0. Introduction

In the dissertation we study the theory of improper intersections in complex analytic geometry. Roughly speaking, this theory investigates intersections of analytic sets or, equivalently, solutions for systems of analytic equations. It is very young in comparison with the theory of intersections of algebraic sets, which deals with systems of polynomial equations. In fact, Draper's paper [13] may be regarded as the first systematic exposition of the analytic theory of proper intersections, whereas the algebraic theory of proper intersections can be traced back near the beginnings of algebraic geometry, and one may link its origin with Isaac Newton's paper *Geometria analytica* of 1680.

Although clear and precise statements did not appear until relatively recently, it is already in the work of Isaac Newton and his contemporaries that two approaches, remaining vital until the present, can be found: the dynamic one, where the multiplicity of a solution is the number of solutions near the given solution when the equations are varied; and the static one, where the multiplicity is obtained without varying the given equations. The definition of the multiplicity $i(X \cdot Y; W)$ of intersection of two varieties X and Y along an irreducible component of the intersection $X \cap Y$ was somewhat problematic. A famous controversy arose about the definition given by the Italian algebraic geometers in the 19th century. They arrived at the following dynamic concept: move the varieties X and Y so that the intersection is proper and transversal, and then $i(X \cdot Y; W)$ is the number of components into which W splits. This dynamic approach was defended in modern algebraic geometry by Severi [50], who linked in [51] both the dynamic and static approaches to the theory of intersections of algebraic sets.

Yet the foregoing definition of intersection multiplicity had not been generally accepted. It was van der Waerden [60] who first developed the rigorous algebraic notion of specialization to make intuitive geometric ideas precise and valid over general ground fields. This enabled him to arrive at a precise algebraic definition of multiplicity for global projective sets (indicating at the same time that the classical Italian definition was essentially correct), and to discover the general version of Bezout's theorem which says that for projective varieties X and Y that meet properly,

$$\deg X \cdot \deg Y = \sum i(X \cdot Y; W) \deg W,$$

where W runs through the irreducible components. Bezout's theorem is a landmark both in the classical algebraic geometry and in the recent theory of improper intersections. Let us mention that originally Bezout only considered proper intersections of n projective hypersurfaces in $\mathbb{P}_n$ that meet in a finite number of points (cf. [7]).

Using the diagonal procedure, one can develop the general notion of intersection multiplicity. This procedure was applied for the first time in modern algebraic geometry by Weil [63], but had already appeared in Pieri [41]. The diagonal construction can be described as follows. In the affine case, consider two copies of $\mathbb{C}^n$: $\mathbb{C}^n_x$ and $\mathbb{C}^n_y$ with coordinates $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$, respectively. If X and Y are two varieties in $\mathbb{C}^n$, the product $X \times Y \subset \mathbb{C}^n_x \times \mathbb{C}^n_y$ is defined by the polynomials of X in the variables x and those of Y in the variables y. The diagonal Δ is defined by $x_1 - y_1, \ldots, x_n - y_n$, and we have the canonical isomorphism

$$X \cap Y \cong (X \times Y) \cap \Delta.$$

The irreducible components W of the left intersection are in a one-to-one correspondence with the irreducible components $W^\Delta := \{(x, x) : x \in W\}$ of the right one. It is easy to check that the left intersection is proper iff so is the right one. The diagonal procedure ensures that the multiplicities of both intersections coincide, which makes the situation easier to handle both from the geometric and algebraic points of view, because the inclusion $\Delta \subset \mathbb{C}^n_x \times \mathbb{C}^n_y$ is a regular imbedding.

The projective counterpart of the diagonal construction is the join construction, originally due to Gaeta [18]. Consider two copies of $\mathbb{P}^n$ with homogeneous coordinates $x = (x_0 : \cdots : x_n)$ and $y = (y_0 : \cdots : y_n)$, respectively. In the join space $\mathbb{P}_{2n+1}$ with homogeneous coordinates $(x_0 : \cdots : x_n : y_0 : \cdots : y_n)$ lies the join $J = J(X, Y)$ of varieties X and Y, defined by the homogeneous polynomials of X in the variables x and those of Y in the variables y; the "diagonal" L is the linear subspace defined by $x_0 - y_0, \ldots, x_n - y_n$. Again we have the canonical isomorphism $X \cap Y \cong J \cap L$, the left intersection is proper iff so is the right one, and the multiplicities of both intersections coincide. It can be easily checked that

$$\deg J(X, Y) = \deg X \cdot \deg Y,$$

whence

$$\deg(X \cdot Y) = \deg J \cdot L = \deg J = \deg X \cdot \deg Y,$$

and thus we have obtained the classical Bezout theorem.

A new definition of intersection multiplicity in terms of completions of local rings was given by Chevalley [9]; therefore his theory included both the analytic and formal cases. Eventually, Samuel [45]—using the characteristic polynomial of an open ideal in a local ring, called the Hilbert–Samuel polynomial—introduced the first definition of multiplicity valid for a general local ring, which was later developed by Nagata [33].

We have given a brief outline of the classical theory of proper intersections. A rapid development of research in the field of improper intersections has been noted recently. Two approaches have arisen within global algebraic geometry: the one developed by Fulton–MacPherson [17] and the other by Stückrad–Vogel [54]. The crucial tool applied in the former is deformation to the normal cone together with the theory of Segre classes, initiated by Segre [49]. The latter is based on a certain intersection algorithm and—as discovered by van Gastel [20]—it leads to a unified theory of intersections with a collection of divisors which incorporates the relevant part of the general theory of Fulton–MacPherson.

Fulton and MacPherson attach to the intersection of a variety V mapping to a scheme Y with a regularly embedded subscheme $X \hookrightarrow Y$ of codimension r, a rational equivalence class of the expected dimension, which is supported on the fibre product $W = X \times_Y V$. Where $V \hookrightarrow Y$ is an embedding, one recovers the usual intersection classes as equivalence classes on the ambient space Y. We now sketch their construction.

Applying the deformation to the normal bundle, one obtains a closed embedding of the normal cone $C_W V$ as a subcone in the normal bundle $N = N_X Y$ of rank r. Consequently, the general problem of determining the intersection product $X \cdot V$ has been reduced to that of a subcone in a vector bundle intersecting the zero section. The central idea is to move the zero section in order to make the intersection proper. This can be done in a canonical way if one passes to rational equivalence. Then the Gysin homomorphism for the intersections with the zero section

$$s : A_k N \to A_{k-r} X$$

(here $A_k X$ denotes the Chow group of k-cycles on X modulo rational equivalence) is the inverse of the flat pull-back homomorphism

$$\pi^* : A_{k-r} X \to A_k N,$$

where $\pi : N \to X$ is the projection of the vector bundle N over X. Consequently, the intersection product $X \cdot V$ can be expressed by the Segre class of the normal cone $C_W V$.

Now we outline the intersection algorithm of Stückrad–Vogel in the set-up of van Gastel. Let V be a purely dimensional scheme (separated and of finite type over a ground field k), $f : V \to Y$ a morphism to a scheme, $\mathcal{L}$ a line bundle on Y, and let $\mathcal{D} = (D_1, \ldots, D_r)$ be a collection of effective Cartier divisors defined by sections $s_1, \ldots, s_r$ of $\mathcal{L}$. Let u_{ij} be indeterminates adjoined to the ground field k and let the new ground field be called K. We shall deal with "generic" divisors D'_j determined by the sections $\sum_{i=1}^{r} u_{ij} s_i$, $j = 1, \ldots, r$. Denote by W the fibre product $V \times_Y (D_1 \cap \ldots \cap D_r)$.

The initial step of the intersection algorithm: Decompose $[V] = \varrho_0 + \alpha_0$, where ϱ_0 is the part of the fundamental cycle $[V]$ supported by W, and α_0 is the rest.

The induction step: Consider the intersection of α_{k-1} with the pull-back of the divisor D'_k, and decompose it as

$$\alpha_{k-1} \cdot f^* D'_k = \varrho_k + \alpha_k,$$

where ϱ_k is the part of the intersection supported by W_K and α_k is the rest.

Observe that all the intersections occurring in the algorithm are proper, and that after r steps everything will be supported by W_K, whence $\alpha_r = 0$. The *Vogel cycle* of the intersection of V with $\mathcal{D}$ over Y is by definition $\sum_k \varrho_k$.

Improper intersections within complex analytic geometry have recently been studied by P. Tworzewski [57]. His theory is based on a local analytic counterpart of the intersection algorithm of Stückrad–Vogel. Some other local counterparts of the intersection algorithm are also investigated from the algebraic point of view by Achilles–Manaresi [2, 3].

The main purpose of the dissertation is to investigate the analytic intersection algorithm from the perspective of the method of deformation to the normal cone. Since in our approach we cannot deal with equivalence classes, the central idea lies in the notion of a filter-regular sequence of elements in a given ring, which is related to the notion of a "generic" collection of divisors (determined by those elements). Under a certain condition of filter-regularity, we are able to apply deformation to the normal cone twice, and thereby deform the initial analytic set V to an algebraic bicone B so that the extended degree of the result of the intersection algorithm (for improper intersections) coincides with the degree sequence of that bicone (cf. [38, 40]). Moreover, the conditions imposed on such "generic" collections of smooth analytic divisors are very strong and of linear character: the first derivatives of the equations of each successive divisor should avoid a finite union of proper linear subspaces (depending on the previous divisors of the collection; cf. Chapt. III, Sect. 2). In this way we reduce the general problem of analytic improper intersections to that of an algebraic bicone intersecting linear hyperplanes. The linear character of this construction is strengthened by the linear sense of the term "generic", described above. Let us still stress that the Stückrad–Vogel algorithm for an algebraic variety intersecting a collection of global divisors is performed in fact for certain "generic" divisors which are defined as linear combinations of the initial ones with coefficients being indeterminates adjoined to the initial ground field. Therefore, the term "generic" has two different meanings for the two intersection algorithms under consideration. Our approach makes the analogy between these two meanings even deeper.

We thus arrive at a linearization procedure wherefrom many consequences concerning extended intersection index and intersection multiplicity for improper intersections in complex analytic geometry are derived. First of all the generalized index for improper intersections can be expressed as the bidegree sequence of a certain algebraic bicone, and the intersection multiplicity at a point P as the Samuel multiplicity at P of the normal cone. Consequently, the intersection multiplicity does not depend on the ambient space. The bidegree sequence of an algebraic bicone can be expressed by the leading coefficients of the Hilbert polynomial of its associated bigraded ring (cf. [61, 62, 3]), by analogy to the way the degree of a cone can be expressed by the leading coefficient of the Hilbert polynomial of its associated graded ring. These formulae are more effective (both in the sense of pure mathematics and the more so of computer algebra) than the original definition of intersection multiplicity, and imply the coincidence of the intersection indices for analytic improper intersections introduced by Tworzewski [57] and those defined for an ideal I in the local algebraic case by Achilles–Manaresi [3] (generalized Samuel multiplicities).

Making use of some properties of normal cones, we prove a generalization (first shown in [38, 40]) of the reduction theorem to the case of analytic improper intersections of analytic sets with submanifolds. The reduction theorem ensures the canonical character of the diagonal procedure for improper intersections, and provides the main step in our proof of a version of Bezout's theorem for improper intersections of algebraic cones (see Sect. 3 in Chapt. III).

Another consequence is that the intersection multiplicity function (which assigns to each point P the multiplicity at P of improper intersection of given two analytic sets V_1 and V_2 on a complex manifold M) is upper semicontinuous in the analytic Zariski topology. Since there is a one-to-one correspondence between the analytically constructible functions from M to $\mathbb{Z}$ and the analytic cycles on M (cf. [57] and Sect. 2 and 3 of

Chapt. III), one can define $V_1 \bullet V_2$ as a unique analytic cycle that corresponds to the intersection multiplicity function. Following Tworzewski, we call $V_1 \bullet V_2$ the (*improper*) *intersection product* of the sets V_1 and V_2. It coincides with the classical product in the case of proper intersections. Yet the intersection product for improper intersections does not fulfil an axiomatics like the one for proper intersections (presented in Sect. 2 of Chapt. I). Therefore one cannot expect the uniqueness of intersection products in the case of improper intersections. Although Bezout's theorem fails to be true in the case of improper intersections in a projective space, we state a version of Bezout's theorem valid for algebraic cones (see Sect. 3 of Chapt. III).

Our linearization procedure for the analytic intersection algorithm allows us to compare the generalized intersection index with the so-called Segre numbers, introduced by Gaffney–Gassler [19] by means of the sequences of polar varieties and Segre cycles, defined inductively. We now sketch their construction. Consider the germ at zero of a reduced closed analytic subspace $(V, 0) \subset (\mathbb{C}^n, 0)$ of pure dimension d, an ideal $I \subset \mathcal{O}_{V,0}$ which defines a nowhere dense subspace $V(I)$ of $(V, 0)$, and the blow-up $Bl_I V$ of V along I:

$$\pi : Bl_I V \to V$$

with exceptional divisor D. We say that a hyperplane H of $Bl_I V$ is *general with respect to* a reduced subspace Z of $Bl_I X$ of pure dimension k if $Z \cap H$ is reduced of dimension $k-1$ and none of its components is contained in D. Using Kleiman's transversality lemma, one can show that there exists a Zariski open subset of hyperplanes of $Bl_I V$ which are general with respect to Z. For a d-tuple $g = (g_1, \ldots, g_d)$ of linear combinations of generators of I, assume that each hyperplane H_k on $Bl_I V$ corresponding to g_k is general with respect to $H_1 \cap \ldots \cap H_{k-1}$; then we also say that g is general. The *polar varieties* and *Segre cycles* of I on V are defined inductively as follows:

$$P_0^g(I, V) := V, \quad P_k^g(I, V) := \pi(H_1 \cap \ldots \cap H_k),$$
$$\Lambda_k^g(I, V) := \pi_*(H_1 \cdots H_{k-1} \cdot D);$$

observe that all the above intersections are proper by construction. The polar varieties are reduced, and the index k gives the codimension of $P_k^g(I, V)$ and $\Lambda_k^g(I, V)$ unless they are empty.

Gaffney–Gassler define the *polar multiplicities* and *Segre numbers* of (I, V) as the multiplicities at zero of the generic polar varieties and generic Segre cycles, respectively. Since the push-forward $\pi_*(D)$ of the exceptional divisor D consists of the one-codimensional subvarieties W of the ideal I counted with multiplicities $(e_I V)_W$ of X along $V(I)$ at W (cf. [16], Chapt. IV, Sect. 3), the polar varieties and Segre cycles for a generic d-tuple $g = (g_1, \ldots, g_d)$ can be redefined as follows (cf. [19], Sect. 2):

$$\begin{aligned} P_k^g(I, V) &= \text{ closure of } V(g_k|_{P_{k-1}^g(I,V)}) \setminus V(I), \\ \Lambda_k^g(I, V) &= [V(g_k|_{P_{k-1}^g(I,V)})] - [P_k^g(I, V)], \\ \Lambda_d^g(I, V) &= [V(g_d|_{P_{d-1}^g(I,V)})]. \end{aligned}$$

Hence, for a generic d-tuple of linear combinations of generators of the ideal $I \subset \mathcal{O}_{V,0}$, the construction of Segre cycles coincides with the analytic intersection algorithm. Segre numbers can always be expressed as the generalized index of an analytic improper intersection (see the corollary to Prop. 5 from Sect. 2, Chapt. III, and also [4]); the cycles determined by the polar varieties correspond to the cycles α_k, and the Segre cycles correspond to the cycles ϱ_k of the analytic intersection algorithm. Segre numbers, together with polar multiplicities, are of great importance for equisingularity theory (cf. [19]): the Segre numbers of the Jacobian ideal are just the Lê numbers of D. Massey [29] and describe equisingularity conditions; they make it possible to generalize Teissier's principle of specialization of integral dependence; their alternating sum is the Euler characteristic of the Milnor fibre; the Whitney conditions are controlled by certain Segre numbers, and the constancy of the polar multiplicities and Segre numbers of the Jacobian ideal is closely related to Whitney stratifications. We expect that the theory of analytic improper intersections may give a contribution to the geometry of singularities. Our feeling is that such a perspective on improper intersections shows the right direction for further research.

E. Cygan [11] discovered that the improper intersection multiplicity is a regular separation exponent for complex analytic sets, and thus it estimates the Łojasiewicz exponent. This observation can be applied in estimating the Łojasiewicz exponent at infinity for polynomial mappings. In particular, it enabled Cygan–Krasiński–Tworzewski [12] to improve Kollár's results concerning the Łojasiewicz exponent.

The dissertation is composed of three chapters. Chapter I is of preparatory nature. There we sketch the classical theory of proper intersections in complex analytic geometry. In Section 2 we give a new axiomatics for this theory. It is based on the continuity of proper intersections under deformation to the normal bundle, which is a special case of the continuity of proper intersections with respect to the convergence of analytic cycles (see e.g. Chirka [10]). The concept of convergence for analytic cycles also plays a significant role in the further parts of the dissertation where we investigate the analytic intersection algorithm from the perspective of deformation to the normal cone. In Section 3, using a method of Stein compact neighbourhoods, we express the index of an analytic complete intersection in terms of the Samuel multiplicities of ideals in the local rings of proper components (the reduction theorem for proper intersections). For a thorough introduction to algebraic and analytic geometry we refer the reader to [1, 14, 22–26, 28, 32, 52].

The last two chapters deal with improper analytic intersections. The local algorithm for analytic improper intersections is introduced in Section 2 of Chapter II. Here both the geometric and algebraic approaches are considered. Section 1 discusses the notion of filter-regularity which later will throw light on the concept of a "generic" collection of divisors; each successive element of the collection should avoid a finite number of prime ideals (depending on the previous elements of the collection). The concept of a "generic" collection is one of the central ideas in geometry. It is essential for the constructions both of the generalized intersection index and Segre numbers. The main purpose of the chapter is to investigate the method of deformation to the normal cone applied to the intersection algorithm. As both geometric and algebraic techniques have been applied, we encounter general (possibly non-reduced) analytic spaces to be deformed. Section 3 recalls briefly a construction of deformation to the normal cone by blowing-up the product of the ambient space and the parametric line $\mathbb{C}$. Similarly to the classical case of proper intersections, this method turns out to be one of the most powerful tools of the theory of analytic improper intersections. In Section 4 we prove the main theorem to the effect that, under certain conditions of filter-regularity, the total result of the intersection algorithm is preserved by deformation to the normal cone.

In Chapter III we elaborate a method of deforming an analytic space to an algebraic bicone, which is a refinement of deformation to the normal cone (applied twice). The main theorem in Section 1 asserts that, under certain conditions of filter-regularity, the multiplicities of the total result of the intersection algorithm are preserved by deformation to that algebraic bicone. The next sections provide many applications which are important for the theory of improper intersections in complex analytic geometry, as for instance the reduction theorem for analytic improper intersections. The intersection product of analytic cycles is constructed in Section 3. It should be emphasized that the product for improper intersections does not fulfil an axiomatics like the one for the theory of proper intersections presented in Chapter I. One can therefore expect several distinct products for the case of improper intersections. Also, Bezout's theorem fails to be true in the case of improper intersections in a projective space. Nevertheless, we state a version of Bezout's theorem valid for algebraic cones. Finally, many examples concerning generalized intersection indices and intersection cycles are given.

I. Classical theory of proper intersections

1. The multiplicity of light mappings and analytic sets. Consider two connected complex manifolds M and N, an analytic subset V of pure dimension d in M, and a holomorphic mapping $f : V \to N$. We recall that if $\dim V = \dim N$ and if f is a proper mapping with finite fibres, then f is an *analytic* (*ramified*) *cover* (cf. [28], Chapt. V, Sect. 7), i.e. there is an analytic subset W nowhere dense in N such that $V \setminus f^{-1}(W)$ is a complex manifold (possibly non-connected) and the restriction

$$f : V \setminus f^{-1}(W) \to N \setminus W$$

is a local biholomorphism, hence a p-sheeted topological cover. We say that p is the *multiplicity* of the analytic cover f. According to Remmert's open mapping theorem, the analytic cover f is an open mapping.

We say that a holomorphic mapping $f : V \to N$ is *light* at a point $P \in V$ if P is an isolated point of its fibre $f^{-1}(f(P))$. Then there are neighbourhoods U and U' of P and $P' := f(P)$ such that $f^{-1}(P') \cap U = \{P\}$ and the restriction $f|U : U \to U'$ is a proper mapping with finite fibres. We call the multiplicity $m_P f$ of the analytic cover $f|U$ (which obviously does not depend on the choice of the neighbourhoods U and U') the *multiplicity* of the mapping f at the point P. It is convenient to define $m_P f := \infty$ whenever f is not light at P.

We now express the multiplicity $m_P f$ in terms of the local rings of P and P'. As the problem is local, we may assume that $P' = 0 \in \mathbb{C}^d$; let $u_1, \dots, u_d$ be the coordinates in $\mathbb{C}^d$. Let $A := \mathcal{O}_{\mathbb{C}^d,0}$ and $B := \mathcal{O}_{V,P}$ be the local rings of P' and P in $\mathbb{C}^d$ and V, respectively; let $\mathfrak{m} = (u_1, \dots, u_d)$ and $\mathfrak{n}$ be their maximal ideals, and put $S := A \setminus \{0\}$. Clearly, A is a regular local ring, which may be identified with a subring of B. According to the Nullstellensatz, the ideal $\mathfrak{m}B$ is $\mathfrak{n}$-primary because P is an isolated point of the fibre $f^{-1}(0)$. By the preparation theorem, B is a finite A-module. Further, A and B are reduced rings and every element $a \in A$, $a \neq 0$, is a non-zero divisor in B. Hence $S^{-1}B$ is the total quotient ring of B; it is obviously a finite-dimensional vector space over the quotient field $S^{-1}A$ of A. The dimension $[B : A]$ of $S^{-1}B$ over $S^{-1}A$ is the maximum number of elements from B which are linearly independent over A. Under this notation, we have the desired formula:

$$m_P f = [B : A];$$

it can be immediately derived from the classical Rückert description lemma (see e.g. [28], Chapt. III, Sect. 3 and Chapt. IV, Sect. 1).

