathbb{C}}$ of Colombeau numbers. More precisely, let

$$\forall n \in \mathbb{N}+2: \quad r_n = \frac{1}{\log n}. \tag{1.2}$$

This gives back Colombeau's algebras of elements with polynomial growth modulo elements of more than polynomial decrease, because

$$\begin{aligned}\limsup_{n\to\infty} |x_n|^{1/\log n} < \infty \ &\Leftrightarrow\ \exists C \in \mathbb{R}_+ : \limsup_{n\to\infty} |x_n|^{1/\log n} = C\\ &\Leftrightarrow\ \exists B,\, \exists n_0,\, \forall n > n_0 : |x_n| \le B^{\log n} = n^{\log B}\\ &\Leftrightarrow\ \exists \gamma \in \mathbb{R} : |x_n| = o(n^\gamma).\end{aligned}$$

On the other hand, $\limsup = 0$ (for the ideal) corresponds to taking $C = 0$ and thus $\forall B > 0$ and $\forall\gamma$ in the last lines.

1.1.3. *Generalized Sobolev algebras.* Another interesting application of this rather simple setting can be obtained by considering Sobolev spaces $E = W^{s,p}(\Omega)$, $s \in \mathbb{N}$, $p \in [1,\infty]$, which are Hilbert spaces for the norm $p_{s,p} = \|\cdot\|_{s,p} = \sum_{|\alpha|\le s} \|\partial^\alpha \cdot\|_{L^p}$. (Elements of this space are distributions with all derivatives of order $|\alpha| \le s$ in $L^p(\Omega)$.)

In order to have an algebra, we can take any $s \in \mathbb{N}$ and $p = \infty$. Then we can apply the construction given previously, with the norm $p = \|\cdot\|_{s,\infty}$ The corresponding Colombeau type algebra is defined by $\mathcal{G}_{W^{s,\infty}} \equiv \mathcal{F}/\mathcal{K}$ where, according to the general definition,

$$\begin{aligned}\mathcal{F} &= \{u \in (W^{s,\infty}(\Omega))^{\mathbb{N}} \mid \limsup_{n\to\infty} \|u_n\|_{s,\infty}^{1/\log n} < \infty\},\\ \mathcal{K} &= \{u \in (W^{s,\infty}(\Omega))^{\mathbb{N}} \mid \limsup_{n\to\infty} \|u_n\|_{s,\infty}^{1/\log n} = 0\}.\end{aligned}$$

Note that also for $n \le 3$, $W^{2,2}(\mathbb{R}^n)$ is an algebra, since we have an inclusion $W^{2,2}(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n)$ (which is continuous). Thus, with

$$\mathcal{F}_{\|\cdot\|_{2,2},r} = \{f \in W^{2,2}(\mathbb{R}^n)^{\mathbb{N}} \mid \limsup_{n\to\infty} \|u_n\|_{2,2}^{r_n} < \infty\}$$

and

$$\mathcal{K}_{\|\cdot\|_{2,2},r} = \{f \in W^{2,2}(\mathbb{R}^n)^{\mathbb{N}} \mid \limsup_{n\to\infty} \|u_n\|_{2,2}^{r_n} = 0\}$$

we obtain the Colombeau algebra $\mathcal{G}_{W^{2,2}(\mathbb{R}^n)}$ (for $r_n \sim 1/\log n$).

By use of the Sobolev lemma, we can construct various Sobolev type algebras [60, 58]. We refer to [79], for example, for an analysis of different domains $\Omega \subset \mathbb{R}^n$ for which Sobolev type lemmas hold for $W^{s,p}(\Omega)$, $s \in \mathbb{N}$, $p \in [1,\infty]$, and that the corresponding space $\mathcal{F}_{\|\cdot\|_{s,p},r}(\Omega)$ can again lead to Sobolev type algebras of generalized functions.

1.1.4. *Comparison results for sequences of weights.* A question arising naturally at this point is whether equivalent sequences of weights (in the classical asymptotic sense) will give rise to identical factor algebras. The answer is affirmative, and we can state the result in the following precise form:

PROPOSITION 4 (equivalent scales). *Let $r = (r_n)_n, s = (s_n)_n$ be two real sequences decreasing to zero. Then*

$$\lim_{n\to\infty} \frac{s_n}{r_n} = C > 0 \ \Rightarrow\ \forall x \in E^{\mathbb{N}} : \|x\|_{p,s} = (\|x\|_{p,r})^C.$$

Whenever this holds, it follows as an immediate consequence that $\mathcal{F}_{p,s} = \mathcal{F}_{p,r}$, $\mathcal{K}_{p,s} = \mathcal{K}_{p,r}$ and therefore also $\mathcal{G}_{p,s} = \mathcal{G}_{p,r}$.

This proposition is a direct consequence of the following

LEMMA 5. *Assume that $r = (r_n)_n, s = (s_n)_n \in \mathbb{R}_+^{\mathbb{N}}$ decrease to zero and satisfy $0 < \liminf_{n\to\infty} s_n/r_n \le \limsup_{n\to\infty} s_n/r_n < \infty$. Then*

$$\forall x \in E^{\mathbb{N}} : \quad \|x\|_{p,s} \in [(\|x\|_{p,r})^{\liminf s_n/r_n}, (\|x\|_{p,r})^{\limsup s_n/r_n}],$$

where the interval has reversed bounds if $\|x\|_{p,r} < 1$.

Proof. Let us first prove the inequality $\|x\|_{p,s} \le (\|x\|_{p,r})^C$ for $\|x\|_{p,r} \ge 1$, where $C = \limsup_{n\to\infty} s_n/r_n$. We have

$$\|x\|_{p,s} = \limsup_{n\to\infty} p(x_n)^{s_n} = \limsup_{n\to\infty} e^{s_n \log p(x_n)} = e^{\limsup s_n \log p(x_n)}.$$

Let us now write $s_n = c_n r_n$, so that $\limsup_{n\to\infty} c_n = C > 0$. For $\log p(x_n) \ge 0$,

$$\limsup_{n\to\infty} s_n \log p(x_n) = \limsup_{n\to\infty} c_n r_n \log p(x_n) \le C \limsup_{n\to\infty} r_n \log p(x_n). \qquad (*)$$

Thus, for $\|x\|_{p,r} \ge 1$,

$$\|x\|_{p,s} \le e^{C \limsup r_n \log p(x_n)} = (\|x\|_{p,r})^C = (\|x\|_{p,r})^{\limsup s_n/r_n}.$$

The other bound of the interval in the lemma is obtained by exchanging r and s. Indeed, this yields

$$\|x\|_{p,r} \le (\|x\|_{p,s})^{\limsup r_n/s_n} \qquad (\text{for } \|x\|_{p,s} \ge 1),$$

and taking this inequality to the power $1/\limsup r_n/s_n = \liminf s_n/r_n$ yields

$$\|x\|_{p,s} \ge (\|x\|_{p,r})^{\liminf s_n/r_n} \qquad (\text{for } \|x\|_{p,s} \ge 1).$$

For $\|x\|_{p,r} < 1$, i.e. $\log p(x_n) < 0$, the direction of the inequality $(*)$ is preserved when $\limsup c_n$ is replaced by $\liminf c_n$. This is most easily checked by first reasoning on the absolute value, $|\limsup(\dots)| = \liminf |\dots|$, and then changing the direction of the inequality, when going to the real negative values. Thus we have instead

$$\|x\|_{p,s} \le (\|x\|_{p,r})^{\liminf s_n/r_n} \qquad (\text{for } \|x\|_{p,r} < 1).$$

and the converse for $\limsup s_n/r_n$, i.e. the claimed lemma. ■

COROLLARY 6. *The previous inequality for the semi-ultranorm in the case of finite upper limit of s/r also implies inclusion relations for the spaces of moderate nets, and the converse inclusion for the ideals, whenever one of the sequences of weights is dominated by the other one*:

$$r = O(s) \ \Rightarrow\ \mathcal{F}_{p,s} \subset \mathcal{F}_{p,r} \ \&\ \mathcal{K}_{p,r} \subset \mathcal{K}_{p,s}.$$

These relations will be used in Section 4, where algebras defined by a whole family of sequences of weights will be considered.

It is also clear that when $\limsup s/r = \infty$ or $\liminf s/r = 0$, we cannot have a nontrivial relation between $\|\!| \cdot |\!\|_{p,r}$ and $\|\!| \cdot |\!\|_{p,s}$ of a quantitative type similar to what precedes.

1.1.5. *Relation to Maddox' sequence spaces.* The spaces $\mathcal{F}_{|\cdot|,r}$ and $\mathcal{K}_{|\cdot|,r}$ defined above are identical to Maddox' sequence spaces $\ell_\infty(r)$ and $c_0(r)$,

$$c_0(r) = \bigcap_{k\in\mathbb{N}} \{x \in \mathbb{C}^{\mathbb{N}} \mid \lim_{n\to\infty} |x_n| k^{1/r_n} = 0\} \quad (= \mathcal{K}_{|\cdot|,r}),$$
$$\ell_\infty(r) = \bigcup_{k\in\mathbb{N}} \{x \in \mathbb{C}^{\mathbb{N}} \mid \sup_{n\in\mathbb{N}} |x_n| k^{-1/r_n} < \infty\} \quad (= \mathcal{F}_{|\cdot|,r}),$$

introduced by Nakano [57], Simons [74] and studied extensively by Maddox and his students [48]–[53]. Indeed,

$$\begin{aligned}\exists k \in \mathbb{N} : \sup_{n\in\mathbb{N}} |x_n| k^{-1/r_n} < \infty \;&\Leftrightarrow\; \exists k \in \mathbb{N} : |x_n| = O(k^{1/r_n})\\ &\Leftrightarrow\; \exists k : \limsup_{n\to\infty} |x_n|^{r_n} \le k \;\Leftrightarrow\; \|\!| x |\!\|_r < \infty,\end{aligned}$$

$$\begin{aligned}\forall k : \lim_{n\to\infty} |x_n| k^{1/r_n} = 0 \;&\Leftrightarrow\; \forall\varepsilon > 0 : |x_n| = o(\varepsilon^{1/r_n})\\ &\Leftrightarrow\; \forall\varepsilon > 0,\, n > n_0 : |x_n|^{r_n} < \varepsilon \;\Leftrightarrow\; \|\!| x |\!\|_r = 0.\end{aligned}$$

In particular, these two types of sequence spaces belong to the well-known classes of echelon and coechelon spaces, for $c_0(r)$ and $\ell_\infty(r)$ respectively [30].

The same characterization can be used for generalized Sobolev spaces as defined in Subsection 1.1.3.

In our case, we shall always require $\lim r_n = 0$ (see also Remark 43). By [23, p. 111] and the fact that for any k there is $\rho > 0$ such that $\sum_{n\in\mathbb{N}} (k/\rho)^{1/r_n} < \infty$, we see that both $\mathcal{F}_{p,r}$ and $\mathcal{K}_{p,r}$ constructed in Subsection 1.1.1 are Montel and Schwartz spaces.

On the other hand, this implies that we never have AD spaces, i.e. the subset of finite sequences will never be dense in $\mathcal{F}_{p,r}$ (but will always be in $\mathcal{K}_{p,r}$).

While the cited and other traditional work on sequence spaces is restricted to the case $(\mathbb{C}, |\cdot|)$, our main work applies to factor algebras constructed from more complicated base spaces (E, p). Nevertheless, all spaces that follow can be described as intersection or union of such echelon (resp. coechelon) spaces. The additional properties we require in our construction of Colombeau type algebras will however simplify the situation with respect to the general abstract theory.

1.2. Locally convex vector spaces and algebras

1.2.1. *Definition.* Consider now a topological algebra E over $\mathbb{C}$, with locally convex structure determined by a family $\mathcal{P}$ of seminorms. We shall assume that

$$\forall p \in \mathcal{P},\, \exists \bar{p} \in \mathcal{P},\, \exists C \in \mathbb{R}_+,\, \forall x, y \in E : \quad p(xy) \le C\bar{p}(x)\bar{p}(y),$$

which implies continuity of multiplication. Now let

$$\mathcal{F}_{\mathcal{P},r} = \{f \in E^{\mathbb{N}} \mid \forall p \in \mathcal{P} : \|\!|f|\!\|_{p,r} < \infty\},$$
$$\mathcal{K}_{\mathcal{P},r} = \{f \in E^{\mathbb{N}} \mid \forall p \in \mathcal{P} : \|\!|f|\!\|_{p,r} = 0\}.$$

Proposition 7.

(i) *$\mathcal{F}_{\mathcal{P},r}$ is a (sub-)algebra of $E^{\mathbb{N}}$, and $\mathcal{K}_{\mathcal{P},r}$ is an ideal of $\mathcal{F}_{\mathcal{P},r}$, thus*

$$\mathcal{G}_{\mathcal{P},r} = \mathcal{F}_{\mathcal{P},r}/\mathcal{K}_{\mathcal{P},r}$$

is an algebra. As before, we also use the notation $\mathcal{G}_r(E,\mathcal{P})$ instead of $\mathcal{G}_{\mathcal{P},r}$, and similarly for $\mathcal{F}$ and $\mathcal{K}$.

(ii) *For every $p \in \mathcal{P}$,*

$$d_{p,r} : E^{\mathbb{N}} \times E^{\mathbb{N}} \to \overline{\mathbb{R}}_+, \qquad (f,g) \mapsto \|\!|f-g|\!\|_{p,r},$$

is an ultrapseudometric on $\mathcal{F}_{\mathcal{P},r}$, and the family $(d_{p,r})_{p\in\mathcal{P}}$ makes $\mathcal{F}_{\mathcal{P},r}$ a topological algebra over $(\mathcal{F}_{|\cdot|,r}, d_{|\cdot|,r})$.

(iii) *For every $p \in \mathcal{P}$,*

$$\widetilde{d}_{p,r} : \mathcal{G}_{\mathcal{P},r} \times \mathcal{G}_{\mathcal{P},r} \to \mathbb{R}_+, \qquad ([f],[g]) \mapsto d_{p,r}(f,g),$$

is an ultrametric on $\mathcal{G}_{\mathcal{P},r}$, where $[f],[g]$ are the classes of $f,g \in \mathcal{F}_{\mathcal{P},r}$. The family of ultrametrics $\{\widetilde{d}_{p,r}\}_{p\in\mathcal{P}}$ defines a topology, identical to the quotient topology, for which $\mathcal{G}_{\mathcal{P},r} = \mathcal{F}_{\mathcal{P},r}/\mathcal{K}_{\mathcal{P},r}$ is a topological algebra over $\mathbb{C}_r = \mathcal{G}_{|\cdot|,r}$.

Proof. (i) If $f,g \in \mathcal{F}$ and $\lambda \in \mathbb{C}$, we have $\forall p \in \mathcal{P} : \|\!|\lambda f + g|\!\|_{p,r} \le \max(\|\!|f|\!\|_{p,r}, \|\!|g|\!\|_{p,r})$, thus $\mathcal{F}_{\mathcal{P},r}$ and $\mathcal{K}_{\mathcal{P},r}$ are $\mathbb{C}$-linear (sub)spaces. Using continuity of multiplication in $(E,\mathcal{P})$, we have $\forall p \in \mathcal{P}$, $\exists \bar{p} \in \mathcal{P} : \|\!|f \cdot g|\!\|_{p,r} \le \|\!|f|\!\|_{\bar{p},r}\|\!|g|\!\|_{\bar{p},r}$ (while the constant C disappears in view of $C^{r_n} \to 1$). Thus $\mathcal{F}_{\mathcal{P},r}$ is a $\mathbb{C}$-subalgebra of $E^{\mathbb{N}}$, and $\mathcal{K}_{\mathcal{P},r}$ is an ideal of $\mathcal{F}_{\mathcal{P},r}$, as claimed.

(ii) The first part of (ii) is for a fixed seminorm and thus a direct consequence of Proposition–Definition 2. Continuity of addition and multiplication in $\mathcal{F}_{\mathcal{P},r}$ are implied by the previous two inequalities. Thus, $\mathcal{F}_{|\cdot|,r}$ is a topological ring, and $\mathcal{F}_{\mathcal{P},r}$ a topological $\mathcal{F}_{|\cdot|,r}$-algebra, because $\forall p \in \mathcal{P}$, $\forall \lambda \in \mathcal{F}_{|\cdot|,r} : \|\!|\lambda f|\!\|_{p,r} \le \|\!|\lambda|\!\|_{|\cdot|,r}\|\!|f|\!\|_{p,r}$.

(iii) The first inequality above also implies the independence of the ultrametric from the representatives of $[f],[g] \in \mathcal{G}_{\mathcal{P},r}$. Finally, by definition, $\mathcal{K}_{\mathcal{P},r}$ is here again the intersection of all neighborhoods of zero, so that $\mathcal{G}_{\mathcal{P},r}$ is nothing else than the associated Hausdorff space. ∎

1.2.2. *Examples*

Example 8 (simplified Colombeau algebra). Take $\Omega \subset \mathbb{R}^n$, $E = \mathcal{C}^\infty(\Omega)$, $\mathcal{P} = \{p_\nu\}_{\nu\in\mathbb{N}}$, with $p_\nu = p_\nu^\nu$, and

$$p_\nu^\mu(f) := \sup_{|\alpha|\le\nu,\, x\in K_\mu} |f^{(\alpha)}(x)|,$$

where $r = 1/\log$ and $(K_\mu)_{\mu\in\mathbb{N}}$ is an increasing sequence of compact sets exhausting Ω.

Then $\mathcal{G}_{\mathcal{P},r} = \mathcal{F}_{\mathcal{P},r}/\mathcal{K}_{\mathcal{P},r}$, where

$$\mathcal{F}_{\mathcal{P},r} = \{(f_n)_n \in \mathcal{C}^\infty(\Omega)^{\mathbb{N}} \mid \forall\nu \in \mathbb{N} : \sup_{n>1} p_\nu(f_n)^{1/\log n} < \infty\},$$
$$\mathcal{K}_{\mathcal{P},r} = \{(f_n)_n \in \mathcal{C}^\infty(\Omega)^{\mathbb{N}} \mid \forall\nu \in \mathbb{N} : \lim_{n\to\infty} p_\nu(f_n)^{1/\log n} = 0\},$$

is just the simplified Colombeau algebra $\mathcal{G}_s(\Omega)$.

In the framework of echelon and coechelon spaces, we put for $k, \nu \in \mathbb{N}^*$,

$$\mathcal{F}_{\mathcal{P},r,\nu,k} = \{f \in \mathcal{C}^\infty(\Omega)^{\mathbb{N}} \mid \sup_{n>1} k^{-\log n} p_\nu(f_n) < \infty\},$$

which is a coechelon type space, and then

$$\mathcal{F}_{\mathcal{P},r,\nu} = \bigcup_{k\in\mathbb{N}} \mathcal{F}_{\mathcal{P},r,\nu,k}, \qquad \mathcal{F}_{\mathcal{P},r} = \bigcap_{\nu\in\mathbb{N}} \mathcal{F}_{\mathcal{P},r,\nu}.$$

On the other hand,

$$\mathcal{K}_{\mathcal{P},r,\nu,k} = \{f \in \mathcal{C}^\infty(\Omega)^{\mathbb{N}} \mid \lim_{n\to\infty} k^{\log n} p_\nu(f_n) = 0\}$$

is a sequence of echelon type spaces, and we let

$$\mathcal{K}_{\mathcal{P},r,\nu} = \bigcap_{k\in\mathbb{N}} \mathcal{K}_{\mathcal{P},r,\nu,k}, \qquad \mathcal{K}_{\mathcal{P},r} = \bigcap_{\nu\in\mathbb{N}} \mathcal{K}_{\mathcal{P},r,\nu}.$$

It is easily seen that these spaces are identical to those above, thus their quotient is again the classical simplified Colombeau algebra $\mathcal{G}_s(\Omega)$.

Consider the space

$$\mathcal{B}^\infty = \left\{\phi \in \mathcal{S}(\mathbb{R}^s) \,\middle|\, \forall\alpha \in \mathbb{N} : \int x^\alpha \phi_i = \delta_{\alpha,0}\right\} \tag{1.3}$$

and fix $\phi \in \mathcal{B}^\infty$. We realize the embedding of $T \in \mathcal{D}'(\Omega)$ into $\mathcal{G}_s(\Omega)$ as

$$i_\phi : \mathcal{D}'(\Omega) \to \mathcal{G}_s(\Omega), \qquad T \mapsto i_\phi(T) = [(\kappa_n T) * \phi_n],$$

where $[f_n] = (f_n)_n + \mathcal{K}_{\mathcal{P},r}$ denotes the class of the representative $(f_n)_n$ in $\mathcal{G}_s(\Omega)$, and where $(\kappa_n)_n \in \mathcal{D}(\Omega)^{\mathbb{N}}$ is a sequence of functions such that $\kappa_n|_{K_n} = 1$, $\operatorname{supp} \kappa_n| \subset K_{n+1}$, where $(K_n)_n$ is an increasing sequence of compact sets exhausting Ω.

Example 9 (temperate Colombeau algebra [31, 69]). We can also describe $\mathcal{G}_\tau(\mathbb{R}^s)$ in this setting. To do so, define

$$p_{\nu,N}(\varphi) = \sup\{(1 + |x|^2)^{-N} |\varphi^{(\alpha)}(x)| \mid x \in \mathbb{R}^s,\ |\alpha| \le \nu\}$$

and

$$\mathcal{F}_{r,\nu,N} = \{f \in \mathcal{C}^\infty(\mathbb{R}^s)^{\mathbb{N}} \mid \|\!|f|\!\|_{p_{\nu,N},r} \le e^N\},$$
$$\mathcal{K}_{r,\nu,N} = \{f \in \mathcal{C}^\infty(\mathbb{R}^s)^{\mathbb{N}} \mid \|\!|f|\!\|_{p_{\nu,N},r} = 0\}.$$

Now, for

$$\mathcal{F}_{\tau,r} = \bigcap_{\nu\in\mathbb{N}} \bigcup_{N\in\mathbb{N}} \mathcal{F}_{r,\nu,N}, \qquad \mathcal{K}_{\tau,r} = \bigcap_{\nu\in\mathbb{N}} \bigcup_{N\in\mathbb{N}} \mathcal{K}_{r,\nu,N}$$

the quotient space $\mathcal{G}_{\tau,r} = \mathcal{F}_{\tau,r}/\mathcal{K}_{\tau,r}$ is once again a topological algebra over $\mathbb{C}_r$, and equal to the classical space $\mathcal{G}_\tau(\mathbb{R}^s)$ for $r_n \sim 1/\log n$.

EXAMPLE 10 (full Colombeau algebra [8, 31]). Let us now introduce the "full" Colombeau algebra, based on the same $(E,\mathcal{P})$ as above. Following Colombeau, for all $q\in\mathbb{N}$ let

$$\mathcal{A}_q=\Big\{\phi\in\mathcal{D}(\mathbb{R}^s)\,\Big|\,\forall\alpha\in\mathbb{N}^s:|\alpha|\le q\Rightarrow\int x^\alpha\phi=\delta_{\alpha,0}\Big\}.$$

Then, for fixed $\nu,N\in\mathbb{N}$ and $\phi\in\mathcal{A}_N$ let

$$\mathcal{F}_{\nu,N,\phi}=\{(f_\varphi)_\varphi\in E^{\mathcal{A}_0}\mid |||(f_{\phi_n})_n|||_{p_\nu,r}\le N\},$$

where $\phi_n=n^s\phi(n\cdot)$. (Here $(f_{\phi_n})_n$ are "extracted sequences" of the elements $(f_\varphi)_\varphi\in E^{\mathcal{D}(\mathbb{R}^s)}$).

As in [8], denote by $\Gamma\subset\mathbb{R}_+^{\mathbb{N}}$ the set of increasing positive sequences going to infinity. Now define, for each $\gamma\in\Gamma$,

$$\mathcal{K}_{\nu,\gamma,q}=\{(f_\varphi)_\varphi\in E^{\mathcal{A}_q}\mid\forall\phi\in\mathcal{A}_q:|||(f_{\phi_n})_n|||_{p_\nu,r}\le\gamma(q)^{-1}\},$$

and

$$\mathcal{F}=\bigcap_{\nu\in\mathbb{N}}\mathcal{F}_\nu,\quad\mathcal{F}_\nu=\bigcup_{N\in\mathbb{N}}\mathcal{F}_{\nu,N},\quad\mathcal{F}_{\nu,N}=\bigcap_{\phi\in\mathcal{A}_N}\mathcal{F}_{\nu,N,\phi},$$

$$\mathcal{K}=\bigcap_{\nu\in\mathbb{N}}\mathcal{K}_\nu,\quad\mathcal{K}_\nu=\bigcup_{\gamma\in\Gamma}\mathcal{K}_{\nu,\gamma},\quad\mathcal{K}_{\nu,\gamma}=\bigcap_{q\in\mathbb{N}}\mathcal{K}_{\nu,\gamma,q}.$$

Then $\mathcal{F}$ is an algebra and $\mathcal{K}$ an ideal of $\mathcal{F}$, and $\mathcal{G}=\mathcal{F}/\mathcal{K}$ is the original full Colombeau algebra.

The original construction of Colombeau for the ideal has been slightly modified in [31], by taking an ideal which can in our notations be written as

$$\mathcal{K}=\bigcap_{\nu,N\in\mathbb{N}}\mathcal{K}_\nu,\quad\mathcal{K}_{\nu,N}=\bigcup_{q\in\mathbb{N}}\mathcal{K}_{\nu,N,q},\quad\mathcal{K}_{\nu,N,q}=\bigcap_{\varphi\in\mathcal{A}_q}\mathcal{F}_{\nu,1/N,\varphi}.$$

If one wants to consider the full Colombeau type algebra which is invariant under the composition with $\mathcal{C}^\infty$-diffeomorphisms [31], one has to consider instead of the above definition of $\mathcal{A}_q$ the following one:

$$\mathcal{A}_q=\Big\{(\phi^n)_n\in\mathcal{C}^\infty(\Omega)^{\mathbb{N}}\mid(\phi^n)_n\text{ is bounded in }\mathcal{D}(\mathbb{R}^n),$$
$$\forall n\in\mathbb{N}:\int\phi^n=1,\ \int x^\alpha\phi^n=o(n^{-q})\Big\}.$$

and the corresponding mollifiers $\phi_n=n\phi^n(n\cdot)$.

EXAMPLE 11. Replacing the spaces $\mathcal{A}_q$ with the space $\mathcal{B}^\infty$ introduced in (1.3), we can avoid the q index in the definition of $\mathcal{K}$. We take

$$\mathcal{F}_{\nu,N,\phi}=\{(f_\varphi)_\varphi\in E^{\mathcal{B}^\infty}\mid |||(f_{\phi_n})_n|||_{p_\nu,r}\le N\},$$

and

$$\mathcal{F}=\bigcap_{\nu\in\mathbb{N}}\mathcal{F}_\nu,\quad\mathcal{F}_\nu=\bigcup_{N\in\mathbb{N}}\mathcal{F}_{\nu,N},\quad\mathcal{F}_{\nu,N}=\bigcap_{\phi\in\mathcal{B}^\infty}\mathcal{F}_{\nu,N,\phi},$$

$$\mathcal{K}=\bigcap_{\nu,N\in\mathbb{N}}\mathcal{K}_\nu,\quad\mathcal{K}_{\nu,N}=\bigcap_{\varphi\in\mathcal{B}^\infty}\mathcal{F}_{\nu,1/N,\varphi}.$$

Then $\mathcal{F}$ is again an algebra and $\mathcal{K}$ an ideal of $\mathcal{F}$. The algebra $\mathcal{G}=\mathcal{F}/\mathcal{K}$ is studied in [69, 71, 72].

