Contents

Abstract

The theory of multi-norms was developed by H. G. Dales and M. E. Polyakov in a memoir that was published in *Dissertationes Mathematicae*. In that memoir, the notion of 'equivalence' of multi-norms was defined. In the present memoir, we make a systematic study of when various pairs of multi-norms are mutually equivalent.

In particular, we study when (p, q)-multi-norms defined on spaces $L^r(\Omega)$ are equivalent, resolving most cases; we have stronger results in the case where $r = 2$. We also show that the standard $[t]$-multi-norm defined on $L^r(\Omega)$ is not equivalent to a (p, q)-multi-norm in most cases, leaving some cases open. We discuss the equivalence of the Hilbert space multi-norm, the (p, q)-multi-norm, and the maximum multi-norm based on a Hilbert space. We calculate the value of some constants that arise.

Several results depend on the classical theory of (q, p)-summing operators.

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1. Introduction

The theory of multi-norms was developed by H. G. Dales and M. E. Polyakov in a memoir [11], which was published in *Dissertationes Mathematicae*. One motivation for the development of this theory was to resolve a question on the injectivity of the Banach left modules $L^p(G)$ over the group algebra $L^1(G)$ of a locally compact group G: indeed, for $p > 1$, $L^p(G)$ is injective if and only if G is amenable [12].

However, the theory of multi-norms developed a life of its own: it is shown in [11] that the theory has connections with tensor norms on the spaces $c_0 \otimes E$, with the theory of (q, p)-summing operators, and with Banach algebras of operators, through the concept of a 'multi-bounded' operator.

In [11], there are many examples of multi-norms based on a normed space. For example, this memoir introduced the maximum and minimum multi-norms, the (p, q)-multi-norm based on a normed space (for $1 \leq p \leq q < \infty$), the standard t-multi-norm based on a space $L^r(\Omega)$ (for $1 \leq r \leq t < \infty$), and the Hilbert multi-norm based on a Hilbert space.

There is a natural notion of 'equivalence' of two multi-norms based on the same normed space, and we find it of interest to establish when various pairs of the known examples are indeed mutually equivalent. This often leads to questions of the equality of various classes of summing operators on certain Banach spaces. However, this relationship to summing operators is not entirely straightforward: results on such operators in the literature seem to give only partial indications. For example, in the case of (p, q)-multi-norms on a Hilbert space H, we would like information about $\Pi_{q,p}(H, c_0)$, but classical results determine $\Pi_{q,p}(H)$.

Some easy results on the equivalences of pairs of multi-norms were given in [11] and in [12]. In the present paper, we shall present a more systematic study of these equivalences.

In Chapter 1, we shall recall some background in functional analysis, including the theory of summing norms and tensor norms. In particular, we shall define the Banach space $(\Pi_{q,p}(E, F), \pi_{q,p})$ of (q, p)-summing operators between Banach spaces E and F.

In Chapter 2, we shall give the definition of a multi-norm, and introduce the notions of the rate of growth $(\varphi_n(E))$ of a multi-norm based on a space E and our notion of the mutual equivalence of two multi-norms based on the same normed space. Two equivalent multi-norms have similar rates of growth, but the converse is, in general, not true. We shall recall the definitions of the maximum and minimum multi-norms, $(\|\cdot\|_n^{\max} : n \in \mathbb{N})$ and $(\|\cdot\|_n^{\min} : n \in \mathbb{N})$, based on a normed space.

We shall define the (p, q)-multi-norm $(\|\cdot\|_n^{(p,q)} : n \in \mathbb{N})$ based on such a space E in the case where $1 \leq p \leq q < \infty$, and we shall related these multi-norms to certain c_0-norms on the algebraic tensor product $c_0 \otimes E$; for example, it is shown in Theorem 2.10 that the (p,p)-multi-norm corresponds to the Chevet–Saphar norm on $c_0 \otimes E$. We shall show in Corollary 2.9 that the multi-norms corresponding to points (p_1, q_1) and (p_2, q_2) are mutually equivalent if and only if the Banach spaces $\Pi_{q_1,p_1}(E', c_0)$ and $\Pi_{q_2,p_2}(E', c_0)$ are the same.

We shall begin to study the relations between (p,q)-multi-norms in §2.5, giving first indications in a diagram on page 20; this diagram follows from standard results on (q,p)-summing operators given by Diestel, Jarchow, and Tonge in the fine text [14]. In Examples 2.16 and 2.17, we shall calculate some explicit (p,q)-multi-norms; these results will be used later to show that certain (p,q)-multi-norms are not mutually equivalent. It was already known that the $(1,1)$-multi-norm is the maximum multi-norm on each normed space.

In §2.6, we shall describe the standard t-multi-norm on a Banach space $L^r(\Omega, \mu)$, where (Ω, μ) is a measure space; these multi-norms played an important role in [11], especially in connection with the theory of multi-bounded operators between Banach lattices. In §2.7, we shall describe the Hilbert multi-norm based on a Hilbert space; in fact, this is equal to the $(2,2)$-multi-norm based on the same space.

Our first aim in Chapter 3 is to determine when two (p,q)-multi-norms based on a space $L^r(\Omega, \mu)$ are mutually equivalent; here $1 \le p \le q < \infty$ and $r \ge 1$. In the case where $r = 1$, complete results are given in §3.1. The case where $r > 1$ is more difficult, and there is a clear distinction between the cases where $r < 2$ and $r \ge 2$. To discuss the question, it is helpful to consider certain curves $\mathcal{C}_c$ and $\mathcal{D}_c$, defined for for $0 \le c < 1$; the union of these curves fills out the 'triangle' $\mathcal{T} = \{(p,q) : 1 \le p \le q\}$. A picture of these curves in the case where $r > 2$ is given on page 32.

We say that two points $P_1 = (p_1, q_1)$ and $P_2 = (p_2, q_2)$ in $\mathcal{T}$ are equivalent if the corresponding (p,q)-multi-norms are equivalent on $L^r(\Omega)$. In Theorem 3.11, we shall show that in the 'upper-left' of our diagram, P_1 and P_2 are mutually equivalent, and that the corresponding multi-norms are equivalent to the minimum multi-norm. It is also shown that, otherwise, P_1 and P_2 are not equivalent whenever they lie on distinct curves $\mathcal{D}_c$. Thus we must turn to consideration of points on the same curve $\mathcal{D}_c$ (for $c < 1/\bar{r}$, where $\bar{r} = \min\{2, r\}$). In §3.6, we shall use Khintchine's inequalities to show that P_1 and P_2 are not equivalent on the space ℓ^r whenever they are not equivalent on ℓ^2, and hence whenever the spaces $\Pi_{q_1,p_1}(\ell^2)$ and $\Pi_{q_2,p_2}(\ell^2)$ are distinct; the latter question is classical, and full results are given in [14]. Thus we are able to resolve most questions of mutual equivalence of (p,q)-multi-norms on $L^r(\Omega, \mu)$. Results in the case where $r \in (1,2)$ are given in Theorem 3.16, and those in the case where $r \ge 2$ are given in Theorem 3.18. Some cases are left open in Theorems 3.16 and 3.18, but a full solution is given in the case where $r = 2$. Some of the remaining cases will be resolved in [7].

Let Ω be a measure space, and take $r \ge 1$. In §3.8, we shall consider the conjecture that the multi-norms $(\|\cdot\|_n^{[t]})$ and $(\|\cdot\|_n^{(p,q)})$ are not mutually equivalent whenever $r > 1$ and $L^r(\Omega)$ is infinite-dimensional. (By Theorem 2.20, $(\|\cdot\|_n^{[q]}) = (\|\cdot\|_n^{(1,q)})$ on $L^1(\Omega)$ for $q \ge 1$.) We shall prove this conjecture for many, but not all, values of p, q, and r in Theorem 3.22. Further results will be given in [7].

Let H be a complex Hilbert space. Then the Hilbert multi-norm, the (p,p)-multi-norms for $p \in [1,2]$, and the maximum multi-norm based on H are all pairwise equivalent. In Chapter 4, we shall discuss these norms in more detail. For example, we know that, for each $p \in [1,2]$, there is a constant C_p such that $\|\boldsymbol{x}\|_n^{\max} = \|\boldsymbol{x}\|_n^{(1,1)} \leq C_p \|\boldsymbol{x}\|_n^{(p,p)}$ for all $\boldsymbol{x} \in H^n$ and all $n \in \mathbb{N}$. In §4.1, we shall show that $2/\sqrt{\pi}$ is the best value of C_2; this is a consequence of the 'Little Grothendieck Theorem'.

In the remainder of Chapter 4, we shall consider the best constant c_n, defined for each fixed $n \in \mathbb{N}$, such that $\|\boldsymbol{x}\|_n^{\max} \leq c_n \|\boldsymbol{x}\|_n^{(2,2)}$ for $\boldsymbol{x} \in H^n$. We shall show that $c_2 = 1$, but that $c_3 > 1$ in the real case; however, a rather long calculation will show that $c_3 = 1$ in the complex case; finally, we shall show in §4.5 that $c_4 > 1$ even in the complex case.

Two points left open in the present work will be resolved in [7]; see Remarks 3.17 and 3.19.

We first give some background to the material of this paper, and recall some definitions from earlier works.

1.1. Basic notation. The natural numbers and the integers are $\mathbb{N}$ and $\mathbb{Z}$, respectively. For $n \in \mathbb{N}$, we set $\mathbb{N}_n = \{1, \dots, n\}$. The complex field is $\mathbb{C}$; the unit circle and open unit disc in $\mathbb{C}$ are $\mathbb{T}$ and $\mathbb{D}$, respectively.

Let $1 \leq p \leq \infty$. Then the conjugate to p is denoted by p', so that $1 \leq p' \leq \infty$ and satisfies $1/p + 1/p' = 1$.

Let (α_n) and (β_n) be two sequences of complex numbers. Then (α_n) and (β_n) are *similar*, written $\alpha_n \sim \beta_n$, if there are constants $C_1, C_2 > 0$ such that

$$C_1|\alpha_n| \leq |\beta_n| \leq C_2|\alpha_n| \quad (n \in \mathbb{N}).$$

An easy form of *Hölder's inequality* gives the following. Let $p, q \in [1, \infty]$ be conjugate indices. Then, for each $n \in \mathbb{N}$ and each $x_1, \dots, x_n, y_1, \dots, y_n \in \mathbb{C}$, we have

$$\sum_{j=1}^{n} |x_j y_j| \leq \left(\sum_{j=1}^{n} |x_j|^p \right)^{1/p} \left(\sum_{j=1}^{n} |y_j|^q \right)^{1/q}. \tag{1.1}$$

Now take $a_1, \dots, a_n \in \mathbb{R}^+$ and r, s with $1 \leq r \leq s$. Then (in the case where $r < s$) we apply (1.1) with $x_j = a_j^r$ and $y_j = 1$ for $j \in \mathbb{N}_n$ and with $p = s/r$ and $q = s/(s-r)$ to see that

$$\frac{1}{n^{1/r}}(a_1^r + \cdots + a_n^r)^{1/r} \leq \frac{1}{n^{1/s}}(a_1^s + \cdots + a_n^s)^{1/s}. \tag{1.2}$$

1.2. Linear and Banach spaces. Let E be a linear space (always taken to be over the complex field, $\mathbb{C}$, unless otherwise stated).

Let C be a convex set in E. An element $x \in C$ is an *extreme point* if $C \setminus \{x\}$ is also convex; the set of extreme points of C is denoted by $\operatorname{ex} C$. Let $x \in C$. Then, to show that $x \in \operatorname{ex} C$, it suffices to show that $u = 0$ whenever $u \in E$ and $x \pm u \in C$.

For a linear space E and $n \in \mathbb{N}$, we denote by E^n the linear space direct product of n copies of E. g Let F be another linear space. Then the linear space of all linear operators from E to F is denoted by $\mathcal{L}(E, F)$. The identity operator on E is I_E, or just I when the space is obvious.

Let E be a normed space. The closed unit ball and unit sphere of E are denoted by $E_{[1]}$ and S_E, respectively, so that $\mathrm{ex}\, E_{[1]} \subset S_E$. We denote the dual space of E by E'; the action of $\lambda \in E'$ on an element $x \in E$ is written as $\langle x, \lambda \rangle$, and the canonical embedding of E into its bidual E'' is $\kappa_E : E \to E''$.

Let E and F be normed spaces. Then $\mathcal{B}(E, F)$ is the normed space of all bounded linear operators from E to F; it is a Banach space whenever F is complete. The ideal of finite-rank operators in $\mathcal{B}(E, F)$ is denoted by $\mathcal{F}(E, F)$. We set $\mathcal{B}(E) = \mathcal{B}(E, E)$, so that $\mathcal{B}(E)$ is a unital normed algebra; it is a Banach algebra whenever E is complete. The *dual* of $T \in \mathcal{B}(E, F)$ is $T' \in \mathcal{B}(F', E')$, so that $\|T'\| = \|T\|$. The closed ideal of $\mathcal{B}(E)$ consisting of the compact operators is denoted by $\mathcal{K}(E)$.

A closed subspace F of a normed space E is λ-*complemented* if there exists $P \in \mathcal{B}(E)$ with $P^2 = P$, with $P(E) = F$, and with $\|P\| \leq \lambda$.

We write $E \cong F$ when two Banach spaces $(E, \|\cdot\|)$ and $(F, \|\cdot\|)$ are *isometrically isomorphic.*

Let (Ω, μ) be a measure space, and take $p \geq 1$. Then we denote by $L^p(\Omega) = L^p(\Omega, \mu)$ (or $L^p(\mu)$) the Banach space of (equivalence classes of) complex-valued, p-integrable functions on Ω, equipped with the norm $\|\cdot\|_p$, which is given by

$$\|f\|_p = \left(\int_\Omega |f(x)|^p \, \mathrm{d}\mu(x) \right)^{1/p} = \left(\int_\Omega |f|^p \, \mathrm{d}\mu \right)^{1/p} \qquad (f \in L^p(\Omega)).$$

We also define the related space $L^\infty(\Omega) = L^\infty(\Omega, \mu)$. All these spaces are Dedekind complete (complex) Banach lattices in the standard way. For some background on Banach lattices that is sufficient for our purposes, see [11, §1.3].

Let c_0 and ℓ^p be the usual Banach spaces of sequences, where $1 \leq p \leq \infty$. We shall write $(\delta_n)_{n=1}^\infty$ for the standard unit Schauder basis for c_0 and ℓ^p (when $p \geq 1$). For $n \in \mathbb{N}$, we write ℓ_n^∞ and ℓ_n^p for the linear space $\mathbb{C}^n$ with the supremum and ℓ^p norms, respectively; we regard each ℓ_n^∞ as a subspace of c_0, and hence regard $(\delta_i)_{i=1}^n$ as a basis for ℓ_n^∞. The space of all continuous functions on a compact Hausdorff space K is denoted by $C(K)$.

We shall several times use the following two results.

Proposition 1.1. *Take $p \geq 1$, and let Ω be a measure space such that $L^p(\Omega)$ is infinite-dimensional. Then there is an isometric lattice homomorphism $J : \ell^p \to L^p(\Omega)$ and a positive contraction of $L^p(\Omega)$ onto $J(\ell^p)$, so that $J(\ell^p)$ is 1-complemented in $L^p(\Omega)$.*

Proof. This is [4, 4.1], for example. ∎

Proposition 1.2. *Let E be an infinite-dimensional Banach space, and take $\varepsilon > 0$ and $n \in \mathbb{N}$. Then there exist $x_1, \dots, x_n \in E$ such that*

$$1 - \varepsilon \leq \|x_n\| \leq 1 \qquad (n \in \mathbb{N})$$

and

$$\left\| \sum_{i=1}^n \alpha_i x_i \right\| \leq \left(\sum_{i=1}^n |\alpha_i|^2 \right)^{1/2} \qquad (\alpha_1, \dots, \alpha_n \in \mathbb{C}).$$

Proof. By Dvoretzky's theorem, E contains near-isometric copies of ℓ_n^2, and this gives the result. Actually, our claim is somewhat weaker, and follows from more elementary arguments, given in [14, Lemma 1.3], for example. ∎

We shall refer to Lorentz sequence spaces. Suppose that $1 \leq p \leq q < \infty$. Then the *Lorentz sequence space* $\ell^{p,q}$ consists of the sequences $x = (x_n) \in c_0$ such that

$$\|x\|_{p,q} = \Big(\sum_{n=1}^{\infty} n^{q/p-1}(x_n^*)^q\Big)^{1/q} < \infty,$$

where x^* is the decreasing re-arrangement of $|x|$; the version based on $\mathbb{N}_n$ is $\ell_n^{p,q}$. For this definition, see [14, p. 207], for example. The spaces $(\ell^{p,q}, \|\cdot\|_{p,q})$ are Banach spaces. In the case where $q = p$, we obtain the usual spaces ℓ^p and ℓ_n^p.

We shall also refer to Schatten classes. Let H be a Hilbert space. For $p \geq 1$, the p-th *Schatten class* $\mathcal{S}_p(H)$ consists of the compact operators $T \in \mathcal{K}(H)$ such that the positive operator $(T^*T)^{p/2}$ has finite trace; the norm $\|\cdot\|_{\mathcal{S}_p}$ on $\mathcal{S}_p(H)$ is given by

$$\|T\|_{\mathcal{S}_p} = \big(\mathrm{tr}((T^*T)^{p/2})\big)^{1/p} \quad (T \in \mathcal{S}_p(H)).$$

Equivalently, $T \in \mathcal{S}_p(H)$ if and only if the operator $|T| = (T^*T)^{1/2}$ is compact and $\lambda = (\lambda_n) \in \ell^p$, where (λ_n) is the (decreasing) sequence of non-zero eigenvalues of $|T|$, counted according to their multiplicities; now $\|T\|_{\mathcal{S}_p} = \|\lambda\|_p$. The space $(\mathcal{S}_p(H), \|\cdot\|_{\mathcal{S}_p})$ is a Banach operator ideal in $\mathcal{B}(H)$; the ideal $\mathcal{S}_2(H)$ coincides with the space of *Hilbert–Schmidt operators* on H, and the corresponding norm is the *Hilbert–Schmidt norm*.

In the case where $2 < p < q < \infty$, the space $\mathcal{S}_{2q/p,q}(H)$ consists of the operators $T \in \mathcal{B}(H)$ such that the above sequence of eigenvalues belongs to the Lorentz sequence space $\ell^{2q/p,q}$, and so satisfies the condition that

$$\|T\|_{\mathcal{S}_{2q/p,q}} = \Big(\sum_{n=1}^{\infty} n^{p/2-1}\lambda_n^q\Big)^{1/q} < \infty.$$

Suppose that H is an infinite-dimensional Hilbert space, and let (e_n) be an orthonormal sequence in H. For $\alpha > 0$, set $T_\alpha e_n = n^{-\alpha}e_n \ (n \in \mathbb{N})$, so that T_α extends to an operator in $\mathcal{B}(H)$ in an obvious way. Then $T_\alpha \in \mathcal{S}_p(H)$ if and only if $\alpha p > 1$. Thus $\mathcal{S}_p(H) \neq \mathcal{S}_q(H)$ whenever $p, q \geq 1$ with $p \neq q$. Further, $T_\alpha \in \mathcal{S}_{2q/p,q}(H)$ if and only if $\alpha > p/2q$, and so $\mathcal{S}_r(H) \neq \mathcal{S}_{2q/p,q}(H)$ whenever $r \neq 2q/p$.

Now suppose that $r = 2q/p$. We take an infinite subset X of $\mathbb{N}$, and define $T \in \mathcal{B}(H)$ by setting $Te_n = n^{-\alpha}e_n \ (n \in X)$ and $Te_n = 0 \ (n \in \mathbb{N} \setminus X)$, where $q\alpha = 1 - p/2$, and again extending T to belong to $\mathcal{B}(H)$. Then $T \in \mathcal{S}_r(H)$ if and only if

$$\sum_{n\in X} n^{2/p-1} < \infty,$$

and so $T \in \mathcal{S}_r(H)$ for a suitably 'sparse' set X, noting that $2/p - 1 < 0$. However, $T \in \mathcal{S}_{2q/p,q}(H)$ if and only if $\sum_{n\in X} 1 < \infty$, and this is never the case for infinite X. Thus it is always true that the spaces $\mathcal{S}_r(H)$ and $\mathcal{S}_{2q/p,q}(H)$ are distinct.

Similarly, the spaces $\mathcal{S}_{2q/p,q}(H)$ corresponding to pairs (p_1, q_1) and (p_2, q_2) are distinct whenever $(p_1, q_1) \neq (p_2, q_2)$.

1.3. Summing norms and summing operators. Let E be a normed space, and let $n \in \mathbb{N}$. Following the notation of [11, 12, 18], we define the *weak p-summing norm* (for $1 \leq p < \infty$) on E^n by

$$\mu_{p,n}(\boldsymbol{x}) = \sup\Big\{ \Big(\sum_{i=1}^{n} |\langle x_i, \lambda\rangle|^p\Big)^{1/p} : \lambda \in E'_{[1]} \Big\},$$

where $\boldsymbol{x} = (x_1, \dots, x_n) \in E^n$. We set $\ell_n^p(E)^w = (E^n, \mu_{p,n})$. It follows from [18, p. 26] that, for each $\boldsymbol{x} = (x_1, \dots, x_n) \in E^n$, we have

$$\mu_{p,n}(\boldsymbol{x}) = \sup\Big\{ \Big\|\sum_{i=1}^{n} \zeta_i x_i\Big\| : \zeta_1, \dots, \zeta_n \in \mathbb{C},\ \sum_{i=1}^{n} |\zeta_i|^{p'} \le 1 \Big\}. \tag{1.3}$$

We also have

$$\mu_{p,n}(\boldsymbol{x}) = \|T_{\boldsymbol{x}} : \ell_n^{p'} \to E\|, \tag{1.4}$$

where $T_{\boldsymbol{x}} : (\beta_1, \dots, \beta_n) \mapsto \sum_{i=1}^n \beta_i x_i$ belongs to $\mathcal{B}(\ell_n^{p'}, E)$. Thus the map $\boldsymbol{x} \mapsto T_{\boldsymbol{x}}$ is an isometric isomorphism from $(E^n, \mu_{p,n})$ onto $\mathcal{B}(\ell_n^{p'}, E)$. Also, let F be another normed space, and take $T \in \mathcal{B}(E, F)$. Then clearly

$$\mu_{p,n}(Tx_1, \dots, Tx_n) \le \|T\| \mu_{p,n}(x_1, \dots, x_n) \quad (x_1, \dots, x_n \in E,\ n \in \mathbb{N}).$$

We note that

$$\mu_{p_1,n}(\boldsymbol{x}) \ge \mu_{p_2,n}(\boldsymbol{x}) \quad (\boldsymbol{x} \in E^n,\ n \in \mathbb{N})$$

whenever $1 \le p_1 \le p_2 < \infty$.

We also define the *weak p-summing norm* of a sequence $\boldsymbol{x} = (x_i)$ of elements in E by

$$\mu_p(\boldsymbol{x}) = \sup\Big\{ \Big(\sum_{i=1}^{\infty} |\langle x_i, \lambda\rangle|^p\Big)^{1/p} : \lambda \in E'_{[1]} \Big\} = \lim_{n\to\infty} \mu_{p,n}(x_1, \dots, x_n);$$

thus $\mu_p(\boldsymbol{x})$ takes values in $[0, \infty]$. The sequences $\boldsymbol{x}$ such that $\mu_p(\boldsymbol{x}) < \infty$ are the *weakly p-summable sequences* in E, and the space of these sequences is $\ell^p(E)^w$; see [14, p. 32] and [24, p. 134], where $\mu_p(\,\cdot\,)$ is denoted by $\|\cdot\|_p^{\mathrm{weak}}$ and $\|\cdot\|_p^w$, respectively. It follows from [18, p. 26] that, for each sequence $\boldsymbol{x} = (x_i)$ in E, we have

$$\mu_p(\boldsymbol{x}) = \sup\Big\{ \Big\|\sum_{i=1}^{\infty} \zeta_i x_i\Big\| : (\zeta_i) \in (\ell^{p'})_{[1]} \Big\}. \tag{1.5}$$

Suppose that $1 \le p \le q < \infty$. We recall from [14, Chapter 10] that an operator T from a normed space E into another normed space F is *(q,p)-summing* if there exists a constant C such that

$$\Big(\sum_{i=1}^{n} \|Tx_i\|^q\Big)^{1/q} \le C\, \mu_{p,n}(x_1, \dots, x_n) \quad (x_1, \dots, x_n \in E,\ n \in \mathbb{N}).$$

The smallest such constant C is denoted by $\pi_{q,p}(T)$. The set of these (q,p)-summing operators, which is denoted by $\Pi_{q,p}(E, F)$, is a linear subspace of $\mathcal{B}(E, F)$ and a normed space when equipped with the norm $\pi_{q,p}$; $(\Pi_{q,p}(E, F), \pi_{q,p})$ is a Banach space when E and F are Banach spaces. In the case where $p = q$, we shall write Π_p and π_p instead of $\Pi_{p,p}$ and $\pi_{p,p}$, respectively. The space (Π_p, π_p) of all *p-summing operators* has been studied by many authors; see [13, 14, 17, 18, 24], for example. In the case where $E = F$, we shall write $\Pi_{q,p}(E)$ instead of $\Pi_{q,p}(E, E)$, $\pi_{q,p}(E)$ instead of $\pi_{q,p}(E, E)$, etc.

