1. Introduction

Let A and B be subsets of $\mathbb{R}^n$. Their sum $A+B$, called the *Minkowski sum* of A and B, is defined by

$$A+B=\{a+b \mid a\in A,\, b\in B\}.$$

Being one of the most fundamental operations on sets in spaces with the addition operation, Minkowski sum has been used, both implicitly and explicitly, in virtually all branches of mathematics. However, there have not been many investigations of the properties of Minkowski sum itself. One notable exception is the pioneering works by H. Brunn, H. Minkowski, and others on the so-called Brunn–Minkowski theory, which compares the volumes of the Minkowski sum and its summands [1, 16, 19].

Meanwhile, mainly due to the convenience of describing various geometric relations, the Minkowski sum has been adopted and used extensively in engineering and computer science. A few examples are mechanical engineering (collision-free path planning [13]), image processing and mathematical morphology [6, 20], computer graphics (metamorphosis [5]), geometric modeling (offset and sweep curve/surface generation [15, 21], computation of CSG operations [18]), and computational geometry [7].

The common problem persistent in all such applications is the efficient computation of Minkowski sums [8, 9, 11]. But the need for dealing with complex geometric objects encountered in real-world applications makes this goal seem far from satisfactory. Thus there naturally arises the need for fundamental geometric and topological analysis of Minkowski sum, which should be more detailed than just comparing volumes.

In this paper, we will investigate some global topological properties of Minkowski sum in relation to the geometric structures of its summands. Although Minkowski sum has a simple definition, it may lead to a lot of complicated phenomena. In general, the Minkowski sum operation does not preserve the topological properties of the sets in the Euclidean space. To give an idea, we first show some examples: see Figures 1, 2 and 3. Note that all the summands in these figures are homeomorphic to the unit disk. But in Figure 1, the result of the Minkowski sum is not simply connected. In Figure 2, the Minkowski sum is not simply connected, and its boundary is not homeomorphic to the unit circle. Worse still, the Minkowski sum has infinitely many "holes" in Figure 3.

These examples show that even when the summands are topologically simple, their Minkowski sum can become quite complex in the topological sense. Especially, Minkowski sum does not preserve even the simplest topological property of the sets in $\mathbb{R}^2$, that is, that of being homeomorphic to the unit disk.

Thus the following natural problem arises:

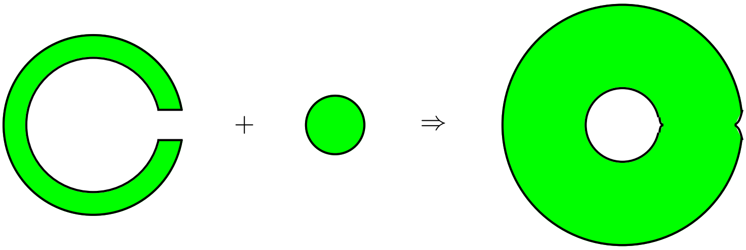


Fig. 1. Multiply connected Minkowski sum

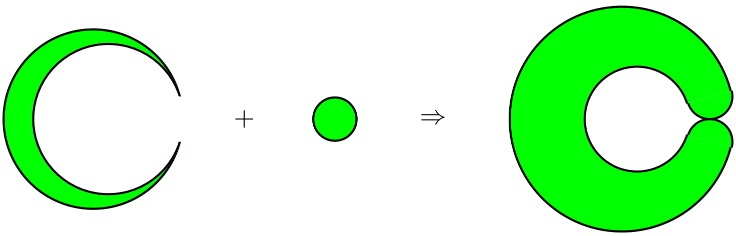
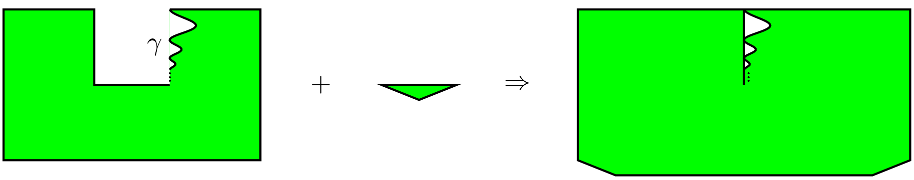


Fig. 2. Minkowski sum with singular boundary



$$\gamma(t) = \left(e^{-1/t}\left(1 + \sin\tfrac{1}{t}\right), t\right),\ t \in \left[0, \tfrac{2}{3\pi}\right]$$

Fig. 3. Minkowski sum with infinitely many holes

Problem 1. *Find a class of sets in $\mathbb{R}^2$ which are homeomorphic to the unit disk, such that the Minkowski sums of sets in that class are always homeomorphic to the unit disk.*

An immediate answer to this problem is the class of all convex sets which are homeomorphic to the unit disk, since it can be shown easily that the Minkowski sum of convex sets is also convex. But a serious drawback of the convexity is that it is too strong: there are too many useful sets which are not convex. So another important problem is:

Problem 2. *Find a class of sets in $\mathbb{R}^2$ which contains all convex sets homeomorphic to the unit disk, and is maximal among all the classes satisfying the condition in Problem* 1.

If we consider two bounded sets A and B in the plane as rigid, mutually impenetrable objects, then the complement of the Minkowski sum $A+B$ in $\mathbb{R}^2$ represents the set of all possible relative positions of the translates of A and $-B$. One such configuration can be continuously moved into another by translation without mutual penetrating if and only if the two configurations are in the same connected component of the complement of $A+B$ in $\mathbb{R}^2$. So, the Minkowski sum $A+B$ is simply connected if and only if any two relative positions of the translates of A and $-B$ can be continuously moved into each other by translation without mutual penetrating, or, in other words, any relative positions can be continuously pulled over to separate A and $-B$ indefinitely.

We will show that there exists an important class of planar domains that we call *semi-convex*, which satisfy the conditions both in Problems 1 and 2. Intuitively speaking, a planar domain is semi-convex if the normal vector field along the boundary does not turn concavely by more than the angle π. We mention that our definition of semi-convexity differs from that introduced in [14]. It is also significantly more general than the usual notion of star-shapedness, and, to the author's knowledge, it is the first among the many variations of convexity which has an optimal property with respect to the Minkowski sum.

In general, the boundary curves of a Minkowski sum are results of the operation called *convolution* on the boundary curves of the summands. The convolution can be considered as a basic building block in analyzing the Minkowski sum of the shapes represented by boundary curves. But there has been few precise mathematical studies on the convolution of curves in the literature. Also, we will observe in Section 2 that the convolution can behave wildly unless we restrict the class of the curves to be convolved, which is a fact not often noted in both theory and practice. So in Section 2, we carefully analyze the mathematical properties of the convolution of curves, and classify the curve classes according to their differential regularity with particular regard to convolution.

Often in practice, the curve pieces used to describe shape boundaries come from specific fixed classes such as the class of rational curves or various classes of splines (e.g., the NURBS curves). However, most of these important curve classes are not closed under convolution, which makes it impossible to represent the Minkowski sum boundary in a uniform manner (i.e., with the curve pieces in the same curve class used to represent the summands), and thus causes serious problems in practice. Meanwhile, it also turns out that the curve classes $\mathcal{C}^{k:l}$ and $\mathcal{C}^{k:l}_c$ introduced in Section 2 are not closed under convolution. These facts imply that the usual conditions on the boundary curves such as rationality or differentiability are not preserved under Minkowski sum. In particular, it is not clear whether the notion of semi-convexity is closed under Minkowski sum, unless we restrict the boundary curves to be in special curve classes. Thus it is a necessary and important problem to find a condition on classes of curves which guarantees closedness under convolution.