2. Projective and inductive limits

2.1. Projective limit. Let $(E^\mu_\nu, p^\mu_\nu)_{\mu,\nu\in\mathbb{N}}$ be a family of seminormed spaces over $\mathbb{C}$ such that

$$\forall \mu,\nu\in\mathbb{N}:\quad E^\mu_{\nu+1}\hookrightarrow E^\mu_\nu,\qquad E^{\mu+1}_\nu\hookrightarrow E^\mu_\nu, \tag{2.1}$$

where $\hookrightarrow$ means continuously embedded. This implies that there exist constants C^μ_ν, $\tilde{C}^\mu_\nu\in\mathbb{R}_+$ such that (3)

$$\forall \mu,\nu\in\mathbb{N}:\quad p^\mu_\nu\le C^\mu_\nu p^\mu_{\nu+1},\qquad p^\mu_\nu\le \tilde{C}^\mu_\nu p^{\mu+1}_\nu. \tag{2.2}$$

In addition, we assume that the spaces $\overleftarrow{E}^\mu=\operatorname{proj\,lim}_{\nu\to\infty}E^\mu_\nu$ are algebras such that

$$\forall \mu,\nu\in\mathbb{N},\ \exists\nu'\in\mathbb{N},\ C>0,\ \forall f,g\in E^\mu_{\nu'}:\ fg\in E^\mu_\nu \text{ and } p^\mu_\nu(fg)\le C p^\mu_{\nu'}(f)p^\mu_{\nu'}(g). \tag{2.3}$$

Then let

$$\overleftarrow{E}=\operatorname*{proj\,lim}_{\mu\to\infty}\overleftarrow{E}^\mu=\operatorname*{proj\,lim}_{\mu\to\infty}\operatorname*{proj\,lim}_{\nu\to\infty}E^\mu_\nu,$$

and define

$$\overleftarrow{\mathcal{F}}_{p,r}=\{f\in\overleftarrow{E}^{\mathbb{N}}\mid \forall\mu,\nu\in\mathbb{N}:\|\!|f|\!\|_{p^\mu_\nu,r}<\infty\},$$
$$\overleftarrow{\mathcal{K}}_{p,r}=\{f\in\overleftarrow{E}^{\mathbb{N}}\mid \forall\mu,\nu\in\mathbb{N}:\|\!|f|\!\|_{p^\mu_\nu,r}=0\}.$$

(Here $p\equiv(p^\mu_\nu)_{\nu,\mu}$ stands (on the l.h.s.) for the whole family of seminorms.) Then Proposition 7 holds, with the slight changes of notations introduced above (see Proposition 13 at the end of the next section).

Remark 12. The representation

$$\overleftarrow{E}=\operatorname*{proj\,lim}_{\mu\to\infty}\overleftarrow{E}^\mu=\operatorname*{proj\,lim}_{\mu\to\infty}\operatorname*{proj\,lim}_{\nu\to\infty}E^\mu_\nu$$

can of course be diagonalized to be given in the form $\overleftarrow{E}=\operatorname{proj\,lim}_{\nu\to\infty}E^\nu_\nu$. But we prefer the former construction because of the following simple motivation: Consider

$$\overleftarrow{\mathcal{F}}^\mu_{p,r}=\{f\in\overleftarrow{E}^{\mathbb{N}}\mid \exists C,\ \forall\nu\in\mathbb{N}:\|\!|f|\!\|_{p^\mu_\nu,r}<C\}$$

where $E^\mu_\nu=\mathcal{C}^\infty(\mathbb{R}^s)$, equipped with the seminorm

$$p^\mu_\nu(f)=\sup_{|\alpha|\le\nu,\,|x|\le\mu}|f^{(\alpha)}(x)|.$$

Then $\overleftarrow{\mathcal{F}}^\infty_{p,r}:=\bigcap_{\mu\in\mathbb{N}}\overleftarrow{\mathcal{F}}^\mu_{p,r}=\mathcal{E}^\infty_M(\mathbb{R}^s)$ in the sense of Oberguggenberger [60], and $\mathcal{G}^\infty(\mathbb{R}^s)$ $=\mathcal{F}^\infty_{p,r}/\mathcal{K}_{p,r}$ is the algebra of regular generalized functions, used for the analysis of local and microlocal properties of Colombeau generalized functions. (This algebra plays for Colombeau's simplified algebra the role of $\mathcal{C}^\infty$ for $\mathcal{D}'$; see Section 2.5 below.)

2.2. Inductive limit. Consider now a family $(E^\mu_\nu, p^\mu_\nu)_{\mu,\nu\in\mathbb{N}}$ of seminormed spaces over $\mathbb{C}$ such that

$$\forall \mu,\nu\in\mathbb{N}:\quad E^\mu_\nu\hookrightarrow E^\mu_{\nu+1},\qquad E^{\mu+1}_\nu\hookrightarrow E^\mu_\nu. \tag{2.4}$$

(3) The following inequalities should be considered to hold on the domain of the right hand side seminorm, seen as a subset of the domain of the left hand side seminorm, through the given embeddings.

This implies that there exist constants $C^\mu_\nu, \tilde{C}^\mu_\nu \in \mathbb{R}_+$ such that

$$\forall \mu, \nu \in \mathbb{N}: \quad p^\mu_{\nu+1} \le C^\mu_\nu p^\mu_\nu, \qquad p^\mu_\nu \le \tilde{C}^\mu_\nu p^{\mu+1}_\nu.$$

Now let

$$\forall \mu \in \mathbb{N}: \quad \overrightarrow{E}^\mu = \operatorname*{ind\,lim}_{\nu\to\infty} E^\mu_\nu.$$

Assume that the spaces $\overrightarrow{E}^\mu$ are algebras such that for every $\mu, \nu \in \mathbb{N}$ there exist $\nu' \in \mathbb{N}$, $\nu' > \nu$, and $C > 0$ such that for all $f, g \in E^\mu_{\nu'}$,

$$fg \in E^\mu_\nu \quad \text{and} \quad p^\mu_\nu(fg) \le C p^\mu_{\nu'}(f) p^\mu_{\nu'}(g).$$

We assume furthermore that for every $\mu \in \mathbb{N}$ this inductive limit is regular, i.e. a set $A \subset \overrightarrow{E}^\mu$ is bounded iff it is contained in some E^μ_ν and bounded there.

Note that (2.4) implies that $\forall \mu \in \mathbb{N}: \overrightarrow{E}^{\mu+1} \hookrightarrow \overrightarrow{E}^\mu$. Now let

$$\overrightarrow{E} := \operatorname*{proj\,lim}_{\mu\to\infty} \overrightarrow{E}^\mu = \operatorname*{proj\,lim}_{\mu\to\infty} \operatorname*{ind\,lim}_{\nu\to\infty} E^\mu_\nu,$$

and define

$$\overrightarrow{\mathcal{F}}_{p,r} = \{ f \in \overrightarrow{E}^{\mathbb{N}} \mid \forall \mu \in \mathbb{N}, \exists \nu \in \mathbb{N} : f \in (E^\mu_\nu)^{\mathbb{N}} \wedge \|f\|_{p^\mu_\nu, r} < \infty \},$$
$$\overrightarrow{\mathcal{K}}_{p,r} = \{ f \in \overrightarrow{E}^{\mathbb{N}} \mid \forall \mu \in \mathbb{N}, \exists \nu \in \mathbb{N} : f \in (E^\mu_\nu)^{\mathbb{N}} \wedge \|f\|_{p^\mu_\nu, r} = 0 \}.$$

Then Proposition 7 holds again with the appropriate change of notations:

PROPOSITION 13.

(i) *Writing* $\overleftrightarrow{\cdot}$ *for both* $\overrightarrow{\cdot}$ *and* $\overleftarrow{\cdot}$, *we have that* $\overleftrightarrow{\mathcal{F}}_{p,r}$ *is an algebra and* $\overleftrightarrow{\mathcal{K}}_{p,r}$ *is an ideal thereof, thus* $\overleftrightarrow{\mathcal{G}}_{p,r} = \overleftrightarrow{\mathcal{F}}_{p,r}/\overleftrightarrow{\mathcal{K}}_{p,r}$ *is an algebra. Instead of* $\overleftrightarrow{\mathcal{G}}_{\mathcal{P},r}$, *we also suggest the notation* $\mathcal{G}_r(\overleftrightarrow{E})$, *and idem for* $\overleftrightarrow{\mathcal{F}}$ *and* $\overleftrightarrow{\mathcal{K}}$.

(ii) *For every* $\mu, \nu \in \mathbb{N}$,

$$d_{p,\mu,\nu} : (E^\mu_\nu)^{\mathbb{N}} \times (E^\mu_\nu)^{\mathbb{N}} \to \overline{\mathbb{R}}_+, \qquad (f,g) \mapsto \|f - g\|_{p^\mu_\nu, r},$$

is an ultrapseudometric on $(E^\mu_\nu)^{\mathbb{N}}$.

(iii) *The above family of ultrapseudometrics makes* $\overleftarrow{\mathcal{G}}_{p,r} = \overleftarrow{\mathcal{F}}_{p,r}/\overleftarrow{\mathcal{K}}_{p,r}$ *a topological algebra over* $\mathbb{C}_r$, *with quotient topology equivalent to the topology defined by the family of ultrametrics* $(\tilde{d}_{p^\mu_\nu})_{\mu,\nu}$, *where* $\tilde{d}_{p^\mu_\nu}([f],[g]) = d_{p^\mu_\nu}(f,g)$, $[f]$ *standing for the class of* f.

(iv) *If* τ_μ *denotes the inductive limit topology on* $\mathcal{F}^\mu_{p,r} = \bigcup_{\nu\in\mathbb{N}}((E^\mu_\nu)^{\mathbb{N}}, d_{\mu,\nu})$, $\mu \in \mathbb{N}$, *then* $\overrightarrow{\mathcal{F}}_{p,r}$ *is a topological algebra for the projective limit topology of the family* $(\mathcal{F}^\mu_{p,r}, \tau_\mu)_\mu$.

Proof. The proof goes again along the same lines, where the above assumption on the regularity of the inductive limits helps to use the same reasoning as before. ∎

EXAMPLE 14. For $\Omega \subset \mathbb{R}^s$, an exhausting sequence of compacts $K_\mu \Subset \Omega$, $\mu \in \mathbb{N}$, and an increasing sequence $(M_n)_n \in \mathbb{R}^{\mathbb{N}}_+$, define the seminorms

$$p^{M,\mu}_\nu : \varphi \mapsto \sup_{\alpha\in\mathbb{N},\, x\in K_\mu} \frac{\nu^{|\alpha|} |\varphi^{(\alpha)}(x)|}{M_{|\alpha|}}$$

(clearly increasing in μ and ν), and $q_\nu^{M,\mu} = p_{1/\nu}^{M,\mu}$ (decreasing in ν). These seminorms are used to define Beurling (resp. Roumieu) type ultradifferentiable functions, which will be studied in some detail in the next chapter.

2.3. Completeness. Without assuming completeness of $\overleftrightarrow{E}$, we have

PROPOSITION 15.

(i) $\overleftarrow{\mathcal{F}}_{p,r}$ *is complete.*

(ii) *If for all* $\mu \in \mathbb{N}$, *a subset of* $\overrightarrow{\mathcal{F}}^{\mu}_{p,r}$ *is bounded iff it is a bounded subset of* $(E^\mu_\nu)^{\mathbb{N}}$ *for some* $\nu \in \mathbb{N}$, *then* $\overrightarrow{\mathcal{F}}_{p,r}$ *is sequentially complete.*

REMARK 16. In the projective limit case, we have a metrisable space, therefore sequential completeness implies completeness. This is not the case for the inductive limit case.

Proof. If $(f^m)_{m\in\mathbb{N}}$ is a Cauchy sequence in $\overleftarrow{\mathcal{F}}_{p,r}$, there exists a strictly increasing sequence $(m_\mu)_{\mu\in\mathbb{N}}$ of integers such that

$$\forall \mu \in \mathbb{N},\ \forall k,\ell \geq m_\mu : \quad \limsup_{n\to\infty} p^\mu_\mu (f^k_n - f^\ell_n)^{r_n} < \frac{1}{2^\mu}.$$

Thus, there exists a strictly increasing sequence $(n_\mu)_{\mu\in\mathbb{N}}$ of integers such that

$$\forall \mu \in \mathbb{N},\ \forall k,\ell \in [m_\mu, m_{\mu+1}],\ \forall n \geq n_\mu : \quad p^\mu_\mu (f^k_n - f^\ell_n)^{r_n} < \frac{1}{2^\mu}.$$

(Restricting k,ℓ to $[m_\mu, m_{\mu+1}]$ allows us to take n_μ independent of k,ℓ.) Let $\boldsymbol{\mu}(n) = \sup\{\mu \mid n_\mu \leq n\}$, and consider the diagonalized sequence

$$\bar{f} = (f_n^{m_{\boldsymbol{\mu}(n)}})_n, \quad \text{i.e.} \quad \bar{f}_n = \begin{cases} f_n^{m_0} & \text{if } n \in [n_0, n_1), \\ \dots & \\ f_n^{m_\mu} & \text{if } n \in [n_\mu, n_{\mu+1}), \\ \dots & \end{cases}$$

Now let us show that $f^m \to \bar{f}$ in $\overleftarrow{\mathcal{F}}_{p,r}$ as $m \to \infty$. Indeed, for ε and $p^{\mu_0}_\nu$ given, choose $\mu > \mu_0, \nu$ such that $1/2^\mu < \frac{1}{2}\varepsilon$. As p^μ_ν is increasing in both indices, we have for $m > m_\mu$ (say $m \in [m_{\mu+s}, m_{\mu+s+1}]$)

$$\begin{aligned} p^{\mu_0}_\nu (f^m_n - \bar{f}_n)^{r_n} &\leq p^\mu_\mu (f^m_n - f_n^{m_{\boldsymbol{\mu}(n)}})^{r_n} \\ &\leq p^\mu_\mu (f^m_n - f_n^{m_{\mu+s+1}})^{r_n} + \sum_{\mu'=\mu+s+1}^{\boldsymbol{\mu}(n)-1} p^{\mu'}_{\mu'} (f_n^{m_{\mu'}} - f_n^{m_{\mu'+1}})^{r_n} \end{aligned}$$

and for $n > n_{\mu+s}$, we have of course $n \geq n_{\boldsymbol{\mu}(n)}$, thus finally

$$p^{\mu_0}_\nu (f^m_n - \bar{f}_n)^{r_n} < \sum_{\mu'=\mu+s}^{\boldsymbol{\mu}(n)} \frac{1}{2^{\mu'}} < \frac{2}{2^\mu} < \varepsilon$$

and therefore $f^m \to \bar{f}$ in $\overleftarrow{\mathcal{F}}$.

For a Cauchy net $(f^m)_m$ in $\overrightarrow{\mathcal{F}}_{p,r}$, the proof requires some additional considerations. We know that for every μ there is $\nu(\mu)$ such that

$$p^\mu_{\nu(\mu)} (f^m_n - f^p_n)^{r_n} < \varepsilon_\mu,$$

where $(\varepsilon_\mu)_\mu$ decreases to zero. For every μ we can choose $\nu(\mu)$ so that $p^{\mu}_{\nu(\mu)} \le p^{\mu+1}_{\nu(\mu+1)}$. Now by the same arguments as above, we prove the completeness in the case of $\overrightarrow{\mathcal{F}}_{p,r}$. ■

2.4. Sheaf theory aspects. Let us now apply concepts of sheaf theory to local and microlocal analysis in generalized function spaces, through the sequence space presentation.

We will investigate under what conditions a generalized algebra $\overleftrightarrow{\mathcal{G}}_{p,r}$ is a (pre-)sheaf, provided that $\overleftrightarrow{E}$ is a (pre-)sheaf. Here, $\overleftrightarrow{E}$ stands for the functor associating to each open set Ω the space $\overleftrightarrow{E}(\Omega)$ constructed according to the preceding sections for a given family $(E^{\mu}_{\nu}(\Omega), p^{\mu}_{\nu,\Omega})$. More details will be given below.

Some definitions are necessary to formulate such statements more precisely and to prove them.

2.4.1. *Preliminary considerations.* Recall that a presheaf F (of objects in a concrete category) on a topological space X is given by

- the association of a set $F(\Omega)$ to each open set Ω of X, and
- for every inclusion of open sets $\Omega' \subset \Omega$, a restriction map $\rho_{\Omega,\Omega'} : F(\Omega) \to F(\Omega')$, $f \mapsto f|_{\Omega'}$, such that
  - for each open set Ω of X, $\rho_{\Omega,\Omega}$ is the identity map on $F(\Omega)$, and
  - for any three open sets $\Omega'' \subset \Omega' \subset \Omega$, we have $\rho_{\Omega',\Omega''} \circ \rho_{\Omega,\Omega'} = \rho_{\Omega,\Omega''}$.

A presheaf F is a sheaf iff the following conditions hold:

(i) Let $(\Omega_i)_i$ be a family of open sets and $(f_i)_i$ a compatible family of sections $f_i \in F(\Omega_i)$, i.e. such that

$$\forall i, j : \quad f_i|_{\Omega_i \cap \Omega_j} = f_j|_{\Omega_i \cap \Omega_j}.$$

Then there exists a section $f \in F(\bigcup_i \Omega_i)$ such that $\forall i : f|_{\Omega_i} = f_i$.

(ii) Let $\Omega = \bigcup_{i \in I} \Omega_i$, $f, g \in F(\Omega)$ and $f|_{\Omega_i} = g|_{\Omega_i}$ for all i. Then $f = g$.

To speak of a sheaf of objects in a given category, one requires that the sets $F(\Omega)$ be objects of this category, and the restrictions be morphisms of the category. We restrict ourselves here to (pre-)sheaves of topological algebras over topological rings, on a paracompact topological space X. Accordingly, the restriction maps must be continuous algebra morphisms.

(Recall that Colombeau type generalized functions are never topological vector spaces, because scalar multiplication with elements of $\mathbb{R}$ or $\mathbb{C}$ is not continuous, as seen in Proposition–Definition 2; they are only topological modules (and algebras, if $\overleftrightarrow{E}$ is so) over the ring of generalized numbers.)

2.4.2. *The presheaf* $\overleftrightarrow{E}$. Let X be a paracompact Hausdorff space. Let us assume that for each (fixed) open set $\Omega \subset X$, the space $\overleftrightarrow{E}(\Omega)$ is constructed as described in the previous sections from a sequence $(E^{\mu}_{\nu}(\Omega), p^{\mu}_{\nu,\Omega})$ satisfying the given inclusion relations. Thus, for every fixed Ω,

$$\overleftarrow{E}(\Omega) = \operatorname*{proj\,lim}_{\mu\to\infty} \overleftarrow{E}^{\mu}(\Omega) = \operatorname*{proj\,lim}_{\mu\to\infty} \operatorname*{proj\,lim}_{\nu\to\infty} E^{\mu}_{\nu}(\Omega),$$

resp.

$$\overleftrightarrow{E}(\Omega) = \operatorname*{proj\,lim}_{\mu\to\infty} \overrightarrow{E}^{\mu}(\Omega) = \operatorname*{proj\,lim}_{\mu\to\infty} \operatorname*{ind\,lim}_{\nu\to\infty} E_\nu^\mu(\Omega).$$

Moreover, we now assume that the spaces $E_\nu^\mu(\Omega)$ are spaces of (at least continuous) functions, defined on Ω, for which we have the (pointwise) restrictions of functions in the usual sense, $f \in E_\nu^\mu(\Omega) \subset C^0(\Omega) \mapsto f|_{\Omega'} \in C^0(\Omega')$. (In what follows, we will study more precisely the question to which $E_{\nu'}^{\mu'}(\Omega')$ this restricted function will belong, in order to find that $\Omega \to \overleftrightarrow{E}(\Omega)$ are indeed sheaves.)

Proposition 17. *Under the above assumptions*, $\overleftrightarrow{E} : \Omega \to \overleftrightarrow{E}(\Omega)$ (*with the pointwise restriction*) *is a presheaf of vector spaces*, *if for any open sets* $\Omega_1 \subset \Omega_2$ *in* X, *we have*

– *in the projective limit case*:

$$\forall \mu,\nu\in\mathbb{N},\ \exists \mu',\nu'\in\mathbb{N},\ \exists C>0,\ \forall f\in E_{\nu'}^{\mu'}(\Omega_2):$$
$$f|_{\Omega_1}\in E_\nu^\mu(\Omega_1) \quad \text{and} \quad p_{\nu,\Omega_1}^{\mu}(f|_{\Omega_1}) \le C p_{\nu',\Omega_2}^{\mu'}(f), \tag{2.5}$$

– *in the inductive limit case*:

$$\forall \mu\in\mathbb{N},\ \exists \mu'\in\mathbb{N},\ \forall \nu'\in\mathbb{N},\ \exists \nu\in\mathbb{N},\ \exists C>0,\ \forall f\in E_{\nu'}^{\mu'}(\Omega_2):$$
$$f|_{\Omega_1}\in E_\nu^\mu(\Omega_1) \quad \text{and} \quad p_{\nu,\Omega_1}^{\mu}(f|_{\Omega_1}) \le C p_{\nu',\Omega_2}^{\mu'}(f). \tag{2.6}$$

Proof. Since the proof for the projective limit case is analogous but much simpler, we only consider the inductive limit case.

Let $f \in \overleftrightarrow{E}(\Omega_2)$. Fix μ. Determine μ' according to condition (2.6). We know that $f \in E_{\nu'}^{\mu'}(\Omega_2)$ for some ν'. Then, by (2.6), there is ν such that $f|_{\Omega_1}$ belongs to $E_\nu^\mu(\Omega_1)$, thus $f|_{\Omega_1} \in \overleftrightarrow{E}(\Omega_1)$. Now, without fixing f from the beginning, one sees that the second condition implies that the (set categorical) restriction is indeed continuous. ∎

2.4.3. *The (pre)sheaf* $\mathcal{G}_r(\overleftrightarrow{E})$. Now we will consider, for each Ω, the algebras $\mathcal{F}_r(\overleftrightarrow{E}(\Omega))$, $\mathcal{K}_r(\overleftrightarrow{E}(\Omega))$ (ideal of $\mathcal{F}_r(\overleftrightarrow{E}(\Omega))$) and $\mathcal{G}_r(\overleftrightarrow{E}(\Omega))$. We keep the hypotheses of the beginning of this subsection.

Proposition 18. *Assume that we have* (2.5) *in the projective limit case* (*resp.* (2.6) *in the inductive limit case*). *Then*:

(i) $\mathcal{F}_r(\overleftrightarrow{E}) : \Omega \to \mathcal{F}_r(\overleftrightarrow{E}(\Omega))$ *is a presheaf of topological* $\mathcal{F}_{|\cdot|,r}$*-algebras*;

(ii) $\mathcal{K}_r(\overleftrightarrow{E}) : \Omega \to \mathcal{K}_r(\overleftrightarrow{E}(\Omega))$ *is a presheaf of ideals of* $\mathcal{F}_r(\overleftrightarrow{E})$, *i.e.*, *a presheaf of topological algebras such that for each* Ω, $\mathcal{K}_r(\overleftrightarrow{E})(\Omega)$ *is an ideal of* $\mathcal{F}_r(\overleftrightarrow{E})(\Omega)$;

(iii) $\mathcal{G}_r(\overleftrightarrow{E}) = \mathcal{F}_r(\overleftrightarrow{E})/\mathcal{K}_r(\overleftrightarrow{E}) : \Omega \to \mathcal{F}_r(\overleftrightarrow{E})(\Omega)/\mathcal{K}_r(\overleftrightarrow{E})(\Omega)$, *is a presheaf of topological* $\mathcal{G}_{|\cdot|,r}(=\overline{\mathbb{K}}_r)$*-algebras*, *for the restriction mapping*

$$\mathcal{G}_r(\overleftrightarrow{E})(\Omega) \ni f \mapsto f|_{\Omega'} = (\tilde f_n|_{\Omega'})_n + \mathcal{K}_r(\overleftrightarrow{E})(\Omega') \in \mathcal{G}_r(\overleftrightarrow{E})(\Omega'),$$

where $(\tilde f_n)_n$ *is any representative of* f.

Proof. Let us start by defining what the restriction mappings are in $\mathcal{F}_r(\overleftrightarrow{E})$. For given $\Omega \supset \Omega_1$, elements f of $\mathcal{F}_r(\overleftrightarrow{E})(\Omega)$ are sequences of functions of $\overleftrightarrow{E}(\Omega)$. They can, by assumption, be componentwise restricted to Ω_1, i.e. we have a function $\tilde\rho_{\Omega,\Omega_1}$ which maps any $f = (f_n)_n \in \mathcal{F}_r(\overleftrightarrow{E}(\Omega)) \subset \overleftrightarrow{E}^{\mathbb{N}}(\Omega)$ into the sequence $f|_{\Omega_1} = (f_n|_{\Omega_1})_n \in \overleftrightarrow{E}(\Omega_1)^{\mathbb{N}}$.

But more precisely, the respective assumptions (2.5) and (2.6) imply that the sequence $f|_{\Omega_1}$ is an element of $\mathcal{F}_r(\overrightarrow{E}(\Omega_1))$ for $f \in \mathcal{F}_r(\overrightarrow{E}(\Omega))$. We will explain this in the inductive limit case; a similar and even simpler explanation holds for the projective limit case.

Let $(f_n)_n \in \mathcal{F}_r(\overrightarrow{E}(\Omega))$. We know that for every μ' there exists ν' such that $\forall n \in \mathbb{N}$: $f_n \in E_{\nu'}^{\mu'}(\Omega)$. Fix μ and determine μ' according to (2.6), and ν' as above. Now, again by (2.6) and ν from this condition, we have $(f_n|_{\Omega_1})_n \in E_\nu^\mu(\Omega_1)^{\mathbb{N}}$. Again by (2.6), we find that $(f_n)_n \in \mathcal{F}_r(\overrightarrow{E}(\Omega_1))$. The same reasoning can be applied to $\mathcal{K}$ instead of $\mathcal{F}$.

The condition $\tilde{\rho}_{\Omega,\Omega} = \mathrm{id}$ and the one on composition of restrictions are immediately checked to hold. Finally, conditions (2.6) (resp. (2.5)) also imply continuity of the restriction mapping.

Thus, $\mathcal{F}_r(\overleftrightarrow{E})$ and $\mathcal{K}_r(\overleftrightarrow{E})$ are presheaves of topological $\mathcal{F}_{|\cdot|,r}$-algebras. Now, again by (2.6), one can prove that for each Ω, $\mathcal{K}_r(\overleftrightarrow{E})(\Omega)$ is an ideal of $\mathcal{F}_r(\overleftrightarrow{E})(\Omega)$, as claimed.