A basic inclusion theorem [14, Theorem 2.8] shows that $\Pi_p(E,F) \subset \Pi_q(E,F)$ whenever $1 \leq p \leq q < \infty$. A more complicated inclusion theorem [14, Theorem 10.4] will be used in Theorem 2.11, given below.

Let us make some obvious remarks about summing operators. Let E, F, and G be Banach spaces, and take $T \in \mathcal{B}(E,F)$ and $1 \leq p \leq q < \infty$. Then:

- $T \in \Pi_{q,p}(E,F)$ if and only if $S \circ T \in \Pi_{q,p}(E,G)$, with equal norm, for any isometry $S : F \to G$;
- $T \in \Pi_{q,p}(E,F)$ if and only if $T \circ P \in \Pi_{q,p}(G,F)$, with equal norm, for any contractive projection $P : G \to E$.

These remarks will be used implicitly at some future points.

The Pietsch domination theorem can be stated in the following way (cf. the discussion after [24, Theorem 6.18]). Take $p \geq 1$. A map $T \in \mathcal{B}(E,F)$ is p-summing if and only if we can find a non-empty, compact Hausdorff space K and a probability measure μ on K, together with operators $V \in \mathcal{B}(E, C(K))$ and $U \in \mathcal{B}(L^p(\mu), \ell^\infty(I))$ such that the following diagram commutes:

$$\begin{tikzcd} E \arrow[r, "T"] \arrow[d, "V"] & F \arrow[r, hook] & \ell^\infty(I) \\ C(K) \arrow[rr] & & L^p(\mu) \arrow[u, "U"'] \end{tikzcd}$$

Here the map $C(K) \to L^p(\mu)$ is the canonical inclusion map, I is a suitable index set, and $\ell^\infty(I)$ can be replaced by any injective Banach space G such that F is isometric to a subspace of G.

Let E and F be normed spaces. Take $n \in \mathbb{N}$, and suppose that $1 \leq p \leq q < \infty$. Then the (q,p)-*summing constants* of the operator $T \in \mathcal{B}(E,F)$ are the numbers

$$\pi_{q,p}^{(n)}(T) := \sup\left\{ \left(\sum_{i=1}^n \|Tx_i\|^{\,q} \right)^{1/q} : x_1, \dots, x_n \in E,\ \mu_{p,n}(x_1, \dots, x_n) \leq 1 \right\}.$$

Further, $\pi_{q,p}^{(n)}(E) = \pi_{p,q}^{(n)}(I_E)$; these are the (q,p)-*summing constants* of the normed space E. We write $\pi_p^{(n)}(T)$ for $\pi_{p,p}^{(n)}(T)$ and $\pi_p^{(n)}(E)$ for $\pi_{p,p}^{(n)}(E)$. It follows that

$$\pi_{q,p}^{(n)}(E) = \sup\left\{ \left(\sum_{i=1}^n \|x_i\|^{\,q} \right)^{1/q} : x_1, \dots, x_n \in E,\ \mu_{p,n}(x_1, \dots, x_n) \leq 1 \right\}. \tag{1.6}$$

Proposition 1.3. *Suppose that $1 \leq p \leq q < \infty$ and that $n \in \mathbb{N}$. Then:*

(i) $\pi_{q,p}^{(n)}(E) \leq n^{1/q}$ *for each normed space E;*
(ii) $\pi_{q,p}^{(n)}(E) = n^{1/q}$ *for each infinite-dimensional normed space E whenever $p \geq 2$;*
(iii) $\pi_{q,p}^{(n)}(E) \geq n^{1/2-1/p+1/q}$ *for each infinite-dimensional normed space E whenever $p \leq 2$;*
(iv) $\pi_{q,p}^{(n)}(\ell^s) = n^{1/q}$ *whenever $s \in [1,\infty]$ and $p \geq \min\{s', 2\}$.*

Proof. (i) This is immediate.

(ii) Take $\varepsilon > 0$, and choose $x_1, \dots, x_n \in E$ to be as specified in Proposition 1.2. For each $\zeta_1, \dots, \zeta_n \in \mathbb{C}$ with $\sum_{i=1}^n |\zeta_i|^{p'} \le 1$, we have

$$\Big\|\sum_{i=1}^n \zeta_i x_i\Big\| \le \Big(\sum_{i=1}^n |\zeta_i|^2\Big)^{1/2} \le \Big(\sum_{i=1}^n |\zeta_i|^{p'}\Big)^{1/p'}$$

because $p' \le 2$. Thus, by equation (1.3), $\mu_{p,n}(x_1, \dots, x_n) \le 1$, and so

$$\pi_{q,p}^{(n)}(E) \ge (1-\varepsilon)n^{1/q}.$$

This holds true for each $\varepsilon > 0$, and so $\pi_{q,p}^{(n)}(E) \ge n^{1/q}$. By (i), $\pi_{q,p}^{(n)}(E) = n^{1/q}$.

(iii) Take $\varepsilon > 0$ and choose $x_1, \dots, x_n \in E$ as in (ii). Now, since $p' \ge 2$, the argument in (ii) shows that $\mu_{p,n}(x_1, \dots, x_n) \le n^{1/2-1/p'}$, and so

$$\pi_{q,p}^{(n)}(E) \ge (1-\varepsilon)n^{1/2-1/p+1/q}$$

for every $\varepsilon > 0$. Hence $\pi_{q,p}^{(n)}(E) \ge n^{1/2-1/p+1/q}$.

(iv) In the case where $p \ge 2$, this follows from (ii). Now suppose that $p \ge s'$. Take $x_j = \delta_j$ $(j \in \mathbb{N}_n)$. As in the proof of (ii), we see that $\mu_{p,n}(x_1, \dots, x_n) \le 1$, and so $\pi_{q,p}^{(n)}(\ell^s) \ge n^{1/q}$. ∎

We shall also need the following simple interpolation result.

Proposition 1.4. *Let E be a normed space. Suppose that $1 \le p \le q_1 < q < q_2 < \infty$, so that*

$$\frac{1}{q} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}$$

for some $\theta \in (0,1)$. Then

$$\pi_{q,p}^{(n)}(E) \le (\pi_{q_1,p}^{(n)}(E))^{1-\theta} \cdot (\pi_{q_2,p}^{(n)}(E))^{\theta} \quad (n \in \mathbb{N}).$$

Proof. Take $x_1, \dots, x_n \in E$ with $\mu_{p,n}(x_1, \dots, x_n) \le 1$. Using a version of Hölder's inequality, we see that

$$\begin{aligned}\Big(\sum_{i=1}^n \|x_i\|^{\,q}\Big)^{1/q} &\le \Big(\sum_{i=1}^n \|x_i\|^{(1-\theta)[q_1/(1-\theta)]}\Big)^{(1-\theta)/q_1} \cdot \Big(\sum_{i=1}^n \|x_i\|^{\theta[q_2/\theta]}\Big)^{\theta/q_2}\\ &\le (\pi_{q_1,p}^{(n)}(E))^{1-\theta} \cdot (\pi_{q_2,p}^{(n)}(E))^{\theta},\end{aligned}$$

which implies the result. ∎

1.4. Tensor norms. Let E and F be linear spaces. Then $E \otimes F$ denotes the algebraic tensor product of E and F.

Let E_1, E_2, F_1, F_2 be linear spaces, and take $S \in \mathcal{L}(E_1, E_2)$ and $T \in \mathcal{L}(F_1, F_2)$. Then $S \otimes T$ denotes the unique linear operator from $E_1 \otimes F_1$ to $E_2 \otimes F_2$ such that

$$(S \otimes T)(x \otimes y) = Sx \otimes Ty \quad (x \in E_1,\ y \in F_1).$$

In particular, we have defined $\lambda \otimes \mu$ whenever λ and μ are linear functionals on E_1 and F_1, respectively.

Now suppose that E and F are normed spaces. We shall discuss various norms on the space $E \otimes F$. For the definitions and properties stated below, see [13, Chapter I], [14], and [24, Section 6.1], for example.

We shall often regard $E \otimes F$ as a linear subspace of $\mathcal{B}(F', E)$ by setting

$$(x \otimes y)(\lambda) = \langle y, \lambda \rangle x \quad (x \in E,\, y \in F,\, \lambda \in F');$$

in this way, we identify $E \otimes F$ with $\mathcal{F}(F', E) \subset \mathcal{B}(F', E)$. Similarly, we can identify $E \otimes F$ with $\mathcal{F}(E', F) \subset \mathcal{B}(E', F)$.

The injective and projective tensor norms on $E \otimes F$ are denoted by $\|\cdot\|_\varepsilon$ and $\|\cdot\|_\pi$, respectively; the completions of $E \otimes F$ with respect to these norms are denoted by

$$(E \check{\otimes} F, \|\cdot\|_\varepsilon) \quad \text{and} \quad (E \widehat{\otimes} F, \|\cdot\|_\pi),$$

respectively.

For $\mu \in (E \widehat{\otimes} F)'$, define T_μ by

$$\langle y, T_\mu x \rangle = \langle x \otimes y, \mu \rangle \quad (x \in E,\, y \in F).$$

Then $T_\mu x \in F'$ $(x \in E)$, $T_\mu \in \mathcal{B}(E, F')$, and the map

$$\mu \mapsto T_\mu, \quad (E \widehat{\otimes} F)' \to \mathcal{B}(E, F'), \tag{1.7}$$

is an isometric isomorphism, and so $(E \widehat{\otimes} F)' \cong \mathcal{B}(E, F')$.

A norm $\|\cdot\|$ on $E \otimes F$ is a *sub-cross-norm* if

$$\|x \otimes y\| \leq \|x\|\,\|y\| \quad (x \in E,\, y \in F)$$

and a *cross-norm* if

$$\|x \otimes y\| = \|x\|\,\|y\| \quad (x \in E,\, y \in F).$$

Further, a sub-cross-norm $\|\cdot\|$ on $E \otimes F$ is a *reasonable cross-norm* if the linear functional $\lambda \otimes \mu$ is bounded and $\|\lambda \otimes \mu\| \leq \|\lambda\|\,\|\mu\|$ for each $\lambda \in E'$ and $\mu \in F'$. In fact, a sub-cross-norm is reasonable if and only if

$$\|z\|_\varepsilon \leq \|z\| \leq \|z\|_\pi \quad (z \in E \otimes F).$$

Let α be a reasonable cross-norm on $E \otimes F$. Then the completion of the normed space $(E \otimes F, \alpha)$ is denoted by $E \widehat{\otimes}_\alpha F$. The map in (1.7) identifies the dual of $E \widehat{\otimes}_\alpha F$ with a linear subspace of $\mathcal{B}(E, F')$.

A *uniform cross-norm* is an assignment of a cross-norm to $E \otimes F$ for all pairs of Banach spaces (E, F), with the property that, for each operator $S \in \mathcal{B}(E_1, E_2)$ and $T \in \mathcal{B}(F_1, F_2)$, the linear map $S \otimes T : E_1 \otimes F_1 \to E_2 \otimes F_2$ is bounded, with norm at most $\|S\|\,\|T\|$, with respect to the assigned norms on $E_1 \otimes F_1$ and $E_2 \otimes F_2$. The projective and injective tensor norms are uniform cross-norms. For further details, see [13, §12.1] and [24, §6.1].

For Banach spaces E and F, the (right) *Chevet–Saphar norm* d_p on $E \otimes F$ is defined as

$$d_p(z) = \inf_{n \in \mathbb{N}} \left\{ \mu_{p',n}(x_1, \ldots, x_n) \left(\sum_{i=1}^n \|y_i\|^p \right)^{1/p} : z = \sum_{i=1}^n x_i \otimes y_i \in E \otimes F \right\};$$

see, for example, [13, Chapter 12] and [24, p. 135]. This norm is a reasonable cross-norm; in fact, it is a uniform cross-norm.

Given a tensor $z \in E \otimes F$, let z^t be the 'flipped' tensor in $F \otimes E$. We define the left Chevet–Sapher norm g_p by $g_p(z) = d_p(z^t)$ [24, p. 135].

Let α be a uniform cross-norm. Following [24, Chapter 7], we define the *Schatten dual tensor norm* α^s by

$$\alpha^s(z) = \sup\{|\langle z, \lambda\rangle| : \lambda \in E' \otimes F',\, \alpha(\lambda) \le 1\} \qquad (z \in E \otimes F),$$

using the obvious dual pairing between $E \otimes F$ and $E' \otimes F'$. In general, this does not lead to a satisfactory duality theory, as it may happen that $(\alpha^s)^s \neq \alpha$. To correct this, we define the *dual tensor norm* α' by first setting $\alpha' = \alpha^s$ on $E \otimes F$ whenever E and F are finite-dimensional spaces, and then extend α' to all Banach spaces by finite generation. The details are technical, and we refer the reader to [13, Chapter II] and [24, Chapter 7] for further information.

We say that a uniform cross-norm α is *totally accessible* if the embedding of $E \otimes F$ into $(E' \mathbin{\widehat{\otimes}}_{\alpha'} F')'$ induces the norm α on $E \otimes F$ for all Banach spaces E and F. That is, α is totally accessible if $(\alpha')^s = \alpha$. In the case where this is true under the additional hypothesis that at least one of the two spaces E or F has the metric approximation property, α is said to be *accessible*. For us, it is important that c_0 has the metric approximation property and that many norms α on spaces $E \otimes F$ are accessible. For example, by [24, Proposition 7.21], g_p is an accessible norm for any p (and hence the same is true of d_p).

Let E and F be normed spaces. A bounded operator $T : E \to F$ is *p-integral* if it gives a bounded linear functional on the space $E \mathbin{\widehat{\otimes}}_{g'_p} F'$, and the *$p$-integral norm* of T, denoted by $i_p(T)$, is defined to be the norm of this functional; see [24, §7.3]. Such maps have a representation theory which is analogous to the Pietsch representation theorem for p-summing operators; see [24, Theorem 7.22], for example. Indeed, we can factor such an operator T as

$$\begin{tikzcd} E \arrow[r, "T"] \arrow[d] & F \arrow[r, "\kappa_F"] & F'' \\ C(K) \arrow[rr] & & L_p(\mu) \arrow[u] \end{tikzcd}$$

Comparing this to the factorisation result above for p-summing maps, we see that the only difference is that here we embed F into its bidual F'', but for a p-summing map, we embed F into an injective space. Thus every p-integral map is p-summing. In the special case where $F = c_0$, we know that $F'' = \ell^\infty$, and so we conclude with the following proposition.

PROPOSITION 1.5. *Let E be a normed space. Then the classes of p-summing and p-integral maps from E to c_0 coincide, with equal norms.* ∎

2. Basic facts on multi-normed spaces

2.1. Multi-normed spaces. The following definition is due to Dales and Polyakov. For a full account of the theory of multi-normed spaces, see [11], and, for further work, see [12]. The main definition is taken from [11, Definition 2.1].

DEFINITION 2.1. Let $(E, \|\cdot\|)$ be a normed space, and let $(\|\cdot\|_n : n \in \mathbb{N})$ be a sequence such that $\|\cdot\|_n$ is a norm on E^n for each $n \in \mathbb{N}$, with $\|\cdot\|_1 = \|\cdot\|$ on $E = E^1$. Then the sequence $(\|\cdot\|_n : n \in \mathbb{N})$ is a *multi-norm* if the following axioms hold (where in each case the axiom is required to hold for all $n \in \mathbb{N}$ and all $x_1, \dots, x_n \in E$):

(A1) $\|(x_{\sigma(1)}, \dots, x_{\sigma(n)})\|_n = \|(x_1, \dots, x_n)\|_n$ for each permutation σ of $\mathbb{N}_n$;
(A2) $\|(\alpha_1 x_1, \dots, \alpha_n x_n)\|_n \leq \max_{i \in \mathbb{N}_n} |\alpha_i| \, \|(x_1, \dots, x_n)\|_n$ $(\alpha_1, \dots, \alpha_n \in \mathbb{C}^n)$;
(A3) $\|(x_1, \dots, x_n, 0)\|_{n+1} = \|(x_1, \dots, x_n)\|_n$;
(A4) $\|(x_1, \dots, x_{n-1}, x_n, x_n)\|_{n+1} = \|(x_1, \dots, x_{n-1}, x_n)\|_n$.

The space E equipped with a multi-norm is a *multi-normed space*, written in full as $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$; we say that the multi-norm is *based* on E.

In the case where E is a Banach space, $(E^n, \|\cdot\|_n)$ is a Banach space for each $n \in \mathbb{N}$, and we refer to a *multi-Banach space*.

Let $(\|\cdot\|_n^1 : n \in \mathbb{N})$ and $(\|\cdot\|_n^2 : n \in \mathbb{N})$ be two multi-norms based on a normed space E. Then, following [11, Definition 2.23], we write

$$(\|\cdot\|_n^1) \leq (\|\cdot\|_n^2)$$

if $\|\boldsymbol{x}\|_n^1 \leq \|\boldsymbol{x}\|_n^2$ for each $\boldsymbol{x} \in E^n$ and $n \in \mathbb{N}$, and write

$$(\|\cdot\|_n^1) = (\|\cdot\|_n^2)$$

if $\|\boldsymbol{x}\|_n^1 = \|\boldsymbol{x}\|_n^2$ for each $\boldsymbol{x} \in E^n$ and $n \in \mathbb{N}$. The multi-norm $(\|\cdot\|_n^2 : n \in \mathbb{N})$ *dominates* a multi-norm $(\|\cdot\|_n^1 : n \in \mathbb{N})$ if there is a constant $C > 0$ such that

$$\|\boldsymbol{x}\|_n^1 \leq C \|\boldsymbol{x}\|_n^2 \quad (\boldsymbol{x} \in E^n,\ n \in \mathbb{N}), \tag{2.1}$$

and, in this case, we write

$$(\|\cdot\|_n^1) \preccurlyeq (\|\cdot\|_n^2).$$

The two multi-norms are *equivalent*, written

$$(\|\cdot\|_n^1) \cong (\|\cdot\|_n^2),$$

if each dominates the other; if the two multi-norms are not equivalent, we shall write $(\|\cdot\|_n^1) \ncong (\|\cdot\|_n^2)$.

We shall be interested in determining when one multi-norm dominates the other (and, in this case, in the best value of the constant C in equation (2.1)) and when two multi-norms are equivalent.

Let $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ be a multi-normed space. For $n \in \mathbb{N}$, define

$$\varphi_n(E) = \sup\{\|(x_1, \dots, x_n)\|_n : x_1, \dots, x_n \in E_{[1]}\}.$$

Then the sequence $(\varphi_n(E))$ is the *rate of growth* corresponding to the multi-norm [11, Definition 3.1]. This sequence depends on both E and the specific multi-norm.

2.2. Multi-norms as tensor norms. In [12], we explained how multi-norms correspond to certain tensor norms. We recall this briefly; details are given in [12, §3].

Definition 2.2. Let E be a normed space. Then a norm $\|\cdot\|$ on $c_0 \otimes E$ is a c_0*-norm* if $\|\delta_1 \otimes x\| = \|x\|$ for each $x \in E$ and if the linear operator $T \otimes I_E$ is bounded on $(c_0 \otimes E, \|\cdot\|)$, with norm at most $\|T\|$, for each $T \in \mathcal{K}(c_0)$.

Similarly, a norm $\|\cdot\|$ on $\ell^\infty \otimes E$ is an ℓ^∞*-norm* if $\|\delta_1 \otimes x\| = \|x\|$ for each $x \in E$ and if $T \otimes I_E$ is bounded on $(\ell^\infty \otimes E, \|\cdot\|)$, with norm at most $\|T\|$, for each $T \in \mathcal{K}(\ell^\infty)$.

By [12, Lemma 3.3], each c_0-norm on $c_0 \otimes E$ and each ℓ^∞-norm on $\ell^\infty \otimes E$ is a reasonable cross-norm.

Suppose that $\|\cdot\|$ is a c_0-norm on $c_0 \otimes E$, and set

$$\|(x_1, \dots, x_n)\|_n = \Big\|\sum_{i=1}^n \delta_i \otimes x_i\Big\| \quad (x_1, \dots, x_n \in E,\ n \in \mathbb{N}).$$

Then $(\|\cdot\|_n : n \in \mathbb{N})$ is a multi-norm based on E.

A more general and detailed version of the following theorem is given as [12, Theorem 3.4].

THEOREM 2.3. *Let E be a normed space. Then the above construction defines a bijection from the family of c_0-norms on $c_0 \otimes E$ to the family of multi-norms based on E.* ∎

We shall be interested in uniform cross-norms, restricted to tensor products of the form $c_0 \otimes E$. This motivates us to give the following definition.

DEFINITION 2.4. A *uniform c_0-norm* is an assignment of a c_0-norm $\|\cdot\|$ to $c_0 \otimes E$ for all Banach spaces E such that the operator $I \otimes T : c_0 \otimes E \to c_0 \otimes F$ is bounded with respect to the two corresponding norms, with norm $\|T\|$, for any normed spaces E and F and each $T \in \mathcal{B}(E, F)$.

Let E be a normed space. As in [11] and [12], there is a *maximum multi-norm* based on E; it is denoted by $(\|\cdot\|_n^{\max} : n \in \mathbb{N})$, and is defined by the property that

$$\|\boldsymbol{x}\|_n \le \|\boldsymbol{x}\|_n^{\max} \quad (\boldsymbol{x} \in E^n,\ n \in \mathbb{N})$$

for every multi-norm $(\|\cdot\|_n : n \in \mathbb{N})$ based on E. This multi-norm corresponds to the projective tensor norm $\|\cdot\|_\pi$ on $c_0 \otimes E$ via the above correspondence. By [11, Theorem 3.33], for each $n \in \mathbb{N}$ and $\boldsymbol{x} = (x_1, \dots, x_n) \in E^n$, we have

$$\|\boldsymbol{x}\|_n^{\max} = \sup\Big\{\Big|\sum_{i=1}^n \langle x_i, \lambda_i\rangle\Big| : \lambda_1, \dots, \lambda_n \in E',\ \mu_{1,n}(\lambda_1, \dots, \lambda_n) \le 1\Big\}. \tag{2.2}$$

The rate of growth sequence corresponding to the maximum multi-norm based on E is intrinsic to E; it is denoted by $(\varphi_n^{\max}(E))$. The value of this sequence for various examples is calculated in [11, §3.6]. For example, for $n \in \mathbb{N}$, we have $\varphi_n^{\max}(\ell^p) = n^{1/p}$ for $p \in [1, 2]$ and $\varphi_n^{\max}(\ell^p) = n^{1/2}$ for $p \in [2, \infty]$ [11, Theorem 3.54]. It is shown in [11, Theorem 3.58] that $\sqrt{n} \le \varphi_n^{\max}(E) \le n$ $(n \in \mathbb{N})$ for each infinite-dimensional Banach space E.

Similarly, there is a *minimum multi-norm* $(\|\cdot\|_n^{\min} : n \in \mathbb{N})$ based on a normed space E. As in [11, Definition 3.2], it is defined by

$$\|(x_1, \dots, x_n)\|_n^{\min} = \max_{i \in \mathbb{N}} \|x_i\| \quad (x_1, \dots, x_n \in E).$$

The minimum multi-norm based on E corresponds to the injective tensor norm $\|\cdot\|_\varepsilon$ on $c_0 \otimes E$ in the above correspondence, and so the minimum multi-norm on $c_0 \otimes E$ is the relative norm on $\mathcal{F}(E', c_0)$ from $(\mathcal{B}(E', c_0), \|\cdot\|)$. Of course, the rate of growth sequence of the minimum multi-norm is the constant sequence 1.