In order to express $m_P f$ by means of multiplicities of ideals, we need the following fundamental (cf. [46] or [65], Chapt. VIII, Sect. 10):

Samuel's Formula. *Consider a local ring A with maximal ideal $\mathfrak{m}$ and an over-ring B which is a finite A-module*; *then B is*, *of course*, *a semilocal ring with a finite number of maximal ideals $\mathfrak{n}_i$. Suppose that no $a \in A$, $a \neq 0$, is a zero divisor in B and that* height $\mathfrak{n}_i = \dim A$. *Then*, *for any $\mathfrak{m}$-primary ideal q in A*, *we have*

$$[B : A] \cdot e(q) = \sum_i [B/\mathfrak{n}_i : A/\mathfrak{m}] \cdot e(qA_{\mathfrak{n}_i})$$

(*here $e(q)$ denotes the multiplicity of the ideal q*).

Remark. If every $a \in A$, $a \neq 0$, is a non-zero divisor in B, then the ring B is equidimensional. If, moreover, the ring A is analytically irreducible (i.e. its completion $\widehat{A}$ in the maximal-adic topology is a domain), then every $a \in \widehat{A}$, $a \neq 0$, is a non-zero divisor in $\widehat{B}$, and thus $\widehat{B}$ is equidimensional as well. Consequently, height $\mathfrak{n}_i = \dim A$ for all i, because we have the canonical isomorphism

$$\widehat{B} \cong \bigoplus_i \widehat{B}_{\mathfrak{n}_i}.$$

In our geometric context, Samuel's formula yields the

Algebraic Formula for the Multiplicity of a Holomorphic Mapping.

$$m_P f = e(\mathfrak{m}B) = e\Big(\sum_{j=1}^{d} u_j B\Big).$$

If the ring B is Cohen–Macaulay (which is true whenever the local ring B is regular or, equivalently, whenever the set V is a complex manifold near the point P), we obtain the following formula for the multiplicity of a light holomorphic mapping f at P:

$$m_P f = e(\mathfrak{m}B) = \text{length } B/\mathfrak{m}B = \dim_{\mathbb{C}} B/\mathfrak{m}B.$$

Now we proceed to recall the notion of multiplicity of an analytic set at a point. Let V be an analytic set of pure dimension d in a domain $D \subset \mathbb{C}^r$ and $P \in V$. Denote by $G(k, r)$ the Grassmann manifold of all k-dimensional vector subspaces in $\mathbb{C}^r$. For any $L \in G(r - d, r)$, let π_L be the projection parallel to L. We define the *multiplicity* $m_P V$ of the set V at P by putting

$$m_P V := \min\{m_P(\pi_L|V) : L \in G(r - d, r)\};$$

$m_P V$ is a finite number as there exist vector subspaces $L \in G(r-d, r)$ such that P is an isolated point of $V \cap (P+L)$. Furthermore, the minimum is attained for generic $L \in G(r-d, r)$, which is formulated precisely in the statement below (see e.g. [10], Chapt. 2, Sect. 11):

Let V be an analytic set of pure dimension d in the vicinity of $0 \in \mathbb{C}^r$ and $L \in G(r-d, r)$. Then the equality

$$m_0(\pi_L|V) = m_0 V$$

holds iff $L \cap C(V, 0) = \{0\}$ (here $C(V, 0)$ denotes the tangent cone to V at 0). The L with the above property form a subset U of $G(r-d, r)$ whose complement is an algebraic subset in $G(r-d, r)$ of codimension ≥ 1, and thus U is an open dense connected subset of $G(r-d, r)$.

We say that an $L \in G(r-d, r)$ *meets* the set V *transversally* at 0 if $L \cap C(V, 0) = \{0\}$. More generally, we say that analytic sets $V_1, \dots, V_k$ of pure dimensions $d_1, \dots, d_k$ in the vicinity of a point $P \in \mathbb{C}^r$ meet transversally at P if their tangent cones $C(V_i, P)$ intersect properly, i.e. if

$$\dim \bigcap_i C(V_i, P) = \sum_i d_i - (k-1)r.$$

Remark. Although the above geometric definition of $m_P V$ is extrinsic, it does not depend on the ambient space. This will be evident once we express $m_P V$ as the multiplicity $e(\mathfrak{m}_P)$ of the maximal ideal $\mathfrak{m}_P$ in the local ring of V at P.

We now recall some algebraic facts concerning the theory of reductions of ideals in local rings (cf. [35]). Consider a local ring A with maximal ideal $\mathfrak{m}$ and with infinite residue class field k. If $\mathfrak{a}$ and $\mathfrak{b}$ are ideals in A, then $\mathfrak{b}$ is called a *reduction* of $\mathfrak{a}$ if $\mathfrak{b} \subset \mathfrak{a}$ and $\mathfrak{b}\mathfrak{a}^s = \mathfrak{a}^{s+1}$ for a positive integer s. Clearly, if $\mathfrak{b}$ is a reduction of $\mathfrak{a}$, then $\mathfrak{a}$ and $\mathfrak{b}$ have the same minimal prime ideals. Moreover,

$$e(\mathfrak{a}A_\mathfrak{p}) = e(\mathfrak{b}A_\mathfrak{p})$$

for every common minimal prime ideal $\mathfrak{p}$ of $\mathfrak{a}$ and $\mathfrak{b}$. Therefore the definition suggests that if $\mathfrak{b}$ is a reduction of $\mathfrak{a}$, then $\mathfrak{b}$ is a simplified version of $\mathfrak{a}$, and that the smaller $\mathfrak{b}$, the more marked the simplification. In fact, every reduction $\mathfrak{b}$ of $\mathfrak{a}$ contains at least one minimal reduction $\mathfrak{c}$ of $\mathfrak{a}$. It turns out that a reduction of an ideal $\mathfrak{a}$ is minimal iff it can be generated by $l(\mathfrak{a})$ elements, where $l(\mathfrak{a})$ is the dimension of the graded local ring

$$\mathfrak{G}(\mathfrak{a}) := \bigoplus_{n=0}^{\infty} \mathfrak{a}^n / \mathfrak{m}\mathfrak{a}^n$$

(*op. cit.*, Sect. 4). If the ideal $\mathfrak{a}$ is $\mathfrak{m}$-primary, then

$$l(\mathfrak{a}) = \dim\ \mathfrak{G}(\mathfrak{a}) = \dim\ A.$$

The statement below (*op. cit.*) is a precise formulation of the assertion that any generic $l(\mathfrak{a})$ elements of the ideal $\mathfrak{a}$ generate a minimal reduction.

Consider a local ring A with maximal ideal $\mathfrak{m}$ and residue class field k, and an ideal $\mathfrak{a} = (u_1, \dots, u_r)$ $(u_i \in A)$; let $l := l(\mathfrak{a})$. Then one can find a finite number of polynomials $P_k(T_{ij}) \in A[T]$ in lr indeterminates T_{ij} $(i = 1, \dots, l,\ j = 1, \dots, r)$ with the following *property*: *for any elements* $a_{ij} \in A$ $(i = 1, \dots, l,\ j = 1, \dots, r)$, *the* l *linear combinations* $v_i = \sum_{j=1}^r a_{ij} u_j$ *generate a minimal reduction of* $\mathfrak{a}$ *iff at least one* $P_k(a_{ij})$ *is a unit in* A (*i.e.* $P_k(a_{ij}) \not\equiv 0 \pmod{\mathfrak{m}}$).

As minimal reductions always exist, *some of the polynomials* $P_k(T_{ij})$ *do not vanish identically modulo* $\mathfrak{m}$. *If the infinite residue class field* k *can be embedded into* A, *then*, *of course*, *one can find scalars* $a_{ij} \in k$ *such that* $P_k(a_{ij}) \not\equiv 0 \pmod{\mathfrak{m}}$.

Now we can readily return to the geometric context:

- V is an analytic set of pure dimension d at $P = 0 \in \mathbb{C}^r$,
- $B = \mathcal{O}_{V,P}$ is the local ring of V at P,
- $\mathfrak{m}_P$ is the maximal ideal of B.

Let $u_1, \dots, u_r$ be the coordinates in $\mathbb{C}^r$. Since the minimum number of generators for minimal reductions of the ideal $\mathfrak{m}_P$ is its height d, one can find a finite number of polynomials $P_k(T_{ij}) \in \mathbb{C}[T]$ in dr indeterminates T_{ij} $(i = 1, \dots, d,\ j = 1, \dots, r)$ with the following property:

For any elements $a_{ij} \in \mathbb{C}$ $(i = 1, \dots, d, j = 1, \dots, r)$, *the* d *linear combinations* $v_i = \sum_{j=1}^r a_{ij} u_j$ *generate a minimal reduction of* $\mathfrak{m}_P$ *iff at least one* $P_k(a_{ij})$ *is different from zero. In particular*, *for such generic linear combinations* $v_1, \dots, v_d$, *we have the equality*

$$e(\mathfrak{m}_P) = e\Big(\sum_{i=1}^d v_i B\Big).$$

On the other hand, the algebraic multiplicity $e(\sum_{i=1}^d v_i B)$ is the multiplicity at P of the linear projection

$$\pi|V : V \ni (u_1, \dots, u_r) \mapsto \Big(\sum_{j=1}^r a_{1j} u_j, \dots, \sum_{j=1}^r a_{dj} u_j\Big) \in \mathbb{C}^d.$$

Hence $m_P V = e(\mathfrak{m}_P)$ because also $m_P V$ coincides with the multiplicity at P of a generic projection (Proposition 1). Summing up, we have obtained the following intrinsic algebraic definition of the multiplicity of an analytic set at a point P:

Let V *be an analytic set of pure dimension at a point* P. *Then the multiplicity* $m_P V$ *of* V *at* P *coincides with the multiplicity* $e(\mathfrak{m}_P)$ *of the maximal ideal* $\mathfrak{m}_P$ *in the local ring of* V *at* P:

$$m_P V = e(\mathfrak{m}_P).$$

We conclude the section with the following well known

WHITNEY'S THEOREM (cf. [64, 10]). *Let* V *be a purely dimensional analytic set*, *and put*

$$V^{(k)} := \{P \in V : m_P V \geq k\} \quad (k = 1, 2, \dots).$$

Then $V^{(k)}$ *are analytic subsets in* V, $V^{(1)} = V$ *and* $V^{(2)} = V^{\mathrm{sing}}$ *is the singular locus of* V.

2. Axioms for proper intersections of analytic cycles. Let M be a complex manifold of dimension m. We recall that *analytic cycles* on M are formal locally finite sums

$$Z = \sum c_i V_i,$$

where V_i are irreducible analytic subsets in M and c_i are integers; local finiteness means that each point P in M has a neighbourhood U that meets only a finite number of V_i with $c_i \neq 0$. The *support* $|Z|$ of the cycle Z is the analytic set

$$|Z| := \bigcup \{V_i : c_i \neq 0\};$$

we call the sets V_i *components* of the cycle Z. We say that Z is a *cycle of pure dimension* d if every analytic set V_i is of dimension d. Analytic cycles of pure dimension d will also be called *d-cycles*. A cycle Z is *positive* if all its coefficients c_i are positive integers. Throughout the paper we are, in fact, interested only in positive analytic cycles.

One can extend, by additivity, the notion of multiplicity at a point P to analytic cycles. Similarly, one can define the multiplicity $m_P f|Z$ of a holomorphic mapping in the vicinity of a point P in the support $|Z|$ of the cycle Z.

We say that analytic cycles $Z_1, \ldots, Z_k$ of pure dimensions $d_1, \ldots, d_k$ (respectively) in the vicinity of a point $P \in M$ *intersect properly at* P if their supports $|Z_j|$ meet properly at P, i.e.

$$\dim \bigcap_j |Z_j| = \sum_j d_j - (k-1)m$$

(and thus the dimension of the intersection is the smallest possible). If the cycles $Z_1, \ldots, Z_k$ meet properly at every point P of $\bigcap_j |Z_j|$, we say that they *intersect properly on* M. Then, according to the classical theory of proper intersections, we can define the intersection product

$$Z_1 \cdot \ldots \cdot Z_k$$

which is a $(\sum_j d_j - (k-1)m)$-cycle on M. As we shall see later, the proper intersections of analytic cycles in open subsets of affine spaces are uniquely determined by the following four axioms:

Basic Axiom. *Let $Z_1, \ldots, Z_k$ be purely dimensional analytic cycles in an open subset of $\mathbb{C}^m$ that intersect properly. Then the operation*

$$(Z_1, \ldots, Z_k) \rightsquigarrow Z_1 \cdot \ldots \cdot Z_k$$

is local (*i.e. the intersection product in open subsets U does in fact depend only on the behaviour of the cycles in U*), *multi-additive*, *commutative and associative in the vicinity of the analytic set $|Z_1| \cap \ldots \cap |Z_k|$.*

Normalization Axiom. *If Z is a d-cycle in an open subset U of $\mathbb{C}^m$, then*

$$(U \times \{0\}) \cdot (Z \times \mathbb{C}^n) = Z \times \{0\}.$$

Invariance Axiom. *Intersection products are invariant under affine isomorphisms*, *i.e. if $Z_1, \ldots, Z_k$ are purely dimensional analytic cycles in an open subset of $\mathbb{C}^m$ that intersect properly and if $F : \mathbb{C}^m \to \mathbb{C}^m$ is an affine isomorphism*, *then*

$$F(Z_1) \cdot \ldots \cdot F(Z_k) = F(Z_1 \cdot \ldots \cdot Z_k).$$

Continuity Axiom. *Consider an open subset U of $\mathbb{C}^s$ and an open neighbourhood Ω of $U \times \{0\} \times \mathbb{C}$ in $U \times \mathbb{C}^r \times \mathbb{C}$. Let $\mathcal{V}$ be an analytic subset in Ω such that every fibre V_λ of $\mathcal{V}$ over $\lambda \in \mathbb{C}$ (in the set-theoretic sense) is of pure dimension d and properly intersects the set $U \times \{0\} \times \{\lambda\}$ in $\Omega \cap (U \times \mathbb{C}^r \times \{\lambda\})$; hence $\mathcal{V}$ is of pure dimension $d+1$. Clearly, if $[\mathcal{V}_\lambda]$ is the d-cycle on $\Omega \cap (U \times \mathbb{C}^r \times \{\lambda\})$ determined by the Cartier divisor $\mathcal{V}_\lambda$ coming from the divisor $(t-\lambda)$ on $\mathbb{C}$, then the support of $[\mathcal{V}_\lambda]$ is the analytic set V_λ. Suppose that, upon identification of the sets $U \times \{0\} \times \{\lambda\}$ with U, all set-theoretic intersections*

$$V_\lambda \cap (U \times \{0\} \times \{\lambda\})$$

coincide. Then all intersection products $U \cdot [\mathcal{V}_\lambda]$, which are analytic $(d-r)$-cycles on U, are equal.

We begin with the observation that, according to the locality and multi-additivity axioms, if $Z_1, \ldots, Z_k$ are purely dimensional analytic cycles meeting properly, then

$$Z_1 \cdot \ldots \cdot Z_k = \sum_i c_i W_i,$$

where W_i are the irreducible branches of $\bigcap_i |Z_i|$ and c_i are integers. We say that

$$c_i =: i(Z_1 \cdot \ldots \cdot Z_k; W_i)$$

is the *intersection index* of $Z_1 \cdot \ldots \cdot Z_k$ along W_i. In other words, intersection products are uniquely determined by intersection indices.

The above four axioms ensure the uniqueness of proper intersections as follows. The basic axiom allows us to consider intersections of only two cycles which consist of purely dimensional analytic subsets of a domain in an affine space. All four axioms enable the diagonal procedure, and thus they reduce the problem to that of a cycle intersecting an affine subspace. Finally, the continuity axiom leads to the intersections described in the normalization axiom. We shall now elaborate this more precisely.

We first demonstrate the uniqueness of proper intersections with affine subspaces. This will be done by means of deformation to the normal cone presented in Section 3 of Chapter III. The concept of deformation to the normal cone, which is of geometric nature, appeared first, however, in an algebraic set-up of Gerstenhaber [21] (the algebra for such deformations was created by Rees [43, 44]); in fact, already Samuel [45] based his algebraic intersection theory on the construction of associated graded rings (which corresponds to that of normal cones) and on a variant of the Hilbert polynomial, called the Hilbert–Samuel polynomial.

Let V be an analytic subset of pure dimension d in a domain D in $\mathbb{C}^m$, and let S be an affine subspace in $\mathbb{C}^m$ which meets V properly. Due to the invariance axiom, we may obviously arrange coordinates in $\mathbb{C}^m$ as follows:

$$\mathbb{C}^m = \mathbb{C}^s_u \times \mathbb{C}^r_v \quad (s+r=m), \quad S = \{(u,v) \in \mathbb{C}^m : v = 0\}.$$

Consider the (ideal-sheaf-theoretic) intersection $V \cap S$ which is of pure dimension $d-r$, and the deformation spaces of $\mathbb{C}^m$ and V with respect to S and $V \cap S$, respectively. They are reduced analytic spaces, and thus we can describe them geometrically as follows:

- the first one may be identified simply with $\mathbb{C}^m$;
- the second one may be identified with the closure $\mathcal{V}$ of the following family of analytic sets parametrized by $\lambda \in \mathbb{P}_1$:

$$\bigcup_\lambda V_\lambda := \{(u, v; 1 : \lambda) \in \mathbb{C}^m : (u, v/\lambda) \in V\}.$$

The normal cone V_∞ (in the set-theoretic sense) is an analytic set of pure dimension d, and thus it coincides with

$$\{(u, v) \in \mathbb{C}^m : (u, 0) \in V,\ v \text{ is arbitrary}\}.$$

The cycle $[\mathcal{V}_\infty]$ determined by the Cartier divisor of the point $\infty \in \mathbb{P}_1$, which is the normal cone $[C_{V\cap S}V]$, is thus of the form

$$\sum_i m_i(W_i \times \mathbb{C}^r),$$

where W_i are the irreducible branches of $V \cap S$ and m_i are positive integers, which are the coefficients of the cycle $[\mathcal{V}_\infty]$ along its branches $W_i \times \mathbb{C}^r$; clearly, the m_i are independent of intersection theory. Therefore the continuity, multi-additivity and normalization axioms yield

$$V \cdot S = \mathcal{V}_\infty \cdot S = \Big(\sum_i m_i W_i \times \mathbb{C}^r\Big) \cdot S = \sum_i m_i W_i,$$

whence the uniqueness follows.

Remark. The coefficients m_i must coincide, of course, with the intersection indices $i(V \cdot S; W_i)$ once such a unique theory of proper intersections is constructed.

In particular, we obtain the following

Claim. *Let P be a regular point of V. If an affine subspace L of dimension $n - d$ intersects V transversally at P, then*

$$i(V \cdot L; P) = 1.$$

(*Also, the converse is true: if an affine subspace meets V properly at an isolated point P with multiplicity* 1, *then P is a regular point of V, and the intersection is transversal—see e.g.* [36, 47]).

Since the diagonal $\Delta \subset D \times D$ is an affine subspace, the proof of uniqueness will be complete if we establish the

Diagonal Procedure. *For any subset V in D, let*

$$V^\Delta := \{(P, P) : P \in V\}$$

be the subset corresponding to V in the diagonal $\Delta = \{(P, P) \in D \times D : P \in D\}$. If V_1, V_2 are two purely dimensional analytic subsets which intersect properly in D and if W is an irreducible branch of $V_1 \cap V_2$, then

$$i(V_1 \cdot V_2; W) = i((V_1 \times V_2) \cdot \Delta; W^\Delta).$$

Indeed, W^Δ is an irreducible branch of the proper intersection $\Delta \cap (V_1 \times D) \cap (D \times V_2)$, and thus the associativity axiom is applicable. We get

$$i(V_1^\Delta \cdot (D \times V_2); W^\Delta) \cdot i(\Delta \cdot (V_1 \times D); V_1^\Delta)$$
$$= i(\Delta \cdot (V_1 \times V_2); W^\Delta) \cdot i((V_1 \times D) \cdot (D \times V_2); V_1 \times V_2).$$

But

$$i(\Delta \cdot (V_1 \times D); V_1^\Delta) = 1 \quad \text{and} \quad i(V_1^\Delta \cdot (D \times V_2); W^\Delta) = i(V_1 \cdot V_2; W),$$

which follows directly from the invariance, associativity and normalization axioms, the easy details being left to the reader. It is therefore sufficient to show that

$$i((V_1 \times D) \cdot (D \times V_2); V_1 \times V_2) = 1.$$

The proof is straightforward: we apply the associativity and normalization axioms as well as the foregoing claim. First observe that if V_1 and an affine subspace L_1 of D meet transversally at an isolated point P_1 of the regular locus of V_1, then

$$i((L_1 \times D) \cdot (V_1 \times D); \{P_1\} \times D) = 1.$$

Indeed, if this intersection index is k, then, for any point $P_2 \in D$, we have

$$(P_1, P_2) = (L_1 \times D) \cdot (V_1 \times P_2) = (L_1 \times D) \cdot ((V_1 \times D) \cdot (D \times P_2))$$
$$= ((L_1 \times D) \cdot (V_1 \times D)) \cdot (D \times P_2) = k \cdot (P_1 \times D) \cdot (D \times P_2) = k \cdot (P_1, P_2),$$

whence $k = 1$, as desired. Similarly,

$$i((D \times V_2) \cdot (D \times L_2); D \times P_2) = 1$$

whenever V_2 and an affine subspace L_2 of D meet transversally at an isolated point P_2 of the regular locus of V_2. Supposing

$$i((V_1 \times D) \cdot (D \times V_2); V_1 \times V_2) = k,$$

we therefore obtain

$$(L_1 \times D) \cdot (V_1 \times D) \cdot (D \times V_2) \cdot (D \times L_2) = (P_1 \times D) \cdot (D \times P_2) = (P_1, P_2).$$

On the other hand,

$$(L_1 \times D) \cdot (V_1 \times D) \cdot (D \times V_2) \cdot (D \times L_2) = (L_1 \times D) \cdot k \cdot (V_1 \times V_2) \cdot (D \times L_2)$$
$$= k \cdot (L_1 \times L_2) \cdot (V_1 \times V_2) = k \cdot (P_1, P_2),$$

whence $k = 1$, and thus the diagonal procedure is established.