In Section 2, we introduce special curve classes, called *Minkowski classes*, which are closed under convolution. An important example of a Minkowski class, denoted by $\mathcal{W}$, is given in Section 3, for which we use Łojasiewicz's structure theorem for real-analytic varieties [12]. It is shown that $\mathcal{W}$ contains practically all the curves used in engineering applications. This in particular means that it is not too restrictive to consider the Minkowski sum only in the category of $\mathcal{M}$-domains for a Minkowski class $\mathcal{M}$. Here, an $\mathcal{M}$*-domain* means a subset in $\mathbb{R}^2$ whose boundary consists of finitely many curves in $\mathcal{M}$.

Note that we consider a fairly general class of domains, including ones with corners on their boundaries. In fact, this is also necessary, since such domains can arise naturally as a result of the Minkowski sum operation on quite nice domains. To handle them, we introduce two concepts: *sector* in Section 4, and *virtual boundary* in Section 5. A sector is a local germ of a domain near a boundary point, whether cornered or not. So, by examining the effect of Minkowski sum on sectors, we can understand the essential and local behavior of Minkowski sum. By integrating these results, we obtain the global result in Section 6 that the set of all $\mathcal{M}$-domains is closed under Minkowski sum for any Minkowski class $\mathcal{M}$, which is a basis for the further closedness result for semi-convexity.

The notion of virtual boundary is a generalization of that of the usual boundary in a way that incorporates corners in a uniform manner. It is defined to be in one-to-one continuous correspondence with the outer normal vectors on the boundary including those at the corners. Together with the analysis of sectors, the notion of virtual boundary enables a uniform and easy treatment of cornered domains, thus reducing the globally complex problem of Minkowski sum to the analysis of a few local genotypes of the sectors.

The notion of *semi-convexity*, which generalizes that of convexity, will be formally introduced in Section 7. Let $\mathcal{M}$ be a Minkowski class. It is proved that the Minkowski sum of any two semi-convex $\mathcal{M}$-domains is homeomorphic to the unit disk, which answers Problem 1 above within the category of $\mathcal{M}$-domains. In Section 8, we prove that for any $\mathcal{M}$-domain which is homeomorphic to the unit disk but is not semi-convex, there exists a semi-convex $\mathcal{M}$-domain such that their Minkowski sum is not homeomorphic to the unit disk. This answers Problem 2 above within the category of $\mathcal{M}$-domains. In fact, it is shown that the set of all semi-convex $\mathcal{M}$-domains is uniquely maximal among all the classes of $\mathcal{M}$-domains which satisfy the condition in Problem 1 and contain all the $\mathcal{M}$-domains called *flag domains*. Finally, we prove in Section 9 that the set of all semi-convex $\mathcal{M}$-domains is closed under Minkowski sum. In proving these results, we will use the Gauss–Bonnet Theorem, translated into the language of virtual boundary, as one of the main tools. In Section 10, we summarize the results in this paper, and discuss some further research directions.

Since semi-convexity is geometric in nature, the properties of semi-convex domains proved in this paper reveal a new relationship between the geometric and topological properties of Minkowski sum. Also, since semi-convexity can easily be checked algorithmically, it is expected to be utilized in various application areas using Minkowski sum.

2. Curves

The boundaries of reasonable domains consist of curves. So, to analyse domains, we first analyse curves. In this section, we define various special curve classes according to their regularity, and study their properties with respect to the operation of *convolution*. In particular, the *Minkowski classes* are introduced, which are defined essentially to be closed under convolution. We also set up some conventions and notations which will be used throughout this paper.

Let $\mathbf{v} = (v_1, v_2)$, $\mathbf{w} = (w_1, w_2)$ be in $\mathbb{R}^2$. We write $\mathbf{v} \mathbin{/\!/} \mathbf{w}$ if either at least one of $\mathbf{v}$ and $\mathbf{w}$ is $0 = (0, 0)$, or $\mathbf{v} = k\mathbf{w}$ for some $k \in \mathbb{R}$. Let $p \in \mathbb{R}^2$ and $r > 0$. By $B_r(p)$, we always denote the *closed* ball in $\mathbb{R}^2$, centered at p and with radius r. The open ball will be denoted by $B_r^o(p)$. The unit circle in $\mathbb{R}^2$ will be denoted by S^1. Thus, $S^1 = \{\mathbf{v} \in \mathbb{R}^2 \mid |\mathbf{v}| = 1\} = \partial B_1(0)$.

2.1. *Convolution*

Definition 2.1 ($C^{k:l}$ curve). Let $k, l = 1, 2, \ldots, \infty, \omega$ (ω for real-analytic), and $k \geq l$. Let $n = 1, 2, \ldots$ A curve $\gamma : (a, b) \to \mathbb{R}^n$ is called a C^k *curve* if there exists a reparametrization $\breve{\gamma} : (\breve{a}, \breve{b}) \to \mathbb{R}^2$ of γ such that $\breve{\gamma}' \neq 0$ on $(\breve{a}, \breve{b})$, and $\breve{\gamma}$ is C^k. A curve $\gamma : [a, b] \to \mathbb{R}^n$ is called a $C^{k:l}$ *curve* if the restriction of γ to (a, b) is a C^k curve, and there exists an extension $\widetilde{\gamma} : (a - \varepsilon, b + \varepsilon) \to \mathbb{R}^n$ of γ for some $\varepsilon > 0$, such that $\widetilde{\gamma}$ is a C^l curve.

Here, it is important to note that $\breve{\gamma}' \neq 0$. Without this condition, a curve γ may not be a C^k curve, even if it is k-times differentiable.

Definition 2.2 (the class $\mathcal{C}^{k:l}$). Let $k, l = 1, 2, \ldots, \infty, \omega$ and $k \geq l$. Then we denote by $\mathcal{C}^{k:l}$ the class of all $C^{k:l}$ curves in $\mathbb{R}^2$ defined on closed intervals, which have *no self-intersections*. An element in $\mathcal{C}^{k:l}$ will be called a $\mathcal{C}^{k:l}$*-curve*.

Note that closed loops are excluded in this definition. The inclusion relations between the classes $\mathcal{C}^{k:l}$ in Figure 4 are immediate from the definition.

$$
\begin{array}{ccccccccc}
\mathcal{C}^{1:1} & & & & & & & & \\
\cup & & & & & & & & \\
\mathcal{C}^{2:1} & \supset & \mathcal{C}^{2:2} & & & & & & \\
\cup & & \cup & & & & & & \\
\vdots & & \vdots & & & & & & \\
\cup & & \cup & & & & & & \\
\mathcal{C}^{\infty:1} & \supset & \mathcal{C}^{\infty:2} & \supset & \ldots & \supset & \mathcal{C}^{\infty:\infty} & & \\
\cup & & \cup & & & & \cup & & \\
\mathcal{C}^{\omega:1} & \supset & \mathcal{C}^{\omega:2} & \supset & \ldots & \supset & \mathcal{C}^{\omega:\infty} & \supset & \mathcal{C}^{\omega:\omega}
\end{array}
\qquad
\begin{array}{ccccccccc}
\mathcal{C}_c^{1:1} & & & & & & & & \\
\cup & & & & & & & & \\
\mathcal{C}_c^{2:1} & \supset & \mathcal{C}_c^{2:2} & & & & & & \\
\cup & & \cup & & & & & & \\
\vdots & & \vdots & & & & & & \\
\cup & & \cup & & & & & & \\
\mathcal{C}_c^{\infty:1} & \supset & \mathcal{C}_c^{\infty:2} & \supset & \ldots & \supset & \mathcal{C}_c^{\infty:\infty} & & \\
\cup & & \cup & & & & \cup & & \\
\mathcal{C}_c^{\omega:1} & \supset & \mathcal{C}_c^{\omega:2} & \supset & \ldots & \supset & \mathcal{C}_c^{\omega:\infty} & \supset & \mathcal{C}_c^{\omega:\omega}
\end{array}
$$

Fig. 4. Inclusion relations for $\mathcal{C}^{k:l}$ and $\mathcal{C}_c^{k:l}$

Remark 2.1. Given a $\mathcal{C}^{k:l}$-curve $\gamma : [a, b] \to \mathbb{R}^2$, we usually assume that it is defined on some slightly larger open interval $(a - \varepsilon, b + \varepsilon)$, and γ is k-times differentiable on (a, b), l-times differentiable on $(a - \varepsilon, b + \varepsilon)$, and $\gamma' \neq 0$ on $(a - \varepsilon, b + \varepsilon)$.