2.3. The (p,q)-multi-norm. We recall the definition of the (p,q)-multi-norm based on a normed space E.

Let E be a normed space, and take $p,q \in [1,\infty)$. Following [11, Definition 4.1.1] and [12, §1], for each $n \in \mathbb{N}$ and each $\boldsymbol{x} = (x_1,\dots,x_n) \in E^n$, we define

$$\|\boldsymbol{x}\|_n^{(p,q)} = \sup\Big\{\Big(\sum_{i=1}^n |\langle x_i,\lambda_i\rangle|^q\Big)^{1/q} : \mu_{p,n}(\boldsymbol{\lambda}) \le 1\Big\},$$

where the supremum is taken over all $\boldsymbol{\lambda} = (\lambda_1,\dots,\lambda_n) \in (E')^n$. It is clear that $\|\cdot\|_n^{(p,q)}$ is a norm on E^n. As noted in [11, Theorem 4.1], $(\|\cdot\|_n^{(p,q)} : n \in \mathbb{N})$ is a multi-norm based on E whenever $1 \le p \le q < \infty$.

Definition 2.5. Let E be a normed space, and suppose that $1 \le p \le q < \infty$. Then the multi-norm $(\|\cdot\|_n^{(p,q)} : n \in \mathbb{N})$ described above is the *(p,q)-multi-norm* over E. The corresponding c_0-norm on $c_0 \otimes E$ is $\|\cdot\|^{(p,q)}$.

The rate of growth sequence corresponding to the above (p,q)-multi-norm is denoted by $(\varphi_n^{(p,q)}(E))$, as in [11, Definition 4.2].

We shall use the following remark, from [11, Proposition 4.3]. Let F be a 1-complemented subspace of a Banach space E, and take $x_1,\dots,x_n \in F$. Then the value of $\|(x_1,\dots,x_n)\|_n^{(p,q)}$ is independent of whether it be calculated with respect to F or E.

Let E and F be normed spaces, and take $T \in \mathcal{B}(E,F)$. Then clearly

$$\|(Tx_1,\dots,Tx_n)\|_n^{(p,q)} \le \|T\|\|(x_1,\dots,x_n)\|_n^{(p,q)} \quad (x_1,\dots,x_n \in E,\ n \in \mathbb{N}).$$

The following theorem refers to *multi-bounded sets* in and *multi-bounded operators* on multi-normed spaces; for background information, see [12], and, in more detail, [11, Chapter 6]. For example, the *multi-bound* of a multi-bounded set B is defined by

$$c_B = \sup_{n\in\mathbb{N}}\{\|(x_1,\dots,x_n)\|_n : x_1,\dots,x_n \in B\}.$$

Theorem 2.6. *Let E be a normed space, and suppose that $1 \le p \le q < \infty$. Then the (p,q)-multi-norm induces the norm on $c_0 \otimes E$ given by embedding $c_0 \otimes E$ into $\Pi_{q,p}(E',c_0)$. This norm is a uniform c_0-norm on $c_0 \otimes E$.*

Proof. We start by observing that [12, Theorem 4.2] shows that the ℓ^1-norm (that is, the dual multi-norm) on $\ell^1 \otimes E'$ norms $c_0 \otimes E$. The converse is also true, so that we have $\ell^1 \otimes E' \subset (c_0 \otimes E)'$, and the embedding is an isometry.

In [12, Definition 5.4], we defined $\mathcal{B}_{p,q}(\ell^1,E)$ to be the set of those $T \in \mathcal{B}(\ell^1,E)$ which are multi-bounded when we take the minimum multi-norm based on ℓ^1 and the (p,q)-multi-norm based on E. The norm on the space $\mathcal{B}_{p,q}(\ell^1,E)$ is denoted by $\alpha_{p,q}$, so that $\alpha_{p,q}(T)$ is equal to the multi-bound c_B of the set $B := \{T(\delta_k) : k \in \mathbb{N}\}$. Thus the natural inclusion of $c_0 \otimes E$ into $\mathcal{B}_{p,q}(\ell^1,E)$ (where we identify $c_0 \otimes E$ with $\mathcal{F}(\ell^1,E)$) induces the (p,q)-multi-norm on $c_0 \otimes E$. It follows from [12, Proposition 5.5] that T belongs to $\mathcal{B}_{p,q}(\ell^1,E)$ if and only if the dual operator T' belongs to $\Pi_{q,p}(E',\ell^\infty)$, with equal norms. The combination of these two results immediately gives the result.

It remains to show that the resulting norm is a uniform c_0-norm. Let $T \in \mathcal{B}(E,F)$, and consider the operator $I \otimes T : c_0 \otimes E \to c_0 \otimes F$. It is easy to see that we have the following commutative diagram:

$$\begin{tikzcd} c_0 \otimes E \arrow[r, "I\otimes T"] \arrow[d] & c_0 \otimes F \arrow[d] \\ \Pi_{q,p}(E', c_0) \arrow[r, "\varphi"] & \Pi_{q,p}(F', c_0) \end{tikzcd}$$

Here φ is the map $S \mapsto S \circ T'$. Since the vertical arrows are isometries, it suffices to show that $\|\varphi\| \le \|T\| = \|T'\|$. But this follows immediately from properties of (q,p)-summing maps; see [14, Proposition 10.2]. ∎

REMARK 2.7. A refinement of the above argument shows that, for each normed space E, the (p,q)-multi-norm based on E' induces the norm on $c_0 \otimes E'$ given by embedding $c_0 \otimes E'$ into $\Pi_{q,p}(E, c_0)$.

It follows immediately from Theorem 2.6 and the closed graph theorem that the (p_1,q_1)- and (p_2,q_2)-multi-norms are equivalent on E whenever

$$\Pi_{q_1,p_1}(E', c_0) = \Pi_{q_2,p_2}(E', c_0);$$

moreover, c_0 can be replaced by any infinite-dimensional $C(K)$-space. The converse is also true; this is a special case of the following theorem.

THEOREM 2.8. *Let E be a Banach space, and take p_1, q_1, p_2, q_2 such that $1 \le p_1 \le q_1 < \infty$ and $1 \le p_2 \le q_2 < \infty$. Suppose that the (p_1,q_1)- and (p_2,q_2)-multi-norms are mutually equivalent on E. Then*

$$\Pi_{q_1,p_1}(E', F) = \Pi_{q_2,p_2}(E', F)$$

for every Banach space F.

Proof. Let F be a Banach space. It is standard that there is an isometry $\varphi : F \to \ell^\infty(I)$ for some index set I. For each finite subset $A \subset I$, let $P_A : \ell^\infty(I) \to c_0$ be the projection map

$$\ell^\infty(I) \to \ell^\infty(A) \subset c_0.$$

Assume towards a contradiction that we have $\Pi_{q_1,p_1}(E', F) \not\subset \Pi_{q_2,p_2}(E', F)$, and take $T \in \Pi_{q_1,p_1}(E', F) \setminus \Pi_{q_2,p_2}(E', F)$. From the definition of the (q,p)-summing norm, we see that

$$\pi_{q,p}(T) = \pi_{q,p}(\varphi \circ T) = \sup_A \pi_{q,p}(P_A \circ \varphi \circ T);$$

here we take the supremum over all finite subsets $A \subset I$. Hence there exists a sequence (A_n) of finite subsets of I such that

$$n \cdot \pi_{q_1,p_1}(T_n) < \pi_{q_2,p_2}(T_n) \quad (n \in \mathbb{N}),$$

where $T_n := P_{A_n} \circ \varphi \circ T : E' \to \ell^\infty(A_n) \subset c_0$.

Take $n \in \mathbb{N}$. Since $T_n \in \mathcal{F}(E', c_0)$, the operator T_n is induced by a tensor $\tau_n \in c_0 \otimes E''$. Remark 2.7 and the previous paragraph then show that

$$n \cdot \|\tau_n\|^{(p_1,q_1)}_{c_0 \otimes E''} = n \cdot \pi_{q_1,p_1}(T_n) < \pi_{q_2,p_2}(T_n) = \|\tau_n\|^{(p_2,q_2)}_{c_0 \otimes E''}.$$

In fact, since A_n is finite, the tensor τ_n can be identified with an element $\boldsymbol{x}_n \in (E'')^{m(n)}$ for some $m(n) \in \mathbb{N}$. Thus, this shows that the identity operator

$$((E'')^{m(n)}, \|\cdot\|_{m(n)}^{(p_1,q_1)}) \to ((E'')^{m(n)}, \|\cdot\|_{m(n)}^{(p_2,q_2)})$$

has norm at least n. By [11, Corollary 4.14], it follows that the identity operator

$$(E^{m(n)}, \|\cdot\|_{m(n)}^{(p_1,q_1)}) \to (E^{m(n)}, \|\cdot\|_{m(n)}^{(p_2,q_2)})$$

has norm at least n. This is true for every $n \in \mathbb{N}$. But this contradicts the assumption that the (p_1, q_1)- and the (p_2, q_2)-multi-norms are equivalent on E. ∎

Corollary 2.9. *Let E be a Banach space, and suppose that $1 \le p_1 \le q_1 < \infty$ and $1 \le p_2 \le q_2 < \infty$. Then the following are equivalent:*

(a) $(\|\cdot\|_n^{(p_1,q_1)} : n \in \mathbb{N}) \cong (\|\cdot\|_n^{(p_2,q_2)} : n \in \mathbb{N})$ *on E;*
(b) $\Pi_{q_1,p_1}(E', c_0) = \Pi_{q_2,p_2}(E', c_0)$. ∎

2.4. The (p,p)-multi-norm. We now give another description of the (p,p)-multi-norm.

Theorem 2.10. *Let E be a normed space. Then the tensor norm on $c_0 \otimes E$ induced from the (p,p)-multi-norm is the Chevet–Saphar norm d_p on $c_0 \otimes E$.*

Proof. By Theorem 2.6, the embedding of $c_0 \otimes E$ into $\Pi_p(E', c_0)$ induces the (p,p)-multi-norm. By Proposition 1.5, $\Pi_p(E', c_0)$ agrees isometrically with the class of p-integral maps from E' to c_0. By definition, the p-integral norm, $i_p(T)$, of a map $T : E' \to c_0$ is the norm of the induced functional on $E' \widehat{\otimes}_{g'_p} \ell^1 = \ell^1 \widehat{\otimes}_{d'_p} E'$. Hence the natural map

$$(c_0 \otimes E, \|\cdot\|^{(p,p)}) \to (\ell^1 \widehat{\otimes}_{d'_p} E')'$$

is an isometry. Since c_0 has the metric approximation property and d_p is an accessible tensor norm, as explained in the introduction, it follows that the $\|\cdot\|^{(p,p)}$-norm on $c_0 \otimes E$ is just the d_p norm, as claimed. ∎

Thus we have another description of the (p,p)-multi-norm based on a normed space E. The value of this result is that it gives an excellent description of the dual space to $(c_0 \otimes E, \|\cdot\|^{(p,p)})$, namely as

$$(c_0 \widehat{\otimes}_{d_p} E)' \cong \Pi_{p'}(c_0, E'),$$

the collection of p'-summing maps from c_0 to E'; see [24, Proposition 6.11]. The maps in $\Pi_{p'}(c_0, E')$ are usually rather well understood.

In the general case where $q \ge p$, we can give an abstract description of the dual space of $(c_0 \otimes E, \|\cdot\|^{(p,q)})$, as [11, §4.1.4], but we lack a good concrete description of this dual space, and this means that we are unable to adapt the arguments of this section to the more general case.

2.5. Relations between (p,q)-multi-norms. Let E be a normed space, and consider the above (p,q)-multi-norms based on E, where $1 \le p \le q < \infty$. It is clear that, for each fixed $p \ge 1$ and $q_1 \ge q_2 \ge p$, we have $(\|\cdot\|_n^{(p,q_1)}) \le (\|\cdot\|_n^{(p,q_2)})$, and, for each fixed $q \ge 1$ and $p_1 \le p_2 \le q$, we have $(\|\cdot\|_n^{(p_1,q)}) \le (\|\cdot\|_n^{(p_2,q)})$. Further, it is proved in [11, Theorem 4.4] that $(\|\cdot\|_n^{(p,p)}) \geq (\|\cdot\|_n^{(q,q)})$ whenever $1 \leq p \leq q < \infty$, and so $(\|\cdot\|_n^{(1,1)})$ is the maximum among these multi-norms; by (2.2), it is the maximum multi-norm.

The following theorem, which follows immediately from Theorem 2.6 and the analogous result for (q,p)-summing operators that is given in [14, Theorem 10.4], for example, gives more information about the relations between (p,q)-multinorms.

To picture the theorem, consider the following. We write $\mathcal{T}$ for the extended 'triangle' given by

$$\mathcal{T} = \{(p,q) \in \mathbb{R}^2 : 1 \leq p \leq q\},$$

and, for $c \in [0,1)$, we consider the curve

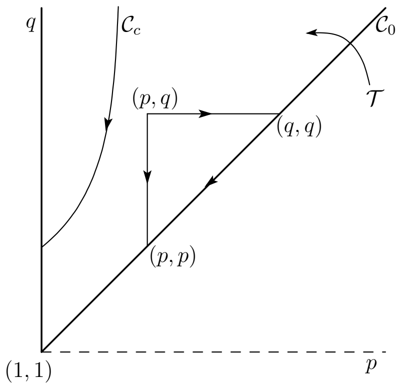
$$\mathcal{C}_c = \{(p,q) \in \mathcal{T} : 1/p - 1/q = c\};$$

we have

$$\mathcal{T} = \bigcup \{\mathcal{C}_c : c \in [0,1)\}.$$

Then the multi-norm $(\|\cdot\|_n^{(p,q)})$ increases as we move down a fixed curve $\mathcal{C}_c$, and it increases when we move to a lower point on a curve to the right.

In the diagram, arrows indicate increasing multi-norms in the ordering $\leq$.



THEOREM 2.11. *Let E be a normed space, and take (p_1,q_1) and (p_2,q_2) in $\mathcal{T}$. Then $(\|\cdot\|_n^{(p_1,q_1)}) \leq (\|\cdot\|_n^{(p_2,q_2)})$ whenever both $1/p_2 - 1/q_2 \leq 1/p_1 - 1/q_1$ and $q_2 \leq q_1$.* ∎

It is easy to see that $\varphi_n^{(p,q)}(E) = \pi_{q,p}^{(n)}(E')$ for each normed space E and each $n \in \mathbb{N}$ [11, Theorem 4.4], and so the following result is immediate from Proposition 1.3.

PROPOSITION 2.12. *Suppose that (p,q) is in $\mathcal{T}$ and that $n \in \mathbb{N}$. Then:*

(i) $\varphi_n^{(p,q)}(E) \leq n^{1/q}$ *for each normed space E;*
(ii) $\varphi_n^{(p,q)}(E) = n^{1/q}$ *for each infinite-dimensional normed space E whenever $p \geq 2$;*
(iii) $\varphi_n^{(p,q)}(E) \geq n^{1/2-1/p+1/q}$ *for each infinite-dimensional normed space E whenever $p \leq 2$;*
(iv) $\varphi_n^{(p,q)}(\ell^r) = n^{1/q}$ *whenever $r \geq 1$ and $p \geq \min\{r,2\}$.* ∎

In Theorem 3.10, we shall improve clause (iv) of the above proposition by giving (asymptotic) values of $\varphi_n^{(p,q)}(\ell^r)$ for all values of p and q with $1 \leq p \leq q < \infty$ in the case where $r > 1$.

COROLLARY 2.13. *Let E be an infinite-dimensional Banach space, and take (p_1, q_1) and (p_2, q_2) in $\mathcal{T}$. Then the (p_1, q_1)- and (p_2, q_2)-multi-norms based on E are not equivalent whenever $p_1, p_2 \geq 2$ and $q_1 \neq q_2$.* ■

Combining the previous proposition with Theorem 2.11, we obtain the following.

COROLLARY 2.14. *Let E be an infinite-dimensional Banach space. Suppose that (p, q) is in $\mathcal{T}$.*

(i) *The (p, q)- and the maximum multi-norms based on E are not equivalent whenever $q > 2$.*
(ii) *The (p, q)- and the minimum multi-norms based on E are not equivalent whenever $1/p - 1/q < 1/2$.*

Proof. (i) Take $p_1 \in (2, q)$. Then

$$(\|\cdot\|_n^{(p,q)}) \leq (\|\cdot\|_n^{(p_1,p_1)}) \leq (\|\cdot\|_n^{(2,2)}).$$

However, $(\|\cdot\|_n^{(p_1,p_1)}) \not\cong (\|\cdot\|_n^{(2,2)})$ on E by Proposition 2.12(ii), and so this implies that $(\|\cdot\|_n^{(p,q)}) \not\cong (\|\cdot\|_n^{\max})$ on E.

(ii) This follows from Proposition 2.12. ■

We shall compare the (p, q)-multi-norms on $L^r(\Omega)$, and, when $r > 1$, we shall compute $\varphi_n^{(p,q)}(\ell^r)$ asymptotically for all other values of p and q later. For these calculations, we shall need to use the following proposition, which is an immediate consequence of Proposition 1.4.

PROPOSITION 2.15. *Let E be a normed space. Suppose that $1 \leq p \leq q_1 < q < q_2 < \infty$, so that*

$$\frac{1}{q} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}$$

for some $\theta \in (0, 1)$. Then

$$\varphi_n^{(p,q)}(E) \leq (\varphi_n^{(p,q_1)}(E))^{1-\theta} \cdot (\varphi_n^{(p,q_2)}(E))^{\theta} \qquad (n \in \mathbb{N}).$$ ■

The following calculations of some specific (p, q)-multi-norms will also be useful.

EXAMPLE 2.16. Let $r \geq 1$, set $s = r'$, and take $(p, q) \in \mathcal{T}$. Then

$$\|(\delta_1, \ldots, \delta_n)\|_n^{(p,q)} = \Big\|\sum_{i=1}^{n} e_i \otimes \delta_i\Big\|_{c_0 \otimes \ell^r} = \Big\|\sum_{i=1}^{n} e_i \otimes \delta_i\Big\|_{\Pi_{q,p}(\ell^s, c_0)} = \pi_{q,p}(I_n)$$

for each $n \in \mathbb{N}$, where I_n is the formal identity map from ℓ_n^s to ℓ_n^∞. Here we are now writing (δ_i) and (e_i) for the standard bases in ℓ^r and c_0, respectively.

The value of $\|(\delta_1, \ldots, \delta_n)\|_n^{(p,q)}$ based on the Banach space ℓ^r is calculated for certain values of p and q in [11, Example 4.8]. We now calculate this value for all $(p, q) \in \mathcal{T}$ by elementary means.

Fix $n \in \mathbb{N}$, and, for $(p, q) \in \mathcal{T}$, write

$$\Delta_n(p, q) = \|(\delta_1, \ldots, \delta_n)\|_n^{(p,q)}. \tag{2.3}$$

Set $s = r'$ and $u = p'$. Then

$$\Delta_n(p,q) = \sup\Big\{ \Big(\sum_{i=1}^n |\lambda_{i,i}|^q\Big)^{1/q} : \lambda_1,\dots,\lambda_n \in \ell_n^s,\ \mu_{p,n}(\lambda_1,\dots,\lambda_n) \le 1\Big\},$$

and so, using (1.4), we see that

$$\Delta_n(p,q) = \sup\Big\{ \Big(\sum_{i=1}^n |\lambda_{i,i}|^q\Big)^{1/q} : (\lambda_{i,j})_{i,j=1}^n \in \mathcal{B}(\ell_n^u, \ell_n^s)_{[1]}\Big\}. \tag{2.4}$$

We now use [21, Proposition 1.c.8], which states the following: Suppose that a matrix $(\lambda_{i,j})_{i,j=1}^n$ defines a contraction from ℓ_n^u to ℓ_n^s. Then the 'diagonal' operator obtained by setting all the off-diagonal terms of our matrix to 0 also defines a contraction between the same spaces. As the sum in (2.4) involves only the terms $\lambda_{i,j}$ with $j = i$, we see that we can make this change without changing the value of $\Delta_n(p,q)$, and thus we can say that

$$\Delta_n(p,q) = \sup\{\|\alpha\|_q : D_\alpha \in \mathcal{B}(\ell_n^u, \ell_n^s)_{[1]}\},$$

where $D_\alpha x = (\alpha_1 x_1, \dots, \alpha_n x_n)$ for each $\alpha, x \in \mathbb{C}^n$.

We *claim* that

$$\Delta_n(p,q) = \begin{cases} n^{1/q} & \text{when } u \le s, \\ n^{1/q+1/u-1/s} & \text{when } u > s \text{ and } 1/q + 1/u \ge 1/s, \\ 1 & \text{when } 1/q + 1/u < 1/s. \end{cases}$$

Indeed, suppose first that $u > s$. Then there exists $t > 1$ such that $1/s = 1/u + 1/t$. Let $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{C}^n$. A version of Hölder's inequality implies that

$$\Big(\sum_{i=1}^n |\alpha_i x_i|^s\Big)^{1/s} \le \Big(\sum_{i=1}^n |\alpha_i|^t\Big)^{1/t} \Big(\sum_{i=1}^n |x_i|^u\Big)^{1/u}$$

for every $(x_i) \in \ell_n^u$. Moreover, equality is attained for a suitable choice of (x_i), and so we see that

$$\|D_\alpha : \ell_n^u \to \ell_n^s\| = \|\alpha\|_t \qquad (\alpha \in \mathbb{C}^n).$$

Thus the problem now is to compute

$$\Delta_n(p,q) = \sup\{\|\alpha\|_q : \alpha \in (\ell_n^t)_{[1]}\}.$$

If $t > q$, the supremum occurs when $\alpha = (\alpha_i)$ is the constant sequence $(n^{-1/t})$, in which case we obtain

$$\Big(\sum_{i=1}^n |\alpha_i|^q\Big)^{1/q} = (n \cdot n^{-q/t})^{1/q} = n^{1/q-1/t} = n^{1/q+1/u-1/s}.$$

If $t \le q$, the supremum occurs at a point mass, in which case we obtain $\|\alpha\|_q = 1$.

Finally, suppose that $u \le s$. Then we see that

$$\Big(\sum_{i=1}^n |\alpha_i x_i|^s\Big)^{1/s} \le \Big(\sum_{i=1}^n |\alpha_i x_i|^u\Big)^{1/u} \le \|\alpha\|_\infty \Big(\sum_{i=1}^n |x_i|^u\Big)^{1/u},$$

and equality occurs when $(x_1, \dots, x_n)$ is a point mass. Thus

$$\|D_\alpha : \ell_n^u \to \ell_n^s\| = \|\alpha\|_\infty \qquad (\alpha \in \mathbb{C}^n).$$

It follows that

$$\Delta_n(p,q) = \sup\{\|\alpha\|_q : \alpha \in (\ell_n^\infty)_{[1]}\} = n^{1/q}.$$

This establishes the claim.

We conclude as follows. Suppose that $r \geq 1$ and $(p,q) \in \mathcal{T}$. Then the (p,q)-multi-norm based on ℓ^r satisfies the following equation for each $n \in \mathbb{N}$:

$$\|(\delta_1,\dots,\delta_n)\|_n^{(p,q)} = \begin{cases} n^{1/r+1/q-1/p} & \text{when } p < r \text{ and } 1/p - 1/q \leq 1/r, \\ 1 & \text{when } 1/p - 1/q > 1/r, \\ n^{1/q} & \text{when } p \geq r. \end{cases} \tag{2.5}$$

We can also write the above formula more concisely as follows:

$$\|(\delta_1,\dots,\delta_n)\|_n^{(p,q)} = n^\alpha \qquad (n \in \mathbb{N}),$$

where

$$\alpha = \left(\frac{1}{q} - \left(\frac{1}{p} - \frac{1}{r}\right)^+\right)^+.$$

Here, $x^+ = \max\{x, 0\}$ for each $x \in \mathbb{R}$. ■

Example 2.17. Suppose that $r \geq 1$ and $(p,q) \in \mathcal{T}$. Set $s = r'$ and $u = p'$, as before.

Fix $n \in \mathbb{N}$. For $i \in \mathbb{N}_n$, take

$$f_i = \frac{1}{n^{1/r}} \sum_{j=1}^n \zeta^{-ij}\delta_j = \frac{1}{n^{1/r}}(\zeta^{-i}, \zeta^{-2i}, \dots, \zeta^{-ni}, 0, 0, \dots) \in \ell^r,$$

where $\zeta = \exp(2\pi \mathrm{i}/n)$, so that $\|f_i\|_{\ell^r} = 1$ $(i \in \mathbb{N}_n)$, and then set $\boldsymbol{f} = (f_1, \dots, f_n)$. Next take $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n)$, where

$$\lambda_i = \sum_{j=1}^n \zeta^{ij}\delta_j = (\zeta^i, \zeta^{2i}, \dots, \zeta^{ni}, 0, 0, \dots) \in \ell^s.$$

Note that

$$\left(\sum_{i=1}^n |\langle f_i, \lambda_i\rangle|^q\right)^{1/q} = n^{1+1/q-1/r}. \tag{2.6}$$

We take $\zeta_1, \dots, \zeta_n \in \mathbb{C}$ with $\sum_{i=1}^n |\zeta_i|^u \leq 1$, and set $z_i = \sum_{j=1}^n \zeta_j \zeta^{ij}$ $(i \in \mathbb{N}_n)$, so that

$$\sum_{i=1}^n |z_i|^2 = n \sum_{i=1}^n |\zeta_i|^2 \quad \text{and} \quad \left\|\sum_{i=1}^n \zeta_i \lambda_i\right\|_{\ell^s} = \left(\sum_{i=1}^n |z_i|^s\right)^{1/s}.$$

Now suppose that $r \geq 2$, so that $1 \leq s \leq 2$.