Remark. The uniqueness of proper intersections implies their invariance under biholomorphisms. The invariance can also be deduced directly from the reduction theorem for proper intersections presented in Section 3.

The above proof of uniqueness suggests how to construct the proper intersections of analytic cycles on a complex manifold. We sketch this in a few lines:

- one must apply the diagonal procedure;
- the index of proper isolated intersection with an affine subspace L is equal to the multiplicity of an affine mapping which determines L;
- the intersection index $i(Z_1 \cdot \ldots \cdot Z_k; W)$ coincides with $i(Z_1 \cdot \ldots \cdot Z_k \cdot L; P)$ whenever L is an affine subspace which meets W transversally at an isolated regular point P.

For a detailed construction of proper intersections, we refer the reader to e.g. Chirka [10], Chapt. II, Sect. 12. His construction is based on the notion of convergence of analytic cycles; we shall make frequent use of it later on. Roughly speaking, a sequence of analytic cycles converges if so do both their supports and their proper isolated intersections with affine spaces; for the notion of convergence of analytic sets and its properties, see also [58]. Proper intersection turns out to be a continuous operation with respect to convergence of analytic cycles (cf. [10], Chapt. II, Sect. 12). It is worth pointing out that our continuity axiom is a special case of the above-mentioned continuity of proper intersections.

REMARK. We shall see in the next section that if M and N are complex manifolds of dimensions m and n, respectively, and if $f : M \to N$ is a holomorphic mapping whose fibres all have dimension $m-n$, then the fibres $f^{-1}(Q)$ regarded as analytic cycles on M ($Q \in N$) form a continuous family, i.e. the $(m-n)$-cycles $f^{-1}(Q_j)$ converge to $f^{-1}(Q_0)$ on M whenever the points Q_j tend to Q_0 in N.

Proper intersections of analytic sets with complete intersections can be calculated by means of the reduction theorem presented in Section 3. Below we state only the special case of isolated intersections (*loc. cit.*):

PROPOSITION 1. *Let V be an analytic set of pure dimension d in the vicinity of a point P in $\mathbb{C}^m$. If an affine subspace L of $\mathbb{C}^m$ meets V properly at the isolated point P, then*

$$i(V \cdot L; P) = m_P(\pi_L|V),$$

where π_L is the projection parallel to L, and $\pi_L|V$ is its restriction to V. More generally, consider a holomorphic mapping near P,

$$f = (f_1, \dots, f_d) : \mathbb{C}^m \to \mathbb{C}^d,$$

whose restriction $f|V : V \to \mathbb{C}^d$ is light at P. Let $D_i = (f_i)$ $(i = 1, \dots, d)$ be the divisors of the functions f_i. Then P is an isolated point of the intersection

$$V \cdot D_1 \cdot \ldots \cdot D_d,$$

and we have the formula

$$i(V \cdot D_1 \cdot \ldots \cdot D_d; P) = m_P f|V = e\Big(\sum_{i=1}^{d} f_i \mathcal{O}_{V,P}\Big).$$

Finally, we recall two important theorems from the theory of analytic proper intersections (*loc. cit.*):

PROPOSITION 2. *Let $Z_1, \dots, Z_k$ be analytic cycles of pure dimensions in a domain $D \subset \mathbb{C}^m$. Suppose $Z_1, \dots, Z_k$ intersect properly at a point P in D, and denote by G the set of those affine subspaces of dimension $m - \dim_P \bigcap_i |Z_i|$ which meet $\bigcap_i |Z_i|$ transversally at P. Then the intersection index*

$$i(Z_1 \cdot \ldots \cdot Z_k \cdot L; P)$$

does not depend on the affine subspace L in G.

PROPOSITION 3. *Let $Z_1, \dots, Z_k$ be analytic cycles of pure dimensions in a domain $D \subset \mathbb{C}^m$. If $Z_1, \dots, Z_k$ intersect properly in D, then for any $P \in \bigcap_i |Z_i|$ we have*

$$m_P(Z_1 \cdot \ldots \cdot Z_k) \geq m_P Z_1 \cdot \ldots \cdot m_P Z_k;$$

moreover, equality holds iff the cycles $Z_1, \dots, Z_k$ *intersect transversally at* P (*i.e. the tangent cones to the supports* $|Z_i|$ *at* P *meet properly*).

REMARK. An immediate corollary to Proposition 3 is the classical Bezout theorem on the intersection of projective cycles to the effect that if $Z_1, \dots, Z_k$ are positive algebraic cycles in the projective space $\mathbb{P}_m$ which meet properly, then

$$\deg(Z_1 \cdot \ldots \cdot Z_k) = \deg Z_1 \cdot \ldots \cdot \deg Z_k.$$

3. Reduction theorem for proper intersections. We begin by presenting the method of compact Stein neighbourhoods. Let

$$K = \overline{\Delta^m}(b) := \{z \in \mathbb{C}^m : |z_i| \le b_i \text{ for } i = 1, \dots, m\}$$

be a compact polydisk in $\mathbb{C}^m$, $b \in \mathbb{R}^m$, $b > 0$, and let $R := \mathcal{O}(K)$ be the ring of global sections of the structure sheaf $\mathcal{O}$ over K. According to Frisch's theorem [15], R is a noetherian ring. Hence and by Cartan's Theorem B, every maximal ideal of R is of the form

$$\mathfrak{m}(P) := \{f(z) \in R : f(P) = 0\},$$

where $P = (a_1, \dots, a_m)$ is a point in K; the ideal $\mathfrak{m}(P)$ is generated by the functions $z_i - a_i$ $(i = 1, \dots, m)$. Moreover, again by Cartan's Theorem B,

$$\{f(z) \in R : (f)_P \in \mathfrak{m}_P^k\} = \mathfrak{m}(P)^k$$

(here $(f)_P$ denotes the germ of f at P, and $\mathfrak{m}_P$ is the maximal ideal of the stalk $\mathcal{O}_P$). The local ring embeddings

$$\mathbb{C}[z_1, \dots, z_m]_{(z_1 - a_1, \dots, z_m - a_m)} \hookrightarrow R_{\mathfrak{m}(P)} \hookrightarrow \mathcal{O}_P$$

are therefore homeomorphic embeddings in the maximal-adic topologies. Consequently, since the ring

$$\mathbb{C}[z_1, \dots, z_m]_{(z_1 - a_1, \dots, z_m - a_m)}$$

is dense in $\mathcal{O}_P$, all the three local rings have a common completion (namely, $\widehat{\mathcal{O}}_P$ which is isomorphic to the formal power series ring over $\mathbb{C}$ in m indeterminates). In particular, $R_{\mathfrak{m}(P)}$ is a regular local ring of dimension m.

We still need the following criterion of Matsumura (cf. [30] or [31], Appendix 40).

JACOBIAN CRITERION FOR A REGULAR RING TO BE EXCELLENT. *Let* k *be a field of characteristic zero, and* R *be a regular ring containing* k *such that, for any maximal ideal* $\mathfrak{m}$ *of* R, *the residue field* $R/\mathfrak{m}$ *is algebraic over* k *and* height $\mathfrak{m} = m$. *If there exist* $x_1, \dots, x_m \in R$ *and derivations* $D_1, \dots, D_m \in \mathrm{Der}_k(R)$ *for which* $D_i x_j = \delta_{ij}$, *then* R *is an excellent ring.*

Using this criterion, we immediately deduce that our ring R of global sections over K is excellent; consequently, the canonical homomorphism

$$R_{\mathfrak{m}(P)} \to \widehat{\mathcal{O}}_P$$

is regular for any maximal ideal $\mathfrak{m}(P)$ of R. Hence the sequence of local ring homomorphisms

$$R_{\mathfrak{m}(P)} \to \mathcal{O}_P \to \widehat{\mathcal{O}}_P$$

implies that the first homomorphism is regular because so is their superposition, and the second one is faithfully flat (cf. [31], Chapt. XIII, Sect. 33).

Summing up, the ring R of global sections is an excellent regular ring and, for any point $P \in K$, the canonical ring homomorphism $R \to \mathcal{O}_P$ is regular. In view of the basic properties of regular homomorphisms and excellent rings, we can therefore deduce the following

PROPOSITION 1. *Consider an analytic subspace V in the vicinity of a compact polydisk K in $\mathbb{C}^m$. Let $\mathcal{I}$ be the coherent ideal sheaf of V, R be the ring of sections over K of the structure sheaf of $\mathbb{C}^m$, $I \subset R$ be the ideal of sections of $\mathcal{I}$ over K and let $A := R/I$. Denote by $\mathcal{O}_V = \mathcal{O}/\mathcal{I}$ the structure sheaf of V. For any $P \in K$, we have the canonical ring homomorphism $R \to \mathcal{O}_P$ such that the image of I generates the stalk $\mathcal{I}_P$. Then the ring A is excellent, and the homomorphism $\iota_P : A \to \mathcal{O}_{V,P}$ thus arising is regular.*

REMARK. Clearly, there is a one-to-one correspondence between the irreducible varieties W of V in the vicinity of the compact polydisk K and the minimal prime ideals $\mathfrak{p}$ in the ring A.

We now apply these important properties of the ring of global sections of the structure sheaf, together with the theorem of transition, to analytic intersection theory. The theorem of transition, stated below, originated in Chevalley [9]; it was next generalized by Samuel [45], and eventually by Nagata [34].

THEOREM OF TRANSITION. *Let A and B be two local rings with maximal ideals $\mathfrak{m}$ and $\mathfrak{n}$, respectively. Suppose B is a flat A-module such that $\operatorname{length}_B(B/\mathfrak{m}B) =: k < \infty$. Then*

$$\operatorname{length}_B(B/\mathfrak{q}B) = k \cdot \operatorname{length}_A(A/\mathfrak{q}) \quad \textit{and} \quad e(\mathfrak{q}B) = k \cdot e(\mathfrak{q})$$

for each $\mathfrak{m}$-primary ideal $\mathfrak{q}$ in A.

PROPOSITION 2. *Take the notation of Proposition 1. Let $\mathfrak{q}$ be an ideal in the ring A and let $\mathfrak{p}$ be a minimal prime ideal of $\mathfrak{q}$; clearly, $\mathfrak{p}$ determines an analytic subvariety W of V in the vicinity of the compact polydisk K. Then, for any point P in W, the ideal in the local ring $\mathcal{O}_{V,P}$ generated by $\mathfrak{p}$ is radical, i.e. it is a finite intersection of prime ideals:*

$$\mathfrak{p}\mathcal{O}_{V,P} = \bigcap_i \mathfrak{p}_{P,i};$$

the prime ideals $\mathfrak{p}_{P,i}$ correspond to the irreducible branches of the germ W_P of W at P. Furthermore, for all i, we have the equality of the multiplicities

$$e(\mathfrak{q}A_{\mathfrak{p}}) = e(\mathfrak{q}(\mathcal{O}_{V,P})_{\mathfrak{p}_{P,i}}).$$

Indeed, the ideal $\mathfrak{p}\mathcal{O}_{V,P}$ is radical because the canonical homomorphism $\iota_P : A \to \mathcal{O}_{V,P}$ is regular by Proposition 1. Consequently, if $\mathfrak{p}$ is a minimal prime ideal of $\mathfrak{q}$, each $\mathfrak{p}_{P,i}$ is a minimal prime ideal of $\mathfrak{q}\mathcal{O}_{V,P}$. Thus the equality of the multiplicities follows directly from the theorem of transition and the fact that $k = 1$.

REMARK. It is easy to check that $\dim A/\mathfrak{p} = \dim \mathcal{O}_{V,P}/\mathfrak{p}_{P,i}$.

COROLLARY. *The multiplicities $e(\mathfrak{q}(\mathcal{O}_{V,P})_{\mathfrak{p}_{P,i}})$ depend neither on $P \in W$ nor on the choice of the irreducible branch of the germ W_P.*

Now we can readily apply the above method of compact Stein neighbourhoods to analytic intersection theory. We prove the important

Reduction Theorem. *Consider an analytic set V of pure dimension d in an open domain U in $\mathbb{C}^m$, and holomorphic functions $f_i : U \to \mathbb{C}$ $(i = 1, \dots, r)$ such that a subvariety W in U is a proper component of the intersection of V and the divisors $D_i = (f_i)$. Let P be any point of W, $\mathfrak{p}$ the prime ideal in the local ring $R := \mathcal{O}_{V,P}$ of any irreducible branch of the germ W_P, and let*

$$\mathfrak{b} = \sum_{i=1}^{r} f_i R;$$

clearly, $\mathfrak{p}$ is a prime ideal of $\mathfrak{b}$. Then the multiplicity of the ideal $\mathfrak{q}$ in the localization of R with respect to $\mathfrak{p}$ depends neither on P nor on the choice of the irreducible branch of the germ W_P, and we have the algebraic formula for the intersection index:

$$i(V \cdot D_1 \cdot \ldots \cdot D_r; W) = e(\mathfrak{b} R_{\mathfrak{p}}).$$

Proof. Notice that the case of an isolated intersection has been considered in Section 2, Proposition 1. The independence from the choice of P and of an irreducible branch of W_P follows immediately from Proposition 2. So pick P that is a regular point of the set-theoretic intersection $V \cap |D_1| \cap \ldots \cap |D_r|$; in particular, P is a regular point of the branch W. Obviously, we may assume that $P = 0$. The general case of the theorem can now be established by means of the two associativity formulae: the one valid within the theory of local rings and the associativity axiom of intersection theory. We recall the former (cf. [46], or [34], Chapt. III, Sect. 24).

Associativity Formula for Multiplicities in Local Rings. *Consider a local ring R with maximal ideal $\mathfrak{m}$, and an $\mathfrak{m}$-primary ideal $\mathfrak{a}$ generated by a system of parameters $a_1, \dots, a_r \in \mathfrak{m}$. For a fixed integer $k = 0, 1, \dots, r$, let $\mathfrak{b}$ be the ideal generated by $a_1, \dots, a_k$, and let $\mathfrak{p}_i$ be the minimal prime ideals of $\mathfrak{b}$ for which* length $\mathfrak{p}_i = k$ *and* $\dim R/\mathfrak{p}_i = r - k$. *Then*

$$e(\mathfrak{a}) = \sum_i e((\mathfrak{a} + \mathfrak{p}_i)/\mathfrak{p}_i) \cdot e(\mathfrak{b} R_{\mathfrak{p}_i}).$$

Remark. If a local ring R is analytically equidimensional (i.e. its completion $\widehat{R}$ in the maximal-adic topology is equidimensional), then every minimal prime ideal of $\mathfrak{b}$ satisfies the above two conditions concerning length and dimension.

Corollary. *If a local ring R is analytically equidimensional and if $\mathfrak{p}$ is a unique minimal prime ideal of $\mathfrak{b}$, then*

$$e(\mathfrak{a}) = e((\mathfrak{a} + \mathfrak{p})/\mathfrak{p}) \cdot e(\mathfrak{b} R_{\mathfrak{p}}).$$

In particular, if $\mathfrak{p}$ is the only isolated primary component of the ideal $\mathfrak{b}$, then

$$e(\mathfrak{a}) = e((\mathfrak{a} + \mathfrak{p})/\mathfrak{p}).$$

In our geometric context, the germs $f_1, \dots, f_r$ form a sequence of parameters of the localization $R_{\mathfrak{p}}$. Pick linear hyperplanes H_j in $\mathbb{C}^m$ $(j = r + 1, \dots, d)$ that meet W transversally at P. If f_j are linear equations for H_j $(j = r + 1, \dots, d)$, then the germs

$f_1, \dots, f_r, f_{r+1}, \dots, f_d$ form a system of parameters of the local ring R, and $f_{r+1}, \dots, f_d$ generate the maximal ideal of the regular local ring $\mathcal{O}_{W,P} = R/\mathfrak{p}$. By the associativity formula for multiplicities in local rings, we get

$$e(\mathfrak{a}) = e((\mathfrak{a} + \mathfrak{p})/\mathfrak{p}) \cdot e(\mathfrak{b}R_{\mathfrak{p}}) = e(\mathfrak{b}R_{\mathfrak{p}}).$$

On the other hand, it follows from Proposition 1 in Section 2 and the associativity axiom that

$$\begin{aligned} e(\mathfrak{a}) &= i(V \cdot D_1 \cdot \ldots \cdot D_r \cdot H_{r+1} \cdot \ldots \cdot H_d; P) \\ &= i(V \cdot D_1 \cdot \ldots D_r; W) \cdot i(W \cdot H_{r+1} \cdot \ldots \cdot H_d; P) \\ &= i(V \cdot D_1 \cdot \ldots D_r; W), \end{aligned}$$

whereby the reduction theorem is proved.

Remark. The reduction theorem makes it possible to express the proper intersections of analytic sets with a collection of effective divisors by means of multiplicities in local rings.

We conclude the section with the following

Proposition 3. *Let $f : M \to N$ be a holomorphic mapping between complex manifolds M and N of dimensions m and n, respectively. Suppose that all fibres of f have common dimension $m - n$, $m \geq n$ (which implies, by Remmert's theorem, that f is an open mapping). Consider the fibres $f^{-1}(Q)$ $(Q \in N)$ of f in the ideal-sheaf-theoretic sense; the induced cycles $[f^{-1}(Q)]$ are the fibres of f regarded as analytic cycles on M. Then the fibres $[f^{-1}(Q)]$ $(Q \in N)$ form a continuous family, i.e. the $(m-n)$-cycles $[f^{-1}(Q_j)]$ converge to $[f^{-1}(Q_0)]$ on M whenever the points Q_j tend to Q_0 in N.*

Remarks. (1) No stronger assumption (as e.g. properness) needs to be imposed on the function f above. Therefore we may, of course, encounter the phenomenon that the fibres $[f^{-1}(Q_j)]$ converge to the zero cycle (and their supports converge to the empty set).

(2) The cycles $[f^{-1}(Q)]$ coincide with those considered in [59], where also a useful concept of an s-parametrization is given (cf. [37]).

The theorem being local with respect to N, we may assume that $N = \mathbb{C}^n$. Set

$$\operatorname{graph}(f) := \{(P, Q) \in M \times \mathbb{C}^n : Q = f(P)\},$$

and let $p : \operatorname{graph}(f) \to M$ be the canonical projection, which is a biholomorphism. Since proper intersection product is a continuous operation, it is sufficient to show that

$$(*) \qquad [f^{-1}(Q)] = p_*(\operatorname{graph}(f) \cdot (M \times \{Q\})).$$

So fix a point $Q = (w_1, \dots, w_n)$; any irreducible branch W of $f^{-1}(Q)$ corresponds to the irreducible branch

$$\operatorname{graph}(f|W) := \{(P, Q) \in W \times \mathbb{C}^n : Q = f(P)\}$$

of $\operatorname{graph}(f) \cdot (M \times \{Q\})$. For P in the regular locus of $f^{-1}(Q)$, let $\mathfrak{p}$ be the prime ideal in the local ring $\mathcal{O}_{M,P}$ of the germ W_P. According to the reduction theorem and using the biholomorphism p, we deduce that the coefficient of the branch W on the right-hand side of $(*)$ is

$$e\Big(\sum_{i=1}^{n}(f_i - w_i)(\mathcal{O}_{M,P})_{\mathfrak{p}}\Big),$$

which is equal to

$$\text{length } (\mathcal{O}_{M,P})_{\mathfrak{p}}/\sum_{i=1}^{n}(f_i - w_i)(\mathcal{O}_{M,P})_{\mathfrak{p}}$$

because the ring $(\mathcal{O}_{M,P})_{\mathfrak{p}}$ is regular, and a fortiori Cohen–Macaulay. Hence this coefficient coincides with that of W on the left-hand side of $(*)$, as asserted.

II. Local algorithm for analytic improper intersections

1. Filter-regular and superficial elements. Throughout this section A always denotes a noetherian ring and I is an ideal in A. For a submodule N of a noetherian A-module M, set

$$\begin{aligned} N :_M \langle I\rangle &:= \bigcup_{n\in\mathbb{N}}(N :_M I^n) = \{m \in M : m\cdot I^n \subset N \text{ for some } n \in \mathbb{N}\} \\ &= N :_M I^N \quad \text{for } N \gg 0. \end{aligned}$$

The operation $N :_M \langle I\rangle$ has the following property:

Let

$$\mathfrak{a} = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_r \cap \mathfrak{q}_{r+1} \cap \ldots \cap \mathfrak{q}_{r+s}$$

be an irredundant primary decomposition of an ideal $\mathfrak{a}$ *in the ring* A *such that* $\mathfrak{p}_i := \sqrt{\mathfrak{q}_i} \not\supset I$ *for* $i = 1,\ldots,r$, *and* $\mathfrak{p}_j := \sqrt{\mathfrak{q}_j} \supset I$ *for* $j = r+1,\ldots,r+s$. *Then*

$$\mathfrak{a} :_A \langle I\rangle = \bigcap_{i=1}^{r}\mathfrak{q}_i.$$

An easy verification is left to the reader.

Proposition 1. *For* $a \in I$, *the following conditions are equivalent*:

(1) $0 :_A (0 :_A a) \supset I^n$ *for some positive integer* n;
(2) $(0 :_A a)_{\mathfrak{p}} = 0$ *for all prime ideals* $\mathfrak{p}$ *in* A *such that* $I \not\subset \mathfrak{p}$;
(3) $a/1 \in A_{\mathfrak{p}}$ *is a non-zero divisor for all prime ideals* $\mathfrak{p}$ *in* A *such that* $I \not\subset \mathfrak{p}$.

The only non-trivial equivalence is (1)⇔(2). So suppose

$$0 :_A (0 :_A a) \supset I^n$$

and let $\mathfrak{p}$ be a prime ideal of A such that $I \not\subset \mathfrak{p}$. Pick $s \in I^n \setminus \mathfrak{p}$. Then $s\cdot(0 :_A a) = 0$, whence $(0 :_A a)_{\mathfrak{p}} = 0$, as desired.