Let $\gamma : [a, b] \to \mathbb{R}^2$ be a $\mathcal{C}^{1:1}$-curve, and let $\widetilde{\gamma} : (a - \varepsilon, b + \varepsilon) \to \mathbb{R}^2$ be a C^1 extension of γ. It is easy to see that the limit

$$\mathbf{v}[\gamma](t) = \lim_{\tau \to t} \frac{\widetilde{\gamma}(\tau) - \widetilde{\gamma}(t)}{|\widetilde{\gamma}(\tau) - \widetilde{\gamma}(t)|}$$

exists in S^1 for every $t \in [a, b]$, and $\mathbf{v}[\gamma] : [a, b] \to S^1$ is continuous. We will denote $\mathbf{v}[\gamma](a)$ also by $\mathbf{v}[\gamma]$. Note that these are independent of the choice of $\widetilde{\gamma}$. Let $\mu : \mathbb{R} \to S^1$ be the covering map defined by $\mu(t) = (\cos t, \sin t)$ for $t \in \mathbb{R}$. Now there exists a continuous function $\theta : [a, b] \to \mathbb{R}$ such that $\mathbf{v}[\gamma](t) = \mu(\theta(t))$ for every $t \in [a, b]$. We call θ an *angle function* of γ. Note that if $\widetilde{\theta}$ is another angle function of γ, then, for some integer n, we have $\widetilde{\theta}(t) = \theta(t) + 2n\pi$ for every $t \in [a, b]$. So the following is well defined:

Definition 2.3 (convex curve). Let $\gamma : [a, b] \to \mathbb{R}^2$ be a $\mathcal{C}^{1:1}$-curve, and let $\theta : [a, b] \to \mathbb{R}$ be an angle function of γ. Then γ is called *convex* if θ is either strictly increasing or strictly decreasing, provided it is not constant. The *signature* of γ, $\sigma(\gamma)$, is defined to be $+$ (resp., $-$) if θ is strictly increasing (resp., strictly decreasing), and 0 if θ is constant.

For $k, l = 1, 2, \ldots, \infty, \omega$ with $k \geq l$, we denote by $\mathcal{C}_c^{k:l}$ the class of all convex curves in $\mathcal{C}^{k:l}$. An element of $\mathcal{C}_c^{k:l}$ will be called a $\mathcal{C}_c^{k:l}$*-curve*.

From the above definition, the inclusion relations between the classes $\mathcal{C}_c^{k:l}$ in Figure 4 are obvious.

DEFINITION 2.4 ($*$-admissible curves). Two $\mathcal{C}_c^{1:1}$-curves γ_1, γ_2 are said to be $*$-*admissible* to each other if $\mathbf{v}[\gamma_1] \mathbin{/\!\!/} \mathbf{v}[\gamma_2]$ and $\sigma(\gamma_1) = \sigma(\gamma_2) \neq 0$.

Note that the $*$-admissibility is a transitive relation. Let $\gamma_i : [a_i, b_i] \to \mathbb{R}^2$, $i = 1, \ldots, n$, be $\mathcal{C}_c^{1:1}$-curves which are $*$-admissible to each other. Let $\widetilde{\theta}_i : [a_i, b_i] \to \mathbb{R}$ be an angle function of γ_i for each i. For each i, define $\theta_i : [a_i, b_i] \to \mathbb{R}$ by $\theta_i(t) = \widetilde{\theta}_i(t)$ if $\mathbf{v}[\gamma_i] = \mathbf{v}[\gamma_1]$, and $\theta_i(t) = \widetilde{\theta}_i(t) + \pi$ if $\mathbf{v}[\gamma_i] = -\mathbf{v}[\gamma_1]$. Then, with no loss of generality, we can assume $\theta_i(a_i) = \theta_1(a_1)$ for each i. Define $\alpha = \min\{\theta_1(b_1), \ldots, \theta_n(b_n)\}$ if $\sigma(\gamma_1) = +$, and $\alpha = \max\{\theta_1(b_1), \ldots, \theta_n(b_n)\}$ if $\sigma(\gamma_1) = -$. Let $h : [0,1] \to \mathbb{R}$ be the linear function with $h(0) = \theta_1(a_1)$ and $h(1) = \alpha$. Now we define $\gamma = \gamma_1 * \ldots * \gamma_n : [0,1] \to \mathbb{R}^2$, the *convolution* of $\gamma_1, \ldots, \gamma_n$, by

$$\gamma(t) = \gamma_1(\theta_1^{-1}(h(t))) + \ldots + \gamma_n(\theta_n^{-1}(h(t)))$$

for $t \in [0,1]$. Note that $\mathbf{v}[\gamma_1](\theta_1^{-1}(h(t))) \mathbin{/\!\!/} \ldots \mathbin{/\!\!/} \mathbf{v}[\gamma_n](\theta_n^{-1}(h(t)))$ for every t.

From the definition, it is clear that the result of convolution does not depend on the order of operations. It is also easy to see that convolutions are continuous curves. But in general, a convolution of $\mathcal{C}_c^{1:1}$-curves can exhibit quite anomalous behavior, and it cannot be expected to be even a $\mathcal{C}^{1:1}$-curve. This can happen even when the terms belong to $\mathcal{C}_c^{\omega:\infty}$, as can be seen from the following example:

EXAMPLE 2.1. For some small $\delta > 0$, let $\gamma_+, \gamma_- : [0, \delta] \to \mathbb{R}^2$ be given by $\gamma_\pm(t) = (t, f_\pm(t))$ for $t \in [0, \delta]$, where $f_\pm : [0, \delta] \to \mathbb{R}$ are defined by

$$f_\pm(t) = \int_0^t \frac{1}{\xi^2} \exp\left(-\frac{1}{\xi}\right) \left[4 \pm \left\{1 + \sqrt{2} \sin\left(\frac{1}{\xi} - \frac{\pi}{4}\right)\right\}\right] d\xi$$

for $t \in [0, \delta]$. Note that

$$0 < \frac{1}{\xi^2} \exp\left(-\frac{1}{\xi}\right) \left[4 \pm \left\{1 + \sqrt{2} \sin\left(\frac{1}{\xi} - \frac{\pi}{4}\right)\right\}\right] \leq (5 + \sqrt{2}) \frac{1}{\xi^2} \exp\left(-\frac{1}{\xi}\right)$$

for every $\xi > 0$. So we have

$$0 < f_\pm(t) \leq (5 + \sqrt{2}) \int_0^t \frac{1}{\xi^2} \exp\left(-\frac{1}{\xi}\right) d\xi = (5 + \sqrt{2}) \exp\left(-\frac{1}{t}\right)$$

for every $t \in (0, \delta]$. This shows that f_+ and f_- are well defined. It is easy to see that $f_\pm$ are real-analytic on $(0, \delta]$, and $\lim_{t \to 0+} f_\pm^{(k)} = 0$ for every $k < \infty$. Hence $\gamma_\pm \in \mathcal{C}^{\omega:\infty}$. Note that