With this, it is immediate to see that the given restriction on $\mathcal{G}_r(\overleftrightarrow{E})$ is well defined (independent of the chosen representative), and the general theory implies that

$$\mathcal{G}_r(\overleftrightarrow{E}) : \Omega \to \mathcal{G}_r(\overleftrightarrow{E}(\Omega)) \equiv \mathcal{F}_r(\overleftrightarrow{E}(\Omega))/\mathcal{K}_r(\overleftrightarrow{E}(\Omega))$$

indeed defines a presheaf. ∎

Example 19. Take $\mathcal{S}$, the presheaf of rapidly decreasing smooth functions on $X = \mathbb{R}^s$. We define, for any open subset $\Omega \subset \mathbb{R}^s$,

$$\forall \mu, \nu \in \mathbb{N} : \quad q_{\nu,\Omega}^\mu(f) = \sup_{x \in \Omega,\, t \le \mu,\, |\alpha| \le \nu} (1 + |x|)^t |f^{(\alpha)}(x)|$$

and set $S_\nu^\mu(\Omega) = \{f \in C^\infty(\Omega) \mid q_{\nu,\Omega}^\mu(f) < \infty\}$. Then

$$\mathcal{S}(\Omega) = \operatorname*{proj\,lim}_{\mu \to \infty} \operatorname*{proj\,lim}_{\nu \to \infty} S_\nu^\mu(\Omega).$$

As property (2.5) clearly holds for the family $(q_{\nu,\Omega}^\mu)_{\nu,\mu,\Omega}$, the corresponding functor $\mathcal{G}_{\mathcal{S},r} = \mathcal{G}_r(\mathcal{S}) : \Omega \to \mathcal{G}_r(\mathcal{S}(\Omega))$ defines a presheaf of rapidly decreasing generalized functions.

Proposition 20. *Assume that for every open $\Omega \subset X$ and every locally finite open covering $(\Omega_\lambda)_\lambda$ of Ω, we have a partition of unity $(\eta_i)_i \in \overleftrightarrow{E}(\Omega)^{\mathbb{N}}$ (that is, there exists a subcover $(\Omega_i)_{i\in\mathbb{N}}$ of $(\Omega_\lambda)_\lambda$ such that $\operatorname{supp}\eta_i \subset \Omega_i$ and $\sum_i \eta_i = 1$ on Ω). Moreover, assume:*

- *in the projective limit case, (2.5) and that for all $\mu, \nu \in \mathbb{N}^*$, there exists a finite subfamily $(\Omega_{i_j})_{j\in\{1,\dots,\ell\}}$ and $(\mu_j)_j, (\nu_j)_j \in (\mathbb{N}^*)^\ell$ such that*

$$\forall f \in E_\nu^\mu(\Omega), \forall j : \quad \eta_{i_j} f \in E_{\nu_j}^{\mu_j}(\Omega_{i_j}) \quad \text{and} \quad p_{\nu,\Omega}^\mu(f) \le \sum_{j=1}^{\ell} p_{\nu_j,\Omega_{i_j}}^{\mu_j}(\eta_{i_j} f), \qquad (2.7)$$

- *in the inductive limit case, (2.6) and that for any $\mu \in \mathbb{N}^*$, there exists a finite subfamily $(\Omega_{i_j})_{j\in\{1,\dots,\ell\}}$ and $(\mu_j)_j \in (\mathbb{N}^*)^\ell$ such that for all $(\nu_j)_j \in (\mathbb{N}^*)^\ell$ there is $\nu \in \mathbb{N}^*$ such that*

$$\forall f \in E_\nu^\mu(\Omega), \forall j : \quad \eta_{i_j} f \in E_{\nu_j}^{\mu_j}(\Omega_{i_j}) \quad \text{and} \quad p_{\nu,\Omega}^\mu(f) \le \sum_{j=1}^{\ell} p_{\nu_j,\Omega_{i_j}}^{\mu_j}(\eta_{i_j} f), \qquad (2.8)$$

where $\Omega = \bigcup_i \Omega_i$.

Then $\mathcal{F}_r(\overleftrightarrow{E})$ *is a fine sheaf, and* $\mathcal{K}_r(\overleftrightarrow{E})$ *is a fine subsheaf thereof. In addition, for every open* Ω *in* X,

$$0 \to \mathcal{K}_r(\overleftrightarrow{E})(\Omega) \to \mathcal{F}_r(\overleftrightarrow{E})(\Omega) \to \mathcal{G}_r(\overleftrightarrow{E})(\Omega) \to 0$$

is an exact sequence, and $\mathcal{G}_r(\overleftrightarrow{E})$ *is a fine sheaf.*

Proof. Consider the inductive limit case and the presheaf $\Omega \to \mathcal{F}_r(\overrightarrow{E})(\Omega)$ (resp. $\Omega \to \mathcal{K}_r(\overrightarrow{E})(\Omega)$). Let $\Omega = \bigcup_{i\in I} \Omega_i$, $(f_n)_n \in \mathcal{F}_r(\overrightarrow{E})(\Omega)$ (resp. $\mathcal{K}_r(\overrightarrow{E})(\Omega)$), and $(f_n|_{\Omega_i})_n = 0$. Then clearly $(f_n)_n = 0$ in the respective sequence spaces over Ω. Since we have assumed that the spaces $E^\mu_\nu(\Omega)$ consist of functions which are at least continuous, their glueing for the second sheaf property leads to a proof showing that the second condition holds for $\Omega \to \mathcal{F}_r(\overrightarrow{E})(\Omega)$ and for $\Omega \to \mathcal{K}_r(\overrightarrow{E})(\Omega)$. Both sheaves are fine since we have partitions of unity, as usual.

Let $(f_n)_n \in \mathcal{F}_r(\overrightarrow{E})(\Omega)$, and $\Omega = \bigcup_{i\in I} \Omega_i$. Assume that $(f_n|_{\Omega_i})_n \in \mathcal{K}_r(\overrightarrow{E})(\Omega_i)$. Then, by taking powers $1/r_n$ on both sides of (2.8), we find that $(f_n)_n \in \mathcal{K}_r(\overrightarrow{E})(\Omega)$. This implies that the short sequence is exact and by the well-known result of sheaf theory, it follows that $\Omega \to \mathcal{G}_r(\overrightarrow{E})(\Omega)$ is a fine sheaf. ■

Example 21 (generalization of Example 8). Take $\mathcal{C}^\infty$, the sheaf of smooth functions on $X = \mathbb{R}^s$, and denote by $\mathcal{O}$ the set of all open subsets of $\mathbb{R}^s$. We can find a family $(K^\Omega_\mu)_{\mu\in\mathbb{N},\,\Omega\in\mathcal{O}}$ of compact subsets of $\mathbb{R}^s$ such that for each $\Omega \in \mathcal{O}$, the sequence $(K^\Omega_\mu)_{\mu\in\mathbb{N}}$ exhausts Ω. We set

$$\forall \mu,\nu \in \mathbb{N}, \forall f \in \mathcal{C}^\nu(\Omega): \quad p^\mu_{\nu,\Omega}(f) = \sup_{x\in K^\Omega_\mu,\,|\alpha|\le\nu} |f^{(\alpha)}(x)|.$$

Then

$$\mathcal{C}^\infty(\Omega) = \operatorname{proj}\lim_{\mu\to\infty} \operatorname{proj}\lim_{\nu\to\infty} E^\mu_\nu(\Omega)$$

where $E^\mu_\nu(\Omega) = \mathcal{C}^\nu(\Omega)$ is equipped with the seminorm $p^\mu_{\nu,\Omega}$. Moreover, we can choose the family $(K^\Omega_\mu)_{\mu\in\mathbb{N},\,\Omega\in\mathcal{O}}$ such that properties (2.5) and (2.7) hold. Thus, $\mathcal{G}_r(\mathcal{C}^\infty) : \Omega \to \mathcal{G}_r(\mathcal{C}^\infty(\Omega))$ defines a fine sheaf. We simply denote it by $\mathcal{G}_r : \Omega \to \mathcal{G}_r(\Omega)$.

For $r_n \sim 1/\log n$, we recover the well known result for the sheaf of Colombeau simplified algebras.

Example 22 (continuation of Example 19). The functor $\mathcal{G}_{\mathcal{S},r} : \Omega \to \mathcal{G}_r(\mathcal{S}(\Omega))$ is not a sheaf. The associated sheaf is $\mathcal{G}_r : \Omega \to \mathcal{G}_r(\Omega)$, as in distribution theory, the associated sheaf to $\mathcal{S}'$ is $\mathcal{D}'$.

Remark 23. By the given theory, it follows that algebras of generalized ultradistributions for non-quasianalytic sequences (M_p) (in our case for $M_p = p!^s$, $s > 1$) constitute fine sheaves. Let us just note that we do not have partitions of unity in spaces of analytic functions. In this case one can use other techniques (theory of holomorphic functions) in order to prove the sheaf properties of the space of holomorphic generalized functions [62].

2.5. Introduction to regularity theory. Our aim is to show how the concept of regular generalized functions introduced in [31, 60] and slightly generalized in [12] fits into our settings. We restrict ourselves here to the case of projective limits, since we want to illustrate the concepts with the example of $\mathcal{G}_r$ (see Example 21), which corresponds to the $\mathcal{C}^\infty$-analysis in the framework of Schwartz's distributions.

2.5.1. *Subspaces of $\mathcal{G}_r(\overleftarrow{E})$ and singular supports*

DEFINITION 24. We say that a subset $\mathcal{R}$ of $\mathbb{R}_+^{\mathbb{N}^2} = \{(C_\nu^\mu)_{\mu,\nu\in\mathbb{N}} \mid C_\nu^\mu \in \mathbb{R}_+\}$ is *regular* iff

$$\forall C \in \mathcal{R}, \forall \mu,\nu \in \mathbb{N}: \quad C_\nu^\mu \le C_\nu^{\mu+1}, \quad C_\nu^\mu \le C_{\nu+1}^\mu, \tag{2.9}$$

$$\forall C \in \mathcal{R}, \forall \kappa \in \mathbb{R}_+, \exists D \in \mathcal{R}, \forall \mu,\nu \in \mathbb{N}^2: \quad \kappa C_\nu^\mu \le D_\nu^\mu, \tag{2.10}$$

$$\forall C_1, C_2 \in \mathcal{R}, \exists D \in \mathcal{R}, \forall \mu,\nu \in \mathbb{N}^2: \quad \max(C_{1,\nu}^\mu, C_{2,\nu}^\mu) \le D_\nu^\mu, \tag{2.11}$$

$$\forall C_1, C_2 \in \mathcal{R}, \exists D \in \mathcal{R}, \forall \mu,\nu \in \mathbb{N}^2: \quad C_{1,\nu}^\mu C_{2,\nu}^\mu \le D_\nu^\mu. \tag{2.12}$$

EXAMPLE 25.

(i) The set $\mathcal{B}$ of bounded sequences, increasing in both indices, is a regular subset of the subset of $\mathbb{R}_+^{\mathbb{N}^2}$ of all sequences increasing in both indices, which is itself regular.

(ii) The set $\mathcal{B}_1$ (resp. $\mathcal{B}_2$) of increasing sequences depending only on μ (resp. ν) is regular.

With the notations and the background of the previous subsection, we set, for any $\Omega \in \mathcal{O}$ and any regular subset $\mathcal{R}$,

$$\mathcal{F}_r^{\mathcal{R}}(\overleftarrow{E}(\Omega)) = \{f \in \overleftarrow{E}^{\mathbb{N}}(\Omega) \mid \exists C \in \mathcal{R}, \forall \mu,\nu \in \mathbb{N}: \|f\|_{p_\nu^\mu,r}^\Omega < C_\nu^\mu\}.$$

PROPOSITION 26. *Assume that property* (2.5) *holds* (*resp. that* $\overleftarrow{E}$ *allows for partitions of unity and that properties* (2.5) *and* (2.7) *hold*). *Then* $\mathcal{F}_r^{\mathcal{R}}(\overleftarrow{E}) : \Omega \to \mathcal{F}_r^{\mathcal{R}}(\overleftarrow{E}(\Omega))$ *defines a subpresheaf* (*resp. subsheaf*) *of subalgebras of* $\mathcal{F}_r(\overleftarrow{E})$.

The algebraic properties of $\mathcal{F}_r^{\mathcal{R}}(\overleftarrow{E}(\Omega))$ come directly from properties (2.10–2.12) in Definition 24, whereas the proof of presheaf (resp. sheaf) properties follows the same lines as in Proposition 18 (resp. Propositions 18 and 20).

Under the assumptions of Proposition 26, the presheaf (resp. sheaf)

$$\mathcal{G}_r^{\mathcal{R}}(\overleftarrow{E}) = \mathcal{F}_r^{\mathcal{R}}(\overleftarrow{E})/\mathcal{K}_r(\overleftarrow{E})$$

is called the *sheaf of* $(r,\mathcal{R})$*-type generalized functions*.

EXAMPLE 27. We consider the sheaf $\mathcal{G}_r$ based on $\mathcal{C}^\infty$, introduced in Example 21, and the regular set $\mathcal{B}_1$ of increasing sequences depending only on μ. Then the subsheaf $\mathcal{G}_r^{\mathcal{B}_1} = \mathcal{G}_r^\infty$ is the sheaf of $\mathcal{G}^\infty$ generalized functions, introduced in [60], and used in local and microlocal study of generalized functions.

EXAMPLE 28. We consider the presheaf $\mathcal{G}_{\mathcal{S},r}$ based on $\mathcal{S}$, introduced in Example 19, and the regular set $\mathcal{B}$ of bounded sequences, increasing in both indices. The subpresheaf $\mathcal{G}_{\mathcal{S},r}^\infty = \mathcal{G}_{\mathcal{S},r}^{\mathcal{B}}$ is used for the characterization of compactly supported $\mathcal{G}^\infty$ generalized functions: A compactly supported generalized function is $\mathcal{G}^\infty$ regular iff its Fourier transform belongs to $\mathcal{G}_{\mathcal{S},r}^\infty(\mathbb{R}^s)$. (See below and [12, 33–35] for more details and applications.)

We assume now that $\overleftarrow{E}$ is a sheaf of algebras and that properties (2.5) and (2.7) hold. Our framework gives the tools for the local study of Colombeau type generalized functions. First, as $\mathcal{G}_r(\overleftarrow{E})$ is a presheaf, the notion of restriction makes sense. Thus, for any regular set $\mathcal{R}$ and $f \in \mathcal{G}_r^{\mathcal{R}}(\overleftarrow{E}(\Omega))$ (Ω an open subset of X), we can define

$$O_{\mathcal{R}}(f) = \{x \in \Omega \mid \exists V \in \mathcal{V}_x : f|_V \in \mathcal{G}_r^{\mathcal{R}}(\overleftarrow{E}(V))\}.$$

From sheaf properties, it follows that $f|_{O_{\mathcal{R}}}$ belongs to $\mathcal{G}_r^{\mathcal{R}}(\overleftarrow{E}(O_{\mathcal{R}}))$ and that $O_{\mathcal{R}}(f)$ is the largest open set of X having this property. We call $O_{\mathcal{R}}(f)$ the (*open*) *set of* $\mathcal{R}$*-regularity* of f and we define

$$\operatorname*{supp\,sing}_{\mathcal{R}}(f) = X \setminus O_{\mathcal{R}}(f).$$

EXAMPLE 29. Returning to Example 27, we define, in particular, the $\mathcal{G}^\infty$ singular support of a generalized function, by choosing $\mathcal{R} = \mathcal{B}_1$.

2.5.2. *Elements of microlocal analysis.* We shall do this study for the case of the sheaf $\mathcal{G}_r$, introduced in Example 21.

Some embedding results. One can show that, for any open subset Ω of $\mathbb{R}^s$, the space $\mathcal{G}_{C,r}(\Omega)$ of compactly supported elements of $\mathcal{G}_r(\Omega)$ is naturally embedded in $\mathcal{G}_{C,r}(\mathbb{R}^s)$, and that $\mathcal{G}_{C,r}(\mathbb{R}^s)$ is embedded in $\mathcal{G}_{\mathcal{S},r}^{\mathcal{B}_2}(\mathbb{R}^s)$. (Recall that $\mathcal{B}_2$ is the set of sequences $(\mu,\nu) \mapsto C_\nu^\mu = C_\nu$, that is, the set of sequences depending only on ν.)

Indeed, for any $f \in \mathcal{G}_{C,r}(\mathbb{R}^s)$, there exists a representative $(f_n)_n \in f$ such that each f_n is supported in the same compact set, which can be included in one of the K_μ. (We refer to Example 21 for the notation, with the simplification $K_\mu = K_\mu^{\mathbb{R}^s}$.) Such a representative is constructed by multiplying any $(g_n)_n \in f$ by a function $\theta \in \mathcal{D}(\mathbb{R}^s)$ satisfying $\theta \equiv 1$ on a neighborhood of $\operatorname{supp}(f)$ and $0 \le \theta \le 1$ elsewhere. Furthermore, for any $(g_n)_n$, the class of $(\theta g_n)_n$ in $\mathcal{G}_{\mathcal{S},r}(\mathbb{R}^s)$ does not depend on the choices of $(g_n)_n$ and θ. We have, with the notations of Examples 19 and 21,

$$\forall \mu,\nu \in \mathbb{N},\ \exists C_\mu > 0,\ \forall f \in \mathcal{D}(\mathbb{R}^s) \text{ with } \operatorname{supp}(f) \subset K_{\mu_0} :$$
$$p_\nu^{\mu_0}(f) \le q_\nu^\mu(f) \le C_\mu p_\nu^{\mu_0}(f). \tag{2.13}$$

From the previous remarks and these inequalities, it is straightforward that the mapping

$$\iota_{C,\mathcal{S}} : \mathcal{G}_{C,r}(\mathbb{R}^s) \to \mathcal{G}_{\mathcal{S},r}(\mathbb{R}^s), \qquad f \mapsto [(f_n)_n]_{\mathcal{S}},$$

(where $(f_n)_n \in f$ is such that each f_n is supported in the same compact set K_{μ_0}) is an injective morphism of algebras.

Furthermore, inequalities (2.13) imply that $\iota_{C,\mathcal{S}}(f) = [(f_n)_n]_{\mathcal{S}}$ satisfies

$$\|\!|\iota_{C,\mathcal{S}}(f)|\!\|_{q_\nu^\mu,r} \le \|\!|f|\!\|_{p_\nu^{\mu_0},r}.$$

Thus, $\iota_{C,\mathcal{S}}(\mathcal{G}_{C,r}(\mathbb{R}^s)) \subset \mathcal{G}_{\mathcal{S},r}^{\mathcal{B}_2}(\mathbb{R}^s)$ as stated above.

Fourier transform. Since the Fourier transform ([4]) $\mathcal{FT} : \mathcal{S}(\mathbb{R}^s) \to \mathcal{S}(\mathbb{R}^s)$ is a linear continuous mapping, there exists a canonical extension (still denoted by $\mathcal{FT}$) defined by

$$\mathcal{FT} : \mathcal{G}_{\mathcal{S},r}(\mathbb{R}^s) \to \mathcal{G}_{\mathcal{S},r}(\mathbb{R}^s), \qquad f \mapsto [(\mathcal{FT}(f_n))_n]_{\mathcal{S}},$$

([4]) We denote the Fourier transform by $\mathcal{FT}$ to avoid confusion with the spaces $\mathcal{F}_{p,r}$ and related functorial notation.

where $(f_n)_n$ is a representative of f. Moreover, $\mathcal{FT}$ is a linear isomorphism, continuous for the topology given by the family of ultranorms $(\|\!|\cdot\|\!|_{q_\nu^\mu,r})_{\mu,\nu}$. (See Section 5 for a more general approach to the problem of extension of maps.)

From now on, we call a subset $\mathcal{R}$ of $\mathbb{R}_+^{\mathbb{N}^2}$ *regular* if it satisfies (2.9–2.12) and

$$\forall C\in\mathcal{R},\ \forall\mu_0,\nu_0\in\mathbb{N}^2,\ \exists D\in\mathcal{R},\ \forall\mu,\nu\in\mathbb{N}^2: C_{\nu+\nu_0}^{\mu+\mu_0}\le D_\nu^\mu. \tag{2.14}$$

For $\mathcal{R}\subset\mathbb{R}_+^{\mathbb{N}^2}$, define

$$\check{\mathcal{R}}=\{C\in\mathbb{R}_+^{\mathbb{N}^2}\mid \exists D\in\mathcal{R},\ \forall\mu,\nu\in\mathbb{N}^2: C_\nu^\mu=D_\mu^\nu\}.$$

One can check that a set $\mathcal{R}$ is regular if and only if $\check{R}$ is regular.

With this, we can formulate the following exchange proposition:

Proposition 30. *Let $\mathcal{R}$ be a regular set. Then*

$$\mathcal{FT}(\mathcal{G}_{\mathcal{S},r}^{\mathcal{R}}(\mathbb{R}^s))=\mathcal{G}_{\mathcal{S},r}^{\check{\mathcal{R}}}(\mathbb{R}^s). \tag{2.15}$$

The proof of Proposition 30 is based on properties of regular sets and on the following classical lemma:

Lemma 31. *For all μ,ν in $\mathbb{N}$, there exists $C_{\mu,\nu}>0$ such that*

$$\forall u\in\mathcal{S}(\mathbb{R}^s):\quad q_\nu^\mu(\mathcal{FT}(u))\le q_\mu^{\nu+s+1}(u).$$

Note that the equality (2.15) also holds for the inverse Fourier transform.

Example 32. Choosing $\mathcal{R}=\mathcal{B}$ gives, in particular, $\mathcal{FT}(\mathcal{G}_{\mathcal{S},r}^{\infty}(\mathbb{R}^s))=\mathcal{G}_{\mathcal{S},r}^{\infty}(\mathbb{R}^s)$, since $\check{\mathcal{B}}=\mathcal{B}$.

The following proposition gives a characterization of regular compactly supported generalized functions by a regularity property of their Fourier transform. This is an analogue in the framework of generalized functions of the similar result asserting that a compactly supported distribution is a smooth function if and only if its Fourier transform (which is *a priori* a slowly increasing function) is rapidly decreasing.

Proposition 33. *Let $\mathcal{R}_2$ be a regular set, formed by sequences depending only on ν. For $f\in\mathcal{G}_{C,r}(\mathbb{R}^s)$, the following two statements are equivalent*:

(i) *f belongs to $\mathcal{G}_r^{\mathcal{R}_2}(\mathbb{R}^s)$,*
(ii) *$\mathcal{FT}(f)$ belongs to $\mathcal{G}_{\mathcal{S},r}^{\check{\mathcal{R}}_2}(\mathbb{R}^s)$.*

Proof. Let $f\in\mathcal{G}_r^{\mathcal{R}_2}(\mathbb{R}^s)$. A closer inspection of the previous embedding results shows that $\mathcal{G}_r^{\mathcal{R}_2}(\mathbb{R}^s)$ is embedded in $\mathcal{G}_{\mathcal{S},r}^{\mathcal{R}_2}(\mathbb{R}^s)$. Using Proposition 30, we see that $\mathcal{FT}(f)$ belongs to $\mathcal{G}_{\mathcal{S},r}^{\check{\mathcal{R}}_2}(\mathbb{R}^s)$. Conversely, if $\mathcal{FT}(f)\in\mathcal{G}_{\mathcal{S},r}^{\check{\mathcal{R}}_2}(\mathbb{R}^s)$, then $f\in\mathcal{G}_{\mathcal{S},r}^{\mathcal{R}_2}(\mathbb{R}^s)$. Since f is compactly supported, we can find K_{μ_0} such that $\operatorname{supp}(f)\subset K_{\mu_0}$. From the left inequality of (2.13), it follows that $f\in\mathcal{G}_r^{\mathcal{R}_2}(\mathbb{R}^s)$. ■

Remark 34. Taking sequences depending only on ν in Proposition 33 is not a loss of generality, since we consider compactly supported generalized functions.

Indeed, take $f\in\mathcal{G}_{C,r}^{\mathcal{R}}(\mathbb{R}^s)$ with $\operatorname{supp}(f)\subset K_{\mu_0}$ and, for all $\mu,\nu\in\mathbb{N}$, $\|\!|f\|\!|_{p_\nu^\mu,r}\le C_\nu^\mu$, $(C_\nu^\mu)_{\mu,\nu}\in\mathcal{R}$. Then, for all $k\ge\mu_0$, we have

$$\|\!|f\|\!|_{p_\nu^\mu,r}=\|\!|f\|\!|_{p_\nu^{\mu_0},r}\le C_\nu^{\mu_0}.$$

Thus, $\mathcal{G}_{C,r}^{\mathcal{R}}(\mathbb{R}^s) = \mathcal{G}_{C,r}^{\mathcal{R}_2}(\mathbb{R}^s)$ with $\mathcal{R}_2 = \{(C_\nu^{\mu_0})_{\mu,\nu} \mid \mu_0 \in \mathbb{N},\ (C_\nu^{\mu})_{\mu,\nu} \in \mathcal{R}\}$. (The inclusion $\supset$ comes from the monotonicity in μ of the sequence $(C_\nu^{\mu})_{\mu,\nu}$.)

EXAMPLE 35. Take $\mathcal{R}_2 = \mathcal{B}$, the set of increasing bounded sequences, defining the sheaf of algebras $\mathcal{G}_r^{\infty}$. Then $\check{\mathcal{R}}_2 = \mathcal{B}$, thus $\mathcal{G}_{\mathcal{S},r}^{\check{\mathcal{R}}_2}(\mathbb{R}^s) = G_{\mathcal{S},r}^{\infty}(\mathbb{R}^s)$. We recover the characterization of $\mathcal{G}^{\infty}$ regular compactly supported mentioned in Example 28.

Microlocalization. Proposition 33 is a basis of local analysis in this approach to Colombeau generalized functions and justifies the following notions.

Notations. Let Ω be an open subset of $\mathbb{R}^s$. For $(x,\xi) \in \Omega \times \mathbb{R}^s \setminus \{0\}$, we denote by

(i) $\mathcal{V}_x$ (resp. $\mathcal{V}_x^{\Gamma}$) the set of all open neighborhoods (resp. open convex conic neighborhoods) of x (resp. ξ),
(ii) $\mathcal{D}_x(\Omega)$ the set of elements of $\mathcal{D}(\Omega)$ not vanishing at x.

From now on, we fix a regular set $\mathcal{R}$. As we are going to investigate the local behavior of generalized functions, we may assume that sequences in $\mathcal{R}$ only depend on ν, according to Remark 34. For $f \in \mathcal{G}_{C,r}(\Omega)$, we set

$$\mathcal{O}_{\mathcal{R}}^{\Gamma}(f) = \{\xi \in \mathbb{R}^s \setminus \{0\} \mid \exists \Gamma \in \mathcal{V}_x^{\Gamma} : \mathcal{FT}(f)|_{\Gamma} \in \mathcal{G}_{\mathcal{S},r}^{\check{\mathcal{R}}}(\Gamma)\}.$$

LEMMA 36. *For $f \in \mathcal{G}_{C,r}(\mathbb{R}^s)$ and $\varphi \in \mathcal{D}(\mathbb{R}^s)$, we have $\mathcal{O}_{\mathcal{R}}^{\Gamma}(f) \subset \mathcal{O}_{\mathcal{R}}^{\Gamma}(\varphi f)$.*

The proof follows the same lines as the one of Lemma 27 in [12].

Let $\mathcal{R}$ be a regular set and Ω a subset of $\mathbb{R}^s$.

DEFINITION 37. A function f in $\mathcal{G}_r(\Omega)$ is said to be $\mathcal{R}$-*microregular* at $(x,\xi) \in \Omega \times \mathbb{R}^s \setminus \{0\}$ if there exist $\varphi \in \mathcal{D}_x(\Omega)$ and $\Gamma \in \mathcal{V}_x^{\Gamma}$ such that $\mathcal{FT}(\varphi f)|_{\Gamma} \in \mathcal{G}_{\mathcal{S},r}^{\check{\mathcal{R}}}(\Gamma)$.