Conversely, suppose condition (2) holds and let

$$0 :_A (0 :_A a) = \bigcap_i \mathfrak{q}_i$$

be an irredundant primary decomposition. If

$$I^n \not\subset 0 :_A (0 :_A a)$$

for all positive integers n, then $I^n \not\subset \mathfrak{q}_i$ for some i and all n, and thus $I \not\subset \mathfrak{p} := \sqrt{\mathfrak{q}_i}$. Since $(0 :_A a)_{\mathfrak{p}} = 0$, there is an $s \in A \setminus \mathfrak{p}$ such that $s \cdot (0 :_A a) = 0$. Then

$$s \in 0 :_A (0 :_A a) \subset \mathfrak{q}_i \subset \mathfrak{p},$$

which is a contradiction, and the proof is complete.

An element $a \in I$ satisfying one of the above equivalent conditions will be called a *filter-regular element* with respect to the ideal I. The notion of filter-regularity was introduced by Schenzel–Ngo Viet Trung–Nguyn Tu Cuong [48] in connection with the theory of generalized Cohen–Macaulay modules, and Stückrad–Vogel [55] applied it to certain generalizations of Buchsbaum modules. The proposition below lists a few other conditions equivalent to filter-regularity; its proof is straightforward.

Proposition 2. *Let*

$$0 = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_r \cap \mathfrak{q}_{r+1} \cap \ldots \cap \mathfrak{q}_{r+s}$$

be an irredundant primary decomposition such that $\mathfrak{p}_i := \sqrt{\mathfrak{q}_i} \not\supset I$ *for* $i = 1, \ldots, r$, *and* $\mathfrak{p}_j := \sqrt{\mathfrak{q}_j} \supset I$ *for* $j = r+1, \ldots, r+s$. *Take* $a \in I$ *and a positive integer* N *such that*

$$I^N \subset \bigcap_{j=r+1}^{r+s} \mathfrak{q}_j.$$

Then the following conditions are equivalent:

(1) a *is filter-regular with respect to* I;
(2) $a \notin \mathfrak{p}$ *for all associated primes* $\mathfrak{p}$ *of* A *which do not contain* I;
(3) $0 :_A a \subset 0 :_A \langle I \rangle$;
(4) $(0 :_A a) \cap I^N = 0$;
(5) $0 :_A \langle a \rangle \subset 0 :_A \langle I \rangle$;
(6) $(0 :_A \langle a \rangle) \cap I^N = 0$.

We now discuss the notion of filter-regularity in the case of the associated graded ring

$$G = G_I(A) = \bigoplus_{n=0}^{\infty} G_I^n(A) = \bigoplus_{n=0}^{\infty} I^n/I^{n+1}.$$

Denote by

$$G^+ = G_I^+(A) = \bigoplus_{n=1}^{\infty} G_I^n(A)$$

the ideal of all elements of G of positive degree. Consider $a \in I$ and its initial form a^* in G. Let s be the degree of a^*; we call the integer s the *order* of a with respect to I. Then a^* is filter-regular with respect to G^+ iff

$$(0 :_G a^*) \cap (G^+)^N = 0$$

for some positive integer N or, equivalently,

$$(I^n :_A a) \cap I^N = I^{n-s}$$

for all $n \geq N + s$. The latter means exactly that a is *superficial* with respect to I (the notion of a superficial element was introduced by Samuel [45]; see also [65], Chapt. VIII or [2, 33, 34]).

PROPOSITION 3. *Assume that the ideal I is contained in the Jacobson radical of the ring A. If an element $a \in I$ is superficial with respect to I, then it is filter-regular with respect to I.*

Indeed, let a^* be the initial form of a in G and s be the degree of a^*. By hypothesis and Proposition 2, there is a positive integer N such that

$$(0 :_G a^*) \cap (G^+)^N = 0$$

or, equivalently,

$$(I^{n+s} :_A a) \cap I^N = I^n \quad \text{for all } n \geq N.$$

Take $x \in I^N$ and suppose that $ax = 0$. Then, by the Krull intersection theorem,

$$x \in \bigcap_{n=N}^{\infty} (I^{n+s} :_A a) \cap I^N = \bigcap_{n=N}^{\infty} I^n = 0.$$

Hence again by Proposition 2, a is filter-regular with respect to I, as desired.

Observe now that for every ideal I in a noetherian ring A and for any $a \in A$, there is a positive integer k such that

$$I^{n+k} :_A a \subset I^n + (0 :_A a)$$

for all positive integers n. This is a direct consequence of the Artin–Rees lemma, according to which there is a positive integer k such that

$$I^{n+k} \cap aA = I^n(I^k \cap aA)$$

for all positive integers n. Therefore, if $x \in I^{n+k} :_A a$, then $ax \in I^{n+k} \cap aA = I^n(I^k \cap aA) \subset aI^n$. Consequently, $ax = ay$ with $y \in I^n$, and thus $x = y + (x - y) \in I^n + (0 :_A a)$, as asserted.

In particular, if a is a non-zero divisor in A, then, for all positive integers n, we have

$$I^{n+k} :_A a \subset I^n.$$

Hence we immediately obtain the following

PROPOSITION 4. *Consider a superficial element $a \in I$ of order s with respect to an ideal I in a noetherian ring A. Let a^* be the initial form of a in the associated graded ring $G = G_I(A)$. If a is a non-zero divisor in A, then*

$$I^{s+n} :_A a = I^n \quad \textit{for } n \gg 0.$$

For an ideal J in A, we denote by

$$G_I(J, A) := \bigoplus_{n=0}^{\infty} (J \cap I^n + I^{n+1})/I^{n+1}$$

the homogeneous ideal in the graded ring $G = G_I(A)$ generated by the initial forms of all elements of J.

The equality $I^{s+n} :_A a = I^n$ always implies $aA \cap I^{s+n} = aI^n$. The converse implication holds whenever a is a non-zero divisor in A. We can therefore state a corollary to Proposition 4.

Corollary 1. *Under the notation of Proposition* 4, *if a is a non-zero divisor in A, then the ideal $G_I(aA, A)$ coincides with a^*G in sufficiently large degrees. Consequently*,

$$G_I(aA, A) :_G \langle G^+ \rangle = a^*G :_G \langle G^+ \rangle.$$

Moreover, the above assertion still holds if we drop the assumption that a is a non-zero divisor (*cf.* [2], *Lemma* 2.5).

From now on we shall always assume that the ideal I is contained in the Jacobson radical of the ring A. We say that a sequence $a_1, \ldots, a_r \in I$ is *filter-regular* with respect to I if for each $k = 1, \ldots, r$ the class of a_k in the ring $A/(a_1, \ldots, a_{k-1})$ is a filter-regular element. In other words, it is required that each element a_k avoid all the associated primes $\mathfrak{p}$ of the ideal $(a_1, \ldots, a_{k-1}) \subset A$ which do not contain I; equivalently, for all $k = 1, \ldots, r$ and every prime ideal $\mathfrak{p}$ of A such that $a_1, \ldots, a_k \in \mathfrak{p}$ and $I \not\subset \mathfrak{p}$, the elements $a_1/1, \ldots, a_k/1 \in A_\mathfrak{p}$ form an $A_\mathfrak{p}$-regular sequence.

We say that a sequence $a_1, \ldots, a_r \in I$ is *superficial* with respect to I if the sequence of initial forms $a_1^*, \ldots, a_r^* \in G = G_I(A)$ is filter-regular with respect to G^+.

Recall now that in the general theory of graded rings, $\mathfrak{a} :_G \langle G^+ \rangle$ is the saturation of a homogeneous ideal $\mathfrak{a}$; the *saturation* of $\mathfrak{a}$ is, by definition, the largest homogeneous ideal $\mathfrak{b}$ in G which coincides with $\mathfrak{a}$ in large degrees (see e.g. [65], Chapt. VII, Sect. 2).

We say that a homogeneous ideal $\mathfrak{a}$ is *saturated* if it is equal to its saturation. The following three conditions are equivalent (*loc. cit.*):

(1) two homogeneous ideals $\mathfrak{a}$ and $\mathfrak{b}$ have the same saturation;

(2) $\mathfrak{a} :_G \langle G^+ \rangle = \mathfrak{b} :_G \langle G^+ \rangle$;

(3) for any relevant prime ideal $\mathfrak{p}$ in G (i.e. $\mathfrak{p}$ does not contain G^+), the localizations of $\mathfrak{a}$ and $\mathfrak{b}$ with respect to $\mathfrak{p}$ coincide.

Let us observe that Corollary 1 extends, by an easy induction on r, to filter-regular sequences.

Corollary 2. *If $a_1, \ldots, a_r \in I$ is a superficial sequence with respect to I, then*

$$G_I((a_1, \ldots, a_r), A) :_G \langle G^+ \rangle = (a_1^*, \ldots, a_r^*) :_G \langle G^+ \rangle;$$

or, equivalently, the homogeneous ideals $G_I((a_1, \ldots, a_r), A)$ and $(a_1^*, \ldots, a_r^*)$ coincide in large degrees. In other words, these homogeneous ideals have the same saturation in $G = G_I(A)$.

Further, Proposition 3 extends to the case of superficial sequences.

Proposition 5. *Assume that the ideal I is contained in the Jacobson radical of the ring A. If a sequence $a_1, \ldots, a_r \in I$ is superficial with respect to I, then it is filter-regular with respect to I.*

The proof is straightforward.

Finally, we recall a lemma of Samuel (cf. [46] or [65], Chapt. VIII, Theorem 22) applied later in Section 4.

Samuel's Lemma. *Consider a one-dimensional local ring A with maximal ideal $\mathfrak{m}$ and an $\mathfrak{m}$-primary ideal $\mathfrak{q}$ in A. If $a \in \mathfrak{q}$ is a superficial element of order one with respect to $\mathfrak{q}$, then $e(\mathfrak{q}) = e(aA)$.*

Proof. Consider first the case where a is a non-zero divisor in A. It follows from Proposition 4 that

$$\mathfrak{q}^n :_A a = \mathfrak{q}^{n-1} \quad \text{for } n \gg 0.$$

We have the following evident equalities:

$$\begin{aligned}
\operatorname{length}(A/\mathfrak{q}^n) &- \operatorname{length}((A/aA)/(\mathfrak{q}/aA)^n) \\
&= \operatorname{length}(A/\mathfrak{q}^n) - \operatorname{length}(A/(\mathfrak{q}^n + aA) \\
&= \operatorname{length}((\mathfrak{q}^n + aA)/\mathfrak{q}^n) = \operatorname{length}(aA/(\mathfrak{q}^n \cap aA)) \\
&= \operatorname{length}(aA/(aA \cdot (\mathfrak{q}^n :_A a))) = \operatorname{length}(A/(\mathfrak{q}^n : Aa)) \\
&= \operatorname{length}(A/\mathfrak{q}^{n-1}),
\end{aligned}$$

whence we get, for sufficiently large n, the equalities

$$e(\mathfrak{q}) = \operatorname{length}(A/\mathfrak{q}^n) - \operatorname{length}(A/\mathfrak{q}^{n-1}) = \operatorname{length}((A/aA)/(\mathfrak{q}/aA)^n).$$

Since the ring A/aA has dimension 0, we have

$$\operatorname{length}((A/aA)/(\mathfrak{q}/aA)^n) = \operatorname{length}(A/aA)$$

for n large. But this is also the length of $a^{n-1}A/a^nA$ because the rings A/aA and $a^{n-1}A/a^nA$ are isomorphic (as a is a non-zero divisor). Hence $e(\mathfrak{q}) = e(aA)$, as asserted.

In the general case, let $\mathfrak{a}$ be the annihilator of a; it has, of course, a finite length l. Consequently, $\mathfrak{q}^n\mathfrak{a} = 0$ for n large since

$$\bigcap_{n=0}^{\infty} (\mathfrak{q}^n \cap \mathfrak{a}) = 0.$$

So consider the quotient ring $\overline{A} := A/\mathfrak{a}$. For any $\mathfrak{m}$-primary ideal $\mathfrak{o}$ in A, set $\overline{\mathfrak{o}} := (\mathfrak{o}+\mathfrak{a})/\mathfrak{a}$. Then

$$\begin{aligned}
\operatorname{length}(\overline{A}/\overline{\mathfrak{o}}^n) &= \operatorname{length}(A/(\mathfrak{a} + \mathfrak{o}^n)) = \operatorname{length}(A/\mathfrak{o}^n) - \operatorname{length}((\mathfrak{a} + \mathfrak{o}^n)/\mathfrak{o}^n) \\
&= \operatorname{length}(A/\mathfrak{o}^n) - \operatorname{length}(\mathfrak{a}/(\mathfrak{a} \cap \mathfrak{o}^n)) = \operatorname{length}(A/\mathfrak{o}^n) - l
\end{aligned}$$

for n large enough. Hence $e(\overline{\mathfrak{o}}) = e(\mathfrak{o})$. In particular, $e(\overline{\mathfrak{q}}) = e(\mathfrak{q})$ and $e(\overline{a}\overline{A}) = e(aA)$, where $\overline{a}$ is the $\mathfrak{a}$-residue of a. Since $\overline{a}$ is superficial of order one with respect to $\overline{\mathfrak{q}}$ and is a non-zero divisor in $\overline{A}$, we have already shown that $e(\overline{\mathfrak{q}}) = e(\overline{a}\overline{A})$. Therefore, $e(\mathfrak{q}) = e(aA)$, and the proof of the lemma is complete.

2. Geometric and algebraic approaches to the intersection algorithm. In Chapter I we referred to convergence of analytic cycles on a complex manifold. We now give the precise definition and some further properties of this notion needed in the theory of analytic improper intersections. The *topology of local uniform convergence* for the family of closed subsets of a complex manifold M (cf. Tworzewski–Winiarski [58]) is determined by the basis which consists of the sets of the form

$$\{F \subset M : F \text{ is closed}, \ F \cap K = \emptyset, \ F \cap U \neq \emptyset \text{ for } U \in S\},$$

where K ranges over all compact subsets and S ranges over all finite families of open subsets in M. The claim below describes local uniform convergence; its proof is straightforward.

Claim. *A sequence of closed subsets F_n converges to a closed set F_0 iff for every point $P \in F_0$ and every neighbourhood U of P, we have $F_n \cap U \neq \emptyset$ for n large enough, and for every compact subset K disjoint from F_0, we have $F_n \cap K = \emptyset$ for n large enough.*

By the *degree* of a 0-cycle $\sum_{i=1}^{r} k_i Q_i$ we mean the sum $\sum_{i=1}^{r} k_i$ of its coefficients. The following proposition (cf. [57]) characterizes the convergence of positive analytic cycles.

Proposition 1. *Consider a sequence Z_n $(n = 0, 1, 2, \ldots)$ of positive p-cycles on a complex manifold M of dimension m. If the supports $|Z_n|$ converge to the support $|Z_0|$, then the following conditions are equivalent*:

(1) *the sequence Z_n converges to the cycle Z_0, i.e. for each regular point P of $|Z_0|$ and any relatively compact submanifold N of dimension $m-p$ meeting $|Z_0|$ transversally at P such that $|Z_0| \cap \overline{N} = \{P\}$, we have*

$$\deg(Z_n \cdot N) = \deg(Z_0 \cdot N) \quad \textit{for } n \gg 0;$$

(2) *for each point P of $|Z_0|$ and any relatively compact submanifold N of dimension $m-p$ such that $|Z_0| \cap N = |Z_0| \cap \overline{N} = \{P\}$, we have*

$$\deg(Z_n \cdot N) = \deg(Z_0 \cdot N) \quad \textit{for } n \gg 0;$$

(3) *for each point P from a given dense subset of regular points of $|Z_0|$ there is a relatively compact submanifold N of dimension $m-p$ meeting $|Z_0|$ transversally at P such that*

$$|Z_0| \cap \overline{N} = \{P\}, \quad \textit{and} \quad \deg(Z_n \cdot N) = \deg(Z_0 \cdot N) \quad \textit{for } n \gg 0.$$

Remark. One must control the intersection of the cycle Z_0 with the whole compact closure $\overline{N}$ of N, because otherwise the intersections $Z_n \cdot N$ could yield some additional points which converge to the border of the submanifold N. Thus, the assumption $|Z_0| \cap N = \{P\}$ would be insufficient.

We saw in Chapter I that proper intersection product is a continuous operation on analytic cycles on a complex manifold. This fact will play a significant role in the proofs of the theorems on deformations to the algebraic cone and to an algebraic bicone (cf. Sect. 4 and Chapt. III, Sect. 1).

Suppose now that S is a closed s-dimensional submanifold of a complex manifold M of dimension m. For a given analytic cycle $Z = \sum_i k_i Z_i$, where the Z_i form a locally finite family of irreducible analytic subsets in M, the part of Z supported by S is the cycle

$$Z^S := \sum_{Z_i \subset S} k_i Z_i.$$

In what follows, we shall need the following

TWORZEWSKI'S LEMMA (*op. cit.*). *Let Z_n $(n = 0, 1, 2, \ldots)$ be a sequence of positive p-cycles on M which converges to the cycle Z_0. If*

$$m_P Z_0^S \leq m_P Z_n^S$$

for a point P of a submanifold S, then there is an open neighbourhood U of P such that

- *$Z_n^S \cap U$ converges to $Z_0^S \cap U$ and*
- *$(Z_n - Z_n^S) \cap U$ converges to $(Z_0 - Z_0^S) \cap U$*

as sequences of analytic cycles in U.

We wish to recall a local algorithm for improper intersections in complex analytic geometry, considered by P. Tworzewski (*op. cit.*). This algorithm is a local analytic counterpart of the Stückrad–Vogel intersection algorithm from global algebraic geometry (cf. [54]). Some other local algebraic counterparts have been investigated by Achilles–Manaresi [2, 3]; their algorithm is carried out for any collection of elements of a local ring A that forms a filter-regular sequence with respect to a given ideal I of A.

For an open subset U of the complex manifold M such that $U \cap S \neq \emptyset$, denote by $\mathcal{H}(U)$ the set of all $\mathcal{H} = (H_1, \ldots, H_r)$, where $r := m - s$ is the codimension of S in M, satisfying the two conditions:

(1) H_k $(k = 1, \ldots, r)$ is a smooth hypersurface in U containing $S \cap U$;

(2) $\bigcap_{k=1}^r T_P H_k = T_P S$ for each point P in $S \cap U$ (here $T_P N$ is the tangent space to the manifold N at the point P).

For a given analytic subset V in M of pure dimension d, let $\mathcal{H}(U, V)$ be the set of all $\mathcal{H} \in \mathcal{H}(U)$ such that

$$((U \setminus S) \cap V) \cap H_1 \cap \ldots \cap H_k$$

is an analytic subset in $U \setminus S$ of pure dimension $d - k$ (or the empty set) for $k = 1, \ldots, r$.

For every $\mathcal{H} = (H_1, \ldots, H_r) \in \mathcal{H}(U, V)$, we define two families of analytic cycles on U by the recursive formula

$$\alpha_{-1} = [V \cap U], \quad H_0 := U,$$
$$\varrho_k = (\alpha_{k-1} \cdot H_k)^S, \quad \alpha_k = \alpha_{k-1} \cdot H_k - \varrho_k.$$

Observe that it follows directly from the definition of $\mathcal{H}(U, V)$ that the above intersection products are proper. Consequently, α_k is a $(d - k)$-cycle on U and ϱ_k is a $(d - k)$-cycle on $S \cap U$; ϱ_k will be called the *result of the kth step* of the intersection algorithm for the collection $\mathcal{H}$. The *total result* of the intersection algorithm is the cycle

$$\varrho = \varrho(V \cdot \mathcal{H}) := \sum_{k=0}^{r} \varrho_k$$

on the submanifold S.

For an analytic cycle $Z = \sum_{i=0}^m Z_i$ on M, where Z_i are i-cycles $(i = 0, \ldots, m)$, the *extended multiplicity* $\text{ext.mult}_P Z$ of Z at a point P is the sequence

$$\text{ext.mult}_P Z := (m_P Z_m, \ldots, m_P Z_0) \in \mathbb{N}^{m+1}.$$

For any point P in S, Tworzewski [57] defined the multi-index

$$\widetilde{g}(P) = \widetilde{g}(V,S)(P) := \min\{\text{ext.mult}_P \varrho(V \cdot \mathcal{H}) : \mathcal{H} \in \mathcal{H}(U,V), P \in U\} \in \mathbb{N}^{s+1},$$

where both neighbourhoods and admissible collections $\mathcal{H}(U,V)$ vary, and the minimum is with respect to the lexicographic ordering. The $(s+1)$-tuple $\widetilde{g}(V,S)(P)$ will be called the *generalized* (or *extended*) *intersection index* of V with the submanifold S at the point P. We call the sum $g(P)$ of the components of $\widetilde{g}(P)$ the *intersection index* (or *multiplicity*) of V with the submanifold S at P. Tworzewski proved that the function

$$S \ni P \mapsto \widetilde{g}(V,S)(P) \in \mathbb{N}^{s+1}$$

is upper semicontinuous in the analytic Zariski topology, while the function

$$S \ni P \mapsto g(V,S)(P) \in \mathbb{N}$$

is analytically constructible (*op. cit.*).

In Chapter III we shall reduce the analytic intersection algorithm to the case of an algebraic bicone intersecting linear hyperplanes (see also [38, 40]). Next we shall prove that the generalized intersection index is realized by a collection of smooth divisors provided that their equations (testing elements in the ring A) satisfy certain conditions of filter-regularity. These conditions imposed on such a "generic" collection of smooth analytic divisors are very strong and of linear character: the first derivatives of the equations of each successive divisor should avoid a finite union of proper linear subspaces (depending on the previous divisors of the collection).

The generalized index for improper intersections will be expressed as the bidegree sequence of a certain algebraic bicone, and intersection multiplicity at a point P as the Samuel multiplicity at P of the normal cone. This will make it possible to derive many consequences important for the theory of improper intersections in complex analytic geometry (see also [38, 40]). In particular, it will turn out that the generalized intersection index for analytic improper intersections by Tworzewski [57] coincides with the indices defined for the local algebraic case by Achilles–Manaresi [3] (generalized Samuel multiplicities) as well as with the Segre numbers by Gaffney–Gassler [19].