In the case where $p \geq 2$, so that $u \leq 2$, we have $\sum_{i=1}^n |\zeta_i|^2 \leq \sum_{i=1}^n |\zeta_i|^u \leq 1$, and so

$$\left\|\sum_{j=1}^n \zeta_j \lambda_j\right\|_{\ell^s} = \|(z_1, \dots, z_n)\|_{\ell^s} = \left(\sum_{j=1}^n |z_j|^s\right)^{1/s}.$$

Hence, by (1.2),

$$\mu_{p,n}(\boldsymbol{\lambda}) \leq \frac{n^{1/s}}{n^{1/2}} \left(\sum_{i=1}^n |z_i|^2\right)^{1/2} \leq n^{1/s}.$$

It follows from (2.6) that

$$\|\boldsymbol{f}\|_n^{(p,q)} \geq \frac{n^{1+1/q}}{n^{1/r+1/s}} = n^{1/q}.$$

In the case where $1 \leq p \leq 2$, so that $u \geq 2$, we have

$$\Big(\sum_{i=1}^n |\zeta_i|^2\Big)^{1/2} \leq \frac{n^{1/2}}{n^{1/u}}\Big(\sum_{i=1}^n |\zeta_i|^u\Big)^{1/u} \leq n^{1/2-1/u},$$

and so

$$\mu_{p,n}(\boldsymbol{\lambda}) \leq \frac{n^{1/s}}{n^{1/2}}\Big(\sum_{i=1}^n |z_i|^2\Big)^{1/2} \leq n^{1/2+1/s-1/u}.$$

Hence, again by (2.6),

$$\|\boldsymbol{f}\|_n^{(p,q)} \geq \frac{n^{1+1/q+1/u}}{n^{1/2+1/s+1/r}} = n^{1/2-1/p+1/q}.$$

It is always true that $\|\boldsymbol{f}\|_n^{(p,q)} \geq 1$.

We conclude that, in the case where $r \geq 2$, we have the following estimates, which hold for each $n \in \mathbb{N}$:

$$\|\boldsymbol{f}\|_n^{(p,q)} \geq \begin{cases} n^{1/2-1/p+1/q} & \text{when } 1 \leq p \leq 2 \text{ and } 1/p - 1/q \leq 1/2, \\ 1 & \text{when } 1/p - 1/q > 1/2, \\ n^{1/q} & \text{when } p \geq 2. \end{cases} \tag{2.7}$$

We shall see from Theorem 3.10, given below, that $\|\boldsymbol{f}\|_n^{(p,q)}$ is always equal to the term on the right-hand side of (2.7) to within a constant independent of n. ∎

2.6. The standard t-multi-norm on L^r-spaces. Let (Ω, μ) be a measure space, take $r \geq 1$, and suppose that $r \leq t < \infty$. In [11, §4.2] and [12, §6], there is a definition and discussion of the standard t-multi-norm on the Banach space $L^r(\Omega)$. We recall the definition.

Take $n \in \mathbb{N}$. For each ordered partition $\boldsymbol{X} = (X_1, \dots, X_n)$ of Ω into measurable subsets and each $f_1, \dots, f_n \in L^r(\Omega)$, we define

$$r_{\boldsymbol{X}}((f_1, \dots, f_n)) = \Big(\sum_{i=1}^n \|P_{X_i} f_i\|^t\Big)^{1/t}.$$

Here $P_{X_i} : f \mapsto f|X_i$ is the projection of $L^r(\Omega)$ onto $L^r(X_i)$, and $\|\cdot\|$ is the L^r-norm. Then we define

$$\|(f_1, \dots, f_n)\|_n^{[t]} = \sup_{\boldsymbol{X}} r_{\boldsymbol{X}}((f_1, \dots, f_n)),$$

where the supremum is taken over all such measurable ordered partitions $\boldsymbol{X}$.

As in [11, §4.2.1], we see that $(\|\cdot\|_n^{[t]} : n \in \mathbb{N})$ is a multi-norm based on $L^r(\Omega)$; it is the *standard t-multi-norm* on $L^r(\Omega)$.

Clearly the norms $\|\cdot\|_n^{[t]}$ decrease as a function of $t \in [r, \infty)$, and so the maximum among these norms is $\|\cdot\|_n^{[r]}$.

For example, by [11, (4.9)], we have

$$\|(f_1,\dots,f_n)\|_n^{[t]} = (\|f_1\|^t + \cdots + \|f_n\|^t)^{1/t} \quad (n \in \mathbb{N})$$

whenever $f_1,\dots,f_n$ in $L^r(\Omega)$ have pairwise disjoint supports, and, in particular,

$$\|(\delta_1,\dots,\delta_n)\|_n^{[t]} = n^{1/t} \quad (n \in \mathbb{N}).$$

As remarked in [11], it appears that the definition of the standard t-multi-norm depends on the concrete representation of the space $L^r(\Omega)$. However, in [11, §4.2.8], there is a definition of an 'abstract t-multi-norm based on a σ-Dedekind complete Banach lattice E', and it is shown in [11, Theorem 4.36] that the standard t-multi-norm based on a Banach lattice $L^r(\Omega)$ depends on only the norm and the lattice structure of the space. For a related result, see [11, Theorem 4.40].

The rate of growth of the standard t-multi-norm based on $L^r(\Omega)$ is denoted by $\varphi_n^{[t]}(L^r(\Omega))$, as in [11, Definition 4.21]. In fact, it is easily seen that

$$\varphi_n^{[t]}(L^r(\Omega)) = n^{1/t} \tag{2.8}$$

for every infinite-dimensional $L^r(\Omega)$-space.

In the special case where $t = r$, we have

$$\|(f_1,\dots,f_n)\|_n^{[r]} = \left(\int_\Omega (|f_1| \vee \cdots \vee |f_n|)^r\right)^{1/r} \tag{2.9}$$

for $n \in \mathbb{N}$ and elements $f_1,\dots,f_n \in L^r(\Omega)$; this is equation (4.12) in [11].

For a Banach space E and $r \geq 1$, the space $L^r(\Omega, E)$ consists of (equivalence classes of) μ-measurable functions $F : \Omega \to E$ such that the function $s \mapsto \|F(s)\|$ on Ω belongs to the space $L^p(\Omega)$; with respect to the norm $\|\cdot\|$ specified by

$$\|F\| = \left(\int_\Omega \|F(s)\|^r \,\mathrm{d}\mu(s)\right)^{1/r} \quad (F \in L^r(\Omega, E)),$$

the space $L^r(\Omega, E)$ is a Banach space. The tensor product $L^r(\Omega) \otimes E$ can be identified with a dense subspace of $L^r(\Omega, E)$. Indeed, $f \otimes x \in L^r(\Omega) \otimes E$ corresponds to the function $s \mapsto f(s)x$. See [13, Chapter 7] and [24, §2.3].

Now take $n \in \mathbb{N}$ and $f_1,\dots,f_n \in L^r(\Omega)$. Then $(f_1,\dots,f_n)$ corresponds to the element $\sum_{i=1}^n \delta_i \otimes f_i$ in $c_0 \otimes L^r(\Omega)$, and hence to the function

$$s \mapsto \sum_{i=1}^n f_i(s)\delta_i \in L^r(\Omega, c_0),$$

and its norm in $L^r(\Omega, c_0)$ is exactly $\|(f_1,\dots,f_n)\|_n^{[r]}$ by equation (2.9).

Thus, in the case where $t = r$, we can regard the standard r-multi-norm on $L^r(\Omega)$ simply as that given by the embedding of $c_0 \otimes L^r(\Omega)$ in $L^r(\Omega, c_0)$.

There seems to be no similarly useful representation of the standard t-multi-norm on $L^r(\Omega)$ in the case where $t > r$.

2.7. The Hilbert multi-norm. We now recall an alternative description of the $(2,2)$-multi-norm based on a Hilbert space. This involves the Hilbert multi-norm that was introduced in [11, §4.1.5].

Let H be a Hilbert space, with inner-product denoted by $[\,\cdot\,,\,\cdot\,]$. For $n \in \mathbb{N}$, we write

$$H = H_1 \oplus_\perp \cdots \oplus_\perp H_n$$

when $H_1, \ldots, H_n$ are pairwise-orthogonal (closed) subspaces of H.

Take $n \in \mathbb{N}$. For each family $\boldsymbol{H} = \{H_1, \ldots, H_n\}$ such that $H = H_1 \oplus_\perp \cdots \oplus_\perp H_n$, we set

$$r_{\boldsymbol{H}}((x_1, \ldots, x_n)) = (\|P_1x_1\|^2 + \cdots + \|P_nx_n\|^2)^{1/2} = \|P_1x_1 + \cdots + P_nx_n\|$$

for $x_1, \ldots, x_n \in H$, where $P_i : H \to H_i$ is the orthogonal projection for $i \in \mathbb{N}_n$. Then we set

$$\|\boldsymbol{x}\|_n^H = \sup_{\boldsymbol{H}} r_{\boldsymbol{H}}(\boldsymbol{x}) \qquad (\boldsymbol{x} \in H^n),$$

where the supremum is taken over all orthogonal decompositions $\boldsymbol{H}$ of H. As in [11, Theorem 4.15], $(\|\cdot\|_n^H : n \in \mathbb{N})$ is a multi-norm based on H; it is called the *Hilbert multi-norm*.

The following result is [11, Theorem 4.19].

THEOREM 2.18. *Let H be a Hilbert space. Then $(\|\cdot\|_n^H) = (\|\cdot\|_n^{(2,2)})$.* ■

2.8. Relations between multi-norms. In this subsection, we shall first summarize some results about the relationships between multi-norms that were already established in [11].

THEOREM 2.19. *Let E be a normed space. Then $(\|\cdot\|_n^{(1,1)}) = (\|\cdot\|_n^{\max})$.*

Proof. This is [11, Theorem 4.6]. ■

THEOREM 2.20. *Take r, t with $1 \le r \le t < \infty$, and let Ω be a measure space. Then*

$$(\|\cdot\|_n^{[t]}) \le (\|\cdot\|_n^{(r,t)}) \qquad \text{on } L^r(\Omega).$$

Moreover, when $r = 1$, these two multi-norms are equal on $L^1(\Omega)$ whenever $t \in [1, \infty)$. Further, $(\|\cdot\|_n^{[1]}) = (\|\cdot\|_n^{\max})$ on $L^1(\Omega)$.

Proof. This combines [11, Theorems 4.22, 4.23, and 4.26]. ■

By (2.8), different standard t-multi-norms on an infinite-dimensional $L^r(\Omega)$ space are not equivalent to each other, and they are never equivalent to the minimum multi-norm; we shall see in Theorem 3.22 that they are never equivalent to the maximum multi-norm.

THEOREM 2.21. *Take $r \ge 1$, and suppose that $r \le t < \infty$. Suppose that either $2 \le r \le t$ or that $1 < r < 2$ and $r \le t < r/(2-r)$. Then the multi-norms $(\|\cdot\|_n^{[t]} : n \in \mathbb{N})$ and $(\|\cdot\|_n^{(r,t)} : n \in \mathbb{N})$ based on ℓ^r are not equivalent.*

Proof. This is [11, Theorem 4.27]. ■

We shall extend and complement the above results in the present memoir.

3. Comparing (p,q)-multi-norms on L^r spaces

In this section, we aspire to determine when two (p,q)-multi-norms based on a space $L^r(\Omega)$ are equivalent; we shall obtain a reasonably complete classification, but cannot give a fully comprehensive account.

3.1. The case where $r = 1$. In this section, we investigate the equivalence of various (p,q)-multi-norms on spaces of the form $L^1(\Omega)$.

By Example 2.16, $(\|\cdot\|_n^{(p_1,q_1)})$ is not equivalent to $(\|\cdot\|_n^{(p_2,q_2)})$ on $L^1(\Omega)$ whenever $L^1(\Omega)$ is infinite-dimensional and $q_1 \neq q_2$ because $\Delta_n(p,q) = n^{1/q}$ $(n \in \mathbb{N})$ for each $(p,q) \in \mathcal{T}$, in the notation of that example; it remains to investigate the case where $q_1 = q_2$.

The following result is [12, Theorem 5.6]. It is also a consequence of Theorem 2.6 and the corresponding result in [23, Corollary 2.5] (see also [14, Theorem 10.9]).

THEOREM 3.1. *Let Ω be a measure space, and take $p,q,s \in \mathbb{R}$ with $1 \leq p < q < s < \infty$. Then*

$$(\|\cdot\|_n^{(p,q)}) \cong (\|\cdot\|_n^{(1,q)}) \not\succcurlyeq (\|\cdot\|_n^{(s,s)}) \quad \text{on } L^1(\Omega). \ \blacksquare$$

The following result shows that the condition '$p < q$' in the above theorem is sharp. Note also that

$$\|(\delta_1,\dots,\delta_n)\|_n^{(q,q)} = \|(\delta_1,\dots,\delta_n)\|_n^{(1,q)} \ \ (= n^{1/q}) \quad (n \in \mathbb{N}),$$

for $q \geq 1$, and so the above equation is not sufficient to enforce the non-equivalence of $(\|\cdot\|_n^{(q,q)})$ and $(\|\cdot\|_n^{(1,q)})$.

THEOREM 3.2. *Let Ω be a measure space such that $L^1(\Omega)$ is infinite-dimensional. Take $q > 1$. Then $(\|\cdot\|_n^{(q,q)}) \geq (\|\cdot\|_n^{(1,q)})$, but $(\|\cdot\|_n^{(q,q)}) \not\cong (\|\cdot\|_n^{(1,q)})$ on $L^1(\Omega)$.*

Proof. First, suppose that our multi-norms are based on ℓ^1.

Take $n \in \mathbb{N}$, and let I_n be the identity map from ℓ_n^∞ to the Lorentz space $\ell_n^{q,1}$. A calculation of Montgomery-Smith [22] (see [9] for a statement of this example) shows that

$$\pi_{q,q}(I_n) \sim n^{1/q}(1+\log n)^{1-1/q}, \qquad \pi_{q,1}(I_n) \sim n^{1/q}.$$

For each $n \in \mathbb{N}$, we can find $m = m(n) \in \mathbb{N}$, with $m(n) \geq n$, and an operator $\varphi_n : \ell_n^{q,1} \to \ell_{m(n)}^\infty$ with

$$(1-1/n)\|x\|_{q,1} \leq \|\varphi_n(x)\|_\infty \leq \|x\|_{q,1} \quad (x \in \ell_n^{q,1}).$$

Let $p_n : \ell^\infty \to \ell_n^\infty$ be the natural projection, and define

$$T_n = \frac{1}{n^{1/q}}\varphi_n \circ I_n \circ p_n : \ell^\infty \to \ell_{m(n)}^\infty \subset c_0.$$

From the definition of the (q,p)-summing norm, it follows that

$$(1-1/n)\frac{1}{n^{1/q}}\pi_{q,p}(I_n) \leq \pi_{q,p}(T_n) \leq \frac{1}{n^{1/q}}\pi_{q,p}(I_n)$$

whenever $1 \leq p \leq q < \infty$. In particular, $\pi_{q,1}(T_n) \sim 1$, but $\pi_{q,q}(T_n) \sim (1+\log n)^{1-1/q}$.

Since $T_n = T_n \circ p_n$, we see that T_n is the image of

$$\boldsymbol{x}_n := \sum_{i=1}^{n} T_n(e_i) \otimes \delta_i$$

via the natural inclusion $c_0 \otimes \ell^1 \hookrightarrow \mathcal{B}(\ell^\infty, c_0)$. The previous paragraph and Theorem 2.6 imply that

$$\|\boldsymbol{x}_n\|^{(q,1)}_{c_0 \otimes \ell^1} \sim 1, \quad \text{but} \quad \|\boldsymbol{x}_n\|^{(q,q)}_{c_0 \otimes \ell^1} \sim (1 + \log n)^{1-1/q}.$$

Hence $(\|\cdot\|_n^{(q,q)}) \not\cong (\|\cdot\|_n^{(1,q)})$ on ℓ^1.

For a general measure space Ω, the result follows from Theorem 1.1. ∎

We summarize the situation for (p,q)-multi-norms based on $L^1(\Omega)$. In this special case, we have a full solution to the question of equivalences.

THEOREM 3.3. *Let Ω be a measure space such that $L^1(\Omega)$ is infinite-dimensional, and suppose that $(p_1, q_1), (p_2, q_2) \in \mathcal{T}$*

(i) *Suppose that $q_2 > q_1$. Then $(\|\cdot\|_n^{(p_1,q_1)}) \not\preccurlyeq (\|\cdot\|_n^{(p_2,q_2)})$, and these multi-norms are not equivalent on $L^1(\Omega)$.*
(ii) *Suppose that $q_2 = q_1 = q$ and $p_2 > p_1$. Then $(\|\cdot\|_n^{(p_2,q)}) \geq (\|\cdot\|_n^{(p_1,q)})$; these multi-norms are equivalent on $L^1(\Omega)$ when also $p_2 < q$, but they are not equivalent to $(\|\cdot\|_n^{(q,q)})$.* ∎

COROLLARY 3.4. *Let Ω be a measure space such that $L^1(\Omega)$ is infinite-dimensional, and suppose that $(p,q) \in \mathcal{T}$. Then the (p,q)-multi-norm on $L^1(\Omega)$ is not equivalent to the minimum multi-norm, and it is equivalent to the maximum multi-norm if and only if $p = q = 1$, in which case, it is actually equal to the maximum multi-norm.* ∎

3.2. The case where $r > 1$. In this case, it is more difficult to determine when the (p,q)-multi-norms are equivalent on $L^r(\Omega)$.

Throughout we suppose that $L^r(\Omega)$ is infinite-dimensional.

In this section, it is convenient to continue to use the earlier notation $\mathcal{C}_c$ for the curve

$$\mathcal{C}_c = \{(p,q) \in \mathcal{T} : 1/p - 1/q = c\},$$

whenever $c \in [0,1)$. This curve is contained in the triangle $\mathcal{T}$.

We shall consider points P_1 and P_2 in $\mathcal{T}$, and shall say P_1 *and* P_2 *are equivalent* (respectively, *not equivalent*) *on* E to mean that the multi-norms $(\|\cdot\|_n^{(p_1,q_1)})$ and $(\|\cdot\|_n^{(p_2,q_2)})$ based on a Banach space E are equivalent (respectively, not equivalent).

The first result, which shows that various pairs of multi-norms are not equivalent, follows directly from Proposition 2.12 and the calculation given in Example 2.16. Indeed, (i) follows from Proposition 2.12(iv), and (ii)–(iv) follow from equation (2.5).

PROPOSITION 3.5. *Let Ω be a measure space, and take $r \geq 1$. Then two points $P_1 \in \mathcal{C}_{c_1}$ and $P_2 \in \mathcal{C}_{c_2}$ are not equivalent on $L^r(\Omega)$ in the following cases:*

(i) $p_1, p_2 \geq \min\{r, 2\}$, *and* $q_1 \neq q_2$;
(ii) $p_1, p_2 \leq r$, $\min\{c_1, c_2\} < 1/r$, *and* $c_1 \neq c_2$;

(iii) $p_1 \le r \le p_2$, *and*

$$\frac{1}{r} - \frac{1}{p_1} + \frac{1}{q_1} \ne \frac{1}{q_2};$$

(iv) $p_1 \ge r \ge p_2$, *and*

$$\frac{1}{r} - \frac{1}{p_2} + \frac{1}{q_2} \ne \frac{1}{q_1}. \ \blacksquare$$

We now concentrate on the (p,p)-multi-norms and the maximum multi-norm on $L^r(\Omega)$.

Let E be a normed space. We recall that it follows from Theorem 2.10 that the dual space of $(c_0 \otimes E, \|\cdot\|^{(p,p)})$ is $\Pi_{p'}(c_0, E')$; the dual of the maximum multi-norm, identified with $(c_0 \widehat{\otimes} E, \|\cdot\|_\pi)$, is $\mathcal{B}(c_0, E')$.

PROPOSITION 3.6. *Let Ω be a measure space. Suppose that*

$$\textit{either} \quad 1 \le p \le 2 \le r < \infty \quad \textit{or} \quad 1 \le p < r < 2.$$

Then $(\|\cdot\|_n^{(p,p)})$ is equivalent to $(\|\cdot\|_n^{\max})$ on $L^r(\Omega)$.

Proof. In the case where $1 \le p \le 2 \le r < \infty$, so that $r' \in (1,2]$, we use [14, Theorem 3.7], which tells us that every operator $T : c_0 \to L^{r'}(\Omega)$ is 2-summing, with $\pi_2(T) \le K_G\|T\|$, where K_G is the Grothendieck constant. Since $\Pi_2(c_0, E) \subset \Pi_{p'}(c_0, E)$ is a norm-decreasing inclusion (for any Banach space E), we conclude that

$$(c_0 \otimes L^r(\Omega), \|\cdot\|^{(p,p)})' = \Pi_{p'}(c_0, L^{r'}(\Omega)) = \mathcal{B}(c_0, L^{r'}(\Omega)) = (c_0 \otimes L^r(\Omega), \|\cdot\|^{\max})',$$

which gives the result.

Similarly, in the case where $1 \le p < r < 2$, so that $r' > 2$, we appeal to [14, Corollary 10.10], which shows in particular that we have $\Pi_{p'}(c_0, L^{r'}(\Omega)) = \mathcal{B}(c_0, L^{r'}(\Omega))$. The result follows. $\blacksquare$

PROPOSITION 3.7. *Let Ω be a measure space such that $L^r(\Omega)$ is infinite-dimensional. Suppose that $1 \le r < 2$. Then $(\|\cdot\|_n^{(r,r)}) \not\cong (\|\cdot\|_n^{\max})$ on $L^r(\Omega)$.*

Proof. Here we appeal to [20, Theorem 7, clause 2], which, using an example of Schwartz [25], shows that $\Pi_s(c_0, \ell^s) \ne \mathcal{B}(c_0, \ell^s)$ for $s > 2$. The required conclusion follows. $\blacksquare$

Thus we have a complete classification of the (p,p)-multi-norms on $L^r(\Omega)$ into equivalence classes, summarized in the following theorem.

THEOREM 3.8. *Let Ω be a measure space such that $L^r(\Omega)$ is infinite-dimensional, where $r \ge 1$. Set $\overline{r} = \min\{2, r\}$. Then:*

(i) $(\|\cdot\|_n^{(q,q)}) \not\cong (\|\cdot\|_n^{(p,p)})$ *on $L^r(\Omega)$ whenever $p, q \ge \overline{r}$ and $p \ne q$;*
(ii) $(\|\cdot\|_n^{(p,p)}) \not\cong (\|\cdot\|_n^{\max})$ *on $L^r(\Omega)$ whenever $p > \overline{r}$;*
(iii) $(\|\cdot\|_n^{(p,p)}) \cong (\|\cdot\|_n^{\max})$ *on $L^r(\Omega)$ whenever $1 \le p < \overline{r}$;*
(iv) $(\|\cdot\|_n^{(1,1)}) = (\|\cdot\|_n^{\max})$ *on $L^r(\Omega)$;*
(v) *if $1 < r < 2$, then $\overline{r} = r$ and $(\|\cdot\|_n^{(r,r)}) \not\cong (\|\cdot\|_n^{\max})$ on $L^r(\Omega)$;*
(vi) *if $r \ge 2$, then $\overline{r} = 2$ and $(\|\cdot\|_n^{(2,2)}) \cong (\|\cdot\|_n^{\max})$ on $L^r(\Omega)$.*

Proof. Notice that (ii) follows by applying (i) with $q = \bar{r}$ and (iv) is just a special case of Theorem 2.19. $\blacksquare$

3.3. The role of Orlicz's theorem. We shall now determine when the (p,q)-multi-norm based on $L^r(\Omega)$ is equivalent to the minimum multi-norm. For this, we shall need a form of Orlicz's theorem. Indeed, a generalization of Orlicz's theorem given in [14, Theorem 10.7] shows that, for each $s \in [1,\infty)$, the identity operator on $L^s(\Omega)$ is $(\tilde{s},1)$-summing, where $\tilde{s} := \max\{s,2\}$. In the case where $s = 2$, so that $\tilde{s} = 2$ also, the $(2,1)$-summing norm of the identity operator on $L^2(\Omega)$ is equal to 1.