$$f_\pm''(t) = \frac{1}{t^4} \exp\left(-\frac{1}{t}\right) \left[(1 - 2t) \left[4 \pm \left\{1 + \sqrt{2} \sin\left(\frac{1}{t} - \frac{\pi}{4}\right)\right\}\right] \mp \sqrt{2} \cos\left(\frac{1}{t} - \frac{\pi}{4}\right)\right]$$

for $t > 0$. So it follows that $f''_{\pm}(t) > 0$ for $t \in (0, \delta]$ if we choose sufficiently small $\delta > 0$. This shows that $\gamma_+, \gamma_- \in \mathcal{C}_c^{\omega:\infty}$, and $\mathbf{v}[\gamma_1] = \mathbf{v}[\gamma_2] = (1, 0)$, $\sigma(\gamma_1) = \sigma(\gamma_2) = +$. Let $f = f_+ - f_-$. Then

$$f(t) = 2\int_0^t \frac{1}{\xi^2} \exp\left(-\frac{1}{\xi}\right)\left\{1 + \sqrt{2}\sin\left(\frac{1}{\xi} - \frac{\pi}{4}\right)\right\} d\xi = 2\exp\left(-\frac{1}{t}\right)\left(1 + \sin\frac{1}{t}\right)$$

for $t \in (0, \delta]$. Let $t_n = (3\pi/2 + 2\pi N + 2n\pi)^{-1}$ for $n = 1, 2, \ldots$, where $(3\pi/2 + 2\pi N + 2\pi)^{-1} \le \delta < (3\pi/2 + 2\pi N)^{-1}$. Let $S = \{(s, t) \in [0, \delta] \times [0, \delta] \mid \gamma_+(s) = \gamma_-(t)\}$. It is easy to see that $S = \{(t_n, t_n) \mid n = 1, 2, \ldots\} \cup \{(0, 0)\}$. Note also that $f'(t_n) = f'_+(t_n) - f'_-(t_n) = 0$ for all n. Now γ_+ and $-\gamma_-$ are in $\mathcal{C}_c^{\omega:\infty}$, and $*$-admissible to each other. Let $\gamma = \gamma_1 * (-\gamma_2) : [0, 1] \to \mathbb{R}^2$. Then, from the above argument, it is easy to see that there exist sequences $a_n, b_n \searrow 0$ with $a_{n+1} < b_n < a_n$ such that $\gamma(a_n) = 0$ and $\gamma(b_n) \neq 0$ for every n. Clearly, this cannot happen for a $\mathcal{C}^{1:1}$-curve. Thus we conclude that $\gamma \notin \mathcal{C}^{1:1}$.

The following lemma shows that the convolution behaves as expected if we know beforehand that it has only a mild regularity, i.e., $\mathcal{C}^{1:1}$.

LEMMA 2.1. *Let $\gamma_i : [a_i, b_i] \to \mathbb{R}^2$, $i = 1, 2$, be two $\mathcal{C}_c^{1:1}$-curves which are $*$-admissible to each other. Let $\gamma = \gamma_1 * \gamma_2$. Suppose $\gamma \in \mathcal{C}^{1:1}$. Then, for any t, t_1, t_2 such that $\gamma_1(t_1)$, $\gamma_2(t_2)$ are summed to the convolution $\gamma_1 * \gamma_2$, we have*

$$\mathbf{v}[\gamma](t) \mathbin{/\!\!/} \mathbf{v}[\gamma_1](t_1) \mathbin{/\!\!/} \mathbf{v}[\gamma_2](t_2).$$

In consequence, γ is in $\mathcal{C}_c^{1:1}$ and is $*$-admissible to γ_1 and γ_2.

Proof. Let $\mathbf{v} = \mathbf{v}[\gamma_1](t_1) = \pm\mathbf{v}[\gamma_2](t_2)$. First, note that

$$\frac{\gamma(\tau) - \gamma(t)}{|\gamma(\tau) - \gamma(t)|} = \frac{1}{|\mathbf{v}_1 + k\mathbf{v}_2|} \cdot \mathbf{v}_1 + \frac{1}{\left|\mathbf{v}_2 + \frac{1}{k}\mathbf{v}_1\right|} \cdot \mathbf{v}_2,$$

where

$$\mathbf{v}_1 = \frac{\gamma_1(\tau_1) - \gamma_1(t_1)}{|\gamma_1(\tau_1) - \gamma_1(t_1)|}, \qquad \mathbf{v}_2 = \frac{\gamma_2(\tau_2) - \gamma_2(t_2)}{|\gamma_2(\tau_2) - \gamma_2(t_2)|}, \qquad k = \frac{|\gamma_2(\tau_2) - \gamma_2(t_2)|}{|\gamma_1(\tau_1) - \gamma_1(t_1)|},$$

and $\gamma(\tau) = \gamma_1(\tau_1) + \gamma_2(\tau_2)$. Let $\mathbf{v} = \lim_{\tau\to t} \mathbf{v}_1 = \pm\lim_{\tau\to t} \mathbf{v}_2$. Then

$$\mathbf{v}[\gamma](t) = \lim_{\tau\to t}\left(\frac{1}{|\mathbf{v}_1 + k\mathbf{v}_2|} \pm \frac{1}{\left|\mathbf{v}_2 + \frac{1}{k}\mathbf{v}_1\right|}\right) \cdot \mathbf{v} = \lim_{\tau\to t} \frac{1 \pm k}{|\mathbf{v}_1 + k\mathbf{v}_2|} \cdot \mathbf{v}.$$

Since we know that $\mathbf{v}[\gamma](t) \in S^1$, it follows that $\mathbf{v}[\gamma](t) = \mathbf{v}$ or $-\mathbf{v}$. Now the rest of the proof follows easily. ∎

2.2. *Minkowski class*

DEFINITION 2.5 (Minkowski class). A subclass $\mathcal{M}$ of $\mathcal{C}_c^{1:1}$ is called a *Minkowski class* if the following two conditions are satisfied:

(1) $\mathcal{M}$ is closed under *restriction*, i.e., if $\gamma : [a, b] \to \mathbb{R}^2$ is in $\mathcal{M}$, then $\gamma|_{[c,d]}$ is also in $\mathcal{M}$ for any $[c, d] \subset [a, b]$.

(2) $\mathcal{M}$ is closed under *initial convolution*, i.e., for any two $*$-admissible $\mathcal{M}$-curves $\gamma_1 : [a_1, b_1] \to \mathbb{R}^2$ and $\gamma_2 : [a_2, b_2] \to \mathbb{R}^2$, the convolution $\gamma_1|_{[a_1, a_1+\varepsilon]} * \gamma_2|_{[a_2, a_2+\varepsilon]}$ is either an $\mathcal{M}$-curve or constant for some $\varepsilon > 0$.

As an example, let $\mathcal{LA}$ be the set of all line segments and circular arcs in $\mathbb{R}^2$. It can be easily checked that $\mathcal{LA}$ is a Minkowski class. In Section 3, we will present a non-trivial Minkowski class $\mathcal{W}$, which is significantly larger than $\mathcal{LA}$.

Let $\gamma_i : [a_i, b_i] \to \mathbb{R}^2$, $i = 1, 2$, be two continuous curves. We say that γ_1, γ_2 have an *intersection* at (s, t) if $\gamma_1(s) = \gamma_2(t)$. We say that γ_1, γ_2 have an *isolated intersection* at (s, t) if $\gamma_1(s) = \gamma_2(t)$ and $\gamma_1(s') \neq \gamma_2(t')$ for every $(s', t') \in (s-\varepsilon, s+\varepsilon)\times(t-\varepsilon, t+\varepsilon)\setminus\{(s, t)\}$ for some $\varepsilon > 0$.

