1. Introduction

The theory of elliptic variational inequalities started in the middle 60's and today is a well developed area of mathematics. It is closely connected with the convexity of the energy functional involved and with the notion of subdifferential in the sense of convex analysis. The existence theory of variational inequalities is based on monotonicity arguments. So, for example, only monotone (possibly multivalued) boundary conditions and stress-strain laws can be studied.

In the case of lack of monotonicity of the underlying law, or equivalently if the corresponding energy functional is nonconvex, the variational expression is no longer a variational inequality. Another type of inequality expression arises in the variational formulation of the problem. These new variational expressions are known in the literature as "hemivariational inequalities" and are based on the generalized subdifferential in the sense of Clarke of locally Lipschitz functionals. Roughly speaking, mechanical problems involving nonmonotone boundary conditions or stress-strain laws derived by nonconvex superpotentials lead to hemivariational inequalities. For concrete applications to problems of continuum mechanics and engineering, we refer to the books of Naniewicz–Panagiotopoulos [56] and Panagiotopoulos [57]. The mathematical theory of hemivariational inequalities can be found in the book of Motreanu–Panagiotopoulos [50].

A typical feature of nonconvex problems is that, while in the convex case the stationary variational inequalities give rise to minimization problems for the energy, in the nonconvex case the problem of the stationarity of the potential emerges and so we look for local critical points (e.g. minima or maxima) of nonsmooth energy functionals. It is therefore reasonable to expect that critical point theory (its nonsmooth variant) can play a prominent role in the analysis of hemivariational inequalities. Indeed, thus far the study of hemivariational inequalities has focused on problems of variational nature and consequently the tools employed came from nonsmooth critical point theory as developed by Chang [16] (for extensions we refer to Kourogenis–Papageorgiou [39] and Kyritsi–Papageorgiou [41]). We refer to the works of Bocea [9], Goeleven–Motreanu–Panagiotopulos [31]–[33], Marzocchi [46], Motreanu–Naniewicz [48], Motreanu–Panagiotopoulos [49], Panagiotopoulos–Radulescu [58] who treat semilinear problems and to the papers of Gasiński–Papageorgiou [24]–[28] who deal with quasilinear problems involving the p-Laplacian operator. Naniewicz [51]–[54] approached hemivariational inequalities using tools from nonlinear analysis such as fixed point theory, Ekeland's variational principle, Galerkin's approximations and nonlinear operators of monotone type (see also Naniewicz [55]).

In this paper we consider larger classes of hemivariational inequalities, not necessarily of variational nature. Using a variety of analytical tools, such as the theory of nonlinear operators of monotone type, multivalued analysis, degree theory, the method of upper and lower solutions, comparison theorems, nonlinear maximum principles and of course nonsmooth critical point theory, we prove existence theorems, multiplicity theorems and obtain positive bounded solutions for a variety of nonlinear hemivariational inequality problems.

In order to make the paper self-contained, in the next section we present the basic mathematical tools that we shall use in the study of our problems.

2. Mathematical background

In our methods of proof, among other things, prominent are the theory of nonlinear operators of monotone type and the techniques of multivalued analysis. So we start the presentation of the relevant mathematical background with a presentation of some basic definitions and facts from these areas of nonlinear analysis. Our main sources are the books of Hu–Papageorgiou [35], [36], Showalter [59] and Zeidler [64].

Let (Ω, Σ) be a measurable space and X a separable Banach space. Throughout this paper we shall use the following notation:

$$\mathcal{P}_{f(c)}(X) = \{A \subseteq X : \text{nonempty, closed (and convex)}\},$$
$$\mathcal{P}_{(w)k(c)} = \{A \subseteq X : \text{nonempty, (weakly) compact (and convex)}\}.$$

Also given $A \subseteq X$ we set $d(x, A) = \inf[\|x - \alpha\| : \alpha \in A]$, the distance of $x \in X$ from A, and for every $x^* \in X^*$, $\sigma(x^*, A) = \sup[(x^*, \alpha) : \alpha \in A]$, the support function of A. The function $x \mapsto d(x, A)$ is Lipschitz continuous with constant 1 (i.e. is nonexpansive) and it is convex if $A \subseteq X$ is a convex subset. The support function $\sigma(\cdot, A) : X^* \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ is sublinear and w^*-lower semicontinuous. Moreover, if $A \in \mathcal{P}_{wk}(X)$, then $\sigma(\cdot, A)$ is continuous for the Mackey topology $m(X, X^*)$ on X^*.

A multifunction $F : \Omega \to 2^X \setminus \{\emptyset\}$ is said to be *graph measurable* if $\operatorname{Gr} F = \{(\omega, x) \in \Omega \times X : x \in F(\omega)\}$ (the graph of F) belongs to the product σ-field $\Sigma \times B(X)$, where $B(X)$ is the Borel σ-field of X. A multifunction $F : \Omega \to \mathcal{P}_f(X)$ is said to be *measurable* if for all $x \in X$ the distance function $\omega \mapsto d(x, F(\omega)) = \inf[\|x - u\| : u \in F(\omega)]$ is measurable. For $\mathcal{P}_f(X)$-valued multifunctions measurability implies graph measurability and moreover, if there is a σ-finite measure μ on (Ω, Σ) with respect to which Σ is complete, then the two notions are equivalent.

Let (Ω, Σ, μ) be a σ-finite measure space and $F : \Omega \to 2^X \setminus \{\emptyset\}$. For $1 \leq p \leq \infty$ let S_F^p be the set of all $L^p(\Omega, X)$-selectors of F, i.e. $S_F^p = \{f \in L^p(\Omega, X) : f(\omega) \in F(\omega)\ \mu\text{-a.e. on } \Omega\}$. This set may be empty. For a graph measurable multifunction F the set S_F^p is nonempty if and only if $\inf\{\|u\| : u \in F(\omega)\} \leq h(\omega)$ μ-a.e. on Ω with $h \in L^p(\Omega)_+$. We can show that S_F^p is closed (resp. convex) in $L^p(\Omega, X)$ if and only if for μ-almost all $\omega \in \Omega$, $F(\omega)$ is closed (resp. convex) in X (for the convexity we also need μ to be nonatomic).

Given a multifunction $F : \Omega \to 2^X \setminus \{\emptyset\}$, by a *measurable selection* of F we mean a measurable function $f : \Omega \to X$ such that $f(\omega) \in F(\omega)$ μ-a.e. on Ω. One of the main results on the existence of measurable selections is the so-called "Yankov–von Neumann–Aumann selection theorem" (see Hu–Papageorgiou [35, p. 158]).

THEOREM 1. *If* (Ω, Σ, μ) *is a* σ*-finite measure space*, X *is a separable Banach space and* $F : \Omega \to 2^X \setminus \{\emptyset\}$ *is a graph measurable multifunction*, *then* F *admits a measurable selection* f.

REMARK. If Σ is μ-complete we can say that $f(\omega) \in F(\omega)$ for all $\omega \in \Omega$.

If (Y, d) is a metric space and $\{C_n\}_{n\ge1}$ is a sequence of nonempty subsets of X, we define

$$\liminf_{n\to\infty} C_n = \{y \in Y : y = \lim y_n,\ y_n \in C_n,\ n \ge 1\} = \{y \in Y : \lim d(y, C_n) = 0\},$$

$$\begin{aligned}\limsup_{n\to\infty} C_n &= \{y \in Y : y = \lim y_{n_k},\ y_{n_k} \in C_{n_k},\ n_1 < \ldots < n_k < \ldots\}\\ &= \{y \in Y : \liminf d(y, C_n) = 0\}.\end{aligned}$$

Evidently, $\liminf_{n\to\infty} C_n \subseteq \limsup_{n\to\infty} C_n$ and both sets are closed (possibly empty). If $\liminf_{n\to\infty} C_n = \limsup_{n\to\infty} C_n = C$, then we say that the C_n's *converge to* C *in the Kuratowski sense* and write $C_n \xrightarrow{K} C$ as $n \to \infty$. If Y is actually a Banach space, then we can also define

$$w\text{-}\limsup_{n\to\infty} C_n = \{y \in Y : y = w\text{-}\lim_k y_{n_k},\ y_{n_k} \in C_{n_k},\ n_1 < \ldots < n_k < \ldots\}.$$

Here w stands for the weak topology on Y. In general a weakly convergent sequence in the Lebesgue–Bochner space $L^p(\Omega, X)$ $(1 \le p < \infty)$ is not pointwise convergent. However, we have the following result which will be useful in what follows:

PROPOSITION 2. *If* (Ω, Σ, μ) *is a* σ*-finite measure space*, X *is a Banach space*, $F(\omega) \in \mathcal{P}_{wk}(X)$ *for all* $\omega \in \Omega$, $\{f_n\}_{n\ge1} \subseteq L^p(\Omega, X)$ $(1 \le p < \infty)$, $f_n(\omega) \in F(\omega)$ μ*-a.e. on* Ω, $n \ge 1$ *and* $f_n \xrightarrow{w} f$ *in* $L^p(\Omega, X)$, *then* $f(\omega) \in \overline{\mathrm{conv}}(w - \limsup_{n\to\infty}\{f_n(\omega)\})$ *for* μ*-almost all* $\omega \in \Omega$.

Let V, Z be two Hausdorff topological spaces. A multifunction $F : V \to 2^Z \setminus \{\emptyset\}$ is said to be *lower semicontinuous* (lsc for short) (resp. *upper semicontinuous*, usc for short) if for every $C \subseteq Z$ closed, the set $F^+(C) = \{v \in V : F(v) \subseteq C\}$ (resp. $F^-(C) = \{v \in V : F(v) \cap C \ne \emptyset\}$) is closed in V. If Z is regular, then a $\mathcal{P}_f(Z)$-valued, usc multifunction F has a closed graph. In fact, if F is $\mathcal{P}_k(Z)$-valued, then we can drop the requirement that Z is regular. The converse is not in general true. However, if $F : V \to \mathcal{P}_f(Z)$ has closed graph and it is locally compact (i.e. for every $v \in V$, there exists a neighborhood $\mathcal{U}$ of v such that $\overline{F(\mathcal{U})} \in \mathcal{P}_k(Z)$), then F is usc. If Z is a metric space and $F : V \to 2^Z \setminus \{\emptyset\}$, then F is lsc if and only if for all $z \in Z$, $v \mapsto \varphi_z(v) = d(z, F(v))$ is upper semicontinuous on V (here d denotes the metric in Z). A multifunction which is both usc and lsc, is said to be *continuous* (or *Vietoris continuous*).

If Z is a metric space and $A, C \subseteq Z$ are two nonempty sets, we define

$$h^*(A, C) = \sup[d(\alpha, C) : \alpha \in A] \qquad \text{(the *excess* of A over C)},$$

$$h(A, C) = \max\{h^*(A, C), h^*(C, A)\} \qquad \text{(the *Hausdorff distance* between A and C)}.$$

We know that h is a generalized metric on $\mathcal{P}_f(Z)$, called the *Hausdorff metric*, and if Z is a complete metric space, then so is $(\mathcal{P}_f(Z), h)$. A multifunction $F : V \to 2^Z \setminus \{\emptyset\}$ is said to be *h-lower semicontinuous* (h-lsc for short) (resp. h-upper semicontinuous, h-usc for short) if for all $v \in V$, the function $u \mapsto h^*(F(v), F(u))$ is continuous at v (resp. the function $u \mapsto h^*(F(u), F(v))$ is continuous at v). In general, h-lsc $\Rightarrow$ lsc and usc $\Rightarrow$ h-usc and the converse implications hold if F is $\mathcal{P}_k(Z)$-valued. A multifunction F which is both h-lsc and h-usc is said to be *h-continuous*. Evidently for $\mathcal{P}_k(Z)$-valued multifunctions continuity and h-continuity are equivalent notions.

Now let X be a reflexive Banach space and X^* its topological dual. A map $A : D \subseteq X \to 2^{X^*}$ is said to be *monotone* if for all $x, y \in D$ and all $x^* \in A(x)$, $y^* \in A(y)$ we have $(x^* - y^*, x - y) \geq 0$ (here by $(\cdot, \cdot)$ we denote the duality brackets for the pair (X, X^*)). If in addition $(x^* - y^*, x - y) = 0$ implies $x = y$, then we say that A is *strictly monotone*. We say that A is *maximal monotone* if the fact that $(x^* - y^*, x - y) \geq 0$ for all $x \in D$ and all $x^* \in A(x)$ implies that $y \in D$ and $y^* \in A(y)$. Stating this in a different way, A is maximal monotone if and only if its graph is maximal with respect to inclusion among the graphs of all monotone maps. It is easy to see that if $A : D \subseteq X \to 2^{X^*}$ is maximal monotone, then $\operatorname{Gr} A$ is sequentially closed in $X \times X_w^*$ and in $X_w \times X^*$ (here by X_w and X_w^* we denote the spaces X and X^* furnished with their respective weak topologies). In the next proposition we provide conditions which make a monotone operator maximal monotone (see Hu–Papageorgiou [35, p. 309]).

Proposition 3. *If X is a reflexive Banach space, $A : D \subseteq X \to 2^{X^*}$ is a monotone map with nonempty, closed and convex values, and for every $x, h \in X$ the multifunction $t \mapsto A(x + th)$ is usc from $[0, 1]$ into X_w^*, then A is maximal monotone.*

An operator $A : X \to X^*$ which is single-valued and everywhere defined on X (i.e. $D = X$) is said to *be demicontinuous* if $x_n \to x$ in X implies that $A(x_n) \xrightarrow{w} A(x)$ in X^*. A monotone, demicontinuous map is maximal monotone.

A map $A : D \subseteq X \to 2^{X^*}$ is said to be *coercive* if D is bounded or D is unbounded and $\inf\{\|x^*\| : x^* \in A(x)\} \to \infty$ as $\|x\| \to \infty$, $x \in D$. A maximal monotone, coercive operator is surjective (see Hu–Papageorgiou [35, p. 322]).

An operator $A : X \to 2^{X^*}$ is said to be *pseudomonotone* if the following conditions are satisfied:

(a) for all $x \in X$, we have $A(x) \in \mathcal{P}_{wkc}(X^*)$;

(b) for every finite-dimensional subspace V of X, $A|_V$ is usc from V into X_w^*;

(c) if $x_n \xrightarrow{w} x$ in X, $x_n^* \in A(x_n)$ for all $n \geq 1$ and $\limsup_{n\to\infty}(x_n^*, x_n - x) \leq 0$, then for every $y \in X$, there exists $x^*(y) \in A(x)$ such that $(x^*(y), x - y) \leq \liminf_{n\to\infty}(x_n^*, x_n - y)$.

If A is bounded (namely it maps bounded sets to bounded sets) and satisfies condition (c), then it satisfies condition (b) as well. An operator $A : D \subseteq X \to 2^{X^*}$ is said to be *generalized pseudomonotone* if $x_n \xrightarrow{w} x$ in X, $x_n^* \xrightarrow{w} x^*$ in X^*, $x_n^* \in A(x_n)$ for $n \geq 1$ and $\limsup(x_n^*, x_n - x) \leq 0$ imply $x^* \in A(x)$ and $(x_n^*, x_n) \to (x^*, x)$. Every maximal monotone operator is generalized pseudomonotone. Also a pseudomonotone operator is generalized pseudomonotone, while the converse is true if the operator has values in $\mathcal{P}_{wkc}(X^*)$ and it is bounded (see Hu–Papageorgiou [35, p. 366]). A pseudomonotone coercive operator is surjective.

As we already mentioned in Section 1, hemivariational inequalities are based on the Clarke subdifferential (see Clarke [18]) and the corresponding nonsmooth critical point theory uses the Clarke subdifferential calculus. So let us recall the basic definitions and facts of Clarke's theory. For further details we refer to Clarke [18]. Let Y be a Banach space and $\varphi : Y \to \mathbb{R}$. We say that φ is *locally Lipschitz* if for every bounded open subset $\mathcal{U} \subseteq Y$, there exists a constant $k > 0$ depending on $\mathcal{U}$ such that $|\varphi(y) - \varphi(u)| \le k\|y - u\|$ for all $y, u \in \mathcal{U}$. It is a well known fact of convex analysis that a proper convex and lower semicontinous function $\psi : Y \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ is locally Lipschitz in the interior of its effective domain $\operatorname{dom}\psi = \{x \in Y : \psi(x) < \infty\}$. In analogy to the directional derivative of a convex function, for a locally Lipschitz function $\varphi : Y \to \mathbb{R}$, we define the generalized directional derivative at $y \in Y$ in the direction $h \in Y$ by

$$\varphi^0(y; h) = \limsup_{y' \to y,\, \lambda \downarrow 0} \frac{\varphi(y' + \lambda h) - \varphi(y')}{\lambda}.$$

It is easy to check that the function $h \mapsto \varphi^0(y; h)$ is sublinear continuous and so by the Hahn–Banach theorem it is the support function of a nonempty, convex and w^*-compact set

$$\partial\varphi(y) = \{y^* \in Y^* : (y^*, h) \le \varphi^0(y; h) \text{ for all } h \in Y\}.$$

The set $\partial\varphi(y)$ is called the *generalized* (*Clarke*) *subdifferential* of φ at $y \in Y$. If $\varphi, \psi : Y \to \mathbb{R}$ are two locally Lipschitz functions, then for all $y \in Y$ and all $\lambda \in \mathbb{R}$ we have $\partial(\varphi + \psi)(y) \subseteq \partial\varphi(y) + \partial\psi(y)$ and $\partial(\lambda\varphi)(y) = \lambda\partial\varphi(y)$. Moreover, if $\varphi : Y \to \mathbb{R}$ is also convex, as we already mentioned, it is locally Lipschitz and the generalized subdifferential coincides with the subdifferential in the sense of convex analysis which is defined by $\partial\varphi(y) = \{y^* \in Y^* : (y^*, z - y) \le \varphi(z) - \varphi(y) \text{ for all } z \in Y\}$. Also if φ is strictly differentiable (in particular if φ is continuously Gateaux differentiable) at $y \in Y$, then $\partial\varphi(y) = \{\varphi'(y)\}$.

Given a locally Lipschitz function $\varphi : Y \to \mathbb{R}$, a point $y \in Y$ is said to be a *critical point* of φ if $0 \in \partial\varphi(y)$. If $\varphi \in C^1(Y)$, then as we just said $\partial\varphi(y) = \{\varphi'(y)\}$ and so this definition of critical point coincides with the classical (smooth) one. It is easy to see that if $y \in Y$ is a local extremum of φ (i.e. a local minimum or a local maximum), then $0 \in \partial\varphi(y)$. From the smooth critical point theory, we know that crucial is a compactness condition, known as the Palais–Smale condition (PS-condition for short). In the present nonsmooth setting this condition takes the following form (see Chang [16]): *A locally Lipschitz function $\varphi : Y \to \mathbb{R}$ satisfies the nonsmooth PS-condition if every sequence $\{y_n\}_{n\ge1} \subseteq Y$ such that $\{\varphi(y_n)\}_{n\ge1}$ is bounded and $m(y_n) = \inf\{\|y_n^*\| : y_n^* \in \partial\varphi(y_n)\} \to 0$ has a strongly convergent subsequence*. Kourogenis–Papageorgiou [39], following the lead in the smooth case of Cerami [15] and Bartolo–Benci–Fortunato [7], proved that a weaker form of the PS-condition suffices to have a deformation lemma and through it derive minimax principles locating critical points. This weaker compactness condition is known as the (*nonsmooth*) *C-condition* and has the form: *Every sequence $\{y_n\}_{n\ge1} \subseteq Y$ such that $\{\varphi(y_n)\}_{n\ge1}$ is bounded and $(1 + \|y_n\|)m(y_n) \to 0$ has a strongly convergent subsequence*. Using this condition, Kourogenis–Papageorgiou [39] obtained a deformation lemma, which was then used to obtain the following basic minimax principle. First a definition:

Definition. Let Z be a Hausdorff space and $C_1 \subseteq C$ and D subsets of Z. We say that C_1 and D *link* in Z if

(a) $C_1 \cap D = \emptyset$,
(b) for every $\vartheta \in C(C, Z)$ with $\vartheta|_{C_1} =$ identity, we have $\vartheta(C) \cap D \neq \emptyset$.

Kourogenis–Papageorgiou [39] proved the following minimax principle:

Theorem 4. *If X is a reflexive Banach space, $C_1 \subseteq C$ and D are nonempty subsets of X with D closed, C_1 and D link in X, $\varGamma_0 = \{\vartheta \in C(C, X) : \vartheta|_{C_1} = identity\}$, $\varphi : X \to \mathbb{R}$ is locally Lipschitz and satisfies the nonsmooth C-condition, $c = \inf_{\vartheta\in\varGamma_0} \sup_{c\in C} \varphi(\vartheta(c))$ and $\sup_{C_1} \varphi \leq \inf_D \varphi$, then $c \geq \inf_D \varphi$ and c is a critical value of φ (i.e. there exists a critical point $x \in X$ of φ such that $\varphi(x) = c$). Moreover, if $c = \inf_D \varphi$ then there exists $x \in D$ such that $x \in K_c = \{x \in X : 0 \in \partial\varphi(x),\ \varphi(x) = c\}$.*

As usual, with appropriate choices of C_1, C and D, Kourogenis–Papageorgiou [39] proved nonsmooth versions of the Mountain Pass Theorem, Generalized Mountain Pass Theorem and Saddle Point Theorem (actually with relaxed boundary conditions).

In our study of problems involving the p-Laplacian operator, we shall need to use the known facts about the spectrum of $(-\varDelta_p, W_0^{1,p}(Z))$. More precisely, let $Z \subseteq \mathbb{R}^N$ be a bounded open domain with a locally Lipschitz boundary $\varGamma$ and consider the following nonlinear eigenvalue problem:

$$\begin{cases} -\operatorname{div}(\|Dx(z)\|^{p-2}Dx(z)) = \lambda|x(z)|^{p-2}x(z) & \text{a.e. on } Z,\\ x|_\varGamma = 0. \end{cases} \tag{1}$$

The least $\lambda \in \mathbb{R}$ for which (1) has a notrivial solution is called the *first eigenvalue* λ_1 of $(-\varDelta_p, W_0^{1,p}(Z))$. The first eigenvalue λ_1 is positive, isolated and simple (i.e. the associated eigenfunctions are constant multiples of each other). Moreover, λ_1 has a variational characterization via the Rayleigh quotient, namely

$$\lambda_1 = \inf\left[\frac{\|Dx\|_p^p}{\|x\|_p^p} : x \in W_0^{1,p}(Z),\ x \not\equiv 0\right]. \tag{2}$$

For details see Anane [3] (where $\varGamma$ is a C^2-manifold) and Lindqvist [45] (the general case). The infimum in (2) is realized at the normalized eigenfunction u_1. Note that if u_1 minimizes the Rayleigh quotient, so does $|u_1|$ and so it follows that u_1 does not change sign on Z. In fact, if $\varGamma$ is a $C^{1,\alpha}$-manifold $(0 < \alpha < 1)$, from the nonlinear regularity theorem of Lieberman [44] we have $u_1 \in C^{1,\beta}(\overline{Z})$, $0 < \beta < 1$, and $u_1(z) \neq 0$ for all $z \in Z$.

So we may always assume that $u_1(z) > 0$ for all $z \in Z$. The Lyusternik–Schnirelmann theory gives, in addition to λ_1, a whole strictly increasing sequence $\{\lambda_k\}_{k\geq 1} \subseteq \mathbb{R}_+$ for which problem (1) has a nontrivial solution. These numbers are defined as follows. We introduce the set $S = \{x \in W_0^{1,p}(Z) : \|Dx\|_p = 1\}$ and the function $\psi : S \to \mathbb{R}_-$ defined by $\psi(x) = -\|x\|^p$. Also let $\mathcal{A}_n = \{K \subseteq S : K \text{ is symmetric, closed and } \gamma(K) \geq n\}$ where γ denotes the Krasnosel'skiĭ $\mathbb{Z}_2$-genus (see Struwe [60, p. 86]). We set

$$c_n = \inf_{K\in\mathcal{A}_n} \sup_{x\in K} \psi(x). \tag{3}$$

The sequence $\{\lambda_n = -1/c_n\}_{n\geq 1}$ is strictly increasing and tends to ∞. These numbers are called the *Lyusternik–Schnirelmann* (or *variational*) eigenvalues of $(-\Delta_p, W_0^{1,p}(Z))$.

When $p = 2$ (linear case), from the spectral analysis of compact self-adjoint operators, we know that the variational eigenvalues are all the eigenvalues of $(-\Delta, H_0^1(Z))$. However, when $p \neq 2$ we do not know if this is the case. Recently Anane–Tsouli [4] proved that if $\lambda_2^* = \inf[\lambda > \lambda_1 : \lambda$ is an eigenvalue of $(-\Delta_p, W_0^{1,p}(Z))]$, then $\lambda_2^* = \lambda_2$, i.e. the second eigenvalue and the second variational eigenvalue coincide. So the spectrum of $(-\Delta_p, W_0^{1,p}(Z))$ starts with two Lyusternik–Schnirelmann (variational) eigenvalues. Define

$$V_k = \{x \in W_0^{1,p}(Z) : -\operatorname{div}(\|Dx(z)\|^{p-2}Dx(z)) = \lambda_k|x(z)|^{p-2}x(z) \text{ a.e. on } Z\} \qquad k \geq 1.$$

These are symmetric, closed cones, but in general are not subspaces of $W_0^{1,p}(Z)$, unless λ_k is simple. Also if $W_n = \bigcup_{k=1}^n V_k$ and $\widehat{W}_n = \bigcup_{k>n} V_n$, then in contrast to the linear case $(p = 2)$, in general we do not have the inequalities

$$\|Dx\|_p^p \leq \lambda_n\|x\|_p^p \quad \text{for } x \in W_n \quad \text{and} \quad \|Dx\|_p^p \geq \lambda_{n+1}\|x\|_p^p \quad \text{for } x \in \widehat{W}_n.$$

This fact is the source of difficulties in constructing linkings in the quasilinear case.

Finally let V, W be Banach spaces and $K : V \to W$. We say that K is

(a) *completely continuous* if $v_n \xrightarrow{w} v$ in V implies $K(v_n) \to K(v)$ in W;
(b) *compact* if it is continuous and maps bounded sets to relatively compact sets.

In general these two notions are distinct. However, if V is reflexive, then complete continuity implies compactness. Moreover, if V is reflexive and K is linear, the two notions are equivalent. Also a multivalued map $G : V \to 2^W \setminus \{\emptyset\}$ is said to be *compact* if it is usc and maps bounded sets in V to relatively compact subsets of W. In this work, we shall need the following multivalued generalization of the classical Leray–Schauder alternative principle, which is due to Bader [6]. So let $G : V \to \mathcal{P}_{wkc}(W)$ be an usc multifunction from V into W_w (= the Banach space W furnished with the weak topology), let $K : W \to V$ be a completely continuous map and set $\Phi = K \circ G$.