Now suppose that $r > 1$, and again set $\bar{r} = \min\{2,r\}$. Set $s = r'$, the conjugate of r, so that

$$\tilde{s} = \max\{s,2\} = \bar{r}\,'.$$

Then, since the identity operator on $L^s(\Omega)$ belongs to $\Pi_{\tilde{s},1}(L^s(\Omega))$, we obtain

$$\mathcal{B}(L^s(\Omega),F) = \Pi_{\tilde{s},1}(L^s(\Omega),F)$$

for each Banach space F; in the case where $r = 2$, we have equality of the norms as well.

It follows from Theorem 2.6 that the tensor norm on $c_0 \otimes L^r(\Omega)$ induced from the $(1,\bar{r}\,')$-multi-norm is equivalent to the injective tensor norm, which is induced by $\mathcal{B}(L^s(\Omega),c_0)$. That is, the $(1,\bar{r}\,')$- and the minimum multi-norms on $L^r(\Omega)$ are equivalent. This and Theorem 2.11 imply the following.

THEOREM 3.9. *Let Ω be a measure space, take $r > 1$, and set $\bar{r} := \min\{r,2\}$. Suppose that $1 \le p \le q < \infty$. Then $(\|\cdot\|_n^{(p,q)}) \cong (\|\cdot\|_n^{\min})$ on $L^r(\Omega)$ whenever $1/p - 1/q \ge 1/\bar{r}$. Moreover $(\|\cdot\|_n^{(p,q)}) = (\|\cdot\|_n^{\min})$ on $L^2(\Omega)$ whenever $1/p - 1/q \ge 1/2$.* ∎

3.4. Asymptotic estimates. The next stage of our analysis is to give a complete asymptotic estimate for $\varphi_n^{(p,q)}(\ell^r)$ for all relevant values of p,q when $r > 1$.

THEOREM 3.10. *Let Ω be a measure space such that $L^r(\Omega)$ is infinite-dimensional, where $r > 1$. Set $\bar{r} = \min\{r,2\}$, and suppose that $1 \le p \le q < \infty$. Then:*

(i) $\varphi_n^{(p,q)}(L^r(\Omega)) \sim 1$ *when* $1/p - 1/q \ge 1/\bar{r}$;
(ii) $\varphi_n^{(p,q)}(L^r(\Omega)) = n^{1/q}$ *when* $p \ge \bar{r}$;
(iii) $\varphi_n^{(p,q)}(L^r(\Omega)) \sim n^{1/\bar{r}-1/p+1/q}$ *when* $1/p - 1/q \le 1/\bar{r}$ *and* $p \le \bar{r}$.

In the case where $r = 2$, all three estimates are actual equalities.

Proof. Statements (i) and (ii) follow from Theorem 3.9 and Proposition 2.12(iv), respectively.

(iii) Suppose now that $1/p - 1/q < 1/\bar{r}$ and that $p < \bar{r}$. Again, we need to consider only the space ℓ^r. By Proposition 2.12(iii) (when $r \ge 2$) or by Example 2.16 (when $r \le 2$), we see that

$$\varphi_n^{(p,q)}(\ell^r) \ge n^{1/\bar{r}-1/p+1/q} \qquad (n \in \mathbb{N}).$$

When $q = p$, we know by Theorem 3.8(iii) that $(\|\cdot\|_n^{(p,p)}) \cong (\|\cdot\|_n^{\max})$, and so

$$\varphi_n^{(p,p)}(\ell^r) \sim \varphi_n^{\max}(\ell^r) \sim n^{1/\bar{r}}$$

by [11, Theorem 3.54]. Thus we need to consider only the case where $q > p$.

Set $q_1 = p$ and $q_2 = p\bar{r}/(\bar{r}-p)$, so that $1/p - 1/q_2 = 1/\bar{r}$. We also see that $q_1 < q < q_2$, and so

$$\frac{1}{q} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2},$$

where $\theta = \bar{r}(1/p - 1/q)$. Using Proposition 2.15, we deduce from (i) and the previous paragraph that

$$\varphi_n^{(p,q)}(\ell^r) \leq (\varphi_n^{(p,p)}(\ell^r))^{1-\theta} \cdot (\varphi_n^{(p,q_2)}(\ell^r))^{\theta} \leq C_r\, n^{(1-\theta)/\bar{r}} = C_r\, n^{1/\bar{r}-1/p+1/q}$$

for all $n \in \mathbb{N}$, where C_r is a constant depending only on r; when $r = 2$, this constant can be taken to be 1.

This completes the proof. ∎

We now obtain the following asymptotic estimates, where $\boldsymbol{f}$ is as in Example 2.17 and the multi-norm is calculated with respect to ℓ^r, where $r \geq 2$:

$$\|\boldsymbol{f}\|_n^{(p,q)} \sim \begin{cases} n^{1/2-1/p+1/q} & \text{when } 1 \leq p \leq 2 \text{ and } 1/p - 1/q \leq 1/2, \\ 1 & \text{when } 1/p - 1/q > 1/2, \\ n^{1/q} & \text{when } p \geq 2. \end{cases} \tag{3.1}$$

It is interesting to see where the maximum rate of growth is attained. Indeed, suppose that $(p,q) \in \mathcal{T}$ and we are considering the rate of growth of the (p,q)-multi-norm on ℓ^r, where $r \geq 1$. Then it follows from equation (2.5) in Example 2.16 that

$$\varphi_n^{(p,q)}(\ell^r) \sim \|(\delta_1, \dots, \delta_n)\|_n^{(p,q)} \quad \text{when } r \leq 2$$

and from equation (2.7) in Example 2.17 that

$$\varphi_n^{(p,q)}(\ell^r) \sim \|(f_1, \dots, f_n)\|_n^{(p,q)} \quad \text{when } r \geq 2,$$

where, for $i \in \mathbb{N}_n$, we are setting

$$f_i = \frac{1}{n^{1/r}} \sum_{j=1}^{n} \zeta^{-ij} \delta_j \quad \text{with } \zeta = \exp(2\pi \mathrm{i}/n).$$

Thus the maximum rate of growth is attained at either $(\delta_1, \dots, \delta_n)$ or at $(f_1, \dots, f_n)$.

3.5. Classification theorem. We now give our main classification result obtained in the case where $r > 1$. For this, let us modify the curves $\mathcal{C}_c$ to obtain curves $\mathcal{D}_c$ for $0 \leq c < 1$ as follows. Set $\bar{r} = \min\{2, r\}$.

(i) The case where $c \in [1/\bar{r}, 1)$: Set $\mathcal{D}_c = \mathcal{C}_c$.

(ii) The case where $c \in [0, 1/r)$: Set $u_c = r/(1 - cr)$, so that $\mathcal{C}_c$ meets the vertical line $p = r$ at (r, u_c). Set

$$\mathcal{D}_c = \{(p,q) \in \mathcal{C}_c : p \in [1, r]\} \cup \{(p, u_c) : p \in [r, u_c]\}.$$

Thus $\mathcal{D}_c$ agrees with $\mathcal{C}_c$ on the interval $[1, r]$ and is the horizontal line $q = u_c$ on the interval $[r, u_c]$. In the case where $r < 2$ and $c \in (1/2, 1/r)$, the point at which the line $q = u_c$ meets the curve $\mathcal{C}_{1/2}$ is denoted by x_c, so that $r < x_c < 2$.

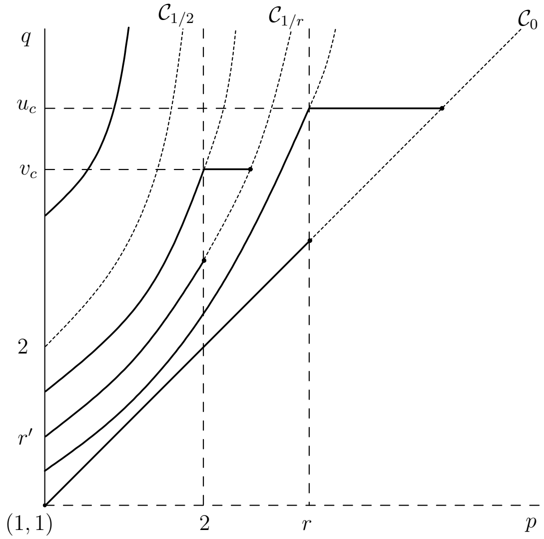
Note that $\mathcal{D}_0$ is the diagonal line segment $\{(p,p) : 1 \leq p \leq r\}$.

(iii) The case where $c \in [1/r, 1/2)$ (which only occurs when $r > 2$): Set $v_c = 2/(1 - 2c)$, so that $\mathcal{C}_c$ meets the vertical line $p = 2$ at $(2, v_c)$, and set $w_c := r v_c/(r - v_c)$, so that the horizontal line $q = v_c$ meets the curve $\mathcal{C}_{1/r}$ at (w_c, v_c). Set

$$\mathcal{D}_c = \{(p,q) \in \mathcal{C}_c : p \in [1, 2]\} \cup \{(p, v_c) : p \in [2, w_c]\}.$$

Thus $\mathcal{D}_c$ agrees with the old curve $\mathcal{C}_c$ on the interval $[1,2]$, and then it becomes the horizontal line $q = v_c$ until this line meets the curve $\mathcal{C}_{1/r}$, at which point it terminates. Note that $\mathcal{D}_{1/r}$ is the curve $\mathcal{C}_{1/r}$ restricted to the interval $[1,2]$.

Note that $\bigcup\{\mathcal{D}_c : 0 \leq c < 1\} = \mathcal{T}$. Note also that, unlike the curves $\mathcal{C}_c$, the curves $\mathcal{D}_c$ depend on the value of r. The case where $r > 2$ is illustrated in the diagram, in which we present in bold the curves $\mathcal{D}_c$ when $c \geq 1/2$, when $c \in (1/r, 1/2)$, when $c = 1/r$, when $c \in (0, 1/r)$, and when $c = 0$.



THEOREM 3.11. *Take $r > 1$, let Ω be a measure space such that $L^r(\Omega)$ is infinite-dimensional, and set $\bar{r} = \min\{2, r\}$. Take $c_1, c_2 \in [0, 1)$, and consider points $P_1 \in \mathcal{D}_{c_1}$ and $P_2 \in \mathcal{D}_{c_2}$.*

(i) *Suppose that $c_1, c_2 \in [1/\bar{r}, 1)$. Then P_1 and P_2 are equivalent (and the corresponding (p,q)-multi-norms are equivalent to the minimum multi-norm) on $L^r(\Omega)$.*
(ii) *Suppose that $c_1 \in [1/\bar{r}, 1)$ and $c_2 \in [0, 1/\bar{r})$. Then P_1 and P_2 are not equivalent on $L^r(\Omega)$.*
(iii) *Suppose that $c_1, c_2 \in [0, 1/\bar{r})$ and that $c_1 \neq c_2$. Then P_1 and P_2 are not equivalent on $L^r(\Omega)$.*

Proof. Clause (i) follows from Theorem 3.9, whereas (ii) follows from Theorem 3.10.

It remains to prove clause (iii). For this, we suppose that $c_1, c_2 \in [0, 1/\bar{r})$ and that $c_1 \neq c_2$.

Assume towards a contradiction that P_1 and P_2 are equivalent on $L^r(\Omega)$.

Case 1: $p_1, p_2 \leq \bar{r}$. In this case, the desired contradiction follows from Theorem 3.10(iii), noting that $P_i \in \mathcal{C}_{c_i}$ for both $i = 1$ and $i = 2$ in this case.

Case 2: $p_1, p_2 \geq \bar{r}$. In this case, we must have $q_1 = q_2$ by Theorem 3.10(ii). From the definition of the curves $\mathcal{D}_c$, this can happen (with $c_1 \neq c_2$) only when $\min\{p_1, p_2\} < r$, and so $r > 2$, and $\min\{c_1, c_2\} < 1/r$. In particular, we must have $\bar{r} = 2$. By Proposition 3.5(i), we must have $\max\{p_1, p_2\} > r$. Thus, without loss of generality, we may suppose that $p_1 > r > p_2 \geq 2$. Proposition 3.5(iv) then implies that

$$\frac{1}{r} - \frac{1}{p_2} + \frac{1}{q_2} = \frac{1}{q_1}.$$

Since $q_1 = q_2$, this implies that $p_2 = r$, a contradiction.

It remains to consider the case where $p_1 < \bar{r} < p_2$; the case where $p_1 > \bar{r} > p_2$ is dealt with similarly. We divide this case further into the following two cases.

Case 3: $r \leq 2$, so that $\bar{r} = r$, and $p_1 < r < p_2$. In this case, it follows from either Theorem 3.10 or Proposition 3.5(iii) that

$$\frac{1}{r} - \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{q_2}.$$

This implies first that $(r, q_2) \in \mathcal{C}_{c_1} \cap \mathcal{D}_{c_1}$, and then that $(p_2, q_2) \in \mathcal{D}_{c_1}$ by the definition of $\mathcal{D}_{c_1}$, a contradiction of the assumption that $c_1 \neq c_2$.

Case 4: $r > 2$, so that $\bar{r} = 2$, and $p_1 < 2 < p_2$. In this case, it follows from Theorem 3.10 that

$$\frac{1}{2} - \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{q_2}.$$

This implies that $(2, q_2) \in \mathcal{C}_{c_1} \cap \mathcal{D}_{c_1}$. So it follows from the definition of $\mathcal{D}_c$ and the assumption that $(p_2, q_2) \notin \mathcal{D}_{c_1}$ that $c_2 < 1/r$. By Proposition 3.5(i), we deduce that $p_2 > r$. But then Proposition 3.5(iii) implies that

$$\frac{1}{r} - \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{q_2},$$

and so $r = 2$, again a contradiction.

This concludes the proof of the theorem. ■

3.6. The role of Khintchine's inequalities. The previous theorem reduces our problem to that of determining the equivalence of two points P_1 and P_2 lying on the same curve $\mathcal{D}_c$, where $c \in [1, 1/\bar{r})$. For further progress, we shall use Khintchine's inequalities, for which see [17, Chapter 12], for example.

Let $n \in \mathbb{N}$. We shall consider $(\varepsilon_{i,j})$ to be a fixed $n \times 2^n$ matrix with entries in $\{-1, 1\}$ such that its 2^n columns range over all possible choices of n-tuples of ± 1. The Khintchine inequality tells us that, for each $r > 1$, there exist constants $A_r, B_r > 0$, depending only on r (but not on n), such that

$$A_r \Big(\sum_{i=1}^{n} |\alpha_i|^2\Big)^{1/2} \leq \Big(\frac{1}{2^n} \sum_{j=1}^{2^n} \Big|\sum_{i=1}^{n} \varepsilon_{i,j}\alpha_i\Big|^r\Big)^{1/r} \leq B_r \Big(\sum_{i=1}^{n} |\alpha_i|^2\Big)^{1/2}$$

for every $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ and every $n \in \mathbb{N}$. These constants are those specified in the next lemma.

LEMMA 3.12. *Let $r > 1$, and take $n \in \mathbb{N}$. Then there exists a linear monomorphism $R_n : \ell^2_n \to \ell^r$ such that*

$$\frac{1}{B_{r'}}\|(x_1,\dots,x_n)\|_n^{(p,q)} \le \|(R_nx_1,\dots,R_nx_n)\|_n^{(p,q)} \le B_r\|(x_1,\dots,x_n)\|_n^{(p,q)}$$

whenever $1 \le p \le q < \infty$ and $x_1,\dots,x_n \in \ell^2_n$.

Proof. Set $s = r'$, the conjugate index to r, so that $1 < s < \infty$. For each $i \in \mathbb{N}_n$, set

$$g_i = \frac{1}{2^{n/r}}(\varepsilon_{i,1},\dots,\varepsilon_{i,2^n},0,0,\dots) \in \ell^r \quad \text{and} \quad \varphi_i = \frac{1}{2^{n/s}}(\varepsilon_{i,1},\dots,\varepsilon_{i,2^n},0,0,\dots) \in \ell^s.$$

The maps $\delta_i \mapsto g_i$ and $\delta_i \mapsto \varphi_i$ extend linearly to linear operators $R : \ell^2_n \to \ell^r$ and $S : \ell^2_n \to \ell^s$, respectively. Moreover, by the Khintchine inequality, we see that

$$A_r\|x\|_{\ell^2} \le \|Rx\|_{\ell^r} \le B_r\|x\|_{\ell^2} \quad \text{and} \quad A_s\|x\|_{\ell^2} \le \|Sx\|_{\ell^s} \le B_s\|x\|_{\ell^2} \qquad (x \in \ell^2_n),$$

so that, in particular, both R and S are linear monomorphisms. It is also the case that

$$\langle Rx, Sy\rangle = \langle x, y\rangle \qquad (x, y \in \ell^2_n),$$

where we identify $(\ell^2_n)' = \ell^2_n$ in an obvious manner.

Take $p, q \in \mathcal{T}$ and take $x_1,\dots,x_n \in \ell^2_n$. We then see that

$$\begin{aligned}
&\|(Rx_1,\dots,Rx_n)\|_n^{(p,q)}\\
&\qquad = \sup\Big\{\Big(\sum_{i=1}^n |\langle Rx_i,\lambda_i\rangle|^q\Big)^{1/q} : \lambda_1,\dots,\lambda_n \in \ell^s,\ \mu_{p,n}(\lambda_1,\dots,\lambda_n) \le 1\Big\}\\
&\qquad = \sup\Big\{\Big(\sum_{i=1}^n |\langle x_i,R'\lambda_i\rangle|^q\Big)^{1/q} : \lambda_1,\dots,\lambda_n \in \ell^s,\ \mu_{p,n}(\lambda_1,\dots,\lambda_n) \le 1\Big\}\\
&\qquad \le \sup\Big\{\Big(\sum_{i=1}^n |\langle x_i,y_i\rangle|^q\Big)^{1/q} : y_1,\dots,y_n \in \ell^2_n,\ \mu_{p,n}(y_1,\dots,y_n) \le B_r\Big\}\\
&\qquad = B_r\|(x_1,\dots,x_n)\|_n^{(p,q)}.
\end{aligned}$$

On the other hand, from the first equation above, we also see that

$$\begin{aligned}
&\|(Rx_1,\dots,Rx_n)\|_n^{(p,q)}\\
&\qquad \ge \sup\Big\{\Big(\sum_{i=1}^n |\langle Rx_i,Sy_i\rangle|^q\Big)^{1/q} : y_1,\dots,y_n \in \ell^2_n,\ \mu_{p,n}(y_1,\dots,y_n) \le 1/B_s\Big\}\\
&\qquad = \sup\Big\{\Big(\sum_{i=1}^n |\langle x_i,y_i\rangle|^q\Big)^{1/q} : y_1,\dots,y_n \in \ell^2_n,\ \mu_{p,n}(y_1,\dots,y_n) \le 1/B_s\Big\}\\
&\qquad = \frac{1}{B_s}\|(x_1,\dots,x_n)\|_n^{(p,q)}.
\end{aligned}$$

Thus, setting $R_n := R$, we obtain the desired operator. ∎

THEOREM 3.13. *Suppose that $(p_1,q_1),(p_2,q_2) \in \mathcal{T}$. Assume that the (p_1,q_1)- and (p_2,q_2)-multi-norms are not equivalent on ℓ^2. Then, for every $r > 1$, the (p_1,q_1)- and (p_2,q_2)-multi-norms are not equivalent on ℓ^r.*

Proof. Take $r > 1$. Without loss of generality, by the assumption, we see that there exist a sequence (α_n) in $(0, \infty)$ with $\alpha_n \nearrow \infty$ and a sequence $(\boldsymbol{x}_n)$ where, for each $n \in \mathbb{N}$, $\boldsymbol{x}_n = (x_{n,1}, \dots, x_{n,n}) \in (\ell^2)^n$, such that

$$\|(x_{n,1}, \dots, x_{n,n})\|_n^{(p_1,q_1)} > \alpha_n \|(x_{n,1}, \dots, x_{n,n})\|_n^{(p_2,q_2)}.$$

It is obvious that we may consider $x_{n,1}, \dots, x_{n,n}$ as belonging to ℓ_n^2. Now set $y_{n,i} = R_n x_{n,i}$ for each $i \in \mathbb{N}_n$, where R_n is the map defined in the previous lemma. We then obtain, for each $n \in \mathbb{N}$, a tuple $(y_{n,1}, \dots, y_{n,n}) \in (\ell^r)^n$ such that

$$\|(y_{n,1}, \dots, y_{n,n})\|_n^{(p_1,q_1)} > \frac{\alpha_n}{B_r B_{r'}} \|(y_{n,1}, \dots, y_{n,n})\|_n^{(p_2,q_2)}.$$

Thus $(\|\cdot\|_n^{(p_1,q_1)})$ and $(\|\cdot\|_n^{(p_1,q_1)})$ are not equivalent on ℓ^r. This completes the proof. ■

COROLLARY 3.14. *Let $(p_1, q_1), (p_2, q_2) \in \mathcal{T}$. Suppose that $\Pi_{q_1,p_1}(\ell^2, F) \neq \Pi_{q_2,p_2}(\ell^2, F)$ for some Banach space F. Then, for each $r > 1$, the (p_1, q_1)- and (p_2, q_2)-multi-norms are not equivalent on ℓ^r.*

Proof. This follows from the previous theorem and Theorem 2.8, using the Riesz representation theorem. ■

3.7. Final classification. Theorem 3.13 suggests that we study more closely the spaces $\Pi_{q,p}(H)$, where H is a Hilbert space, and this we shall do to obtain the final classification that we can achieve.

We first state some results that identify $\Pi_{q,p}(H)$. Clause (i) of the following theorem combines Corollaries 3.16 and 4.13 of [14], and the remaining clauses are stated on page 207 of [14]. In fact, clauses (ii) and (iii) of the following theorem originate in [19, Theorem 2] (where this result is attributed to Mityagin), and (iv) is from [5] and [6, Theorem 3]. Recall that $\mathcal{S}_r(H)$ and $\mathcal{S}_{2q/p,q}(H)$ were defined in §1.2.

THEOREM 3.15. *Let H be a Hilbert space, and take $(p, q) \in \mathcal{T}$.*

(i) *Suppose that $p = q$. Then $\Pi_p(H) = \Pi_2(H) = \mathcal{S}_2(H)$.*
(ii) *Suppose that $p \leq 2$ and $1/p - 1/q < 1/2$. Then $\Pi_{q,p}(H) = \mathcal{S}_r(H)$, where*

$$1/r = 1/q - 1/p + 1/2.$$

(iii) *Suppose that $1/p - 1/q \geq 1/2$. Then $\Pi_{q,p}(H) = \mathcal{B}(H)$.*
(iv) *Suppose that $2 < p < q < \infty$. Then $\Pi_{q,p}(H) = \mathcal{S}_{2q/p,q}(H)$.* ■

In connection with clause (i), we note that the exact constants that determine the relations between the π_p-norm and the π_2-norm on (real and complex) Hilbert spaces of various dimensions are calculated in [15].

Recall that the point $x_c \in (r, u_c)$ was defined on page 31.

THEOREM 3.16. *Take $r \in (1, 2)$, and let Ω be a measure space such that $L^r(\Omega)$ is infinite-dimensional. Suppose that two distinct points $P_1 = (p_1, q_1)$ and $P_2 = (p_2, q_2)$ in $\mathcal{T}$ are equivalent on $L^r(\Omega)$. Then one of the following cases must occur.*

(i) *The points P_1 and P_2 both lie in the region*

$$\{(p, q) \in \mathcal{T} \colon 1/p - 1/q \geq 1/r\};$$

in this case, the (p,q)-multi-norms corresponding to points in this region are all equivalent to the minimum multi-norm on $L^r(\Omega)$.

(ii) *The points P_1 and P_2 both lie on the same curve $\{(p,q) \in \mathcal{D}_c\colon 1/p - 1/q \geq 1/2\}$ for some $c \in [1/2, 1/r)$. Further, $p_1, p_2 \in [1, x_c]$.*

(iii) *The points P_1 and P_2 both lie on the same curve $\{(p,q) \in \mathcal{C}_c\colon 1 \leq p \leq r\}$ for some $c \in (0, 1/2)$.*

(iv) *The points P_1 and P_2 both lie on the line segment $\{(p,p)\colon 1 \leq p < r\}$; in this case, the (p,p)-multi-norms corresponding to points on this line segment are all equivalent to the maximum multi-norm on $L^r(\Omega)$.*

Proof. By Theorems 3.8 and 3.11, all that remains to be considered is the case where P_1 and P_2 both lie on the same curve $\mathcal{D}_c$, where $0 < c < 1/r$. Without loss of generality, we may suppose that $p_1 < p_2$, and so $p_1 < q_1$ and

$$1/p_1 - 1/q_1 \geq 1/p_2 - 1/q_2.$$

Case 1: $c \in [1/2, 1/r)$ and $1/p_i - 1/q_i < 1/2$ for both $i = 1, 2$. Then, by Theorem 3.15(i), (ii), or (iv), we have, for each $i = 1, 2$,

$$\Pi_{q_i,p_i}(\ell^2) = \text{ either } \mathcal{S}_{2q_i/p_i,q_i}(\ell^2) \text{ or } \mathcal{S}_{r_i}(\ell^2),$$

where $1/r_i = 1/2 - 1/p_i + 1/q_i$. Note that, since $c \geq 1/2$ and $P_1 \neq P_2$, we must have $1/p_1 - 1/q_1 \neq 1/p_2 - 1/q_2$, and so $r_1 \neq r_2$. Thus we see, by a remark on page 9, that $\Pi_{q_1,p_1}(\ell^2) \neq \Pi_{q_2,p_2}(\ell^2)$, and hence, by Corollary 3.14, P_1 and P_2 are not equivalent on ℓ^r. This contradicts the hypothesis, and so this case cannot occur.