Now, we wish to present an algebraic version of the local intersection algorithm for analytic improper intersections (see also [38, 40]). We shall understand the algebraic objects such as local rings, graded rings, ideals, non-zero divisors, etc., also in their local geometric meaning, i.e. as germs of analytic sets, cones, analytic subsets, divisors, etc. determined by them. This abuse of language is very convenient and is not misleading because the complementary algebro-geometric techniques point to an appropriate context. In particular, if A is the local ring of a given analytic set at a fixed point P, any ideal $\mathfrak{a}$ in A determines an analytic cycle $[\mathfrak{a}]$ near P:

$$[\mathfrak{a}] := \sum \text{length}(A_{\mathfrak{p}}/\mathfrak{q}A_{\mathfrak{p}}) \cdot [\mathfrak{p}],$$

where the sum is taken over all isolated primary ideals $\mathfrak{q}$ of $\mathfrak{a}$ and $\mathfrak{p} = \sqrt{\mathfrak{q}}$; here $[\mathfrak{p}]$ denotes the (local) analytic cycle determined by the ideal $\mathfrak{p}$ (with coefficient 1). We thus take into account only non-embedded components of $\mathfrak{a}$ with geometric multiplicities. By $\text{mult}[\mathfrak{a}]$

we mean the multiplicity of the cycle $[\mathfrak{a}]$ at P:

$$\mathrm{mult}[\mathfrak{a}] = \sum \mathrm{length}(A_{\mathfrak{p}}/\mathfrak{q}A_{\mathfrak{p}}) \cdot e(A/\mathfrak{p}).$$

In Tworzewski's algorithm, two conditions are imposed on the collections (H_k): one extrinsic with respect to the set V about their tangent spaces, and one intrinsic to the effect that for each k the set

$$(V \setminus S) \cap H_1 \cap \ldots \cap H_k$$

is of pure dimension $d-k$ near P. The reduction theorem for proper intersections (see Sect. 3 of Chapt. I) will enable us to describe the intersection algorithm algebraically in terms of the local ring A of the set V at P, and the ideal $I \subset A$ of the submanifold S. Instead of considering collections (H_k), however, one can deal with their equations a_k regarded as elements of A; the equations a_k involved in the intersection algorithm will be called *testing elements*. The intrinsic condition means that each element a_k avoids all the minimal primes of the ideal $(a_1, \ldots, a_{k-1})$ which do not contain I. In our algebraic approach, one can disregard the extrinsic condition imposed on collections of hypersurfaces and strengthen the intrinsic one; namely, it is required that each element a_k avoid all the associated primes of the ideal $(a_1, \ldots, a_{k-1})$ which do not contain I (i.e. testing elements a_i are supposed to be filter-regular with respect to the ideal I). In other words, for all $k = 1, \ldots, r$ and for every prime ideal $\mathfrak{p}$ of A such that $a_1, \ldots, a_k \in \mathfrak{p}$ and $I \not\subset \mathfrak{p}$, the elements $a_1/1, \ldots, a_k/1 \in A_{\mathfrak{p}}$ form an $A_{\mathfrak{p}}$-regular sequence.

For a collection of testing elements $a_1, \ldots, a_r$, we put

$$\mathfrak{a}_{-1} := (0), \quad a_0 := 0,$$

and inductively

$$\mathfrak{a}_{k-1}(a) + a_k A = \mathfrak{a}_k(a) \cap \mathfrak{r}_k(a),$$

where $\mathfrak{a}_k(a)$ (resp. $\mathfrak{r}_k(a)$) is the intersection of those associated primary ideals of an irredundant primary decomposition of the ideal $\mathfrak{a}_{k-1}(a) + a_k A$ whose prime ideals do not contain (resp. do contain) the ideal I. Clearly, the isolated primary ideals of $\mathfrak{r}_k(a)$, and thus the cycles $[\mathfrak{r}_k(a)]$, do not depend on the above-mentioned primary decompositions. The independence from the family of ideals $\mathfrak{a}_i(a)$, however, is an immediate consequence of the algebraic formula from Section 1:

$$\mathfrak{a}_k(a) = (\mathfrak{a}_{k-1}(a) + a_k A) :_A \langle I \rangle.$$

The ideals $\mathfrak{r}_k(a)$ are the ideal-theoretic results of the local intersection algorithm; they are counterparts of the results ϱ_k of the intersection algorithm for analytic cycles. What connects both geometric and algebraic approaches to the intersection algorithm is thus the theory of analytic cycles: $\varrho_k(a) = [\mathfrak{r}_k(a)]$. We shall elaborate this more precisely in Section 4. A remarkable advantage of using the theory of positive analytic cycles is that we get at our disposal a dynamic concept of convergence for which proper intersection is a continuous operation (see Chapter I).

REMARKS. (1) It follows directly from the construction that for any prime ideal $\mathfrak{p}$ in A with $I \not\subset \mathfrak{p}$, $\mathfrak{r}_k(a)A_{\mathfrak{p}} = A_{\mathfrak{p}}$, and thus by easy induction on k that

$$(a_1, \ldots, a_k)A_{\mathfrak{p}} = \mathfrak{a}_k(a)A_{\mathfrak{p}}.$$

Furthermore, if $a_1, \dots, a_r$ is a filter-regular sequence, then a_k is not a zero divisor in $A/\mathfrak{a}_{k-1}(a)$. Otherwise a_k would belong to an associated prime $\mathfrak{p}$ of $\mathfrak{a}_{k-1}(a)$, and therefore $a_k/1$ would be a zero divisor in $A_\mathfrak{p}/\mathfrak{a}_{k-1}(a)A_\mathfrak{p}$. Since $I \not\subset \mathfrak{p}$ by construction,

$$A_\mathfrak{p}/\mathfrak{a}_{k-1}(a)A_\mathfrak{p} = A_\mathfrak{p}/(a_1, \dots, a_{k-1})A_\mathfrak{p}$$

and $a_1/1, \dots, a_k/1$ would not form an $A_\mathfrak{p}$-regular sequence, contradicting filter-regularity.

(2) Through dealing with subvarieties together with coefficients, cycle calculus is a geometric tool by means of which—on a par with the ideal-theoretic approach—one can describe intersections of algebraic or analytic sets. Although the intuitive geometric notion of a cycle (called a virtual variety) had been introduced in algebraic geometry by Severi, who had also initiated the theory of rational and algebraic equivalence of cycles, it was Weil [63] who first gave a systematic exposition of the subject in a precise algebraic language. In the theory of proper intersections, the reduction theorem together with the additivity and associativity formulae (see Chapter I) make it possible to work interchangeably with effective divisors and their intersection cycles, or with their ideals, provided that those effective divisors form a regular sequence (locally, in the vicinity of an irreducible proper component of the intersection). In the next section, we apply this fact to calculate the result of the intersection algorithm.

3. Deformation to the normal cone in analytic geometry. We begin with recalling the notions of analytic and projective cones. Consider an analytic space X and a coherent ideal sheaf $\mathcal{I}$ on $X \times \mathbb{C}_t^{r+1}$ with $r \geq 0$. For any $\lambda \in \mathbb{C}$, $\lambda \neq 0$, the mapping

$$\phi_\lambda : \mathbb{C}_t^{r+1} \ni t \mapsto \lambda t \in \mathbb{C}_t^{r+1}$$

determines a biholomorphism

$$\phi_\lambda : X \times \mathbb{C}_t^{r+1} \to X \times \mathbb{C}_t^{r+1}.$$

We say that the ideal sheaf $\mathcal{I}$ is *homogeneous* with respect to the affine space $\mathbb{C}_t^{r+1}$ (or with respect to the variables t) if it satisfies the following *homogeneity condition*:

$$\phi_\lambda^* \mathcal{I} = \mathcal{I} \quad \text{for all } \lambda \neq 0.$$

The following proposition is due to Cartan [8]:

A coherent ideal sheaf $\mathcal{I}$ on $X \times \mathbb{C}_t^{r+1}$ is homogeneous with respect to the variables t iff each point $P \in X$ has a neighbourhood U such that $\mathcal{I}$ is generated in $U \times \mathbb{C}_t^{r+1}$ by a finite number of global holomorphic functions in $U \times \mathbb{C}_t^{r+1}$ which are homogeneous polynomials (forms) with respect to the coordinates t.

The analytic subspace C of $X \times \mathbb{C}_t^{r+1}$ determined by the homogeneous ideal sheaf $\mathcal{I}$, together with the canonical projection $p : C \to X$, will be called an *analytic cone* over X. The space X will be called the *vertex space* of the cone C.

For every point $P \in X$, the ideal

$$\mathfrak{I}_P := \mathcal{I}_{(P,0)} \cap \mathcal{O}_{X,P}[t]$$

(where $\mathcal{O}_{X,P}$ is the stalk at P of the structure sheaf $\mathcal{O}_X$) is a homogeneous ideal in the polynomial ring $\mathcal{O}_{X,P}[t]$ which generates the stalk $\mathcal{I}_{(P,0)}$ at $(P,0)$ of the homogeneous ideal sheaf $\mathcal{I}$. We call the quotient ring $\mathcal{O}_{X,P}[t]/\mathfrak{I}_P$ the *graded ring* of the analytic cone C at P.

The stalks

$$\mathcal{K}_P := \mathcal{I}_{(P,0)} \cap \mathcal{O}_{X,P} = \mathfrak{I}_P \cap \mathcal{O}_{X,P}$$

form a coherent ideal sheaf $\mathcal{K}$ on X; let W be the analytic subspace of X determined by $\mathcal{K}$. Then there is a factorization

$$C \to W \hookrightarrow X$$

of the canonical projection p, and W is embeddable into C:

$$W \cong W \times \{0\} \hookrightarrow C;$$

we call $W \hookrightarrow C$ the *zero-section embedding* of the cone C.

Let X be an analytic space. Every homogeneous coherent ideal sheaf $\mathcal{I}$ on $X \times \mathbb{C}_t^{r+1}$ induces, by projectivization, a coherent ideal sheaf on $X \times \mathbb{P}_r$. In this manner we obtain a correspondence between the analytic cones C in $X \times \mathbb{C}_t^{r+1}$, and the analytic subspaces of $X \times \mathbb{P}_r$, called *projective cones* on X; the projective cone that corresponds to a cone C will be denoted by $\mathbb{P}(C)$.

Conversely, let $G := \mathbb{C}^{r+1} \setminus \{0\}$ $(r \geq 0)$ and

$$\varrho : X \times G \to X \times \mathbb{P}_r$$

be the canonical mapping. Then, for any coherent ideal sheaf $\mathcal{J}$ on $X \times \mathbb{P}_r$, $\varrho^* \mathcal{J}$ is a coherent ideal sheaf on $X \times G$ which satisfies the homogeneity condition. A relative, i.e. parameter-dependent, version of Chow's theorem (cf. [14]) says that every coherent ideal sheaf $\mathcal{I}$ on $X \times G$ which satisfies the homogeneity condition extends to a homogeneous ideal sheaf on $X \times \mathbb{C}_t^{r+1}$.

Let X be an analytic space, and $\mathcal{I}$ be a coherent ideal sheaf on $X \times \mathbb{C}_t^{r+1}$ which is homogeneous with respect to the variables t. Consider a homogeneous primary decomposition of $\mathcal{I}$ (cf. [53]). Then the *saturation* $\overline{\mathcal{I}}$ is the intersection of those primary homogeneous ideal sheaves whose associated subvarieties are not contained in $X \times \{0\}$. The saturation $\overline{\mathcal{I}}$ can also be expressed in terms of gap sheaves:

$$\overline{\mathcal{I}} = \mathcal{I}[X \times \{0\}],$$

where $\mathcal{I}[X \times \{0\}]$ denotes the gap sheaf of $\mathcal{I}$ with respect to the analytic subset $X \times \{0\}$ in $X \times \mathbb{C}^{r+1}$.

Let C be the analytic cone on X determined by $\mathcal{I}$. Geometrically speaking, the saturation $\overline{\mathcal{I}}$ corresponds to the closure (cf. [14]) of the analytic space $C \cap (X \times G)$ in $X \times \mathbb{C}_t^{r+1}$.

We call a homogeneous ideal sheaf $\mathcal{I}$ on $X \times \mathbb{C}_t^{r+1}$ *saturated* if $\mathcal{I} = \overline{\mathcal{I}}$. Clearly, two homogeneous ideal sheaves $\mathcal{I}_1$ and $\mathcal{I}_2$ induce the same ideal sheaf on $X \times \mathbb{P}_r$ iff they have the same saturation. We say that an analytic cone C on X is *relevant* if it is determined by a saturated homogeneous ideal sheaf.

The saturation $\overline{\mathcal{I}}$ can also be described by means of the graded rings of the cone X at the points of X. For any point $P \in X$, the ideal

$$\mathfrak{I}_P := \mathcal{I}_{(P,0)} \cap \mathcal{O}_{X,P}[t]$$

is a homogeneous ideal in the polynomial ring $\mathcal{O}_{X,P}[t]$; its saturation is the homogeneous ideal

$$\overline{\mathfrak{I}}_P := \{g(z,t) \in \mathcal{O}_{X,P}[t] : t^\alpha g(z,t) \in \mathfrak{I}_P \text{ for } |\alpha| \gg 0\}.$$

It is easy to check that the family of homogeneous ideals $\overline{\mathfrak{I}}_P$ determines the saturation $\overline{\mathcal{I}}$ of the homogeneous ideal sheaf $\mathcal{I}$.

For any coherent ideal sheaf $\mathcal{J}$ on $X \times \mathbb{P}_r$, the pull-back

$$\mathcal{I} := \varrho^* \mathcal{J}$$

is a coherent ideal sheaf on $X \times G$ which satisfies the homogeneity condition and extends—by Chow's theorem with parameter—to a unique saturated homogeneous ideal sheaf $\widetilde{\mathcal{I}}$ on $X \times \mathbb{C}_t^{r+1}$. Clearly, $\widetilde{\mathcal{I}}$ coincides with the saturation of every homogeneous extension of $\mathcal{I}$.

Every homogeneous extension of $\mathcal{I}$ determines, by projectivization, the initial ideal sheaf $\mathcal{J}$. Summing up, in geometric language, there is a one-to-one correspondence between the relevant analytic cones on X and the projective cones on X such that an analytic cone C corresponds to its projectivization $\mathbb{P}(C)$. Conversely, the analytic cone which corresponds to the projective cone $\mathbb{P}(C)$ is the unique relevant extension of the analytic space

$$\varrho^{-1}(\mathbb{P}(C)) \subset X \times G.$$

Notice that $\varrho^{-1}(\mathbb{P}(C)) = C \setminus (X \times \{0\})$ (here the symbol ϱ^{-1} stands for the inverse image space operation) as the holomorphic mapping ϱ is flat.

We now recall the notion of a blowing-up of an analytic space. Consider an analytic subspace X of an analytic space Y given by a coherent ideal sheaf $\mathcal{K}$ in the structure sheaf $\mathcal{O}$ of Y. For any $P \in Y$, take a finite number of germs $f(z) = (f_0(z), \dots, f_r(z))$ that generate the stalk $\mathcal{K}_P$. Let

$$\mathfrak{I}_P \subset \mathcal{O}_P[t] \quad (\text{where } t = (t_0, \dots, t_r))$$

be the ideal generated by those forms $F(z,t) \in \mathcal{O}_P[t]$ for which $F(z, f(z)) = 0$, and G_P be the graded quotient ring $\mathcal{O}_P[t]/\mathfrak{I}_P$. This definition of G_P does not depend (up to isomorphism) on the choice of the generators of $\mathcal{K}_P$ because of the canonical isomorphism of graded rings

$$G_P \cong \bigoplus_{n=0}^{\infty} \mathcal{K}_P^n.$$

We may assume that the germs $f_i(z)$ $(i = 0, 1, \dots, r)$ are represented locally in a neighbourhood U by holomorphic functions which generate every stalk $\mathcal{K}_P$ with $P \in U$. The family of homogeneous ideals $\mathfrak{I}_P \subset \mathcal{O}_P[t]$ with $P \in U$ generates an ideal sheaf $\mathcal{I}$ on the product $U \times \mathbb{C}_t^{r+1}$, which turns out to be coherent and homogeneous. Indeed, we may assume that U is a semianalytic compact Stein subset of Y. According to Frisch's theorem [15], the ring $\mathcal{O}(U)$ of sections of the structure sheaf $\mathcal{O}$ over U is noetherian. Therefore the homogeneous ideal generated by those forms $F(z,t) \in \mathcal{O}(U)[t]$ for which $F(z, f(z)) = 0$ is generated by a finite number of forms $F_i(z,t)$. Hence by Cartan's Theorem A, the forms $F_i(z,t)$ generate the ideal sheaf $\mathcal{I}$, which is the desired result.

We have thus constructed an analytic cone

$$C(X \cap U, U) \subset U \times \mathbb{C}_t^{r+1},$$

and a projective cone

$$Bl_{X\cap U} U \subset U \times \mathbb{P}_r.$$

Since the graded rings G_P are independent of the choice of the generators of the ideal sheaf $\mathcal{K}$ near P, one can easily deduce that such local constructions can be glued together to form analytic spaces $C(X, Y)$ and $Bl_X Y$; the former is an analytic cone and the latter is a projective cone. We have the canonical holomorphic mappings

$$p : C(X, Y) \to Y \quad \text{and} \quad \pi : Bl_X Y \to Y;$$

locally, they are the restrictions of the projections

$$p : U \times \mathbb{C}_t^{r+1} \to U \quad \text{and} \quad \pi : U \times \mathbb{P}_r \to U,$$

respectively. The pair $(Bl_X Y,\ \pi : Bl_X Y \to Y)$ will be called the *blow-up* of the analytic space Y *along* X (or *with center* X). The inverse image $\pi^{-1}(X)$ is a Cartier divisor on $Bl_X Y$, called the *exceptional divisor* of the blow-up $Bl_X Y$.

REMARKS. (1) The blow-up $\pi : Bl_X Y \to Y$ is an isomorphism away from X. Under the notation of the above local construction, if $P \notin X$ whence obviously

$$f_k(P) \neq 0 \quad \text{for some } k = 0, 1, \ldots, r,$$

then in a neighbourhood U of P we have the inclusion

$$Bl_{X\cap U} U \subset U \times U_k,$$

and $Bl_{X\cap U} U$ is isomorphic to the graph of the holomorphic mapping

$$(f_0(z)/f_k(z), \ldots, f_{k-1}(z)/f_k(z), 1, f_{k+1}(z)/f_k(z), \ldots, f_r(z)/f_k(z)).$$

(2) If the analytic subspace X is nowhere dense in Y, then the restriction of the blow-up $\pi : Bl_X Y \to Y$ to each irreducible component is a holomorphic mapping onto an irreducible component of Y which is a biholomorphism over the complement of X. Thus the blow-up $Bl_X Y$ is a purely dimensional analytic space whenever so is Y.

(3) The blow-up $Bl_X Y$ of an analytic space Y along a Cartier divisor X may be canonically identified with Y, because the graded rings of the blow-up at the points P of Y are isomorphic to the polynomial rings $\mathcal{O}_P[t]$ in one variable t.

Blowing-ups can be characterized by the following universal property.

Let X be an analytic subspace of an analytic space Y. For every holomorphic mapping $\phi : Z \to Y$ of analytic spaces such that $\phi^{-1}(X)$ is a Cartier divisor on Z, there exists a unique holomorphic mapping $\psi : Z \to Bl_X Y$ which makes the following diagram commutative:

$$\begin{tikzcd} Z \arrow[rr, "\psi"] \arrow[dr, "\phi"'] & & Bl_X Y \arrow[dl, "\pi"] \\ & Y & \end{tikzcd}$$

We now proceed to define normal cones. Consider a subspace X of an analytic space Y, and the analytic cone $p : C(X,Y) \to Y$ defined before; let $C_XY := p^{-1}(X)$. The *normal cone* to X in the analytic space Y is the analytic cone

$$p : C_XY \to X.$$

Clearly, the graded ring of C_XY at $P \in X$ is

$$\bigoplus_{n=0}^{\infty} \mathcal{K}_P^n/\mathcal{K}_P^{n+1},$$

where $\mathcal{K}$ is the ideal sheaf of the analytic subspace X in Y. The normal cone C_XY induces the exceptional divisor of the blow-up Bl_XY (which is a projective cone on X). This definition will be clarified from the viewpoint of deformation theory (cf. [16]).

Let $\widetilde{Y}$ and $\mathcal{Y}$ be the blow-ups of Y and $Y \times \mathbb{P}_1$ along X and $X \times \{\infty\}$, respectively; let

$$\pi : \mathcal{Y} \to Y \times \mathbb{P}_1 \quad \text{and} \quad \varrho : \mathcal{Y} \to \mathbb{P}_1$$

be the canonical projections. It is easy to check that $\widetilde{Y}$ can be canonically embedded as a closed analytic subspace of $\mathcal{Y}$, and that the blow-up of $X \times \mathbb{P}_1$ along the Cartier divisor $X \times \{\infty\}$, which may be identified with $X \times \mathbb{P}_1$, is canonically embeddable into $\mathcal{Y}$. Let $C := C_XY$ be the normal cone to X in Y. It is clear that the normal cone

$$C_{X\times\{\infty\}}(Y \times \mathbb{P}_1)$$

is canonically isomorphic to $C \oplus 1$. The analytic cone C may be identified with the complement to $\mathbb{P}(C)$ in $\mathbb{P}(C \oplus 1)$.