THEOREM 5. *If V, W, Φ are as above and Φ is compact, then one of the following alternatives holds*:

(a) *$S = \{v \in V : v \in \beta\Phi(v)$ for some $\beta \in (0,1)\}$ is unbounded, or*
(b) *Φ has a fixed point $v \in V$ (i.e. $v \in \Phi(v)$).*

3. Strongly nonlinear hemivariational inequalities

In this section we study a very general nonlinear hemivariational inequality, where the differential operator is multivalued and depends on both x and Dx, and the right hand side nonlinearity is also dependent on both x and Dx. More specifically, let $Z \subseteq \mathbb{R}^N$ be a bounded open domain with a C^1 boundary Γ. In this section the problem under consideration is the following:

$$(4) \qquad \begin{cases} -\operatorname{div} \alpha(z, x(z), Dx(z)) - \partial j(z, x(z)) \ni f(z, x(z), Dx(z)) & \text{a.e. on } Z, \\ x|_\Gamma = 0. \end{cases}$$

Here $\alpha : Z \times \mathbb{R} \times \mathbb{R}^N \to 2^{\mathbb{R}^N} \setminus \{\emptyset\}$ and for every $x \in W_0^{1,p}(Z)$, by $\operatorname{div} \alpha(z, x(z), Dx(z))$ we understand the set

$$\{\operatorname{div} v(z) : v \in L^q(Z, \mathbb{R}^N),\ v(z) \in \alpha(z, x(z), Dx(z)) \text{ a.e. on } Z\}$$

with $1/p + 1/q = 1$ (hence $v \in S^q_{\alpha(\cdot, x(\cdot), Dx(\cdot))}$). By a *solution* of (4) we mean a function $x \in W_0^{1,p}(Z)$ for which there exist $v \in S^q_{\alpha(\cdot, x(\cdot), Dx(\cdot))}$ and $u^* \in S^q_{\partial j(\cdot, x(\cdot))}$ such that

$$-\operatorname{div} v(z) - u^*(z) = f(z, x(z), Dx(z)) \quad \text{a.e. on } Z.$$

Our hypotheses on the data of (4) are the following:

$\boxed{H(\alpha)_1}$ $\alpha : Z \times \mathbb{R} \times \mathbb{R}^N \to \mathcal{P}_{kc}(\mathbb{R}^N)$ is a multifunction such that

(i) $(z, x, \xi) \mapsto \alpha(z, x, \xi)$ is graph measurable;
(ii) for almost all $z \in Z$ and all $x \in \mathbb{R}$, $\xi \mapsto \alpha(z, x, \xi)$ is strictly monotone;
(iii) for almost all $z \in Z$, $(x, \xi) \mapsto \alpha(z, x, \xi)$ has closed graph, while for almost all $z \in Z$ and all $\xi \in \mathbb{R}^N$, $x \mapsto \alpha(z, x, \xi)$ is lsc;
(iv) for almost all $z \in Z$, all $x \in \mathbb{R}$, all $\xi \in \mathbb{R}^N$ and all $v \in \alpha(z, x, \xi)$ we have
$$\|v\| \le b_1(z) + c_1(|x|^{p-1} + \|\xi\|^{p-1}) \quad \text{with } b_1 \in L^q(Z)_+,\ c_1 > 0,\ p \ge 2;$$
(v) for almost all $z \in Z$, all $x \in \mathbb{R}$, all $\xi \in \mathbb{R}^N$ and all $v \in \alpha(z, x, \xi)$ we have
$$(v, \xi)_{\mathbb{R}^N} \ge \eta_1 \|\xi\|^p - \eta_2 \quad \text{with } \eta_1, \eta_2 > 0.$$

Remark. A possibility for the multifunction $\alpha(z, x, \xi)$ is when $\alpha(z, x, \xi) = \vartheta(z, x)\partial\psi(z, \xi)$ where $\vartheta(z, x)$ is a Carathéodory function (i.e. measurable in $z \in Z$, continuous in $x \in \mathbb{R}$), $\vartheta \ge 0$ and ψ is also a Carathéodory function, strictly convex and not necessarily differentiable in $\xi \in \mathbb{R}^N$. Here $\partial\psi(z, \xi)$ denotes the subdifferential in the sense of convex analysis with respect to the ξ-variable. Suppose that $\xi \mapsto \partial\psi(z, \xi)$ has $(p-1)$-growth and it is coercive. Then this $\alpha(z, x, \xi)$ satisfies the hypotheses $H(\alpha)_1$.

$\boxed{H(j)_1}$ $j : Z \times \mathbb{R} \to \mathbb{R}$ is a function such that

(i) for all $x \in \mathbb{R}$, $z \mapsto j(z, x)$ is measurable;
(ii) for almost all $z \in Z$, $x \mapsto j(z, x)$ is locally Lipschitz;
(iii) for almost all $z \in Z$, all $x \in \mathbb{R}$ and all $u^* \in \partial j(z, x)$ we have
$$|u^*| \le b_2(z) + c_2|x|^{r-1} \quad \text{with } b_2 \in L^{r'}(Z),\ 1/r + 1/r' = 1,\ c_2 > 0$$
and
$$1 \le r < p^* = \begin{cases} \dfrac{Np}{N-p} & \text{if } p < N, \\ \infty & \text{otherwise;} \end{cases}$$
(iv) there exists $\vartheta \in L^\infty(Z)_+$ such that
$$\limsup_{|x| \to \infty} \frac{u^*}{|x|^{p-2}x} \le \vartheta(z)$$
uniformly for almost all $z \in Z$ and all $u^* \in \partial j(z, x)$, and $\vartheta(z) \le \lambda_1 \eta_1$ a.e. on Z, with strict inequality on a set of positive Lebesgue measure (here λ_1 is the first eigenvalue of $(-\Delta_p, W_0^{1,p}(Z))$ and η_1 is as in hypothesis $H(\alpha)_1$(v)).

$\boxed{H(f)_1}$ $f : Z \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a function such that

(i) for all $(x, \xi) \in \mathbb{R} \times \mathbb{R}^N$, $z \mapsto f(z, x, \xi)$ is measurable;
(ii) for almost all $z \in Z$, $(x, \xi) \mapsto f(z, x, \xi)$ is continuous;
(iii) for almost all $z \in Z$, all $x \in \mathbb{R}$ and all $\xi \in \mathbb{R}^N$ we have

$$|f(z, x, \xi)| \leq b_3(z) + c_3(|x|^{\theta-1} + \|\xi\|^{\theta-1})$$

with $b_3 \in L^q(Z)$, $c_3 > 0$, $1 \leq \theta < p$.

We consider the multivalued operator $V : W_0^{1,p}(Z) \to \mathcal{P}_{wkc}(W^{-1,q}(Z))$ defined by

$$V(x) = \{-\operatorname{div} v : v \in S^q_{\alpha(\cdot, x(\cdot), Dx(\cdot))}\}.$$

Also for fixed $x \in W_0^{1,p}(Z)$ consider the auxiliary operator $K_x : W_0^{1,p}(Z) \to 2^{W^{-1,q}(Z)}$ defined by

$$K_x(y) = \{-\operatorname{div} u : u \in S^q_{\alpha(\cdot, x(\cdot), Dy(\cdot))}\}.$$

Lemma 6. *If hypotheses $H(\alpha)_1$ hold, then for every $x \in W_0^{1,p}(Z)$, $y \mapsto K_x(y)$ is maximal monotone.*

Proof. By hypothesis $H(\alpha)_1$(i), for any $x, y \in W_0^{1,p}(Z)$, the multifunction $z \mapsto \alpha(z, x(z), Dy(z))$ is measurable and so Theorem 1 and hypothesis $H(\alpha)_1$(iv) guarantee that $S^q_{\alpha(\cdot, x(\cdot), Dy(\cdot))}$ is nonempty and also it is bounded, closed and convex. Therefore the multivalued operator K_x has nonempty, weakly compact and convex values. Moreover, K_x is monotone (hypothesis $H(\alpha)_1$(ii)), so Proposition 3 asserts that in order to show the desired maximality of $K_x(\cdot)$, it suffices to show that for every $y, h \in W_0^{1,p}(Z)$, the multifunction $t \mapsto K_x(y + th)$ is usc from $[0, 1]$ into $W^{-1,q}(Z)_w$. To this end, if $C \subseteq W^{-1,q}(Z)$ is a nonempty, weakly closed set and $M_{y,h}(t) = K_x(y + th)$, we have to show that

$$M^-_{y,h}(C) = \{t \in [0, 1] : M_{y,h}(t) \cap C \neq \emptyset\}$$

is closed in $[0, 1]$. So let $t_n \in M^-_{y,h}(C)$, $n \geq 1$, and suppose $t_n \to t$. Let $v_n^* \in M_{y,h}(t_n) \cap C$, $n \geq 1$. From the definition of K_x we have $v_n^* = -\operatorname{div} u_n$ with $u_n \in S^q_{\alpha(\cdot, x(\cdot), D(y+t_n h)(\cdot))}$. From hypothesis $H(\alpha)_1$(iv) we deduce that $\{u_n\}_{n\geq1} \subseteq L^q(Z, \mathbb{R}^N)$ is bounded. This fact enables us to assume that $u_n \xrightarrow{w} u$ in $L^q(Z, \mathbb{R}^N)$. Now Proposition 2 combined with the closedness of $\operatorname{Gr} \alpha(z, x(z), \cdot)$ for almost all $z \in Z$ (see hypothesis $H(\alpha)_1$(iii)) implies

$$u(z) \in \overline{\operatorname{conv}} \limsup_{n\to\infty} \alpha(z, x(z), D(y + t_n h)(z)) \subseteq \alpha(z, x(z), D(y + th)(z)) \quad \text{a.e. on } Z.$$

Note that $-\operatorname{div} u_n \xrightarrow{w} -\operatorname{div} u$ in $W^{-1,q}(Z)$ and so $v_n^* \xrightarrow{w} v^* = -\operatorname{div} u \in M_{y,h}(t)$. Hence we have $-\operatorname{div} u \in K_x(y + th) \cap C$, i.e. $t \in M^-_{y,h}(C)$. So the set $M^-_{y,h}(C)$ is closed and this proves the maximality of the monotone operator K_x. ∎

Using this lemma, we can prove the following result about V.

Proposition 7. *If hypotheses $H(\alpha)_1$ hold, then V is a multivalued operator of type $(S)_+$.*

Proof. Let $\{x_n\} \subseteq W_0^{1,p}(Z)$ and $v_n^* \in V(x_n)$, $n \geq 1$, be two sequences such that

$$x_n \xrightarrow{w} x \quad \text{in } W_0^{1,p}(Z) \quad \text{and} \quad \limsup_{n\to\infty} \langle v_n^*, x_n - x\rangle \leq 0.$$

Here by $\langle\cdot,\cdot\rangle$ we denote the duality brackets for the pair $(W_0^{1,p}(Z), W^{-1,q}(Z))$. We have

$$v_n^* = -\operatorname{div} v_n \quad \text{ with } v_n \in S^q_{\alpha(\cdot,x_n(\cdot),Dx_n(\cdot))},\ n \geq 1.$$

By hypothesis $H(\alpha)_1$(iv), we see that $\{v_n\}_{n\geq 1} \subseteq L^q(Z,\mathbb{R}^N)$ is bounded and so by passing to a subsequence if necessary, we may assume that $v_n \xrightarrow{w} v$ in $L^q(Z,\mathbb{R}^N)$. Hence $v_n^* = -\operatorname{div} v_n \xrightarrow{w} -\operatorname{div} v$ in $W^{-1,q}(Z)$ as $n\to\infty$.

Let $y \in W_0^{1,p}(Z)$ and consider the multifunction $K : Z \to \mathcal{P}_{kc}(\mathbb{R}^N)$ defined by $K(z) = \alpha(z, x(z), Dy(z))$. Because of hypothesis $H(\alpha)_1$(i), K is measurable and so by Theorem 1 it admits measurable selectors which belong to $L^q(Z,\mathbb{R}^N)$ (hypothesis $H(\alpha)_1$(iv)). Let $w \in L^q(Z,\mathbb{R}^N)$ be such a selector (i.e. $w \in S^q_{\alpha(\cdot,x(\cdot),Dy(\cdot))}$). For each $n \geq 1$, we introduce the multifunction $L_n : Z \to 2^{\mathbb{R}^N}$ defined by

$$L_n(z) = \{\xi \in \alpha(z, x_n(z), Dy(z)) : \|w(z)-\xi\| = d(w(z), \alpha(z,x_n(z),Dy(z)))\}.$$

Clearly, for almost all $z \in Z$, $L_n(z) \neq \emptyset$ and by redefining L_n on a Lebesgue-null set, we may assume without any loss of generality that $L_n(z) \neq \emptyset$ for all $z \in Z$. Then we have

$$\operatorname{Gr} L_n = \operatorname{Gr} \alpha(\cdot, x_n(\cdot), Dy(\cdot)) \cap \{(z,\xi) \in Z \times \mathbb{R}^N : \eta_n(z,\xi) = 0\},$$

where $\eta_n(z,\xi) = \|w(z)-\xi\| - d(w(z), \alpha(z,x_n(z),Dy(z)))$.

Note that $\operatorname{Gr} \alpha(\cdot,x_n(\cdot),Dy(\cdot)) \in \mathcal{L}\times B(\mathbb{R}^N)$ with $\mathcal{L}$ being the Lebesgue σ-field of Z and $B(\mathbb{R}^N)$ the Borel σ-field of $\mathbb{R}^N$ (hypothesis $H(\alpha)_1$(i)). Also η_n is a Carathéodory function (i.e. η_n is measurable in $z \in Z$ and continuous in $\xi \in \mathbb{R}^N$), hence it is jointly measurable (see Hu–Papageorgiou [35, p. 142]). Therefore we infer that $\operatorname{Gr} L_n \in \mathcal{L}\times B(\mathbb{R}^N)$. Using Theorem 1 we obtain $w_n \in S^q_{\alpha(\cdot,x_n(\cdot),Dy(\cdot))}$ (hypothesis $H(\alpha)_1$(iv)), $n \geq 1$, such that $w_n(z) \in L_n(z)$ a.e. on Z. So

$$\|w(z)-w_n(z)\| = d(w(z), \alpha(z,x_n(z),Dy(z))) \quad \text{ a.e. on } Z,$$

hence

$$\|w(z)-w_n(z)\| \leq h^*(\alpha(z,x(z),Dy(z)), \alpha(z,x_n(z),Dy(z))) \quad \text{ a.e. on } Z. \tag{5}$$

From the compact embedding of $W_0^{1,p}(Z)$ in $L^p(Z)$ (Sobolev embedding theorem) and by passing to a subsequence if necesesary, we have $x_n \to x$ in $L^p(Z)$ and $x_n(z) \to x(z)$ a.e. on Z.

Since by hypothesis $H(\alpha)_1$(iii), $\alpha(z,\cdot,Dy(z))$ is lsc and it has compact values in $\mathbb{R}^N$, it is h-lsc and so $h^*(\alpha(z,x(z),Dy(z)),\ \alpha(z,x_n(z),Dy(z))) \to 0$ on Z. Hence from (5) it follows that $w_n(z) \to w(z)$ a.e. on Z. From the extended dominated convergence theorem (see for example Hu–Papageorgiou [35, p. 907]) we have $w_n \to w$ in $L^q(Z,\mathbb{R}^N)$. Exploiting the monotonicity of $\alpha(z,x_n(z),\cdot)$ (hypothesis $H(\alpha)_1$(iii)), we have

$$\begin{aligned}
0 &\leq \int_Z (v_n(z)-w_n(z), Dx_n(z)-Dy(z))_{\mathbb{R}^N}\,dz \\
&= \int_Z (v_n(z), Dx_n(z)-Dx(z))_{\mathbb{R}^N}\,dz + \int_Z (v_n(z), Dx(z)-Dy(z))_{\mathbb{R}^N}\,dz \\
&\quad + \int_Z (w_n(z), Dy(z)-Dx_n(z))_{\mathbb{R}^N}\,dz \\
&= \langle v_n^*, x_n - x\rangle + \int_Z (v_n(z), Dx(z)-Dy(z))_{\mathbb{R}^N}\,dz + \int_Z (w_n(z), Dy(z)-Dx_n(z))_{\mathbb{R}^N}\,dz.
\end{aligned}$$

By hypothesis, $\limsup_{n\to\infty}\langle v_n^*, x_n - x\rangle \le 0$. Also recall that $v_n \xrightarrow{w} v$ and $w_n \to w$ in $L^q(Z,\mathbb{R}^N)$. So in the limit as $n\to\infty$ we obtain

$$0 \le \int_Z (v(z), Dx(z) - Dy(z))_{\mathbb{R}^N}\,dz + \int_Z (w(z), Dy(z) - Dx(z))_{\mathbb{R}^N}\,dz$$

and hence

$$0 \le \langle -\operatorname{div} v - (-\operatorname{div} w), x - y\rangle.$$

But $(y, -\operatorname{div} w) \in \operatorname{Gr} K_x$ was arbitrary. Since by Lemma 6, K_x is maximal monotone, it follows that $-\operatorname{div} v \in K_x(x)$ and hence $v \in S^q_{\alpha(\cdot,x(\cdot),Dx(\cdot))}$.

As above by a measurable selection argument involving Theorem 1, we obtain $u_n \in S^q_{\alpha(\cdot,x_n(\cdot),Dx(\cdot))}$, $n \ge 1$, such that $u_n \to v$ in $L^q(Z,\mathbb{R}^N)$ as $n\to\infty$. Let $u_n^* = -\operatorname{div} u_n$, $v^* = -\operatorname{div} v$. From the choice of the sequences $\{x_n\}_{n\ge1} \subseteq W_0^{1,p}(Z)$ and $\{v_n^*\}_{n\ge1} \subseteq W^{-1,q}(Z)$, we have

$$\limsup_{n\to\infty}\langle v_n^* - v^*, x_n - x\rangle \le 0,$$

hence

$$\limsup_{n\to\infty}[\langle v_n^* - u_n^*, x_n - x\rangle + \langle u_n^* - v^*, x_n - x\rangle] \le 0$$

and so

$$\limsup_{n\to\infty}\langle v_n^* - u_n^*, x_n - x\rangle + \liminf_{n\to\infty}\langle u_n^* - v^*, x_n - x\rangle \le 0.$$

Because $u_n \to v$ in $L^q(Z,\mathbb{R}^N)$, we have $u_n^* = -\operatorname{div} u_n \to -\operatorname{div} v = v^*$ in $W^{-1,q}(Z)$ and so $\lim_{n\to\infty}\langle u_n^* - v^*, x_n - x\rangle = 0$. Thus we obtain

$$\limsup_{n\to\infty}\langle v_n^* - u_n^*, x_n - x\rangle \le 0. \tag{6}$$

From the monotonicity of $\alpha(z, x_n(z), \cdot)$ (hypothesis $H(\alpha)_1$(ii)), we have

$$\langle v_n^* - u_n^*, x_n - x\rangle = \int_Z (v_n(z) - u_n(z), Dx_n(z) - Dx(z))_{\mathbb{R}^N}\,dz \ge 0,$$

which implies

$$\liminf_{n\to\infty}\langle v_n^* - u_n^*, x_n - x\rangle \ge 0. \tag{7}$$

From (6) and (7) it follows that $\langle v_n^* - u_n^*, x_n - x\rangle \to 0$. Since

$$\langle v_n^* - u_n^*, x_n - x\rangle = \int_Z (v_n(z) - u_n(z), Dx_n(z) - Dx(z))_{\mathbb{R}^N}\,dz \to 0 \quad \text{as } n\to\infty$$

and the integrand is nonnegative (by the monotonicity of $\alpha(z, x_n(z), \cdot)$), by passing to a subsequence if necessary, we may assume that

$$\beta_n(z) = (v_n(z) - u_n(z), Dx_n(z) - Dx(z))_{\mathbb{R}^N} \to 0 \quad \text{a.e. on } Z \text{ as } n\to\infty$$

and

$$|\beta_n(z)| \le k_1(z) \quad \text{a.e. on } Z \text{ for all } n \ge 1 \text{ with } k_1 \in L^1(Z).$$

Because of hypotheses $H(\alpha)_1$(iv) and (v), we may choose a measurable set $N \subseteq Z$ with $|N| = 0$ (here $|\cdot|$ denotes the Lebesgue measure on $\mathbb{R}^N$) such that for all $z \in Z\setminus N$, we have

$$\begin{aligned} k_1(z) &\ge (v_n(z) - u_n(z), Dx_n(z) - Dx(z))_{\mathbb{R}^N} \\ &\ge \eta_1[\|Dx_n(z)\|^p + \|Dx(z)\|^p] - 2\eta_2 \\ &\quad - \|Dx_n(z)\|(b_1(z) + c_1|x_n(z)|^{p-1} + c_1\|Dx(z)\|^{p-1}) \\ &\quad - \|Dx(z)\|(b_1(z) + c_1|x_n(z)|^{p-1} + c_1\|Dx_n(z)\|^{p-1}). \end{aligned} \tag{8}$$

By passing to a subsequence if necessary, we may assume that $|x_n(z)| \leq k_2(z)$ a.e. on Z, $n \geq 1$, with $k_2 \in L^p(Z)$. So from (8) it follows that for all $z \in Z \setminus N$, $\{Dx_n(z)\}_{n\geq 1} \subseteq \mathbb{R}^N$ is bounded. Hence by passing to a subsequence if necessary (the subsequence in general will depend on $z \in Z \setminus N$), we may assume that $Dx_n(z) \to \widehat{\xi}(z)$. Fix $z \in Z \setminus N$. We can find $g_n(z) \in \alpha(z, x(z), \widehat{\xi}(z))$ such that

$$\begin{aligned} \|v_n(z) - g_n(z)\| &= d(v_n(z), \alpha(z, x(z), \widehat{\xi}(z))) \\ &\leq h^*(\alpha(z, x_n(z), Dx_n(z)), \alpha(z, x(z), \widehat{\xi}(z))). \end{aligned} \tag{9}$$

From the definition of h^* (see Section 2), we can find $s_n(z) \in \alpha(z, x_n(z), Dx_n(z))$, $n \geq 1$, such that $d(s_n(z), \alpha(z, x(z), \widehat{\xi}(z))) = h^*(\alpha(z, x_n(z), Dx_n(z)), \alpha(z, x(z), \widehat{\xi}(z)))$.

Note that $\{s_n(z)\}_{n\geq 1} \subseteq \mathbb{R}^N$ is bounded and so we may assume that $s_n(z) \to s(z)$ in $\mathbb{R}^N$ as $n \to \infty$. Because $(x_n(z), Dx_n(z), s_n(z)) \in \operatorname{Gr} \alpha(z, \cdot, \cdot)$, $n \geq 1$, in the limit as $n \to \infty$ we obtain $(x(z), \widehat{\xi}(z), s(z)) \in \operatorname{Gr} \alpha(z, \cdot, \cdot)$ (hypothesis $H(\alpha)_1$(iii)). Thus $h^*(\alpha(z, x_n(z), Dx_n(z)), \alpha(z, x(z), \widehat{\xi}(z))) \to 0$ as $n \to \infty$ and so from (9) we have $\|v_n(z) - g_n(z)\| \to 0$. Note that $\{g_n(z)\}_{n\geq 1} \subseteq \alpha(z, x(z), \widehat{\xi}(z)) \in \mathcal{P}_{kc}(\mathbb{R}^N)$ and so we may assume that $g_n(z) \to \widehat{g}(z) \in \alpha(z, x(z), \widehat{\xi}(z))$. Therefore finally we have $v_n(z) \to \widehat{g}(z) \in \alpha(z, x(z), \widehat{\xi}(z))$ for all $z \in Z \setminus N$.

Recall that for all $z \in Z \setminus N$, we have $(v_n(z) - u_n(z), Dx_n(z) - Dx(z))_{\mathbb{R}^N} \to 0$ as $n \to \infty$. So in the limit as $n \to \infty$ we obtain $(\widehat{g}(z) - v(z), \widehat{\xi}(z) - Dx(z))_{\mathbb{R}^N} = 0$ with $\widehat{g}(z) \in \alpha(z, x(z), \widehat{\xi}(z))$ and $v(z) \in \alpha(z, x(z), Dx(z))$. Exploiting the strict monotonicity of $\alpha(z, x(z), \cdot)$ we obtain $\widehat{\xi}(z) = Dx(z)$ for all $z \in Z \setminus N$. Therefore for the original sequence $\{Dx_n(z)\}_{n\geq 1}$ we have $Dx_n(z) \to Dx(z)$ for all $z \in Z \setminus N$. Recall that $Dx_n \xrightarrow{w} Dx$ in $L^p(Z, \mathbb{R}^N)$. Also from (8) we have

$$\begin{aligned} \eta_1 \|Dx_n(z)\|^p \leq k_1(z) &+ \eta_1 \|Dx(z)\|^p + 2\eta_2 \\ &+ \|Dx_n(z)\| (b_1(z) + c_1 |x_n(z)|^{p-1} + c_1 \|Dx(z)\|^{p-1}) \\ &+ \|Dx(z)\| (b_1(z) + c_1 |x_n(z)|^{p-1} + c_1 \|Dx_n(z)\|^{p-1}), \quad z \in Z \setminus N. \end{aligned}$$

Using Young's inequality with $\varepsilon > 0$ small enough, we obtain

$$\begin{aligned} \eta_3(\varepsilon) \|Dx_n(z)\|^p \leq k_1(z) &+ \eta_1 \|Dx(z)\|^p + 2\eta_2 \\ &+ \eta_4(\varepsilon)(b_1(z)^q + c_1^q |x_n(z)|^p + c_1^q \|Dx(z)\|^p) \\ &+ b_1(z) \|Dx(z)\|^p + c_1 |x_n(z)|^{p-1} \|Dx(z)\|^p + \eta_5(\varepsilon) \|Dx(z)\|^p \end{aligned}$$

and hence $\{\|Dx_n(\cdot)\|^p\}_{n\geq 1}$ is uniformly integrable. Invoking the extended dominated convergence theorem, we obtain $Dx_n \to Dx$ in $L^p(Z, \mathbb{R}^N)$ and so $x_n \to x$ in $W_0^{1,p}(Z)$. Therefore V is of type $(S)_+$. ∎

Let $G : W_0^{1,p}(Z) \to \mathcal{P}_{wkc}(L^{r'}(Z))$ be defined by $G(x) = S^{r'}_{\partial j(\cdot, x(\cdot))}$ and let $N_f : W_0^{1,p}(Z) \to L^q(Z)$ be the Nemytskiĭ operator corresponding to the function f, i.e. $N_f(x)(\cdot) = f(\cdot, x(\cdot), Dx(\cdot))$. We introduce the multivalued operator $R : W_0^{1,p}(Z) \to \mathcal{P}_{wkc}(W^{-1,q}(Z))$ defined by $R(x) = V(x) - G(x) - N_f(x)$.