The next lemma shows an important property of Minkowski classes:

Lemma 2.2. *Any two γ_1, γ_2 in a Minkowski class $\mathcal{M}$ cannot have infinitely many isolated intersections.*

Proof. With no loss of generality, assume $a_1 = a_2 = 0$, where $\gamma_i : [a_i, b_i] \to \mathbb{R}^2$ for $i = 1, 2$. Suppose γ_1 and γ_2 have infinitely many isolated intersections. Since $[a_1, b_1] \times [a_2, b_2]$ is compact, there exists an accumulation point of the isolated intersections, which we can assume to be $\gamma_1(0) = \gamma_2(0)$. Also, we can assume $\gamma_1(0) = \gamma_2(0) = 0$ and $\mathbf{v}[\gamma_1] = (1, 0)$. Since $\gamma_1(0) = \gamma_2(0)$ is an accumulation point of the isolated intersections, we can also assume that $\mathbf{v}[\gamma_2] = \mathbf{v}[\gamma_1] = (1, 0)$ and $\sigma(\gamma_1) = \sigma(\gamma_2) = +$. Thus, for $i = 1, 2$, we can write $\gamma_i(t) = (t, f_i(t))$ for small $t \geq 0$, where f_i is a C^1 function such that $f_i(0) = f_i'(0) = 0$, and f_i' is strictly increasing. Now there exists a sequence $t_n \searrow 0$ such that γ_1 and γ_2 have an isolated intersection at (t_n, t_n) for every n. If $f_1'(t_n) = f_2'(t_n)$ except for at most finitely many n's, then the convolution $\gamma = \gamma_1 * (-\gamma_2)$ would not be in $\mathcal{C}^{1:1}$, which can be seen from the argument in Example 2.1. So we can assume $f_1'(t_n) \neq f_2'(t_n)$ for every n. We can also assume that $f_1(t) \neq f_2(t)$ if $t \neq t_n$ for any n. In this case, it is easy to see that $\gamma(t_n)$'s are in the regions D_1 and D_3 alternating with n, where $D_1 = \{(x, y) \in \mathbb{R}^2 \mid x > 0,\ y > 0\}$ and $D_3 = \{(x, y) \in \mathbb{R}^2 \mid x < 0,\ y < 0\}$. But this is impossible, since γ should be in $\mathcal{M}$, and thus in $\mathcal{C}_c^{1:1}$. ■

Remark 2.2. Example 2.1 shows that two $\mathcal{C}_c^{\omega:\infty}$-curves can have infinitely many isolated intersections, which implies that $\mathcal{C}_c^{k:l}$ is not a Minkowski class for $k, l = 1, 2, \ldots, \infty, \omega$, $k \geq l$, except for $\mathcal{C}_c^{\omega:\omega}$. Later, we will also see that $\mathcal{C}_c^{\omega:\omega}$ is not a Minkowski class.

Let $\gamma_i : [a_i, b_i] \to \mathbb{R}^2$, $i = 1, 2$, be two one-to-one continuous curves. We write $\gamma_1 \approx \gamma_2$ if there exist $a_i < c_i \leq b_i$ for $i = 1, 2$ and a homeomorphism $h : [a_1, c_1] \to [a_2, c_2]$ such that $h(a_1) = a_2$ and $\gamma_1(t) = \gamma_2(h(t))$ for every $t \in [a_1, c_1]$. We write $\gamma_1 \sim \gamma_2$ if γ_1 can be moved to a curve $\widetilde{\gamma}_1$ by a rigid motion in the plane so that $\widetilde{\gamma}_1 \approx \gamma_2$. Note that both the relations $\approx$ and $\sim$ are symmetric and transitive.

Let $\gamma_i : [a_i, b_i] \to \mathbb{R}^2$, $i = 1, 2$, be two $\mathcal{C}_c^{1:1}$-curves. Note that, with appropriate rigid motions in the plane, we can always move γ_1 and γ_2 to obtain curves $\widetilde{\gamma}_1$, $\widetilde{\gamma}_2$ respectively so that $\widetilde{\gamma}_1(a_1) = \widetilde{\gamma}_2(a_2) = 0$, $\mathbf{v}[\widetilde{\gamma}_1] = \mathbf{v}[\widetilde{\gamma}_2] = (1, 0)$ and $\sigma(\widetilde{\gamma}_1), \sigma(\widetilde{\gamma}_2) \geq 0$. We write $\gamma_1 \rhd \gamma_2$ (resp., $\gamma_1 \lhd \gamma_2$) if there exist continuous functions $f_1, f_2 : [0, \varepsilon] \to \mathbb{R}$, for some $\varepsilon > 0$, such that the graph of f_i is contained in the image of $\widetilde{\gamma}_i$ for $i = 1, 2$, and $f_1(x) > f_2(x)$ (resp., $f_1(x) < f_2(x)$) for every $x \in (0, \varepsilon]$.

Let $\mathcal{M}$ be a Minkowski class. As an important consequence of Definition 2.5 and Lemma 2.2, note that, given any γ_1, γ_2 in $\mathcal{M}$, there are only three possibilities: either $\gamma_1 \rhd \gamma_2$, or $\gamma_1 \lhd \gamma_2$, or $\gamma_1 \sim \gamma_2$. Suppose γ_1 and γ_2 are $*$-admissible to each other. Then the convolution $\gamma = \gamma_1 * \gamma_2$ is *initially constant* (that is, constant for some interval from the start) if and only if $\gamma_1 \sim \gamma_2$ and $\mathbf{v}[\gamma_1] = -\mathbf{v}[\gamma_2]$. For the remaining cases, γ is initially in $\mathcal{M}$, and the next lemma shows the relation between γ and γ_1, γ_2 with regard to the above relations $\rhd$, $\lhd$ and $\sim$. See Figure 5 for the illustration of these results.

LEMMA 2.3 (convolution in Minkowski class). *Let $\mathcal{M}$ be a Minkowski class, and let $\gamma_i : [a_i, b_i] \to \mathbb{R}^2$, $i = 1, 2$, be two $\mathcal{M}$-curves which are $*$-admissible to each other. Let γ be an initial piece of the convolution $\gamma_1 * \gamma_2$, which is either in $\mathcal{M}$, or is a constant.*

(1) *Suppose $\mathbf{v}[\gamma_1] = \mathbf{v}[\gamma_2]$. Then γ is always in $\mathcal{M}$, $\mathbf{v}[\gamma] = \mathbf{v}[\gamma_1] = \mathbf{v}[\gamma_2]$, $\sigma(\gamma) = \sigma(\gamma_1) = \sigma(\gamma_2)$, and $\gamma \lhd \gamma_1$, $\gamma \lhd \gamma_2$.*

(2) *Suppose $\mathbf{v}[\gamma_1] = -\mathbf{v}[\gamma_2]$. Then γ is constant if and only if $\gamma_1 \sim \gamma_2$. If $\gamma_1 \rhd \gamma_2$ (resp., $\gamma_1 \lhd \gamma_2$), then $\gamma \in \mathcal{M}$, $\mathbf{v}[\gamma] = \mathbf{v}[\gamma_2]$ (resp., $\mathbf{v}[\gamma] = \mathbf{v}[\gamma_1]$), $\sigma(\gamma) = \sigma(\gamma_1) = \sigma(\gamma_2)$, and $\gamma \rhd \gamma_2$ (resp., $\gamma \rhd \gamma_1$).*

Proof. With no loss of generality, assume that $a_1 = a_2 = 0$, $\gamma_1(0) = \gamma_2(0) = 0$, $\mathbf{v}[\gamma_1] = (1, 0)$, and $\sigma(\gamma_1) = \sigma(\gamma_2) = +$. There are two possibilities for $\mathbf{v}[\gamma_2]$: $(1, 0)$ and $(-1, 0)$. We can assume $\gamma_1(t) = (t, f_1(t))$, $\gamma_2(t) = (\pm t, \pm f_2(t))$ ($\pm$ depending on the direction of $\mathbf{v}[\gamma_2]$) for small $t \geq 0$, where f_i is a C^1 function such that $f_i(0) = f'(0) = 0$ and f_i' is strictly increasing for $i = 1, 2$. Since either $\gamma_1 \rhd \gamma_2$, $\gamma_1 \sim \gamma_2$, or $\gamma_1 \lhd \gamma_2$, we can assume that either $f_1(t) > f_2(t)$, $f_1(t) = f_2(t)$, or $f_1(t) < f_2(t)$ for every small $t > 0$.