Case 2: $c \in [1/2, 1/r)$ and $1/p_2 - 1/q_2 < 1/2 \leq 1/p_1 - 1/q_1$. Then, by Theorem 3.15, we have

$$\Pi_{q_2,p_2}(\ell^2) = \text{ either } \mathcal{S}_{2q_2/p_2,q_2}(\ell^2) \text{ or } \mathcal{S}_{r_2}(\ell^2),$$

where $1/r_2 = 1/2 - 1/p_2 + 1/q_2$. On the other hand, the same theorem implies that $\Pi_{q_1,p_1}(\ell^2) = \mathcal{B}(\ell^2)$. So again we see that $\Pi_{q_1,p_1}(\ell^2) \neq \Pi_{q_2,p_2}(\ell^2)$, and hence, by Corollary 3.14, P_1 and P_2 are not equivalent on ℓ^r. This contradicts the hypothesis, and so this case cannot occur.

We have shown in the above two cases that we cannot have both $1/p_2 - 1/q_2 < 1/2$ and $c \in [1/2, 1/r)$, and so necessarily $1 \leq p_1 \leq p_2 \leq x_c$ when $c \in [1/2, 1/r)$.

Case 3: $c \in (0, 1/2)$. Now $1/p_i - 1/q_i < 1/2$ for each $i = 1, 2$, and so, by Theorem 3.15, we have

$$\Pi_{q_i,p_i}(\ell^2) = \text{ either } \mathcal{S}_{2q_i/p_i,q_i}(\ell^2) \text{ or } \mathcal{S}_{r_i}(\ell^2),$$

for each $i = 1, 2$, where $1/r_i = 1/2 - 1/p_i + 1/q_i$. Note that r_1 and r_2 cannot both be equal to 2. Thus, since P_1 and P_2 are equivalent on ℓ^r, by Corollary 3.14, we must have $\Pi_{q_1,p_1}(\ell^2) = \Pi_{q_2,p_2}(\ell^2)$. The only way this can happen, by the remark on page 9, is when $p_i \leq 2$ $(i = 1, 2)$ and $1/p_1 - 1/q_1 = 1/p_2 - 1/q_2$. By the definition of the curve $\mathcal{D}_c$, this can happen only if $p_i \leq r$ $(i = 1, 2)$.

The three cases above complete the proof. ∎

REMARK 3.17. In [7], it will be shown that P_1 and P_2 are equivalent whenever $1 < r < 2$ and both points lie on the same curve $\mathcal{C}_c$ for some $c \in (0, 1/r)$. Thus we know in every case whether P_1 and P_2 are equivalent, save for the case where both points lie on the same horizontal line $q = u_c$ and where $r \leq p_1 < p_2 \leq x_c$. In this case, $\Pi_{q_1,p_1}(\ell^2) = \Pi_{q_2,p_2}(\ell^2)$, a necessary condition for equivalence of the two multi-norms by Theorems 2.8 and 3.13. ■

THEOREM 3.18. *Take $r \geq 2$, and let Ω be a measure space such that $L^r(\Omega)$ is infinite-dimensional. Suppose that two distinct points $P_1 = (p_1, q_1)$ and $P_2 = (p_2, q_2)$ in $\mathcal{T}$ are equivalent on $L^r(\Omega)$. Then one of the following cases must occur.*

(i) *The points P_1 and P_2 both lie in the region*

$$\{(p, q) \in \mathcal{T} \colon 1/p - 1/q \geq 1/2\};$$

in this case, the (p, q)-multi-norms corresponding to points in this region are all equivalent to the minimum multi-norm on $L^r(\Omega)$.

(ii) *The points P_1 and P_2 both lie on the same curve $\{(p, q) \in \mathcal{C}_c \colon 1 \leq p \leq 2\}$ for some $c \in (0, 1/2)$.*

(iii) *The points P_1 and P_2 both lie on the line segment $\{(p, p) \colon 1 \leq p \leq 2\}$; in this case, the (p, p)-multi-norms corresponding to points on this line segment are all equivalent to the maximum multi-norm on $L^r(\Omega)$.*

Proof. As in Theorem 3.16, all that remains to be considered is the case where P_1 and P_2 both lie on the same curve $\mathcal{D}_c$, where $0 < c < 1/2$. We again need to consider only the space ℓ^r. For this, suppose without loss of generality that $p_1 < p_2$, and so $p_1 < q_1$. Assume towards a contradiction that $p_2 > 2$. Then, by Theorem 3.15(i) or (iv), we have

$$\Pi_{q_2,p_2}(\ell^2) = \text{ either } \mathcal{S}_{2q_2/p_2,q_2}(\ell^2) \text{ or } \mathcal{S}_2(\ell^2).$$

First, suppose that $p_1 > 2$. Then, by Theorem 3.15(iv), we have

$$\Pi_{q_1,p_1}(\ell^2) = \mathcal{S}_{2q_1/p_1,q_1}.$$

But we know that $\mathcal{S}_{2q_1/p_1,q_1}(\ell^2)$ is never equal to either $\mathcal{S}_{2q_2/p_2,q_2}(\ell^2)$ or $\mathcal{S}_2(\ell^2)$, and so P_1 and P_2 are not equivalent on ℓ^r by Corollary 3.14.

Second, suppose that $p_1 \leq 2$. Then $\Pi_{q_1,p_1}(\ell^2) = \mathcal{S}_{r_1}(\ell^2)$ by Theorem 3.15(ii), where $1/r_1 = 1/2 - c$, and so $r_1 > 2$. Thus again we see that $\Pi_{q_1,p_1}(\ell^2) \neq \Pi_{q_2,p_2}(\ell^2)$ by a remark on page 9.

In both cases, we arrive at a contradiction to the assumption that P_1 and P_2 are equivalent on ℓ^r. Therefore $p_2 \leq 2$, and the proof is completed. ■

In the Hilbert spaces case, i.e. when $r = 2$, using Theorems 2.18, we see that the (p, p)-multi-norms corresponding to points in the clause (iii) above are all equivalent to the Hilbert space multi-norm.

REMARK 3.19. There remains the case where P_1, P_2 both lie on a curve $\mathcal{C}_c$ such that $0 \leq c < 1/2$ and $p_1, p_2 \in [1, 2]$. Then again $\Pi_{q_1,p_1}(\ell^2) = \Pi_{q_2,p_2}(\ell^2)$, a necessary condition for equivalence by Theorems 2.8 and 3.13. In [7], it will be shown that P_1 and P_2 are indeed equivalent whenever $r \geq 2$ and both points lie on the same curve $\mathcal{C}_c$ for some $c \in (0, 1/2)$. Thus we have a complete classification whenever $r \geq 2$. ■

3.8. The relation with standard t-multi-norms. Let Ω be a measure space, and take $r \geq 1$. Then we have defined the standard t-multi-norm $(\|\cdot\|_n^{[t]})$ on $L^r(\Omega)$ whenever $t \geq r$, and we have defined the (p,q)-multi-norm $(\|\cdot\|_n^{(p,q)})$ on $L^r(\Omega)$ whenever $(p,q) \in \mathcal{T}$. We *conjecture* that $(\|\cdot\|_n^{[t]}) \not\cong (\|\cdot\|_n^{(p,q)})$ whenever $r > 1$ and $L^r(\Omega)$ is infinite-dimensional.

The first result proves rather more than the conjecture, but only in the special case in which $t = r$. In the following theorem, we suppose that $c_0 \otimes L^r(\Omega) \subset L^r(\Omega, c_0)$ has the norm from $L^r(\Omega, c_0)$ corresponding to the standard r-multi-norm based on $L^r(\Omega)$ in the manner explained above.

THEOREM 3.20. *Let Ω be a measure space, and take $r > 1$. Suppose that $L^r(\Omega)$ is an infinite-dimensional space. Then the c_0-norm on $c_0 \otimes L^r(\Omega)$ induced by the standard r-multi-norm $(\|\cdot\|_n^{[r]} : n \in \mathbb{N})$ based on $L^r(\Omega)$ is not equivalent to any uniform c_0-norm.*

Proof. The following theorem is proved in [13, Section 7.3]. Suppose that $S \in \mathcal{B}(L^r(\Omega))$. Then the operator $I \otimes S : c_0 \otimes L^r(\Omega) \to c_0 \otimes L^r(\Omega)$ extends to a bounded operator on $L^r(\Omega, c_0)$ if and only if S is *regular*, in the sense that it is a linear combination of positive operators on the Banach lattice $L^r(\Omega)$. However, since $L^r(\Omega)$ is an infinite-dimensional space, not all the operators $S \in \mathcal{B}(L^r(\Omega))$ are regular.

Indeed, for a concrete example of an operator in $\mathcal{B}(L^r(\Omega))$ which is not regular, we follow [13, Section 7.6]. Set $s = r'$, and let $S : \ell^s(\mathbb{Z}) \to \ell^s(\mathbb{Z})$ be the discrete Hilbert transform given by

$$S(\delta_k) = \sum_{m \neq k} \frac{1}{m-k} \delta_m \qquad (k \in \mathbb{N}).$$

Then S is bounded on $\ell^s(\mathbb{Z})$, but $I \otimes S$ is not bounded on the space $\ell^1 \otimes \ell^s(\mathbb{Z}) \subset \ell^s(\mathbb{Z}, \ell^1)$. By duality, we see that $I \otimes S'$ is not bounded on the space $c_0 \otimes \ell^r(\mathbb{Z}) \subset \ell^r(\mathbb{Z}, c_0)$. In the case where $L^r(\Omega)$ is infinite-dimensional, this latter space contains a 1-complemented copy of $\ell^r(\mathbb{Z})$, and so we obtain an example of an operator on $L^r(\Omega)$ that is not regular.

For a stronger example, it is shown by Arendt and Voigt [4] that the subalgebra of regular operators on $L^r(\Omega)$ is not even dense in $\mathcal{B}(L^r(\Omega))$ whenever $r > 1$ and $L^r(\Omega)$ is infinite-dimensional.

We conclude that the standard r-multi-norm cannot be equivalent to any uniform c_0-norm on $c_0 \otimes L^r(\Omega)$. ∎

COROLLARY 3.21. *Let Ω be a measure space, and take $r > 1$. Suppose that $L^r(\Omega)$ is an infinite-dimensional space. Then the standard r-multi-norm is not equivalent to the maximum or minimum multi-norms or to any (p,q)-multi-norm on $L^r(\Omega)$ for $(p,q) \in \mathcal{T}$.*

Proof. This follows from the theorem because the projective, injective, and Chevet–Saphar norms are uniform c_0-norms. ∎

Again suppose that Ω is a measure space. Since $L^1(\Omega)$ is Dedekind complete as a Banach lattice, it follows from a remark on page 13 of [2] that every order-bounded operator on $L^1(\Omega)$ is regular. Since $L^1(\Omega)$ is an AL-space, and hence a KB-space (see [2]), it follows from [2, Theorem 15.3] that every bounded operator on $L^1(\Omega)$ is order-bounded. Thus, in this case, every $S \in \mathcal{B}(L^1(\Omega))$ is regular. Thus the argument of the above proof does not apply. Indeed, the conclusion of the preceding paragraph does not hold: by Theorem 2.20, $(\|\cdot\|_n^{[q]}) = (\|\cdot\|_n^{(1,q)})$ on $L^1(\Omega)$ for every $q \geq 1$ (cf. Theorem 3.3).

The following theorem subsumes Theorem 2.21 and part of Corollary 3.21.

THEOREM 3.22. *Let Ω be a measure space, and take $r > 1$, where $L^r(\Omega)$ is infinite-dimensional. Suppose that $t \geq r$ and that $(p,q) \in \mathcal{T}$, and assume that*

$$(\|\cdot\|_n^{(p,q)}) \cong (\|\cdot\|_n^{[t]}) \quad \text{on } L^r(\Omega).$$

Then $r < 2$, $t \geq 2r/(2-r)$, and (p,q) lies on the same curve $\mathcal{D}_c$ as (r,t) with $p \leq 2t/(2+t)$, so that $1/p - 1/q \geq 1/2$. Moreover, we must also have $(\|\cdot\|_n^{[t]}) \cong (\|\cdot\|_n^{(r,t)})$ on $L^r(\Omega)$.

Proof. We need to consider only the space ℓ^r. Set $\bar{r} = \min\{r, 2\}$, as before. By (2.8), $\varphi_n^{[t]}(\ell^r) = n^{1/t}$, and so it follows from Theorem 3.10 that one of the following must happen:

(i) either $p \geq \bar{r}$ and $q = t$;
(ii) or $p \leq \bar{r}$ and $1/t = 1/\bar{r} - 1/p + 1/q$.

Let $n \in \mathbb{N}$, and take $\boldsymbol{g} = (g_1, \dots, g_n) \in (\ell^r)^n$ to be as in the proof of Lemma 3.12. Then we see that

$$\|\boldsymbol{g}\|_n^{[t]} = \frac{1}{2^{n/r}} \sup\{(m_1^{t/r} + \cdots + m_k^{t/r})^{1/t} : m_1 + \cdots + m_k = 2^n\}.$$

Since $t/r \geq 1$, we have $m_1^{t/r} + \cdots + m_k^{t/r} \leq 2^{nt/r}$, and so $\|\boldsymbol{g}\|_n^{[t]} \leq 1$. On the other hand, Lemma 3.12 tells us that

$$\|\boldsymbol{g}\|_n^{(p,q)} \sim \|(\delta_1, \dots, \delta_n)\|_n^{(p,q)},$$

where (δ_k) is the standard basis sequence for ℓ^2. These and Example 2.16 imply that $1/p - 1/q \geq 1/2$.

The previous two paragraphs now imply the claimed result. ∎

Thus $(\|\cdot\|_n^{(p,q)})$ is not equivalent to $(\|\cdot\|_n^{[t]})$ on $L^r(\Omega)$ in each of the following cases:

(i) $r \geq 2$;
(ii) $1 < r < 2$ and $t < 2r/(2-r)$;
(iii) $1/p - 1/q < 1/2$;
(iv) (p,q) and (r,t) lie on different curves $\mathcal{D}_c$.

Moreover, our conjecture would be established if we could prove that $\|\cdot\|_n^{[t]} \not\cong \|\cdot\|_n^{(r,t)}$ on ℓ^r for any $t \geq r$; this is open only when $1 < r < 2$ and $t \geq 2r/(2-r)$. Some further partial results will be given in [7]; in particular, it will be proved that, in certain special cases, our conjecture on page 38 is false.

4. The Hilbert space multi-norm

4.1. Equivalent norms. Let H be a (complex) Hilbert space with inner product denoted by $[\cdot,\cdot]$, and take $p \in [1,2]$. We know from Propositions 3.6, 2.19, and 2.18 that there is a constant C_p such that

$$\|\boldsymbol{x}\|_n^H = \|\boldsymbol{x}\|_n^{(2,2)} \leq \|\boldsymbol{x}\|_n^{(p,p)} \leq \|\boldsymbol{x}\|_n^{\max} = \|\boldsymbol{x}\|_n^{(1,1)} \leq C_p \|\boldsymbol{x}\|_n^{(p,p)} \quad (\boldsymbol{x} \in H^n)$$

for all $n \in \mathbb{N}$. Our first theorem gives the best value of C_2.

THEOREM 4.1. *Let H be an infinite-dimensional, complex Hilbert space. Then*

$$\|\boldsymbol{x}\|_n^H = \|\boldsymbol{x}\|_n^{(2,2)} \leq \|\boldsymbol{x}\|_n^{\max} \leq \frac{2}{\sqrt{\pi}}\|\boldsymbol{x}\|_n^{(2,2)} \quad (\boldsymbol{x} \in H^n,\ n \in \mathbb{N});$$

the constant $2/\sqrt{\pi}$ is best-possible in this inequality.

Proof. By Theorem 2.10, the $(2,2)$-multi-norm on H is the Chevet–Saphar norm d_2 on $c_0 \otimes H$. Thus the dual space of $(c_0 \otimes H, \|\cdot\|^H)$ is the space $\Pi_2(c_0, H') = \Pi_2(c_0, \overline{H})$, where $\overline{H}$ is the conjugate of H. Thus, by duality, the claim is equivalent to showing that

$$\|T\| \leq \pi_2(T) \leq \frac{2}{\sqrt{\pi}}\|T\| \quad (T \in \mathcal{B}(c_0, H)).$$

The 'Little Grothendieck Theorem' says that every $T \in \mathcal{B}(\ell_n^\infty, H)$ is 2-summing, with $\pi_2(T) \leq (2/\sqrt{\pi}\,)\|T\|$ for each $n \in \mathbb{N}$. See [13, Theorem 11.11] for the estimate, and [13, Section 20.19], where it is shown that this constant is the best possible (when the scalars are the complex numbers). In particular, we see that each operator $T \in \mathcal{B}(c_0, H)$ is such that $\pi_2(T) \leq (2/\sqrt{\pi}\,)\|T\|$; it follows that

$$\sup\{\pi_2(T)/\|T\| : T \in \mathcal{B}(c_0, H)\} = 2/\sqrt{\pi},$$

and this gives the required estimate. ∎

The function $p \mapsto C_p,\ [1,2] \to [1, 2/\sqrt{\pi}]$, is increasing, with $C_1 = 1$ and $C_2 = 2/\sqrt{\pi}$; we do not have a formula for C_p.

4.2. Equivalence at level n. We now consider the best constant c_n, defined for each $n \in \mathbb{N}$, such that

$$\|\boldsymbol{x}\|_n^{\max} \leq c_n\|\boldsymbol{x}\|_n^{(2,2)} \quad (\boldsymbol{x} \in H^n).$$

We know that (c_n) is an increasing sequence in $[1, 2/\sqrt{\pi}\,]$ with $c_1 = 1$ and that

$$\lim_{n\to\infty} c_n = 2/\sqrt{\pi}.$$

We wonder: which is the smallest value of n such that $c_n > 1$? The first fact that we can offer is that $c_2 = c_3 = 1$, so that

$$\|\boldsymbol{x}\|_n^{\max} = \|\boldsymbol{x}\|_n^{(2,2)} = \|\boldsymbol{x}\|_n^H \quad (\boldsymbol{x} \in H^n)$$

for $n = 1, 2, 3$.

We start with some preliminary results. The following is a slight generalization of [18, Proposition 2.8]. In the result, we define r by

$$\frac{1}{r} = \frac{1}{p} - \frac{1}{2} = \frac{1}{2} - \frac{1}{p'}$$

in the case where $1 < p < 2$, so that $p = 2r/(r+2)$.

PROPOSITION 4.2. *Let $1 \leq p < \infty$, and let $(x_1, \dots, x_n)$ be an orthogonal n-tuple in a Hilbert space. Then*

$$\mu_{p,n}(x_1, \dots, x_n) = \begin{cases} \max\{\|x_i\| : i \in \mathbb{N}_n\} & (p \geq 2), \\ \left(\sum_{i=1}^n \|x_i\|^r\right)^{1/r} & (1 < p < 2), \\ \left(\sum_{i=1}^n \|x_i\|^2\right)^{1/2} & (p = 1). \end{cases}$$

Proof. We calculate simply that

$$\mu_{p,n}(x_1,\dots,x_n) = \sup\Big\{\Big\|\sum_{i=1}^n \alpha_i x_i\Big\| : \sum_{i=1}^n |\alpha_i|^{p'} \le 1\Big\}$$
$$= \sup\Big\{\Big(\sum_{i=1}^n |\alpha_i|^2\|x_i\|^2\Big)^{1/2} : \sum_{i=1}^n |\alpha_i|^{p'} \le 1\Big\}.$$

Suppose that $p \ge 2$. Then $p' \le 2$, and we see that the supremum is attained when $(\alpha_i) = (\delta_{i,i_0})$ for some $i_0 \in \mathbb{N}_n$.

Next, suppose that $1 < p < 2$, so that $2 < p' < \infty$. We set $R = r/2$, so that $R = p'/(p'-2)$ and $R' = p'/2 > 1$. Then, by ℓ^R-$\ell^{R'}$-duality, we see that

$$\Big(\sum_{i=1}^n \|x_i\|^{2R}\Big)^{1/R} = \sup\Big\{\sum_{i=1}^n |\alpha_i|^2\|x_i\|^2 : \sum_{i=1}^n |\alpha_i|^{2R'} \le 1\Big\} = \mu_{p,n}(x_1,\dots,x_n)^2$$

because $2R' = p'$, and hence

$$\Big(\sum_{i=1}^n \|x_i\|^r\Big)^{1/r} = \mu_{p,n}(x_1,\dots,x_n).$$

Suppose that $p = 1$. Then we are really taking the supremum over the collection of sequence (α_i) such that $|\alpha_i| \le 1$ $(i \in \mathbb{N}_n)$, and the result follows immediately.

The result follows in each case. ∎

Let H be a Hilbert space, and take r with $2 \le r \le \infty$. For $n \in \mathbb{N}$, we denote by $S_n^r \subset H^n$ the set of all orthogonal n-tuples $(x_1,\dots,x_n) \in H^n$ with $\sum_{i=1}^n \|x_i\|^r = 1$. In particular, we have

$$S_n^2 = \{(x_i) \in H^n : (x_i) \text{ orthogonal and } \|x_1 + \cdots + x_n\| = 1\}.$$

By Proposition 4.2, with r as defined for some $p \in (1,2)$, we have

$$\overline{\langle S_n^r\rangle} \subset (H^n, \mu_{p,n})_{[1]}.$$

That is, the closed convex hull of S_n^r is a subset of the closed unit ball of H^n equipped with the norm $\mu_{p,n}$. By Proposition 4.2, this result also holds when $p = 1$ and $r = 2$, and it holds for $r = \infty$ and any $p \ge 2$. For us, it is actually these cases which are of most interest:

$$\overline{\langle S_n^\infty\rangle} \subset (H^n, \mu_{2,n})_{[1]}, \qquad \overline{\langle S_n^2\rangle} \subset (H^n, \mu_{1,n})_{[1]}.$$

The Russo–Dye theorem [10, Theorem 3.2.18(iii)] can be used to show that the closed unit ball $(H^n, \mu_{2,n})_{[1]}$ is precisely $\overline{\langle S_n^\infty\rangle}$. Thus we could ask: for which $n \in \mathbb{N}$ is it true that $\overline{\langle S_n^2\rangle} = (H^n, \mu_{1,n})_{[1]}$? We shall show shortly that this is equivalent to asking if the Hilbert multi-norm and the maximum multi-norm agree at level n.

LEMMA 4.3. *Let H be a Hilbert space, and suppose that $2 \le r < \infty$ and $n \in \mathbb{N}$. Then*

$$S_n^r \subset \operatorname{ex} \overline{\langle S_n^r\rangle}.$$

In the case where H is a finite-dimensional, $S_n^r = \operatorname{ex} \overline{\langle S_n^r\rangle}$.

Proof. Let X be the space H^n with the norm $\|\cdot\|$ given by

$$\|(x_i)\|_X = \Big(\sum_{i=1}^n \|x_i\|^r\Big)^{1/r} \qquad ((x_i) \in X).$$

Then S_n^r is a subset of the closed unit ball of X, and hence also $\overline{\langle S_n^r \rangle}$ is a subset of the closed unit ball of X. The space X is strictly convex (see, for example, [8]).

Assume towards a contradiction that $(y_i) \in S_n^r$, but that (y_i) is not an extreme point of $\overline{\langle S_n^r \rangle}$, so that we can find $x, z \in \overline{\langle S_n^r \rangle}$ with $x \neq z$ and $2y = x + z$. We then have

$$1 = \|y\|_X \leq \frac{1}{2}(\|x\|_X + \|z\|_X) \leq 1,$$

and so $\|x\|_X = \|z\|_X = 1$. By the strict convexity of X, we have $\|(x+z)/2\| < 1$ because $x \neq z$, a contradiction, as required.