Proposition 8. *If hypotheses $H(\alpha)_1, H(j)_1$ and $H(f)_1$ hold, then R is pseudomonotone.*

Proof. Since R is defined on all of $W_0^{1,p}(Z)$ and clearly it is bounded with closed convex values, it suffices to show that R is generalized pseudomonotone (see Section 2). So suppose $x_n \xrightarrow{w} x$ in $W_0^{1,p}(Z)$, $x_n^* \xrightarrow{w} x^*$ in $W^{-1,q}(Z)$, $x_n^* \in R(x_n)$, $n \geq 1$ and $\limsup_{n\to\infty}\langle x_n^*, x_n - x\rangle \leq 0$. We have to show that $x^* \in R(x)$ and $\langle x_n^*, x_n\rangle \to \langle x^*, x\rangle$.

From the definition of R we have

$$x_n^* = v_n^* - g_n - N_f(x_n) \quad \text{with } v_n^* \in V(x_n),\ g_n \in G(x_n),\ n \geq 1.$$

Because of hypothesis $H(j)_1$(iii), $\{g_n\}_{n\geq 1} \subseteq L^{r'}(Z)$ is bounded and so we may assume that $g_n \xrightarrow{w} g$ in $L^{r'}(Z)$. Also from the compact embedding of $W_0^{1,p}(Z)$ into $L^p(Z)$ we have $x_n \to x$ in $L^p(Z)$ and by passing to a subsequence if necessary, we may assume that $x_n(z) \to x(z)$ a.e. on Z. Exploiting the fact that $\operatorname{Gr}\partial j(z,\cdot)$ is closed and using Proposition 2, we obtain

$$g(z) \in \overline{\operatorname{conv}} \limsup_{n\to\infty} \partial j(z, x_n(z)) \subseteq \partial j(z, x(z)) \quad \text{a.e. on } Z,$$

which implies

$$g \in S^{r'}_{\partial j(\cdot, x(\cdot))} = G(x).$$

Moreover, since $1 \leq r < p^*$, we see that $W_0^{1,p}(Z)$ is embedded compactly in $L^r(Z)$ and so $x_n \to x$ in $L^r(Z)$. Hence we have $\langle g_n, x_n - x\rangle = \int_Z g_n(z)(x_n - x)(z)\,dz \to 0$ as $n \to \infty$. Also because of hypothesis $H(f)_1$(iii), $\{N_f(x_n)\}_{n\geq 1} \subseteq L^q(Z)$ is bounded and so $\langle N_f(x_n), x_n - x\rangle = \int_Z f(z, x_n(z), Dx_n(z))(x_n - x)(z)\,dz \to 0$ as $n \to \infty$. Thus finally we obtain

$$\limsup_{n\to\infty}\langle v_n^*, x_n - x\rangle \leq \limsup_{n\to\infty}\langle x_n^*, x_n - x\rangle \leq 0. \tag{10}$$

But from Proposition 7 we know that V is a multivalued operator of type $(S)_+$. So from (10) it follows that $x_n \to x$ in $W_0^{1,p}(Z)$. Then from hypotheses $H(f)_1$ and the dominated convergence theorem we have $N_f(x_n) \to N_f(x)$ in $L^q(Z)$ (and so in $W^{-1,q}(Z)$ as well). Since $g_n \xrightarrow{w} g$ in $L^{r'}(Z)$ and $L^{r'}(Z)$ is embedded continuously in $W^{-1,q}(Z)$ (recall that $1 \leq r < p^*$), we have $g_n \xrightarrow{w} g$ in $W^{-1,q}(Z)$. Also $v_n^* = -\operatorname{div} v_n$ with $v_n \in S^q_{\alpha(\cdot, x_n(\cdot), Dx_n(\cdot))}$, $n \geq 1$. As before $v_n \xrightarrow{w} v$ in $L^q(Z, \mathbb{R}^N)$ and so $v_n^* = \operatorname{div} v_n \xrightarrow{w} -\operatorname{div} v = v^*$ in $W^{-1,q}(Z)$ as $n \to \infty$. Since $x_n \to x$ in $W_0^{1,p}(Z)$, via Proposition 2 and the closedness of $\operatorname{Gr}\alpha(z,\cdot,\cdot)$ for almost all $z \in Z$ (hypothesis $H(\alpha)_1$(iii)), we have

$$v(z) \in \overline{\operatorname{conv}} \limsup \alpha(z, x_n(z), Dx_n(z)) \subseteq \alpha(z, x(z), Dx(z)) \quad \text{a.e. on } Z,$$

hence $v^* \in V(x)$. Therefore in the limit as $n \to \infty$, we obtain

$$x^* = v^* - g - N_f(x) \quad \text{with } v^* \in V(x),\ g \in G(x), \quad \text{hence} \quad x^* \in R(x).$$

Also since $x_n \to x$ in $W_0^{1,p}(Z)$, we have $\langle x_n^*, x_n\rangle \to \langle x^*, x\rangle$ and so we conclude that the operator R is pseudomonotone. ∎

Proposition 9. *If hypotheses $H(\alpha)_1$, $H(j)_1$ and $H(f)_1$ hold, then R is coercive.*

Proof. We begin the proof by establishing the following claim:

Claim. *There exists $\beta > 0$ such that for all $x \in W_0^{1,p}(Z)$, $x \not\equiv 0$, we have*

$$\psi(x) = \eta_1\|Dx\|^p - \int_Z \vartheta(z)|x(z)|^p\,dz \geq \beta\|Dx\|_p^p. \tag{11}$$

Remark that by virtue of the variational expression for $\lambda_1 > 0$ (see (2)) and the hypothesis on ϑ (hypothesis $H(f)_1$(iii)), we have $\psi \geq 0$. Suppose that the claim were not true. Then we could find $\{x_n\}_{n\geq 1} \subseteq W_0^{1,p}(Z)$ with $\|Dx_n\|_p = 1$ such that $\psi(x_n) \downarrow 0$. By Poincaré's inequality, $\{x_n\}_{n\geq 1} \subseteq W_0^{1,p}(Z)$ is bounded and so we may assume that $x_n \xrightarrow{w} x$ in $W_0^{1,p}(Z)$. Hence $x_n \to x$ in $L^p(Z)$. Then from the weak lower semicontinuity of the norm in a Banach space, we obtain

$$0 \geq \eta_1 \|Dx\|_p^p - \int_Z \vartheta(z)|x(z)|^p\, dz$$

and so

$$\lambda_1 \eta_1 \|x\|_p^p \geq \int_Z \vartheta(z)|x(z)|^p\, dz \geq \eta_1 \|Dx\|_p^p \quad \text{(hypothesis } H(j)_1\text{(iv))}. \tag{12}$$

Using (2) (the variational characterization of $\lambda_1 > 0$), we have

$$\lambda_1 \|x\|_p^p = \|Dx\|_p^p, \quad \text{hence} \quad x = \pm u_1 \quad \text{or} \quad x = 0.$$

Recall that u_1 is the normalized principal eigenfunction of $(-\Delta_p, W_0^{1,p}(Z))$. Since $\psi(x_n) = \eta_1 - \int_Z \vartheta(z)|x_n(z)|^p\, dz$, $n \geq 1$, in the limit as $n \to \infty$ we have $\eta_1 = \int_Z \vartheta(z)|x(z)|^p\, dz$ and hence $x \not\equiv 0$. So $x = \pm u_1$. Recalling that $u_1(z) > 0$ for all $z \in Z$ and taking into account hypothesis $H(j)_1$(iv), we derive from (12) that

$$\lambda_1 \eta_1 \|u_1\|_p^p > \int_Z \vartheta(z)|u_1(z)|^p\, dz \geq \eta_1 \|Du_1\|_p^p,$$

which contradicts (2). Therefore the claim is true and (11) holds.

Next by hypothesis $H(j)_1$(iv) we can find $M_1 > 0$ such that for almost all $z \in Z$ and all $u^* \in \partial j(z,x)$ we have

$$u^* \leq (\vartheta(z) + \lambda_1\beta/2)|x|^{p-2}x \quad \text{if } x \geq M_1$$

and

$$u^* \geq (\vartheta(z) + \lambda_1\beta/2)|x|^{p-2}x \quad \text{if } x \leq -M_1.$$

On the other hand, by hypothesis $H(j)_1$(iii), for almost all $z \in Z$, all $|x| \leq M_1$ and all $u^* \in \partial j(z,x)$ we have

$$|u^*| \leq \gamma_1(z) \quad \text{with } \gamma_1 \in L^{r'}(Z).$$

So finally we can write that for almost all $z \in Z$ and all $u^* \in \partial j(z,x)$ we have

$$u^* \leq (\vartheta(z) + \lambda_1\beta/2)|x|^{p-2}x + \gamma_1(z) \quad \text{if } x \geq 0$$

and

$$u^* \geq (\vartheta(z) + \lambda_1\beta/2)|x|^{p-2}x - \gamma_1(z) \quad \text{if } x \leq 0.$$

Then for $x^* \in R(x)$ we have $x^* = v^* - u^* - N_f(x)$ with $v^* \in V(x)$ and $u^* \in G(x)$. So

$$\langle x^*, x\rangle = \langle v^*, x\rangle - \int_Z u^* x\, dz - \int_Z f(z,x,Dx)x\, dz.$$

We have $v^* = -\operatorname{div} v$ with $v \in S^q_{\alpha(\cdot,x(\cdot),Dx(\cdot))}$ and so

$$\begin{aligned}\langle v^*, x\rangle = \langle -\operatorname{div} v, x\rangle &= \int_Z (v(z), Dx(z))_{\mathbb{R}^N}\, dz \\ &\geq \eta_1 \|Dx\|_p^p - \eta_2|Z| \quad \text{(hypothesis } H(\alpha)_1\text{(v))}.\end{aligned}$$

Also we have

$$\int\limits_Z u^*x\,dz = \int\limits_{\{x>0\}} u^*x\,dz + \int\limits_{\{x<0\}} u^*x\,dz \le \int\limits_Z \left(\vartheta(z) + \frac{\lambda_1\beta}{2}\right)|x|^p\,dz + \|\gamma_1\|_{r'}\|x\|_r$$

and

$$\int\limits_Z f(z,x,Dx)x\,dz \le \|b_3\|_q\|x\|_p + c_4\|x\|_p^\theta + c_5\|Dx\|_p^\theta \quad \text{for some } c_4, c_5 > 0.$$

Thus finally we obtain

$$\begin{aligned}\langle x^*, x\rangle &\ge \eta_1\|Dx\|_p^p - \int\limits_Z \vartheta(z)|x(z)|^p\,dz - \frac{\lambda_1\beta}{2}\|x\|_p^p - \|\gamma_1\|_{r'}\|x\|_r \\ &\quad - \|b_3\|_q\|x\|_p - c_4\|x\|_p^\theta - c_5\|Dx\|_p^\theta \\ &\ge \frac{\beta}{2}\|Dx\|_p^p - c_6\|Dx\|_p - c_7\|Dx\|_p^\theta \quad \text{for some } c_6, c_7 > 0.\end{aligned}$$

In the last inequality we have used (11), the variational expression for λ_1 (see (2)) and the Sobolev embedding theorem. Since $\theta < p$, it follows that R is coercive. ∎

Now we are ready for the existence theorem for the problem (4).

THEOREM 10. *If hypotheses $H(\alpha)_1$, $H(j)_1$ and $H(f)_1$ hold, then problem* (4) *has a solution.*

Proof. From Propositions 8 and 9 we know that R is pseudomonotone and coercive, thus it is surjective (see Section 2). So we can find $x \in W_0^{1,p}(Z)$ such that $0 \in R(x)$. Hence there exist $v \in S^q_{\alpha(\cdot,x(\cdot),Dx(\cdot))}$ and $u^* \in G(x)$ such that $-\operatorname{div} v = u^* + N_f(x) \in L^s(Z)$, $s = \min\{r', q\}$. Therefore $-\operatorname{div}\ v(z) = u^*(z) + f(z,x(z),Dx(z))$ a.e. on Z and so x is a solution of (4). ∎

4. Method of upper-lower solutions

We continue our investigation of problem (4). However now we drop the growth hypothesis $H(f)_1$(iii) and replace it by the assumption that there exists an ordered pair of upper and lower solutions for problem (4). Therefore our approach is based on the method of upper-lower solutions coupled with suitable truncation and penalization techniques. For previous works on nonlinear second order elliptic equations using the method of upper-lower solutions, we refer to the papers of Deuel–Hess [21], Mawhin–Schmitt [47], Carl–Dietrich [12], Delgado–Suarez [20] and the references therein. However, for the general nonlinear hemivariational problems like (4), to our knowledge, nothing was done before. Only the paper of Carl–Dietrich [12] considers problems with discontinuities. More precisely, the map α is single-valued and independent of x (it depends only on $z \in Z$ and the gradient Dx), $j(z,x) = \int_0^x g(z,r)\,dr$, with $g : Z \times \mathbb{R} \to \mathbb{R}$ a measurable function which is locally bounded in $x \in \mathbb{R}$ and $f \equiv 0$. In general problems with discontinuities are a special case of hemivariational inequalities in which the generalized potential j is the indefinite integral of a locally bounded function. So our work here is the first application of the method of upper-lower solutions to strongly nonlinear hemivariational inequalities.

We start with the definition of upper and lower solutions.

Definition. (a) A function $\varphi \in W^{1,p}(Z)$, $\varphi|_\Gamma \geq 0$ is an *upper solution* for problem (4) if there exists $u_+^* \in S^q_{\partial j(\cdot,\varphi(\cdot))}$ such that

$$\langle v^*, y\rangle - \int_Z f(z,\varphi(z),D\varphi(z))y(z)\,dz \geq \int_Z u_+^*(z)y(z)\,dz \tag{13}$$

for all $v^* \in V(\varphi)$ and all $y \in W_0^{1,p}(Z)_+$.

(b) A function $\psi \in W^{1,p}(Z)$, $\psi|_\Gamma \leq 0$ is a *lower solution* for problem (4) if there exists $u_-^* \in S^q_{\partial j(\cdot,\psi(\cdot))}$ such that

$$\langle v^*, y\rangle - \int_Z f(z,\psi(z),D\psi(z))y(z)\,dz \leq \int_Z u_-^*(z)y(z)\,dz \tag{14}$$

for all $v^* \in V(\psi)$ and all $y \in W_0^{1,p}(Z)_+$.

We assume the existence of an ordered pair (φ,ψ). Namely:

$\boxed{H_0}$ There exist an upper solution φ and a lower solution ψ for problem (4) such that $\psi(z) \leq \varphi(z)$ a.e. on Z.

Also our hypotheses on the generalized potential $j(z,x)$ and the nonlinearity f are modified as follows:

$\boxed{H(j)_2}$ $j : Z \times \mathbb{R} \to \mathbb{R}$ is a function such that

(i) for all $x \in \mathbb{R}$, $z \mapsto j(z,x)$ is measurable;
(ii) for almost all $z \in Z$, $x \mapsto j(z,x)$ is locally Lipschitz;
(iii) for almost all $z \in Z$, and all $x \in [\psi(z),\varphi(z)]$ and all $u^* \in \partial j(z,x)$ we have

$$|u^*| \leq b_2(z) \qquad \text{with } b_2 \in L^{r'}(Z),\ 1 \leq r < p^*,\ 1/r + 1/r' = 1.$$

$\boxed{H(f)_2}$ $f : Z \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a function such that

(i) for all $(x,\xi) \in \mathbb{R} \times \mathbb{R}^N$, $z \mapsto f(z,x,\xi)$ is measurable;
(ii) for almost all $z \in Z$, $(x,\xi) \mapsto f(z,x,\xi)$ is continuous;
(iii) for almost all $z \in Z$, all $x \in [\psi(z),\varphi(z)]$ and all $\xi \in \mathbb{R}^N$ we have

$$|f(z,x,\xi)| \leq b_3(z) + c_3\|\xi\|^{p-1} \qquad \text{with } b_3 \in L^q(Z),\ c_3 > 0,\ 1 \leq r < p^*.$$

We introduce the truncation function $\tau : W^{1,p}(Z) \to W^{1,p}(Z)$ defined by

$$\tau(x)(z) = \begin{cases} \varphi(z) & \text{if } \varphi(z) \leq x(z), \\ x(z) & \text{if } \psi(z) \leq x(z) \leq \varphi(z), \\ \psi(z) & \text{if } x(z) \leq \psi(z). \end{cases}$$

It is easy to check that τ is continuous. Also we introduce the penalty function $\beta : Z \times \mathbb{R} \to \mathbb{R}$ defined by

$$\beta(z,x) = \begin{cases} |x|^{p-2}x - |\varphi(z)|^{p-2}\varphi(z) & \text{if } \varphi(z) < x, \\ 0 & \text{if } \psi(z) \leq x \leq \varphi(z), \\ |x|^{p-2}x - |\psi(z)|^{p-2}\psi(z) & \text{if } x < \psi(z). \end{cases}$$

Clearly this is a Carathéodory function, for almost all $z \in Z$ and all $x \in \mathbb{R}$ we have $|\beta(z,x)| \le b_4(z) + c_4|x|^{p-1}$ with $b_4 \in L^q(Z)$, $c_4 > 0$ and we can easily verify that

$$\int_Z \beta(z,x(z))x(z)\,dz \ge c_4\|x\|_p^p - c_5 \quad \text{for some } c_4, c_5 > 0.$$

We consider the following auxiliary nonlinear elliptic problem:

$$\text{(15)} \qquad \begin{cases} -\operatorname{div}\alpha(z,\tau(x),Dx) - \partial j(z,\tau(x)(z)) \ni f(z,\tau(x),D\tau(x)) - \varrho\beta(z,x) & \text{a.e. on } Z, \\ x|_\Gamma = 0, \qquad \varrho > 0. \end{cases}$$

PROPOSITION 11. *If hypotheses $H(\alpha)_1$, $H(j)_2$ and H_0 hold, then for large $\varrho > 0$ problem* (15) *has a solution $x \in W_0^{1,p}(Z)$.*

Proof. As in Section 3 our method is based on the theory of nonlinear operators of monotone type. To this end, let $V_1 : W_0^{1,p}(Z) \to \mathcal{P}_{wkc}(W^{-1,q}(Z))$ be the multivalued operator defined by

$$V_1(x) = \{-\operatorname{div} v : v \in S^q_{\alpha(\cdot,\tau(x)(\cdot),Dx(\cdot))}\}.$$

Arguing as in the proof of Proposition 7 we can check that V_1 is of type $(S)_+$.

Also let $N_f^1 : W_0^{1,p}(Z) \to L^{r'}(Z)$ and $B : W_0^{1,p}(Z) \to L^q(Z)$ be the nonlinear operators defined by

$$N_f^1(x)(\cdot) = f(\cdot,\tau(x)(\cdot),D\tau(x)(\cdot)) \quad \text{and} \quad B(x)(\cdot) = \beta(\cdot,x(\cdot)).$$

By virtue of hypotheses $H(f)_2$ and the properties of the penalty function β, we see that N_f^1 and B are both continuous.

Finally, let $Q : Z \times \mathbb{R} \to \mathcal{P}_{fc}(\mathbb{R})$ be the multifunction defined by

$$Q(z,x) = \begin{cases} (-\infty, u_+^*(z)], & \varphi(z) < x, \\ \mathbb{R}, & \psi(z) \le x \le \varphi(z), \\ [u_-^*(z), \infty), & x < \psi(z), \end{cases}$$

where $u_+^* \in S^q_{\partial j(\cdot,\varphi(\cdot))}$, $u_-^* \in S^q_{\partial j(\cdot,\psi(\cdot))}$ satisfy (13), (14) respectively. As before let $G(x) = S^q_{\partial j(\cdot,x(\cdot))}$ and let $G_1 : W_0^{1,p}(Z) \to \mathcal{P}_{wkc}(L^q(Z))$ be defined by

$$G_1(x) = G(\tau(x)) \cap S^q_{Q(\cdot,x(\cdot))}.$$

CLAIM 1. $\operatorname{Gr} G_1$ *is sequentially closed in* $W_0^{1,p}(Z) \times L^{r'}(Z)_w$.

Indeed, let $(x_n,u_n) \in \operatorname{Gr} G_1$, $n \ge 1$, with $(x_n,u_n) \to (x,u)$ in $W_0^{1,p}(Z) \times L^{r'}(Z)_w$. Then $\tau(x_n) \to \tau(x)$ in $W^{1,p}(Z)$. By passing to a subsequence if necessary we may assume that $x_n(z) \to x(z)$ and $\tau(x_n)(z) \to \tau(x)(z)$ a.e. on Z. It follows from Proposition 2 that

$$\begin{aligned} u(z) &\in \overline{\operatorname{conv}} \limsup_{n\to\infty} [\partial j(z,\tau(x_n)(z)) \cap Q(z,x_n(z))] \\ &\subseteq \overline{\operatorname{conv}} [\limsup_{n\to\infty} \partial j(z,\tau(x_n)(z)) \cap \limsup_{n\to\infty} Q(z,x_n(z))] \quad \text{a.e. on } Z. \end{aligned}$$

But both the multifunctions $\partial j(z,\cdot), Q(z,\cdot)$ are closed convex valued with closed graph, so

$$\limsup_{n\to\infty} \partial j(z,\tau(x_n)(z)) \subseteq \partial j(z,\tau(x)(z)) \quad \limsup_{n\to\infty} Q(z,x_n(z)) \subseteq Q(z,x(z)) \quad \text{a.e. on } Z$$

and finally, $u(z) \in \partial j(z,\tau(x)(z)) \cap Q(z,x(z))$ a.e. on Z. Hence, $u \in G_1(x)$ and this proves Claim 1.

Let $R_1 : W_0^{1,p}(Z) \to \mathcal{P}_{wkc}(W^{-1,q}(Z))$ be defined by

$$R_1(x) = V_1(x) + \varrho B(x) - G_1(x) - N_f^1(x).$$

CLAIM 2. *R_1 is pseudomonotone. Moreover, for large $\varrho > 0$, it is coercive.*

The pseudomonotonicity of R_1 follows just as the pseudomonotonicity of R in Proposition 8 by using the fact that $\operatorname{Gr} G_1$ is sequentially closed in $W_0^{1,p}(Z) \times L^{r'}(Z)_w$ (see Claim 1). So let us prove the coercivity of R_1. To this end let $x \in W_0^{1,p}(Z)$ and $x^* \in R_1(x)$. Then

$$x^* = v^* + \varrho B(x) - u^* - N_f^1(x) \quad \text{with } v^* \in V_1(x) \text{ and } u^* \in G_1(x),$$

which implies

$$\langle x^*, x\rangle = \langle v^*, x\rangle + \varrho \int_Z \beta(z, x(z))x(z)\,dz - \int_Z u^* x\,dz - \int_Z f(z, \tau(x), D\tau(x))x(z)\,dz.$$

From hypothesis $H(\alpha)_1$(v) we have

$$\langle v^*, x\rangle = \int_Z (v(z), Dx(z))_{\mathbb{R}^N}\,dz \ge \eta_1 \|Dx\|_p^p - \eta_2 \quad \text{with } v \in S^q_{\alpha(\cdot,\tau(x)(\cdot),Dx(\cdot))}.$$

Also from hypothesis $H(f)_2$(iii) and since

$$D\tau(x)(z) = \begin{cases} D\varphi(z) & \text{if } \varphi(z) < x(z), \\ Dx(z) & \text{if } \psi(z) \le x(z) \le \varphi(z), \\ D\psi(z) & \text{if } x(z) < \psi(z), \end{cases}$$

(see Evans–Gariepy [23, p. 130]), we get

$$\langle N_f^1(x), x\rangle = \int_Z f(z, \tau(x), D\tau(x))x(z)\,dz \le c_7 \|x\|_{1,p}^{p-1} \|x\|_p + c_8 \|x\|_p + c_9$$

with $c_7, c_8, c_9 > 0$.

Applying Young's inequality with $\varepsilon > 0$, we obtain

$$\|x\|_{1,p}^{p-1} \|x\|_p \le \frac{1}{\varepsilon^p p} \|x\|_p^p + \frac{\varepsilon^q}{q} \|x\|_{1,p}^p \quad (\text{recall that } p - 1 = p/q),$$

which implies that

$$\langle N_f^1(x), x\rangle \le c_7 \frac{1}{\varepsilon^p p} \|x\|_p^p + c_7 \frac{\varepsilon^q}{q} \|x\|_{1,p}^p + c_8 \|x\|_p + c_9.$$

Recall that from the properties of the penalty function β we have

$$\varrho \int_Z \beta(z, x(z))x(z)\,dz \ge \varrho c_{10} \|x\|_p^p - \varrho c_{11} \quad \text{with } c_{10}, c_{11} > 0.$$

Moreover, from hypothesis $H(j)_2$(iii) and the compact embedding of $W_0^{1,p}(Z)$ in $L^r(Z)$ we have

$$\int_Z u^* x\,dz \le c_{12} \|x\|_{1,p} \quad \text{with } c_{12} > 0.$$

So finally via Poincaré's inequality, we obtain

$$\langle x^*, x\rangle \ge (c_{13} - c_7 \varepsilon^q/q) \|x\|_{1,p}^p + (\varrho c_{10} - c_7/(\varepsilon^p p)) \|x\|_p^p - c_{12} \|x\|_{1,p} - c_{14}(\varrho).$$

First choose $\varepsilon > 0$ so that $c_{13} > c_7\varepsilon^q/q$ and then based on this choice of ε, choose $\varrho > 0$ large so that $\varrho c_{10} > c_7/(\varepsilon^p p)$. With these choices, the last inequality asserts that the operator R_1 is coercive, as claimed.

Now R_1 being pseudomonotone and coercive, it is surjective and so we can find $x \in W_0^{1,p}(Z)$ such that $0 \in R_1(x)$. This is the desired solution of (15). ∎

Using Proposition 11, we can produce a solution for problem (4).

THEOREM 12. *If hypotheses $H(\alpha)_1$, $H(j)_2$, $H(f)_2$ and H_0 hold, then problem* (4) *has a solution $x \in W_0^{1,p}(Z)$ such that $\psi \le x \le \varphi$.*

Proof. Let $x \in W_0^{1,p}(Z)$ be a solution of (15) (Proposition 11). We shall show that x belongs to the order interval $[\psi, \varphi]$. We have

$$(16)\qquad v^* + \varrho B(x) = u^* + N_f^1(x) \quad \text{ with } v^* = -\operatorname{div} v,\ v \in S^q_{\alpha(\cdot,\tau(x)(\cdot),Dx(\cdot))},\ u^* \in G_1(x).$$

Meanwhile, $\psi \in W^{1,p}(Z)$ is a lower solution to the original problem (4) and $u_-^* \in S^q_{\partial j(\cdot,\psi(\cdot))}$ has been chosen so that (14) is satisfied for all $v_1^* \in V(\psi)$. Moreover, the fact that $\psi|_\Gamma \le 0$ guarantees that $(\psi - x)^+ \in W_0^{1,p}(Z)_+$.