Consider first the case when $\mathbf{v}[\gamma_2] = (1, 0)$. By Lemma 2.1, it is clear that $\gamma \in \mathcal{M}$, $\mathbf{v}[\gamma] = (1, 0)$ and $\sigma(\gamma) = +$. So we can write $\gamma(t) = (t, f(t))$ for small $t \geq 0$, where f is a C^1 function such that $f(0) = f'(0) = 0$ and f' is strictly increasing for small t. Since γ is in $\mathcal{M}$, we can see that, for $i = 1, 2$, $f(t)$ is either greater than, equal to, or less than $f_i(t)$ for every small $t > 0$.

Now, for any small $t > 0$, we can take small $t_1, t_2 > 0$ such that $t = t_1 + t_2$, $f_1'(t_1) = f'(t_2)$, and $f(t) = f_1(t_1) + f_2(t_2)$. By Lemma 2.1, $f'(t) = f_1'(t_1) = f'(t_2)$. Since $t > t_1, t_2$ and f_1', f_2' are strictly increasing, we have $f_1'(t), f_2'(t) > f'(t)$. Thus $f_i(t) > f(t)$, $i = 1, 2$, for every small $t > 0$, which implies that $\gamma \lhd \gamma_1$ and $\gamma \lhd \gamma_2$. This shows (1).

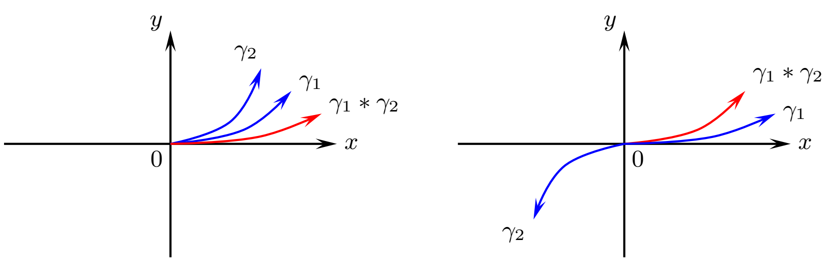


Fig. 5. Convolutions of $\mathcal{M}$-curves

Now consider the case when $\mathbf{v}[\gamma_2] = (-1, 0)$. Obviously, γ is constant if and only if $\gamma_1 \sim \gamma_2$. So assume $\gamma_1 \not\sim \gamma_2$. Then either $\gamma_1 \rhd \gamma_2$ or $\gamma_1 \lhd \gamma_2$. Suppose $\gamma_1 \lhd \gamma_2$. By Lemma 2.1, either $\mathbf{v}[\gamma] = (1, 0)$ or $\mathbf{v}[\gamma] = (-1, 0)$. If $\mathbf{v}[\gamma] = (-1, 0)$, then we must have $f_1'(t) > f_2'(t)$ for small $t > 0$, since f_1', f_2' are strictly increasing. It follows that $f_1(t) > f_2(t)$ for sufficiently small $t > 0$, which contradicts the assumption that $\gamma_1 \lhd \gamma_2$. So $\mathbf{v}[\gamma] = (1, 0)$. Since $\gamma \in \mathcal{M}$ and $\sigma(\gamma) = +$, we can assume $\gamma(t) = (t, f(t))$ for small $t \geq 0$, where f is a

C^1 function such that $f(0) = f'(0) = 0$, and f' is strictly increasing for small $t > 0$. Now for any small $t > 0$, we can take small $t_1, t_2 > 0$ such that $t = t_1 - t_2$, $f_1'(t_1) = f'(t_2)$, and $f(t) = f_1(t_1) - f_2(t_2)$. By Lemma 2.1, $f'(t) = f_1'(t_1) = f'(t_2)$. Since $t < t_1$ and f_1' is strictly increasing, we have $f_1'(t) < f'(t)$, and thus $f_1(t) < f(t)$ for small $t > 0$. This implies that $\gamma \rhd \gamma_1$. By a symmetric argument, we can also show that $\mathbf{v}[\gamma] = \mathbf{v}[\gamma_2]$ and $\gamma \rhd \gamma_2$, when $\gamma_1 \rhd \gamma_2$. Thus we have shown (2). ■

3. The class $\mathcal{W}$

In this section, we present an important example of a Minkowski class, called $\mathcal{W}$, which is large enough to contain practically all the important curves such as the NURBS curves. We will need the following proposition which is part of Łojasiewicz's Structure Theorem for real-analytic varieties ([10], [12]).

Proposition 1 (S. Łojasiewicz). *Let $\Phi : U \to \mathbb{R}$ be a real-analytic function on an open set $U \ni 0$ in $\mathbb{R}^n$, $n \geq 1$, and let $Z = \{(x_1, \ldots, x_n) \in U \mid \Phi(x_1, \ldots, x_n) = 0\}$. Then there exist $T \in SO(n, \mathbb{R})$ and an open set $N \ni 0$ such that the set $Z \cap N$ can be decomposed as*

$$Z \cap N = V^0 \cup \ldots \cup V^{n-1},$$

where each V^k can be decomposed again as

$$V^k = \bigcup_{i=1}^{p_k} \Gamma_i^k$$

for some $0 \leq p_k < \infty$. Here, each Γ_i^0 is a point, and for each Γ_i^k with $k \geq 1$, there exist a connected open set $U_i^k \in \mathbb{R}^k$ and real-analytic functions $\xi_{i,k+1}^k, \ldots \xi_{i,n}^k$ on U_i^k such that

$$\Gamma_i^k = T\cdot\{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid (x_1, \ldots, x_k) \in U_i^k, x_j = \xi_{i,j}^k(x_1, \ldots, x_k) \text{ for } j = k+1, \ldots, n\}.$$

In fact, what we really need is the following consequence of the above proposition.

Corollary 1. *Let $\Phi : U \to \mathbb{R}$ be a real-analytic function on an open set $U \ni 0$ in $\mathbb{R}^n$, $n \geq 1$, and let $Z = \{(x_1, \ldots, x_n) \in U \mid \Phi(x_1, \ldots, x_n) = 0\}$. Then there exists an open neighborhood N of 0 in $\mathbb{R}^2$ such that the set $Z \cap N$ has a finite number of connected components.*

By using the above result, we first see how convolution behaves in the class $\mathcal{C}_c^{\omega:\omega}$. Here, we define $\mathbf{v} \times \mathbf{w} = v_1 w_2 - v_2 w_1$ for $\mathbf{v} = (v_1, v_2), \mathbf{w} = (w_1, w_2) \in \mathbb{R}^2$. Note that $\mathbf{v} \mathbin{/\!/} \mathbf{w}$ if and only if $\mathbf{v} \times \mathbf{w} = 0$.