Now suppose that H is finite-dimensional. Then the set S_n^r is closed, and so, by Mil'man's converse to the Krein–Mil'man theorem, $S_n^r = \mathrm{ex}\,\overline{\langle S_n^r \rangle}$. ∎

Finally, we show the link with the Hilbert multi-norm. In the result, we identify (anti-linearly) the dual space of H^n with H^n; a sequence (x_n) in a Hilbert space is *orthogonal* if $[x_i, x_j] = 0$ whenever $i \neq j$.

THEOREM 4.4. *Let H be a Hilbert space, and let $n \in \mathbb{N}$. Then:*

(i) *the unit ball of the dual of $(H^n, \|\cdot\|_n^H)$ is $\overline{\langle S_n^2 \rangle}$;*
(ii) *the unit ball of the dual of $(H^n, \|\cdot\|_n^{\max})$ is the unit ball of $(H^n, \mu_{1,n})$.*

In particular, $\|\cdot\|_n^H = \|\cdot\|_n^{\max}$ on H^n whenever $S_n^2 = \mathrm{ex}((H^n, \mu_{1,n})_{[1]})$.

Proof. For (i), let $y = (y_i) \in H^n$ be an orthogonal family with $\sum_{i=1}^n \|y_i\|^2 \leq 1$. Let $x = (x_i) \in H^n$ satisfy $\|x\|_n^H \leq 1$, and then choose a family $(P_i)_{i=1}^n$ of mutually orthogonal projections summing to I_H with $P_i(y_i) = y_i$ $(i \in \mathbb{N}_n)$. Then

$$[x, y] = \Big|\sum_{i=1}^n [x_i, P_i(y_i)]\Big| \leq \Big(\sum_{i=1}^n \|P_i(x_i)\|^2\Big)^{1/2} \cdot \Big(\sum_{i=1}^n \|y_i\|^2\Big)^{1/2} \leq \|x\|_n^H \leq 1.$$

Thus the norm of y as a functional on $(H^n, \|\cdot\|_n^H)$ is at most 1. We conclude that $S_n^2 \subset (H^n, \|\cdot\|_n^H)'_{[1]}$, and hence $\overline{\langle S_n^2 \rangle} \subset (H^n, \|\cdot\|_n^H)'_{[1]}$.

Conversely, assume towards a contradiction that $\overline{\langle S_n^2 \rangle} \subsetneq (H^n, \|\cdot\|_n^H)'_{[1]}$. Then there exists $x \in (H^n, \|\cdot\|_n^H)'_{[1]}$ such that a small open ball about x is disjoint from $\overline{\langle S_n^2 \rangle}$. By the Hahn–Banach theorem, there exists $z = (z_i) \in H^n$ and $\gamma \in \mathbb{R}$ such that

$$\Re\Big(\sum_{i=1}^n [z_i, x_i]\Big) < \gamma \leq \Re\Big(\sum_{i=1}^n [z_i, y_i]\Big) \qquad ((y_i) \in \overline{\langle S_n^2 \rangle}).$$

Since $\overline{\langle S_n^2 \rangle}$ is absolutely convex, we see that $\gamma < 0$, and so actually

$$-\Re\Big(\sum_{i=1}^n [z_i, x_i]\Big) > |\gamma| \geq \Big|\sum_{i=1}^n [z_i, y_i]\Big| \qquad ((y_i) \in \overline{\langle S_n^2 \rangle}).$$

Now observe that

$$\sup\Big\{\Big|\sum_{i=1}^n [z_i, y_i]\Big| : (y_i) \in \overline{\langle S_n^2 \rangle}\Big\}$$

is greater than or equal to

$$\sup\Big|\sum_{i=1}^{n}[z_i, y_i]\Big|$$

with the supremum taken over all orthogonal sequences in H with $\sum_{i=1}^{n}\|y_i\|^2 \le 1$, and that this supremum is equal to

$$\sup\Big|\sum_{i=1}^{n}\alpha_i[z_i, e_i]\Big|$$

taken over all orthonormal sequences (e_i) in H and all sequences (α_i) with $\sum_{i=1}^{n}|\alpha_i|^2 \le 1$. In its turn, this supremum is equal to

$$\sup\Big(\sum_{i=1}^{n}|[z_i, e_i]|^2\Big)^{1/2}$$

taken over all orthonormal sequences (e_i) in H, and hence, finally, to $\|(z_i)\|_n^H$. Thus

$$-\Re\Big(\sum_{i=1}^{n}[z_i, x_i]\Big) > \|(z_i)\|_n^H.$$

But this contradicts the fact that $(x_i) \in (H^n, \|\cdot\|_n^H)'_{[1]}$. Thus (i) holds.

For (ii), we know that $(H^n, \|\cdot\|_n^{\max}) \cong \ell_n^\infty \widehat{\otimes} H$, and that the dual of the latter space is $\ell_n^1 \check{\otimes} H' = \mathcal{B}(H, \ell_n^1)$. By definition, the space $(H^n, \mu_{1,n})$ can be identified with $\mathcal{B}(H, \ell_n^1)$, and so (ii) follows.

In conclusion, it follows that $\|\cdot\|_n^H = \|\cdot\|_n^{\max}$ if and only if $\overline{\langle S_n^2\rangle} = (H^n, \mu_{1,n})_{[1]}$. By the previous lemma, this equality holds whenever $S_n^2 = \operatorname{ex}(H^n, \mu_{1,n})_{[1]}$. ■

We shall show that indeed $S_n^2 = \operatorname{ex}(H^n, \mu_{1,n})_{[1]}$ when $n = 2$ or $n = 3$; thus, in these cases, we have a description of the dual space of $(H^n, \mu_{1,n})$, which may be of independent interest.

4.3. Calculation of c_2. We begin with an elementary result that shows that $c_2 = 1$.

THEOREM 4.5. *Let H be a complex Hilbert space. Then $\|\cdot\|_2^H = \|\cdot\|_2^{\max}$ on H^2.*

Proof. It is sufficient to prove the result in the case where the dimension of H is at least 2.

Set $L := (H^2, \mu_{1,2})_{[1]}$, and recall that

$$\mu_{1,2}(y_1, y_2) = \sup\{\|\zeta_1 y_1 + \zeta_2 y_2\| : \zeta_1, \zeta_2 \in \mathbb{T}\} \quad (y_1, y_2 \in H).$$

Let $(y_1, y_2) \in \operatorname{ex} L$. By replacing y_1 and y_2 by $\eta_1 y_1$ and $\eta_2 y_2$, respectively, for suitable $\eta_1, \eta_2 \in \mathbb{T}$, we may suppose that $\|y_1 + y_2\| = 1$, and so

$$\|y_1\|^2 + \|y_2\|^2 + 2\Re(\zeta_1\overline{\zeta_2}[y_1, y_2]) \le \|y_1\|^2 + \|y_2\|^2 + 2\Re[y_1, y_2] = 1$$

for each $\zeta_1, \zeta_2 \in \mathbb{T}$. We have $\Re(\zeta[y_1, y_2]) \le \Re[y_1, y_2]$ $(\zeta \in \mathbb{T})$, and so $[y_1, y_2] \ge 0$.

Assume towards a contradiction that $[y_1, y_2] > 0$.

Choose $u \in H$ with $\|u\| = 1$ such that $[y_1, u] = [u, y_2]$, and then choose $\varepsilon > 0$ with $\varepsilon^2 < [y_1, y_2]$. Set $w_1 = y_1 + \varepsilon u$ and $w_2 = y_2 - \varepsilon u$. Then we have

$$\|w_1\|^2 + \|w_2\|^2 = \|y_1\|^2 + \|y_2\|^2 + 2\varepsilon^2$$

because $[y_1, u] = [u, y_2]$, and

$$\Re\,(\zeta_1\overline{\zeta}_2([y_1, y_2] - \varepsilon^2)) \le [y_1, y_2] - \varepsilon^2 = [w_1, w_2]$$

for each $\zeta_1, \zeta_2 \in \mathbb{T}$ because $[y_1, y_2] - \varepsilon^2 > 0$. Hence

$$\|\zeta_1 w_1 + \zeta_2 w_2\| \le \|w_1 + w_2\| = 1 \qquad (\zeta_1, \zeta_2 \in \mathbb{T}),$$

and so $(y_1 + \varepsilon u,\ y_2 - \varepsilon u) \in L$. Similarly, $(y_1 - \varepsilon u,\ y_2 + \varepsilon u) \in L$. However

$$2(y_1, y_2) = (y_1 + \varepsilon u,\ y_2 - \varepsilon u) + (y_1 - \varepsilon u,\ y_2 + \varepsilon u).$$

It follows that (y_1, y_2) is not an extreme point of L, the required contradiction.

We have shown that $(y_1, y_2) \in S_2^2$. Thus $\operatorname{ex} L \subset S_2^2$, and so $L \subset \overline{\langle S_2^2\rangle}$. This implies that $L = \overline{\langle S_2^2\rangle}$ (and, by Lemma 4.3, we must also have $\operatorname{ex} L = S_2^2$.) ∎

4.4. Calculation of c_3. Next we consider the case where $n = 3$. In fact, there is now a difference between real and complex Hilbert spaces.

PROPOSITION 4.6. *Let H be a real Hilbert space of dimension at least 3. Then $\|\cdot\|_3^H$ and $\|\cdot\|_3^{\max}$ are not equal on H^3.*

Proof. It is sufficient to consider H to be the real 3-dimensional Hilbert space $\ell_3^2(\mathbb{R})$. Set $L = (H^3, \mu_{1,3})_{[1]}$.

For $y_1, y_2, y_3 \in H$, we now have

$$\mu_{1,3}(y_1, y_2, y_3) = \sup\{\|t_1 y_1 + t_2 y_2 + t_3 y_3\| : t_1, t_2, t_3 \in \{\pm 1\}\}.$$

Consider the vectors

$$y_1 = \frac{1}{\sqrt{11}}(1, 0, 0), \qquad y_2 = \frac{1}{\sqrt{11}}(1, 1, 0), \qquad y_3 = \frac{1}{\sqrt{11}}(-1, 2, 1).$$

We see that $[y_1, y_2] = [y_2, y_3] = 1/11$ and $[y_1, y_3] = -1/11$, and so

$$\|y_1 + y_2 + y_3\|^2 = \sum_{j=1}^{3} \|y_j\|^2 + 2\sum_{i<j} [y_i, y_j] = \frac{1}{11}(9 + 2 \cdot 1) = 1.$$

For each $t_1, t_2, t_3 \in \{\pm 1\}$, we have $t_1 t_2 - t_1 t_3 + t_2 t_3 \le 1$, and so it follows immediately that $\mu_{1,3}(y_1, y_2, y_3) = 1$, showing that $(y_1, y_2, y_3) \in L$. Note that the expression $t_1 t_2 - t_1 t_3 + t_2 t_3$ takes its maximum value of 1 when $t_1 = t_2 = t_3 = 1$, when $t_1 = t_2 = 1$ and $t_3 = -1$, and when $t_1 = 1$ and $t_2 = t_3 = -1$.

We *claim* that $\boldsymbol{y} := (y_1, y_2, y_3)$ is an extreme point of L.

Assume towards a contradiction that there exists $\boldsymbol{u} \in H^3$ with $\boldsymbol{u} \neq 0$ such that $\boldsymbol{y} \pm \boldsymbol{u} \in L$, say $\boldsymbol{u} = (u_1, u_2, u_3)$, with $u_1, u_2, u_3 \in H$.

Take $t_1, t_2, t_3 \in \{\pm 1\}$ with $t_1 t_2 - t_1 t_3 + t_2 t_3 = 1$. Then clearly $\|t_1 y_1 + t_2 y_2 + t_3 y_3\| = 1$. However

$$\|t_1(y_1 + u_1) + t_2(y_2 + u_2) + t_3(y_3 + u_3)\| \le 1$$

and

$$\|t_1(y_1 - u_1) + t_2(y_2 - u_2) + t_3(y_3 - u_3)\| \le 1.$$

Since H is strictly convex, it follows that $t_1 u_1 + t_2 u_2 + t_3 u_3 = 0$. By taking the various possibilities for t_1, t_2, t_3 such that $t_1 t_2 - t_1 t_3 + t_2 t_3 = 1$ specified above, we see that

$u_1+u_2+u_3=0$, that $u_1+u_2-u_3=0$, and that $u_1-u_2-u_3=0$. Thus $u_1=u_2=u_3=0$, a contradiction. Hence $(y_1,y_2,y_3)\in \operatorname{ex} L$.

Since $\{y_1,y_2,y_3\}$ is manifestly not an orthogonal set in H, it follows that $\boldsymbol{y}$ is not in the set S_3^2, and so the two multi-norms are not equal. ∎

We shall now show that we obtain a different result from the above in the case where H is a complex Hilbert space. Indeed $\|\cdot\|_3^H=\|\cdot\|_3^{\max}$ on each complex Hilbert space H. But now the (elementary) calculations seem to be much more challenging.

Lemma 4.7. *Take $(y_1,y_2,y_3)\in H^3$. Suppose that $(\xi_1,\xi_2,\xi_3)\in\mathbb{T}^3$ is such that*

$$\|\xi_1y_1+\xi_2y_2+\xi_3y_3\|=\max\{\|\eta_1y_1+\eta_2y_2+\eta_3y_3\|:(\eta_1,\eta_2,\eta_3)\in\mathbb{T}^3\}.$$

Then

$$\Im[\xi_1y_1,\xi_2y_2]=\Im[\xi_2y_2,\xi_3y_3]=\Im[\xi_3y_3,\xi_1y_1].$$

Proof. We see that $\|\eta_1y_1+\eta_2y_2+\eta_3y_3\|$ for $(\eta_1,\eta_2,\eta_3)\in\mathbb{T}^3$ attains its maximum at the point $(\eta_1,\eta_2,\eta_3)=(\xi_1,\xi_2,\xi_3)$ whenever

$$\Re(\eta_1\overline{\eta_2}[y_1,y_2])+\Re(\eta_2\overline{\eta_3}[y_2,y_3])+\Re(\eta_3\overline{\eta_1}[y_3,y_1])$$

attains its maximum at $(\eta_1,\eta_2,\eta_3)=(\xi_1,\xi_2,\xi_3)$.

Next set $[y_1,y_2]=a\exp(\mathrm{i}\alpha)$, $[y_2,y_3]=b\exp(\mathrm{i}\beta)$, and $[y_3,y_1]=c\exp(\mathrm{i}\gamma)$, where $a,b,c\geq 0$ and $\alpha,\beta,\gamma\in\mathbb{R}$. Also, take $t_1,t_2,t_3\in\mathbb{R}$ with $\eta_i=\exp(\mathrm{i}t_i)$ for $i=1,2,3$. Then the fact that the real-valued function

$$F:(t_1,t_2,t_3)\mapsto a\cos(t_1-t_2+\alpha)+b\cos(t_2-t_3+\beta)+c\cos(t_3-t_1+\gamma)$$

attains its maximum at (t_1,t_2,t_3) implies that

$$0=\frac{\partial F}{\partial t_1}(t_1,t_2,t_3)=\frac{\partial F}{\partial t_2}(t_1,t_2,t_3)=\frac{\partial F}{\partial t_3}(t_1,t_2,t_3),$$

and hence that

$$a\sin(t_1-t_2+\alpha)=b\sin(t_2-t_3+\beta)=c\sin(t_3-t_1+\gamma).$$

This gives the specified equations. ∎

In the following lemmas, A is the angle at the vertex A of the triangle ABC, and BC is the length of the side from B to C, etc. In the first two lemmas, ABC is a triangle (if such a triangle exists) with $BC=1/a$, $CA=1/b$, and $AB=1/c$, where $a,b,c>0$. Further, we shall consider the function

$$F:(r,s,t)\mapsto a\cos r+b\cos s+c\cos t,\qquad \mathbb{R}^3\to\mathbb{R}.$$

Lemma 4.8. *Consider F_π to be the restriction of F to the set*

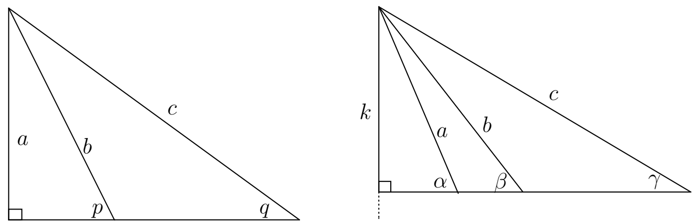
$$\{(r,s,t)\in\mathbb{R}^3:r+s+t\equiv\pi \pmod{2\pi}\}.$$

(i) *Suppose that the triangle ABC exists. Then F_π attains its maximum at exactly two points $(r,s,t)=(A,B,C)$ or $(r,s,t)=(-A,-B,-C)$ $(\mathrm{mod}\ 2\pi)$.*
(ii) *Suppose that the triangle ABC does not exist and that $a\leq b\leq c$. Then F_π attains its maximum at exactly the point $(r,s,t)=(\pi,0,0)$ $(\mathrm{mod}\ 2\pi)$.*

Proof. This is elementary. ∎

Lemma 4.9. *Suppose that $M \not\equiv \pi \pmod{2\pi}$, and consider F_M to be the restriction of F to the set*
$$\{(r,s,t) \in \mathbb{R}^3 : r+s+t \equiv M \pmod{2\pi}\}.$$
Then F_M attains its maximum at exactly one tuple $(r,s,t) \pmod{2\pi}$.

Proof. Without loss of generality, we may suppose that $a \le b \le c$. The case where $M = 0 \pmod{2\pi}$ is obvious. Replacing M by $M + 2k\pi$ or $2k\pi - M$, if necessary, we may suppose that $0 < M < \pi$. Note that the maximum of F_M is at least
$$a\cos M + b + c > b + c - a \ge c.$$
Set
$$p = \arcsin(a/b) \quad \text{and} \quad q = \arcsin(a/c),$$
so that we have the picture below.



Suppose that (r,s,t) is any point where F_M attains its maximum; say $r,s,t \in (-\pi,\pi]$. We have seen that (r,s,t) must satisfy
$$a\sin r = b\sin s = c\sin t \quad \text{as well as} \quad r+s+t \equiv M \pmod{2\pi}. \tag{4.1}$$
Set $h = a\sin r$. Then $h \ne 0$ and
$$\cos r = \pm\sqrt{1-\frac{h^2}{a^2}}, \quad \cos s = \pm\sqrt{1-\frac{h^2}{b^2}}, \quad \text{and} \quad \cos t = \pm\sqrt{1-\frac{h^2}{c^2}}.$$
Since $a \le b \le c$ and $F_M(r,s,t) > c$, we deduce that
$$\cos s = \sqrt{1-\frac{h^2}{b^2}} \quad \text{and} \quad \cos t = \sqrt{1-\frac{h^2}{c^2}},$$
so that $s = \arcsin(h/b)$ and $t = \arcsin(h/c)$. In particular, we must have
$$|s| \le p \quad \text{and} \quad |t| \le q.$$

Assume toward a contradiction that $h < 0$. In the case where $\cos r \ge 0$, we see that $r,s,t \in [-\pi/2, 0)$ and so $r+s+t = M - 2\pi$. This implies that
$$p + q \ge \frac{3\pi}{2} - M > \frac{\pi}{2}.$$
In particular, we must have $1/b^2 + 1/c^2 > 1/a^2$, so that ABC is an (acute) triangle. Since $3\pi/2 \le 2\pi + r < \pi - s - t \le 2\pi$, we see that
$$F_M(r,s,t) = F_M(2\pi + r, s, t) < F(\pi - s - t, s, t) \le F(A,B,C) < F(A',B',C'),$$

where the second inequality follows from Lemma 4.8 and where $A' \in (0, A), B' \in (0, B)$, and $C' \in (0, C)$ are such that $A' + B' + C' = M$ (this is possible since $0 < M < \pi$). This contradicts the assumption that F_M attains its maximum at (r, s, t).

In the case where $\cos r < 0$, we see that $r \in (-\pi, -\pi/2)$, whereas $s, t \in [-\pi/2, 0)$, and so $r + s + t = M - 2\pi$. It follows that $\pi > -r > M - \pi - r = \pi + s + t > 0$, and so

$$F_M(r, s, t) < a\cos(\pi + s + t) + b\cos(-s) + c\cos(-t) = F(\pi + s + t, -s, -t).$$

Choosing $u \in (0, \pi + s + t)$, $v \in (0, -s)$, and $w \in (0, -t)$ such that $u + v + w = M$, which is possible since $0 < M < \pi$, the above implies that

$$F_M(r, s, t) < F_M(u, v, w).$$

This again contradicts the assumption that F_M attains its maximum at (r, s, t).

Thus we must have $h > 0$, so that $r \in (0, \pi)$, $s \in (0, p]$, and $t \in (0, q]$. We consider the following two cases.

Case 1: $M \le \pi/2 + p + q$. Assume toward a contradiction that $\cos r < 0$. Then $r \in (\pi/2, \pi)$, whereas $s, t \in (0, \pi/2]$, and so $r + s + t = M$. Consider the function g defined by

$$g(k) = \pi - \arcsin\left(\frac{k}{a}\right) + \arcsin\left(\frac{k}{b}\right) + \arcsin\left(\frac{k}{c}\right) \qquad (0 \le k \le a).$$

Then $g(h) = M$ and $g(a) = \pi/2 + p + q$. If $p + q \ge \pi/2$, then $g(h) < \pi \le g(a)$, and so there exists $k \in (h, a]$ such that $g(k) = \pi$. But this means that $\pi - \arcsin(k/a)$, $\arcsin(k/b)$, and $\arcsin(k/c)$ are three angles of a triangle whose sides are $1/a$, $1/b$ and $1/c$. In particular, this implies that ABC is a triangle with $A \ge \pi/2$, so that $1/a^2 \ge 1/b^2 + 1/c^2$, which means that $p + q \le \pi/2$. Thus we must have $p + q \le \pi/2$ anyways, so that $1/a^2 \ge 1/b^2 + 1/c^2$.

We see that

$$g'(k) = -\frac{1}{\sqrt{a^2 - k^2}} + \frac{1}{\sqrt{b^2 - k^2}} + \frac{1}{\sqrt{c^2 - k^2}},$$

and, for $k \in (0, a)$, since $1/a^2 \ge 1/b^2 + 1/c^2$, we have

$$\begin{aligned} g''(k) &= -\frac{k}{(a^2 - k^2)^{3/2}} + \frac{k}{(b^2 - k^2)^{3/2}} + \frac{k}{(c^2 - k^2)^{3/2}} \\ &< -\frac{k/a^3}{\left(1 - \dfrac{k^2}{a^2}\right)^{3/2}} + \frac{k/b^3}{\left(1 - \dfrac{k^2}{a^2}\right)^{3/2}} + \frac{k/c^3}{\left(1 - \dfrac{k^2}{a^2}\right)^{3/2}} < 0. \end{aligned}$$

Note that $g'(0) > 0$ and $g'(a) = -\infty$. So we see that there exists a unique $k_0 \in (0, a)$ such that $g'(k_0) = 0$, g is strictly increasing on $(0, k_0)$, and g is strictly decreasing on (k_0, a). In particular, since $h \in (0, a)$, we must have $M = g(h) > \min\{g(0), g(a)\} = \pi/2 + p + q$; a contradiction of the assumption of Case 1.

Thus we must have $\cos r \ge 0$, and so $r, s, t \in (0, \pi/2]$. Hence (r, s, t) must be the unique triple (α, β, γ) that satisfies (4.1) and such that $\alpha, \beta, \gamma \in (0, \pi/2]$ (see the picture).

Case 2: $M > \pi/2 + p + q$. In this case, there exists no triple (α, β, γ) that satisfies (4.1) and such that $\alpha, \beta, \gamma \in (0, \pi/2]$, and so $r \in (\pi/2, \pi]$. It follows that $r + s + t = M$. We also see from the assumption that $p + q < \pi/2$, so that $1/a^2 > 1/b^2 + 1/c^2$. Consider the function g defined as in Case 1. Then $g(h) = M$. We again find a unique $k_0 \in (0, a)$

such that g is strictly increasing on $(0, k_0)$, and g is strictly decreasing on (k_0, a). Since $g(a) = \pi/2 + p + q < M < g(0) = \pi$, h is the unique point $l \in (k_0, a)$ such that $g(l) = M$. This shows that (r, s, t) is the unique triple (mod 2π) at which F_M attains its maximum. ■

We summarize the above lemmas in the setting of our problem as follows.