Now fix $v_1^* \in V(\psi)$, i.e.

$$v_1^* = -\operatorname{div} v_1, \qquad v_1 \in S^q_{\alpha(\cdot,\psi(\cdot),D\psi(\cdot))}$$

and use $(\psi - x)^+$ as testing function in both (14), (16) to obtain

$$(17)\qquad \int_Z (v(z), D(\psi - x)^+(z))_{\mathbb{R}^N}\,dz - \int_Z f(z,\tau(x),D\tau(x))(\psi - x)^+\,dz \\ + \varrho\int_Z \beta(z,x)(\psi - x)^+\,dz = \int_Z u^*(\psi - x)^+\,dz$$

and

$$(18)\qquad \int_Z (v_1(z), D(\psi - x)^+(z))_{\mathbb{R}^N}\,dz - \int_Z f(z,\psi,D\psi)(\psi - x)^+\,dz \le \int_Z u_-^*(\psi - x)^+\,dz.$$

By using the definition of $\tau(x)$ and of $D\tau(x)$ (see the proof of Proposition 11), it is easy to check that

$$\int_Z f(z,\tau(x),D\tau(x))(\psi - x)^+\,dz = \int_Z f(z,\psi,D\psi)(\psi - x)^+\,dz,$$

so subtracting (17) from (18) we have

$$\int_Z (v_1(z) - v(z), D(\psi - x)^+(z))_{\mathbb{R}^N}\,dz - \varrho\int_Z \beta(z,x)(\psi - x)^+\,dz \le \int_Z (u_-^* - u^*)(\psi - x)^+\,dz.$$

Set

$$Z_+ = \{z \in Z : x(z) < \psi(z)\}.$$

From the definition of the multivalued map G_1 we see that $u^*(z) \ge u_-^*(z)$ a.e. on Z_+, so that

$$\int_Z (u_-^* - u^*)(\psi - x)^+\,dz \le 0.$$

Moreover, for almost all $z \in Z_+$, we have

$$v_1(z) \in \alpha(z,\psi(z),D\psi(z)), \quad v(z) \in \alpha(z,\tau(x)(z),Dx(z)) = \alpha(z,\psi(z),Dx(z)),$$

which implies

$$\int_Z (v_1(z) - v(z), D(\psi - x)^+(z))_{\mathbb{R}^N}\, dz = \int_{Z_+} (v_1(z) - v(z), D\psi(z) - Dx(z))_{\mathbb{R}^N}\, dz \geq 0$$

(recall that the mapping $\xi \mapsto \alpha(z, x, \xi)$ is monotone). Consequently,

$$\begin{aligned}(19)\qquad 0 \leq \int_Z \beta(z, x)(\psi - x)^+\, dz &= \int_Z (|x|^{p-2}x - |\psi|^{p-2}\psi)(\psi - x)^+\, dz \\ &\leq -c_{15} \int_Z |(\psi - x)^+|^p\, dz \quad \text{for some } c_{15} > 0.\end{aligned}$$

Here we used the elementary inequality which says that for all $a, c \in \mathbb{R}$,

$$(|a|^{p-2}a - |c|^{p-2}c)(a - c) \geq 2^{2-p}|a - c|^p.$$

This is another way to say that $a \mapsto |a|^p/p$ is a strongly convex function.

From (19) we obtain

$$(\psi - x)^+(z) = 0 \quad \text{a.e. on } Z, \quad \text{hence} \quad \psi \leq x.$$

In a similar fashion we show that $x \leq \varphi$. Therefore finally $x \in [\psi, \varphi]$ and so $\tau(x) = x$, $D\tau(x) = Dx$, $\beta(z, x) = 0$, $G_1(x) = G(x)$ and these imply that x is a solution of (4). ∎

Now that we have established the existence of at least one solution in the order interval $[\psi, \varphi]$, we ask the question of whether among all these solutions there is a maximum and a minimum solution for the pointwise ordering on $W_0^{1,p}(Z)$. Such solutions (if they exist) are known as *extremal solutions* of (4) in the order interval. For semilinear equations and classical solutions, this problem was investigated by Amann [2] and Stuart [61] (problems with discontinuities), and for certain quasilinear equations and weak solutions by Carl–Heikkila–Laksmikantham [13] (problems with discontinuities). In [13] the second order quasilinear differential operator is of the form $-\sum_{i,j=1}^N D_i(\alpha_{ij}(z, x)D_j x)$, i.e. the operator is mildly nonlinear, since the gradient of x enters linearly.

Here we extend the aforementioned works by establishing the existence of extremal solutions in the order interval $[\psi, \varphi]$ for a particular case of problem (1) with suitable monotone structure (variational inequality). So the problem under consideration is the following:

$$(20)\qquad \begin{cases} -\operatorname{div} \alpha(z, x(z), Dx(z)) \in \partial j(z, x(z)) + f(z, x(z)) & \text{a.e. on } Z, \\ x|_\Gamma = 0. \end{cases}$$

Our hypotheses on the data of (20) are the following:

$\boxed{H(\alpha)_2}$ $\alpha : Z \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a function such that

(i) for every $x \in R$ and $\xi \in \mathbb{R}^N$, $z \mapsto \alpha(z, x, \xi)$ is measurable;
(ii) for almost all $z \in Z$, $(x, \xi) \mapsto \alpha(z, x, \xi)$ is continuous;
(iii) for almost all $z \in Z$ and all $x \in \mathbb{R}$, $\xi \mapsto \alpha(z, x, \xi)$ is monotone;
(iv) for almost all $z \in Z$, all $x, x' \in \mathbb{R}$ and all $\xi \in \mathbb{R}^N$ we have

$$\|\alpha(z, x, \xi)\| \leq b_1(z) + c_1(|x|^{p-1} + \|\xi\|^{p-1}) \quad \text{with } b_1 \in L^q(Z),\ c_1 > 0$$

and

$$\|\alpha(z, x, \xi) - \alpha(z, x', \xi)\| \leq [c_2(|x| + |x'| + \|\xi\|)^{p-1} + k_1(z)]|x - x'|,$$

with $c_2 > 0$, $k_1 \in L^q(Z)$;

(v) for almost all $z \in Z$, all $x \in \mathbb{R}$ and all $\xi \in \mathbb{R}^N$ we have

$$(\alpha(z,x,\xi),\xi)_{\mathbb{R}^N} \geq \eta_1\|\xi\|^p - k_2(z) \quad \text{with } \eta_1 > 0,\ k_2 \in L^1(Z).$$

$\boxed{H(j)_3}$ $j : Z \times \mathbb{R} \to \mathbb{R}$ is a function such that

(i) for all $x \in \mathbb{R}$, $z \mapsto j(z,x)$ is measurable;

(ii) for almost all $z \in Z$, $x \mapsto j(z,x)$ is also concave (hence it is also locally Lipschitz);

(iii) for almost all $z \in Z$, all $x \in \mathbb{R}$ and all $u^* \in \partial j(z,x)$ we have

$$|u^*| \leq b_2(z) + c_3|x|^{p-1} \quad \text{with } b_2 \in L^q(Z),\ c_3 > 0.$$

$\boxed{H(f)_3}$ $f : Z \times \mathbb{R} \to \mathbb{R}$ is a function such that $f(\cdot,\psi(\cdot)), f(\cdot,\varphi(\cdot)) \in L^q(Z)$ and

(i) f is N-measurable, i.e. for all $x : Z \to \mathbb{R}$ measurable functions $z \mapsto f(z,x(z))$ is measurable;

(ii) for almost all $z \in Z$, $x \mapsto f(z,x)$ is decreasing and continuous.

Remark. If f is jointly measurable, then clearly it is N-measurable. More general conditions implying N-measurability involve the so-called Shragin functions.

We start our analysis of problem (20) with the following auxiliary result.

Proposition 13. *If hypotheses $H(\alpha)_2$, $H(j)_3$, $H(f)_3$ hold and $x, y \in W_0^{1,p}(Z)$ are solutions of* (20), *then $v = \min\{x,y\} \in W_0^{1,p}(Z)$ and $w = \max\{x,y\} \in W_0^{1,p}(Z)$ are both solutions of* (20).

Proof. By definition there exist $u_1^*, u_2^* \in L^q(Z)$ such that

$$-\operatorname{div}\alpha(z,x(z),Dx(z)) = u_1^*(z) + f(z,x(z)), \quad u_1^*(z) \in \partial j(z,x(z)) \text{ a.e. on } Z \tag{21}$$

and

$$-\operatorname{div}\alpha(z,y(z),Dy(z)) = u_2^*(z) + f(z,y(z)), \quad u_2^*(z) \in \partial j(z,y(z)) \text{ a.e. on } Z. \tag{22}$$

First we show that for every $\vartheta \in C_0^1(Z)$ we have

$$\begin{aligned}\int_{\{y<x\}} (\alpha(z,x,Dx),D\vartheta)_{\mathbb{R}^N}\,dz - \int_{\{y<x\}} u_1^*\vartheta\,dz - \int_{\{y<x\}} f(z,x)\vartheta\,dz \\ = \int_{\{y<x\}} (\alpha(z,y,Dy),D\vartheta)_{\mathbb{R}^N}\,dz - \int_{\{y<x\}} u_2^*\vartheta\,dz - \int_{\{y<x\}} f(z,y)\vartheta\,dz.\end{aligned} \tag{23}$$

To show this equality, for every $\varepsilon > 0$ we introduce the truncation function $\eta_\varepsilon : \mathbb{R} \to \mathbb{R}$ defined by

$$\eta_\varepsilon(t) = \begin{cases} \varepsilon & \text{if } \varepsilon \leq t, \\ t & \text{if } |t| \leq \varepsilon, \\ -\varepsilon & \text{if } t \leq -\varepsilon. \end{cases}$$

Note that $\eta_\varepsilon(x-y)^+\vartheta \in W_0^{1,p}(Z)$, where $\eta_\varepsilon(x-y)^+(z) = \eta_\varepsilon((x-y)^+(z))$ (see Evans–Gariepy [23, p. 130]). Using this as our test function we obtain

$$0 = \int_Z (\alpha(z,x,Dx) - \alpha(z,y,Dy), D(\eta_\varepsilon(x-y)^+\vartheta))_{\mathbb{R}^N}\, dz$$
$$- \int_Z (u_1^* - u_2^*)\eta_\varepsilon(x-y)^+\vartheta\, dz - \int_Z (f(z,x) - f(z,y))\eta_\varepsilon(x-y)^+\vartheta\, dz.$$

Recall that

$$D(\eta_\varepsilon(x-y)^+\vartheta) = \vartheta D(\eta_\varepsilon(x-y)^+) + \eta_\varepsilon(x-y)^+ D\vartheta$$
$$= \begin{cases} D(x-y)\vartheta + (x-y)D\vartheta & \text{on } \{0 < x-y < \varepsilon\}, \\ \varepsilon D\vartheta & \text{on } \{\varepsilon \le x-y\}, \\ 0 & \text{on } \{x-y<0\}. \end{cases}$$

Thus we obtain

$$0 = \int_{\{0<x-y<\varepsilon\}} (\alpha(z,x,Dx) - \alpha(z,y,Dy), D(x-y))_{\mathbb{R}^N}\vartheta\, dz$$
$$+ \int_Z (\alpha(z,x,Dx) - \alpha(z,y,Dy), D\vartheta)_{\mathbb{R}^N}\eta_\varepsilon(x-y)^+\, dz$$
$$- \int_Z (u_1^* - u_2^*)\eta_\varepsilon(x-y)^+\vartheta\, dz - \int_Z (f(z,x) - f(z,y))\eta_\varepsilon(x-y)^+\vartheta\, dz.$$

We divide by $\varepsilon > 0$. Thus we have

$$\frac{1}{\varepsilon}\int_{\{0<x-y<\varepsilon\}} (\alpha(z,x,Dx) - \alpha(z,y,Dy), D(x-y))_{\mathbb{R}^N}\vartheta\, dz \tag{24}$$
$$= \int_Z (\alpha(z,y,Dy) - \alpha(z,x,Dx), D\vartheta)_{\mathbb{R}^N}\, \frac{\eta_\varepsilon(x-y)^+}{\varepsilon}\, dz$$
$$+ \int_Z (u_1^* - u_2^*)\,\frac{\eta_\varepsilon(x-y)^+}{\varepsilon}\,\vartheta\, dz + \int_Z (f(z,x) - f(z,y))\,\frac{\eta_\varepsilon(x-y)^+}{\varepsilon}\,\vartheta\, dz.$$

We estimate the left hand side of (24). Indeed, acting on (21) and (22) with the test function $\eta_\varepsilon(x-y)^+ \in W_0^{1,p}(Z)$ and then subtracting we obtain

$$\int_Z (\alpha(z,x,Dx) - \alpha(z,y,Dy), D\eta_\varepsilon(x-y)^+)_{\mathbb{R}^N}\, dz$$
$$= \int_{\{0<x-y<\varepsilon\}} (\alpha(z,x,Dx) - \alpha(z,y,Dy), D(x-y))_{\mathbb{R}^N}\, dz$$
$$= \int_Z (u_1^* - u_2^*)\eta_\varepsilon(x-y)^+\, dz + \int_Z (f(z,x) - f(z,y))\eta_\varepsilon(x-y)^+\, dz.$$

Due to the monotonicity of $-\partial j(z,\cdot)$ (hypothesis $H(j)_3$(ii)), we have

$$\int_Z (u_1^* - u_2^*)\eta_\varepsilon(x-y)^+\, dz \le 0.$$

Similarly by hypothesis $H(f)_3$(ii) we have

$$\int_Z (f(z,x) - f(z,y))\eta_\varepsilon(x-y)^+\, dz \le 0.$$

Thus we obtain

$$\int_{\{0<x-y<\varepsilon\}} (\alpha(z,x,Dx)-\alpha(z,y,Dy),D(x-y))_{\mathbb{R}^N}\,dz \le 0,$$

hence

$$\int_{\{0<x-y<\varepsilon\}} (\alpha(z,x,Dx)-\alpha(z,x,Dy),D(x-y))_{\mathbb{R}^N}\,dz \le \int_{\{0<x-y<\varepsilon\}} (\alpha(z,y,Dy)-\alpha(z,x,Dy),D(x-y))_{\mathbb{R}^N}\,dz. \tag{25}$$

Using hypothesis $H(\alpha)_2$(iv) we have

$$\begin{aligned}\Big|\int_{\{0<x-y<\varepsilon\}} (\alpha(z,y,Dy)-\alpha(z,x,Dy),D(x-y))_{\mathbb{R}^N}\,dz\Big| &\le \int_{\{0<x-y<\varepsilon\}} [c_2(|x|+|y|+\|Dy\|)^{p-1}+k_1(z)]\cdot|x-y|\cdot\|D(x-y)\|\,dz\\ &\le \varepsilon \int_{\{0<x-y<\varepsilon\}} \mu_1(z)\|D(x-y)\|\,dz\end{aligned}\tag{26}$$

where

$$\mu_1(\cdot)=c_2(|x(\cdot)|+|y(\cdot)|+\|Dy(\cdot)\|)^{p-1}+k_1(\cdot)\in L^q(Z).$$

Using this inequality in (25), observing that the left hand side of that inequality is nonnegative (hypothesis $H(\alpha)_2$(iii)) and then dividing by ε, we obtain

$$\begin{aligned}0 &\le \frac{1}{\varepsilon}\int_{\{0<x-y<\varepsilon\}} (\alpha(z,x,Dx)-\alpha(z,x,Dy),D(x-y))_{\mathbb{R}^N}\,dz\\ &\le \int_{\{0<x-y<\varepsilon\}} \mu_1(z)\|D(x-y)\|\,dz \to 0 \quad \text{as } \varepsilon\downarrow 0\end{aligned}\tag{27}$$

(remark that $\{0<x-y<\varepsilon\}\to\{x=y\}$ as $\varepsilon\downarrow 0$). Therefore we have

$$\begin{aligned}\Big|\frac{1}{\varepsilon}\int_{\{0<x-y<\varepsilon\}} (\alpha(z,x,Dx)-\alpha(z,y,Dy),D(x-y))_{\mathbb{R}^N}\vartheta\,dz\Big| &\le \Big|\frac{1}{\varepsilon}\int_{\{0<x-y<\varepsilon\}} (\alpha(z,x,Dx)-\alpha(z,x,Dy),D(x-y))_{\mathbb{R}^N}\vartheta\,dz\Big|\\ &\quad+\Big|\frac{1}{\varepsilon}\int_{\{0<x-y<\varepsilon\}} (\alpha(z,x,Dy)-\alpha(z,y,Dy),D(x-y))_{\mathbb{R}^N}\vartheta\,dz\Big|\\ &\le \frac{\|\vartheta\|_\infty}{\varepsilon}\int_{\{0<x-y<\varepsilon\}} (\alpha(z,x,Dx)-\alpha(z,x,Dy),D(x-y))_{\mathbb{R}^N}\,dz\\ &\quad+\|\vartheta\|_\infty\int_{\{0<x-y<\varepsilon\}} \mu_1(z)\|D(x-y)\|\,dz \to 0 \quad \text{as } \varepsilon\downarrow 0\end{aligned}\tag{28}$$

(see (26) and (27)).

Note that
$$\frac{\eta_\varepsilon(x-y)^+}{\varepsilon}(z) \to \mathcal{X}_{\{y<x\}}(z) \quad \text{a.e. on } Z \text{ as } \varepsilon \downarrow 0$$
and for all $\varepsilon > 0$, $0 \le \eta_\varepsilon(x-y)^+/\varepsilon \le 1$ a.e. on Z. So from the Lebesgue dominated convergence theorem as $\varepsilon \downarrow 0$ we have

$$\int_Z (\alpha(z,y,Dy) - \alpha(z,x,Dx), D\vartheta)_{\mathbb{R}^N} \frac{\eta_\varepsilon(x-y)^+}{\varepsilon}\, dz \to \int_{\{y<x\}} (\alpha(z,y,Dy) - \alpha(z,x,Dx), D\vartheta)_{\mathbb{R}^N}\, dz, \tag{29}$$

$$\int_Z (u_1^* - u_2^*) \frac{\eta_\varepsilon(x-y)^+}{\varepsilon} \vartheta\, dz \to \int_{\{y<x\}} (u_1^* - u_2^*)\vartheta\, dz, \tag{30}$$

$$\int_Z (f(z,x) - f(z,y)) \frac{\eta_\varepsilon(x-y)^+}{\varepsilon} \vartheta\, dz \to \int_{\{y<x\}} (f(z,x) - f(z,y))\vartheta\, dz. \tag{31}$$

Returning to (24), by passing to the limit as $\varepsilon \downarrow 0$ and using (28)–(31) we obtain
$$\begin{aligned} 0 = &\int_{\{y<x\}} (\alpha(z,y,Dy) - \alpha(z,x,Dx), D\vartheta)_{\mathbb{R}^N}\, dz \\ &+ \int_{\{y<x\}} (u_1^* - u_2^*)\vartheta\, dz + \int_{\{y<x\}} (f(z,x) - f(z,y))\vartheta\, dz. \end{aligned}$$
From this equality we get (23). In fact since $C_0^1(Z)$ is dense in $W_0^{1,p}(Z)$, we deduce that (23) is true for all $\vartheta \in W_0^{1,p}(Z)$.

Let $v = \min\{x,y\} \in W_0^{1,p}(Z)$ and set $\widehat{u}^* = \mathcal{X}_{\{x\le y\}} u_1^* + \mathcal{X}_{\{y<x\}} u_2^* \in S^q_{\partial j(\cdot, v(\cdot))}$. For all $\vartheta \in W_0^{1,p}(Z)$ we have
$$\begin{aligned} &\int_Z (\alpha(z,v,Dv), D\vartheta)_{\mathbb{R}^N}\, dz - \int_Z \widehat{u}^*\vartheta\, dz - \int_Z f(z,v)\vartheta\, dz \\ &\quad = \int_{\{x\le y\}} (\alpha(z,x,Dx), D\vartheta)_{\mathbb{R}^N}\, dz + \int_{\{y<x\}} (\alpha(z,y,Dy), D\vartheta)_{\mathbb{R}^N}\, dz \\ &\qquad - \int_{\{x\le y\}} u_1^*\vartheta\, dz - \int_{\{y<x\}} u_2^*\vartheta\, dz - \int_{\{x\le y\}} f(z,x)\vartheta\, dz - \int_{\{y<x\}} f(z,y)\vartheta\, dz \\ &\quad = \int_{\{x\le y\}} (\alpha(z,x,Dx), D\vartheta)_{\mathbb{R}^N}\, dz + \int_{\{y<x\}} (\alpha(z,x,Dx), D\vartheta)_{\mathbb{R}^N}\, dz \\ &\qquad - \int_{\{x\le y\}} u_1^*\vartheta\, dz - \int_{\{y<x\}} u_1^*\vartheta\, dz - \int_{\{x\le y\}} f(z,x)\vartheta\, dz - \int_{\{y<x\}} f(z,x)\vartheta\, dz \\ &\quad = \int_Z (\alpha(z,x,Dx), D\vartheta)_{\mathbb{R}^N}\, dz - \int_Z u_1^*\vartheta\, dz - \int_Z f(z,x)\vartheta\, dz = 0, \end{aligned}$$
hence
$$\int_Z (\alpha(z,v,Dv), D\vartheta)_{\mathbb{R}^N}\, dz - \int_Z \widehat{u}^*\vartheta\, dz - \int_Z f(z,v)\vartheta\, dz = 0 \quad \text{for all } \vartheta \in W_0^{1,p}(Z)$$

and so

$$\begin{cases} -\operatorname{div}\alpha(z,v(z),Dv(z)) = \widehat{u}^*(z) + f(z,v(z)) \in \partial j(z,v(z)) + f(z,v(z)) & \text{a.e. on } Z,\\ v|_\Gamma = 0. \end{cases}$$

Therefore $v = \min\{x,y\} \in W_0^{1,p}(Z)$ is a solution of (20). Similarly we show that $w = \max\{x,y\} \in W_0^{1,p}(Z)$ is a solution of (20). ■

Using this proposition, we can now establish the existence of extremal solutions in the order interval $[\psi,\varphi]$.

THEOREM 14. *If hypotheses $H(\alpha)_2$, $H(j)_3$, $H(f)_3$ and H_0 hold, then problem* (20) *has extremal solutions in the order interval* $[\psi,\varphi]$.

Proof. Let $S = \{x \in [\psi,\varphi] : x \in W_0^{1,p}(Z)\,\text{is a solution of (20)}\}$. From Theorem 12 we know that $S \neq \emptyset$. Let $\mathcal{C}$ be a chain in S (i.e. a linearly (totally) ordered subset of S; on $W_0^{1,p}(Z)$ we consider the usual pointwise ordering induced by $L^p(Z)_+$, i.e. if $x,y \in W_0^{1,p}(Z)$, then $x \le y$ if and only if $x(z) \le y(z)$ a.e. on Z). Note that $S \subseteq L^p(Z)$ is order bounded and so we can define $w = \sup\mathcal{C}$. Since the order on $W_0^{1,p}(Z)$ is the pointwise order inherited from the Banach lattice $L^p(Z)$, from Corollary 7 on p. 336 of Dunford–Schwartz [22], we can find a sequence $\{x_n\}_{n\ge1} \subseteq \mathcal{C}$ such that $x_n \xrightarrow{w} w$ in $W_0^{1,p}(Z)$ (because of hypothesis $H(\alpha)_2$(v)) and $x_n \to w$ in $L^p(Z)$ as $n \to \infty$ (recall that $W_0^{1,p}(Z)$ is compactly embedded in $L^p(Z)$ by the Sobolev embedding theorem). Exploiting the lattice structure of S we can assume that $x_n(z) \uparrow w(z)$ a.e. on Z. For every $n \ge 1$ we have

$$A(x_n) = u_n^* + N_f(x_n)$$

where $u_n^* \in S^q_{\partial j(\cdot,x_n(\cdot))}$, $A : W_0^{1,p}(Z) \to W^{-1,q}(Z)$ is the nonlinear operator defined by

$$\langle A(x),y\rangle = \int_Z (\alpha(z,x,Dx),Dy)_{\mathbb{R}^N}\,dz$$

and $N_f(x_n)(\cdot) = f(\cdot,x_n\cdot)$. Using a simplified version of the proof of Proposition 7, we can show that A is bounded, pseudomonotone. By virtue of hypothesis $H(f)_3$(ii) and since $x_n(z) \uparrow w(z)$ a.e. on Z, we have $f(z,x_n(z)) \downarrow f(z,w(z))$ a.e. on Z. Because $N_f(\psi), N_f(\varphi) \in L^q(Z)$, we have $N_f(w) \in L^q(Z)$ and from the monotone convergence theorem we have $N_f(x_n) \to N_f(w)$ in $L^q(Z)$. Also from hypothesis $H(j)_3$(iii), $\{u_n^*\}_{n\ge1} \subseteq L^q(Z)$ is bounded and so we may assume that $u_n^* \xrightarrow{w} u^*$ in $L^q(Z)$. Using Proposition 2 and the closedness of the graph of $\partial j(z,\cdot)$ we have $u^* \in S^q_{\partial j(\cdot,w(\cdot))}$. Finally, because A is bounded we may assume that $A(x_n) \xrightarrow{w} v^*$ in $W^{-1,q}(Z)$. But A being pseudomonotone, it is generalized pseudomonotone (see Section 2) and so $v^* = A(x)$. Therefore in the limit as $n \to \infty$, we have

$$A(w) = u^* + N_f(w) \quad \text{with } u^* \in S^q_{\partial j(\cdot,w(\cdot))}, \quad \text{hence} \quad w \in S.$$

Invoking Zorn's lemma, we infer that S has a maximal element $x_M \in S$. Proposition 13 implies that x_M is the greatest element of S. Similarly we produce $x_m \in S$ which is the smallest element of S. Evidently $\{x_m, x_M\}$ are the desired extremal solutions of (20). ■

5. Bounded solutions of definite sign

In this section we consider a hemivariational inequality driven by the p-Laplacian operator. Using some auxiliary problems, the nonlinear maximum principle and the method of upper-lower solutions, we shall establish the existence of bounded positive and negative solutions for the problem.