We set, for f in $\mathcal{G}_r(\Omega)$,

$$\mathcal{O}_{\mathcal{R},x}^{\Gamma}(f) = \bigcup_{\varphi \in \mathcal{D}_x(\Omega)} \mathcal{O}_{\mathcal{R}}^{\Gamma}(\varphi f) = \{\xi \in \mathbb{R}^s \setminus \{0\} \mid f \text{ is } \mathcal{R} \text{ microregular at } (x,\xi)\},$$

$$\Sigma_{\mathcal{R},x}^{\Gamma}(f) = \bigcap_{\varphi \in \mathcal{D}_x(\Omega)} \mathcal{O}_{\mathcal{R}}^{\Gamma}(\varphi f) = (\mathbb{R}^s \setminus \{0\}) \setminus \mathcal{O}_{\mathcal{R},x}^{\Gamma}(f).$$

DEFINITION 38. For f in $\mathcal{G}_r(\Omega)$, the set

$$WF_{\mathcal{R}}(f) = \{(x,\xi) \in \Omega \times (\mathbb{R}^s \setminus \{0\}) \mid \xi \in \Sigma_{\mathcal{R},x}^{\Gamma}(f)\}$$

is called the $\mathcal{R}$ *wavefront* of f.

The following proposition establishes the link between the $\mathcal{R}$ wavefront and the $\mathcal{R}$ singular support of f.

PROPOSITION 39. *For f in $\mathcal{G}_r(\Omega)$, the projection on the first component of $WF_{\mathcal{R}}(f)$ is equal to* $\operatorname{supp}\operatorname{sing}_{\mathcal{R}}(f)$.

The proof follows the same lines as the one of Lemma 8.1.1 in [32] which concerns the same result for the $\mathcal{C}^{\infty}$ wavefront of a distribution. The key point is given by Lemma 36 or its analogue for the distributional case.

EXAMPLE 40. Taking $\mathcal{R} = \mathcal{B}$, the set of bounded sequences, we recover the $\mathcal{G}^{\infty}$ wavefront of a Colombeau generalized function.

3. Embeddings

We already showed through examples that various definitions of Colombeau algebras $\bar{\mathbb{C}}$ and $\mathcal{G}$ can be realized through sequence spaces corresponding to the sequence $r_n = 1/\log n$. The embedding of Schwartz distributions and of smooth functions into $\mathcal{G}$ is well-known (see Example 8 and [31, 59]). It is also well-known that the multiplication of smooth functions embedded into $\mathcal{G}$ is the usual multiplication, i.e. it commutes with the (canonical "constant") embedding.

In this section we deal with some classes of ultradistributions and periodic hyperfunctions. We will apply the general construction given in Section 2, and now study embeddings and the multiplication of regular elements embedded into the corresponding sequence space.

3.1. General remarks on embeddings of duals. Under mild assumptions on $\overleftrightarrow{E}$, we show that our algebras of classes of sequences contain embedded elements of strong dual spaces $\overleftrightarrow{E}'$. First, we consider the embedding of the delta distribution. We show that general assumptions on test spaces and on a delta sequence lead to the unboundedness of this sequence in $\overleftrightarrow{E}$.

We assume that $\overleftrightarrow{E}$ is dense and continuously embedded in one of the following spaces: $F = \mathcal{C}^0(\mathbb{R}^s)$, the space of continuous functions with the projective topology given by sup norms on the balls $B(0,n)$, $n \in \mathbb{N}^*$, or $F = \mathcal{K}(\mathbb{R}^s) = \operatorname{ind\,lim}_{n\to\infty}(\mathcal{K}_n, \|\cdot\|_\infty)$, where

$$\mathcal{K}_n = \{\psi \in \mathcal{C}(\mathbb{R}^s) \mid \operatorname{supp}\psi \subset B(0,n)\}.$$

(Recall that $\mathcal{K}'(\mathbb{R}^s)$ is the space of Radon measures.)

In both cases we have $\delta \in F'$ and therefore also $\delta \in \overleftrightarrow{E}'$.

Proposition 41. *Consider a sequence $(\delta_n)_n \in \overleftrightarrow{E}^{\mathbb{N}}$, converging weakly to δ in $\overleftrightarrow{E}'$, i.e. for all $\psi \in \overleftrightarrow{E}$ the integral $\int_{\mathbb{R}^s} \delta_n(x)\psi(x)\,dx$ is defined and tends to $\psi(0)$ as $n \to \infty$. Then $(\delta_n)_n$ cannot be bounded in $\overleftrightarrow{E}$ in any of the following cases:*

(i) *$F = \mathcal{C}^0(\mathbb{R}^s)$ and $\forall n \in \mathbb{N} : \delta_n \in F'$ and*

$$\exists M > 0,\ \forall n \in \mathbb{N} : \quad \sup_{|x|>M} |\delta_n(x)| < M.$$

(ii) *$F = \mathcal{K}(\mathbb{R}^s)$ and there exists a compact set K such that $\forall n \in \mathbb{N}^* : \operatorname{supp}\delta_n \subset K$.*

(iii) *$\overleftrightarrow{E}$ is sequentially weakly dense in $\overleftrightarrow{E}'$ and*

1. *every $\phi \in \overleftrightarrow{E}$ defines an element of F' by $\psi \mapsto \int_{\mathbb{R}^s} \phi(x)\psi(x)\,dx$,*
2. *if $(\phi_n)_n$ is a bounded sequence in $\overleftrightarrow{E}$, then $\sup_{n\in\mathbb{N},\, x\in\mathbb{R}^s} |\phi_n(x)| < \infty$.*

Proof. We will give the proof for (i) and (iii).

(i) Let us show that $(\delta_n)_n$ is not bounded in $\overleftrightarrow{E}$. First, consider $\overleftarrow{E}$. Boundedness of $(\delta_n)_n$ in $\overleftarrow{E}$ would imply: $\forall \mu \in \mathbb{N}$, $\forall \nu \in \mathbb{N}$, $\exists C_1 > 0$, $\forall n \in \mathbb{N} : p^\mu_\nu(\delta_n) < C_1$. Continuity of $\overleftarrow{E} \hookrightarrow \mathcal{C}^0(\mathbb{R}^s)$ gives

$$\forall k \in \mathbb{N},\ \exists \mu \in \mathbb{N},\ \exists \nu \in \mathbb{N},\ \exists C_2 > 0,\ \forall \psi \in \overleftarrow{E} : \quad \sup_{|x|<k} |\psi(x)| \le C_2 p^\mu_\nu(\psi).$$

It follows that $\exists C > 0$, $\forall n \in \mathbb{N} : \sup_{x\in\mathbb{R}^s} |\delta_n(x)| < C$, which is impossible. To show this, take $\psi \in \mathcal{C}^0(\mathbb{R}^s)$ positive and such that $\psi(0) = C + 1$ and $\int \psi < 1$. The assumption

$\delta_n \in F'$ implies that it acts on $\mathcal{C}^0(\mathbb{R}^s)$ by $\psi \mapsto \int \delta_n(x)\psi(x)\,dx$. This gives $C+1 = \psi(0) = \lim_{n\to\infty} |\int \delta_n \psi\,dx| \le C$.

For $\overrightarrow{E}$, simply exchange $\forall\nu \leftrightarrow \exists\nu$ in the above.

(iii) Assumption 2 and boundedness of $(\delta_n)_n$ in $\overset{\leftrightarrows}{E}$ would imply that $\exists C > 0, \forall n \in \mathbb{N}: \sup_{x\in\mathbb{R}^s} |\delta_n(x)| < C$. Then, by assumption 1 we conclude the proof as in (i). ∎

Remark 42. One can take for $\overset{\leftrightarrows}{E}$ one of Schwartz test function spaces or the Beurling or Roumieau test function space of ultradifferentiable functions. Since the delta distribution lives on all functions which are continuous at zero, one can also consider F and $\overset{\leftrightarrows}{E}$ to consist of holomorphic functions with appropriate topologies. This was the reason for considering $\mathcal{C}^0$, although there are many classes of test spaces which would imply the necessary accommodation of conditions of the previous assertion.

Thus, the appropriate choice of a sequence r decreasing to 0 appears to be important to have at least δ embedded into the corresponding algebra. It can be chosen such that for all $\mu \in \mathbb{N}$ and all $\nu \in \mathbb{N}$ (resp. some $\nu \in \mathbb{N}$ in the $\overrightarrow{E}$ case), $\limsup_{n\to\infty} p_\nu^\mu(\delta_n)^{r_n} = A_\nu^\mu$ and $\exists \mu_0, \nu_0 : A_{\nu_0}^{\mu_0} \neq 0$.

So the embedding of duals into corresponding algebras is realized on the basis of two demands:

(i) $\overset{\leftrightarrows}{E}$ is weakly sequentially dense in $\overset{\leftrightarrows}{E}'$.

(ii) There exists a sequence $(r_n)_n$ decreasing to zero such that for all $f \in \overset{\leftrightarrows}{E}'$ and corresponding sequence $(f_n)_n$ in $\overset{\leftrightarrows}{E}$, with $f_n \to f$ weakly in $\overset{\leftrightarrows}{E}'$, we have for all μ and all ν (resp. some ν), $\limsup_{n\to\infty} p_\nu^\mu(f_n)^{r_n} < \infty$.

Remark 43. In the definition of our sequence spaces $\overrightarrow{\mathcal{F}}_{p,r}$ (resp. $\overleftarrow{\mathcal{F}}_{p,r}$), we assumed $r_n \searrow 0$ as $n \to \infty$. (Later, we will have *families* of sequences decreasing to 0.)

In principle, one could consider more general sequences of weights. For example, if $r_n \in (\alpha, \beta)$, $0 < \alpha < \beta$, then $\overset{\leftrightarrows}{E}$ can be embedded, in the set-theoretical sense, via the canonical map $f \mapsto (f)_n$ $(f_n = f)$. If $r_n \to \infty$, $\overset{\leftrightarrows}{E}$ is no more included in $\overset{\leftrightarrows}{\mathcal{F}}_{p,r}$.

In the case we are considering $(r_n \to 0)$, the induced topology on $\overset{\leftrightarrows}{E}$ is obviously a discrete topology. But this is necessarily so, since we want to have "divergent" sequences in $\overset{\leftrightarrows}{\mathcal{F}}_{p,r}$. Thus, in order to have an appropiate topological algebra containing "δ", it is unavoidable that our generalized topological algebra induces a discrete topology on the original algebra $\overset{\leftrightarrows}{E}$.

In some sense, in our construction this is the price to pay, in analogy to Schwartz' impossibility statement for multiplication of distributions [73].

3.2. Colombeau ultradistributions of Gevrey class. In [67], we constructed Colombeau type algebras of ultradistributions with general sequences M_p, $p \in \mathbb{N}$, satisfying assumptions (M.1), (M.2) and (M.3)$'$ ([40], [65]). Here, we will consider the case $M_p = p!^m$, where $m > 1$. In some sense, we will simplify the situation considered in [67], but at the same time improve significantly the assertions of [67]. To do so, we cast the whole theory into the sequence space framework of this paper.

In the next example, we give the realisation of the ring of ultracomplex numbers through the quotient of corresponding sequence spaces.

EXAMPLE 44. Consider the sequence $\forall n \in \mathbb{N}^* : r_n = 1/n^{1/m}$ with some fixed $m > 0$. With this sequence and $E = \mathbb{C}$, $p = |\cdot|$ (absolute value), one obtains the ultracomplex numbers $\mathcal{F}_{|\cdot|,r} / \mathcal{K}_{|\cdot|,r} = \overline{\mathbb{C}}^{p!^m}$; cf. [67] ($m > 1$), [77] ($m \le 1$). We will use the notation $\mathcal{F}_{|\cdot|,r} = \mathcal{E}_0^{p!^m}$, $\mathcal{K}_{|\cdot|,r} = \mathcal{N}_0^{p!^m}$.

Now we will apply our constructions of Section 2. For the function space $E = \mathcal{C}^\infty(\mathbb{R}^s)$, we define the following sequences of seminorms, for all $\mu, \nu \in \mathbb{R}_+$ and $m > 1$:

$$p_\nu^{m,\mu}(f) = \sup_{|x|\le\mu,\, \alpha\in\mathbb{N}^s} \frac{\nu^{|\alpha|}}{\alpha!^m} |f^{(\alpha)}(x)|, \qquad q_\nu^{m,\mu} = p_{1/\nu}^{m,\mu},$$

and let, for $\mu, \nu \in \mathbb{N}$, $E_\nu^\mu = E_{p_\nu^{m,\mu}}$ (resp. $E_\nu^\mu = E_{q_\nu^{m,\mu}}$) be the subset of E on which the given seminorm is finite.

For the first case, we clearly have $E_\nu^{\mu+1} \hookrightarrow E_\nu^\mu$, $E_{\nu+1}^\mu \hookrightarrow E_\nu^\mu$ for any $\mu, \nu \in \mathbb{N}$, and for the second case, we have $E_\nu^{\mu+1} \hookrightarrow E_\nu^\mu$, $E_\nu^\mu \hookrightarrow E_{\nu+1}^\mu$ for any $\mu, \nu \in \mathbb{N}$.

Denote by $D_{p_\nu^{m,\mu}}$ (resp. $D_{q_\nu^{m,\mu}}$) the subspace of $E_{p_\nu^{m,\mu}}$ (resp. $E_{q_\nu^{m,\mu}}$) consisting of smooth functions supported by the ball $\{|x| \le \nu\}$.

Recall (cf. [41]) that

$$\mathcal{E}^{(m)} = \operatorname*{proj\,lim}_{\mu\to\infty} \mathcal{E}^{(m,\mu)} = \operatorname*{proj\,lim}_{\mu\to\infty} \operatorname*{proj\,lim}_{\nu\to\infty} E_{p_\nu^{m,\mu}},$$
$$\mathcal{D}^{(m)} = \operatorname*{ind\,lim}_{\mu\to\infty} \mathcal{D}^{(m,\mu)} = \operatorname*{ind\,lim}_{\mu\to\infty} \operatorname*{proj\,lim}_{\nu\to\infty} D_{p_\nu^{m,\mu}},$$

resp.

$$\mathcal{E}^{\{m\}} = \operatorname*{proj\,lim}_{\mu\to\infty} \mathcal{E}^{\{m,\mu\}} = \operatorname*{proj\,lim}_{\mu\to\infty} \operatorname*{ind\,lim}_{\nu\to\infty} E_{q_\nu^{m,\mu}},$$
$$\mathcal{D}^{\{m\}} = \operatorname*{ind\,lim}_{\mu\to\infty} \mathcal{D}^{\{m,\mu\}} = \operatorname*{ind\,lim}_{\mu\to\infty} * \operatorname*{ind\,lim}_{\nu\to\infty} D_{q_\nu^{m,\mu}}.$$

These are spaces of ultradifferentiable functions of Beurling, respectively Roumieu type; their duals are spaces of compactly supported Beurling ultradistributions and (general) Beurling ultradistributions, respectively of compactly supported Roumieu ultradistributions and (general) Roumieu ultradistributions.

Take $m > 1$, $m' > 0$, $r_n = n^{-1/m'}$, and let $f = (f_n)_n$ be a sequence of smooth functions on $\mathbb{R}^s$. Let

$$\|\!|f|\!\|_{p_\nu^{m,\mu},m'} = \limsup_{n\to\infty} [p_\nu^{m,\mu}(f_n)]^{n^{-1/m'}},$$
$$\|\!|f|\!\|_{q_\nu^{m,\mu},m'} = \limsup_{n\to\infty} [q_\nu^{m,\mu}(f_n)]^{n^{-1/m'}}.$$

DEFINITION 45. The sets of exponential growth order ultradistribution nets and null nets of Beurling type are defined, respectively, by

$$\overleftarrow{\mathcal{F}}_{p,r} = \mathcal{E}_{\exp}^{(p!^m, p!^{m'})} = \{f = (f_n)_n \mid \forall\mu,\, \forall\nu : \|\!|f|\!\|_{p_\nu^{m,\mu},m'} < \infty\},$$
$$\overleftarrow{\mathcal{K}}_{p,r} = \mathcal{N}^{(p!^m, p!^{m'})} = \{f = (f_n)_n \mid \forall\mu,\, \forall\nu : \|\!|f|\!\|_{p_\nu^{m,\mu},m'} = 0\}.$$

The sets of exponential growth order ultradistribution nets and null nets of Roumieu type are defined, respectively, by

$$\vec{\mathcal{F}}_{q,r} = \mathcal{E}_{\exp}^{\{p!^m,p!^{m'}\}} = \{f = (f_n)_n \mid \forall \mu, \exists \nu : \|\!|f|\!\|_{q_\nu^{m,\mu},m'} < \infty\},$$
$$\vec{\mathcal{K}}_{q,r} = \mathcal{N}^{\{p!^m,p!^{m'}\}} = \{f = (f_n)_n \mid \forall \mu, \exists \nu : \|\!|f|\!\|_{q_\nu^{m,\mu},m'} = 0\}.$$

Recall [41] that an operator of the form $P(D) = \sum_{k\in\mathbb{N}} a_k D^k$ is called an *ultradifferential operator of class* (m) (resp. *of class* $\{m\}$) if there exist $h > 0$, $B > 0$ (resp. for every $h > 0$ there exists $B > 0$) such that

$$\forall k \in \mathbb{N} : \quad |a_k| \le B h^{|k|}/k!^m. \tag{3.1}$$

Proposition 46.

(i) $\mathcal{E}_{\exp}^{(p!^m,p!^{m'})}$ *and* $\mathcal{E}_{\exp}^{\{p!^m,p!^{m'}\}}$ *are algebras under pointwise multiplication, and* $\mathcal{N}^{(p!^m,p!^{m'})}$ *(resp.* $\mathcal{N}^{\{p!^m,p!^{m'}\}}$*) are ideals of these algebras.*

(ii) *The pseudodistances induced by* $\|\!|\cdot|\!\|_{p_\nu^{m,\mu},m'}$ *(resp.* $\|\!|\cdot|\!\|_{q_\nu^{m,\mu},m'}$*) are ultrapseudometrics on the respective domains.*

(iii) $\mathcal{E}_{\exp}^{(p!^m,p!^{m'})}$ *(resp.* $\mathcal{E}_{\exp}^{\{p!^m,p!^{m'}\}}$*) are closed under the action of any ultradifferential operator of class* (m) *(resp. of class* $\{m\}$*).*

The Colombeau ultradistribution algebras $\mathcal{G}^{(p!^m,p!^{m'})}$ and $\mathcal{G}^{\{p!^m,p!^{m'}\}}$ are defined by

$$\overleftarrow{\mathcal{G}}_{p,r} = \mathcal{G}^{(p!^m,p!^{m'})} = \mathcal{E}_{\exp}^{(p!^m,p!^{m'})}/\mathcal{N}^{(p!^m,p!^{m'})},$$
$$\vec{\mathcal{G}}_{p,r} = \mathcal{G}^{\{p!^m,p!^{m'}\}} = \mathcal{E}_{\exp}^{\{p!^m,p!^{m'}\}}/\mathcal{N}^{\{p!^m,p!^{m'}\}}.$$

These topological algebras are also invariant under the actions of ultradifferential operators of respective classes (m) and $\{m\}$ [41].

Proposition 47. *Let* $m' \ge m'' > 0$. *Then*

$$\mathcal{E}_{\exp}^{\{p!^m,p!^{m'})} \subset \mathcal{E}_{\exp}^{\{p!^m,p!^{m''})}, \qquad \mathcal{N}^{\{p!^m,p!^{m''})} \subset \mathcal{N}^{\{p!^m,p!^{m'})}$$

where we introduced the notation $^{\{\dots)}$ *for either* $^{\{\dots\}}$ *or* $^{(\dots)}$*. Moreover, the injection* $\mathcal{E}_{\exp}^{\{p!^m,p!^{m'})} \hookrightarrow \mathcal{E}_{\exp}^{\{p!^m,p!^{m''})}$ *is continuous. However, we do* ***not*** *have injections of the factor spaces, i.e.* $\mathcal{G}^{\{p!^m,p!^{m'})} \not\hookrightarrow \mathcal{G}^{\{p!^m,p!^{m''})}$*, but we do have natural embeddings of quotient* vector spaces,

$$\mathcal{G}_{\exp}^{\{p!^m,p!^{m''})} = \mathcal{E}_{\exp}^{\{p!^m,p!^{m'})}/\mathcal{N}^{\{p!^m,p!^{m'})} \hookrightarrow \mathcal{E}_{\exp}^{\{p!^m,p!^{m''})}/\mathcal{N}^{\{p!^m,p!^{m'})}$$

and algebras

$$\mathcal{E}_{\exp}^{\{p!^m,p!^{m'})}/\mathcal{N}^{\{p!^m,p!^{m''})} \hookrightarrow \mathcal{E}_{\exp}^{\{p!^m,p!^{m''})}/\mathcal{N}^{\{p!^m,p!^{m''})} = \mathcal{G}_{\exp}^{\{p!^m,p!^{m''})}$$

The left hand side of the last display is thus a subalgebra of $\mathcal{G}_{\exp}^{\{p!^m,p!^{m''})}$*, with the property that association with respect to the subspace* $\mathcal{N}^{\{p!^m,p!^{m'})}$ *(see Section* 6*) is compatible with multiplication.*

Proof. The inclusion relation is easy to see. The given injection is continuous, since the topology of the space on the left is stronger than the one on the right. We do not have injections of the factor spaces, since the ideals satisfy the converse inclusion relations: Necessarily, if the space of moderate sequences on the right hand side is bigger (such that it can contain sequences from the l.h.s.), then the ideal on the r.h.s. is smaller than the ideal on the l.h.s.. Thus, the image of the ideal on the left, under the canonical injection, is not included in the ideal on the r.h.s., which means that the injection map cannot be well-defined on the quotient algebras. The algebra embedding is possible since $\mathcal{N}^{\{p!^m, p!^{m''})}$ is also an ideal of the smaller $\mathcal{E}_{\exp}^{\{p!^m, p!^{m'})}$. ∎

REMARK 48. Clearly, one can define spaces of Colombeau ultradistributions on an open subset Ω of $\mathbb{R}^n$. As in the case of conventional Colombeau generalized functions, one can prove that $\Omega \to \mathcal{G}^{\{\cdot,\cdot)}(\Omega)$ constitutes a sheaf which is fine but not flabby (cf. [21, 41] for the definitions and proofs of these properties in ultradistribution spaces).

EXAMPLE 49. We just mention the interesting approach of [3] to ultradistribution generalized functions. Consider the seminorms $p_\nu : \varphi \mapsto \sup_{|\alpha|\le\nu,\, |x|\le\nu} |\varphi^{(\alpha)}(x)|$ and let, for $s > 1$, $r_n^{(s)} = 1/n^{1/s}$ and

$$\mathcal{F}_{p,r^{(s)}}(\Omega) = \{f \in (\mathcal{C}^\infty(\Omega))^{\mathbb{N}} \mid \forall\nu \in \mathbb{N} : \|\!|f|\!\|_{p_\nu, r^{(s)}} < \infty\},$$
$$\mathcal{K}_{p,r^{(s)}}(\Omega) = \{f \in (\mathcal{C}^\infty(\Omega))^{\mathbb{N}} \mid \forall\nu \in \mathbb{N} : \|\!|f|\!\|_{p_\nu, r^{(s)}} = 0\}.$$

With this construction and mollifiers from $\mathcal{S}^{\{s\}}$, embeddings of $\mathcal{D}^{\{2s-1\}}$ and $\mathcal{E}^{\{2s-1\}}$ into the corresponding algebra $\mathcal{G}_{p,r^{(2s-1)}}(\Omega)$ are considered in [3].

3.2.1. *Mollifiers.* The problem of embeddings of various generalized function spaces into corresponding Colombeau type algebras is closely related to the choice of sequences of mollifiers, sequences of appropriately smooth functions converging to the delta distribution. For the embedding of Schwartz distributions and $\mathcal{C}^\infty$, the problem is trivial, while for ultradistributions and ultradifferentiable functions it is essential. The same holds for periodic hyperfunctions of the next subsection.

In the theorems to follow, mollifiers will be constructed by elements of spaces $\Sigma_{\rm der}$ and $\Sigma^{\rm pow}$.

DEFINITION 50. $\Sigma^{\rm pow}$ consists of the smooth functions φ on $\mathbb{R}$ such that for some $b > 0$,

$$\sigma^b(\varphi) = \sup_{\beta\in\mathbb{N},\, x\in\mathbb{R}} \frac{|x^\beta \varphi(x)|}{b^\beta \beta!} < \infty.$$

$\Sigma_{\rm der}$ consists of the smooth functions φ on $\mathbb{R}$ such that for some $b > 0$,

$$\sigma_b(\varphi) = \sup_{\alpha\in\mathbb{N},\, x\in\mathbb{R}} \frac{|\varphi^{(\alpha)}(x)|}{b^\alpha \alpha!} < \infty.$$

Both spaces are endowed with the respective inductive topologies.

Let $m > 1$. Let ϕ^n, $n \in \mathbb{N}$, be a bounded net in $\Sigma^{\rm pow}$ (resp. in $\Sigma_{\rm der}$) such that

$$\forall n \in \mathbb{N} : \int_{\mathbb{R}} \phi^n(t)\, dt = 1, \qquad \int_{\mathbb{R}} t^j \phi^n(t)\, dt = 0, \qquad j = 1, \ldots, [n^{1/m}] + 1.$$

Then $(\phi_n)_n$ is called a *net of* $\{m, \mathrm{pow}\}$*-mollifiers* (resp. $\{m, \mathrm{der}\}$*-mollifiers*), where

$$\forall n \in \mathbb{N}^* : \quad \phi_n = n\phi^n(n\cdot).$$

The essential novelty compared to the construction of ultradistribution algebras of generalized functions in [67] is contained in the previous definition and the next lemma:

LEMMA 51.

(i) *Let* $\forall n\in\mathbb{N}^*$, $x\in\mathbb{R}: h_n(x)=\exp\big(n^2-\sqrt[n]{n^{2n}+x^{2n}}\big)$. *Then*, $\forall n\in\mathbb{N}^*: h_n(0)=1$, $\forall\alpha\in\{1,\dots,2n-1\}: h_n^{(\alpha)}(0)=0$, *and*

$$\exists r>0,\ \exists C>0:\quad \sup_{\alpha,n\in\mathbb{N}}\frac{|h_n^{(\alpha)}(x)|}{r^\alpha\alpha!}<C. \tag{3.2}$$

Moreover, for a given $m>1$, *there exists a function* $g:\mathbb{N}^*\to\mathbb{N}^*$ *so that* (5)

$$\phi^n=\frac{1}{2\pi}\,\mathcal{FT}(h_{g(n)}),\qquad n\in\mathbb{N}^*$$

defines a net of $\{m,\mathrm{pow}\}$*-mollifiers.*

(ii) *Let*

$$\forall n\in\mathbb{N}^*,\ x\in\mathbb{R}:\quad k_n(x)=\exp(-x^{2n}).$$

Then

$$\forall n\in\mathbb{N}^*: k_n(0)=1,\qquad \forall\alpha\in\{1,\dots,2n-1\}: k_n^{(\alpha)}(0)=0,$$

and there exist $r>0$ *and* $C>0$ *such that*

$$\sup_{\beta\in\mathbb{N},\,n\in\mathbb{N}^*}\frac{|x^\beta k_n(x)|}{r^\beta\beta!}<C. \tag{3.3}$$

Moreover, for a given $m>1$ *there exists a function* $g:\mathbb{N}^*\to\mathbb{N}^*$ *so that*

$$\phi^n=\frac{1}{2\pi}\,\mathcal{FT}(k_{g(n)}),\qquad n\in\mathbb{N}^*,$$

defines a net of $\{m,\mathrm{der}\}$*-mollifiers.*

Proof. (i) Clearly, $\widehat{h}_n=\mathcal{FT}(h_n)$ satisfies $\int\widehat{h}_n=1$ and $\int x^m\widehat{h}_n=0$ whenever $1\le m\le 2n-1$, for all $n\in\mathbb{N}^*$.