The blow-up $\mathcal{Y}$ is isomorphic away from $\lambda = \infty$ to the graph of the holomorphic mapping

$$Y \times \mathbb{P}_1 \ni (z; 1/\lambda : 1) \mapsto (\lambda f_1(z), \ldots, \lambda f_r(z)) \in \mathbb{C}^r,$$

and hence

$$\varrho^{-1}(\mathbb{C}) = \varrho^{-1}(\mathbb{P}_1 \setminus \{\infty\}) \cong Y \times \mathbb{C}.$$

In particular, the fibre $\mathcal{Y}_\lambda$ of $\mathcal{Y}$ over $\lambda \neq \infty$ is the graph of the holomorphic mapping

$$Y \ni z \mapsto (\lambda f_1(z), \ldots, \lambda f_r(z)).$$

Summing up, we have constructed an analytic space $\mathcal{Y}$ together with a closed embedding ι of $X \times \mathbb{P}_1$ into $\mathcal{Y}$, and a holomorphic mapping $\varrho : \mathcal{Y} \to \mathbb{P}_1$ such that the diagram

$$\begin{tikzcd} X \times \mathbb{P}_1 \arrow[rr, "\iota"] \arrow[dr, "\mathrm{pr}"'] & & \mathcal{Y} \arrow[dl, "\varrho"] \\ & \mathbb{P}_1 & \end{tikzcd}$$

is commutative and:

- over $\mathbb{C} = \mathbb{P}_1 \setminus \{\infty\}$, $\varrho^{-1}(\mathbb{C}) \cong Y \times \mathbb{C}$ and the embedding ι is, under this identification, the trivial embedding $X \times \mathbb{C} \hookrightarrow Y \times \mathbb{C}$;
- the fibre $\mathcal{Y}_\infty$ of the projection ϱ over ∞ is the sum of two effective Cartier divisors:

$$\mathcal{Y}_\infty = \mathbb{P}(C \oplus 1) + \widetilde{Y};$$

the embedding ι_∞ of $X = X \times \{\infty\}$ into $\mathcal{Y}_\infty$ is the zero-section embedding of X into C followed by the canonical open embedding of C into its projective completion $\mathbb{P}(C \oplus 1)$.

Therefore, putting $\mathcal{Y}^\circ := \mathcal{Y} \setminus \widetilde{Y}$, one has a family of embeddings of X:

$$\begin{tikzcd} X \times \mathbb{P}_1 \arrow[rr, "\iota"] \arrow[dr, "\mathrm{pr}"'] & & \mathcal{Y}^\circ \arrow[dl, "\varrho^\circ"] \\ & \mathbb{P}_1 & \end{tikzcd}$$

which deforms the given embedding of X into Y to the zero-section embedding of X into $C := C_X Y$ (the analytic cone $C := C_X Y$ is identified with the complement to $\mathbb{P}(C)$ in $\mathbb{P}(C \oplus 1)$). We call such an analytic family of embeddings *deformation to the normal cone*; the analytic space

$$\mathcal{Y}^\circ = \mathcal{Y}^\circ_X(Y)$$

will be called the *deformation space* of Y with respect to X.

REMARKS. (1) In Proposition 1, the divisors $\mathbb{P}(C \oplus 1)$ and $\widetilde{Y}$ intersect in the projective cone $\mathbb{P}(C)$, which is embedded as the hyperplane at infinity in $\mathbb{P}(C \oplus 1)$ and as the exceptional divisor in $\widetilde{Y}$.

(2) The foregoing construction demonstrates that the normal cone $C_X Y$ is of pure dimension d whenever so is the analytic space Y.

Deformation to the normal cone is additive in the following sense:

Let Y be an analytic space and Y_i its irreducible components with geometric multiplicities m_i. If X is a closed analytic subspace of Y, and $X_i := X \cap Y_i$ (ideal-sheaf-theoretic intersection), then the deformation spaces $\mathcal{Y}_i^\circ$ of Y_i with respect to X_i are subvarieties in the deformation space $\mathcal{Y}^\circ$ which are the irreducible components of $\mathcal{Y}^\circ$ with geometric multiplicities m_i. Moreover, the exceptional divisor in $\mathcal{Y}^\circ$ restricts to the exceptional divisor in $\mathcal{Y}_i^\circ$.

Finally, one can extend by additivity the notion of a normal cone, and construct normal cones in an analytic cycle; such a normal cone is, of course, an analytic cycle as well. The constructions of cycles and normal cones commute provided that the analytic space Y is of pure dimension:

With the above notation, suppose the analytic space Y is purely dimensional. Then the analytic cycle $[C_X Y]$ induced by the normal cone $C_X Y$ coincides with the analytic cycle

$$\sum_i m_i \cdot [C_{X_i} Y_i].$$

This follows directly from the additivity of deformation to the normal cone and from the well known fact concerning Cartier and Weil divisors on purely dimensional analytic spaces:

Suppose D is an effective Cartier divisor on an analytic space Y of pure dimension n. Let Y_i be the irreducible components of Y with geometric multiplicities m_i, and D_i

be the restrictions of D *to* Y_i*. Then we have the following equality of Weil divisors, i.e. analytic* $(n-1)$*-cycles on* Y*, determined by those Cartier divisors*:

$$[D] = \sum_i m_i \cdot [D_i].$$

4. Deformation and the analytic intersection algorithm. We begin by elaborating precisely the local intersection algorithm presented in Section 2. It is formulated in the language of analytic cycles (see also [38, 39]). As before, for a cycle α near a point P on a complex manifold M and a closed submanifold S, α^S denotes the part of the cycle α supported by S; the cycle α^S consists of the cycles $c \cdot [\mathfrak{p}]$ (where $\mathfrak{p}$ is a prime ideal in the local ring of M at P, and c is an integer) occurring in α such that $\mathfrak{p}$ contains the ideal of the submanifold S. Given an algebraic subset V of pure dimension d in M, A denotes the local ring of V at a point $P \in V$ and $I \subset A$ is the ideal of the submanifold S. In our algebraic context, we shall assume that the testing elements $a_1, \dots, a_r \in I$ $(r = m - s$ is the codimension of S in $M)$ are filter-regular with respect ideal I; then, of course,

$$\sqrt{\sum_{k=1}^{r} a_k A} = \sqrt{I} \quad \text{and} \quad \mathfrak{a}_r(a) = A.$$

We define two families of cycles by the recursive formula

$$\alpha_{-1} = [0] = [V], \quad a_0 = 0,$$
$$\varrho_k(a) = (\alpha_{k-1}(a) \cdot [a_k])^S, \quad \alpha_k(a) = \alpha_{k-1}(a) \cdot [a_k] - \varrho_k(a).$$

Notice that all the above intersections of cycles are proper by filter-regularity; consequently, $\alpha_k(a)$ and $\varrho_k(a)$ are $(d-k)$-cycles, where $d = \dim A$ is the dimension of the analytic set V. We call $\varrho_k(a)$ the *result of the kth step* of the intersection algorithm. In view of the remarks from the end of Section 2,

$$\alpha_{k-1}(a) = [\mathfrak{a}_{k-1}(a)]$$

and $\varrho_k(a)$ is the part supported by S of the intersection of the cycle $[\mathfrak{a}_{k-1}(a)]$ and the divisor $[a_k]$, because $a_1, \dots, a_{k-1}$ is a regular sequence in the vicinity of any prime of the ideal $(a_1, \dots, a_{k-1})$ that does not contain the ideal I. Furthermore, since a_k is not a zero divisor in $A/\mathfrak{a}_{k-1}(a)$, $\varrho_k(a)$ is the part of the cycle $[\mathfrak{a}_{k-1}(a) + a_k A]$ supported by S, and thus

$$\varrho_k(a) = [\mathfrak{r}_k(a)].$$

Clearly,

$$\varrho_k(a) = [\mathfrak{r}_k(a)] = \sum \operatorname{length}(A_{\mathfrak{p}}/(\mathfrak{a}_{k-1}(a) + a_k A)A_{\mathfrak{p}}) \cdot [\mathfrak{p}]$$

where the sum is taken over all minimal primes of $\mathfrak{a}_{k-1}(a) + a_k A$ that contain I (obviously, the height of such prime ideals $\mathfrak{p}$ is k). We may thus conclude that under the assumption of filter-regularity it is possible to work interchangeably with ideals and with the analytic cycles determined by them.

By the *multiplicity* of the cycle ϱ_k we mean the number

$$\operatorname{mult} \varrho_k(a) = \sum \operatorname{length}(A_{\mathfrak{p}}/(\mathfrak{a}_{k-1}(a) + a_k A)A_{\mathfrak{p}}) \cdot e(A/\mathfrak{p}).$$

The *total result* of the intersection algorithm is the cycle

$$\varrho(a) := \sum_k \varrho_k(a);$$

by its *multiplicity* $\operatorname{mult} \varrho(a)$ we mean the sum $\sum_k \operatorname{mult} \varrho_k(a)$.

We now proceed to establish the main objective of Chapter II, namely the theorem on deformation to the normal cone to the effect that the total result of the local intersection algorithm is invariant under deformation to the normal cone whenever the testing elements form a superficial sequence.

We first list the algebro-geometric notation concerning the intersection algorithm:

- S is a closed submanifold of dimension s of a complex manifold M of dimension m;
- the algorithm is carried out in the vicinity of a fixed point P in S, whence we may assume that M is the germ of $\mathbb{C}^m$ at $P = 0$:

  $$M = \mathbb{C}^m = \mathbb{C}^s_u \times \mathbb{C}^r_v,$$

  where $r + s = m$, $u = (u_1, \ldots, u_s)$ and $v = (v_1, \ldots, v_r)$ are the coordinates in $\mathbb{C}^s$ and in $\mathbb{C}^r$, respectively, and

  $$S = \{(u, v) : v = 0\} \subset \mathbb{C}^s_u \times \mathbb{C}^r_v;$$
- V is the germ of an analytic set of pure dimension d at the point $P = 0 \in \mathbb{C}^m$;
- A is the local ring of V at P and $I \subset A$ is the ideal of the ideal-theoretic intersection $V \cap S$;
- we may, of course, identify the normal cone $C_S M$ to S in M with $M = \mathbb{C}^m$, and then the normal cone $C := C_{V\cap S} V$ to $V \cap S$ in V is an analytic subcone in $\mathbb{C}^m$;
- the graded ring of C is $G = G_I(A) := \bigoplus_{n=0}^{\infty} I^n/I^{n+1}$;
- the zero-section (vertex space) of C (which is just the ideal-theoretic intersection $V \cap S$) corresponds to the ideal G^+ of elements of positive degree in G.

We investigated the general notion of normal cones in analytic spaces (regarded both as analytic spaces and as analytic cycles) in Section 3. We recall, however, this construction in the case of the normal cone $C = C_{V\cap S} V$ in the analytic set V, wherefrom the algorithm starts. Notice that in the subsequent steps of the algorithm we encounter analytic spaces which may not be reduced ones and which are to be deformed. The geometric description of the deformation space is particularly simple in the case of analytic sets. In order to deform the analytic set V to the normal cone $C := C_{V\cap S} V$, one can define the following analytic family of analytic sets:

$$\bigcup_t V_t = \{(u, v; 1 : t) \in \mathbb{C}^m \times \mathbb{P}_1 : (u, v/t) \in V\}$$

parametrized by $t \in \mathbb{C} \setminus \{0\}$; the deformation space $\mathcal{V}$ for V is the closure of the above family in $\mathbb{C}^m \times \mathbb{P}_1$. From the geometric viewpoint, the normal cone to $V \cap \{v = 0\} = V \cap S$ in V may be identified with the fibre $\mathcal{V}_\infty$ over $\infty \in \mathbb{P}_1$. It should be emphasized that the spaces $\mathcal{V}_t \cap S$ do not vary and coincide with the zero-section (vertex space) $V \cap S$ of the normal cone. Moreover, all the fibres V_t for $t \in \mathbb{C}$, $t \neq 0$, are isomorphic to V.

Next consider an analytic subset W of V near P; let $\mathfrak{a} \subset A$ be its ideal. We can construct the deformation space $\mathcal{W}$ for W in the same manner as for V. Then $\mathcal{W} \subset \mathcal{V}$

and the normal cone $\mathcal{W}_\infty$ to $W \cap S$ in W is a subcone of $\mathcal{V}_\infty$ determined by the leading ideal

$$G_I(\mathfrak{a}, A) = \bigoplus_{n=0}^{\infty} (\mathfrak{a} \cap I^n + I^{n+1})/I^{n+1} \subset G$$

of the ideal $\mathfrak{a}$. The quotient ring

$$G/G_I(\mathfrak{a}, A) = \bigoplus_{n=0}^{\infty} I^n/(\mathfrak{a} \cap I^n + I^{n+1})$$

coincides, of course, with the associated graded ring $G_I(A/\mathfrak{a})$ of the quotient ring $A/\mathfrak{a}$.

Remark. With the above notation, if the analytic germ W is of pure dimension k, so is the normal cone $\mathcal{W}_\infty$ to $W \cap S$ in W (see Sect. 3).

The notion of normal cone can be defined in the same manner for an analytic subspace W of V near P (it means that the ideal $\mathfrak{a} \subset A$ of W may not be radical). One may consider the above normal cones also as analytic cycles $[\mathcal{W}_\infty]$ on the cone $\mathcal{V}_\infty$ (from the ideal-theoretic point of view they are determined by the ideals $G_I(\mathfrak{a}, A)$ of the graded ring G). Therefore one can extend, by additivity, the above definitions and construct the normal cone in an analytic cycle.

The constructions of cycles and normal cones commute (see Sect. 3) provided that the analytic subspace W is of pure dimension (i.e. all minimal primes of its ideal $\mathfrak{a}$ are of the same height). A remarkable advantage of using the theory of positive analytic cycles is that we get at our disposal a dynamic concept of convergence for which proper intersection is a continuous operation (see Chapt. I).

The convergence of cycles is a notion of local character. With the previous notation, the family of cycles $[\mathcal{W}_t]$ converges to the cycle $[\mathcal{W}_\infty]$ determined by the normal cone. In what follows, we shall say that $[\mathcal{W}_\infty]$ or $[G_I(\mathfrak{a}, A)]$ are the *limit cycles* for $[W]$ or $[\mathfrak{a}]$, respectively.

We can now pass to the main theorem of this chapter (see also [38, 39]), which is a geometric counterpart of Theorem 3.3 from [3].

Theorem on Deformation to the Normal Cone. *Under the notation of the intersection algorithm, consider a superficial sequence of elements $a_1, \dots, a_r \in I \setminus I^2$ of order one with respect to the ideal I. Then the results $\varrho(a)$ and $\varrho(a^*)$ of the intersection algorithm for $a_1, \dots, a_r \in I$ and for their initial forms $a_1^*, \dots, a_r^* \in G^+$, coincide.*

Proof. First observe that the initial forms $a_1^*, \dots, a_r^*$ in the associated graded ring $G = G_I(A)$ are also filter-regular with respect to the ideal G^+ (Sect. 1, Prop. 5), and thus one can perform the intersection algorithm for both sequences. We show by induction on k that the results $\varrho_k(a)$ and $\varrho_k(a^*)$ coincide, and that $\alpha_k(a^*)$ is the limit cycle for $\alpha_k(a)$. For $k = 0$ this follows from the fact that no component of the limit cycle for $\alpha_0(a)$ lies on $S = \{v = 0\} \subset M = \mathbb{C}^m$; its proof is straightforward. Now, assuming the assertion to hold for $k - 1$, we prove it for k. By continuity of proper intersections of positive analytic cycles, the cycle

$$\alpha_{k-1}(a^*) \cdot [a_k^*] = \alpha_k(a^*) + \varrho_k(a^*)$$

is the limit cycle for

$$\alpha_{k-1}(a) \cdot [a_k] = \alpha_k(a) + \varrho_k(a).$$

Therefore every minimal prime ideal $\mathfrak{p}$ of $\mathfrak{r}_k(a)$ in A corresponds to a minimal prime ideal $\mathfrak{p}^*$ of $\mathfrak{r}_k(a^*)$ in G via the equality

$$\mathfrak{p}/I = \mathfrak{p}^*/G^+ = \mathfrak{p}^* \cap G^0.$$

Moreover, we have the converse correspondence. For if $\mathfrak{p}^*$ is a minimal prime ideal of $\mathfrak{r}_k(a^*)$ and $\mathfrak{p}$ is the ideal in A described by the above equality, then $\mathfrak{p}$ is a minimal prime ideal of $\mathfrak{r}_k(a)$. Indeed, by induction hypothesis, the limit cycle $[G(\mathfrak{a}_{k-1}(a), A)]$ equals $\alpha_{k-1}(a^*) = [\mathfrak{a}_{k-1}(a^*)]$, whence $\mathfrak{p}^*$ contains the homogeneous ideal $G(\mathfrak{a}_{k-1}(a), A)$, and thus

$$(\mathfrak{a}_{k-1}(a) + I)/I \subset \mathfrak{p}^* \cap G^0 = \mathfrak{p}/I.$$

Consequently, $\mathfrak{a}_{k-1}(a) + I \subset \mathfrak{p}$, and a fortiori $\mathfrak{a}_{k-1}(a) + a_k A \subset \mathfrak{p}$. Since $\dim A/\mathfrak{p} = \dim G/\mathfrak{p}^* = \dim A - k$, the cycle $[\mathfrak{p}]$ occurs in $\varrho_k(a)$, as asserted.

Further, the ideals $\mathfrak{a}_{k-1}(a^*)$ and $G_I(\mathfrak{a}_{k-1}(a), A)$ have the same minimal primes, because $[G(\mathfrak{a}_{k-1}(a), A)] = [\mathfrak{a}_{k-1}(a^*)]$ by induction hypothesis. Hence

$$a_k^* \notin G_I(\mathfrak{a}_{k-1}(a), A) \quad \text{or} \quad a_k^* \notin (\mathfrak{a}_{k-1}(a) + I^2)/I^2,$$

and thus the class of a_k in $A/\mathfrak{a}_{k-1}(a)$ is an element of order one with respect to the image of I. Therefore the initial form of the class of a_k in $A/\mathfrak{a}_{k-1}(a)$ is determined by a_k^*.

For any minimal prime $\mathfrak{p}$ of the ideal $\mathfrak{r}_k(a)$, we have the following canonical isomorphisms:

$$G_{\mathfrak{p}^*} \cong G \otimes_A A_{\mathfrak{p}} \cong G_{IA_{\mathfrak{p}}}(A_{\mathfrak{p}}) \quad \text{and} \quad G(\mathfrak{a}_{k-1}(a), A)G_{\mathfrak{p}^*} \cong G_{IA_{\mathfrak{p}}}(\mathfrak{a}_{k-1}(a)A_{\mathfrak{p}}, A_{\mathfrak{p}}).$$

Since $\mathfrak{p}^*$ contains a minimal prime of $G_I(\mathfrak{a}_{k-1}(a), A)$, one can deduce, by repeating *mutatis mutandis* the above reasoning for the above localizations, that the class of a_k in $A_{\mathfrak{p}}/\mathfrak{a}_{k-1}(a)A_{\mathfrak{p}}$ is also an element of order one with respect to the image of I. In particular, the initial form of the class of a_k in $A_{\mathfrak{p}}/\mathfrak{a}_{k-1}(a)A_{\mathfrak{p}}$ is again determined by a_k^*.

Now it can be easily checked by induction on i that

$$(a_1^*, \ldots, a_i^*) :_G \langle G^+ \rangle = G_I(\mathfrak{a}_i(a), A) :_G \langle G^+ \rangle = G_I((a_1, \ldots, a_i), A) :_G \langle G^+ \rangle$$

or, equivalently, that all the homogeneous ideals

$$G_I((a_1, \ldots, a_i), A), \quad (a_1^*, \ldots, a_i^*), \quad \text{and} \quad G_I(\mathfrak{a}_i(a), A)$$

coincide in large degrees (compare Cor. 2 to Prop. 4 of Sect. 1), and hence they have the saturation $\mathfrak{a}_i(a^*)$ in common. Both the homogeneous ideals $G_I(\mathfrak{a}_i(a), A)$ and $\mathfrak{a}_i(a^*)$ therefore have the same associated primes that do not contain G^+. The class of a_k in $A/\mathfrak{a}_{k-1}(a)$ is thus a superficial element with respect to the image of I. In view of the foregoing canonical isomorphisms, one can again deduce that the class of a_k in $A_{\mathfrak{p}}/\mathfrak{a}_{k-1}(a)A_{\mathfrak{p}}$ is a superficial element with respect to the image of I.

We use the above fact to conclude that the coefficients of the cycles $\varrho_k(a)$ and $\varrho_k(a^*)$ at $\mathfrak{p}$ and $\mathfrak{p}^*$, respectively, are equal. We must show that

$$e(a_k A_{\mathfrak{p}}/\mathfrak{a}_{k-1}(a)A_{\mathfrak{p}}) = e(a_k^* G_{\mathfrak{p}^*}/\mathfrak{a}_{k-1}(a^*)G_{\mathfrak{p}^*}).$$

It follows from Samuel's lemma (cf. Sect. 2) that the coefficients to be compared are

$$e(I \cdot A_{\mathfrak{p}}/\mathfrak{a}_{k-1}(a)A_{\mathfrak{p}}) \quad \text{and} \quad e(G^+ \cdot G_{\mathfrak{p}^*}/\mathfrak{a}_{k-1}(a^*)G_{\mathfrak{p}^*}).$$

But $[G(\mathfrak{a}_{k-1}(a), A)] = [\mathfrak{a}_{k-1}(a^*)]$ by induction hypothesis, and hence

$$\begin{aligned} e(G^+ \cdot G_{\mathfrak{p}^*}/\mathfrak{a}_{k-1}(a^*)G_{\mathfrak{p}^*}) &= e(G^+ \cdot G_{\mathfrak{p}^*}/G(\mathfrak{a}_{k-1}(a), A)G_{\mathfrak{p}^*}) \\ &= e(G^+ \cdot G_{IA_{\mathfrak{p}}}(A_{\mathfrak{p}})/G_{IA_{\mathfrak{p}}}(\mathfrak{a}_{k-1}(a)A_{\mathfrak{p}}, A_{\mathfrak{p}}) \\ &= e(I \cdot A_{\mathfrak{p}}/\mathfrak{a}_{k-1}(a)A_{\mathfrak{p}}) \end{aligned}$$

(the first equality can be derived from the equality of the cycles by means of the additivity formula), which is the desired conclusion.

We have thus proved that $\varrho_k(a^*) = \varrho_k(a)$, which also implies that $[G_I(\mathfrak{a}_k(a), A)]^S = 0$, because the limit cycle $[G_I(\mathfrak{a}_k(a), A)]$ for $[\mathfrak{a}_k(a)]$ cannot contribute to the cycle $\varrho_k(a^*)$. Therefore $[\mathfrak{a}_k(a^*)]$ is the limit cycle for $[\mathfrak{a}_k(a)]$:

$$[\mathfrak{a}_k(a^*)] = [G_I(\mathfrak{a}_k(a), A)],$$

which completes the proof of the theorem.