The problem under consideration in this section is the following:

$$\begin{cases} -\operatorname{div}(\|Dx(z)\|^{p-2}Dx(z)) \in \partial j(z,x(z)) + f(z,x(z)) & \text{a.e. on } Z, \\ x|_\Gamma = 0, \quad 2 \le p < \infty. \end{cases} \tag{32}$$

Our hypotheses on the data of (32) are the following:

$\boxed{H(j)_4}$ $j : Z \times \mathbb{R} \to \mathbb{R}$ is a function such that

(i) for all $x \in \mathbb{R}$, $z \mapsto j(z,x)$ is measurable;
(ii) for almost all $z \in Z$, $x \mapsto j(z,x)$ is locally Lipschitz;
(iii) for almost all $z \in Z$, all $x \in \mathbb{R}$ and all $u^* \in \partial j(z,x)$ we have

$$|u^*| \le a_1(z) + c_1|x|^{r-1}$$
$$\text{with } a \in L^{r'}(Z),\ 1/r + 1/r' = 1,\ 1 \le r < p^*,\ c_1 > 0;$$

(iv) there exists $\vartheta \in L^\infty(Z)$ such that $\vartheta(z) \le 0$ on Z with strict inequality on a set of positive Lebesgue measure and $\limsup_{x\to\infty} u^*/x^{p-1} \le \vartheta(z)$ uniformly for almost all $z \in Z$ and all $u^* \in \partial j(z,x)$;
(v) $\liminf_{x\to 0^+} u^*/x^{p-1} > 0$ uniformly for almost all $z \in Z$ and all $u^* \in \partial j(z,x)$.

$\boxed{H(f)_4}$ $f : Z \times \mathbb{R} \to \mathbb{R}$ is a function such that $f(z,0) = 0$ a.e. on Z and

(i) for all $x \in \mathbb{R}$, $z \mapsto f(z,x)$ is measurable;
(ii) for almost all $z \in Z$, $x \mapsto f(z,x)$ is continuous;
(iii) for almost all $z \in Z$ and all $x \ge 0$ we have

$$0 \le f(z,x) \le a_2(z) + |x|^{r-1}$$
$$\text{with } a_2 \in L^{r'}(Z),\ 1/r + 1/r' = 1,\ 1 \le r < p^*,\ c_2 > 0;$$

(iv) $\limsup_{x\to+\infty} f(z,x)/x^{p-1} < \lambda_1$ and $\liminf_{x\to 0^+} f(z,x)/x^{p-1} > \lambda_1$ uniformly for almost all $z \in Z$.

It appears that our results in this section are the first results on the existence of positive and negative solutions for nonlinear hemivariational inequalities.

By hypothesis $H(j)_4$(iv), given $\varepsilon > 0$ we can find $M_1 > 0$ such that for almost all $z \in Z$, all $x \ge M_1 > 0$ and all $u^* \in \partial(z,x)$ we have

$$u^* \le (\vartheta(z) + \varepsilon)x^{p-1}. \tag{33}$$

On the other hand from hypothesis $H(j)_4$(iii) for almost all $z \in Z$, all $0 \le x < M_1$ and all $u^* \in \partial(z,x)$ we have

$$|u^*| \le \gamma_1(z) \quad \text{with } \gamma_1 \in L^q(Z). \tag{34}$$

Combining (33), (34) we see that for almost all $z \in Z$, all $x \geq 0$ and all $u^* \in \partial(z,x)$ we have

$$u^* \leq (\vartheta(z) + \varepsilon)x^{p-1} + \gamma_2(z) \quad \text{with } \gamma_2 \in L^q(Z). \tag{35}$$

Let $g : Z \times \mathbb{R} \to \mathbb{R}$ be the Carathéodory function defined by

$$g(z,x) = \begin{cases} f(z,x) & \text{if } x \geq 0, \\ 0 & \text{if } x \leq 0. \end{cases}$$

We consider the following auxiliary boundary value problem:

$$\begin{cases} -\operatorname{div}(\|Dx(z)\|^{p-2}Dx(z)) - g(z,x(z)) = (\vartheta(z)+\varepsilon)|x(z)|^{p-2}x(z) + \gamma_2(z) \\ \hfill \text{a.e. on } Z, \\ x|_\Gamma = 0. \end{cases} \tag{36}$$

We start by solving (36).

Proposition 15. *If hypotheses $H(j)_4$ and $H(f)_4$ hold, then for all $\varepsilon > 0$ small, problem* (36) *has a solution $\varphi \in C^1(\overline{Z})$ such that $\varphi(z) > 0$ for all $z \in Z$ and $\frac{\partial\varphi}{\partial n}(z') < 0$ for all $z' \in \Gamma$ (here by n we denote the outward normal on the boundary Γ of Z).*

Proof. Let $A : W_0^{1,p}(Z) \to W^{-1,q}(Z)$ be the nonlinear operator defined by

$$\langle A(x), y\rangle = \int_Z \|Dx(z)\|^{p-2}(Dx(z), Dy(z))_{\mathbb{R}^N}\, dz \quad \text{for all } x,y \in W_0^{1,p}(Z).$$

It is easy to check that A is monotone demicontinuous, hence maximal monotone (see Section 2). Also let $N_g : L^r(Z) \to L^{r'}(Z)$ be the Nemytskiĭ operator corresponding to the function g, i.e. $N_g(x)(\cdot) = g(\cdot, x(\cdot))$. By Krasnosel'skiĭ's theorem N_g is bounded, continuous. Also exploiting the compact embedding of $W_0^{1,p}(Z)$ into $L^r(Z)$ (hence of $L^{r'}(Z)$ into $W^{-1,q}(Z)$ as well) it follows that $N_g : W_0^{1,p}(Z) \to W^{-1,q}(Z)$ is a completely continuous (hence compact as well) operator. Similarly we define $J_{\vartheta,\varepsilon} : L^p(Z) \to L^q(Z)$ by $J_{\vartheta,\varepsilon}(x)(\cdot) = (\vartheta(\cdot)+\varepsilon)|x(\cdot)|^{p-2}x(\cdot)$. In the same way we deduce that $J_{\vartheta,\varepsilon}$ viewed as a map from $W_0^{1,p}(Z)$ into $W^{-1,q}(Z)$ is completely continuous (hence compact as well). Since maximal monotone maps defined everywhere and completely continuous are pseudomonotone and the sum of pseudomonotone maps is still pseudomonotone (see Hu–Papageorgiou [35, p. 368]), it follows that $x \mapsto V(x) = A(x) - N_g(x) - J_{\vartheta,\varepsilon}(x)$ is pseudomonotone and bounded.

As in the proof of Proposition 9 (see inequality (11)), we can show that there exists $\beta > 0$ such that

$$\|Dx\|_p^p - \int_Z (\lambda_1 + \vartheta(z))|x(z)|^p\, dz \geq \beta\|Dx\|_p^p \quad \text{for all } x \in W_0^{1,p}(Z). \tag{37}$$

Then for all $x \in W_0^{1,p}(Z)$, we have

$$\begin{aligned} \langle V(x), x\rangle &= \langle A(x), x\rangle - \int_Z g(z,x(z))x(z)\, dz - \int_Z (\vartheta(z)+\varepsilon)|x(z)|^p\, dz \\ &= \|Dx\|_p^p - \int_Z g(z,x(z))x(z)\, dz - \int_Z \vartheta(z)|x(z)|^p\, dz - \varepsilon\|x\|_p^p. \end{aligned} \tag{38}$$

By hypotheses $H(f)_4$(iii) and (iv), for almost all $z \in Z$ and all $x \geq 0$, we have

$$g(z,x) = f(z,x) \leq \lambda_1|x|^{p-2}x + \gamma_3(z) \quad \text{with } \gamma_3 \in L^{r'}(Z)_+.$$

So we can write that

$$-\int_Z g(z,x(z))x(z)\,dz \ge -\lambda_1\|x\|_p^p - \|\gamma_3\|_1.$$

Using this in (38) we obtain

$$\begin{aligned}\langle V(x),x\rangle &= \|Dx\|_p^p - \int_Z(\lambda_1+\vartheta(z))|x(z)|^p\,dz - \varepsilon\|x\|_p^p - \|\gamma_3\|_1\\ &\ge (\beta-\varepsilon/\lambda_1)\|Dx\|_p^p - \|\gamma_3\|_1 \quad \text{(see (37) and (2)).}\end{aligned}$$

So if we choose $0<\varepsilon<\beta\lambda_1$, we see that the pseudomonotone operator V is coercive, thus it is surjective. Therefore we can find $\varphi\in W_0^{1,p}(Z)$ such that $V(\varphi)=\gamma_2$. As before we can verify that φ solves problem (36).

Next let $\varphi^-=\max\{-\varphi,0\}\in W_0^{1,p}(Z)_+$ be our test function. Also recall that

$$D\varphi^-(z)=\begin{cases}-D\varphi(z) & \text{a.e. on } \{\varphi>0\},\\ 0 & \text{a.e. on } \{\varphi\le 0\}\end{cases}$$

(see Evans–Gariepy [23, p. 130]). We have

$$\begin{aligned}\langle V(\varphi),\varphi^-\rangle &= -\|D\varphi\|_p^p - \int_Z g(z,\varphi(z))\varphi^-(z)\,dz + \int_Z(\vartheta(z)+\varepsilon)|\varphi^-(z)|^p\,dz\\ &= \int_Z \gamma_2(z)\varphi^-(z)\,dz \ge 0,\end{aligned}$$

because $\gamma_2\ge 0$. Note that $\int_Z g(z,\varphi(z))\varphi^-(z)\,dz = \int_{\{\varphi<0\}} g(z,\varphi(z))\varphi^-(z)\,dz = 0$ (recall the definition of g). So we have

$$0\le\langle V(\varphi),\varphi^-\rangle = -\|D\varphi^-\|_p^p + \int_Z(\vartheta(z)+\varepsilon)|\varphi^-(z)|^p\,dz,$$

hence

$$\lambda_1\|\varphi^-\|_p^p \le \|D\varphi^-\|_p^p \le \int_Z(\vartheta(z)+\varepsilon)|\varphi^-(z)|^p\,dz \le \varepsilon\|\varphi^-\|_p^p$$

(recall that $\vartheta(z)\le 0$ a.e. on Z) and so

$$0\le(\varepsilon-\lambda_1)\|\varphi^-\|_p^p.$$

If $\varepsilon<\min\{\lambda_1,\lambda_1\beta\}$, from this last inequality we have $\varphi^-=0$ hence $\varphi\ge 0$.

From Theorem 7.1, p. 286, of Ladyzhenskaya–Uraltseva [42], we have $\varphi\in L^\infty(Z)$ and from the nonlinear regularity theorem of Lieberman [44], we have $\varphi\in C^{1,\varepsilon}(\overline{Z})$ for some $0<\varepsilon<1$. Since $\gamma_2\ge 0$ we have (here $\Delta_p\varphi=\operatorname{div}(\|D\varphi\|^{p-2}D\varphi)$ is the p-Laplacian)

$$-\Delta_p\varphi(z) - g(z,\varphi(z)) - (\vartheta(z)+\varepsilon)|\varphi(z)|^{p-2}\varphi(z)\ge 0 \quad \text{a.e. on } Z,$$

which implies

$$\Delta_p\varphi(z)\le\|\vartheta+\varepsilon\|_\infty\varphi(z)^{p-1} \quad \text{a.e. on } Z$$

(recall that $g(z,\varphi(z))\ge 0$ a.e. on Z, from the definition of g and hypothesis $H(f)_4$(iii)).

Invoking Theorem 5 of Vazquez [63], we infer that $\varphi(z)>0$ for all $z\in Z$ and $\frac{\partial\varphi}{\partial n}(z')<0$ for all $z'\in\Gamma$. This completes the proof of the proposition. ∎

From hypotheses $H(j)_4$(iv) and $H(f)_4$(iv), we know that there exist $\delta > 0$ and $\xi > \lambda_1$ such that for almost all $z \in Z$, all $0 < x \le \delta$ and all $u^* \in \partial j(z,x)$ we have

$$0 < u^* \quad \text{and} \quad \xi x^{p-1} \le f(z,x). \tag{39}$$

Let $u_1 \in C^1(\overline{Z}), u_1 > 0$, be the normalized principal eigenfunction of $(-\Delta_p, W_0^{1,p}(Z))$. Evidently we can find $0 < \xi_1 < 1$ such that $0 < \xi_1 u_1 \le \delta$ for all $z \in \overline{Z}$. Also let $\varphi \in C^1(\overline{Z})$ be the solution of the auxiliary problem (36) obtained in Proposition 15. Because of the properties of φ established in that proposition, we know that there exists $\xi_2 > 1$ such that

$$\xi_1 u_1(z) < \xi_2 \varphi(z) \quad \text{for all } z \in Z,$$

hence that

$$v(z) = \frac{\xi_1}{\xi_2} u_1(z) < \varphi(z) \quad \text{for all } z \in Z \text{ (note that } \xi_1/\xi_2 < 1).$$

We have $v \in C^{1,\varepsilon}(\overline{Z}), v(z) > 0$ for all $z \in Z$. For any $y \in W_0^{1,p}(Z)$, $y \ge 0$ and any $u^* \in S^{r'}_{\partial j(\cdot, v(\cdot))}$ we have

$$\begin{aligned}\langle -\Delta_p v, y\rangle &- \int_Z g(z,v(z))y(z)\,dz - \int_Z u^*(z)y(z)\,dz \\ &= \int_Z \|Dv(z)\|^{p-2}(Dv(z),Dy(z))_{\mathbb{R}^N}\,dz - \int_Z f(z,v(z))y(z)\,dz - \int_Z u^*(z)y(z)\,dz.\end{aligned}$$

Since

$$v(z) = \frac{\xi_1}{\xi_2} u_1(z) \in (0,\delta] \quad \text{for all } z \in Z,$$

from (39) we have $u^*(z) > 0$ a.e. on Z and so $\int_Z u^*(z)y(z)\,dz \ge 0$. Also again from (39) we have $\xi v(z)^{p-1} \le f(z,v(z))$ a.e. on Z. So finally we can write that

$$\begin{aligned}\int_Z \|Dv(z)\|^{p-2}&(Dv(z),Dy(z))_{\mathbb{R}^N}\,dz - \int_Z f(z,v(z))y(z)\,dz - \int_Z u^*(z)y(z)\,dz \\ &\le \int_Z \|Dv(z)\|^{p-2}(Dv(z),Dy(z))_{\mathbb{R}^N}\,dz - \xi\int_Z v(z)^{p-1}y(z)\,dz \\ &= \left(\frac{\xi_1}{\xi_2}\right)^{p-1} \int_Z (\lambda_1 - \xi)(u_1(z))^{p-1}y(z)\,dz < 0\end{aligned}$$

(recall the definition of v and that $\xi > \lambda_1$). Therefore we infer that $v \in C^{1,\varepsilon}(\overline{Z})$ is a lower solution of (32).

On the other hand since φ is a solution of the auxiliary problem (36), for all $y \in W_0^{1,p}(Z)$, $y \ge 0$ and all $u^* \in S^{r'}_{\partial j(\cdot,\varphi(\cdot))}$ we have

$$\begin{aligned}\int_Z \|D\varphi(z)\|^{p-2}&(D\varphi(z),Dy(z))_{\mathbb{R}^N}\,dz - \int_Z g(z,\varphi(z))y(z)\,dz \\ &= \int_Z (\vartheta(z)+\varepsilon)|\varphi(z)|^{p-2}\varphi(z)y(z)\,dz + \int_Z \gamma_2(z)y(z)\,dz,\end{aligned}$$

which implies that

$$\int_Z \|D\varphi(z)\|^{p-2}(D\varphi(z), Dy(z))_{\mathbb{R}^N}\,dz - \int_Z f(z,\varphi(z))y(z)\,dz$$
$$= \int_Z (\vartheta(z)+\varepsilon)|\varphi(z)|^{p-2}\varphi(z)y(z)\,dz + \int_Z \gamma_2(z)y(z)\,dz \geq \int_Z u^*(z)y(z)\,dz \quad \text{(see (35))}.$$

Therefore we infer that $\varphi \in C^1(\overline{Z})$ is an upper solution of (32). Also we have $v(z) < \varphi(z)$ for all $z \in Z$. Now working with the upper-lower solution pair (φ, v) as in Theorem 12, through truncation and penalization techniques, we can have the following existence theorem.

THEOREM 16. *If hypotheses $H(j)_4$ and $H(f)_4$ hold, then problem* (32) *has a solution $x \in W_0^{1,p}(Z) \cap L^\infty(Z)$ such that $x(z) > 0$ a.e. on Z.*

In fact if on the functions $j(z,\cdot)$ and $f(z,\cdot)$ we impose a behaviour at 0^- and $-\infty$, similar to that assumed at 0^+ and ∞, we can have a multiplicity result for problem (32). More precisely we can show that problem (32) has at least two nontrivial solutions, one strictly positive and the other strictly negative. The new hypotheses on j and f are the following:

$\boxed{H(j)_5}$ $j : Z \times \mathbb{R} \to \mathbb{R}$ is a function which satisfies hypotheses $H(j)_4$(i)–(iii) and

(iv) there exists $\vartheta \in L^\infty(Z)$ such that $\vartheta(z) \leq 0$ a.e. on Z with strict inequality on a set of positive Lebesgue measure and $\limsup_{|x|\to\infty} u^*/(|x|^{p-2}x) \leq \vartheta(z)$ uniformly for almost all $z \in Z$ and all $u^* \in \partial j(z,x)$;

(v) $\liminf_{x\to 0} u^*/(|x|^{p-2}x) > 0$ uniformly for almost all $z \in Z$ and all $u^* \in \partial j(z,x)$.

$\boxed{H(f)_5}$ $f : Z \times \mathbb{R} \to \mathbb{R}$ is a function which satisfies hypotheses $H(f)_4$(i)–(iii) and

(iv) $\limsup_{|x|\to\infty} f(z,x)/(|x|^{p-2}x) < \lambda_1$ and $\liminf_{x\to 0} f(z,x)/(|x|^{p-2}x) > \lambda_1$ uniformly for almost all $z \in Z$.

Repeating the previous analysis, this time on the negative semiaxis $\mathbb{R}_-$, we obtain a bounded strictly negative solution. Combining this result with Theorem 16, we obtain the following multiplicity result.

THEOREM 17. *If hypotheses $H(j)_5$ and $H(f)_5$ hold, then problem* (32) *has at least two solutions $x, y \in W_0^{1,p}(Z) \cap L^\infty(Z)$ such that $y(z) < 0 < x(z)$ a.e. on Z.*

6. Hemivariational inequalities at resonance

In this section we study nonlinear hemivariational inequalities at resonance using hypotheses of Landesman–Lazer type. We have two existence theorems. The first uses a more standard Landesman–Lazer type condition and our approach is based on Theorem 5 (i.e. it is degree-theoretic). The second theorem is about problems driven by the p-Laplacian differential operator and uses a generalized Landesman–Lazer type condition, first suggested by Tang [62] in the context of second order semilinear ordinary differential equations.

The first problem that we shall examine in this section is the following:

$$\begin{cases} -\operatorname{div}\alpha(z,Dx(z)) - \lambda^*|x(z)|^{p-2}x(z) \in \partial j(z,x(z)) & \text{a.e. on } Z,\\ x|_\Gamma = 0, \quad \lambda^* = \lambda_1 c_1, \quad c_1 > 0, \quad 2 \le p < \infty. \end{cases} \tag{40}$$

Our hypotheses on the functions $\alpha(z,\xi)$ and $j(z,x)$ of problem (40) are the following:

$\boxed{H(\alpha)_3}$ $\alpha : Z \times \mathbb{R}^N \to \mathbb{R}^N$ is a function such that $\alpha(z,0) = 0$ a.e. on Z and

(i) for every $\xi \in \mathbb{R}^N$, $z \mapsto \alpha(z,\xi)$ is measurable;
(ii) for almost all $z \in Z$, $\xi \mapsto \alpha(z,\xi)$ is continuous and strictly monotone;
(iii) for almost all $z \in Z$ and all $\xi \in \mathbb{R}^N$ we have

$$\|\alpha(z,\xi)\| \le b(z) + c\|\xi\|^{p-1},$$

where $b \in L^q(Z)_+$ and $c > 0$;
(iv) for almost all $z \in Z$ and all $\xi \in \mathbb{R}^N$ we have $(\alpha(z,\xi),\xi)_{\mathbb{R}^N} \ge c_1\|\xi\|^p$.

$\boxed{H(j)_6}$ $j : Z \times \mathbb{R} \to \mathbb{R}$ is a function such that

(i) for all $x \in \mathbb{R}$, $z \mapsto j(z,x)$ is measurable;
(ii) for almost all $z \in Z$, $x \mapsto j(z,x)$ is locally Lipschitz;
(iii) for almost $z \in Z$, all $x \in \mathbb{R}$ and all $u^* \in \partial j(z,x)$, we have $|u^*| \le \beta(z)$ with $\beta \in L^q(Z)$ $(1/p + 1/q = 1)$;
(iv) if $g_1(z,x) = \inf[u^* : u^* \in \partial j(z,x)]$ and $g_2(z,x) = \sup[u^* : u^* \in \partial j(z,x)]$, there exist $g_-, g_+ \in L^1(Z)$ such that $\liminf_{x\to-\infty} g_1(z,x) = g_-(z)$, $\limsup_{x\to\infty} g_2(z,x) = g_+(z)$ for almost all $z \in Z$ and $\int_Z g_+(z)u_1(z)\,dz < 0 < \int_Z g_-(z)u_1(z)\,dz$.

Remark. By redefining j on a Lebesgue-null set, we may assume that j is Borel measurable and for all $z \in Z$, $j(z,\cdot)$ is locally Lipschitz. Recall from Section 2 that

$$\begin{aligned} j^0(z,x;h) &= \limsup_{x'\to x,\ \lambda\downarrow 0} \frac{j(z,x'+\lambda h) - j(z,x')}{\lambda} = \inf_{\varepsilon>0} \sup_{\substack{|x'-x|<\varepsilon\\ 0<\lambda<\varepsilon}} \frac{j(z,x'+\lambda h) - j(z,x')}{\lambda} \\ &= \inf_{n\ge1} \sup_{\substack{|x'-x|<1/n\\ 0<\lambda<1/n\\ x',\lambda\in\mathbb{Q}}} \frac{j(z,x'+\lambda h) - j(z,x')}{\lambda}. \end{aligned}$$

From this it follows that the function $(z,x,h) \mapsto j^0(z,x;h)$ is Borel measurable (note that because of hypotheses $H(j)_6$(i) and (ii), the function $j(z,x)$ being Carathéodory, it is jointly measurable). Since $\partial j(z,x) = \{u^* \in \mathbb{R} : u^*\,h \le j^0(z,x;h) \text{ for all } h \in \mathbb{R}\}$, we have $\operatorname{Gr}\partial j = \{(z,x,u^*) \in Z\times\mathbb{R}\times\mathbb{R} : u^* \in \partial j(z,x)\} \in B(Z\times\mathbb{R}\times\mathbb{R}) = B(Z)\times B(\mathbb{R})\times B(\mathbb{R})$, with $B(Z)$ (resp. $B(\mathbb{R})$) being the Borel σ-field of Z (resp. of $\mathbb{R}$). For every $\mu \in \mathbb{R}$ we have

$$\{(z,x) \in Z\times\mathbb{R} : g_1(z,x) < \mu\} = \operatorname{proj}_{Z\times\mathbb{R}}(\operatorname{Gr}\partial j \cap (Z\times\mathbb{R}\times(-\infty,\mu))).$$

Since the subdifferential multifunction ∂j is compact-valued, from Theorem II.1.22, p. 146, of Hu–Papageorgiou [35], we deduce that

$$\mathrm{proj}_{Z\times\mathbb{R}}(\mathrm{Gr}\,\partial j\cap(Z\times\mathbb{R}\times(-\infty,\mu)))\in B(Z\times\mathbb{R})=B(Z)\times B(\mathbb{R})$$

and hence that g_1 is Borel measurable. Similarly we can show that g_2 is Borel measurable. Therefore hypothesis $H(j)_6$(iv) is a reasonable requirement on the generalized potential $j(z,x)$.

For problem (40) we have the following existence theorem.

THEOREM 18. *If hypotheses $H(\alpha)_3$ and $H(f)_6$ hold, then problem* (40) *has a solution $x\in W_0^{1,p}(Z)$.*

Proof. Let $A:W_0^{1,p}(Z)\to W^{-1,q}(Z)$ be the nonlinear operator defined by

$$\langle A(x),y\rangle=\int_Z(\alpha(z,Dx),Dy)_{\mathbb{R}^N}\,dz.$$

We can easily verify that A is strictly monotone, demicontinuous, hence maximal monotone as well. Let $\widehat{A}$ be the restriction of A on $L^q(Z)$, i.e. $\widehat{A}:D\subseteq L^p(Z)\to L^q(Z)$ defined by $\widehat{A}(x)=A(x)$ for all $x\in D=\{x\in W_0^{1,p}(Z):A(x)\in L^q(Z)\}$ (recall $L^q(Z)\subseteq W^{-1,q}(Z)$). To this end let $J:L^p(Z)\to L^q(Z)$ be the nonlinear operator defined by $J(x)(\cdot)=|x(\cdot)|^{p-2}x(\cdot)$. Evidently J is continuous and strictly monotone (hence maximal monotone). We claim that in order to obtain the maximality of the monotone operator $\widehat{A}$ it suffices to show that $R(\widehat{A}+J)=L^q(Z)$, i.e. $\widehat{A}+J$ is surjective. Indeed suppose for the moment that $\widehat{A}+J$ is surjective and let $y\in L^p(Z)$, $v\in L^q(Z)$ be such that for all $x\in D$, we have

$$(\widehat{A}(x)-v,x-y)_{pq}\geq 0. \tag{41}$$

Here by $(\cdot,\cdot)_{pq}$ we denote the duality brackets for the pair $(L^p(Z),L^q(Z))$ i.e. $(x,y)_{pq}=\int_Z x(z)y(z)\,dz$. Since we have assumed that $\widehat{A}+J$ is surjective, we can find $x_1\in D$ such that $\widehat{A}(x_1)+J(x_1)=v+J(y)$. So if in (41) we set $x=x_1\in D$, we obtain

$$(v+J(y)-J(x_1)-v,x_1-y)_{pq}\geq 0,$$

hence

$$(J(y)-J(x_1),x_1-y)_{pq}\geq 0.$$

Because J is strictly monotone, from the above inequality it follows that $y=x_1\in D$ and $v=\widehat{A}(x_1)$, which proves the maximality of $\widehat{A}$ (see Section 2). Therefore it remains to show that $R(\widehat{A}+J)=L^q(Z)$. Note that the operator $A+J:W_0^{1,p}(Z)\to W^{-1,q}(Z)$ is maximal monotone (see Hu–Papageorgiou [35, p. 319]), coercive, hence it is surjective. So given any $h\in L^q(Z)$, we can find $x\in W_0^{1,p}(Z)$ such that $A(x)+J(x)=h$. Hence $A(x)=h-J(x)\in L^q(Z)$ and from the definition of $\widehat{A}$ we have $x\in D$ and $A(x)=\widehat{A}(x)$. Since $h\in L^q(Z)$ was arbitrary we conclude that $\widehat{A}+J$ is surjective. Therefore $\widehat{A}$ is maximal monotone and strictly monotone.