The function $\mathbb{C}\ni z\mapsto\sqrt[n]{n^{2n}+z^{2n}}$ has singularities at $z=ne^{i\pi(2k+1)/(2n)}$. The nearest one to the real axis x has the imaginary part $n\sin\frac{\pi}{2n}$, greater than 1 for all $n>1$. So for every $x\in\mathbb{R}$, the circle $z=x+e^{i\theta}$, $\theta\in[0,2\pi)$, lies in the domain of analyticity of h_n $(n>1)$. Applying Cauchy's integral formula, we have

$$\forall x\in\mathbb{R},\ \forall n>n_0:\quad |h_n^{(\alpha)}(x)|=\left|\frac{\alpha!}{2\pi i}\int\limits_{|\zeta-x|=1/2}\frac{h_n(\zeta)\,d\zeta}{(\zeta-x)^{\alpha+1}}\right|$$
$$\le 2^\alpha\alpha!\max_{\theta\in[0,2\pi]}|h_n(x+e^{i\theta}/2)|.$$

We will prove that there exists a constant $C>0$ such that

$$\forall n\in\mathbb{N}^*,x\in\mathbb{R}:\quad \mathrm{Re}\left(n^2-n^2\sqrt[n]{1+\left(\frac{x+e^{i\theta}/2}{n}\right)^{2n}}\right)<C, \tag{3.4}$$

such that $|h_n(\dots)|\le e^C$.

CASE 1: $\left|\frac{x+e^{i\theta}/2}{n}\right|\ge\frac{3}{4}$. Let $x+e^{i\theta}/2=\rho(\cos\phi+i\sin\phi)$. For n large enough, since $|x|>(3n-2)/4$, we have $\sin\phi\le 2/(3n)$, and $2n\sin\phi\le 4/3$, so that for some n_0 and $n>n_0$ we obtain $2n\phi\le 4/3+\varepsilon\le\pi/2$. This implies $\mathrm{Re}\big(1+\big(\frac{x+e^{i\theta}/2}{n}\big)^{2n}\big)>1$ and (3.4).

(5) We recall that $\mathcal{FT}$ denotes the Fourier transform.

Case 2: $\left|\frac{x+e^{i\theta}/2}{n}\right| \le \frac{3}{4}$. We use

$$\operatorname{Re} \sqrt[n]{1+\left(\frac{x+e^{i\theta}/2}{n}\right)^{2n}} \ge \sqrt[n]{1-\left(\frac{3}{4}\right)^{2n}} \ge 1-\frac{(3/4)^{2n}}{n}-o(n^{-2})$$

(for n large enough). Again, this implies (3.4) and we have proved that

$$\forall x\in\mathbb{R}, n\in\mathbb{N}^*: \quad \max_{\theta\in[0,2\pi]} |h_n(x+e^{i\theta}/2)| \le 1.$$

This proves (3.2). If $g(n)=\frac{1}{2}[n^{1/(m-1)}]+1$ $(n>n_0)$, then one can easily prove that $\phi^n=\frac{1}{2\pi}\mathcal{FT}(h_{g(n)})$, $n>n_0$, defines a net of $\{m,\mathrm{pow}\}$-mollifiers.

(ii) Again, we have $\int \widehat{k}_n=1$, $\int x^m\widehat{k}_n=0$ $\forall m\le 2n-1$, $n\in\mathbb{N}^*$. Estimating $x^\beta k_n(x)$ separately for $|x|\le 2$ and $|x|>2$ one can easily prove (3.3). Taking the same function g as in (i), we finish the proof of (ii). ■

3.2.2. *Embeddings of ultradifferentiable functions and ultradistributions*

Proposition 52. *Assume $m>1$.*

(i) *Let $\rho>0$ be such that $m-\rho>1$. Let $\psi\in\mathcal{D}^{(m)}$ (resp. $\psi\in\mathcal{D}^{\{m-\rho\}}$). Let $(\phi_n)_n$ be a net of $\{m,\mathrm{pow}\}$-mollifiers. Then*

$$(\psi*\phi_n-\psi)_n\in\mathcal{N}^{(p!^m,p!^m)} \qquad (\phi_n=n\phi^n(n\cdot)),$$
$$(\textit{resp. } (\psi*\phi_n-\psi)_n\in\mathcal{N}^{\{p!^m,p!^m\}}).$$

(ii) *Let $f\in\mathcal{E}'^{(m)}$ (resp. $f\in\mathcal{E}'^{\{m\}}$) and $(\phi_n)_n$ a net of $\{m,\mathrm{der}\}$-mollifiers. Then $(f*\phi_n)\in\mathcal{E}_{\exp}^{(p!^m,p!^{m-1})}$ (resp. $(f*\phi_n)\in\mathcal{E}_{\exp}^{\{p!^m,p!^{m-1}\}}$).*

(iii) *If $(\phi_n)_n$ and $(\phi'_n)_n$ are nets of $\{m,\mathrm{pow}\}$-mollifiers, then*

$$\forall\psi\in\mathcal{D}^{(m)}: (\langle f*\phi_n-f*\phi'_n,\psi\rangle)_n\in\mathcal{N}_0^{p!^m},$$
$$(\textit{resp. } \forall\psi\in\mathcal{D}^{\{m-\rho\}}: (\langle f*\phi_n-f*\phi'_n,\psi\rangle)_n\in\mathcal{N}_0^{p!^m}).$$

Remark 53. If $\psi\in\mathcal{D}^{(m)}$, $m>1$, then $(\psi)_n\in\mathcal{E}^{(p!^m,p!^{m'})}$ for every $m'>0$. Fix a net $(\phi_n)_n$ of $\{m,\mathrm{pow}\}$-mollifiers. The embedding $\mathcal{D}^{(m)}\hookrightarrow\mathcal{E}^{(p!^m,p!^{m'})}$ can be realized through $\psi\mapsto(\psi*\phi_n)_n$ as well as through $\psi\mapsto(\psi)_n$. This is a consequence of assertion (i). A similar conclusion follows for $\mathcal{D}^{\{m-\rho\}}$.

Assertion (ii) characterizes the embedding of elements in $\mathcal{E}'^{(m)}$ (resp. $\mathcal{E}'^{\{m\}}$) into the corresponding algebra by regularizations by $\{m,\mathrm{der}\}$-mollifiers.

The present situation shows again the complexity of the problem of finding suitable mollifiers for a given algebra of generalized functions.

Proof of Proposition 52. (i) Assume $\operatorname{supp}\psi\subset[-\mu,\mu]$. Since $\psi*\phi_n-\psi=0$ for $|x|>\mu$, $n>n_0$, we assume in this proof $x\in[-\mu,\mu]$, $n>n_0$.

First, we prove the assertion for the Beurling case; the Roumieu case is treated in a similar way. Let $s\in\mathbb{N}$. We have

$$\begin{aligned}(\psi*\phi_n-\psi)^{(s)}(x)&=\int_{\mathbb{R}}(\psi^{(s)}(x+t/n)-\psi^{(s)}(x))\phi^n(t)\,dt\\&=\int_{\mathbb{R}}\bigg(\sum_{p=0}^{N-1}\frac{t^p}{n^pp!}\,\psi^{(p+s)}(x)+\frac{t^N}{n^NN!}\,\psi^{(N+s)}(\xi)-\psi^{(s)}(x)\bigg)\phi^n(t)\,dt,\end{aligned}$$

where $x \leq \xi \leq x + t/n$. Let $N = [n^{1/m}] + 1$ as in the definition of $\{m, \mathrm{pow}\}$-mollifiers. We have

$$(\psi * \phi_n - \psi)^{(s)}(x) = \int\limits_{\mathbb{R}} \frac{t^N}{n^N N!}\, \psi^{(N+s)}(\xi)\phi^n(t)\, dt.$$

Let $d > 1$ be such that $\sigma^d(\phi^n) < \infty$. Then

$$\left|\frac{\nu^s}{s!^m}\,(\psi * \phi_n - \psi)^{(s)}(x)\right| \leq \int\limits_{\mathbb{R}} \frac{1}{(N+s)!^m}\, |\psi^{(N+s)}(\xi)|\, \frac{\nu^s (N+s)!^m}{n^N s!^m N!}\, t^N |\phi^n(t)|\, dt.$$

We will use $N!^m \leq (N^N)^m$, $(N+s)! \leq e^{N+s} N! s!$ and $1/n^N \leq 2^N/N^{Nm}$. This gives

$$\left|\frac{\nu^s}{s!^m}\,(\psi * \phi_n - \psi)^{(s)}(x)\right| \leq \int\limits_{\mathbb{R}} \frac{(2e(\nu + d)))^{N+s}}{(N+s)!^m}\, |\psi^{(N+s)}(\xi)|\, \frac{N!^m}{N^{mN}}\, \frac{|t|^N}{d^N N!}\, |\phi^n(t)|\, dt.$$

Let $l > 1$. Inserting $e^{-lN} e^{lN}$, with $\nu_0 = 2le(\nu + d)$, we have

$$\left|\frac{r^s}{s!^m}\,(\psi * \phi_n - \psi)^{(s)}(x)\right| \leq 2^{-lN} p^{m,\mu}_{\nu_0}(\psi)\sigma^d(\phi^n).$$

Now we use $e^{-lN} \sim e^{-ln^{1/m}}$ as $n \to \infty$. This implies that for every $\nu > 0$ and $l > 0$, there exists $C > 0$ such that

$$\left|\frac{\nu^s}{s!^m}\,(\psi * \phi_n - \psi)^{(s)}(x)\right| \leq Ce^{-ln^{1/m}}.$$

Taking the supremum over all s and x, we obtain

$$|\!|\!|\psi * \phi_n - \psi|\!|\!|_{p^{m,\mu}_{\nu}, m} = 0.$$

Now, we prove the assertion for the Roumieu case.

Let $d > 1$ be such that $\sigma^d(\phi^n) < \infty$ and $h > 0$ such that $p^{m-\rho,\mu}_{e^{m-\rho}h}(\psi) < \infty$. We have, as above,

$$\begin{aligned}
&\left|\frac{\nu^s}{s!^m}\,(\psi * \phi_n - \psi)^{(s)}(x)\right| \\
&\qquad \leq \int\limits_{\mathbb{R}} \frac{|\psi^{(N+s)}(\xi)|}{(N+s)!^{m-\rho}}\, \frac{\nu^s (N+s)!^{m-\rho}}{n^N s!^m N!}\, t^N |\phi^n(t)|\, dt \\
&\qquad \leq \int\limits_{\mathbb{R}} \frac{(he^{m-\rho})^{N+s} |\psi^{(N+s)}(\xi)|}{(N+s)!^{m-\rho}}\, \frac{N!^m}{N^{Nm}}\, \frac{(h\nu)^s s!^{m-\rho} (dh)^N}{s!^m N!^\rho}\, \frac{|t|^N}{d^N N!}\, |\phi^n(t)|\, dt.
\end{aligned}$$

Let $l > 1$. Note that

$$\sup\left\{\frac{(h\nu)^s s!^{m-\rho}}{s!^m}\,\middle|\, s \in \mathbb{N}\right\} < \infty, \qquad \sup\left\{\frac{(dhe^l)^N}{N!^\rho}\,\middle|\, N \in \mathbb{N}\right\} < \infty.$$

As above we have, with suitable $C > 0$ (inserting $e^{-lN} e^{lN}$),

$$\left|\frac{\nu^s}{s!^m}\,(\psi * \phi_n - \psi)^{(s)}(x)\right| \leq Ce^{-lN} p^{m-\rho,\mu}_{e^{m-\rho}h}(\psi)\sigma^d(\phi^n).$$

Again as above we finish the proof.

(ii) We will give the proof in the Beurling case. The proof in the Roumieu case is similar.

Recall [40] that if $f \in \mathcal{E}'^{(m)}$, then there exists an ultradifferential operator of class (m), $P(D) = \sum_{k=0}^{\infty} a_k D^k$, $\mu_0 > 0$ and continuous functions F_k, $\operatorname{supp} F_k \subset [-\mu_0, \mu_0]$, $k \in \mathbb{N}$, with the property $\sup_{k\in\mathbb{N},\, x\in\mathbb{R}} |F_k(x)| \le M$ such that $f = \sum_{k=0}^{\infty} a_k D^k F_k$.

This implies

$$\forall x \in \mathbb{R}: \quad f * \phi_n(x) = \sum_{k=0}^{\infty} (-1)^k a_k n^k \int_{\mathbb{R}} F_k(x + t/n) D^k \phi^n(t)\, dt,$$

where $(\phi_n)_n$ is a net of $\{m, \mathrm{der}\}$-mollifiers such that $\sigma_b(\phi^n) < \infty$ and a_k, $k \in \mathbb{N}$, satisfy (3.1). For the same reason as in part (i), we take $x \in [-\mu, \mu]$, $\mu > \mu_0$ and $n > n_0$. Let $\nu > 1$ be given and $s \in \mathbb{N}$. We have

$$\begin{aligned}
\frac{\nu^p}{p!^m} |f^{(p)} * \phi_n(x)| &= \left| \sum_{k=0}^{\infty} (-1)^k a_k n^{k+p} \frac{\nu^p}{p!^m} \int_{\mathbb{R}} F_k(x + t/n) D^{k+p} \phi^n(t)\, dt \right| \\
&\le \sum_{k=0}^{\infty} B \frac{\nu^p h^k n^{k+p}}{k!^m p!^m} \int_{\mathbb{R}} |F_k(x + t/n)|\, |D^{k+p} \phi^n(t)|\, dt \\
&\le \sum_{k=0}^{\infty} B \frac{(\nu h)^{p+k} n^{k+p}}{(k+p)!^m} \int_{\mathbb{R}} |F_k(x + t/n)|\, |D^{k+p} \phi^n(t)|\, dt \\
&\le \sum_{k=0}^{\infty} \frac{1}{2^k} B \frac{(2eb\nu h)^{p+k} n^{k+p}}{(k+p)!^{m-1}} \int_{\mathbb{R}} \frac{|F(x + t/n)|}{b^{k+p}(k+p)!} |D^{k+p} \phi^n(t)|\, dt \\
&\le C e^{(2eb\nu h n)^{1/(m-1)}} \sigma_b(\phi^n).
\end{aligned}$$

This proves that $f * \phi_n \in \mathcal{E}_{\exp}^{(p!^m, p!^{m-1})}$.

Let us prove (for the Beurling case) that

$$\langle f, (\check{\phi}_n - \check{\phi}'_n) * \psi \rangle \in \mathcal{N}_0^{p!^m},$$

where $\check{\phi}(t) = \phi(-t)$. By continuity, we know that there exist $\mu \in \mathbb{N}$, $\nu > 0$ and $C > 0$ such that

$$\begin{aligned}
|\langle f, (\check{\phi}_n - \check{\phi}'_n) * \psi \rangle| &\le C p_\nu^{\mu,m}((\check{\phi}_n - \check{\phi}'_n) * \psi) \\
&\le C[p_\nu^{\mu,m}(\check{\phi}_n * \psi - \psi) + p_\nu^{\mu,m}(\check{\phi}'_n * \psi - \psi)]. \qquad (3.5)
\end{aligned}$$

By the first part of the proposition, we have

$$\psi * \phi_n - \psi,\, \psi * \phi'_n - \psi \in \mathcal{N}^{(p!^m, p!^m)}.$$

This implies that for every $k > 0$, there exists $C > 0$ such that for every $n \in \mathbb{N}$, both summands in (3.5) are less than or equal to $Ce^{-kn^{1/m}}$. ∎

3.3. Generalized hyperfunctions on the circle. In this subsection, we will analyze the sequence space realization of the algebra of Colombeau generalized periodic hyperfunctions [77]. As in the previous subsection, we use the construction from Section 2 (through a “proj ind” type space). Here, Fourier expansions will be the main tool for the analysis.

3.3.1. *Basic spaces of functions on the circle.* First, we recall and specify the relevant material related to hyperfunctions on the unit circle $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$. More details can be found in [4], [39] and mainly in [56]. Let $\Omega_\lambda = \{z \in \mathbb{C} \mid 1/\lambda < |z| < \lambda\}$ where

$\lambda > 1$. We denote by $\mathcal{O}_\lambda$ the Banach space of bounded holomorphic functions in Ω_λ with the sup norm on Ω_λ. The space of analytic functions on $\mathbb{T}$ is $\mathcal{A}(\mathbb{T}) = \operatorname{ind}\lim_{\lambda\to 1} \mathcal{O}_\lambda$.

The space $\mathcal{E}'(\mathbb{T})$ of Schwartz distributions on $\mathbb{T}$ is the strong dual of the space $\mathcal{E}(\mathbb{T})$ of smooth functions on $\mathbb{T}$. To each function $f \in \mathcal{E}(\mathbb{T})$ is associated in a canonical way a function $\tilde{f}$ defined on $\mathbb{R}$ by $\tilde{f}(t) = f(e^{it})$. We set $\|\tilde{f}\|_\infty = \sup_{t\in\mathbb{R}} |\tilde{f}(t)|$.

For $f \in \mathcal{A}(\mathbb{T})$, the coefficient $\widehat{T}(k)$ of z^k in the Laurent expansion of f is its kth Fourier coefficient. Complex numbers $(c_k)_{k\in\mathbb{Z}}$ are the Fourier coefficients of some analytic function if and only if $|\!|\!|c|\!|\!|^{\pm}_{(\cdot)^{-1}} < 1$, with

$$|\!|\!|(c_k)_k|\!|\!|^{\pm}_{(\cdot)^{-\nu}} \equiv \limsup_{k\to\infty} (\max(|c_k|, |c_{-k}|))^{k^{-\nu}},$$

equal to the maximum of $|\!|\!|(c_k)_{k\in\mathbb{N}}|\!|\!|_r$ and $|\!|\!|(c_{-k})_{k\in\mathbb{N}}|\!|\!|_r$ with $r = (r_k) = (k^{-1})$.

Let $m \in [0, 1)$ and $\nu > 0$. We set

$$\mathcal{A}_{m,\nu}(\mathbb{T}) = \left\{ f \in \mathcal{A}(\mathbb{T}) \,\middle|\, q_\nu^{m,\infty}(f) := \sup_{t\in\mathbb{R},\, \alpha\in\mathbb{N}} \frac{|\tilde{f}^{(\alpha)}(t)|}{\nu^\alpha \alpha!^m} < \infty \right\}.$$

If $\nu' > \nu$ then $q_{\nu'}^{m,\infty}(f) \le q_\nu^{m,\infty}(f)$. Hence we define

$$\mathcal{A}_m(\mathbb{T}) = \operatorname*{ind\,lim}_{\nu\to\infty} \mathcal{A}_{m,\nu}(\mathbb{T}) \quad \text{and} \quad \mathcal{A}_1(\mathbb{T}) = \operatorname*{ind\,lim}_{m\to 1} \mathcal{A}_m(\mathbb{T}).$$

When $m \ne 0$, in contrast to $\mathcal{A}_{m,\nu}(\mathbb{T})$, $\mathcal{A}_m(\mathbb{T})$ is a subalgebra of $\mathcal{A}(\mathbb{T})$. Clearly, $\mathcal{A}_1(\mathbb{T})$ is also a subalgebra of $\mathcal{A}(\mathbb{T})$.

For $k \in \mathbb{Z}$ we set $e_k(z) = z^k$. It is immediate to see that e_k belongs to any space $\mathcal{A}_{m,\nu}(\mathbb{T})$. The kth Fourier coefficient of $T \in \mathcal{E}'(\mathbb{T})$ is given by $\widehat{T}(k) = \overline{T(e_k)}$ and $T = \sum_{k\in\mathbb{Z}} \widehat{T}(k) z^k$ in the topology of $\mathcal{E}'(\mathbb{T})$. For a sequence $(A_k)_k$ of complex numbers to be the sequence of Fourier coefficients of a *distribution*, it is necessary and sufficient that $|\!|\!|A|\!|\!|^{\pm}_{1/\log} < \infty$. Moreover, T acts on $f \in \mathcal{E}(\mathbb{T})$ by $T(f) = \sum_{k\in\mathbb{Z}} \overline{\widehat{T}(k)}\, \widehat{f}(k)$.

The space $\mathcal{B}(\mathbb{T})$ of hyperfunctions on the circle is the topological dual $\mathcal{A}'(\mathbb{T})$ of $\mathcal{A}(\mathbb{T})$. For $k \in \mathbb{Z}$ and $H \in \mathcal{B}(\mathbb{T})$, the kth Fourier coefficient of H is $\widehat{H}(k) = \overline{H(e_k)}$, and $H = \sum_{k\in\mathbb{Z}} \widehat{H}(k) z^k$ holds in the topology of $\mathcal{B}(\mathbb{T})$. A sequence $(B_k)_k$ of complex numbers is the sequence of Fourier coefficients of some hyperfunction if and only if $|\!|\!|B|\!|\!|^{\pm}_{(\cdot)^{-1}} \le 1$. If $f \in \mathcal{A}(\mathbb{T})$, then $H(f) = \sum_{k\in\mathbb{Z}} \overline{\widehat{H}(k)}\, \widehat{g}(k)$.

The convolution $S * T$ of two hyperfunctions S and T is given by

$$(S * T)(z) = \sum_{k\in\mathbb{Z}} \widehat{S}(k)\widehat{T}(k) z^k,$$

for z belonging to some neighborhood of $\mathbb{T}$. It is seen that $S * f \in \mathcal{A}(\mathbb{T})$ if $S \in \mathcal{B}(\mathbb{T})$ and $f \in \mathcal{A}(\mathbb{T})$. In the same way $S * f \in \mathcal{E}(\mathbb{T})$ if $S \in \mathcal{E}'(\mathbb{T})$ and $f \in \mathcal{E}(\mathbb{T})$.

3.3.2. *Fourier expansion in* $\mathcal{A}_{m,\nu}(\mathbb{T})$. The following lemma will be useful.

LEMMA 54. *Let* $m \in (0, 1)$ *and* $\rho > e/2$. *The function*

$$\varphi : t \mapsto \rho^{-t} t^{m(t+1/2)} e^{-mt}, \qquad t \in (0, \infty),$$

reaches its minimum at a unique point t_ρ *such that* $1/2 < t_\rho < \rho^{1/m} - 1/2$, *and we have*

$$\sqrt{\rho}\, e^{-m(1/2+\rho^{1/m})} < \varphi(t_\rho) < \varphi(\rho^{1/m} - 1/2) \le \sqrt{\rho}\, e^{-m\rho^{1/m}}.$$

Moreover, $\varphi(\rho^{1/m} + 1/2) < \sqrt{e\rho}\, e^{-m\rho^{1/m}}$.

Proof. The derivative of $\psi = \ln\varphi$ is given by $\psi'(t) = -\ln\rho + m(\ln t + \frac{1}{2t})$, and satisfies $\psi'(t) = 0 \Leftrightarrow te^{1/2t} = \rho^{1/m}$. Since $te^{1/2t} \geq e/2$, it follows that there exists a unique point $t_\rho \in (1/2,\infty)$ such that $t_\rho e^{1/2t_\rho} = \rho^{1/m}$, because $\rho > e/2$. This yields $\rho^{1/m} - t_\rho = t_\rho(e^{1/2t_\rho} - 1)$, and, using $x < e^x - 1 < xe^x$ for $x \neq 0$, the claimed inequalities on t_ρ. Writing $\ln(\rho^{1/m} + 1/2) = (1/m)\ln\rho + \ln(1 + 1/2\rho^{1/m})$ gives

$$\psi\bigg(\rho^{1/m} + \frac{1}{2}\bigg) = \frac{1}{2}\ln\rho + m(\rho^{1/m} + 1)\ln\bigg(1 + \frac{1}{2\rho^{1/m}}\bigg) - m\bigg(\rho^{1/m} + \frac{1}{2}\bigg).$$

We find

$$\varphi(t_\rho) = \sqrt{\rho}\, e^{-m(t_\rho + 1/2 + 1/4t_\rho)} \quad \text{and} \quad t_\rho + \frac{1}{2} + \frac{1}{4t_\rho} < \rho^{1/m} + \frac{1}{2},$$

showing that $\sqrt{\rho}\, e^{-m(1/2+\rho^{1/m})} < \varphi(t_\rho)$. Since $\ln(1+1/2\rho^{1/m}) \leq 1/2\rho^{1/m}$, it follows that

$$\psi\bigg(\rho^{1/m} + \frac{1}{2}\bigg) \leq \frac{1}{2}\ln\rho + m\bigg(\frac{1}{2\rho^{1/m}} - \rho^{1/m}\bigg).$$

Using $\rho > e/2$ and $m \in (0,1)$, we find $m/2\rho^{1/m} < 1/2$ and thus $\varphi(\rho^{1/m} + 1/2) \leq \sqrt{e\rho}e^{-m\rho^{1/m}}$.

We show in the same way that $\varphi(\rho^{1/m} - 1/2) \leq \sqrt{\rho}\, e^{-m\rho^{1/m}}$. ■

We give growth conditions on the Fourier coefficients of elements of $\mathcal{A}_{m,\nu}(\mathbb{T})$ for $m \in [0,1)$.

PROPOSITION 55. *Let $f \in \mathcal{A}(\mathbb{T})$ and $m \in (0,1)$.*

(i) *If $f \in \mathcal{A}_{m,\nu}(\mathbb{T})$ then*

$$\|(\hat{f}(k))_k\|^{\pm}_{(\cdot)^{-1/m}} \leq e^{-m/\nu^{1/m}}.$$

Conversely, if the above condition holds, then $f \in \mathcal{A}_{m,\nu'}(\mathbb{T})$ for all $\nu' > \nu$.

(ii) *$f \in \mathcal{A}_m(\mathbb{T})$ if and only if*

$$\|(\hat{f}(k))_k\|^{\pm}_{(\cdot)^{-1/m}} < 1.$$

(iii) *$f \in \mathcal{A}_{0,\nu}(\mathbb{T})$ if and only if $\hat{f}(k) = 0$ for $|k| > \nu$.*

(iv) *$f \in \mathcal{A}_0(\mathbb{T})$ if and only if $(\hat{f}(k))_{k\in\mathbb{Z}}$ has finite support.*

(v) *For all $f \in \mathcal{A}_1(\mathbb{T})$ there exists $g \in \mathcal{O}(\mathbb{C}^*)$ such that $g|_\mathbb{T} = f$.*

Proof. Let $f \in \mathcal{A}_{m,\nu}(\mathbb{T})$ with $0 < m < 1$. For all $\alpha \in \mathbb{N}$, $\tilde{f}^{(\alpha)}(t) = \sum_{p\in\mathbb{Z}}(ip)^\alpha \hat{f}(p)e^{ipt}$. It follows that $\int_{-\pi}^{\pi} \tilde{f}^{(\alpha)}(t)^{-ikt}dt = 2\pi(ik)^\alpha \hat{f}(k)$, thus there is a positive constant C_1 such that $|k|^\alpha|\hat{f}(k)| \leq C_1\nu^\alpha\alpha!^m$.