We present in Chapter III some refinements and applications of the foregoing method of deformation to the normal cone, which are important for the theory of improper intersections in complex analytic geometry.

III. Generalized index and intersection cycles

1. Deformation to an algebraic bicone. In this section we present a method of deforming an analytic space to an algebraic bicone (see also [38, 40]). The basic idea is deformation to the normal cone (see Sects. 3 and 4 of Chapt. II) and the continuity of intersections for limit cycles. Such a deformation to the normal cone applied twice makes it possible to reduce the problem of analytic improper intersections to that of an algebraic bicone intersecting linear hyperplanes.

We adopt the notation introduced in Section 4 of Chapter II, where we deformed the analytic set V to the normal cone $C := C_{V \cap S} V$:

- $S := \{(u, v) \in \mathbb{C}^s_u \times \mathbb{C}^r_v : v = 0\}$ is in the vicinity of the point $P = 0$ an s-dimensional affine subspace in $M := \mathbb{C}^m = \mathbb{C}^s_u \times \mathbb{C}^r_v$ $(m = s + r)$;
- V is the germ of an analytic set of pure dimension d at $P = 0 \in \mathbb{C}^m$;
- A is the local ring of V at P, $\mathfrak{m} \subset A$ is the maximal ideal of A, and $I \subset A$ is the ideal of the ideal-theoretic intersection $V \cap S$;
- the graded ring of the cone C is $G = G_I(A) := \bigoplus_{n=0}^{\infty} I^n/I^{n+1}$;
- the zero-section (vertex space) of C (which is just the ideal-theoretic intersection $V \cap S$) corresponds to the ideal G^+ of elements of positive degree in G.

We shall now deform the cone C to the normal cone B to the fibre of C over the point P of the vertex space. In geometric terms of the ambient manifold $M = \mathbb{C}^m$, we may describe the procedure as before in Section 4 of Chapter II, but with the coordinates u

and v interchanged. In other words, the deformation space is now the closure in $\mathbb{C}^m \times \mathbb{P}_1$ of the following family of analytic sets parametrized by $t \in \mathbb{C} \setminus \{0\}$:

$$\bigcup_t C_t = \{(u, v; 1 : t) \in \mathbb{C}^m \times \mathbb{P}_1 : (u/t, v) \in C\}.$$

The graded ring $\mathfrak{B}$ of B is the associated graded module of the A-module G with respect to the maximal ideal $\mathfrak{m}$ of A. Clearly, $\mathfrak{B}$ has the structure of a bigraded ring $\mathfrak{B} = \bigoplus_{i,j=0}^{\infty} \mathfrak{B}_{ij}$ where

$$\mathfrak{B}_{ij} = G^i_{\mathfrak{m}}(G^j_I(A)) = (\mathfrak{m}^i I^j + I^{j+1})/(\mathfrak{m}^{i+1} I^j + I^{j+1}).$$

The bigraded ring $\mathfrak{B}$ is also the associated graded ring of G with respect to the homogeneous maximal ideal

$$\mathfrak{M} = \mathfrak{m} G^0 \oplus G^+$$

corresponding to the point P. Indeed,

$$\mathfrak{M}^n = \mathfrak{m}^n G^0 \oplus \mathfrak{m}^{n-1} G^1 \oplus \ldots \oplus \mathfrak{m} G^{n-1} \oplus G^{\geq n}$$

whence

$$\mathfrak{M}^n/\mathfrak{M}^{n+1} \cong (\mathfrak{m}^n/\mathfrak{m}^{n+1}) G^0 \oplus (\mathfrak{m}^{n-1}/\mathfrak{m}^n) G^1 \oplus \ldots \oplus (G^n/\mathfrak{m} G^n) \cong \bigoplus_{i+j=n} \mathfrak{B}_{ij}.$$

Accordingly, the cone B is an algebraic cone in $\mathbb{C}^m$, namely the tangent cone to C at the point P. The above observations can also be easily deduced by analysing the deformation space. We can thus conclude that the tangent cone to C at P, which is obviously an algebraic cone in $\mathbb{C}^m$, has the structure of a bicone and coincides with B.

We still need a proposition which is similar to Proposition 3 from Section 1 of Chapter II.

PROPOSITION. *With the above notation, take an element $a \in I$ and suppose that its canonical image $\overline{a}$ in $\mathfrak{B}_{01} = I/\mathfrak{m} I$ is a filter-regular element with respect to $\mathfrak{B}_{01}$. Then a is a superficial element of order one with respect to the ideal I.*

We have to show that a^* is filter-regular with respect to G^+, where a^* denotes the initial form of a in the associated graded ring G (see Sect. 1 of Chapt. II). In other words, we are to show that

$$(0 : a^*) \cap G^{\geq n} = 0 \quad \text{whenever} \quad (0 : \overline{a}) \cap \bigoplus_{i \geq 0,\, j \geq n} \mathfrak{B}_{ij} = 0.$$

So pick $x \in G^{\geq n}$ such that $a^* x = 0$. Then, by our assumption, the image of x in

$$\bigoplus_{j \geq n} \mathfrak{B}_{0j} = A/\mathfrak{m} \otimes G^{\geq n}$$

must be 0. Hence $x \in \mathfrak{m} G^{\geq n}$, and thus we may consider the image of x in

$$\bigoplus_{j \geq n} \mathfrak{B}_{1j} = \mathfrak{m}/\mathfrak{m}^2 \otimes G^{\geq n}.$$

Again by our assumption, this image is zero, whence $x \in \mathfrak{m}^2 G^{\geq n}$. Proceeding by induction we conclude that

$$x \in \bigcap_{i=0}^{\infty} \mathfrak{m}^i G^{\geq n} = 0$$

(by the Krull intersection theorem), which completes the proof.

One immediately obtains the following

Corollary. *If $a_1, \ldots, a_r \in I$ are elements whose canonical images in $\mathfrak{B}_{01} = I/\mathfrak{m}I$ are a filter-regular sequence with respect to $\mathfrak{B}_{01}$, then they are a superficial sequence of elements of order one with respect to the ideal I.*

We are now in a position to prove the main result of the section (see also [38, 40]), which is a geometric counterpart of Theorem 4.1 from [3].

Theorem on Deformation to an Algebraic Bicone. *Under the notation of the intersection algorithm, assume that the canonical images $\overline{a}_1, \ldots, \overline{a}_r \in \mathfrak{B}_{01}$ of $a_1, \ldots, a_r \in I$ in $\mathfrak{B}_{01} = I/\mathfrak{m}I$ form a filter-regular sequence with respect to $\mathfrak{B}_{01}$. By the above corollary, $a_1, \ldots, a_r$ is a superficial sequence of elements of order one with respect to I. Then the multiplicities of the results of the intersection algorithm for the sequences $\overline{a}$, a^* and a coincide*:

$$\operatorname{mult} \varrho_k(\overline{a}) = \operatorname{mult} \varrho_k(a^*) = \operatorname{mult} \varrho_k(a) \quad \textit{for } k = 0, 1, \ldots, r.$$

Proof. First notice that the assumption $\overline{a}_k \neq 0$ in $\mathfrak{B}_{01} = I/\mathfrak{m}I$ or, equivalently, $a_k \in I \setminus \mathfrak{m}I$, means in terms of the ambient manifold $M = \mathbb{C}^m$ that the representatives of the elements a_k on M have non-trivial linear parts in the variables v. In the analytic intersection algorithm, the intersections with $[\overline{a}_k]$ are therefore the hyperplane sections determined by those linear parts.

We prove by induction on k that $\varrho_k(\overline{a})$, $\alpha_k(\overline{a})$ are the limit cycles for $\varrho_k(a^*)$, $\alpha_k(a^*)$, respectively, and that

$$\operatorname{mult} \varrho_k(\overline{a}) = \operatorname{mult} \varrho_k(a^*), \quad \operatorname{mult} \alpha_k(\overline{a}) = \operatorname{mult} \alpha_k(a^*).$$

According to the theorem on deformation to the normal cone, this will complete the proof. We shall need an elementary

Lemma. *Let C be an irreducible analytic germ at $P = 0 \in \mathbb{C}^m = \mathbb{C}^s_u \times \mathbb{C}^r_v$ which is a cone with respect to the variables v. If C is not contained in $S := \{(u, v) \in \mathbb{C}^m : v = 0\}$, then no irreducible component Z of the tangent cone to C at P is contained in S.*

Indeed, each $(u_0, v_0) \in Z$ is the limit of a sequence $(t_n u_n, t_n v_n)$ where $(u_n, v_n) \in C$ and t_n tends to ∞. For generic (u_0, v_0), we have $v_n \neq 0$ for $n \gg 0$ because C is not contained in S. Since C is a cone with respect to the variables v, we may, of course, assume that $\|u_n\| = \|v_n\|$. Consequently, taking a subsequence if necessary, we see that some line $\{(u_0, \lambda v) : \lambda \in \mathbb{C}\}$ with $v \neq 0$ lies on Z, as asserted.

We can now return to our inductive assertion. The case $k = 0$ follows directly from the above lemma, because B coincides with the tangent cone to C at P. Now, assuming the assertion to hold for $k-1$, we prove it for k. By continuity of proper intersections of positive analytic cycles,

$$\alpha_{k-1}(\overline{a}) \cdot [\overline{a}_k] = \alpha_k(\overline{a}) + \varrho_k(\overline{a})$$

is the limit cycle for

$$\alpha_{k-1}(a^*) \cdot [a_k^*] = \alpha_k(a^*) + \varrho_k(a^*).$$

Hence by the lemma, one can easily deduce that both $\alpha_k(\overline{a})$ and $\varrho_k(\overline{a})$ are the limit cycles for $\alpha_k(a^*)$ and $\varrho_k(a^*)$, respectively.

As the multiplicity at a fixed point cannot decrease for limit cycles, we get

$$\operatorname{mult} \alpha_k(\overline{a}) \geq \operatorname{mult} \alpha_k(a^*) \quad \text{and} \quad \operatorname{mult} \varrho_k(\overline{a}) \geq \operatorname{mult} \varrho_k(a^*).$$

On the other hand, we have

$$\operatorname{mult} \alpha_{k-1}(a^*) \cdot [a_k^*] \geq \operatorname{mult} \alpha_{k-1}(a^*) \quad \text{and} \quad \operatorname{mult} \alpha_{k-1}(\overline{a}) \cdot [\overline{a}_k] = \operatorname{mult} \alpha_{k-1}(\overline{a})$$

because $\alpha_{k-1}(\overline{a})$ is a cone with vertex at P and $[\overline{a}_k]$ is a hyperplane section (apply Bezout's theorem). Combining the above inequalities, we conclude that

$$\operatorname{mult} \alpha_k(\overline{a}) = \operatorname{mult} \alpha_k(a^*) \quad \text{and} \quad \operatorname{mult} \varrho_k(\overline{a}) = \operatorname{mult} \varrho_k(a^*),$$

which completes the proof.

As an immediate consequence we obtain

Corollary. *With the assumptions of the preceding theorem, the multiplicities of the total results of the intersection algorithm for the sequences $\overline{a}$ and a coincide*:

$$\operatorname{mult} \varrho(\overline{a}) = \operatorname{mult} \varrho(a).$$

The next section gives many further consequences of the theorem, which are important for the theory of improper intersections in complex analytic geometry.

2. Applications to the theory of improper intersections. Using the analytic intersection algorithm, P. Tworzewski defined pointwise some indices that form the so-called generalized intersection index and intersection multiplicity (which is the sum of those indices) of an analytic subset with a submanifold. The classical diagonal procedure makes it possible to extend the notion of intersection multiplicity to arbitrary analytic subsets of a complex manifold. He next proved that this pointwise defined intersection multiplicity is a constructible function in the analytic Zariski topology, whereon is based his construction of an intersection cycle of two analytic cycles that meet improperly. The above constructions turned out to be intrinsic (i.e. independent of the ambient manifold; cf. Rams [42]) and bi-additive.

On the other hand, Achilles and Manaresi [3] investigated a local counterpart of the Stückrad–Vogel algorithm for any collection of elements of a local ring A that forms a filter-regular sequence with respect to a given ideal I of A. They combined the method of deformation to the normal cone, valid whenever this collection forms a superficial sequence with respect to I (the notion of superficiality being introduced by Samuel [45], Chapt. II; see also [65], Chapt. VIII) with a variant of the Hilbert polynomial for a bigraded ring (elaborated by van der Waerden [61]), and introduced a multiplicity sequence of that bigraded ring, which corresponds to the bidegree sequence defined by van der Waerden [62] for biprojective sets. They also posed the problem whether, in the analytic case, the intersection indices coincide with their multiplicity sequence (called generalized Samuel multiplicities). Here we provide an affirmative answer (first given in [38, 40]). Furthermore, in the corollary to the linear testing theorem (Proposition 5), we relate the generalized intersection index to the so-called Segre numbers, introduced inductively by Gaffney–Gassler [19] by means of the sequences of polar varieties and Segre cycles (cf. [4]).

First we express the generalized index for improper intersections as the bidegree sequence of a certain algebraic bicone, and the intersection multiplicity at a point P as the Samuel multiplicity at P of the normal cone. We prove that the generalized intersection index is realized by a collection of smooth divisors provided that their equations (testing elements in the ring A) satisfy certain conditions of filter-regularity, which turn out to be of linear character: the first derivatives of the equations of each successive divisor should avoid a finite union of proper linear subspaces (depending on the previous divisors of the collection).

Let us recall that the bidegree sequence of an algebraic bicone can be expressed by the leading coefficients of the Hilbert polynomial of its associated bigraded ring (cf. [61, 62, 3]), by analogy to the way the degree of a cone can be expressed by the leading coefficient of the Hilbert polynomial of its associated graded ring. These formulae are more effective (both in the sense of pure mathematics and the more so of computer algebra) than the original definition of intersection multiplicity by Tworzewski [57].

In what follows, we shall derive many consequences for the theory of improper intersections in complex analytic geometry (see also [38, 40]), such as for instance:

- the coincidence between the intersection indices for analytic improper intersections by Tworzewski [57], those defined for an ideal I in the local algebraic case by Achilles–Manaresi [3] (generalized Samuel multiplicities) and the Segre numbers for an ideal I by Gaffney–Gassler [19];
- the intrinsic and additive character of the generalized intersection index (Proposition 3);
- the upper semicontinuity (not merely constructibility) in the analytic Zariski topology of the multiplicity function (Proposition 4);
- the linear testing theorem (Proposition 5);
- a generalization of the classical reduction theorem to the case of analytic improper intersections which ensures the canonical character of the diagonal procedure (stated and proved in the next section);
- a version of Bezout's theorem for improper intersections of algebraic cones (stated and proved in the next section).

We now return to the notion of the bidegree sequence of an algebraic bicone. The bicone B has several bidegrees $g_{k,d-k}$ defined as follows (see e.g. [38, 40]):

$g_{k,d-k}$ is the number of intersection points (counted with multiplicities) of B (or of the algebraic cycle $[B]$ induced by B) with k generic hyperplanes in the variables v and $d-k$ generic hyperplanes in the variables u.

The bicone B in $\mathbb{C}^s_u \times \mathbb{C}^r_v$ determines a biprojective cycle $[B']$ on $\mathbb{P}_{s-1} \times \mathbb{P}_{r-1}$, which has several bidegrees $g'_{k,d-k-2}$, defined similarly to $g_{k,d-k}$ (cf. [61, 62]). They can be expressed as the leading coefficients of the Hilbert polynomial of the associated bigraded ring (cf. [61, 3]). Consider the Chow ring

$$\mathbb{Z}[\sigma, \tau]/(\sigma^s, \tau^r) \quad \text{of} \quad \mathbb{P}_{s-1} \times \mathbb{P}_{r-1}$$

(here σ and τ are the classes modulo rational equivalence of hyperplanes in projective spaces $\mathbb{P}_{s-1}$ and $\mathbb{P}_{r-1}$, respectively). The bidegrees $g'_{k,d-k-2}$ coincide with the coefficients of the class of the biprojective cycle $[B']$ in the Chow ring:

$$[B'] = \sum_k g'_{k,d-k-2} \cdot \sigma^{s-k-1} \tau^{r-d+k+1}.$$

Remark. The purely algebraic theory of bidegrees of bihomogeneous ideals was initiated by Lasker [27], and then developed by van der Waerden [61]. They elaborated a variant of the Hilbert polynomial for a bigraded ring. Next Samuel [45] based his algebraic intersection theory on the construction of associated graded rings (which corresponds to that of normal cones) and on a variant of the Hilbert polynomial, called the Hilbert–Samuel polynomial. The importance of the Hilbert polynomials for geometry lies in the fact that they make it possible to establish a relation between the degree of an algebraic variety, both in projective and biprojective spaces, and the degree of the intersection of this variety with divisors (a generalization of Bezout's theorem).

Clearly, we have $g_{k,d-k} = g'_{k-1,d-k-1}$. Since the results of the intersection algorithm

$$\varrho_k(\overline{a}) \subset S = \{(u, v) : v = 0\} \subset \mathbb{C}^m = \mathbb{C}^s_u \times \mathbb{C}^r_v$$

are the cycles of cones with respect to the variables u, cut out of the bicone B by the linear equations in the variables v that determine the elements $\overline{a}_i$, the multiplicities $\operatorname{mult} \varrho_k(\overline{a})$ are independent of filter-regular sequences $\overline{a}_1, \ldots, \overline{a}_r$ and we have

$$\operatorname{mult} \varrho_k(\overline{a}) = g_{k,d-k} \quad \text{for } k = 0, \ldots, r,\ k \le d,$$

and

$$\varrho_k(\overline{a}) = 0 \quad \text{for } k > d.$$

Notice that the definition of the bicones $\alpha_0(\overline{a})$ and $\varrho_0(\overline{a})$ does not involve the sequence a_i at all. It follows from the classical Bezout theorem that

$$\operatorname{mult} \alpha_k(\overline{a}) = \operatorname{mult}(\alpha_k(\overline{a}) \cdot [\overline{a}_{k+1}]) = \operatorname{mult} \alpha_{k+1}(\overline{a}) + \operatorname{mult} \varrho_{k+1}(\overline{a})$$

for all $k = -1, 0, 1, \ldots, r-1$. Since $\alpha_r(\overline{a}) = 0$, we get

$$\deg B = \operatorname{mult} \alpha_{-1}(\overline{a}) = \sum_{k=0}^{r} \operatorname{mult} \varrho_k(\overline{a}) = \sum_{k=0}^{r} g_{k,d-k}$$

(obviously, in the above sum of bidegrees only the terms $g_{k,d-k}$ with $d - k \ge 0$ are to be retained). Summing up, we obtain

Proposition 1. *If the canonical images $\overline{a}_1, \ldots, \overline{a}_r \in \mathfrak{B}_{01}$ of $a_1, \ldots, a_r \in I$ in $\mathfrak{B}_{01} = I/\mathfrak{m}I$ are a filter-regular sequence with respect to $\mathfrak{B}_{01}$, then the multiplicities* $\operatorname{mult} \varrho_k(a)$ *($k = 0, 1, \ldots, r$) do not depend on the elements a_i, and coincide with the bidegree sequence of the bicone B. Furthermore, the multiplicity* $\operatorname{mult} \varrho(a)$ *of the total result of the* *intersection algorithm equals the multiplicity at P of both the algebraic bicone B and the normal cone C.*

The double deformation we presented in Section 1 may be regarded as a kind of linearization procedure. Indeed, under a certain algebraic condition of filter-regularity, it allows us to reduce the analytic intersection algorithm for a collection $\mathcal{H} = (H_1, \dots, H_r)$ of smooth divisors near P, to the case of an algebraic bicone B intersecting a collection of linear hyperplanes, namely, the tangent hyperplanes $T_P H_k$ $(k = 1, \dots, r)$. Now we wish to emphasize that those algebraic conditions on $\mathcal{H}$ are also of linear character; they are in fact linear conditions on the tangent spaces $T_P H_k$ or, equivalently, linear conditions on the first derivatives of the equations for the smooth divisors H_k. We now indicate this more precisely.

We take the previous notation:

$$M = \mathbb{C}^m = \mathbb{C}^s_u \times \mathbb{C}^r_v, \quad S = \{(u,v) : v = 0\} \subset \mathbb{C}^s_u \times \mathbb{C}^r_v$$

and V is the germ of an analytic set of pure dimension d at $P = 0 \in \mathbb{C}^m$. The assumption of filter-regularity from the theorem on deformation to an algebraic bicone means in terms of the ambient manifold $M = \mathbb{C}^m$ that the local equations $f_k(u,v)$ of H_k near $P = 0$ are of the form

$$f_k(u,v) = \sum_{i=1}^{r} c_{ki} v_i + \sum_{i=1}^{r} v_i g_{ki}(u,v),$$

where $g_{ki}(u,v)$ are analytic functions at zero of order ≥ 1, and for each k not all coefficients c_{ki} $(i = 1, \dots, r)$ vanish. Then the equations $f^*_k(u,v)$ of H_k after the first deformation are

$$f^*_k(u,v) = \sum_{i=1}^{r} c_{ki} v_i + \sum_{i=1}^{r} v_i g_{ki}(u,0).$$

Similarly, the equations $\overline{f}_k(u,v)$ of H_k after the first deformation are

$$\overline{f}_k(u,v) = \sum_{i=1}^{r} c_{ki} v_i.$$

Therefore, our conditions of filter-regularity mean that, for each k, the equation $\overline{f}_k(u,v) = \sum_{i=1}^{r} c_{ki} v_i$ should avoid a finite number of prime ideals in the ring of convergent power series in u, v which do not contain the ideal $(v_1, \dots, v_r)$; the prime ideals to be avoided depend, however, on the divisors $H_1, \dots, H_{k-1}$ chosen before. But the traces of those prime ideals on the $\mathbb{C}$-vector space of linear forms in v form a finite union of proper linear subspaces. We thus arrive at a finite number of linear conditions on the first derivatives $c_{ki} = \partial f_k / \partial v_i(P)$; namely, at a successive kth step of the intersection algorithm, the partial derivatives $c_{k,i}$ $(i = 1, \dots, r)$ should avoid a finite union of proper linear subspaces in $\mathbb{C}^r_c$ with $c = (c_1, \dots, c_r)$.