Next let $V=\widehat{A}+J:D\subseteq L^p(Z)\to L^q(Z)$. This map is maximal monotone, strictly monotone and coercive. So $V^{-1}:L^q(Z)\to D\subseteq W_0^{1,p}(Z)$ is a well defined operator.

CLAIM 1. *V^{-1} is completely continuous.*

Let $v_n \xrightarrow{w} v$ in $L^q(Z)$ and set $x_n = V^{-1}(v_n),\ n \geq 1$. We have

$$V(x_n) = v_n,$$

hence

$$\widehat{A}(x_n) + J(x_n) = v_n,$$

therefore

$$(\widehat{A}(x_n), x_n)_{pq} + (J(x_n), x_n)_{pq} = (v_n, x_n)_{pq}$$

and so

$$c_1 \|Dx_n\|_p^p + \|x_n\|_p^p \leq \|v_n\|_q \|x_n\|_p \quad \text{(hypothesis } H(\alpha)_3\text{(iii))},$$

which implies that $\{x_n\}_{n\geq 1} \subseteq W_0^{1,p}(Z)$ is bounded.

By passing to a subsequence if necessary, we may assume that $x_n \xrightarrow{w} x$ in $W_0^{1,p}(Z)$ and $x_n \to x$ in $L^p(Z)$. Note that for all $n \geq 1$, $(x_n, v_n) \in \operatorname{Gr} V$ and because V is maximal monotone, its graph is sequentially closed in $L^p(Z) \times L^q(Z)_w$. Thus in the limit as $n \to \infty$ we have $(x, v) \in \operatorname{Gr} V$ hence $x = V^{-1}(v)$. Also for all $n \geq 1$ we have

$$(\widehat{A}(x_n), x_n - x)_{pq} + (J(x_n), x_n - x)_{pq} = (v_n, x_n - x)_{pq}$$

so that

$$\langle A(x_n), x_n - x\rangle + (J(x_n), x_n - x)_{pq} = (v_n, x_n - x)_{pq}.$$

Because $\{J(x_n)\}_{n\geq 1}, \{v_n\}_{n\geq 1} \subseteq L^q(Z)$ are bounded and $x_n \to x$, we have

$$(J(x_n), x_n - x)_{pq}, (v_n, x_n - x)_{pq} \to 0 \quad \text{as } n \to \infty,$$

hence

$$\lim \langle A(x_n), x_n - x\rangle = 0.$$

But from Proposition 7 we know that A is of type $(S)_+$. So it follows that $x_n \to x$ in $W_0^{1,p}(Z)$ as $n \to \infty$. This proves the complete continuity of V.

Now let $G : L^p(Z) \to \mathcal{P}_{wkc}(L^q(Z))$ be the multifunction defined by $G(x) = S^q_{\partial j(\cdot, x(\cdot))}$.

Claim 2. *G is usc from $L^p(Z)$ into $L^q(Z)_w$.*

Since G is bounded, it is locally compact (recall that on $L^q(Z)$ we consider the weak topology). So in order to prove the claim it suffices to show that $\operatorname{Gr} G$ is sequentially closed in $L^p(Z) \times L^q(Z)_w$ (see Section 2). So let $(x_n, u_n^*) \in \operatorname{Gr} G,\ n \geq 1$, and assume that $x_n \to x$ in $L^p(Z)$ and $u_n^* \xrightarrow{w} u^*$ in $L^q(Z)$ as $n \to \infty$. By passing to further subsequences if necessary, we may assume that $x_n(z) \to x(z)$ a.e. on Z and so by Proposition 2 we have

$$u^*(z) \in \overline{\operatorname{conv}} \limsup_{n\to\infty} \{u_n^*(z)\} \subseteq \partial j(z, x(z)) \quad \text{a.e. on } Z,$$

the last inclusion following from the closedness of the graph of $x \mapsto \partial j(z, x)$. So $(x, u^*) \in \operatorname{Gr} G$ and this proves the desired upper semicontinuity of G from $L^p(Z)$ into $L^q(Z)_w$.

Let $G_1 = G + (\lambda^* + 1)J|_{W_0^{1,p}(Z)}$. Exploiting the continuous (in fact compact) embedding of $W_0^{1,p}(Z)$ into $L^p(Z)$, we see that G_1 is usc from $W_0^{1,p}(Z)$ into $L^q(Z)_w$ (see Claim 2). Consider the inclusion

$$x \in V^{-1} G_1(x). \tag{42}$$

Because of Claims 1 and 2, to solve (42) we can use Theorem 5, provided we show that

$$S = \{x \in W_0^{1,p}(Z) : x \in tV^{-1}G_1(x) \text{ for some } 0 < t < 1\}$$

is bounded uniformly in t. Suppose that this is not the case. Then we can find $x_n \in D$ and $t_n \in (0,1)$, $n \geq 1$, such that $\|x_n\| \to \infty$, $t_n \to t$ as $n \to \infty$ and for all $n \to \infty$ we have

$$V\left(\frac{1}{t_n}\, x_n\right) = w_n, \qquad w_n \in G_1(x_n), \qquad w_n = u_n^* + (\lambda^* + 1)J(x_n) \qquad \text{with } u^* \in G(x_n),$$

hence

$$A\left(\frac{1}{t_n}\, x_n\right) + J\left(\frac{1}{t_n}\, x_n\right) = u_n^* + (\lambda^* + 1)J(x_n). \tag{43}$$

Because of hypothesis $H(j)_6$(iii), $\{u_n^*\} \subseteq L^q(Z)$ is bounded and so we may assume that $u_n^* \xrightarrow{w} u^*$ in $L^q(Z)$ as $n \to \infty$. Let $y_n = x_n/\|x_n\|$, $n \geq 1$. Since $\|y_n\| = 1$, $n \geq 1$, we may assume that $y_n \xrightarrow{w} y$ in $W_0^{1,p}(Z)$ and $y_n \to y$ in $L^p(Z)$ as $n \to \infty$. We take the duality brackets in $(W_0^{1,p}(Z), W^{-1,q}(Z))$ of (43) with $y_n/\|x_n\|^{p-1}$, $n \geq 1$. We have

$$\begin{aligned}\left\langle \frac{1}{\|x_n\|^{p-1}}\, A\left(\frac{1}{t_n}\, x_n\right), y_n \right\rangle &+ \left\langle \frac{1}{\|x_n\|^{p-1}}\, J\left(\frac{1}{t_n}\, x_n\right), y_n \right\rangle \\ &= \left\langle \frac{u_n^*}{\|x_n\|^{p-1}}, y_n \right\rangle + (\lambda^* + 1)\left\langle \frac{1}{\|x_n\|^{p-1}}\, J(x_n), y_n \right\rangle,\end{aligned}$$

hence

$$\left\langle \frac{1}{\|x_n\|^{p-1}}\, A\left(\frac{1}{t_n}\, x_n\right), y_n \right\rangle + \frac{1}{t_n^{p-1}}\, (J(y_n), y_n)_{pq} = \left(\frac{u_n^*}{\|x_n\|^{p-1}}, y_n\right)_{pq} + (\lambda^* + 1)(J(y_n), y_n)_{pq},$$

therefore

$$c_1\|Dy_n\|_p^p + \|y_n\|_p^p \leq t_n^{p-1}\left(\frac{u_n^*}{\|x_n\|^{p-1}}, y_n\right)_{pq} + t_n^{p-1}(\lambda^* + 1)\|y_n\|_p^p$$

and so

$$c_1\|Dy_n\|_p^p \leq t_n^{p-1}\left(\frac{u_n^*}{\|x_n\|^{p-1}}, y_n\right)_{pq} + t_n^{p-1}\lambda^*\|y_n\|_p^p \tag{44}$$

(since $0 < t_n < 1$).

Note that $u_n^*/\|x_n\|^{p-1} \to 0$ in $L^q(Z)$. So we have

$$c_1 \limsup_{n\to\infty} \|Dy_n\|_p^p \leq t^{p-1}\lambda^*\|y\|_p^p \leq c_1\|Dy\|_p^p,$$

since $0 \leq t \leq 1$ and $\lambda^* = \lambda_1 c_1$ (see (2)).

On the other hand, since $Dy_n \xrightarrow{w} Dy$ in $L^p(Z, \mathbb{R}^N)$ as $n \to \infty$, from the weak lower semicontinuity of the norm in a Banach space we have $\|Dy\|_p \leq \liminf_{n\to\infty} \|Dy_n\|_p$. Therefore it follows that $\|Dy_n\|_p \to \|Dy\|_p$ as $n \to \infty$. Combining this with the fact that $Dy_n \xrightarrow{w} Dy$ in $L^p(Z, \mathbb{R}^N)$ and since the space $L^p(Z, \mathbb{R}^N)$ has the Kadec–Klee property (being uniformly convex) we deduce that $Dy_n \to Dy$ in $L^p(Z, \mathbb{R}^N)$ (see Hu–Papageorgiou [35, p. 28]) and so $y_n \to y$ in $W_0^{1,p}(Z)$ as $n \to \infty$. From (44), by passing to the limit as $n \to \infty$, we obtain

$$\|Dy\|_p^p \leq t^{p-1}\lambda_1\|y\|_p^p,$$

hence

$$t = 1 \quad \text{and} \quad \|Dy\|_p^p = \lambda_1 \|y\|_p^p \quad \text{(see (2) and recall that } \lambda^* = \lambda_1 c_1). \tag{45}$$

Remark that since $\|y_n\| = 1$ and $y_n \to y$ in $W_0^{1,p}(Z)$, we have $\|y\| = 1$ and so $y \neq 0$. Therefore from (45) it follows that $y = \pm u_1$. Assume without loss of generality that $y = u_1$. We have

$$c_1 \|Dy_n\|_p^p + (1 - t_n^{p-1}(\lambda^* + 1))\|y_n\|_p^p \leq \frac{t_n^{p-1}}{\|x_n\|^{p-1}} (u_n^*, y_n)_{pq}.$$

Because $t_n \to 1^-$ and $\lambda^* = \lambda_1 c_1$, using (2) we see that

$$c_1 \|Dy_n\|_p^p + (1 - t_n^{p-1}(\lambda^* + 1))\|y_n\|_p^p > 0 \quad \text{for all } n \geq 1$$

therefore

$$\frac{t_n^{p-1}}{\|x_n\|^{p-1}} (u_n^*, y_n)_{pq} > 0 \quad \text{for all } n \geq 1$$

and so

$$(u_n^*, y_n)_{pq} = \int_Z u_n^*(z) y_n(z)\, dz > 0 \quad \text{for all } n \geq 1. \tag{46}$$

Since $y = u_1, y_n(z) \to u_1(z)$ a.e. on Z and $u_1(z) > 0$ for all $z \in Z$, we have $x_n(z) \to \infty$ a.e. on Z as $n \to \infty$. Since $u_n^* \in S^q_{\partial j(\cdot, x_n(\cdot))}$, from the definition of the function g_2 we have

$$u_n^*(z) \leq g_2(z, x_n(z)) \quad \text{a.e. on } Z$$

and thus

$$u^*(z) \leq g_+(z) \quad \text{a.e. on } Z \quad \text{(see Proposition 2).} \tag{47}$$

Passing to the limit in (46), we obtain

$$0 \leq \int_Z u^*(z) u_1(z)\, dz \leq \int_Z g_+(z) u_1(z)\, dz,$$

which contradicts hypothesis $H(j)_6$(iv). The argument is similar if we assume that $y = -u_1$. Therefore it follows that S is bounded and so we can apply Theorem 5 and obtain $x \in W_0^{1,p}(Z)$ which solves the fixed point problem (42). Thus we have

$$V(x) \in G_1(x),$$

hence

$$A(x) + J(x) \in G(x) + (\lambda^* + 1) J(x),$$

therefore

$$A(x) - \lambda^* J(x) \in G(x)$$

and so

$$\begin{cases} -\operatorname{div} \alpha(z, Dx(z)) - \lambda^* |x(z)|^{p-2} x(z) \in \partial j(z, x(z)) & \text{a.e. on } Z, \\ x|_\Gamma = 0. \end{cases}$$

Hence $x \in W_0^{1,p}(Z)$ is a solution of problem (32). ■

Remark. In Goeleven–Motreanu–Panagiotopoulos [33] the authors study semilinear hemivariational inequalities at resonance (i.e. $\alpha(z, \xi) = \xi$, see Theorem 5.1. in [33]) using Landesman–Lazer conditions. However, they impose rather restrictive hypotheses on the nonsmooth potential $j(z,x)$. Namely they assume that there exists a continuous map $W : L^2(Z) \to L^2(Z)$ which satisfies $W(x)(z) \in \partial j(z,x(z))$ a.e. on Z. This hypothesis is very close to assuming that the Clarke subdifferential of the locally Lipschitz integral functional $I_j(x) = \int_Z j(z,x(z))\,dz$ admits a continuous selector (in fact, if for almost all $z \in Z$, $j(z,\cdot)$ is regular, i.e. $j'(z,x;h) = j^0(z,x;h)$ for all $x,h \in L^2(Z)$ with $j'(z,x;\cdot)$ being the usual directional derivative at x, then the two hypotheses are equivalent). Recalling that for a locally Lipschitz functional the generalized subdifferential is usc from the Banach space X into its dual X^* equipped with the weak* topology, we realize that the hypothesis about the existence of W is very strong. In [33] (Proposition 5.3), the authors give a sufficient condition for the existence of such a map W. However, that condition is still problematic since it requires that the multifunction $x \mapsto \partial I_j(x)$ is continuous from $L^2(Z)$ into itself. In contrast our result does not require this restrictive hypothesis and it concerns nonlinear problems not necessarily of variational form.

For problems of variational form, driven by the p-Laplacian, we can employ a more general form of the Landesman–Lazer condition, introduced recently by Tang [62] for semilinear (i.e. $p = 2$) ordinary differential equations with a smooth nonlinearity. Our result extends also the recent ones for C^1 energy functionals by Arcoya–Orsina [5] and Bouchala–Drábek [10] and for hemivariational inequalities by Gasiński–Papageorgiou [24], [25].

So we consider the following problem:

$$\begin{cases} -\operatorname{div}\alpha(z,Dx(z)) - \lambda_1|x(z)|^{p-2}x(z) \in \partial j(z,x(z)) & \text{a.e. on } Z,\\ x|_\Gamma = 0, \quad 2 \le p < \infty. \end{cases} \tag{48}$$

As before we introduce the functions $g_1(z,x) = \min[u^* : u^* \in \partial j(z,x)]$ and $g_2(z,x) = \max[u^* : u^* \in \partial j(z,x)]$. We have already seen that g_1, g_2 are both measurable functions. Then we introduce the functions

$$G_1(z,x) = \begin{cases} \dfrac{pj(z,x)}{x} - g_1(z,x) & \text{if } x \neq 0,\\ 0 & \text{if } x = 0, \end{cases}$$

$$G_2(z,x) = \begin{cases} \dfrac{pj(z,x)}{x} - g_2(z,x) & \text{if } x \neq 0,\\ 0 & \text{if } x = 0. \end{cases}$$

The generalized Landesman–Lazer conditions will be formulated in terms of these functions. Our hypotheses on the nonsmooth nonlinearity $j(z,x)$ are the following:

$\boxed{H(j)_7}$ $j : Z \times \mathbb{R} \to \mathbb{R}$ is a function such that $j(z,0) = 0$, a.e. on Z and

(i) for all $x \in \mathbb{R}$, $z \mapsto j(z,x)$ is measurable;
(ii) for almost all $z \in Z$, $x \mapsto j(z,x)$ is locally Lipschitz;
(iii) for each $M > 0$ there exists $a_M \in L^q(Z)$ such that for almost all $z \in Z$, all $|x| \le M$ and all $u^* \in \partial j(z,x)$ we have $|u^*| \le a_M(z)$;
(iv) $\lim_{|x|\to\infty} u^*/|x|^{p-1} = 0$ uniformly for almost all $z \in Z$ and all $u^* \in \partial j(z,x)$;
(v) there exist functions $G_1^-, G_2^+ \in L^q(Z)$ such that uniformly for almost all $z \in Z$ we have

$$G_1^-(z) = \limsup_{x\to-\infty} G_1(z,x) \quad \text{and} \quad G_2^+(z) = \liminf_{x\to\infty} G_2(z,x)$$

and

$$\int_Z G_1^-(z)u_1(z)\,dz < 0 < \int_Z G_2^+(z)u_1(z)\,dz.$$

We introduce the locally Lipschitz energy functional $\varphi : W_0^{1,p}(Z) \to \mathbb{R}$ defined by

$$\varphi(x) = \frac{1}{p}\,\|Dx\|_p^p - \frac{\lambda_1}{p}\,\|x\|_p^p - \int_Z j(z,x(z))\,dz.$$

PROPOSITION 19. *If hypotheses $H(j)_7$ hold, then φ satisfies the nonsmooth PS-condition.*

Proof. Let $\{x_n\}_{n\geq 1} \subseteq W_0^{1,p}(Z)$ be a sequence such that

$$|\varphi(x_n)| \leq M_1 \quad \text{for all } n \geq 1 \text{ with } M_1 > 0 \text{ and } m(x_n) \to 0 \text{ as } n \to \infty.$$

Recall (see Section 2) that $m(x_n) = \inf\{\|u^*\| : u^* \in \partial\varphi(x_n)\}$, $n \geq 1$. Since $\partial\varphi(x_n)$ is weakly compact and the norm in a Banach space is weakly lower semicontinuous, we can find $x_n^* \in \partial\varphi(x_n)$ such that $m(x_n) = \|x_n^*\|$, $n \geq 1$. We have $x_n^* = A(x_n) - u_n^*$, with $A : W_0^{1,p}(Z) \to W^{-1,q}(Z)$ the nonlinear operator defined by $\langle A(x), y\rangle = \int_Z \|Dx(z)\|^{p-2}(Dx(z), Dy(z))_{\mathbb{R}^N}\,dz$ and $u_n^* \in L^q(Z)$, $u_n^*(z) \in \partial j(z, x_n(z))$ a.e. on Z, $n \geq 1$. Recall that A is monotone demicontinuous, hence maximal monotone.

We claim that $\{x_n\}_{n\geq 1} \subseteq W_0^{1,p}(Z)$ is bounded. Suppose that this is not the case. Then by passing to a subsequence if necessary, we may assume that $\|x_n\| \to \infty$. Let $y_n = x_n/\|x_n\|$, $n \geq 1$. Evidently $\|y_n\| = 1$ for all $n \geq 1$ and so by passing to a further subsequence if necessary we may assume that

$$y_n \xrightarrow{w} y \quad \text{in } W_0^{1,p}(Z), \quad y_n \to y \quad \text{in } L^p(Z), \quad y_n(z) \to y(z) \quad \text{a.e. on } Z$$

and

$$|y_n(z)| \leq k(z) \quad \text{a.e. on } Z, \text{ for all } n \geq 1, \text{ with } k \in L^q(Z)$$

(see for example Kufner–John–Fučík [40, p. 74]). By hypothesis $H(j)_7$(iv), given $\varepsilon > 0$ we can find $M_2 = M_2(\varepsilon) > 0$ such that for almost all $z \in Z$, all $|x| \geq M_2$ and all $u^* \in \partial j(z,x)$ we have

$$|u^*| \leq \varepsilon|x|^{p-1}.$$

On the other hand from hypothesis $H(j)_7$(iii) for almost all $z \in Z$, all $|x| \leq M_2$ and all $u^* \in \partial j(z,x)$ we have

$$|u^*| \leq a_\varepsilon(z) = a_{M_2(\varepsilon)} \quad \text{with } a_\varepsilon \in L^q(Z).$$

So for almost all $z \in Z$, all $x \in \mathbb{R}$ and all $u^* \in \partial j(z,x)$ we have

$$|u^*| \leq a_\varepsilon(z) + \varepsilon|x|^{p-1}. \tag{49}$$

Using Lebourg's mean value theorem (see Lebourg [43] or Clarke [18, p. 41]), for almost all $z \in Z$ and all $x \in \mathbb{R}$ we have

$$|j(z,x)| \leq |j(z,0)| + a_\varepsilon(z)|x| + \varepsilon|x|^p \leq \beta_\varepsilon(z) + 2\varepsilon|x|^p \quad \text{with } \beta_\varepsilon \in L^1(Z)_+.$$

Then we have

$$\begin{aligned}\left|\int_Z \frac{j(z,x_n(z))}{\|x_n\|^p}\,dz\right| &\le \frac{1}{\|x_n\|^p}\int_Z \beta_\varepsilon(z)\,dz + 2\varepsilon\|y_n\|_p^p \\ &\le \frac{1}{\|x_n\|^p}\int_Z \beta_\varepsilon(z)\,dz + 2\varepsilon \to 2\varepsilon \quad \text{ as } n\to\infty,\end{aligned}$$

hence

$$\limsup_{n\to\infty}\left|\int_Z \frac{j(z,x_n(z))}{\|x_n\|^p}\,dz\right| \le 2\varepsilon.$$

Letting $\varepsilon \downarrow 0$ we conclude that

$$\int_Z \frac{j(z,x_n(z))}{\|x_n\|^p}\,dz \to 0 \quad \text{ as } n\to\infty.$$

From the choice of the sequence $\{x_n\}_{n\ge1} \subseteq W_0^{1,p}(Z)$ we have

$$\frac{1}{p}\,\|Dy_n\|_p^p - \frac{\lambda_1}{p}\,\|y_n\|_p^p - \int_Z \frac{j(z,x_n(z))}{\|x_n\|^p}\,dz = \frac{\varphi(x_n)}{\|x_n\|^p} \le \frac{M_1}{\|x_n\|^p}. \tag{50}$$

Passing to the limit as $n\to\infty$ we obtain

$$\|Dy\|_p^p \le \lambda_1\|y\|_p^p$$

and so

$$y = \pm u_1 \quad \text{or} \quad y = 0 \qquad \text{(see (2))}.$$

But from (50) and (2) we have

$$\limsup \|Dy_n\|_p^p \le \lambda_1\|y\|_p^p \le \|Dy\|_p^p \le \liminf \|Dy_n\|_p^p,$$

which implies that

$$\|Dy_n\|_p \to \|Dy\|_p.$$

Since $Dy_n \xrightarrow{w} Dy$ in $L^p(Z,\mathbb{R}^N)$, as before via the Kadec–Klee property we have $y_n \to y$ in $W_0^{1,p}(Z)$ and so $y \ne 0$. Therefore $y = \pm u_1$ and without any loss of generality, we may assume that $y = u_1$ (the analysis is similar if we assume that $y = -u_1$). From the choice of the sequence $\{x_n\}_{n\ge1} \subseteq W_0^{1,p}(Z)$, we have

$$\langle x_n^*, y_n\rangle - \frac{p\varphi(x_n)}{\|x_n\|} \le \varepsilon_n + \frac{pM_1}{\|x_n\|} \quad \text{ with } \varepsilon_n \downarrow 0,$$

hence

$$\int_Z \frac{pj(z,x_n(z))}{\|x_n\|}\,dz - \int_Z u_n^*(z)y_n(z)\,dz \le \varepsilon_n + \frac{pM_1}{\|x_n\|}.$$

Set

$$h_n(z) = \begin{cases} \dfrac{j(z,x_n(z))}{x_n(z)} & \text{if } x_n(z) \ne 0, \\ 0 & \text{if } x_n(z) = 0. \end{cases}$$

Also since $u_n^*(z) \in \partial j(z,x_n(z))$ a.e. on Z we have $g_1(z,x_n(z)) \le u_n^*(z) \le g_2(z,x_n(z))$ a.e. on Z. Thus we can write that

$$
\begin{aligned}
(51)\qquad \varepsilon_n + \frac{pM_1}{\|x_n\|} &\geq \int_Z \frac{pj(z,x_n(z))}{\|x_n\|}\,dz - \int_Z u_n^*(z)y_n(z)\,dz \\
&\geq \int_Z ph_n(z)y_n(z)\,dz - \int_{\{y_n<0\}} g_1(z,x_n(z))y_n(z)\,dz \\
&\quad - \int_{\{y_n>0\}} g_2(z,x_n(z))y_n(z)\,dz \\
&= \int_{\{y_n<0\}} G_1(z,x_n(z))y_n(z)\,dz + \int_{\{y_n>0\}} G_2(z,x_n(z))y_n(z)\,dz
\end{aligned}
$$

(we have also used the hypothesis $j(z,0)=0$, a.e. on Z).

Recall that $y=u_1$ and $u_1(z)>0$ for all $z\in Z$. So $x_n(z)\to\infty$ a.e. on Z. Therefore if we denote by $|\cdot|$ the Lebesgue measure on $\mathbb{R}^N$, we have $|\{y_n>0\}|\to|Z|$ and $|\{y_n<0\}|\to 0$ as $n\to\infty$. Also note from the definition of G_1^- and G_2^+ that given $\varepsilon>0$ we can find $M_3=M_3(\varepsilon)>0$ such that for almost all $z\in Z$ we have

$$
G_1(z,x)\leq G_1^-(z)+\varepsilon \quad \text{for all } x<-M_3, \qquad G_2(z,x)\geq G_2^+(z)-\varepsilon \quad \text{for all } x>M_3
$$

and

$$
|G_1(z,x)|,|G_2(z,x)|\leq a_{M_3}(z) \qquad \text{for all } x\in[-M_3,M_3]
$$

(in order to get the last inequality we also employed Lebourg's mean value theorem).