Using Stirling's formula, $\alpha! = \alpha^{\alpha+1/2}e^{-\alpha}\sqrt{2\pi}(1+\varepsilon_\alpha)$, $\varepsilon_\alpha \searrow 0$, we find a positive constant C_2 such that

$$\forall\alpha \in \mathbb{N}^*, \forall k \in \mathbb{Z}: \quad |k|^\alpha|\hat{f}(k)| \leq C_2\nu^\alpha\alpha^{m(\alpha+1/2)}e^{-m\alpha}.$$

It follows that

$$\forall\alpha \in \mathbb{N}^*, \forall k \in \mathbb{Z}^*: \quad |\hat{f}(k)| \leq C_2(\nu/|k|)^\alpha\alpha^{m(\alpha+1/2)}e^{-m\alpha}.$$

Using the notations of Lemma 54 and taking $\rho = |k|/\nu$ with $|k| > e\nu/2$ yields $|\hat{f}(k)| \leq C_2\varphi(t)$ for all $t \in \mathbb{N}^*$ and we have $\varphi(\rho^{1/m} + 1/2) \leq \sqrt{\rho e}\, e^{-m\rho^{1/m}}$. Since φ increases on $[\rho^{1/m} - 1/2, \rho^{1/m} + 1/2]$ which contains a positive integer α_ρ, we obtain $|\hat{f}(k)| \leq$

$C_2\varphi(\alpha_\rho) \le C_2\sqrt{\rho e}\, e^{-m\rho^{1/m}}$ for $|k| > e\nu/2$. Hence, there exists a positive constant C such that $\forall k \in \mathbb{Z}^* : |\hat{f}(k)| \le C\sqrt{|k|}e^{-\gamma|k|^{1/m}}$. As $\|\sqrt{k}\| = 1$, we have the inequality of (i).

Conversely, assume that f satisfies the condition of (i). For all $\alpha \in \mathbb{N}$, we have $\tilde{f}^{(\alpha)}(t) = \sum_{k\in\mathbb{Z}}(ik)^\alpha \hat{f}(k)e^{ikt}$. Let $\nu' > \nu$. Choose ν' such that $\nu' > \nu'' > \nu$ and set $\beta' = m/(\nu')^{1/m}$, $\beta'' = m/(\nu'')^{1/m}$. It follows that for $\alpha \neq 0$,

$$\forall k \in \mathbb{Z}^* : \quad |\hat{f}(k)| \le C\sqrt{|k|}e^{-\beta'|k|^{1/m}}.$$

This last inequality gives

$$\|\tilde{f}^{(\alpha)}\|_\infty \le C\Big(\sum_{k\in\mathbb{Z}} e^{-(\beta''-\beta')|k|^{1/m}}\Big) \sup_{k\in\mathbb{Z}} |k|^{\alpha+1/2}e^{-\beta'|k|^{1/m}}.$$

Let $\phi(t) = t^{\alpha+1/2}e^{-\beta t^{1/m}}$, $t \ge 0$. A simple study of ϕ shows that

$$\sup_{t\ge0} \phi(t) = \phi(\nu'(\alpha+1/2)^m) = (\nu'(\alpha+1/2))^{m(\alpha+1/2)}e^{-m(\alpha+1/2)}.$$

Since $\big(\frac{\alpha+1/2}{\alpha}\big)^{m(\alpha+1/2)}$ is bounded, using Stirling's formula, we get a positive constant C_1 such that for all $\alpha \in \mathbb{N}$, $\|\tilde{f}^{(\alpha)}\|_\infty \le C_1(\nu')^\alpha\alpha!^m$, showing that $f \in \mathcal{A}_{m,\nu'}(\mathbb{T})$ and proving (i).

Let $f \in \mathcal{A}_m(\mathbb{T})$. Then $f \in \mathcal{A}_{m,\nu}(\mathbb{T})$ for some $\nu > 0$ and the inequality follows from (i) and $e^{-m/\nu^{1/m}} < 1$. Conversely, if $\|(\hat{f}(k))_k\|^{\pm}_{(\cdot)^{-1/m}} < 1$, then there exists $\nu > 0$ such that $\|(\hat{f}(k))_k\|^{\pm}_{(\cdot)^{-1/m}} \le e^{-m/\nu^{1/m}}$. From (i), it follows that $f \in \mathcal{A}_{m,\nu'}(\mathbb{T})$ for $\nu' > \nu$. Hence $f \in \mathcal{A}_m(\mathbb{T})$, proving (ii).

Let $f \in \mathcal{A}_{0,\nu}(\mathbb{T})$. The above shows that there exists $C_1 > 0$ such that $|k|^\alpha|\hat{f}(k)| \le C_1\nu^\alpha$. Keeping the same notations, we find $|\hat{f}(k)| \le C_1(1/\rho)\nu^\alpha\alpha!^m$ for all $k \in \mathbb{Z}^*$ and all $\alpha \in \mathbb{N}$. If $|k| > \nu$, then $1/\rho < 1$, and letting $\alpha \to \infty$ yields $\hat{f}(k) = 0$.

Conversely, assume that $\hat{f}(k) = 0$ for $|k| > \nu$. Then $f(z) = \sum_{|k|\le\nu} \hat{f}(k)z^k$ for all $z \in \mathbb{C}^*$. It follows that for all $\alpha \in \mathbb{N}$, $\|\tilde{f}^{(\alpha)}\|_\infty \le (\sum_{|k|\le\nu} |\hat{f}(k)|)\nu^\alpha$, that is, $f \in \mathcal{A}_{0,\nu}(\mathbb{T})$, proving (iii).

Claim (iv) follows from (iii) straightforwardly.

Claims (ii) and (iv) show that for $f \in \mathcal{A}_1(\mathbb{T})$ the series $\sum_{k\in\mathbb{Z}} \hat{f}(k)z^k$ converges absolutely for any $z \in \mathbb{C}^*$, proving (v). ∎

3.3.3. *Duality and embeddings.* This section is devoted to the study of the algebras $\mathcal{A}_m(\mathbb{T})$ and $\mathcal{A}_1(\mathbb{T})$ together with the associated dual spaces $\mathcal{A}'_m(\mathbb{T})$ and $\mathcal{A}'_1(\mathbb{T})$ for $m \in (0,1)$.

Let $f \in \mathcal{A}_m(\mathbb{T})$. There exists $\nu > 0$ such that $f \in \mathcal{A}_{m,\nu}(\mathbb{T})$. By Proposition 55, there exists $C_1 > 0$ such that for all $k \in \mathbb{Z}^*$, $|\hat{f}(k)| \le C_1\sqrt{|k|}e^{-\gamma|k|^{1/m}}$ with $\gamma = 1/\nu^{1/m}$.

PROPOSITION 56. *For $m \in (0,1)$, $\mathcal{A}_0(\mathbb{T})$ is a dense subset of $\mathcal{A}_m(\mathbb{T})$.*

Proof. Let $f \in \mathcal{A}_m(\mathbb{T})$. There exists $\nu > 0$ such that $f \in \mathcal{A}_{m,\nu}(\mathbb{T})$. From the proof of Proposition 55, there exists $C_1 > 0$ such that for all $k \in \mathbb{Z}^*$, $|\hat{f}(k)| \le C_1\sqrt{|k|}e^{-\gamma|k|^{1/m}}$ with $\gamma = 1/\nu^{1/m}$. For $n \in \mathbb{N}$, let $f_n(z) = \sum_{|k|\le n} \hat{f}(k)z^k$. Clearly, for each n, $f_n \in \mathcal{A}_0(\mathbb{T})$. We prove that $\lim_{n\to\infty} f_n = f$ in $\mathcal{A}_m(\mathbb{T})$. Let $\nu' > \nu$ and set $\rho = |k|/\nu$, $\gamma' = m/(\nu')^{1/m}$

$< \gamma$ and $e^{-(\gamma-\gamma')/2} = \varepsilon$. It follows that $f, f - f_n \in \mathcal{A}_{m,\nu}(\mathbb{T}) \subset \mathcal{A}_{m,\nu'}(\mathbb{T})$, and

$$q_{\nu'}^{m,\infty}(f - f_n) = \sup_{t\in\mathbb{R},\, \alpha\in\mathbb{N}} \frac{|\sum_{|k|>n}(ik)^\alpha \hat{f}(k)e^{ikt}|}{(\nu')^\alpha \alpha!^m}.$$

By the growth condition on $|\hat{f}(k)|$ we get

$$q_{\nu'}^{m,\infty}(f - f_n) \le C_1 \sup_{\alpha\in\mathbb{N}} \frac{\sum_{|k|>n} |k|^{\alpha+1/2} e^{-\gamma|k|^{1/m}}}{(\nu')^\alpha \alpha!^m}.$$

Writing $e^{-\gamma|k|^{1/m}} = \varepsilon^{2|k|^{1/m}} e^{-\gamma'|k|^{1/m}}$ yields

$$q_{\nu'}^{m,\infty}(f - f_n) \le C_1 \varepsilon^{n^{1/m}} \sup_{\alpha\in\mathbb{N}} \frac{\sum_{|k|>n} \varepsilon^{|k|^{1/m}} |k|^{\alpha+1/2} e^{-\gamma'|k|^{1/m}}}{(\nu')^\alpha \alpha!^m}.$$

Let $\rho' = |k|/\nu'$. From Stirling's formula, we find a positive constant C_2 such that

$$q_{\nu'}^{m,\infty}(f - f_n) \le C_2 \varepsilon^{n^{1/m}} \sum_{|k|>n} \varepsilon^{|k|^{1/m}} \sqrt{|k|} \sup_{\alpha\in\mathbb{N}} (\rho'^\alpha \alpha^{-m(\alpha+1/2)} e^{m\alpha}) e^{-\gamma'|k|^{1/m}}.$$

Substituting ρ' for ρ in Lemma 54 leads to

$$\sup_{\alpha\in\mathbb{N}} \rho'^\alpha \alpha^{-m(\alpha+1/2)} e^{m\alpha} = \sup_{\alpha\in\mathbb{N}} \frac{1}{\varphi(\alpha)} \le \frac{1}{\varphi(t_{\rho'})} \le (\rho')^{-1/2} e^{-m/2} e^{mt_{\rho'}}.$$

Since $t_{\rho'} < (\rho')^{1/m}$ and $m(\rho')^{1/m} = \gamma'|k|^{1/m}$, it follows that

$$\sup_{\alpha\in\mathbb{N}} (\rho')^\alpha \alpha^{-m(\alpha+1/2)} e^{m\alpha} \le (\rho')^{-1/2} e^{-m/2} e^{\gamma'|k|^{1/m}}.$$

Hence, there exists a positive constant C_3 such that $q_{\nu'}^{m,\infty}(f - f_n) \le C_3 \varepsilon^{n^{1/m}}$, showing that $\lim_{n\to\infty} q_{\nu'}^{m,\infty}(f - f_n) = 0$, whence $\lim_{n\to\infty} f_n = f$ in $\mathcal{A}_m(\mathbb{T})$. ∎

For $T \in \mathcal{A}'_m(\mathbb{T})$ and $k \in \mathbb{Z}$, we define the kth Fourier coefficient of T by $\hat{T}(k) = \overline{T(e_k)}$. Obviously, any sequence $(a_k)_k$ of complex numbers is the sequence of Fourier coefficients of $T \in \mathcal{A}'_0(\mathbb{T})$ such that $T(f) = \sum_{k\in\mathbb{Z}} \overline{\hat{T}(k)}\, \hat{f}(k)$ for $f \in \mathcal{A}_0(\mathbb{T})$. We have the following

PROPOSITION 57. *Let $T \in \mathcal{A}'_0(\mathbb{T})$ and $m \in (0,1)$.*

(i) *$T \in \mathcal{A}'_m(\mathbb{T})$ if and only if*

$$\|(\hat{T}(k))_k\|^{\pm}_{(\cdot)^{-1/m}} \le 1. \tag{3.6}$$

Moreover, for all $f \in \mathcal{A}_m(\mathbb{T})$, $T(f) = \sum_{k\in\mathbb{Z}} \overline{\hat{T}(k)} \hat{f}(k)$.

(ii) *$T \in \mathcal{A}'_1(\mathbb{T})$ if and only if*

$$\forall \nu > 1 : \quad \|(\hat{T}(k))_k\|^{\pm}_{(\cdot)^{-\nu}} \le 1. \tag{3.7}$$

(iii) *Let $\hat{T}^*(\mathbb{Z}) = \{k \in \mathbb{Z} \mid \hat{T}(k) \neq 0\}$. Then $T \in \mathcal{A}'_m(\mathbb{T}) \setminus \mathcal{A}_0(\mathbb{T})$ if and only if*

$$\limsup_{k\in\hat{T}^*(\mathbb{Z}),\, |k|\to\infty} \frac{\ln[\ln(1 + |\hat{T}(k)|)]}{\ln|k|} \le 1. \tag{3.8}$$

Proof. Let $T \in \mathcal{A}'_m(\mathbb{T})$. Then $T \in \mathcal{A}'_{m,\nu}(\mathbb{T})$ for all $\nu > 0$. It follows that there is $C_1 > 0$ such that for all $k \in \mathbb{Z}$, $|\hat{T}(k)| \le C_1 q_\nu^{m,\infty}(e_k)$. Since $q_\nu^{m,\infty}(e_k) = \sup_{\alpha\in\mathbb{N}} |k|^\alpha/\nu^\alpha \alpha!^m$, by use of Stirling's formula there is $C_2 > 0$ such that

$$|\hat{T}(k)| \leq C_2 \sup_{\alpha\in\mathbb{N}} (|k|/\nu)^\alpha \alpha^{-m(\alpha+1/2)} e^{m\alpha}.$$

From the end of the proof of Proposition 56, there exists $C > 0$ such that

$$|\hat{T}(k)| \leq C e^{m|k|^{1/m}/\nu^{1/m}}.$$

It follows that

$$\|(\hat{T}(k))_k\|^{\pm}_{(\cdot)^{-1/m}} \leq e^{m/\nu^{1/m}},$$

and letting $\nu \to \infty$ yields $\|(\hat{T}(k))_k\|^{\pm}_{(\cdot)^{-1/m}} \leq 1$.

Let $f \in \mathcal{A}_m(\mathbb{T})$. With the notations of the proof of Proposition 56, $f_n \to f$ in $\mathcal{A}_m(\mathbb{T})$ as $n \to \infty$. Therefore the continuity of T gives $T(f) = \lim_{n\to\infty} \sum_{|k|\leq n} \overline{\hat{T}(k)}\, \hat{f}(k)$. The growth conditions on $\hat{f}(k)$ and $\hat{T}(k)$ show that the series with general term $\hat{T}(k)\hat{f}(k)$ converges absolutely; hence $T(f) = \sum_{k\in\mathbb{Z}} \overline{\hat{T}(k)}\, \hat{f}(k)$.

Conversely, assume that $T \in \mathcal{A}'_0(\mathbb{T})$ satisfies the given inequality and let $\mu > 0$. Since $\|(\hat{T}(k))_k\|^{\pm}_{(\cdot)^{-1/m}} < e^\mu$, it follows that there is $D > 0$ such that for all $k \in \mathbb{Z}$, $|\hat{T}(k)| < De^{\mu|k|^{1/m}}$. This last growth condition enables us to define $T(f) = \sum_{k\in\mathbb{Z}} \overline{\hat{T}(k)}\, \hat{f}(k)$ for $f \in \mathcal{A}_m(\mathbb{T})$. Clearly, T is a linear form on $\mathcal{A}_m(\mathbb{T})$. We show the continuity of T on each $\mathcal{A}_{m,\nu}(\mathbb{T})$.

Let $f \in \mathcal{A}_{m,\nu}(\mathbb{T})$. For all $\alpha \in \mathbb{N}$, we have $|k|^\alpha |\hat{f}(k)| \leq \|\tilde{f}^{(\alpha)}\|_\infty$ for all $k \in \mathbb{Z}$. From the definition of $q^{m,\infty}_\nu$ it follows that $\|\tilde{f}^{(\alpha)}\|_\infty \leq \nu^\alpha \alpha!^m q^{m,\infty}_\nu(f)$, whence

$$\forall k \in \mathbb{Z}^* : \quad |\hat{f}(k)| \leq \inf_{\alpha\in\mathbb{N}} \frac{\nu^\alpha \alpha!^m}{|k|^\alpha}\, q^{m,\infty}_\nu(f).$$

From Lemma 54, there exists a positive constant C_1 such that

$$\inf_{\alpha\in\mathbb{N}} \frac{\nu^\alpha \alpha!^m}{|k|^\alpha} \leq C_1 |k|^{1/2} e^{-\gamma|k|^{1/m}}, \qquad \gamma = 1/\nu^{1/m},\ k \in \mathbb{Z}^*.$$

Let $D > 0$ be such that $|\hat{T}(k)| \leq De^{\gamma|k|^{1/m}/2}$ for all $k \in \mathbb{Z}$. We then have, for some constant $C > 0$,

$$|T(f)| \leq \Big(C \sum_{k\in\mathbb{Z}} |k|^{1/2} e^{-\gamma|k|^{1/m}/2}\Big) q^{m,\infty}_\nu(f),$$

proving the continuity of T on $\mathcal{A}_{m,\nu}(\mathbb{T})$ for all $\nu > 0$. Hence, $T \in \mathcal{A}'_m(\mathbb{T})$.

From (i), $T \in \mathcal{A}'_1(\mathbb{T})$ if and only if $\|(\hat{T}(k))_k\|^{\pm}_{(\cdot)^{-1/m}} \leq 1$ for all $m \in (0,1)$; writing $1/m = \nu$ gives (ii).

Let $T \in \mathcal{A}'_1(\mathbb{T}) \setminus \mathcal{A}_0(\mathbb{T})$ and $\nu > 1$. From (ii), there exists $n_0 \in \mathbb{N}$ such that $|\hat{T}(k)| < e^{|k|^\nu/2}$ for $|k| > n_0$. It follows that $1 + |\hat{T}(k)| < e^{|k|^\nu}$ for $|k| > n_0$. If $k \in \hat{T}^*(\mathbb{Z})$ and $|k| > n_0$, then

$$\frac{\ln[\ln(1 + |\hat{T}(k)|)]}{\ln|k|} < \nu.$$

This being true for all $\nu > 1$, it follows that the inequality of (iii) is true. Conversely, assume that (3.8) holds. Then $\hat{T}^*(\mathbb{Z})$ is not finite and consequently $T \neq \mathcal{A}_0(\mathbb{T})$. Let $\nu > 1$. From (3.8), we have $\ln(\ln(1 + |\hat{T}(k)|)) < \nu \ln|k|$ for $k \in \hat{T}^*(\mathbb{Z})$ and $|k|$ large enough. This means that $|\hat{T}|^{1/|k|^\nu} < 1$ for $|k|$ large enough and $k \in \hat{T}^*(\mathbb{Z})$. It follows that T satisfies (3.7), proving (iii). ∎

Proposition 58. *For all* $m \in [0,1)$, $\mathcal{A}_m(\mathbb{T}) \hookrightarrow \mathcal{A}(\mathbb{T})$. *Consequently*, $\mathcal{A}_1(\mathbb{T}) \hookrightarrow \mathcal{A}(\mathbb{T}) \hookrightarrow \mathcal{E}(\mathbb{T})$ *and* $\mathcal{E}'(\mathbb{T}) \hookrightarrow \mathcal{B}(\mathbb{T}) \hookrightarrow \mathcal{A}_1'(\mathbb{T})$.

Proof. We claim that for all $m \in [0,1)$ and $\nu > 0$, there exists $\lambda > 1$ such that $\mathcal{A}_{m,\nu}(\mathbb{T}) \hookrightarrow \mathcal{O}_\lambda$. Let $f \in \mathcal{A}_{m,\nu}(\mathbb{T}), \alpha \in \mathbb{N}$ and $k \in \mathbb{Z}$. We have $|k|^\alpha |\hat{f}(k)| \le \nu^\alpha \alpha!^m p_{m,\nu}(f)$. It follows that

$$\frac{1}{\alpha!}\left(\frac{|k|}{\nu}\right)^\alpha |\hat{f}(k)| \le \alpha!^{m-1} q_\nu^{m,\infty}(f).$$

Due to $m-1<0$, summing over $\alpha \in \mathbb{N}$ yields

$$e^{|k|/\nu}|\hat{f}(k)| \le \Big(\sum_{\alpha=0}^{\infty} \alpha!^{m-1}\Big) q_\nu^{m,\infty}(f).$$

Hence $|\hat{f}(k)| \le C_1 \mu^{-|k|} q_\nu^{m,\infty}(f)$ with $C_1 = \sum_{\alpha\in\mathbb{N}} \alpha!^{m-1}$ and $\mu = e^{1/\nu} > 1$. Consequently, if $1 < \lambda < \mu$, then $f \in \mathcal{O}_\lambda$ and

$$\|f\|_{L^\infty(C_\lambda)} \le \sum_{k\in\mathbb{Z}} |\hat{f}(k)|\lambda^{|k|} \le \Big(C_1 \sum_{k\in\mathbb{Z}} (\lambda\mu^{-1})^{|k|}\Big) q_\nu^{m,\infty}(f),$$

proving our claim.

Let V denote a convex neighborhood of zero in $\mathcal{A}(\mathbb{T})$. Then for all $\lambda > 1$, $V \cap \mathcal{O}_\lambda$ is a neighborhood of zero in $\mathcal{O}_\lambda$. Let $\nu > 0$ and choose λ such that $1 < \lambda < e^{1/\nu}$. From $\mathcal{A}_{m,\nu}(\mathbb{T}) \hookrightarrow \mathcal{O}_\lambda$, it follows that there exists a neighborhood U of zero in $\mathcal{A}_{m,h}(\mathbb{T})$ such that $U \subset V \cap \mathcal{O}_\lambda \subset \mathcal{O}_\lambda$, showing that $\mathcal{A}_m(\mathbb{T}) \hookrightarrow \mathcal{A}(\mathbb{T})$ and then $\mathcal{A}_1(\mathbb{T}) \hookrightarrow \mathcal{A}(\mathbb{T})$.

Since $\mathcal{A}_1(\mathbb{T}) \hookrightarrow \mathcal{A}(\mathbb{T}) \hookrightarrow \mathcal{E}(\mathbb{T})$, these embeddings being with dense image, it follows straightforwardly that $\mathcal{E}'(\mathbb{T}) \hookrightarrow \mathcal{B}(\mathbb{T}) \hookrightarrow \mathcal{A}_1'(\mathbb{T})$. ∎

3.3.4. *The algebra* $\mathcal{G}_{H,r}(\mathbb{T})$ *of generalized hyperfunctions.* Throughout the rest of this subsection, let $r = (r_n)_n$ be an arbitrary sequence of positive numbers such that $r_n \searrow 0$.

For $n \in \mathbb{N}$, we set $\varphi_{1/r_n}(z) = \sum_{|k|\le 1/r_n} z^k$. We have $\varphi_{1/r_n} * \varphi_{1/r_n} = \varphi_{1/r_n}$ and $\lim_{n\to\infty} \varphi_{1/r_n} = \delta$ in $\mathcal{E}'(\mathbb{T})$. If $H \in \mathcal{B}(\mathbb{T})$, then $H * \varphi_{1/r_n} = \sum_{|k|\le 1/r_n} \hat{H}(k) z^k$ and consequently $\lim_{n\to\infty} H * \varphi_{1/r_n} = H$ in $\mathcal{B}(\mathbb{T})$.

It is easily seen that $\lim_{n\to\infty} \|\varphi_{1/r_n}\|_{L^\infty(\mathbb{T})} = \infty$. More generally, in analogy to Proposition 41, we have:

Proposition 59. *Let* $(\psi_n)_n$ *denote a sequence of elements of* $\mathcal{A}(\mathbb{T})$ *such that* $\lim_{n\to\infty} \psi_n = \delta$ *in* $\mathcal{B}(\mathbb{T})$. *Then* $(\psi_n)_n$ *cannot be bounded in* $\mathcal{A}(\mathbb{T})$.

Proof. Assume that, contrary to the assertion, $(\psi_n)_n$ is bounded in $\mathcal{A}(\mathbb{T})$. Consequently, $\exists C > 0$, $\forall n \in \mathbb{N} : \|\psi_n\|_{L^\infty(\mathbb{T})} \le C$. Since $\lim_{n\to\infty} \psi_n = \delta$ in $\mathcal{B}(\mathbb{T})$, for all $\varphi \in \mathcal{A}(\mathbb{T})$ we have

$$\lim_{n\to\infty} \frac{1}{2\pi} \int_0^{2\pi} \overline{\psi_n(e^{it})}\, \varphi(e^{it})\, dt = \varphi(1).$$

By Cauchy–Schwarz's inequality,

$$\left|\frac{1}{2\pi}\int_0^{2\pi}\overline{\psi_n(e^{it})}\,\varphi(e^{it})\,dt\right| \le \|\psi_n\|_{L^\infty(\mathbb{T})}\|\varphi\|_{L^2(\mathbb{T})}.$$

It follows that $\forall\varphi\in\mathcal{A}(\mathbb{T}) : |\varphi(1)| \le C\|\varphi\|_{L^2(\mathbb{T})}$. Let m and s denote respectively an integer such that $m > (C^2-1)/2$ and a positive constant. Define $\varphi\in\mathcal{A}(\mathbb{T})$ by $\hat\varphi(k) = 0$ for $|k| > m$, and $\hat\varphi(k) = s$ if $|k| \le m$. Then we have

$$\varphi(1) = \sum_{|k|\le m}\hat\varphi(k) = \sqrt{2m+1}\Big(\sum_{|k|\le m}\hat\varphi(k)^2\Big)^{1/2}.$$

This means that

$$\varphi(1) = \sqrt{2m+1}\,\|\varphi\|_{L^2(\mathbb{T})} > C\|\varphi\|_{L^2(\mathbb{T})},$$

which is a contradiction. ■

Let $\mathcal{X}(\mathbb{T}) = \mathcal{A}_1(\mathbb{T})^{\mathbb{N}}$ be the set of sequences of functions $(f_n)_n$ with $f_n\in\mathcal{A}_1(\mathbb{T})$. Let $\lambda > 1$. For $f\in\mathcal{A}_1(\mathbb{T})$, we set

$$q^\lambda(f) = \|f\|_{L^\infty(\Omega_\lambda)}.$$

If $f = (f_n)_n \in \mathcal{X}(\mathbb{T})$, we define

$$|\!|\!|f|\!|\!|_{q^\lambda,r} := \limsup_{n\to\infty} q^\lambda(f)^{r_n}.$$

We define the subsets $\mathcal{X}_r(\mathbb{T})$ and $\mathcal{N}_r(\mathbb{T})$ of $\mathcal{X}(\mathbb{T})$ as follows:

$$\begin{aligned}\vec{\mathcal{F}}_{q,r} &= \mathcal{X}_r(\mathbb{T}) = \{f = (f_n)_n\in\mathcal{X}(\mathbb{T}) \mid \exists\lambda>1 : |\!|\!|f|\!|\!|_{q^\lambda,r} < \infty\},\\ \vec{\mathcal{K}}_{q,r} &= \mathcal{N}_r(\mathbb{T}) = \{f = (f_n)_n\in\mathcal{X}(\mathbb{T}) \mid \exists\lambda>1 : |\!|\!|f|\!|\!|_{q^\lambda,r} = 0\}.\end{aligned}$$

As shown in the general case, $\mathcal{X}_r(\mathbb{T})$ is an algebra for usual termwise operations and $\mathcal{N}_r(\mathbb{T})$ is an ideal of $\mathcal{X}_r(\mathbb{T})$. For $f\in\mathcal{A}_1(\mathbb{T})$ and $\lambda > 1$ we set

$$\hat q^\lambda(f) = \sup_{k\in\mathbb{Z}}\lambda^{|k|}|\hat f(k)|.$$

The above two spaces have the following Fourier characterization:

Proposition 60. *Let $f = (f_n)_n\in\mathcal{X}(\mathbb{T})$. Then*:

(i) $f\in\mathcal{X}_r(\mathbb{T})$ *if and only if*

$$\exists\lambda>1:\quad |\!|\!|f|\!|\!|_{\hat q^\lambda,r} < \infty.$$

(ii) $f\in\mathcal{N}_e(\mathbb{T})$ *if and only if*

$$\exists\lambda>1:\quad |\!|\!|f|\!|\!|_{\hat q^\lambda,r} = 0.$$

Proof. Let $f = (f_n)_n\in\mathcal{X}_r(\mathbb{T})$. Take $\lambda > 1$ such that $|\!|\!|f|\!|\!|_{q^\lambda,r} < \infty$. By the hypothesis, there exist $a > 0$ and $\eta\in\mathbb{N}$ such that $q^\lambda(f_n)^{r_n} < a$ for $n > \eta$. From Cauchy's inequalities in Ω_λ we obtain $|\hat f_n(k)| \le q^\lambda(f_n)\lambda^{-|k|}$, whence $|\hat f_n(k)|^{r_n} \le a\lambda^{-|k|r_n}$ for all $k\in\mathbb{Z}$ and $n > \eta$. It follows that $|\!|\!|f|\!|\!|_{\hat q^\lambda,r} < \infty$.