Lemma 3.1. *Let $\gamma_i : [a_i, b_i] \to \mathbb{R}^2$, $i = 1, \ldots, n$, be $\mathcal{C}_c^{\omega:\omega}$-curves which are $*$-admissible to each other. Then, for some $\varepsilon_1, \ldots, \varepsilon_n > 0$, $\gamma = \gamma_1|_{[a_1, a_1+\varepsilon_1]} * \ldots * \gamma_n|_{[a_n, a_n+\varepsilon_n]}$ is either constant, or is a $\mathcal{C}_c^{\omega:1}$-curve which is $*$-admissible to each γ_i.*

Proof. We assume $a_1 = \ldots = a_n = 0$, $\gamma_1(0) = \ldots = \gamma_n(0) = 0$, $\sigma(\gamma_1) = \ldots = \sigma(\gamma_n) = +$, and $\mathbf{v}[\gamma_1] = (1, 0)$. For each i, let $\widetilde{\theta}_i$ be the angle function of γ_i such that $\widetilde{\theta}_i(0) = 0$ or π, and define $\theta_i : [0, b_i] \to \mathbb{R}$ by $\theta_i = \widetilde{\theta}_i$ if $\mathbf{v}[\gamma_i] = (1, 0)$, and $\theta_i = \widetilde{\theta}_i - \pi$ if $\mathbf{v}[\gamma_i] = (-1, 0)$. Then θ_i is strictly increasing and $\theta_i(0) = 0$ for every i. Take small $0 < \varepsilon_i \leq b_i$ for each i such that $\theta_1(\varepsilon_1) = \ldots = \theta_n(\varepsilon_n)$. Let $\alpha = \theta_1(\varepsilon_1)$. Since γ_i's are in $\mathcal{C}_c^{\omega:\omega}$, we view each γ_i

as defined and real-analytic on $(-\delta, \varepsilon_i]$ for some $\delta > 0$. We can also assume that each γ_i is unit-speed.

Let $U = (-\delta, \varepsilon_1) \times \ldots \times (-\delta, \varepsilon_n) \subset \mathbb{R}^n$. Then $F : U \to \mathbb{R}$ and $G : U \to \mathbb{R}^2$ defined by

$$F(x_1, \ldots, x_n) = \sum_{j \neq k}^{n} |\gamma_j'(x_j) \times \gamma_k'(x_k)|^2, \qquad G(x_1, \ldots, x_n) = \sum_{i=1}^{n} \Big(\prod_{p \neq i}^{n} \kappa_p(x_p) \Big) \gamma_i'(x_i)$$

are real-analytic on U. Here, for each i, $\kappa_i : (-\delta, \varepsilon_i] \to \mathbb{R}$ is the curvature function of γ_i, i.e., $\kappa_i(x_i) = \gamma_i'(x_i) \times \gamma_i''(x_i)$. Let Z_F be the zero set of F in U. Let $Q = [0, \varepsilon_1) \times \ldots \times [0, \varepsilon_n)$. Then it is easy to see that $Z_F \cap Q = \{\zeta(t) \mid t \in [0, \alpha)\}$, where the one-to-one map $\zeta : [0, \alpha] \to \mathbb{R}^n$ is defined by $\zeta(t) = (\theta_1^{-1}(t), \ldots, \theta_n^{-1}(t))$.

Note that $\kappa_i(x_i) = \theta_i'(x_i)$ for $x_i \in [0, \varepsilon_i]$ for each i. So $(\theta_i^{-1})'(t) = 1/\kappa_i(\theta_i^{-1}(t))$ for $t \in [0, \alpha]$ for every i. Since γ_i is real-analytic on $(-\delta, \varepsilon_i]$, we can take ε_i small enough so that κ_i does not vanish on $(0, \varepsilon_i]$. So θ_i^{-1} is real-analytic on $(0, \alpha]$ for each i, and hence ζ is real-analytic on $(0, \alpha]$. Note that $\gamma(t) = \gamma_1(\theta_1^{-1}(t)) + \ldots + \gamma_n(\theta_n^{-1}(t))$ for $t \in [0, \alpha]$. So γ is also real-analytic on $(0, \alpha]$. Now

$$\gamma'(t) = \sum_{i=1}^{n} \gamma_i'(\theta_i^{-1}(t)) \, \frac{1}{\kappa_i(\theta_i^{-1}(t))} = \frac{1}{\prod_{i=1}^{n} \kappa_i(\theta_i^{-1}(t))} \, G(\zeta(t)).$$

Note that $|G \circ \zeta|^2$ is a real-analytic function on $(0, \alpha]$. If $|G \circ \zeta|^2 \equiv 0$ on $(0, \alpha]$, then γ is constant. Suppose $|G \circ \zeta|^2 \not\equiv 0$ on $(0, \alpha]$. Let $S = \{t \in (0, \alpha] \mid |G \circ \zeta|^2(t) = 0\}$. Suppose S has infinitely many elements. Since $|G \circ \zeta|^2$ is real-analytic on $(0, \alpha]$, there exists a sequence $t_k \searrow 0$ in $(0, \alpha)$ such that $S = \{t_k \mid k = 1, 2, \ldots\}$. Define the real-analytic function $\varPhi$ by $\varPhi = F + |G|^2$ on U. Let $Z_\varPhi$ be the zero set of $\varPhi$ in U. By Corollary 1, there exists an open connected neighborhood N of 0 in U such that $Z_\varPhi \cap N$ has a finite number of connected components. Let $\mathbf{x}_k = \zeta(t_k)$ for $k = 1, 2, \ldots$ Since $t_k \searrow 0$ and $\zeta(0) = 0$, infinitely many $\mathbf{x}_k$'s are in N. Denote them again by $\mathbf{x}_k$, $k = 1, 2, \ldots$ Then $Z_\varPhi \cap N \cap Q = \{\mathbf{x}_k \mid k = 1, 2, \ldots\} \cup \{0\}$. This means that $Z_\varPhi \cap N$ has infinitely many isolated points, which contradicts Corollary 1. Thus S is finite. Now we can take ε_i's small enough again such that $\gamma'(t)$ never vanishes on $(0, \alpha]$. So γ on $(0, \alpha]$ is a C^ω curve. Note that $\gamma'(t) \mathbin{/\!\!/} \gamma_i'(\theta_i^{-1}(t))$ for every $t \in (0, \alpha]$ and $i = 1, \ldots, n$. So γ is convex, C^1 on $[0, \alpha]$, $\mathbf{v}[\gamma] \mathbin{/\!\!/} (1, 0)$, and $\sigma(\gamma) = +$. We can take ε_i's smaller still so that γ is one-to-one. Thus we have proved that γ is a $\mathcal{C}_c^{\omega:1}$-curve $*$-admissible to each γ_i. ∎

We have seen that convolutions of any $\mathcal{C}_c^{\omega:\omega}$-curves belong to $\mathcal{C}_c^{\omega:1}$. In fact, this is the best we can get. A convolution of $\mathcal{C}_c^{\omega:\omega}$-curves may not even be a $\mathcal{C}_c^{\omega:2}$-curve, which can be seen from the following example:

Example 3.1. Let

$$\gamma_1(t) = \left(t, \frac{1}{2}t^2\right), \quad t \in [0, 1], \qquad \gamma_2(\theta) = (-\sin\theta, \cos\theta), \quad \theta \in [0, \pi/4].$$

Then $\gamma_1, \gamma_2 \in \mathcal{C}_c^{\omega:\omega}$. It is easy to see that, with some reparametrization,

$$\gamma(\theta) = \left(\tan\theta - \sin\theta, \frac{1}{2}\tan^2\theta + \cos\theta \right), \qquad \theta \in [0, \pi/4],$$

where $\gamma = \gamma_1 * \gamma_2$. From this, we can show that

$$\lim_{\theta \searrow 0} \frac{|\gamma'(\theta) \times \gamma''(\theta)|}{|\gamma'(\theta)|^3} = \infty.$$

So the curvature of γ blows up at $\theta = 0$, which is impossible for a $\mathcal{C}_c^{\omega:2}$-curve. Thus $\gamma \notin \mathcal{C}_c^{\omega:2}$.

Note that Example 3.1 shows that the class $\mathcal{C}_c^{\omega:\omega}$ is not a Minkowski class.

Now we define the curve class $\mathcal{W}$, which is an example of a Minkowski class.