Therefore, by passing to the limit in (51), we obtain

$$
\int_Z G_2^+(z)u_1(z)\,dz\leq 0,
$$

which contradicts hypothesis $H(j)_7$(v).This contradiction implies that $\{x_n\}_{n\geq1}\subseteq W_0^{1,p}(Z)$ is bounded and so we may assume that $x_n\xrightarrow{w}x$ in $W_0^{1,p}(Z)$ and $x_n\to x$ in $L^p(Z)$ as $n\to\infty$. We have

$$
\langle A(x_n),x_n-x\rangle-\lambda_1\int_Z |x_n|^{p-2}x_n(x_n-x)\,dz-\int_Z u_n^*(x_n-x)\,dz=\langle x_n^*,x_n-x\rangle\leq\varepsilon_n\|x_n-x\|.
$$

Note that $\lambda_1\int_Z |x_n|^{p-2}x_n(x_n-x)\,dz, \int_Z u_n^*(x_n-x)\,dz\to 0$ as $n\to\infty$ (see hypothesis $H(j)_7$(iii)). Therefore we get

$$
\limsup_{n\to\infty}\langle A(x_n),x_n-x\rangle\leq 0.
$$

But A being maximal monotone, it is generalized pseudomonotone (see Section 2) and so we have $\langle A(x_n),x_n\rangle\to\langle A(x),x\rangle$, hence $\|Dx_n\|_p\to\|Dx\|_p$. As before, via the Kadec–Klee property, we conclude that $x_n\to x$ in $W_0^{1,p}(Z)$, which finishes the proof of the proposition. ∎

Now we consider the following direct sum decomposition of $W_0^{1,p}(Z)$:

$$
W_0^{1,p}(Z)=\mathbb{R}u_1\oplus V
$$

with V being a topological complement of $\mathbb{R}u_1$ (for example we can have $V=\{v\in W_0^{1,p}(Z):\int_Z|u_1|^{p-2}u_1v\,dz=0\}$. Recall that for all $v\in V$ we have $\lambda^*\|v\|_p^p\leq\|Dv\|_p^p$ for some $\lambda^*>\lambda_1$ (see for example Anane–Tsouli [4]).

PROPOSITION 20. *If hypotheses $H(j)_7$ hold, then $\varphi(tu_1) \to -\infty$ as $|t| \to \infty$.*

Proof. Using (2) we have

$$\varphi(tu_1) = -\int_Z j(z, tu_1(z))\, dz.$$

Recall that given $\varepsilon > 0$ we can find $M_3 = M_3(\varepsilon) > 0$ such that for all $z \in Z \setminus N$, $|N| = 0$ and for all $x > M_3$ we have

$$k_\varepsilon^+(z) = G_2^+(z) - \varepsilon \le G_2(z, x),$$

hence

$$\frac{G_2(z,x)}{x^p} \ge \frac{k_\varepsilon^+(z)}{x^p} = \frac{d}{dx}\left(-\frac{1}{p-1} \cdot \frac{k_\varepsilon^+(z)}{x^{p-1}}\right). \tag{52}$$

For all $z \in Z \setminus N$, all $x > M_3$ and all $u^* \in \partial j(z,x)$ we have

$$\frac{G_2(z,x)}{x^p} = \frac{pj(z,x)}{x^{p+1}} - \frac{g_2(z,x)}{x^p} \le \frac{pj(z,x)}{x^{p+1}} - \frac{u^*}{x^p}.$$

From Clarke [18, p. 48] we know that for $z \in Z \setminus N$ and for $x > M_3$, the function $x \mapsto j(z,x)/x^p$ is locally Lipschitz and

$$\partial\left(\frac{j(z,x)}{x^p}\right) \subseteq \frac{\partial j(z,x)x^p - pj(z,x)x^{p-1}}{x^{2p}} = \frac{\partial j(z,x)}{x^p} - \frac{pj(z,x)}{x^{p+1}}.$$

Therefore for all $z \in Z \setminus N$, $|N| = 0$, all $x > M_3$ and all $v^* \in \partial(j(z,x)/x^p)$ we have

$$v^* \le \frac{g_2(z,x)}{x^p} - \frac{pj(z,x)}{x^{p+1}} = -\frac{1}{x^p} G_2(z,x),$$

hence

$$v^* \le \frac{d}{dx}\left(\frac{1}{p-1} \cdot \frac{k_\varepsilon^+(z)}{x^{p-1}}\right) \quad \text{(see (52))}.$$

Since as we already remarked for $z \in Z \setminus N$, $|N| = 0$, the function $x \mapsto j(z,x)/x^p$ is locally Lipschitz on (M_3, ∞), it is differentiable at every $x \in (M_3, \infty) \setminus D(z)$, $|D(z)| = 0$ (in this case $|\cdot|$ stands for the Lebesgue measure on $\mathbb{R}$). We set

$$v_0^*(z,x) = \begin{cases} \dfrac{d}{dx}\left(\dfrac{j(z,x)}{x^p}\right) & \text{if } x \in (M_3, \infty) \setminus D(z), \\ 0 & \text{otherwise.} \end{cases}$$

Therefore for fixed $z \in Z \setminus N$ and $x \in (M_3, \infty) \setminus D(z)$, we have $v_0^*(z,x) \in \partial(j(z,x)/x^p)$ and so

$$v_0^*(z,x) = \frac{d}{dx}\left(\frac{j(z,x)}{x^p}\right) \le \frac{d}{dx}\left(\frac{1}{p-1} \cdot \frac{k_\varepsilon^+(z)}{x^{p-1}}\right). \tag{53}$$

Let $y < v$ with $y, v \in (M_3, \infty)$. We integrate (53) over the interval $[y, x]$. We obtain

$$\frac{j(z,v)}{v^p} - \frac{j(z,y)}{y^p} \le \frac{k_\varepsilon^+(z)}{p-1}\left(\frac{1}{v^{p-1}} - \frac{1}{y^{p-1}}\right). \tag{54}$$

Now by using hypotheses $H(j)_7$(iii), (iv) in conjunction with Lebourg's mean value theorem we can easily verify that for all $z \in Z \setminus N$, $|N| = 0$,

$$\lim_{v \to +\infty} \frac{j(z,v)}{v^p} = 0.$$

So if in (54) we let $v \to \infty$, we obtain

$$\frac{j(z,y)}{y^p} \geq \frac{k_\varepsilon^+(z)}{p-1} \cdot \frac{1}{y^{p-1}}, \qquad y > M_3,$$

hence

$$\frac{j(z,y)}{y} \geq \frac{k_\varepsilon^+(z)}{p-1}, \qquad y > M_3,$$

and thus

$$\liminf_{y\to+\infty} \frac{j(z,y)}{y} \geq \frac{k_\varepsilon^+(z)}{p-1}. \tag{55}$$

In a similar fashion if $k_\varepsilon^-(z) = G_1^-(z) + \varepsilon$, we can show that

$$\limsup_{y\to-\infty} \frac{j(z,y)}{y} \leq \frac{k_\varepsilon^-(z)}{p-1}. \tag{56}$$

Now suppose that the proposition were not true. Then we could find $\{t_n\}_{n\geq 1} \subseteq \mathbb{R}$ and $M_4 > 0$ such that $|t_n| \to \infty$ and $\varphi(t_n u_1) \geq -M_4$ for all $n \geq 1$. Assume that $t_n \to \infty$. Then

$$-\int_Z j(z, t_n u_1(z))\, dz = \varphi(t_n u_1) \geq -M_4, \qquad n \geq 1,$$

hence

$$-\int_Z \frac{j(z, t_n u_1(z))}{t_n u_1(z)}\, u_1(z)\, dz \geq -\frac{M_4}{t_n}, \qquad n \geq 1.$$

For arbitrary $\varepsilon > 0$, (55) via Fatou's lemma gives

$$\frac{1}{p-1} \int_Z (G_2^+(z) - \varepsilon) u_1(z)\, dz \leq 0,$$

which implies

$$\int_Z G_2^+(z) u_1(z)\, dz \leq 0 \qquad (\text{letting } \varepsilon \downarrow 0),$$

a contradiction to hypothesis $H(j)_7$(v). If $t_n \to -\infty$, the reasoning is similar using this time (56). From these contradictions it follows that the proposition is true. ■

In the next proposition we show that the locally Lipschitz functional φ is coercive on the topological complement V.

PROPOSITION 21. *If hypotheses $H(j)_7$ hold, then $\varphi|_V$ is coercive, i.e. $\varphi(v) \to \infty$ as $\|v\| \to \infty$, $v \in V$.*

Proof. Recall that for almost all $z \in Z$ and all $x \in \mathbb{R}$, we have

$$|j(z,x)| \leq \beta_\varepsilon(z) + 2\varepsilon |x|^p \qquad \text{with } \beta_\varepsilon \in L^1(Z).$$

Hence for every $v \in V$, we have

$$\varphi(v) = \frac{1}{p}\, \|Dv\|_p^p - \frac{\lambda_1}{p}\, \|v\|_p^p - \int_Z j(z, v(z))\, dz.$$

Recall that $\|Dv\|_p^p \geq \lambda^* \|v\|_p^p$ with $\lambda_1 < \lambda^*$. So we obtain

$$\varphi(v) \geq \frac{1}{p}\left(1 - \frac{\lambda_1}{\lambda^*}\right)\|Dv\|_p^p - \|\beta_\varepsilon\|_1 - \frac{2\varepsilon}{\lambda^*}\|Dv\|_p^p. \tag{57}$$

Choose $\varepsilon > 0$ so that $\lambda_1 + 2\varepsilon < \lambda^*$. From (57) and Poincaré's inequality, we conclude that $\varphi|_V$ is coercive. ∎

Propositions 19–21 lead to the following existence theorem for problem (48).

THEOREM 22. *If hypotheses* $H(j)_7$ *hold, then problem* (48) *has a solution* $x \in W_0^{1,p}(Z)$.

Proof. By Proposition 21 we can find $M_5 > 0$ such that $\varphi(v) \geq -M_5$ for all $v \in V$. Also, because of Proposition 20, there exists $t \in \mathbb{R} \setminus \{0\}$ such that $\varphi(\pm tu_1) < -M_5$. Let $C_1 = \{\pm tu_1\}$, $C = [-tu_1, tu_1]$ (i.e. $C_1 = \partial C$) and $D = V$. Then it is easy to see that C_1 and D link in $W_0^{1,p}(Z)$ (see Kourogenis–Papageorgiou [39, Theorem 7] and Struwe [60, pp. 116–117]). So we apply Theorem 4 and obtain $x \in W_0^{1,p}(Z)$ such that $0 \in \partial\varphi(x)$. We have

$$A(x) - \lambda|x|^{p-2}x = u^* \quad \text{with } u^* \in S^q_{\partial j(\cdot,x(\cdot))}.$$

As before we can verify that x is a solution of (48). ∎

REMARK. The following function $j(x)$ (for simplicity we have dropped the z-dependence) satisfies the generalized Landesman–Lazer condition $H(j)_7$, but not the standard one used by Arcoya–Orsina [5] (smooth problems). We have

$$j(x) = \begin{cases} 4x + \sin x - \ln(1+x^2) & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ e^x - x - 1 & \text{if } x < 0, \end{cases}$$

hence

$$\partial j(x) = \begin{cases} 4 + \cos x - \dfrac{2x}{1+x^2} & \text{if } x > 0, \\ [0,5] & \text{if } x = 0, \\ e^x - 1 & \text{if } x < 0. \end{cases}$$

A simple calculation reveals that $G_2^+ = 4p - 3 > 0 > G_1^- = -(p-1)$.

7. Problems which cross λ_1

In this section we continue the study of nonlinear hemivariational inequalities driven by the p-Laplacian. Again the problem is in variational form and so we can use the arsenal of critical point theory. Existence theorems based on critical point theory (smooth or nonsmooth) involve asymptotic conditions for the potential function at $\pm\infty$, which are controlled by the principal eigenvalue λ_1. When we have crossing of λ_1, then the geometry of the "Mountain Pass Theorem" (see Theorem 6 of Kourogenis–Papageorgiou [39], a consequence of Theorem 4 if we choose C_1, C and D appropriately) fails and we need to look for critical points of linking type. However, in this nonlinear situation, for the reasons explained in Section 2, the construction of linkings presents serious technical difficulties and requires special techniques.

Here we prove a theorem for a nonlinear hemivariational inequality in which the generalized potential $j(z,x)$ near the origin is in some sense below the first eigenvalue λ_1, while asymptotically at $\pm\infty$, it is in the same sense in the interval $[\lambda_1,\lambda_2)$, λ_2 being the second eigenvalue of $(-\Delta_p, W_0^{1,p}(Z))$ (see Section 2).

The problem under consideration is now the following:

$$\begin{cases} -\operatorname{div}(\|Dx(z)\|^{p-2}Dx(z)) \in \partial j(z,x(z)) & \text{a.e. on } Z,\\ x|_\Gamma = 0, \quad 2 \le p < \infty. \end{cases} \tag{58}$$

Our hypotheses on the nonsmooth potential $j(z,x)$ are the following:

$\boxed{H(j)_8}$ $j : Z\times\mathbb{R}\to\mathbb{R}$ is a function such that

(i) for all $x\in\mathbb{R}$, $z\mapsto j(z,x)$ is measurable;
(ii) for almost all $z\in Z$, $x\mapsto j(z,x)$ is locally Lipschitz;
(iii) for almost all $z\in Z$, all $x\in\mathbb{R}$ and all $u^*\in\partial j(z,x)$ we have

$$|u^*| \le a_1(z) + c_1|x|^{r-1} \quad \text{with } a_1\in L^{r'}(Z),\ 1/r+1/r'=1,\ c_1>0;$$

(iv) there exist $a>0$ and $0<\mu<p^*$ such that $\limsup_{|x|\to\infty} u^*x - pj(z,x)/|x|^\mu \le -a$ uniformly for almost all $z\in Z$ and all $u^*\in\partial j(z,x)$;
(v) there exists $\beta\in L^\infty(Z)_+$ such that $\beta(z)\le\lambda_1$ a.e. with strict inequality on a set of positive Lebesgue measure such that $\limsup_{x\to 0} pj(z,x)/|x|^p = \beta(z)$ uniformly for almost all $z\in Z$;
(vi) $\lambda_1 \le \liminf_{|x|\to\infty} pj(z,x)/|x|^p \le \limsup_{|x|\to\infty} pj(z,x)/|x|^p < \lambda_2$ uniformly for almost all $z\in Z$.

Remark. Hypotheses $H(j)_8$(v) and (vi) imply that the quotient $pj(z,x)/|x|^p$ “crosses” the principal eigenvalue λ_1 as x moves from 0 to $\pm\infty$. Hypothesis $H(j)_8$(iv) was first employed in the context of smooth problems by Costa–Magalhaes [19] and essentially is a variation of the classical Ambrosetti–Rabinowitz condition for semilinear smooth problems (see for example Struwe [60, p. 102]). In [19], in addition to having a smooth potential (i.e. $j(z,x)\in C^1$), it was also assumed that either $\liminf_{|x|\to\infty} pj(z,x)/|x|^p > \lambda_1$ uniformly for almost all $z\in Z$ (see Theorem 1 in [19]) or $\lim_{|x|\to\infty} pj(z,x)/|x|^p = \lambda_1$ uniformly for almost all $z\in Z$ (see Theorem 2 in [19]). So our work generalizes in different ways that of Costa–Magalhaes [19].

Let $\varphi : W_0^{1,p}(Z)\to\mathbb{R}$ be the locally Lipschitz functional defined by

$$\varphi(x) = \frac{1}{p}\|Dx\|_p^p - \int_Z j(z,x(z))\,dz.$$

Proposition 23. *If hypotheses $H(j)_8$ hold, then φ satisfies the nonsmooth C-condition.*

Proof. Let $\{x_n\}_{n\ge1}\subseteq W_0^{1,p}(Z)$ be a sequence such that

$$|\varphi(x_n)|\le M_1 \quad \text{for all } n\ge 1 \quad\text{and}\quad (1+\|x_n\|)m(x_n)\to 0 \quad \text{as } n\to\infty.$$

As before we can find $x_n^*\in\partial\varphi(x_n)$ such that $\|x_n^*\| = m(x_n)$, $n\to\infty$. Also let $A : W_0^{1,p}(Z)\to W^{-1,q}(Z)$ be the maximal monotone operator corresponding to the p-Laplacian, i.e. $\langle A(x),y\rangle = \int_Z \|Dx\|^{p-2}\,(Dx,\ Dy)_{\mathbb{R}^N}\,dz$ for all $x,y\in W_0^{1,p}(Z)$. We have

$x_n^* = A(x_n) - u_n^*$ with $u_n^* \in L^{r'}(Z)$, $u_n^*(z) \in \partial j(z, x_n(z))$ a.e. on Z, $n \geq 1$. From the choice of the sequence $\{x_n\}_{n\geq 1} \subseteq W_0^{1,p}(Z)$ we have

$$-\|Dx_n\|_p^p + \int_Z pj(z, x_n(z))\, dz \leq pM_1$$

and

$$\langle A(x_n), x_n\rangle - \int_Z u_n^* x_n\, dz \leq \varepsilon_n \quad \text{with } \varepsilon_n \downarrow 0.$$

Adding the two inequalities, and because $\langle A(x_n), x_n\rangle = \|Dx_n\|_p^p$, we have

$$(59)\qquad -\int_Z (u_n^*\, x_n - pj(z, x_n))\, dz \leq pM_1 + \varepsilon_n \leq M_2 \quad \text{for some } M_2 > 0 \text{ and all } n \geq 1.$$

Hypothesis $H(j)_8$ (iv) implies that there exists $M_3 > 0$ such that for almost all $z \in Z$, all $|x| \geq M_3$ and all $u^* \in \partial j(z, x)$ we have

$$(60)\qquad -\frac{a}{2}\,|x|^\mu \geq u^* x - pj(z, x).$$

On the other hand, as before, from hypothesis $H(j)_8$(iii) and Lebourg's mean value theorem, for almost all $z \in Z$, all $|x| \leq M_3$ and all $u^* \in \partial j(z, x)$ we have

$$(61)\qquad u^* x - pj(z, x) \leq a_3(z).$$

From (60) and (61) it follows that for almost all $z \in Z$ and all $x \in \mathbb{R}$ we have

$$\frac{a}{2}\,|x|^\mu - a_4(z) \leq -(u^* x - pj(z, x)) \quad \text{with } a_4 \in L^1(Z)_+.$$

Using this estimate in (59) we obtain

$$\|x_n\|_\mu^\mu \leq M_4 \quad \text{for some } M_4 > 0 \text{ and all } n \geq 1$$

and so

$$\{x_n\}_{n\geq 1} \subseteq L^\mu(Z) \quad \text{is bounded.}$$

Let $\eta = \min[p^*, p(\max(N, p) + \mu)/\max(N, p)]$. If $N \leq p$, then $p^* = \infty$ and so we have $\eta = p + \mu$. Hence $\max\{p, \mu\} < \eta$. If $N > p$, then $p^* = p(N + p^*)/N > p(N + \mu)/N$ and so it follows that $\eta = p(N + \mu)/N$ and because by hypothesis $\mu < p^*$ we check that $\mu < p(N + \mu)/N$. Therefore we always have

$$\max\{p, \mu\} < \eta = p\,\frac{\max(N, p) + \mu}{\max(N, p)}.$$

Let $s > 0$ be such that $\max\{p, \mu\} < s \leq \eta \leq p^*$. Then by hypotheses $H(j)_8$(iii) and (vi) we can find $a_5 \in L^1(Z)_+$ and $a_6 > 0$ such that for almost all $z \in Z$ and all $x \in \mathbb{R}$ we have

$$(62)\qquad j(z, x) \leq a_5(z) + a_6|x|^s.$$

Let

$$\vartheta = \begin{cases} \dfrac{p^*(s - \mu)}{s(p^* - \mu)} & \text{if } N > p,\\ 1 - \mu/s & \text{if } N \leq p. \end{cases}$$

We have $0 < \vartheta < 1$ and $1/s = (1-\vartheta)/\mu + \vartheta/p^*$. From the interpolation inequality (see for example Showalter [59, p. 45]) we have

$$\|x_n\|_s \le \|x_n\|_\mu^{1-\vartheta} \|x_n\|_{p^*}^{\vartheta} \le M_5 \|x_n\|_{p^*}^{\vartheta} \le M_6 \|Dx_n\|_p^{\vartheta} \tag{63}$$

for some $M_5, M_6 > 0$ and all $n \ge 1$.

Here we have used the continuous embedding of $W_0^{1,p}(Z)$ in $L^{p^*}(Z)$ (Sobolev embedding theorem) and the Poincaré inequality.

From hypothesis $H(j)_8$(v), given $\varepsilon > 0$ we can find $0 < \delta \le 1$ such that for almost all $z \in Z$ and all $|x| \le \delta$, we have

$$pj(z,x) \le (\beta(z) + \varepsilon)\ |x|^p.$$

This fact combined with hypotheses $H(j)_8$(iii) and (vi), and because $s > p$, implies that for almost all $z \in Z$ and all $x \in \mathbb{R}$ we have

$$pj(z,x) \le (\beta(z)+\varepsilon)|x|^p + a_7|x|^s + a_8 \quad \text{for some } a_7, a_8 > 0. \tag{64}$$

Recall that for all $n \ge 1$, we have

$$\|Dx_n\|_p^p - \int_Z pj(z, x_n(z))\, dz \le pM_1,$$

which implies

$$\|Dx_n\|_p^p \le pM_1 + \int_Z (\beta(z)+\varepsilon)|x_n(z)|^p\, dz + a_7\|x_n\|_s^s + a_9, \quad a_9 > 0 \quad \text{(see (64))}$$

and hence

$$\|Dx_n\|_p^p - \int_Z \beta(z)|x_n(z)|^p\, dz - \varepsilon\|x_n\|_p^p \le pM_1 + a_7\|x_n\|_s^s + a_9.$$

From the proof of Proposition 9 (see (11)) we know that

$$\|Dx_n\|_p^p - \int_Z \beta(z)|x_n(z)|^p\, dz \ge \xi\|Dx_n\|_p^p \quad \text{for some } \xi > 0 \text{ and all } n \ge 1.$$

So we have

$$(\xi - \varepsilon/\lambda_1)\|Dx_n\|_p^p \le a_{10} + a_7\|x_n\|_s^s \quad \text{for some } a_{10} > 0 \text{ and all } n \ge 1.$$

Choose $\varepsilon < \xi\lambda_1$ and use (63) and the continuous embedding of $W_0^{1,p}(Z)$ into $L^s(Z)$ (recall that $s < p^*$) to obtain

$$\|Dx_n\|_p^p \le a_{11}\|Dx_n\|_p^{\vartheta s} + a_{10} \quad \text{for some } a_{11} > 0 \text{ and all } n \ge 1.$$

Elementary algebra shows that $\vartheta s < p$ and so from the above inequality and Poincaré's inequality we conclude that $\{x_n\} \subseteq W_0^{1,p}(Z)$ is bounded. So we may assume that $x_n \xrightarrow{w} x$ in $W_0^{1,p}(Z)$, $x_n \to x$ in $L^p(Z)$. We have

$$\langle A(x_n), x_n - x\rangle - \int_Z u_n^*(x_n - x)\, dz \le \varepsilon_n\|x_n - x\|$$

and therefore

$$\limsup \langle A(x_n), x_n - x\rangle \le 0.$$

Then as before via the generalized pseudomonotonicity of A and the Kadec–Klee property, we have $x_n \to x$ in $W_0^{1,p}(Z)$. ∎

In the next proposition we show that φ is anti-coercive on $\mathbb{R}u_1$.

PROPOSITION 24. *If hypotheses $H(j)_8$ hold, then $\varphi(tu_1) \to -\infty$ as $|t| \to \infty$.*

Proof. Let $\sigma(z,x) = j(z,x) - \lambda_1|x|^p/p$. Clearly for all $x \in \mathbb{R}$, $z \mapsto \sigma(z,x)$ is measurable, while for almost all $z \in Z$, $x \mapsto \sigma(z,x)$ is locally Lipschitz. Also for almost all $z \in Z$ and all $x > 0$ the function $x \mapsto \sigma(z,x)/x^p$ is locally Lipschitz and

$$\partial\bigg(\frac{\sigma(z,x)}{x^p}\bigg) = \frac{x\partial j(z,x) - pj(z,x)}{x^{p+1}} \quad \text{(see the proof of Proposition 20).}$$

From hypothesis $H(j)_8$(iv) we know that $\sup[u^*x - pj(z,x) : u^* \in \partial j(z,x)] \to -\infty$ as $x \to \infty$ uniformly for almost all $z \in Z$. So given $\gamma_1 > 0$ we can find $M_7 > 0$ such that for almost all $z \in Z$, all $x > M_7$ and all $u^* \in \partial j(z,x)$, we have

$$u^*x - pj(z,x) < -\gamma_1 \quad \text{hence} \quad \partial(\sigma(z,x)/x^p) < -\gamma_1/x^{p+1}.$$

Arguing as in the proof of Proposition 20, we obtain for all $v, y \in (M_7, \infty)$, $y < v$,

$$\frac{\sigma(z,v)}{v^p} - \frac{\sigma(z,y)}{y^p} \le \frac{\gamma_1}{p}\bigg(\frac{1}{v^p} - \frac{1}{y^p}\bigg).$$

Let $v \to \infty$ and use hypothesis $H(j)_8$(vi) to obtain

$$\frac{\sigma(z,y)}{y^p} \ge \frac{\gamma_1}{p}\,\frac{1}{y^p}$$

and thus

$$\sigma(z,y) \ge \gamma_1/p \quad \text{for all } y \in (M_7, \infty).$$

So if $t > 0$ is large enough, we have

$$\varphi(tu_1) = \frac{t^p}{p}\,\|Du_1\|_p^p - \int_Z j(z, tu_1(z))\,dz \le \frac{t^p}{p}\,\|Du_1\|_p^p - \frac{\lambda_1 t^p}{p}\,\|u_1\|_p^p - \frac{\gamma_1}{p}\,|Z|.$$

Since $\gamma_1 > 0$ was arbitrary, we conclude that $\varphi(tu_1) \to -\infty$ as $t \to \infty$. In a similar fashion we can show that $\varphi(tu_1) \to -\infty$ as $t \to -\infty$. ∎

Let $V = \{x \in W_0^{1,p}(Z) : \|Dx\|_p^p = \lambda_2\|x\|_p^p\}$. In general this is not a subspace of $W_0^{1,p}(Z)$ but only a closed cone.

PROPOSITION 25. *If hypotheses $H(j)_8$ hold, then $\varphi(v) \to \infty$ as $\|v\| \to \infty$, $v \in V$.*

Proof. By hypothesis $H(j)_8$(vi), we can find $a_{13} < \lambda_2$ and $M_8 > 0$ such that for almost all $z \in Z$ and all $|x| > M_8$ we have $pj(z,x) \le a_{13}|x|^p$. So if $v \in V$, we have

$$\begin{aligned}
\varphi(v) &= \frac{1}{p}\,\|Dv\|_p^p - \int_Z j(z,v(z))\,dz \\
&= \frac{1}{p}\,\|Dv\|_p^p - \int_{\{|v|>M_8\}} j(z,v(z))\,dz - \int_{\{|v|\le M_8\}} j(z,v(z))\,dz \\
&\ge \frac{1}{p}\,\|Dv\|_p^p - \frac{a_{13}}{p}\,\|v\|_p^p - a_{14} \quad \text{for some } a_{14} > 0 \\
&= \frac{1}{p}\bigg(1 - \frac{a_{13}}{\lambda_2}\bigg)\|Dv\|_p^p - a_{14}.
\end{aligned}$$

Because $a_{13} < \lambda_2$, from the last inequality we conclude that $\varphi|_V$ is coercive. ∎

Now we are ready for the existence theorem for problem (58).