The next proposition expresses the generalized index and multiplicity for an improper analytic intersection in algebraic terms of the degrees of the algebraic cones arising in the deformation process from Section 1.

Proposition 2. *The generalized index for the intersection of the analytic set V with the submanifold S at the point P coincides with the bidegree sequence of the bicone B, and the intersection multiplicity coincides with the multiplicity at P of both the algebraic bicone B and the normal cone C.*

To prove Proposition 2, we have to show that the bidegree sequence of B is equal to the minimum (with respect to the lexicographical ordering) of the extended multiplicities of the total results of the intersection algorithm for admissible collections of smooth divisors. So consider an admissible collection $H_1, \dots, H_r$ which realizes this minimum multiplicity. Since the conditions imposed on divisors in Proposition 1 are of linear character (as described above), one can find r sequences of hypersurfaces $H_k^{(n)}$ ($n \in \mathbb{N}$, $k = 1, \dots, r$) obtained by arbitrarily small rotations of the H_k around $P = 0 \in M = \mathbb{C}^m$, such that their equations $a_1^{(n)}, \dots, a_r^{(n)} \in I$ satisfy the assumptions of Proposition 1 for all $n \in \mathbb{N}$.

Accordingly, H_k is the limit cycle of $H_k^{(n)}$ as n tends to ∞. Denote by α_k and ϱ_k ($k = 0, 1, \dots, r$) the analytic cycles resulting from the intersection algorithm for the divisors $H_1, \dots, H_r$; then $\varrho = \sum_{k=0}^{r} \varrho_k$ is the total result of the algorithm. From Tworzewski's lemma it follows by induction on k that α_k and ϱ_k are the limit cycles of $\alpha_k(a^{(n)})$ and $\varrho_k(a^{(n)})$, respectively, and thus the multiplicities $\operatorname{mult} \varrho_k(a)$ form the bidegree sequence of B.

Indeed, the case $k = 0$ is obvious. Assuming the assertion to hold for $k-1$, we prove it for k. By continuity of proper intersections of positive analytic cycles, $\alpha_{k-1} \cdot H_k = \alpha_k + \varrho_k$ is the limit cycle of

$$\alpha_{k-1}(a^{(n)}) \cdot [a_k^{(n)}] = \alpha_k(a^{(n)}) + \varrho_k(a^{(n)}).$$

But

$$\operatorname{mult} \varrho_k = \operatorname{mult}(\alpha_{k-1} \cdot H_k)^S \le \operatorname{mult} \varrho_k(a^{(n)}) = \operatorname{mult}(\alpha_{k-1}(a^{(n)}) \cdot [a_k^{(n)}])^S$$

by the minimum multiplicity condition. Hence and from Tworzewski's lemma (see [57] and Sect. 2 of Chapt. II), our induction assertion follows immediately. The proof of Proposition 2 is complete.

Remark. The foregoing proposition implies that the intersection indices for analytic improper intersections defined by Tworzewski [57] coincide with those defined by Achilles–Manaresi [3] (cf. the corollary to Proposition 5); this solves a problem posed in the latter paper.

Since the construction of the normal cone $C_{V \cap S} V$ is intrinsic and additive with respect to V (see Sect. 3 of Chapt. II), one can immediately deduce from Proposition 2 the intrinsic and additive character of improper intersections, stated below.

Proposition 3. *The generalized index and multiplicity for an improper intersection of an analytic set V with a submanifold S are intrinsic* (*do not depend on the ambient space*) *and additive with respect to V.*

Since the multiplicity function that assigns to a point P the multiplicity at P of a positive analytic cycle is upper semicontinuous in the analytic Zariski topology (cf. [64, 10]), we also obtain the following

Proposition 4. *The multiplicity function that assigns to a point P the intersection multiplicity at P of an analytic subset V with a submanifold S is upper semicontinuous in the analytic Zariski topology* (*and not merely analytically constructible*; *cf.* [57]).

Now we can readily show that, for an improper intersection of a purely dimensional analytic subset V with a linear subspace S, we are able to compute the generalized intersection index through testing the intersection merely by linear hyperplanes; this solves a problem posed by P. Tworzewski (cf. [56]).

Proposition 5 (Linear Testing Theorem). *Let M be an affine space*, *S a vector subspace of M*, *and V a purely dimensional analytic subset on M. Then the generalized index for the intersection of V with S at $P = 0$ is realized by the intersection algorithm for generic collections of linear hyperplanes in M.*

Indeed, we have seen after the statement of Proposition 1 that our double deformation is a kind of linearization procedure, and that the conditions of filter-regularity imposed on testing elements are in fact linear conditions on their first derivatives. Therefore, it is possible to pick a generic collection of linear forms in the ambient vector space M whose images a_i in the local ring A of V at $P = 0$ satisfy the assumptions of the theorem on deformation to an algebraic bicone. This collection of linear forms determines the collection of linear hyperplanes we are looking for.

Corollary. *The Segre numbers and the generalized Samuel multiplicities of an ideal I in the local ring $A = \mathcal{O}_{V,0}$ of an analytic germ $(V, 0) \subset (\mathbb{C}^n, 0)$ can always be expressed as the generalized index of an improper intersection.*

For if $f = (f_1, \ldots, f_r)$ are generators of I, then the analytic set

$$W := \operatorname{graph} f = \{(u, f(u)) : u \in V\} \subset \mathbb{C}^{n+r} = \mathbb{C}^n_u \times \mathbb{C}^r_v$$

is isomorphic to V, and the ideal in $\mathcal{O}_{W,0}$ that corresponds to I is generated by $v = (v_1, \ldots, v_r)$. We may thus assume that $V = W$, and that I is generated by $v_1, \ldots, v_r$. Since the polar varieties and Segre cycles for a generic d-tuple $g = (g_1, \ldots, g_d)$ of linear combinations of $v_1, \ldots, v_r$ can be determined by the formulae (cf. [19], Sect. 2)

$$\begin{aligned} P^g_k(I, V) &= \text{closure of } V(g_k|_{P^g_{k-1}(I,V)}) \setminus V(I), \\ \Lambda^g_k(I, V) &= [V(g_k|_{P^g_{k-1}(I,V)})] - [P^g_k(I, V)], \\ \Lambda^g_d(I, V) &= [V(g_d|_{P^g_{d-1}(I,V)})], \end{aligned}$$

it follows immediately that, for a generic d-tuple g of linear combinations of $v_1, \ldots, v_r$, the cycles determined by the polar varieties equal the cycles α_k, and the Segre cycles equal the cycles ϱ_k, resulting from the analytic intersection algorithm for the collection of testing elements g. Therefore, in view of the linear testing theorem (Proposition 5), the Segre numbers of the ideal I (which are the multiplicities at zero of such generic Segre cycles) coincide with the generalized index of intersection of our set V with the linear subspace $S = \{(u, v) \in \mathbb{C}^n_u \times \mathbb{C}^r_v : v = 0\}$, which completes the proof.

Further conclusions concerning the generalized intersection index and intersection cycles (such as the reduction theorem for improper analytic intersections or Bezout's theorem for improper intersections of algebraic cones) will be presented in the next section.

3. Intersection cycles and Bezout's theorem. Using the diagonal procedure, one can define the generalized intersection index and intersection multiplicity of arbitrary analytic sets in a complex manifold. This procedure was applied for the first time in modern algebraic geometry by Weil [63], but had already appeared in Pieri [41]. The canonical character of the diagonal procedure is ensured by the reduction theorem. The reduction theorem for improper intersections in complex analytic geometry was first stated and proved in [38, 40]; it is a local analytic counterpart of the reduction theorem from global projective algebraic geometry proved by Achilles–Vogel [6]. The proof is based on the algebraic formula which expresses intersection multiplicity in terms of normal cones (Prop. 2 of Sect. 2) as well as on some properties of the normal cones involved.

Reduction Theorem for Improper Intersections. *The generalized indices and multiplicities for the intersections of an analytic set V with a submanifold S at the point P, and of the analytic set $V \times S$ with the diagonal $\Delta_M := \{(z,z) : z \in M\} \subset M \times M$ at the point (P,P), coincide.*

Proof. We may, of course, assume that $M = \mathbb{C}^m = \mathbb{C}^s_u \times \mathbb{C}^r_v$, and that

$$S = \{(u,v) \in \mathbb{C}^m = \mathbb{C}^s_u \times \mathbb{C}^r_v : v = 0\}$$

is a linear subspace of M. Then the reduction theorem for improper intersections is a direct consequence of the fact that the normal cone C' to the subset $(V \times S) \cap \Delta_M$ in $V \times S$ is isomorphic to the product of the normal cone C to $V \cap S$ in V by S. The assertion about the normal cones can be immediately derived from the analogous statement concerning deformation spaces. The deformation space $\mathcal{V}$ of C is the closure of the analytic family of analytic sets parametrized by $t \in \mathbb{C} \setminus \{0\}$:

$$\{(u,v,tv;1:t) \in \mathbb{C}^s_u \times \mathbb{C}^r_v \times \mathbb{C}^r_z \times \mathbb{P}_1 : (u,v) \in V\}$$

(see Sect. 3 of Chapt. II); similarly, the deformation space $\mathcal{V}'$ of the normal cone to $(V \times S) \cap \Delta_M$ in $V \times S$ is the closure of the analytic family of analytic sets

$$\{(u,v,u',0,t(u-u'),tv;1:t) \in \mathbb{C}^s_u \times \mathbb{C}^r_v \times \mathbb{C}^s_{u'} \times \mathbb{C}^r_{v'} \times \mathbb{C}^s_w \times \mathbb{C}^r_z \times \mathbb{P}_1 : \\ (u,v) \in V,\ u' \text{ is an arbitrary element of } \mathbb{C}^s\}.$$

Then the projection

$$\mathcal{V}' \to \mathcal{V} \times \mathbb{C}^s_w, \quad (u,v,u',0,w,z;1:t) \mapsto (u,v,w,z;1:t),$$

is a biholomorphism; the inverse mapping is given by the formula

$$\mathcal{V} \times \mathbb{C}^s_w \to \mathcal{V}', \quad (u,v,w,z;1:t) \mapsto (u,v,u-w/t,0,w,z;1:t).$$

Thus the deformation space $\mathcal{V}'$ is isomorphic to $\mathcal{V} \times \mathbb{C}^s$, whence the normal cone C' is isomorphic to $C \times \mathbb{C}^s$, which completes the proof.

The above reduction theorem for improper intersections concerns only the intersection of analytic sets with a submanifold. Yet, the classical reduction theorem for proper intersections (where one considers complete intersections; cf. [47], Chapt. II, Sect. 5.7) fails in the case of improper intersections (cf. [5], Example 6.6). We now recall this example which demonstrates that the extension of the Samuel reduction theorem to improper intersections may fail for an arbitrary non-smooth complete intersection. We begin with an evident observation about intersecting affine subspaces.

Example 1. Let V_1 and V_2 be any two affine subspaces of an affine space $M = \mathbb{C}^m$. Then the intersection multiplicity at any point P of the intersection of V_1 with V_2 (possibly improper) is one.

Example 2. Let $M = \mathbb{C}^3$, $V_1 = \{0\} \subset M$, V_2 be the union of two lines $\{z_1 = z_2 = 0\}$ and $\{z_1 = z_3 = 0\}$ in M, and let $P = 0 \in M$. Then the intersection multiplicity of V_1 and V_2 at P is 2, while the multiplicity of the ideal I of V_2 in the local ring A of V_1 at P is obviously 1.

The next example indicates that the minimum of the degrees of the total results of the intersection algorithm for all admissible collections of divisors may be less than the degree of the generalized intersection index.

Example 3. Let

$$M = \mathbb{C}^3 = \mathbb{C}_u \times \mathbb{C}_v^2, \quad S = \{(u, v) \in \mathbb{C}_u \times \mathbb{C}_v^2 : v_1 = v_2 = 0\}$$

and

$$V = \{(u, v) \in \mathbb{C}_u \times \mathbb{C}_v^2 : v_2^2 - v_1 u^2 = 0\}.$$

Then it is easy to check that the generalized index of intersection of V with the line S is $(0, 1, 2)$, and its degree is 3. On the other hand, the intersection of V with the divisor $H_1 = (v_1)$ is the 1-cycle $2 \cdot S$, and thus the total result of the intersection algorithm is the cycle $2 \cdot S$ of degree $2 < 3$.

Consider now two analytic subsets V_1 and V_2 in a complex manifold M. According to the diagonal procedure—similarly to the classical case of proper intersections—the *intersection multiplicity*

$$d(P) = d(V_1, V_2; P)$$

of V_1 and V_2 at a point P will be defined as the intersection multiplicity at (P, P) of $V_1 \times V_2$ and the diagonal $\Delta_M := \{(z, z) : z \in M\} \subset M \times M$. The canonical character of this procedure in the case of improper intersections is ensured, as previously mentioned, by our reduction theorem. As an immediate consequence of Proposition 4 from Section 2, we obtain

Proposition 1. *The multiplicity function*

$$M \ni P \mapsto d(V_1, V_2; P) \in \mathbb{N}$$

is upper semicontinuous in the analytic Zariski topology.

Before we define the intersection cycle of arbitrary analytic subsets in a complex manifold M, we recall the following

Proposition 2 (cf. [57], Sect. 2). *Let N be a complex manifold. Then the multiplicity function $Z \rightsquigarrow \{P \mapsto m_P Z\}$ is a one-to-one correspondence between the analytic cycles Z on N and the analytically constructible functions $f : N \to \mathbb{N}$.*

The *intersection product* of two analytic sets V_1 and V_2 is a unique analytic cycle $V_1 \bullet V_2$ such that

$$m_P(V_1 \bullet V_2) = d(V_1, V_2; P)$$

for all P in M. The definition extends by additivity to the case of analytic cycles. We can now reformulate Proposition 3 from Section 2 as follows.

PROPOSITION 3. *The generalized index and multiplicity for an improper intersection of two analytic sets V_1 and V_2 are intrinsic* (*do not depend on the ambient space*) *and bi-additive with respect to V_1 and V_2. Consequently*, *improper intersection product is a bi-additive operation on analytic cycles.*

PROPOSITION 4. *The intersection product $V_1 \bullet V_2$ coincides with the classical intersection product $V_1 \cdot V_2$ whenever the analytic sets V_1 and V_2 meet properly in the manifold M.*

For the proof, we may assume that $V_1 = V$ is an analytic set of pure dimension and $V_2 = S$ is a submanifold in M, because both the intersection products are constructed by means of the diagonal procedure. Further, since the problem is of local character, we may assume, as before, that $S = \{(u,v) \in \mathbb{C}^m = \mathbb{C}^s \times \mathbb{C}^r : v = 0\}$ is an affine subspace of $M = \mathbb{C}^m$. But we saw in Section 2 of Chapter I that then the normal cone $[C] = [C_{V \cap S} V]$, regarded as an analytic cycle, is of the form

$$[C] = \sum_i i(V \cdot S; W_i)[W_i \times \mathbb{C}^r],$$

where W_i are the irreducible branches of $V \cap S$, and the integer $i(V \cdot S; W_i)$ is the classical intersection index along W_i. Hence Proposition 3 follows immediately.

The following example indicates that the associativity formula fails for improper intersections.

EXAMPLE 4. Set $M = \mathbb{C}^2$, $P := (0,0) \in M$, $V_1 := \{P\}$, $V_2 := \{(u,v) \in M : v = 0\}$, and let $V_3 := \{(u,v) \in M : v^2 - u^3 = 0\}$ be the Neil parabola. Then $V_1 \bullet V_2 = \{P\}$ and $V_2 \bullet V_3 = V_2 \cdot V_3 = 3\{P\}$. Hence

$$(V_1 \bullet V_2) \bullet V_3 = \{P\} \bullet V_3 = 2\{P\}, \quad \text{while} \quad V_1 \bullet (V_2 \bullet V_3) = V_1 \bullet 3\{P\} = 3\{P\}.$$

The next example will show that the general Bezout theorem does not hold true in the case of improper intersections. Nevertheless, we have the following version of Bezout's theorem for improper intersections of algebraic cones.

PROPOSITION 5 (Bezout's theorem for algebraic cones). *Let Z_1 and Z_2 be two cycles of algebraic cones in the affine space $\mathbb{C}^{n+1}$*; *clearly*, *the improper intersection product $Z_1 \bullet Z_2$ is also a cycle of algebraic cones. Then*

$$\deg Z_1 \bullet Z_2 = \deg Z_1 \cdot \deg Z_2$$

(*by the degree* $\deg C$ *of an algebraic cone C we mean its multiplicity at zero*).

Due to bi-additivity, we may assume that $Z_1 = V_1$ and $Z_2 = V_2$ are irreducible bicones. It is easy to verify that

$$\deg(V_1 \times V_2) = \deg V_1 \cdot \deg V_2$$

(cf. Sect. 2). Therefore it follows from the reduction theorem for improper intersections that we may restrict ourselves to the case of an algebraic cone $V = V_1$ intersecting a linear subspace $S = V_2$. Then by construction $\deg(V \bullet S) = m_0(V \bullet S)$. When we perform the intersection algorithm for an admissible collection of hyperplanes H_k containing S, we obtain via the classical Bezout theorem

$$\operatorname{mult} \alpha_{k-1} = \operatorname{mult}(\alpha_{k-1} \cdot H_k) = \operatorname{mult} \alpha_k + \operatorname{mult} \varrho_k.$$

Hence

$$m_0(V \bullet S) := \sum_k \operatorname{mult} \varrho_k = \operatorname{mult} \alpha_{-1} = m_0 V = \deg V,$$

and the proof is complete.

EXAMPLE 5. It is evident that the degree of improper intersection of two projective sets may be less than the product of their degrees, as in the case of an empty intersection. However, we can get the opposite inequality as well. An example is the self-intersection of a surface in $\mathbb{P}_3$ with sufficiently many isolated singularities (in comparison with the degree of the surface). We now wish to examine a cubic surface V in $\mathbb{P}_3$ whose singular locus consists of only four nodes (i.e. non-degenerate isolated singular points of multiplicity 2). In homogeneous coordinates $(z_0 : z_1 : z_2 : z_3)$, V is given by the equation

$$z_0 z_1 z_2 + z_1 z_2 z_3 + z_2 z_3 z_0 + z_3 z_0 z_1 = 0;$$

the nodes of V are

$$P_0 = (1:0:0:0), \quad P_1 = (0:1:0:0), \quad P_2 = (0:0:1:0), \quad P_3 = (0:0:0:1).$$

Recall that, according to the Morse theorem, in the vicinity of a node P the surface V is analytically equivalent to the tangent cone $C_P V$. Consequently, the multiplicity $d(V, V; P)$ of self-intersection of V at P coincides with that of $C_P V$ at P. Hence and by the foregoing Bezout theorem, $d(V, V; P) = 4$. For our cubic surface, we get

$$V \bullet V = V + \sum_{i=0}^{3} 3 \cdot \{P_i\}.$$

We have $\deg V \bullet V = 3 + 4 \cdot 3 = 15 > (\deg V)^2 = 9$.

When we pass to the cone C that determines our cubic surface V, we obtain

$$C \bullet C = C + \sum_{i=0}^{3} 3 \cdot L_i + k \cdot \{0\},$$

where L_i are the lines in $\mathbb{C}^4$ corresponding to the points $P_i \in \mathbb{P}_3$ $(i = 0, 1, 2, 3)$ and $k \in \mathbb{Z}$ is an integer. It follows from Bezout's theorem for algebraic cones (Proposition 5) that

$$9 = \deg(C \bullet C) = 3 + 12 + k,$$

whence $k = -6$. This shows that the intersection product of two analytic sets need not be a positive cycle.

Finally, using deformation to an algebraic bicone we discuss an intersection product (communicated by P. Tworzewski) which is not a positive cycle.

Example 6. Consider the affine space $M = \mathbb{C}^6$, the linear subspace

$$S = \{(u, v) \in \mathbb{C}^6 = \mathbb{C}^4_u \times \mathbb{C}^2_v : v_1 = v_2 = 0\}$$

and the algebraic subset

$$V = \{(u, v) \in \mathbb{C}^6 = \mathbb{C}^4_u \times \mathbb{C}^2_v : v_1 u_1 u_2 - v_2 u_3 u_4 = 0\}.$$

Since V is a bicone with respect to the variables u and v, V coincides with the bicone B constructed in the theorem on deformation to an algebraic bicone for $P = 0 \in S$ (see Sect. 1). The intersection multiplicity $d(V, S; P)$ is thus equal to the multiplicity of the bicone V at P, whence $d(V, S; P) = 3$. More generally, in order to calculate the intersection multiplicity at a point

$$Q = (a_1, a_2, a_3, a_4, 0, 0) \in S,$$

we make the translation of variables by the vector $\vec{Q} = (a_1, a_2, a_3, a_4, 0, 0)$. The equation of V in the new variables is

$$v_1(u_1 - a_1)(u_2 - a_2) - v_2(u_3 - a_3)(u_4 - a_4) = 0.$$

Therefore an easy analysis of the bicone B constructed for the point $Q \in S$ yields

$$d(V, S; Q) = 1 \quad \text{iff} \quad (a_1 a_2 \neq 0 \text{ or } a_3 a_4 \neq 0)$$

and

$$d(V, S; Q) = 2 \quad \text{iff} \quad (a_1 a_2 = 0 \text{ and } a_3 a_4 = 0 \text{ and } Q \neq P).$$

Hence it follows immediately that

$$V \bullet S = S + \sum_{i=1}^{2} \sum_{j=3}^{4} H_{ij} - \sum_{k=1}^{4} L_k + 2 \cdot \{P\},$$

where

$$H_{ij} = \{u \in S : u_i = u_j = 0\} \quad (i = 1, 2;\ j = 3, 4)$$

and

$$L_k = \{u \in S : u_l = 0 \text{ for } l = 1, 2, 3, 4;\ l \neq k\} \quad (k = 1, 2, 3, 4).$$

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