Let $(y_1, y_2, y_3) \in L$, where $L = (H^3, \mu_{1,3})_{[1]}$. For $(\eta_1, \eta_2, \eta_3) \in \mathbb{T}^3$, set

$$N(\eta_1, \eta_2, \eta_3) = \|\eta_1 y_1 + \eta_2 y_2 + \eta_3 y_3\|$$

and

$$F(\eta_1, \eta_2, \eta_3) = \Re(\eta_1 \overline{\eta_2}[y_1, y_2]) + \Re(\eta_2 \overline{\eta_3}[y_2, y_3]) + \Re(\eta_3 \overline{\eta_1}[y_3, y_1]),$$

so that N and F attain their maxima at the same tuple(s) (η_1, η_2, η_3).

We shall now use square-bracket notation $[\eta_1, \eta_2, \eta_3]$ to denote the class of all tuples $(\zeta\eta_1, \zeta\eta_2, \zeta\eta_3)$ $(\zeta \in \mathbb{T})$; we shall also call $[\eta_1, \eta_2, \eta_3]$ a 'tuple', with the understanding that we are identifying all those $[\zeta\eta_1, \zeta\eta_2, \zeta\eta_3]$ for which $\zeta \in \mathbb{T}$.

Set

$$a = |\,[y_1, y_2]\,|, \qquad b = |\,[y_2, y_3]\,|, \qquad c = |\,[y_1, y_2]\,|,$$

and then set $M = \arg[y_1, y_2] + \arg[y_2, y_3] + \arg[y_3, y_1]$.

Suppose that $a, b, c > 0$. Then, by the previous three lemmas (and inspecting their proofs as well), we have $\max F(\eta_1, \eta_2, \eta_3) > \max\{a, b, c\}$, and there are the following cases:

I. $M \equiv 0 \pmod{2\pi}$: N attains its maximum at the unique $[\xi_1, \xi_2, \xi_3]$ in $\mathbb{T}^3$ satisfying the conditions that $\xi_1\overline{\xi_2}[y_1, y_2] > 0$, that $\xi_2\overline{\xi_3}[y_2, y_3] > 0$, and that $\xi_3\overline{\xi_1}[y_3, y_1] > 0$. (Actually, if any two of these inequalities hold, then the third must also hold.)

II. $M \equiv \pi \pmod{2\pi}$ and $1/a$, $1/b$, and $1/c$ are the sides of a triangle: N attains its maximum at those $[\xi_1, \xi_2, \xi_3]$ in $\mathbb{T}^3$ such that

$$\Im(\xi_1\overline{\xi_2}[y_1, y_2]) = \Im(\xi_2\overline{\xi_3}[y_2, y_3]) = \Im(\xi_3\overline{\xi_1}[y_3, y_1]) =: k \neq 0.$$

There are exactly 2 such tuples $[\xi_1, \xi_2, \xi_3]$, and, moreover, for one such tuple, $k > 0$ and, for the other, $k < 0$.

III. $M \equiv \pi \pmod{2\pi}$ and $1/a$, $1/b$, $1/c$ cannot be the sides of any triangle: N attains its maximum at the unique $[\xi_1, \xi_2, \xi_3]$ in $\mathbb{T}^3$ such that

$$\Im(\xi_1\overline{\xi_2}[y_1, y_2]) = \Im(\xi_2\overline{\xi_3}[y_2, y_3]) = \Im(\xi_3\overline{\xi_1}[y_3, y_1]) = 0.$$

IV. $0 < M < \pi \pmod{2\pi}$: N attains its maximum at the unique $[\xi_1, \xi_2, \xi_3]$ in $\mathbb{T}^3$ such that

$$\Im(\xi_1\overline{\xi_2}[y_1, y_2]) = \Im(\xi_2\overline{\xi_3}[y_2, y_3]) = \Im(\xi_3\overline{\xi_1}[y_3, y_1]) =: k > 0.$$

V. $\pi < M < 2\pi \pmod{2\pi}$: N attains maximum at the unique $[\xi_1, \xi_2, \xi_3]$ in $\mathbb{T}^3$ such that

$$\Im(\xi_1\overline{\xi_2}[y_1, y_2]) = \Im(\xi_2\overline{\xi_3}[y_2, y_3]) = \Im(\xi_3\overline{\xi_1}[y_3, y_1]) =: k < 0.$$

Now take $(y_1, y_2, y_3) \in L$, and suppose that N attains its maximum on $\mathbb{T}^3$ at the point $(\xi_1, \xi_2, \xi_3) \in \mathbb{T}^3$. Consider the elements $u = (u_1, u_2, u_3) \in H^3$ with $u \neq 0$, if any, such that

$$(y_1 + \varepsilon u_1, y_2 + \varepsilon u_2, y_3 + \varepsilon u_3) \in L$$

for $\varepsilon = -1$ and $\varepsilon = 1$, and hence, by convexity, for all $\varepsilon \in [-1, 1]$. Since

$$\xi_1 y_1 + \xi_2 y_2 + \xi_3 y_3 \in \operatorname{ex} H_{[1]},$$

it follows that $\xi_1 u_1 + \xi_2 u_2 + \xi_3 u_3 = 0$. So, for each $\varepsilon \in [-1, 1]$, the function

$$(\eta_1, \eta_2, \eta_3) \mapsto \|\eta_1(y_1 + \varepsilon u_1) + \eta_2(y_2 + \varepsilon u_2) + \eta_3(y_3 + \varepsilon u_3)\|, \qquad \mathbb{T}^3 \to \mathbb{R}^+,$$

also attains its maximum at (ξ_1, ξ_2, ξ_3). Lemma 4.7 then implies that

$$\Im(\xi_1\overline{\xi_2}[y_1 + \varepsilon u_1, y_2 + \varepsilon u_2]) = \Im(\xi_2\overline{\xi_3}[y_2 + \varepsilon u_2, y_3 + \varepsilon u_3]) = \Im(\xi_3\overline{\xi_1}[y_3 + \varepsilon u_3, y_1 + \varepsilon u_1]).$$

Since $\xi_1 u_1 + \xi_2 u_2 + \xi_3 u_3 = 0$, the coefficients of ε^2 are equal. Comparing the coefficients of ε, we see that the above equalities are equivalent to

$$[u_i, \xi_1 y_1 + \xi_2 y_2 + \xi_3 y_3] = 0 \qquad (i = 1, 2, 3).$$

Theorem 4.10. *Let H be a complex Hilbert space. Then* $\operatorname{ex}(H^3, \mu_{1,3})_{[1]} = S_3^2$, *and* $\|\cdot\|_3^H = \|\cdot\|_3^{\max}$ *on* H^3.

Proof. It is sufficient to consider only the case where H has dimension at least 3.

Let $(y_1, y_2, y_3) \in \operatorname{ex} L$, where $L = (H^3, \mu_{1,3})_{[1]}$ as before. For $(\eta_1, \eta_2, \eta_3) \in \mathbb{T}^3$, we define $N(\eta_1, \eta_2, \eta_3)$ and $F(\eta_1, \eta_2, \eta_3)$, and then a, b, c, M, as before.

Suppose that N attains its maximum, which is 1, at $[\xi_1, \xi_2, \xi_3]$ in $\mathbb{T}^3$. Let (u_1, u_2, u_3) in H^3 be non-zero and such that

$$\begin{aligned} \xi_1 u_1 + \xi_2 u_2 + \xi_3 u_3 &= 0 \\ [u_i, \xi_1 y_1 + \xi_2 y_2 + \xi_3 y_3] &= 0 \qquad (i \in \mathbb{N}_3). \end{aligned}$$

In the case where N attains maximum at another (different) tuple $[\zeta_1, \zeta_2, \zeta_3]$ in $\mathbb{T}^3$, we require, further, that (u_1, u_2, u_3) also satisfies

$$\begin{aligned} \zeta_1 u_1 + \zeta_2 u_2 + \zeta_3 u_3 &= 0 \\ [u_i, \zeta_1 y_1 + \zeta_2 y_2 + \zeta_3 y_3] &= 0 \qquad (i \in \mathbb{N}_3). \end{aligned}$$

It is easy to see that such (u_1, u_2, u_3) always exists.

For each $\varepsilon \in \mathbb{R}$, set $y_{i,\varepsilon} = y_i + \varepsilon u_i$. For $(\eta_1, \eta_2, \eta_3) \in \mathbb{T}^3$, set

$$N_\varepsilon(\eta_1, \eta_2, \eta_3) = \|\eta_1 y_{1,\varepsilon} + \eta_2 y_{2,\varepsilon} + \eta_3 y_{3,\varepsilon}\|$$

and

$$F_\varepsilon(\eta_1, \eta_2, \eta_3) = \Re(\eta_1\overline{\eta_2}[y_{1,\varepsilon}, y_{2,\varepsilon}]) + \Re(\eta_2\overline{\eta_3}[y_{2,\varepsilon}, y_{3,\varepsilon}]) + \Re(\eta_3\overline{\eta_1}[y_{3,\varepsilon}, y_{1,\varepsilon}]).$$

Finally, set

$$a_\varepsilon = |\,[y_{1,\varepsilon}, y_{2,\varepsilon}]\,|, \qquad b_\varepsilon = |\,[y_{2,\varepsilon}, y_{3,\varepsilon}]\,|, \qquad c_\varepsilon = |\,[y_{3,\varepsilon}, y_{1,\varepsilon}]\,|,$$

and set $M = \arg[y_{1,\varepsilon}, y_{2,\varepsilon}] + \arg[y_{2,\varepsilon}, y_{3,\varepsilon}] + \arg[y_{3,\varepsilon}, y_{1,\varepsilon}]$. Then we see that

$$N_\varepsilon(\xi_1, \xi_2, \xi_3) = N_\varepsilon(\zeta_1, \zeta_2, \zeta_3) = 1,$$

and, from the discussion preceding this theorem, we have

$$\begin{aligned} \Im(\xi_1\overline{\xi_2}[y_{1,\varepsilon}, y_{2,\varepsilon}]) = \Im(\xi_2\overline{\xi_3}[y_{2,\varepsilon}, y_{3,\varepsilon}]) = \Im(\xi_3\overline{\xi_1}[y_{3,\varepsilon}, y_{1,\varepsilon}]) &=: I_\varepsilon \\ \Im(\zeta_1\overline{\zeta_2}[y_{1,\varepsilon}, y_{2,\varepsilon}]) = \Im(\zeta_2\overline{\zeta_3}[y_{2,\varepsilon}, y_{3,\varepsilon}]) = \Im(\zeta_3\overline{\zeta_1}[y_{3,\varepsilon}, y_{1,\varepsilon}]) &=: J_\varepsilon. \end{aligned}$$

(The above equalities about $[\zeta_1, \zeta_2, \zeta_3]$ are considered only when the relevant tuple exists.)

First, we *claim* that, in the case where both $I_0 = 0$ and $F(\xi_1, \xi_2, \xi_3) > 0$, for $|\varepsilon|$ sufficiently small, the sign of I_ε and the sign of

$$\Im([y_{1,\varepsilon}, y_{2,\varepsilon}][y_{2,\varepsilon}, y_{3,\varepsilon}][y_{3,\varepsilon}, y_{1,\varepsilon}]) = \Im\big(\xi_1\overline{\xi_2}[y_{1,\varepsilon}, y_{2,\varepsilon}]\xi_2\overline{\xi_3}[y_{2,\varepsilon}, y_{3,\varepsilon}]\xi_3\overline{\xi_1}[y_{3,\varepsilon}, y_{1,\varepsilon}]\big)$$

are the same. Indeed, since $I_0 = 0$, this can be verified by considering the cases where the coefficients of ε or ε^2 in I_ε are non-zero. This claim implies that, in the case where both $I_0 = 0$ and $F(\xi_1, \xi_2, \xi_3) > 0$:

(i) $0 < M_\varepsilon < \pi \pmod{2\pi}$ implies that $I_\varepsilon > 0$;
(ii) $\pi < M_\varepsilon < 2\pi \pmod{2\pi}$ implies that $I_\varepsilon < 0$;
(iii) $M_\varepsilon \equiv 0$ or $\pi \pmod{2\pi}$ implies that $I_\varepsilon = 0$.

Assume toward a contradiction that $a, b, c > 0$. Then, for $|\varepsilon|$ sufficiently small, we have $a_\varepsilon, b_\varepsilon, c_\varepsilon > 0$. As discussed above, there are five cases:

Case 1: (y_1, y_2, y_3) falls in class **I**. Then, for sufficiently small $|\varepsilon|$, we also have

$$\Re(\xi_1\overline{\xi_2}[y_{1,\varepsilon}, y_{2,\varepsilon}]) > 0, \quad \Re(\xi_2\overline{\xi_3}[y_{2,\varepsilon}, y_{3,\varepsilon}]) > 0, \quad \Re(\xi_3\overline{\xi_1}[y_{3,\varepsilon}, y_{1,\varepsilon}]) > 0,$$
$$F_\varepsilon(\xi_1, \xi_2, \xi_3) > \max\{a_\varepsilon, b_\varepsilon, c_\varepsilon\}, \quad \text{and} \quad M_\varepsilon \text{ is `close' to } 0 \bmod 2\pi.$$

By the claim, if $0 < M_\varepsilon < \pi \pmod{2\pi}$, then $I_\varepsilon > 0$, so that $(y_{1,\varepsilon}, y_{2,\varepsilon}, y_{3,\varepsilon})$ belongs to class **IV**, and N_ε attains its maximum at $[\xi_1, \xi_2, \xi_3]$. If $\pi < M_\varepsilon < 2\pi \pmod{2\pi}$, then $I_\varepsilon < 0$, so that $(y_{1,\varepsilon}, y_{2,\varepsilon}, y_{3,\varepsilon})$ belongs to class **V**, and N_ε attains its maximum at $[\xi_1, \xi_2, \xi_3]$. Finally, if $M_\varepsilon = 0 \pmod{2\pi}$, then $I_\varepsilon = 0$, so that $(y_{1,\varepsilon}, y_{2,\varepsilon}, y_{3,\varepsilon})$ belongs to class **I**, and N_ε again attains its maximum at $[\xi_1, \xi_2, \xi_3]$. Thus we always have $(y_{1,\varepsilon}, y_{2,\varepsilon}, y_{3,\varepsilon}) \in L$ for $|\varepsilon|$ sufficiently small, and so (y_1, y_2, y_3) cannot be an extreme point of L.

Case 2: (y_1, y_2, y_3) falls into class **II**. Suppose that $I_0 > 0$ and $J_0 < 0$. Then, for sufficiently small $|\varepsilon|$, we also have

$$1/a_\varepsilon, 1/b_\varepsilon, 1/c_\varepsilon \text{ are the sides of a triangle}, \quad I_\varepsilon > 0, \quad J_\varepsilon < 0,$$
$$F_\varepsilon(\xi_1, \xi_2, \xi_3) > \max\{a_\varepsilon, b_\varepsilon, c_\varepsilon\}, \quad F_\varepsilon(\zeta_1, \zeta_2, \zeta_3) > \max\{a_\varepsilon, b_\varepsilon, c_\varepsilon\},$$
$$\text{and} \quad M_\varepsilon \text{ is `close' to } \pi \bmod 2\pi.$$

If $0 < M_\varepsilon < \pi \pmod{2\pi}$, then $(y_{1,\varepsilon}, y_{2,\varepsilon}, y_{3,\varepsilon})$ belongs to class **IV**, and N_ε attains its maximum at $[\xi_1, \xi_2, \xi_3]$. If $\pi < M_\varepsilon < 2\pi \pmod{2\pi}$, then $(y_{1,\varepsilon}, y_{2,\varepsilon}, y_{3,\varepsilon})$ belongs to class **V**, and N_ε attains its maximum at $[\zeta_1, \zeta_2, \zeta_3]$. Finally, if $M_\varepsilon = \pi \pmod{2\pi}$, then $(y_{1,\varepsilon}, y_{2,\varepsilon}, y_{3,\varepsilon})$ belongs to class **II**, and N_ε attains its maximum at both $[\xi_1, \xi_2, \xi_3]$ and $[\zeta_1, \zeta_2, \zeta_3]$. Thus we always have $(y_{1,\varepsilon}, y_{2,\varepsilon}, y_{3,\varepsilon}) \in L$ for $|\varepsilon|$ sufficiently small, and so (y_1, y_2, y_3) cannot be an extreme point of L.

The other cases where (y_1, y_2, y_3) falls into classes **III, IV, V** can be covered by similar arguments to obtain contradictions.

Thus we have proved that one of a, b, c must be 0. Say $a = 0$. Assume toward a contradiction that $b, c > 0$. Then we see that N attains its maximum at the unique $[\xi_1, \xi_2, \xi_3]$ in $\mathbb{T}^3$ such that $\xi_2\overline{\xi_3}[y_2, y_3] > 0$ and $\xi_3\overline{\xi_1}[y_3, y_1] > 0$. We also see easily that $F(\xi_1, \xi_2, \xi_3) > \max\{a = 0, b, c\}$. If $a_\varepsilon \equiv 0$, then obviously, when $|\varepsilon|$ is sufficiently small N_ε again attains its maximum at $[\xi_1, \xi_2, \xi_3]$, and so $(y_{1,\varepsilon}, y_{2,\varepsilon}, y_{3,\varepsilon}) \in L$, hence (y_1, y_2, y_3) cannot be an extreme point of L. So $a_\varepsilon \neq 0$ for sufficiently small $|\varepsilon|$. Again, we can argue as above, checking $(y_{1,\varepsilon}, y_{2,\varepsilon}, y_{3,\varepsilon})$ against each of the classes **I** and **III**–**V** (we can avoid class **II**) and the case where $[y_{1,\varepsilon}, y_{2,\varepsilon}] = 0$ to arrive at a contradiction.

Now we have proved that two of a, b, or c must be 0. We can now argue as above to show that all a, b, c are 0. Hence $(y_1, y_2, y_3) \in S_3^2$.

Thus we have proved that $\operatorname{ex} L \subset S_3^2$. This implies that $L \subset \langle S_3^2 \rangle \subset L$. Hence $\operatorname{ex} L = S_3^2$ and the proof is complete. ■

4.5. Calculation of c_4. We can give some information about the constant c_4.

THEOREM 4.11. *Let H be a complex Hilbert space of dimension at least* 3. *Then $\|\cdot\|_n^H$ is not equal to $\|\cdot\|_n^{\max}$ on H^n for every $n \geq 4$.*

Proof. It is sufficient to consider the case where $n = 4$ and $H = \ell_3^2$. Set $L := (H^4, \mu_{1,4})_{[1]}$. Set $x_1 = (1, 0, 0)$, $x_2 = (-1, 2, 0)$, $x_3 = (-1, -1, 3)$, and $x_4 = (-1, -1, -1)$. Then we have $[x_i, x_j] = -1$ for every $i, j \in \mathbb{N}_4$ with $i \neq j$. For each $(\xi_1, \xi_2, \xi_3, \xi_4) \in \mathbb{T}^4$, we have

$$\Re \sum_{i<j} \xi_i \xi_j \geq -2,$$

with the minimum attained at those $(\xi_1, \xi_2, \xi_3, \xi_4) \in \mathbb{T}^4$ for which $\xi_1 + \cdots + \xi_4 = 0$, and so it follows that the function

$$(\xi_1, \xi_2, \xi_3, \xi_4) \mapsto \|\xi_1 x_1 + \cdots + \xi_4 x_4\|, \qquad \mathbb{T}^4 \to \mathbb{R},$$

attains its maximum at each $(\xi_1, \xi_2, \xi_3, \xi_4) \in S$, where we set

$$\begin{aligned} S &:= \{(\xi_1, \xi_2, \xi_3, \xi_4) \in \mathbb{T}^4 \colon \xi_1 + \cdots + \xi_4 = 0\} \\ &= \{(\xi_1, \xi_2, -\xi_1, -\xi_2) \text{ and } (\xi_1, \xi_2, -\xi_2, -\xi_1) : \xi_1, \xi_2 \in \mathbb{T}\}. \end{aligned}$$

Let $\boldsymbol{y} = (y_1, \ldots, y_4)$ be a scaling of $(x_1, \ldots, x_4)$ such that $\mu_{1,4}((y_1, \ldots, y_4)) = 1$. In particular, $\boldsymbol{y} \in L \setminus S_4^2$. We also have

$$\|\xi_1 y_1 + \cdots + \xi_4 y_4\| \leq 1$$

for every $(\xi_1, \xi_2, \xi_3, \xi_4) \in \mathbb{T}^4$, and the equality is attained whenever $(\xi_1, \xi_2, \xi_3, \xi_4) \in S$.

Suppose that $\boldsymbol{u} = (u_1, \ldots, u_4) \in H^4$ is such that $\boldsymbol{y} \pm \boldsymbol{u} \in L$. Then, for every $(\xi_1, \xi_2, \xi_3, \xi_4) \in \mathbb{T}^4$ and every $\varepsilon \in [-1, 1]$, we have

$$\|\xi_1(y_1 + \varepsilon u_1) + \cdots + \xi_4(y_4 + \varepsilon u_4)\| \leq 1.$$

In particular, for each $(\xi_1, \xi_2, \xi_3, \xi_4) \in S$, since $\xi_1 y_1 + \cdots + \xi_4 y_4$, being of norm 1, is an extreme point of $H_{[1]}$, we obtain

$$\xi_1 u_1 + \cdots + \xi_4 u_4 = 0.$$

This implies that $u_1 = \cdots = u_4 =: u$.

Fix an $\varepsilon \in \mathbb{R}$ with $|\varepsilon|$ sufficiently small so that $a_i, b_i > 0$ and $A_i, B_i \in (\pi/2, 3\pi/2)$ $(i \in \mathbb{N}_3)$ can be chosen to satisfy the following equations:

$$\begin{gathered} a_i \exp(\mathrm{i}A_i) = [y_i + \varepsilon u, y_4 + \varepsilon u] \quad (i = 1, 2, 3), \qquad b_1 \exp(\mathrm{i}B_1) = [y_2 + \varepsilon u, y_3 + \varepsilon u], \\ b_2 \exp(\mathrm{i}B_2) = [y_3 + \varepsilon u, y_1 + \varepsilon u], \qquad \text{and} \qquad b_3 \exp(\mathrm{i}B_3) = [y_1 + \varepsilon u, y_2 + \varepsilon u]; \end{gathered}$$

this can be done since $[y_i, y_j] < 0$ for every $i, j \in \mathbb{N}_4$ with $i \neq j$. Using

$$\xi_i = \exp(\mathrm{i}\alpha_i) \quad (i \in \mathbb{N}_3), \qquad \text{and} \qquad \xi_4 = 1,$$

the previous paragraph then implies that the function $f : \mathbb{R}^3 \to \mathbb{R}$ defined by

$$f(\alpha_1, \alpha_2, \alpha_3) := a_1 \cos(\alpha_1 + A_1) + a_2 \cos(\alpha_2 + A_2) + a_3 \cos(\alpha_3 + A_3) \\ + b_1 \cos(\alpha_2 - \alpha_3 + B_1) + b_2 \cos(\alpha_3 - \alpha_1 + B_2) + b_3 \cos(\alpha_1 - \alpha_2 + B_3),$$

attains its maximum at $(\alpha, \pi, \alpha + \pi)$ and $(\pi, \alpha, \alpha + \pi)$ for every $\alpha \in \mathbb{R}$. In particular, these triples must be solutions of the equations

$$0 = \frac{\partial f}{\partial \alpha_1}(\alpha_1, \alpha_2, \alpha_3) = \frac{\partial f}{\partial \alpha_2}(\alpha_1, \alpha_2, \alpha_3) = \frac{\partial f}{\partial \alpha_3}(\alpha_1, \alpha_2, \alpha_3).$$

This implies that $A_i = B_i = \pi$ $(i \in \mathbb{N}_3)$ and $a_1 = a_2 = a_3 = b_1 = b_2 = b_3$.

Thus we have shown that, for each $\varepsilon \in \mathbb{R}$ with sufficiently small $|\varepsilon|$, all the numbers

$$[y_i + \varepsilon u, y_j + \varepsilon u] \quad (i, j \in \mathbb{N}_4,\ i \neq j)$$

are equal to the same negative real number. Thus, the numbers

$$[y_i, u] + [u, y_j] \quad (i, j \in \mathbb{N}_4,\ i \neq j)$$

are all equal, and since $\boldsymbol{y} = (y_1, \dots, y_4)$ is a scaling of $(x_1, \dots, x_4)$, we deduce that

$$[u, x_1] = [u, x_2] = [u, x_3] = [u, x_4].$$

Solving these linear equations, we obtain $u = 0$. This implies that $\boldsymbol{y}$ is an extreme point of L. Hence $S_4^2 \subsetneq \operatorname{ex} L$, and so $\|\cdot\|_4^H \neq \|\cdot\|_4^{\max}$ on H^4. ∎

The above calculation shows that $1 < c_4 \leq c_n \leq 2/\sqrt{\pi}$ for all $n \geq 4$. However, we have not calculated the actual value of c_4, or of any c_n for $n \geq 4$.

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