THEOREM 26. *If hypotheses $H(j)_8$ hold, then problem* (58) *has a solution $x \in W_0^{1,p}(Z)$.*

Proof. From Proposition 25 we know that

$$-\infty < \xi = \inf[\varphi(v) : v \in V] = \min[\varphi(v) : v \in V].$$

On the other hand Proposition 24 implies that for $t^* > 0$ large enough and $y = t^*\, u_1$ we have

$$\varphi(\pm y) < \xi.$$

Recall from Section 2 that $S = \{x \in W_0^{1,p}(Z) : \|Dx\|_p = 1\}$, $\psi(x) = -\|x\|_p^p$ for $x \in S$ and $c_2 = \inf_{K\in\mathcal{A}_2} \sup_{x\in K} \psi(x)$. Set $\mathcal{U} = \{x \in S : -\psi(x) > -c_2\}$. Clearly $\mathcal{U} \subseteq S$ is open and because

$$\|\pm u_1\|_p^p = \frac{1}{\lambda_1}\,\|Du_1\|_p^p = \frac{1}{\lambda_1} = -c_1 > -c_2,$$

we deduce that $\pm u_1 \in \mathcal{U}$. We claim that $\pm u_1$ belong to different path-connected components. Suppose that this is not the case. So we can find $\vartheta : [0,1] \to \mathcal{U}$ continuous map such that $\vartheta(0) = u_1$ and $\vartheta(1) = -u_1$. Let $L = \vartheta([0,1]) \cup (-\vartheta([0,1]))$. Evidently $L \subseteq \mathcal{U}$ and L is compact, symmetric and so $\gamma(L) > 1$, i.e. $L \in \mathcal{A}_2$ (γ is the Krasnosel'skiĭ $\mathbb{Z}_2$-genus). From the definition of $\mathcal{U}$ and since $L \subseteq \mathcal{U}$ we have $\sup[\psi(x) = -\|x\|_p^p : x \in L] < c_2$, which contradicts the definition of c_2 (see Section 2). So indeed u_1 and $-u_1$ belong to different path-connected components of $\mathcal{U}$. Let E be the path-connected component of $\mathcal{U}$ containing u_1, i.e. $u_1 \in E$. Then $-E$ is the path-connected component of $\mathcal{U}$ containing $-u_1$. We set $W = t^*E$ and $\Sigma = W \cup (-W)$. Since $\lambda_2 = -1/c_2$ (Section 2) we have $\|Dw\|_p^p < \lambda_2\|w\|_p^p$ for all $w \in \Sigma$ and $\|Dw\|_p^p = \lambda_2\|w\|_p^p$ for $w \in \partial\Sigma$. Therefore $\partial\Sigma \subseteq V$. Let $C = [-y, y] = \{g \in W_0^{1,p}(Z) : g = \kappa(-y) + (1-\kappa)y,\ \kappa \in [0,1]\}$, $C_1 = \{-y, y\}$ and $D = V$.

CLAIM. *The sets C_1 and V link in $W_0^{1,p}(Z)$.*

Recall that $\varphi(\pm y) < \xi = \inf_V \varphi$. So $E_1 \cap V = \emptyset$. Also let $\vartheta_1 \in C\ (C, W_0^{1,p}(Z))$ be such that $\vartheta_1|_{C_1} =$ identity. Then $\vartheta_1(C) \cap V \supseteq \vartheta_1(C) \cap \partial\Sigma \neq \emptyset$ (since $\pm y \in \partial\Sigma$). This proves the claim.

Therefore because of the claim and Proposition 23, we can apply Theorem 4 and obtain $x \in W_0^{1,p}(Z)$ such that $\xi \le \varphi(x)$ and $0 \in \partial\varphi(x)$. So $A(x) = u^*$ with $u^* \in L^{r'}(Z)$, $u(z) \in \partial j(z, x(z))$ a.e. on Z. As before we can check that x is a solution of (58). ■

REMARK. It is easy to check that the function

$$j(z,x) = \begin{cases} \beta(z)|x|^p/p - x\ln|x| & \text{if } |x| \le 1,\\ \xi|x|^p/p - a|x| + \beta(z)/p & \text{if } |x| > 1 \end{cases}$$

with $\lambda_1 \le \xi < \lambda_2$, $a > 0$ and $\beta \in L^\infty(Z)_+$ as in hypothesis $H(j)_8$(v) satisfies conditions $H(j)_8$. In this case $r = p$ and $\mu = 1$.

8. The nonhomogeneous Neumann problem

In this final section of the paper we consider a nonhomogeneous Neumann problem for a hemivariational inequality driven by the p-Laplacian. Compared to the Dirichlet problem, the study of nonlinear Neumann problems is lagging behind. We refer to the works of Huang [37], Binding–Drábek–Huang [8], Hu–Kourogenis–Papageorgiou [34] and the references therein.

Here we study a nonlinear elliptic hemivariational inequality with nonhomogeneous Neumann boundary condition (in contrast to the aforementioned references where the boundary condition is the homogeneous Neumann condition). So the problem under consideration is the following:

$$\begin{cases} -\operatorname{div}(\|Dx(z)\|^{p-2}Dx(z)) \in \partial j(z,x(z)) \quad \text{a.e. on } Z, \\ \dfrac{\partial x}{\partial n_p}(z) \in \partial k(z,x(z)) \quad \text{a.e. on } \Gamma, \qquad 2 \le p < \infty. \end{cases} \tag{65}$$

In (65),

$$\frac{\partial x}{\partial n_p}(z) = \|Dx(z)\|^{p-2}(Dx(z), n(z))_{\mathbb{R}^N}, \qquad z \in \Gamma,$$

with $n(z)$ being the outer normal vector at $z \in \Gamma$. For the generalized potential $j(z,x)$ we assume Landesman–Lazer type conditions (see Arcoya–Orsina [5] and Section 6).

The precise conditions on the function $j(z,x)$ are the following:

$\boxed{H(j)_9}$ $j : Z \times \mathbb{R} \to \mathbb{R}$ is a function such that $j(\cdot,0) \in L^1(Z)$ and

(i) for all $x \in \mathbb{R}$, $z \mapsto j(z,x)$ is measurable;

(ii) for almost all $z \in Z$, $x \mapsto j(z,x)$ is locally Lipschitz;

(iii) there exists $a \in L^q(Z)$ such that for almost all $z \in Z$, all $x \in \mathbb{R}$ and all $u^* \in \partial j(z,x)$ we have

$$|u^*| \le a(z);$$

(iv) there exist functions $f_+, f_- \in L^q(Z)$ such that $g_1(z,x), g_2(z,x) \to f_+(z)$ a.e. on Z as $x \to \infty$ and $g_1(z,x), g_2(z,x) \to f_-(z)$ a.e. on Z as $x \to -\infty$. Here as before $g_1(z,x) = \min\{u^* : u^* \in \partial j(z,x)\}$ and $g_2(z,x) = \max\{u^* : u^* \in \partial j(z,x)\}$. Also $\int_Z f_-(z)\,dz < 0 < \int_Z f_+(z)\,dz$.

Our conditions on the boundary nonlinearity $k(z,x)$ are the following:

$\boxed{H(k)}$ $k : \Gamma \times \mathbb{R} \to \mathbb{R}_+$ is a function such that $k(\cdot,0) \in L^1(\Gamma)$ and

(i) for all $x \in \mathbb{R}$, $z \mapsto k(z,x)$ is measurable;

(ii) for almost all $z \in \Gamma$, $x \mapsto k(z,x)$ is locally Lipschitz (on Γ we consider the $(N-1)$-dimensional Hausdorff (surface) measure);

(iii) for almost all $z \in \Gamma$, all $x \in \mathbb{R}$ and all $v^* \in \partial k(z,x)$ we have

$$|v^*| \le a_1(z) + c_1|x|^{r-1}, \qquad a_1 \in L^{r'}(Z),\ c_1 > 0,\ 1 \le r < p,\ 1/r + 1/r' = 1;$$

(iv) there exists $M > 0$ such that for almost all $z \in \Gamma$, all $|x| \ge M$ and all $v^* \in \partial k(z,x)$ we have

$$pk(z,x) - v^*x \ge 0.$$

Let $\gamma : W^{1,p}(Z) \to L^p(\Gamma)$ be the trace operator. We consider the following locally Lipschitz functional:

$$\varphi(x) = \frac{1}{p}\|Dx\|_p^p - \int\limits_Z j(z,x(z))\,dz - \int\limits_\Gamma k(z,\gamma(x)(z))\,d\sigma \qquad \text{for } x \in W^{1,p}(Z).$$

PROPOSITION 27. *If hypotheses $H(j)_9$ and $H(k)$ hold, then φ satisfies the nonsmooth PS-condition.*

Proof. Let $\{x_n\}_{n\geq 1} \subseteq W_0^{1,p}(Z)$ be a sequence such that

$$|\varphi(x_n)| \leq M_1 \quad \text{for all } n \geq 1 \quad \text{and} \quad m(x_n) \to 0 \quad \text{as } n \to \infty.$$

As before we take $x_n^* \in \partial\varphi(x_n)$ such that $\|x_n^*\| = m(x_n)$ for all $n \geq 1$. If $\widehat{I}_k : L^p(\Gamma) \to \mathbb{R}$ is defined by $\widehat{I}_k(h) = \int_\Gamma k(z, h(z))\, d\sigma$, $h \in L^p(\Gamma)$, this is a locally Lipschitz integral functional (see Clarke [18, p. 83] or Hu–Papageorgiou [36, p. 313]). Set $I_k = \widehat{I}_k \circ \gamma : W^{1,p}(Z) \to \mathbb{R}$. Then I_k is locally Lipschitz as well and from the chain rule of the generalized subdifferential calculus (see Clarke [18, pp. 45–46]) we have $\partial I_k(x) \subseteq \gamma^*\, \partial\widehat{I}_k(\gamma(x))$ for all $x \in W^{1,p}(Z)$. So finally we have

$$x_n^* = A(x_n) - u_n^* - \gamma^*(v_n^*)$$

with $u_n^* \in L^q(Z)$, $u_n^*(z) \in \partial j(z, x_n(z))$ a.e. on Z and $v_n^* \in L^q(\Gamma)$, $v_n^*(z) \in \partial k(z, \gamma(x_n)(z))$ a.e. on Γ.

We shall prove that $\{x_n\}_{n\geq 1} \subseteq W^{1,p}(Z)$ is bounded. Suppose that this is not the case. We may assume that $\|x_n\| \to \infty$. Set $y_n = x_n/\|x_n\|$, $n \geq 1$. Then we may assume that $y_n \xrightarrow{w} y$ in $W^{1,p}(Z)$ and $y_n \to y$ in $L^p(Z)$. From this choice of the sequence $\{x_n\}_{n\geq 1} \subseteq W_0^{1,p}(Z)$ we have

$$\frac{1}{p}\|Dx_n\|_p^p - \int_Z j(z, x_n(z))\, dz - \int_\Gamma k(z, \gamma(x_n)(z))\, d\sigma \leq M_1.$$

Divide by $\|x_n\|^p$. We obtain

$$\frac{1}{p}\|Dy_n\|_p^p - \int_Z \frac{j(z, x_n(z))}{\|x_n\|^p}\, dz - \int_\Gamma \frac{k(z, \gamma(x_n)(z))}{\|x_n\|^p}\, d\sigma \leq \frac{M_1}{\|x_n\|^p}. \tag{66}$$

From hypothesis $H(j)_9$(iii) and Lebourg's mean value theorem we deduce that for almost all $z \in Z$ and all $x \in \mathbb{R}$ we have $|j(z,x)| \leq a_2(z) + c_2|x|$ with $a_2 \in L^1(Z)$, $c_2 > 0$. Similarly from hypothesis $H(k)$(iii) and Lebourg's mean value theorem we see that, for almost all $z \in \Gamma$ and all $x \in \mathbb{R}$, we have $|k(z,x)| \leq a_3(z) + c_3|x|^r$ with $a_3 \in L^1(Z)$, $c_3 > 0$. Since $r < p$ and $p \geq 2$, we see that

$$\int_Z \frac{j(z, x_n(z))}{\|x_n\|^p}\, dz, \int_\Gamma \frac{k(z, \gamma(x_n)(z))}{\|x_n\|^p}\, d\sigma \to 0 \quad \text{as } n \to \infty.$$

So by passing to the limit as $n \to \infty$, we obtain

$$\|Dy\|_p = 0, \quad \text{hence} \quad y = \xi \in \mathbb{R}.$$

Note that $Dy_n \to 0$ in $L^p(Z, \mathbb{R}^N)$ and so $y_n \to \xi$ in $W^{1,p}(Z)$. Since $\|y_n\| = 1$ for all $n \geq 1$, we deduce that $\xi \neq 0$. Assume without any loss of generality that $\xi > 0$ (the analysis is similar if we assume that $\xi < 0$).

From the choice of the sequence $\{x_n\}_{n\geq 1} \subseteq W^{1,p}(Z)$ we have

$$|p\varphi(x_n)| \leq pM_1 \quad \text{and} \quad |\langle x_n^*, x_n\rangle| \leq \varepsilon_n\|x_n\|, \quad n \geq 1, \text{ with } \varepsilon_n \downarrow 0.$$

So we have

$$-\|Dx_n\|_p^p + \int_Z pj(z, x_n(z))\,dz + \int_\Gamma pk(z, \gamma(x_n)(z))\,d\sigma \le pM_1 \tag{67}$$

and

$$\|Dx_n\|_p^p - \int_Z u_n^*(z)x_n(z)\,dz - \int_\Gamma v_n^*(z)\gamma(x_n)(z)\,d\sigma \le \varepsilon_n\|x_n\|. \tag{68}$$

Adding (67) and (68), we obtain

$$\int_Z (pj(z, x_n(z)) - u_n^*(z)x_n(z))\,dz + \int_\Gamma (pk(z, \gamma(x_n)(z)) - v_n^*(z)\gamma(x_n)(z))\,d\sigma \le pM_1 + \varepsilon_n\|x_n\|. \tag{69}$$

Consider the first integral on the left hand side of (69). Divide by $\|x_n\|$. We get

$$\int_Z \left(\frac{pj(z, x_n(z))}{\|x_n\|} - u_n^*(z)y_n(z)\right) dz.$$

Since $\xi > 0$ we have $x_n(z) \to \infty$ a.e. on Z. For given $0 < \varepsilon < 1$, from Lebourg's mean value theorem, we have for all $z \in Z \setminus N$, $|N| = 0$,

$$j(z, x_n(z)) = j(z, \varepsilon x_n(z)) + w_n^*(z)\ (1 - \varepsilon)\ x_n(z)$$

where

$$w_n^*(z) \in \partial j(z, w_n(z)) \quad \text{and} \quad w_n(z) = (1 - t_n)x_n(z) + t_n\varepsilon x_n(z) \quad \text{for } 0 < t_n < 1,\ n \ge 1.$$

We may assume that $x_n(z) > 0$ (recall $x_n(z) \to \infty$ for all $z \in Z \setminus N$, $|N| = 0$). So $w_n(z) = x_n(z) - t_n$, $(1 - \varepsilon)x_n(z) \ge x_n(z) - (1 - \varepsilon)x_n(z) = \varepsilon x_n(z)$, from which we infer that $w_n(z) \to \infty$ as $n \to \infty$ and so by hypothesis $H(j)_9$(iv) we have $w_n^*(z) \to f_+(z)$ as $n \to \infty$. Let $n_0 = n_0(\varepsilon, z) \ge 1$ be such that if $n \ge n_0$, we have $x_n(z) > 0$ and $|w_n^*(z) - f_+(z)| < \varepsilon$. So for $n \ge n_0$ we have

$$\frac{pj(z, x_n(z))}{x_n(z)} = \frac{pj(z, \varepsilon x_n(z))}{x_n(z)} + \frac{pw_n^*(z)(1 - \varepsilon)x_n(z)}{x_n(z)}.$$

Using the growth of $j(z, \cdot)$ established earlier, we have

$$\frac{-pa_2(z) - pc_2\varepsilon x_n(z)}{x_n(z)} + \frac{p(-\varepsilon + f_+(z))(1 - \varepsilon)x_n(z)}{x_n(z)} \le \frac{pj(z, x_n(z))}{x_n(z)} \le \frac{pa_2(z) + pc_2\varepsilon x_n(z)}{x_n(z)} + \frac{p(\varepsilon + f_+(z))(1 - \varepsilon)x_n(z)}{x_n(z)}.$$

Because $x_n(z) \to \infty$ as $n \to \infty$ and $\varepsilon > 0$ was arbitrary, we infer that

$$\frac{pj(z, x_n(z))}{x_n(z)} \to pf_+(z) \quad \text{a.e. on } Z.$$

Therefore finally we have

$$\frac{1}{\|x_n\|}\int_Z (pj(z, x_n(z)) - u_n^*(z)x_n(z))\,dz \to (p - 1)\xi \int_Z f_+(z)\,dz. \tag{70}$$

Now we examine the second integral on the left hand side of (69). Using hypotheses $H(k)$(iii) and (iv), we have

$$\begin{aligned}\int_{\Gamma} (pk(z,\gamma(x_n)(z)) - v_n^*(z)\gamma(x_n)(z))\,d\sigma \\ &= \int_{\{|\gamma(x_n)|\geq M\}} (pk(z,\gamma(x_n)(z)) - v_n^*(z)\gamma(x_n)(z))\,d\sigma \\ &\quad + \int_{\{|\gamma(x_n)|<M\}} (pk(z,\gamma(x_n)(z)) - v_n^*(z)\gamma(x_n)(z))\,d\sigma \\ &\geq \int_{\{|\gamma(x_n)|<M\}} (pk(z,\gamma(x_n)(z)) - v_n^*(z)\gamma(x_n)(z))\,d\sigma = \beta \in \mathbb{R}\end{aligned}$$

and hence

$$\frac{1}{\|x_n\|}\int_{\Gamma} (pk(z,\gamma(x_n)(z)) - v_n^*(z)\gamma(x_n)(z))\,d\sigma \geq \frac{\beta}{\|x_n\|} \to 0 \quad \text{as } n\to\infty. \tag{71}$$

Returning to (69), dividing by $\|x_n\|$, passing to the limit as $n\to\infty$ and using (70) and (71) we obtain

$$(p-1)\xi\int_Z f_+(z)\,dz \leq 0,$$

a contradiction to hypothesis $H(j)_9$(iv). The argument is similar if we assume that $\xi < 0$ and we reach the inequality $(p-1)\,\xi\,\int_Z f_-(z)\,dz \leq 0$, again a contradiction to hypothesis $H(j)_9$(iv).

Therefore it follows that $\{x_n\}_{n\geq 1} \subseteq W^{1,p}(Z)$ is bounded and so we may assume that $x_n \xrightarrow{w} x$ in $W^{1,p}(Z)$ and $x_n \to x$ in $L^p(Z)$. As in previous proofs, exploiting the fact that A is generalized pseudomonotone (being maximal monotone) and the Kadec–Klee property of $L^p(Z,\mathbb{R}^N)$ (which is uniformly convex), we obtain $x_n \to x$ in $W_0^{1,p}(Z)$ as $n\to\infty$. So φ satisfies the nonsmooth PS-condition. ∎

We consider the direct sum decomposition

$$W^{1,p}(Z) = \mathbb{R}\oplus V$$

with $V = \{v \in W^{1,p}(Z) : \int_Z v(z)\,dz = 0\}$. We examine the behavior of φ on each component of the direct sum.

Proposition 28. *If hypotheses $H(j)_9$ and $H(k)$ hold, then $\varphi|_V$ is coercive.*

Proof. For every $v\in V$ we have

$$\begin{aligned}\varphi(v) &= \frac{1}{p}\|Dv\|_p^p - \int_Z j(z,v(z))\,dz - \int_{\Gamma} k(z,\gamma(x)(z))\,d\sigma \\ &\geq \frac{1}{p}\|Dv\|_p^p - \|a_2\|_1 - c_2\|v\|_p - \|a_3\|_1 - c_4\|x\|_p^r \quad \text{for some } c_4>0 \text{ (since } r<p).\end{aligned}$$

Using the Poincaré–Wirtinger inequality (see for example Hu–Papageorgiou [36, p. 866]) we have

$$\varphi(v) \geq \frac{1}{p}\|Dv\|_p^p - c_5\|Dx\|_p^r - c_6 \quad \text{for some } c_5,c_6>0. \tag{72}$$

Because $r < p$, and recalling that the Poincaré–Wirtinger inequality implies that $\|Dv\|_p$ is an equivalent norm on V, from (72) we infer that $\varphi|_V$ is coercive. ∎

PROPOSITION 29. *If hypotheses* $H(j)_9$ *and* $H(k)$ *hold, then* $\varphi(\xi) \to -\infty$ *as* $|\xi| \to \infty$, $\xi \in \mathbb{R}$.

Proof. Since $k \geq 0$ (hypothesis $H(k)$) for all $\xi \in \mathbb{R}$ we have

$$\varphi(\xi) = -\int_Z j(z,\xi)\,dz - \int_\Gamma k(z,\xi)\,d\sigma \leq -\int_Z j(z,\xi)\,dz,$$

hence

$$\varphi(\xi) \leq -\xi \int_Z \frac{j(z,\xi)}{\xi}\,dz \quad \text{for } \xi \neq 0. \tag{73}$$

From the proof of Proposition 27 we know that for almost all $z \in Z$ we have

$$\frac{j(z,\xi)}{\xi} \to f_\pm(z) \quad \text{as } \xi \to \pm\infty.$$

Thus from (73), the dominated convergence theorem and hypothesis $H(j)_9$(iv) we conclude that $\varphi(\xi) \to -\infty$ as $|\xi| \to \infty$, $\xi \in \mathbb{R}$. ∎

Now we are ready for the existence theorem concerning problem (65).

THEOREM 30. *If hypotheses* $H(j)_9$ *and* $H(k)$ *hold, then problem* (65) *has a solution in* $W^{1,p}(Z)$.

Proof. Propositions 27–29 permit the application of the nonsmooth Saddle Point Theorem (see Theorem 7 in Kourogenis–Papageorgiou [39] or the proof of Theorem 22), which gives $x \in W^{1,p}(Z)$ such that $0 \in \partial\varphi(x)$ (critical point of φ). We have

$$A(x) - u^* = \gamma^*(v^*) \tag{74}$$

with $u^* \in L^q(Z)$, $u^*(z) \in \partial j(z, x(z))$ a.e. on Z and $v^* \in L^q(\Gamma)$, $v^*(z) \in \partial k(z, \gamma(x)(z))$ a.e. on Z. Then for every $\vartheta \in C_0^\infty(Z)$ we have

$$\langle A(x), \vartheta\rangle - \int_Z u^*\vartheta\,dz = \int_\Gamma v^*\gamma(\vartheta)\,d\sigma = 0 \quad (\text{since } \gamma(\vartheta) = 0).$$

From the representation theorem for the elements of $W^{-1,q}(Z) = (W_0^{1,p}(Z))^*$ (see for example Adams [1, p. 50] or Hu–Papageorgiou [36, p. 866]), we see that $-\operatorname{div}(\|Dx\|^{p-2}Dx) \in W^{-1,q}(Z)$. We have

$$\langle A(x), \vartheta\rangle = \int_Z \|Dx\|^{p-2}(Dx, D\vartheta)_{\mathbb{R}^N}\,dz = -\langle \operatorname{div}(\|Dx\|^{p-2}Dx), \vartheta\rangle.$$

Hence

$$\langle -\operatorname{div}(\|Dx\|^{p-2}Dx), \vartheta\rangle = \langle u^*, \vartheta\rangle.$$

Since $C_0^\infty(Z)$ is dense in $W_0^{1,p}(Z)$, the predual of $W^{-1,q}(Z)$, we obtain

$$-\operatorname{div}(\|Dx(z)\|^{p-2}Dx(z)) = u^*(z) \in \partial j(z, x(z)) \quad \text{a.e. on } Z. \tag{75}$$

From the nonlinear Green's formula of Kenmochi [38] and Casas–Fernandez [14] we have

$$\frac{\partial x}{\partial n_p} \in W^{-1/q,q}(\Gamma) = (W^{1/q,p}(\Gamma))^*$$

and for all $y \in W^{1,p}(Z)$,

$$\int_Z \|Dx\|^{p-2}(Dx, Dy)_{\mathbb{R}^N}\, dz + \int_Z \operatorname{div}(\|Dx\|^{p-2}Dx)y\, dz = \left\langle \frac{\partial x}{\partial n_p}, \gamma(y) \right\rangle_{\Gamma} \tag{76}$$

where by $\langle \cdot, \cdot \rangle_{\Gamma}$ we denote the duality brackets for the pair $(W^{1/q,p}(\Gamma), W^{-1/q,q}(\Gamma))$. From (74) we have

$$\int_Z \|Dx\|^{p-2}(Dx, Dy)_{\mathbb{R}^N}\, dz = \int_Z u^* y\, dz + \langle v^*, \gamma(y) \rangle_{\Gamma}$$

(recall that $R(\gamma) = W^{1/q,p}(\Gamma)$, see Kufner–John–Fučík [40, p. 338]). Using this in (76) and taking into account (72) we obtain

$$\langle v^*, \gamma(y) \rangle_{\Gamma} = \left\langle \frac{\partial x}{\partial n_p}, \gamma(y) \right\rangle_{\Gamma}.$$

As we already said $\gamma(W^{1,\,p}(Z)) = W^{1/q,\,p}(\Gamma)$. Therefore

$$\frac{\partial x}{\partial n_p} = v^*(z) \in \partial k(z, \gamma(x)(z)) \qquad \text{a.e. on } \Gamma.$$

So $x \in W^{1,\,p}(Z)$ solves problem (65). ■

Remark. It will be very interesting to know if we can replace the Landesman–Lazer condition of this theorem by the more general one used in Theorem 22. Much more work remains to be done for the nonlinear Neumann problem (for both the smooth (C^1) and the nonsmooth (locally Lipschitz) cases). For hemivariational inequalities further research should include systems of semilinear or quasilinear hemivariational inequalities, multiplicity results using possibly a nonsmooth version of the Saddle Point Reduction Technique and of the local Linking Theorem (see Chang [17] and Brezis–Nirenberg [11]), obstacle problems and other problems with unilateral constraints (in this direction some progress was made recently by Kyritsi–Papageorgiou [41]) and second order nonlinear periodic systems with nonsmooth potential (see Gasiński–Papageorgiou [30]).

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