Conversely, let $f = (f_n)_n\in\mathcal{X}(\mathbb{T})$ and suppose that for some $\lambda > 1$, $|\!|\!|f|\!|\!|_{\hat q^\lambda,r} < \infty$. It follows that there exists $a > 0$ such that $\hat q^\lambda(f_n)^{r_n} < a$ for n large enough. Then we have $|\hat f_n(k)| < a^{1/r_n}\lambda^{-|k|}$ for all $k\in\mathbb{Z}$ and $n > \eta_0$ for some η_0. Consequently, if $s = \sqrt{r}$, we may find $C(s) > 0$ such that $q^s(f_n) \le C(s)a^{1/r_n}$ for $n > \eta_0$, showing that $\|f\|_{q^\lambda,r} < \infty$ and proving (i).

Part (ii) can be proved in a similar way. ■

We now give a version of the algebra of generalized hyperfunctions on the circle which is an improvement of the ones given in [76, 77, 78].

DEFINITION 61. The algebra of generalized hyperfunctions on $\mathbb{T}$, associated to the sequence r, is the factor algebra

$$\vec{\mathcal{G}}_{q,r} = \mathcal{G}_{H,r}(\mathbb{T}) = \mathcal{X}_r(\mathbb{T})/\mathcal{N}_r(\mathbb{T}).$$

If $f \in \mathcal{A}(\mathbb{T})$, then $f(z) = \sum_{k\in\mathbb{Z}} \widehat{f}(k)z^k$ in some Ω_λ. We define

$$(\partial_\theta f)(z) = \sum_{k\in\mathbb{Z}} (ik)\widehat{f}(k)z^k.$$

We also consider the usual derivative of a holomorphic function defined by

$$\frac{df}{dz}(z) = f'(z) = \sum_{k\in\mathbb{Z}} (k+1)\widehat{f}(k+1)z^k.$$

It is seen that d/dz and ∂_θ are connected by $(\partial_\theta f)(z) = izf'(z)$.

These two differential operators being defined componentwise on $(f_n)_n \in \mathcal{X}_r(\mathbb{T})$, by the above proposition it is seen that $\mathcal{X}_r(\mathbb{T})$ and $\mathcal{N}_r(\mathbb{T})$ are invariant under these operators. Consequently, this enables us to equip $\mathcal{G}_{H,r}(\mathbb{T})$ with two differential structures in an obvious way.

3.3.5. *Embedding of $\mathcal{B}(\mathbb{T})$ and $\mathcal{A}(\mathbb{T})$ in $\mathcal{G}_{H,r}(\mathbb{T})$.* The space $\mathcal{B}(\mathbb{T})$ can be embedded in $\mathcal{G}_{H,r}(\mathbb{T})$ in such a way that the usual multiplication of $\mathcal{A}_1(\mathbb{T})$ is preserved:

PROPOSITION 62. *Let*

$$\bar{\mathbf{i}} : \mathcal{B}(\mathbb{T}) \to \mathcal{H}(\mathbb{T}), \quad H \mapsto [H * \varphi_{1/r_n}], \qquad \bar{\mathbf{i}}_0 : \mathcal{A}_1(\mathbb{T}) \to \mathcal{H}(\mathbb{T}), \quad f \mapsto [f].$$

Then $\bar{\mathbf{i}}$ *is a linear embedding and* $\bar{\mathbf{i}}_0$ *is an injective morphism of differential algebras such that*

$$\bar{\mathbf{i}}|_{\mathcal{A}_1(\mathbb{T})} = \bar{\mathbf{i}}_0.$$

Moreover, for any $H \in \mathcal{B}(\mathbb{T})$,

$$\bar{\mathbf{i}}\left(\frac{dH}{dz}\right) = \frac{d}{dz}(\bar{\mathbf{i}}(H)) \quad \text{and} \quad \bar{\mathbf{i}}(\partial_\theta H) = \partial_\theta(\bar{\mathbf{i}}(H)).$$

Proof. The claims on $\bar{\mathbf{i}}_0$ and the last part of the proposition are easy to prove. Let us focus on the properties of the first part relating to $\bar{\mathbf{i}}$. The linearity of $\bar{\mathbf{i}}$ is quite obvious. Let $H \in \mathcal{B}(\mathbb{T})$ and set $h = (h_n)_n$ with $h_n = H * \varphi_{1/r_n}$. From Proposition 55(iv), we have $h_n \in \mathcal{A}_0(\mathbb{T})$, and so $h \in \mathcal{X}(\mathbb{T})$.

Now take $\lambda > 1$. From the property of the Fourier coefficients of H, there exists $C > 0$ such that $|\hat{H}(k)| \le C\lambda^{|k|}$ for all $k \in \mathbb{Z}$. It follows that $\lambda^{|k|}|\hat{h}_n(k)| \le C\lambda^{2/r_n}$, showing that $\|h\|_{\hat{q}^\lambda,r} \le \lambda^2$. By Proposition 60, $h \in \mathcal{X}_r(\mathbb{T})$. It is sufficient to consider restrictions to the spaces $\mathcal{A}_m(\mathbb{T})$ with $0 < m < 1$. Let $f \in \mathcal{A}_m(\mathbb{T})$ with $0 < m < 1$. There is $\lambda > 1$ such that $f(z) = \sum_{k\in\mathbb{Z}} \hat{f}(k)z^k$ for $1/\lambda \le |z| \le \lambda$. Then we have $\bar{\mathbf{i}}_0(f) - \bar{\mathbf{i}}(f) = [f_n]$ where $f_n = f - f * \varphi_{1/r_n}$, that is, $f_n(z) = \sum_{|k|>1/r_n} \hat{f}(k)z^k$. Thus $(f_n)_n \in \mathcal{O}_\lambda$.

We claim that $(f_n)_n \in \mathcal{N}_r(\mathbb{T})$. From Proposition 55, there is $C > 0$ such that for all $k \in \mathbb{Z}^*$, $|\hat{f}(k)| \le C\sqrt{|k|}e^{-\gamma|k|^{1/m}}$ where $\gamma = m/h^{1/m}$. For $|k| > 1/r_n$, writing

$$e^{-\gamma|k|^{1/m}} \le e^{-\frac{\gamma}{2}|k|^{1/m}} e^{-\frac{\gamma}{2}(\frac{1}{r_n})^{1/m}},$$

it follows that

$$\lambda^{|k|}|\hat{f}_n(k)| \le (C\lambda^{|k|}\sqrt{|k|}e^{-\frac{\gamma}{2}|k|^{1/m}})e^{-\frac{\gamma}{2}(\frac{1}{r_n})^{1/m}} \qquad \text{for } |k| > 1/r_n.$$

Since $C\sqrt{|k|}e^{-\frac{\gamma}{2}|k|^{1/m}}$ is bounded with respect to k, we obtain

$$|\!|\!| f |\!|\!|_{\hat{q}^\lambda, r} = \lim_{n\to\infty} e^{-\frac{\gamma}{2}(\frac{1}{r_n})^{1/m-1}} = 0,$$

proving our claim. ∎

An element of $\bar{\mathbf{i}}(\mathcal{B}(\mathbb{T}))$ is called a $\mathcal{G}_{H,r}(\mathbb{T})$ hyperfunction.

4. Sequences of scales and asymptotic algebras

4.1. Sequences of scales

Definition 63. Consider a sequence $r = (r^m)_m$ of positive sequences $(r^m_n)_n$ decreasing to zero, i.e. such that

$$\forall m, n \in \mathbb{N}: \quad r^m_{n+1} \le r^m_n, \qquad \lim_{n\to\infty} r^m_n = 0,$$

which satisfy in addition one of the following conditions:

$$\forall m, n \in \mathbb{N}: \quad r^{m+1}_n \ge r^m_n \tag{4.1}$$

or

$$\forall m, n \in \mathbb{N}: \quad r^{m+1}_n \le r^m_n. \tag{4.2}$$

Then let

$$\begin{aligned} &\text{in case (4.1)}, \quad \overleftrightarrow{\mathcal{F}}_{p,r} = \bigcap_{m\in\mathbb{N}} \overleftrightarrow{\mathcal{F}}_{p,r^m}, \quad \overleftrightarrow{\mathcal{K}}_{p,r} = \bigcup_{m\in\mathbb{N}} \overleftrightarrow{\mathcal{K}}_{p,r^m}, \\ &\text{in case (4.2)}, \quad \overleftrightarrow{\mathcal{F}}_{p,r} = \bigcup_{m\in\mathbb{N}} \overleftrightarrow{\mathcal{F}}_{p,r^m}, \quad \overleftrightarrow{\mathcal{K}}_{p,r} = \bigcap_{m\in\mathbb{N}} \overleftrightarrow{\mathcal{K}}_{p,r^m}, \end{aligned}$$

where $p = (p^\mu_\nu)_{\nu,\mu}$.

Proposition 64. *In both cases of the above definition, $\overleftrightarrow{\mathcal{F}}_{p,r}$ is an algebra and $\overleftrightarrow{\mathcal{K}}_{p,r}$ an ideal of $\overleftrightarrow{\mathcal{F}}_{p,r}$. Thus, $\overleftrightarrow{\mathcal{G}}_{p,r} = \overleftrightarrow{\mathcal{F}}_{p,r}/\overleftrightarrow{\mathcal{K}}_{p,r}$ is an algebra.*

Proof. Let us start with (4.1). For $r^{m+1} \ge r^m$, we have $|\!|\!| f |\!|\!|_{r^{m+1}} \le |\!|\!| f |\!|\!|_{r^m}$ if $p(f_n) \le 1$, hence $\mathcal{K}_{p,r^{m+1}} \supset \mathcal{K}_{p,r^m}$. Conversely, $\mathcal{F}_{p,r^{m+1}} \subset \mathcal{F}_{p,r^m}$. Thus, intersection for $\mathcal{F}$ and union for $\mathcal{K}$ make sense, and $\mathcal{F}_{p,r}$ is obviously a subalgebra. To see that $\mathcal{K}_{p,r}$ is an ideal, take $(k, f) \in \mathcal{K}_{p,r} \times \mathcal{F}_{p,r}$. Then for some m we have $k \in \mathcal{K}_{p,r^m}$, but also $f \in \mathcal{F}_{p,r^m}$, in which $\mathcal{K}_{p,r^m}$ is an ideal. Thus, $k \cdot f \in \mathcal{K}_{p,r^m} \subset \mathcal{K}_{p,r}$.

Now we turn to (4.2). The same reasoning gives now $\mathcal{K}_{p,r^{m+1}} \subset \mathcal{K}_{p,r^m}$ and $\mathcal{F}_{p,r^{m+1}} \supset \mathcal{F}_{p,r^m}$, justifying the definitions of $\mathcal{F}_{p,r}$ and $\mathcal{K}_{p,r}$. Moreover, because of this inclusion property, $\mathcal{F}_{p,r}$ is indeed a subalgebra. To prove that $\mathcal{K}_{p,r}$ is an ideal, take $(k, f) \in$

$\mathcal{K}_{p,r} \times \mathcal{F}_{p,r}$, i.e. $k \in \mathcal{K}_{p,r^{m''}}$ for all m'', and $f \in \mathcal{F}_{p,r^{m'}}$ for some m' . We have to show that $k \cdot f \in \mathcal{K}_{p,r^m}$ for all m. So let m be given.

If $m < m'$, then $\mathcal{K}_{p,r^{m'}} \subset \mathcal{K}_{p,r^m}$, thus $k \cdot f \in \mathcal{K}_{p,r^{m'}} \cdot \mathcal{F}_{p,r^{m'}} \subset \mathcal{K}_{p,r^{m'}} \subset \mathcal{K}_{p,r^m}$.

If $m' < m$, we use $\mathcal{F}_{p,r^{m'}} \subset \mathcal{F}_{p,r^m}$ to get $k \cdot f \in \mathcal{K}_{p,r^m} \cdot \mathcal{F}_{p,r^{m'}} \subset \mathcal{K}_{p,r^m} \cdot \mathcal{F}_{p,r^m} \subset \mathcal{K}_{p,r^m}$. ∎

Example 65.

$$r_n^m = \begin{cases} 1 & \text{if } n \le m \\ 0 & \text{if } n > m \end{cases}$$

(with the convention that $0^0 = 0$) gives Egorov-type algebras, where the "subalgebra" contains everything and the ideal contains only stationary null sequences.

Example 66. $r_n^m = 1/|\log a_m(n)|$, where $(a_m : \mathbb{N} \to \mathbb{R}_+)_{m\in\mathbb{Z}}$ is an asymptotic scale, i.e. $\forall m \in \mathbb{N} : a_{m+1} = o(a_m)$, $a_{-m} = 1/a_m$, $\exists M : a_M = o(a_m^2)$. This gives back the asymptotic algebras of [16], cf. Section 4.3.

4.2. $(\mathcal{C}, \mathcal{E}, \mathcal{P})$-algebras. Let us now show how a quite large class of $(\mathcal{C}, \mathcal{E}, \mathcal{P})$-algebras [55] fits well into the above setting. First, let us recall that $(\mathcal{C}, \mathcal{E}, \mathcal{P})$-algebras are based on a vector space $\mathcal{E}$ with a filtering family $\mathcal{P}$ of seminorms, and a ring $\mathcal{C} = A/I$ of generalized numbers. Here, I is an ideal of A, which is a subring of $\mathbb{K}^\Lambda$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and Λ is some indexing set. Both A and I must be solid as rings, i.e. $\forall s \in \mathbb{K}^\Lambda$: $(\exists r \in A, \forall \lambda \in \Lambda : |s_\lambda| \le |r_\lambda|) \Rightarrow s \in A$, and idem for I. Then the $(\mathcal{C}, \mathcal{E}, \mathcal{P})$-algebra is defined as $\mathcal{G}_{\mathcal{C},\mathcal{E},\mathcal{P}} = \mathcal{E}_A/\mathcal{E}_I$, with

$$\mathcal{E}_X = \{f \in \mathcal{E}^\Lambda \mid \forall p \in \mathcal{P} : p \circ f \in X\}$$

(where $p \circ f \equiv (\lambda \mapsto p(f_\lambda)) = (p(f_\lambda))_\lambda \in (\mathbb{R}_+)^\Lambda \subset \mathbb{K}^\Lambda$). In other words, the function spaces $\mathcal{E}_A$ and $\mathcal{E}_I$ are determined by $\mathcal{C} = A/I$, by selecting the *functions* with the same respective growth properties as the "constants".

It is clear that this is too general to be written in the above setting of sequence spaces, mainly because there is no relation between A and I: one could in principle take $I = A$, and thus $\mathcal{E}_I = \mathcal{E}_A$ independent of $(\mathcal{E}, \mathcal{P})$, but this is impossible in the present construction.

4.3. Asymptotic algebras. However, in many known applications one can restrict oneself to some subclass of these algebras. As the first example and most important case, let us consider asymptotic algebras [16]. Here, A and I are defined by an asymptotic scale ([6]) $\mathbf{a} = (a_m : \Lambda \to \mathbb{R}_+)_{m\in\mathbb{Z}}$:

$$A_{\mathbf{a}} = \{s \in \mathbb{K}^\Lambda \mid \exists m \in \mathbb{Z} : s = o(a_m)\},$$
$$I_{\mathbf{a}} = \{s \in \mathbb{K}^\Lambda \mid \forall m \in \mathbb{Z} : s = o(a_m)\}.$$

Recall that $\mathbf{a}$ must satisfy: $\forall m \in \mathbb{Z} : a_{m+1} = o(a_m)$, $a_{-m} = 1/a_m$, $\exists M \in \mathbb{Z} : a_M = o(a_m^2)$. Some examples that have proved to be useful are:

([6]) The set Λ is supposed to have a base of filters $\mathcal{B}$, which the $o(\cdot)$ notation refers to. In Section 4.2, $\forall\lambda \in \Lambda$ could also be replaced by $\exists\Lambda_0 \in \mathcal{B}$, $\forall\lambda \in \Lambda_0$.

(i) $\Lambda = \mathbb{N}$ and $a_m(\lambda) = 1/\lambda^m$: This leads to Colombeau's generalized numbers and algebras.

(ii) $\Lambda = \mathbb{N}$ and $a_m(\lambda) = 1/\exp^m(\lambda)$ for $m \in \mathbb{N}^*$, where $\exp^m$ is the m-fold iterated exp function: This gives the so-called exponential algebras [16].

(iii) $r_n^m = 1/n^{m/(m-1)}$: This is related to ultradistribution spaces, and will be discussed in detail in a separate publication.

Proposition 67. *Suppose that the family $\mathcal{P}$ of seminorms can be chosen in the form $\mathcal{P} = (p_\nu^\mu)_{\mu,\nu\in\mathbb{N}}$ fitting into our scheme of inductive or projective limit (Section 2.1 or 2.2). Then asymptotic algebras can be described in our formulation by choosing the sequence of weights $r^m = 1/|\log a_m|$ (i.e. $r_\lambda^m = 1/|\log a_m(\lambda)|$).*

Proof. We will show that $\mathcal{E}_I = \mathcal{K}_{\mathcal{P},r}$ and $\mathcal{E}_A = \mathcal{F}_{\mathcal{P},r}$ for $r^m = 1/|\log a_m|$. In view of the definitions, this amounts to showing the equivalences

$$\forall p, \underset{(\exists)}{\forall} a_m : p \circ f = o(a_m) \ \Leftrightarrow\ \forall p, \underset{(\exists)}{\forall} r^m : \|f\|_{p,r^m} \underset{(<\infty)}{=0} .$$

$\mathcal{E}_A \subset \mathcal{F}_{\mathcal{P},r}$: Let $f \in \mathcal{E}_A$. Thus, $\forall p \in \mathcal{P}$, $\exists m : p \circ f = o(a_m)$. We can assume $a_m > 1$ is such that $r^m = 1/\log a_m \Leftrightarrow a_m = e^{1/r^m}$. Thus $p \circ f = o(e^{1/r^m})$. But $p \circ f < e^{1/r^m} \Rightarrow (p \circ f)^{r^m} < e$, thus $\limsup\,(p \circ f)^{r^m} < \infty$ and $f \in \mathcal{F}_{\mathcal{P},r}$.

$\mathcal{F}_{\mathcal{P},r} \subset \mathcal{E}_A$: If $f \in \mathcal{F}_{\mathcal{P},r}$, then $\forall p \in \mathcal{P}$, $\exists \bar{m} : \limsup\,(p \circ f)^{1/|\log a_{\bar{m}}|} < \infty$. With

$$(p \circ f)^{1/|\log a_m|} \le C \ \Leftrightarrow\ p \circ f \le (a_m)^{\log C} \qquad (a_m, C > 1)$$

we have: $\exists C > 0$, $\exists \Lambda_0$, $\forall \lambda \in \Lambda_0 : p(f_\lambda) \le (a_{\bar{m}}(\lambda))^{|\log C|}$. Thus, using the third property of scales, $\exists m : p \circ f = o(a_m)$.

$\mathcal{E}_I \subset \mathcal{K}_{\mathcal{P},r}$: For $f \in \mathcal{E}_I$, we have $p \circ f = o(a_{\bar{m}})$ for all $\bar{m}$. Take $m \in \mathbb{N}$. Now, for any $q \in \mathbb{N}$, $\exists \hat{m} : a_{\hat{m}} = o(a_m^q)$ and $p \circ f = o(a_{\hat{m}})$. Using $a_m = e^{-1/r^m}$, $a_{\hat{m}} = o(a_m{}^q) = o((e^{-1/r_m})^q) = o((e^{-q})^{1/r^m})$, i.e., $(p \circ f)^{r^m} \le e^{-q}$ on some Λ_0. As q was arbitrary, we have $(p \circ f)^{r^m} \to 0$ and thus $f \in \mathcal{K}_{\mathcal{P},r}$.

$\mathcal{K}_{\mathcal{P},r} \subset \mathcal{E}_I$: For $f \in \mathcal{K}_{\mathcal{P},r}$, we have, for all $\bar{m}$, $\limsup p(f_\lambda)^{1/|\log a_{\bar{m}}|} = 0$, i.e.,

$$\forall C > 0, \exists \Lambda_0, \forall \lambda \in \Lambda_0 : \quad p(f_\lambda)^{1/|\log a_{\bar{m}}|} < C.$$

With $a_m, C < 1$, this gives $p(f_\lambda) \le C^{|\log a_{\bar{m}}|} = a_{\bar{m}}{}^{|\log C|}$. Now, to show that $f \in \mathcal{E}_I$, take any m. Let $\bar{m} = m + 1$ and $C = 1/e$. Then $\exists \Lambda_0, \forall \lambda \in \Lambda_0 : p(f_\lambda) < a_{\bar{m}}(\lambda)$. But $a_{\bar{m}} = a_{m+1} = o(a_m)$, thus $p \circ f = o(a_m)$.

Remark 68. We presented our construction only for the case where $\Lambda = \mathbb{N}$. But the same can be done for an arbitrary set Λ of indices equipped with a base of filters, which is all we need to define the ultranorms and associated spaces. In applications, it may be more convenient to take $\Lambda = (0, 1]$ or more complicated indices, with two or more parameters which can be numbers but also functions (mollifiers) or similar.

4.4. Algebras with infra-exponential growth. A second interesting subclass is the $(\mathcal{C}, \mathcal{E}, \mathcal{P})$-algebras of the form

$$A = \{s \in \mathbb{K}^\Lambda \mid \forall \sigma < 0 : s = o(a_\sigma)\},$$
$$I = \{s \in \mathbb{K}^\Lambda \mid \exists \sigma > 0 : s = o(a_\sigma)\},$$

where $\mathbf{a} = (a_\sigma)_{\sigma\in\mathbb{R}}$ is again a scale (i.e. $\forall \sigma > \rho : a_\sigma = o(a_\rho)$, etc.), but indexed by a real number. (Note that here A is given as intersection and I as union of sets: that is why this case is not covered by the previous one.)

For example (again with $\Lambda = \mathbb{N}$),

$$a_\sigma := \lambda \mapsto 1/\exp(\sigma\lambda)$$

gives the so-called algebras with infra-exponential growth [17], pertaining to the embedding of periodic hyperfunctions in $(\mathcal{C}, \mathcal{E}, \mathcal{P})$-algebras.

These algebras can be obtained by taking $\mathcal{F} = \{f \mid \|\|f\|\|_r \le 1\}$ and $\mathcal{K} = \{f \mid \|\|f\|\|_r < 1\}$, with $r_n = 1/n$. (As the norm is compared to 1, all scales $r_\sigma = 1/|\log a_\sigma|$ (i.e. $r_\sigma(\lambda) = 1/|\sigma\lambda|$) are equivalent. More details on this "dual" construction where $(<\infty, =0)$ is replaced by $(\le 1, <1)$ are left to a separate publication.)

5. Functorial properties

In this section, we want to investigate conditions sufficient to extend mappings on the topological factor algebras constructed as before. Consider for example $\varphi : E \to F$ where (E, P) and (F, Q) are spaces equipped with families P and Q of seminorms. In this section we shall denote by $\mathcal{F}_{\Pi,r}(\cdot)$, $\mathcal{K}_{\Pi,r}(\cdot)$ and $\mathcal{G}_{\Pi,r}(\cdot)$ the spaces defined as above, where $\cdot$ stands for E or F and Π stands for P or Q.

Suppose that φ satisfies the following hypotheses:

$$\begin{aligned} (F_1): &\quad & f \in \mathcal{F}_{P,r}(E) &\ \Rightarrow\ \varphi(f) \in \mathcal{F}_{Q,r}(F), \\ (F_2): &\quad & f \in \mathcal{F}_{P,r}(E),\, h \in \mathcal{K}_{P,r}(E) &\ \Rightarrow\ \varphi(f+h) - \varphi(f) \in \mathcal{K}_{Q,r}(F), \end{aligned}$$

where we write $\varphi(f) := (\varphi(f_n))_n$. Then we can consider the following

Definition 69. Under the above hypothesis, we define the r-*extension* of φ by

$$\Phi := \mathcal{G}_r(\varphi) := \begin{pmatrix} \mathcal{G}_{P,r}(E) \to \mathcal{G}_{Q,r}(F) \\ \quad [f] \mapsto \varphi(f) + \mathcal{K}_{Q,r}(F) \end{pmatrix},$$

where f is any representative of $[f] = f + \mathcal{K}_{P,r}(E)$.

The above are of course very general conditions for a map to be well defined on a factor space. In fact, they do not depend on how the spaces $\mathcal{F}_{P,r}(E)$ and $\mathcal{K}_{P,r}(E)$ are defined. In particular, here r can also be a family of sequences $(r^m)_m$, and E can be of proj-proj or ind-proj type.

Example 70. Consider a linear mapping $u \in \mathcal{L}(E, F)$, continuous for (P, Q). Fix $q \in Q$. As u is continuous, there exists $p = p_{(q)}$ such that

$$\exists c, \forall x \in E : \quad q(u(x)) \le c p_{(q)}(x).$$

Thus, for all $f, h \in E^{\mathbb{N}}$,

$$\limsup\,(p_{(q)}(f_n))^{r_n} < \infty \;\Rightarrow\; \limsup\,(q(u(f_n)))^{r_n} < \infty,$$
$$\limsup\,(p_{(q)}(h_n))^{r_n} = 0 \;\Rightarrow\; \limsup(q(u(h_n)))^{r_n} = 0.$$

This example shows how we can define moderate or compatible maps with respect to the "scale" r. In fact, the concrete definitions will depend on the monotonicity properties of the family (r^m) of sequences of weights, according to which $\mathcal{F}_{p,r} = \bigcup \mathcal{F}_{p,r^m}$ and $\mathcal{K}_{p,r} = \bigcap \mathcal{K}_{p,r^m}$ (for $r^{m+1} \le r^m$), or $\mathcal{F}_{p,r} = \bigcap \mathcal{F}_{p,r^m}$ and $\mathcal{K}_{p,r} = \bigcup \mathcal{K}_{p,r^m}$ (for $r^{m+1} \ge r^m$).

For example, recall that asymptotic algebras correspond to the first case: the property $a_{m+1} = o(a_m)$ gives $\log a_{m+1} < \log a_m$, or equivalently $|\log a_{m+1}| > |\log a_m|$, i.e. $r^{m+1} < r^m$ for $r^m = 1/|\log a_m|$.

The analysis of continuity (in the sense of $\|\cdot\|_{p,r}$) shows that the following definitions are convenient:

Definition 71. The map $g : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be *r-moderate* iff it is increasing and

$$\begin{cases} \forall m \in \mathbb{N},\, \exists M \in \mathbb{N},\, \forall x \in \mathbb{R}_+ : & \sup\limits_{n\in\mathbb{N}}\,(g(x^{1/r_n^m}))^{r_n^M} < \infty \quad (r^{m+1} \le r^m),\\ \forall M \in \mathbb{N},\, \exists m \in \mathbb{N},\, \forall x \in \mathbb{R}_+ : & \sup\limits_{n\in\mathbb{N}}\,(g(x^{1/r_n^m}))^{r_n^M} < \infty \quad (r^{m+1} \ge r^m). \end{cases}$$

The map $h : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be *r-compatible* iff it is increasing and

$$\begin{cases} \forall M \in \mathbb{N},\, \exists m \in \mathbb{N} : & (h(x^{1/r_n^m}))^{r_n^M} \underset{x\to 0}{\longrightarrow} 0 \text{ uniformly in } n \quad (r^{m+1} \le r^m),\\ \forall m \in \mathbb{N},\, \exists M \in \mathbb{N} : & (h(x^{1/r_n^m}))^{r_n^M} \underset{x\to 0}{\longrightarrow} 0 \text{ uniformly in } n \quad (r^{m+1} \ge r^m). \end{cases}$$

Proposition 72. *The above definition of an r-moderate map g is equivalent to*

$$g \text{ increasing, and } \begin{cases} \forall m,\, \exists M : & g(\mathcal{F}^+_{r^m}) \subset \mathcal{F}^+_{r^M} \quad (r^{m+1} \le r^m),\\ \forall M,\, \exists m : & g(\mathcal{F}^+_{r^m}) \subset \mathcal{F}^+_{r^M} \quad (r^{m+1} \ge r^m), \end{cases}$$

where $\mathcal{F}^+_{r^m} = \mathbb{R}^{\mathbb{N}}_+ \cap \mathcal{F}_{|\cdot|,r^m}$ are "moderate" sequences of nonnegative numbers.

The definition of an r-compatible map h can be written as

$$h \text{ increasing and } \begin{cases} \forall M,\, \exists m : & \|h(C)\|_M \to 0 \text{ as } \|C\|_m \to 0 \quad (r^{m+1} \le r^m),\\ \forall m,\, \exists M : & \|h(C)\|_M \to 0 \text{ as } \|C\|_m \to 0 \quad (r^{m+1} \ge r^m), \end{cases}$$

or equivalently,

$$h \text{ continuous at } 0, \text{ increasing, and } \begin{cases} \forall M,\, \exists m : & h(\mathcal{K}^+_{r^m}) \subset \mathcal{K}^+_{r^M} \quad (r^{m+1} \le r^m),\\ \forall m,\, \exists M : & h(\mathcal{K}^+_{r^m}) \subset \mathcal{K}^+_{r^M} \quad (r^{m+1} \ge r^m). \end{cases}$$

Proof. We have

$$\begin{aligned} g(\mathcal{F}^+_{r^m}) \subset \mathcal{F}^+_{r^M} &\Leftrightarrow \forall C \in \mathbb{R}^{\mathbb{N}}_+ : (\|C\|_m < \infty \Rightarrow \|g(C)\|_M < \infty)\\ &\Leftrightarrow \forall C \in \mathbb{R}^{\mathbb{N}}_+ : [(\exists x > 0,\, \forall n : C_n \le x^{1/r_n^m}) \Rightarrow \sup_n g(C_n)^{r_n^M} < \infty]. \end{aligned}$$

As g is increasing, one can replace C_n by x^{1/r_n^m}, and since the sequence C_n was arbitrary, we finally have

$$g(\mathcal{F}^+_{r^m}) \subset \mathcal{F}^+_{r^M} \;\Leftrightarrow\; \forall x \in \mathbb{R}_+ : \sup_n g(x^{1/r_n^m})^{r_n^M} < \infty.$$

For h, again take $C_n = x^{1/r_n^m}$ such that $x \to 0 \Leftrightarrow \|C\|_m \to 0$. Clearly, the first form implies that h is continuous at 0, so both instances of $\|\ldots\| \to 0$ can equivalently be replaced by $\|\ldots\| = 0$. Thus we have $\forall M, \exists m$ (resp. $\forall m, \exists M$) : $C \in \mathcal{K}_m \Rightarrow h(C) \in \mathcal{K}_M$, which means $h(\mathcal{K}_m) \subset \mathcal{K}_M$. ∎

Lemma 73. *If g is r-moderate, then $g(\mathcal{F}_r^+) \subset \mathcal{F}_r^+$; if h is r-compatible, then $h(\mathcal{K}_r^+) \subset \mathcal{K}_r^+$.*

Proof. Consider first the case $r^{m+1} \le r^m$, where $\mathcal{F}_r^+ = \bigcup \mathcal{F}_{r^m}^+$ and $\mathcal{K}_r^+ = \bigcap \mathcal{K}_{r^m}^+$. We have $\forall m, \exists M$: $g(\mathcal{F}_{r^m}^+) \subset \mathcal{F}_{r^M}^+$, thus $\forall m : g(\mathcal{F}_{r^m}^+) \subset \bigcup_M \mathcal{F}_{r^M}^+ = \mathcal{F}_r^+$, which is equivalent to $\bigcup_m g(\mathcal{F}_{r^m}^+) = g(\mathcal{F}_r^+) \subset \mathcal{F}_r^+$. Similarly, $\forall M, \exists m : h(\mathcal{K}_{r^m}^+) \subset \mathcal{K}_{r^M}^+$ implies $\forall M : \bigcap_m h(\mathcal{K}_m) = h(\mathcal{K}_r^+) \subset \mathcal{K}_{r^M}^+$, whence $h(\mathcal{K}_r^+) \subset \bigcap_M \mathcal{K}_{r^M}^+ = \mathcal{K}_r^+$.

In the second case, $r^{m+1} \ge r^m$, where $\mathcal{F}_r^+ = \bigcap \mathcal{F}_{r^m}^+$ and $\mathcal{K} = \bigcup \mathcal{K}_{r^m}^+$, the proofs for $g(\mathcal{F}_r^+)$, $h(\mathcal{K}_r^+)$ are identical to the proofs for $h(\mathcal{K}_r^+)$, $g(\mathcal{F}_r^+)$ in the first case. ∎

Now we give the definition, valid for both the above cases, characterizing maps that extend canonically to $\mathcal{G}_r$:

Definition 74. The map $\varphi : (E, P) \to (F, Q)$ is said to be *continuously r-temperate* iff

(α) $\exists r$-moderate g, $\forall q \in Q$, $\exists p \in P$, $\forall f \in E : q(\varphi(f)) \le g(p(f))$,

(β) $\exists r$-moderate g, $\exists r$-compatible h, $\forall q \in Q$, $\exists p \in P$, $\forall f \in E$, $\forall k \in E$:

$$q(\varphi(f+k) - \varphi(f)) \le g(p(f))h(p(k)).$$

Proposition 75. *Any continuously r-temperate map φ extends canonically to*

$$\Phi = \mathcal{G}_r(\varphi) : \mathcal{G}_{P,r}(E) \to \mathcal{G}_{Q,r}(F).$$

Furthermore, this extension is continuous for the topologies $(\mathcal{G}_{P,r}(E), (\|\cdot\|_{p,r})_{p\in P})$ and $(\mathcal{G}_{Q,r}(F), (\|\cdot\|_{q,r})_{q\in Q})$.

Proof. The proof has two parts: first, the well-definedness of the extension; secondly, the continuity of Φ. As a preliminary remark, observe that $\mathcal{F}_{P,r^m} = \{f \mid \forall p \in P : p(f) \in \mathcal{F}_{r^m}^+\}$, and idem for $\mathcal{K}$. This, and the fact that $\mathcal{K}_{r^m}$ is an ideal in $\mathcal{F}_{r^m}$ (and $\mathcal{F}_{r^m}^+ \cdot \mathcal{K}_{r^m}^+ \subset \mathcal{K}_{r^m}^+$), help us to write the proof using the preceding two characterizations of moderate and compatible maps.

First part of the proof: We will show that (α) implies (F_1), and (β) gives (F_2). Using the respective definitions of moderateness and compatibility, the proof will be different for the two cases $r^{m+1} \le r^m$ and $r^{m+1} \ge r^m$.

Let us start with the case $r^{m+1} \le r^m$, where $\mathcal{F}_{P,r} = \bigcup \mathcal{F}_{P,r^m}$ and $\mathcal{K}_{P,r} = \bigcap \mathcal{K}_{P,r^m}$.

Concerning (F_1), we have $f \in \mathcal{F}_{P,r}(E) \Leftrightarrow \exists m, \forall p : p(f) \in \mathcal{F}_{r^m}^+$. By ($\alpha$), there is g such that $\exists M : g(\mathcal{F}_{r^m}^+) \subset \mathcal{F}_{r^M}^+$, and $\forall q : q(\varphi(f)) \le g(p(f)) \in g(\mathcal{F}_{r^m}^+)$, thus $\exists M, \forall q : q(\varphi(f)) \in \mathcal{F}_{r^M}^+$, that is, $\varphi(f) \in \mathcal{F}_{Q,r}(F)$.

Concerning (F_2), take $f \in \mathcal{F}$ and $k \in \mathcal{K}$, i.e. $\exists m, \forall p : p(f) \in \mathcal{F}_{r^m}^+$ and $\forall m', \forall p : p(k) \in \mathcal{K}_{r^{m'}}^+$. Now fix M and q. With (β), there exists g such that $\forall m, \exists M' : g(\mathcal{F}_{r^m}^+) \subset \mathcal{F}_{r^{M'}}^+$, and there is h such that $\forall M'', \exists m' : h(\mathcal{K}_{r^{m'}}^+) \subset \mathcal{K}_{r^{M''}}^+$. We use this for $M'' = \max(M, M')$, such that $\mathcal{K}_{r^{M''}}^+ \subset \mathcal{K}_{r^{M'}}^+$ and $\mathcal{K}_{r^{M''}}^+ \subset \mathcal{K}_{r^M}^+$. Finally, there exists p such that

$$q(\varphi(f+k) - \varphi(f)) \le g(p(f))h(p(k)) \in g(\mathcal{F}_{r^m}^+)h(\mathcal{K}_{r^{m'}}^+) \subset \mathcal{F}_{r^{M'}}^+ \cdot \mathcal{K}_{r^{M''}}^+.$$

If $M' \le M$, this is in $\mathcal{F}^+_{r^{M'}} \cdot \mathcal{K}^+_{r^M} \subset \mathcal{F}^+_{r^M} \cdot \mathcal{K}^+_{r^M} \subset \mathcal{K}^+_{r^M}$. If $M < M'$, this is in $\mathcal{F}^+_{r^{M'}} \cdot \mathcal{K}^+_{r^{M'}} \subset \mathcal{K}^+_{r^{M'}} \subset \mathcal{K}^+_{r^M}$, because the $\mathcal{K}^+_{r^m}$ form a decreasing sequence. Thus, $\varphi(f+k) - \varphi(f) \in \mathcal{K}_{Q,r}(F)$.

Now we turn to the case $r^{m+1} \ge r^m$, where $\mathcal{F} = \bigcap \mathcal{F}_m$ et $\mathcal{K} = \bigcup \mathcal{K}_m$. Let us show (F_1). We have $f \in \mathcal{F}_{P,r}(E) \Leftrightarrow \forall m, \forall p : p(f) \in \mathcal{F}^+_{r^m}$. By (α), there exists g such that $\forall M, \exists m : g(\mathcal{F}^+_{r^m}) \subset \mathcal{F}^+_{r^M}$, and $\forall q, \exists p : q(\varphi(f)) \le g(p(f)) \in g(\mathcal{F}^+_{r^m})$, thus $\forall M, \forall q : q(\varphi(f)) \in \mathcal{F}^+_{r^M}$, i.e. $\varphi(f) \in \mathcal{F}_{Q,r}(F)$.

Finally, (F_2): Take $f \in \mathcal{F}$ and $k \in \mathcal{K}$, i.e. $\forall m, \forall p : p(f) \in \mathcal{F}^+_{r^m}$ and $\exists m', \forall p : p(k) \in \mathcal{K}^+_{r^{m'}}$. Now fix q. With (β), there exists h such that $\forall m', \exists M : h(\mathcal{K}^+_{r^{m'}}) \subset \mathcal{K}^+_{r^M}$; there is g such that $\forall M, \exists m : g(\mathcal{F}^+_{r^m}) \subset \mathcal{F}^+_{r^M}$, and there exists p such that

$$q(\varphi(f+k) - \varphi(f)) \le g(p(f))h(p(k)) \in g(\mathcal{F}^+_{r^m})h(\mathcal{K}^+_{r^{m'}}) \subset \mathcal{F}^+_{r^M} \cdot \mathcal{K}^+_{r^M} \subset \mathcal{K}^+_{r^M},$$

thus $\varphi(f+k) - \varphi(f) \in \mathcal{K}_{Q,r}(F)$.

Second part of the proof: continuity of Φ. We must show that

$$\forall q \in Q : \|\varphi(f+k) - \varphi(f)\|_{q,r^M} \to 0 \quad \text{when} \quad \forall p \in P : \|k\|_{p,r^m} \to 0$$

and this for all M (resp. for some M), in respective cases. The proof is analogous to the above proof of (F_2), up to replacing $p(f) \in \mathcal{F}^+_{r^m}$ by $\|f\|_{p,m} \le K$, $p(k) \in \mathcal{K}^+_{r^m}$ by $\|k\|_{p,m} \le \varepsilon$, and consequent changes. ∎

6. Association in $\mathcal{G}$

We will introduce different types of association, according to what has already been considered in the literature on generalized function spaces. Generally speaking, we will adopt the following terminology: *strong association* is expressed directly on the level of the factor algebra, while *weak association* will be defined in terms of a duality product, and thus with respect to a certain test function space.

Association in Colombeau type generalized numbers. To start with, recall that Colombeau generalized numbers $[x]$ and $[y]$ are said to be *associated*, $[x] \approx [y]$, iff

$$x_n - y_n \underset{n\to\infty}{\longrightarrow} 0 \quad (\text{in } \mathbb{C}).$$

This can also be expressed by considering the subset of null sequences, $N = \{x \in \mathbb{C}^{\mathbb{N}} \mid \lim x_n = 0\}$, and by defining $[x] \approx [y] \Leftrightarrow x - y \in N$.

As any element j of the ideal satisfies $j_n \to 0$, this is clearly independent of the representative. In other words, it is well defined because $I \subset N$.

6.1. The general concept of $\mathcal{J}, X$-association. The following general concept of association allows us to recover all known notions of association, and encompass some new constructions we shall consider below. The definitions of this subsection can be formulated in a general way for any kind of quotient space of type $\mathcal{G} = \mathcal{F}/\mathcal{K}$, where $\mathcal{K}$ is an ideal of any subalgebra $\mathcal{F}$ of any sequence space of $\overleftrightarrow{E}$ type, for example. The independence from the specific choice under consideration justifies dropping the indices of $\mathcal{G}$, $\mathcal{F}$ and $\mathcal{K}$. When it becomes necessary to distinguish between spaces of numbers and spaces of functions, we append the indices specifying the seminorm, e.g. $\mathcal{K}_{r,p}$ denotes the ideal in the space of functions (p being the net of seminorms defining the topology on the base space), whereas $\mathcal{K}_{r,|\cdot|}$ is the ideal in the space of numbers.

Definition 76 ($\mathcal{J}, X$-association). Let $\mathcal{J}$ be an additive subgroup of $\mathcal{F}$ containing the ideal $\mathcal{K}$ of $\mathcal{F}$, and X a set of generalized numbers. Then two elements $F, G \in \mathcal{G} = \mathcal{F}/\mathcal{K}$ are called $\mathcal{J}, X$-*associated*,

$$F \underset{\mathcal{J},X}{\approx} G \quad \text{iff} \quad \forall x \in X : x \cdot (F - G) \in \mathcal{J}/\mathcal{K}.$$

For $X = \{1\}$, we simply write

$$F \underset{\mathcal{J}}{\approx} G \;\Leftrightarrow\; F - G \in \mathcal{J}/\mathcal{K}.$$

Remark 77. As $\mathcal{J}$ is not an ideal, association is not compatible with multiplication in $\mathcal{F}$ (not even by generalized numbers, only by elements of E). However, in the case of differential algebras, $\mathcal{J}$ is usually chosen such that $\underset{\mathcal{J},X}{\approx}$ is stable under differentiation.

Example 78. Usual association of generalized numbers, as recalled above, is obtained for $\mathcal{J} = N$, the set of null sequences:

$$[x] \approx [y] \;\Leftrightarrow\; [x] \underset{N}{\approx} [y].$$

As already mentioned, all elements of the ideal $\mathcal{K}$ tend to zero, i.e. $\mathcal{K} \subset N$, as needed for well-definedness at the level of the factor algebra.

6.2. Strong association. As mentioned, strong association is defined directly in terms of the ultranorm (or ultrametric) of elements of the factor space.

Definition 79. For $s \in \mathbb{R}_+$, *strong s-association* is defined by

$$F \overset{s}{\simeq} G \;\Leftrightarrow\; F \underset{\mathcal{J}^{(s)}_{\mathcal{P},r}}{\approx} G$$

with

$$\mathcal{J}^{(s)}_{\mathcal{P},r} = \{f \in \mathcal{F} \mid \forall p \in \mathcal{P} : \| f \|_{p,r} < e^{-s}\}, \tag{6.1}$$

which is equivalent to

$$F \overset{s}{\simeq} G \;\Leftrightarrow\; \forall p \in \mathcal{P} : \widetilde{d}_{p,r}(F, G) < e^{-s}.$$

For $s = 0$, we write $F \simeq G$ and simply call them *strongly associated.*

Remark 80. If one has $F \overset{s}{\simeq} G$ for all $s \geq 0$, then $F = G$. Indeed, this means that $F - G$ is in the intersection of all balls of positive radius, which is equal to $\mathcal{K} = 0_{\mathcal{G}}$.

6.3. Weak association in $\overleftrightarrow{\mathcal{G}}_{p,r}$. In contrast to the above, weak association is defined by comparing sequences of *numbers* (not *functions*), obtained by means of a duality product

$$\langle \cdot, \cdot \rangle : \overleftrightarrow{E} \times \boldsymbol{D} \to \mathbb{C},$$

where $\boldsymbol{D}$ is a test function space such that $E \hookrightarrow \boldsymbol{D}'$ (for example $\boldsymbol{D} = \mathcal{D}$ for $E = \mathcal{C}^\infty$). The subset $\mathcal{J}$ defining the association will then be of the form

$$\mathcal{J} = J_M := \{f \in \overleftrightarrow{E}^{\mathbb{N}} \mid \forall \psi \in \boldsymbol{D} : (\langle f_n, \psi \rangle)_n \in M\}, \tag{6.2}$$

where M is some $\mathbb{C}$-linear subspace of $\mathbb{C}^{\mathbb{N}}$, like e.g. $M = N$, the sequences of zero limit.

EXAMPLE 81. For the choices given above, $\boldsymbol{D} = \mathcal{D}$, $E = \mathcal{C}^\infty$ and $M = N$, in the case of Colombeau's algebra, we get the usual, so-called weak association $[f] \approx [g] \Leftrightarrow f_n - g_n \to 0$ in $\mathcal{D}'$.

Again, this is independent of the representatives, because $\mathcal{J} \supset \mathcal{K}_{r,p}$. To see this, consider $j \in \mathcal{K}_{r,p}$. Then for any $\varepsilon > 0$ there is n_0 such that for $n > n_0$,

$$|\langle j_n, \psi\rangle| \leq \varepsilon^{1/r_n} \int |\phi| \underset{n\to\infty}{\longrightarrow} 0.$$

Thus, $\langle f_n, \psi\rangle \underset{n\to\infty}{\longrightarrow} 0 \Leftrightarrow \langle f_n + j_n, \psi\rangle \underset{n\to\infty}{\longrightarrow} 0$.

This example is a special case of the definition given below.

EXAMPLE 82. Taking $M = 0_{\mathbb{C}_r} = \mathcal{K}_{|\cdot|,r}$, we obtain the *weak equality* in $\mathcal{G}(\Omega)$ considered for example in [59]:

$$\forall f, g \in \mathcal{G}(\Omega) : f \underset{(w)}{=} g \;\Leftrightarrow\; \forall \psi \in \mathcal{D}(\Omega) : \int (f(x) - g(x))\psi(x)\, dx = 0 \in \mathbb{C}_r.$$

As already mentioned, all elements of the ideal $\mathcal{K}$ tend to zero, i.e. $\mathcal{K} \subset N$, as needed for well-definedness at the level of the factor algebra.

DEFINITION 83. *s-$\boldsymbol{D}'$-association* is defined by

$$F \overset{s}{\underset{\boldsymbol{D}}{\approx}} G \;\Leftrightarrow\; F \underset{\mathcal{J}_N, X_s}{\approx} G$$

with $X_s = \{[(e^{s/r_n})_n]\}$ for $s \in \mathbb{R}$.

Note that this generalized number is always of the same form, but depends in each case on the sequence $(r_n)_n$ defining the topology.

EXAMPLE 84. In Colombeau's case, $r = 1/\log$, we have $X_s = \{[(n^s)_n]\}$. For $s = 0$ ($X_0 = \{1\}$), we get the already mentioned weak association.

For $s \neq 0$, $[f] \overset{s}{\underset{\mathcal{D}}{\approx}} [g] \Leftrightarrow n^s(f_n - g_n) \to 0$ in $\mathcal{D}'$. This also has already been considered (with $\boldsymbol{D} = \mathcal{D}$), for example in [55] (where it was denoted by $\underset{s}{\approx}$). This association is of course stronger than the simple weak association (again because association is not compatible with multiplication even only by generalized numbers).

As an extension of this example, consider $\mathcal{J}$ as above, and $X = \{[(n^s)_n]\}_{s\in\mathbb{N}}$. This means that

$$[f] \approx [g] \;\Leftrightarrow\; \forall s \in \mathbb{N} : \lim n^s(f_n - g_n) = 0 \text{ in } \mathcal{D}'.$$

While for generalized *numbers*, this equation amounts to strict equality, this is not the case for generalized *functions*. Indeed, consider $\phi_1, \phi_2 \in \mathcal{S}(\mathbb{R})$ such that $\forall \alpha \in \mathbb{N}$: $\int x^\alpha \phi_i = \delta_{\alpha,0}$ (the space of such functions is also denoted $\mathcal{B}^\infty$), but $\phi_1(0) \neq \phi_2(0)$. Let $f_{i,n} = n\phi_i(n\cdot)$. Then $f_1 = [(f_{1,n})] \neq f_2 = [(f_{2,n})]$, but $f_1 \approx f_2$. Indeed, for any $\psi \in \mathcal{D}(\mathbb{R})$, $\int (f_{1,n} - f_{2,n})\psi = \int (\phi_1 - \phi_2)\psi(\cdot/n)$, and expanding ψ in a Maclaurin series gives the expected result.

The same constructions can be applied to generalized Sobolev space (Subsection 1.1.3) and to the full Colombeau algebra (Example 10).

In the case of ultradistributions, we take $\boldsymbol{D} = \mathcal{D}^{(m)}$ and $e^{s/r_n} = \exp[sn^{1/m'}]$ for the Beurling case, and analogously in the Roumieu case.

For periodic hyperfunctions (with $\boldsymbol{D} = \mathcal{A}(\mathbb{T})$) this is also a new construction.

DEFINITION 85. *Weak s-association* is defined for any $s \in \mathbb{R}$ by

$$F \overset{(s)}{\simeq} G \;\Leftrightarrow\; F \underset{\mathcal{J}_{(s)}}{\approx} G$$

where

$$\mathcal{J}_{(s)} = \{f \in E^{\mathbb{N}} \mid \forall\psi \in \boldsymbol{D} : \limsup_{n\to\infty} |\langle f_n, \psi\rangle|^{r_n} < e^{-s}\}.$$

It is obtained from the general setting (6.2) by observing that $\mathcal{J}_{(s)} = J_M$ with

$$M = \mathcal{J}^{(s)}_{|\cdot|,r} = \{c \in \mathbb{C}^{\mathbb{N}} \mid \|c\|_{|\cdot|,r} < e^{-s}\}.$$

For $s = 0$, we write $F \overset{\text{sw}}{\approx} G$ and call F and G *strong-weak associated.*

REMARK 86. Let us consider some details concerning the structure of strong-weak association. In the following we will write $|\cdot|_r = \|\cdot\|_{|\cdot|,r}$, i.e.

$$|c|_r = \limsup_{n\to\infty} |c_n|^{r_n}.$$

First, let us remark that $\mathcal{I}_{|\cdot|,r} = \{c \in \mathbb{C}^{\mathbb{N}} \mid |c|_r < 1\}$ is an ideal in the subalgebra $\mathcal{H}_{|\cdot|,r} = \{c \in \mathbb{C}^{\mathbb{N}} \mid |c|_r \leq 1\}$ of $\mathbb{C}^{\mathbb{N}}$.

Let us now consider the topology on $\mathbb{C}^{\mathbb{N}}$ induced by the $|\cdot|_r$-norm. We have

$$|c|_r \leq a \;\Leftrightarrow\; \forall b > a, \exists n_0, \forall n > n_0 : |c_n| \leq b^{1/r_n}.$$

For $b > 1$, $b = 1$ and $b < 1$, the limit of the r.h.s. is respectively ∞, 1 and 0. This means that:

(i) If $|c|_r < 1$, then $\lim c_n = 0$. Thus, all elements of the open unit ball are associated to zero. This is very similar to classical results relating to ultrametric spaces and weak topology.

(ii) If $\lim c_n = 0$, then $\forall b > 1, \exists n_0, \forall n \geq n_0 : |c_n| \leq b^{1/r_n} \to \infty$, and thus $|c|_r \leq 1$: All elements associated to zero are in the *closed* unit ball. (Recall in this context that in ultrametric spaces, open balls and closed balls are open-and-closed sets.)

(iii) When $|c|_r = 1$, the sequence (c_n) can have any limit in $\mathbb{R}_+ \cup \{\infty\}$, or none at all. Indeed, for any null sequence (r_n), the sequences $c_n = r_n$ (resp. $c_n = 1/r_n$) have limits 0 (resp. ∞), while $|c|_r = 1$, since

$$|c_n|^{r_n} = \exp(\pm r_n \log r_n) \underset{n\to\infty}{\longrightarrow} 1 \qquad (\text{because } x \log x \underset{x\to 0}{\longrightarrow} 0).$$

PROPOSITION 87. *Weak s-association implies s-$\boldsymbol{D}'$-association, but conversely s-$\boldsymbol{D}'$-association only implies weak s'-association for all $s' < s$.*

Proof. This follows from $|c|_r < 1 \Rightarrow \lim c_n = 0 \Rightarrow |c|_r \leq 1$, with $c_n = \langle f_n, \psi\rangle e^{s/r_n}$. As discussed in point (iii) of the above remark, for $|c|_r = 1$, nothing can be concluded about the limit of (c_n).

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