Definition 3.1 (the class $\mathcal{W}$). $\mathcal{W}$ is the set of all straight line segments and all $\mathcal{C}_c^{\omega:1}$-curves which are of the form $\gamma_1 * \ldots * \gamma_n$ for some $\gamma_1, \ldots, \gamma_n$ in $\mathcal{C}_c^{\omega:\omega}$, $n \geq 1$.

As an easy consequence of Lemma 3.1, we have the following fact:

Theorem 3.1. *$\mathcal{W}$ is a Minkowski class.*

Proof. First, it is obvious that $\mathcal{W}$ satisfies condition (1) in Definition 2.5. Let $\gamma_1, \gamma_2 \in \mathcal{W}$ be $*$-admissible to each other. Then $\gamma_1 = \alpha_1 * \ldots * \alpha_m$ and $\gamma_2 = \beta_1 * \ldots * \beta_n$ for some $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n \in \mathcal{C}_c^{\omega:\omega}$. By the definition of convolution, $\gamma_1 * \gamma_2 = \alpha_1 * \ldots * \alpha_m * \beta_1 * \ldots * \beta_n$. Now from Lemma 3.1, condition (2) in Definition 2.5 is satisfied. ∎

Note that $\mathcal{W}$ is the smallest Minkowski class containing $\mathcal{C}_c^{\omega:\omega}$. Now we explore the relations of $\mathcal{W}$ with other curve classes. Note first that $\mathcal{C}_c^{\omega:\omega} \subset \mathcal{W} \subset \mathcal{C}_c^{\omega:1}$ by definition. Example 2.1 and Lemma 2.2 show that $\mathcal{W} \neq \mathcal{C}_c^{\omega:1}$. Example 3.1 shows $\mathcal{W} \neq \mathcal{C}_c^{\omega:\omega}$. So $\mathcal{C}_c^{\omega:\omega} \subsetneq \mathcal{W} \subsetneq \mathcal{C}_c^{\omega:1}$. Examples 2.1 and 3.1 also show respectively that $\mathcal{C}_c^{\omega:\infty} \not\subset \mathcal{W}$ and $\mathcal{W} \not\subset \mathcal{C}_c^{\omega:2}$. Moreover, Example 3.2 below shows that $\mathcal{W} \cap (\mathcal{C}_c^{\omega:n} \setminus \mathcal{C}_c^{\omega:n+1}) \neq \emptyset$ for every $1 \leq n < \infty$. Combining all these, Figure 6 shows the inclusion relations between $\mathcal{W}$ and other curve classes.

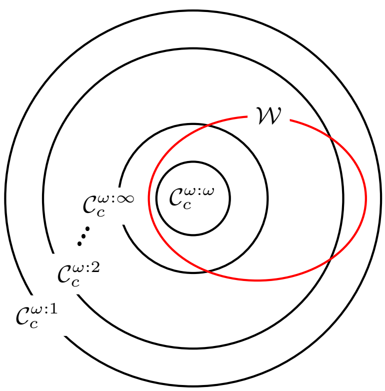


Fig. 6. Inclusion relations for $\mathcal{W}$

Example 3.2. Let $n \geq 1$ be an integer. For some small $0 < T < 1$, let $\gamma_1(t) = (t, f(x))$, $\gamma_2(t) = (-t, -g(t))$ for $t \in [0, T]$, where

$$f(t) = \int_0^t (\tau - \tau^2)^{2n}\, d\tau, \qquad g(t) = \frac{1}{2n+1}\, t^{2n+1}.$$

Clearly, $\gamma_1, \gamma_2 \in \mathcal{C}_c^{\omega:\omega}$. Putting $f'(t) = g'(s)$, we have $s = t - t^2$. So, with reparametrization, we have

$$\gamma(t) = \gamma_1(t) + \gamma_2(s) = \left(t^2, \int_0^t (\tau - \tau^2)^{2n}\, d\tau - \frac{1}{2n+1}\,(t - t^2)^{2n+1}\right),$$

where $\gamma = \gamma_1 * \gamma_2$. Let

$$F(t) = \int_0^t (\tau - \tau^2)^{2n}\, d\tau - \frac{1}{2n+1}\,(t - t^2)^{2n+1}.$$

By Lemma 3.1, we know that γ is in $\mathcal{C}_c^{\omega:1}$. Note that, for $1 \le k < \infty$, γ is in $\mathcal{C}_c^{\omega:k}$ for $k = 1, 2, \ldots$ if and only if the limit $\lim_{t \searrow 0} d^k F/du^k$ exists, where $u = t^2$. Now

$$F = \int_0^{\sqrt{u}} (\tau - \tau^2)^{2n}\, d\tau - \frac{1}{2n+1}\,(\sqrt{u} - u)^{2n+1}.$$

So

$$\begin{aligned}\frac{dF}{du} &= (\sqrt{u} - u)^{2n} - (\sqrt{u} - u)^{2n}\left(\frac{1}{2\sqrt{u}} - 1\right) = (t - t^2)^{2n} - (t - t^2)^{2n}\left(\frac{1}{2t} - 1\right)\\ &= -\frac{1}{2}\,t^{2n-1} + \text{higher order terms in } t.\end{aligned}$$

Note that $dt^m/du = \frac{1}{2} m t^{m-2}$ for every integer m. So, for each $k = 1, 2, \ldots$, we have

$$\frac{d^k F}{du^k} = a_k t^{2n+1-2k} + \text{higher order terms in } t,$$

for some $a_k \neq 0$. It follows that $\lim_{t \searrow 0} d^n F/du^n = 0$ and $\lim_{t \searrow 0} d^{n+1} F/du^{n+1} = -\infty$. This shows that $\gamma \in \mathcal{C}_c^{\omega:n} \setminus \mathcal{C}_c^{\omega:n+1}$.

4. Sectors and domains

We will now define the exact meaning of the word *domain* used in this paper. With our definition, the domains can be of fairly general shape. For example, ones consisting only of curve segments, which cannot be regarded as domains in the conventional sense, are also included. Our analysis of domains and their Minkowski sums will be based on the global integration of various local results. The *sector* introduced below is a basic local object we will use.

Let $\mathcal{C}$ be a class of curves in $\mathcal{C}^{1:1}$. We say that $\mathcal{C}$ is *closed under restriction* if, for every $\gamma : [a, b] \to \mathbb{R}^2$ in $\mathcal{C}$, $\gamma|_{[c,d]}$ is also in $\mathcal{C}$ for every $[c, d] \subset [a, b]$. We will only consider the curve classes which are closed under restriction. Note that $\mathcal{C}^{k:l}$, $\mathcal{C}_c^{k:l}$ for $k, l = 1, 2, \ldots, \infty, \omega$, $k \ge l$ and every Minkowski class satisfy this condition.

Definition 4.1 (sector). Let $\mathcal{C}$ be a class of curves in $\mathcal{C}^{1:1}$ which is closed under restriction. A closed set S in $\mathbb{R}^2$ is called a *$\mathcal{C}$-sector* with center $p \in \mathbb{R}^2$ and radius $r > 0$ if S is bounded by three continuous curves α, β, and γ which satisfy the following conditions:

(1) $\alpha : [a_1, a_2] \to B_r(p)$ and $\beta : [b_1, b_2] \to B_r(p)$ are $\mathcal{C}$-curves such that $\alpha(a_1) = \beta(b_1) = p$.

(2) The functions $\varrho_\alpha : [a_1, a_2] \to [0, r]$ and $\varrho_\beta : [b_1, b_2] \to [0, r]$ defined by $\varrho_\alpha(t) = |\alpha(t) - p|$ and $\varrho_\beta(t) = |\beta(t) - p|$ are homeomorphisms.

(3) Either $\alpha([a_1, a_2]) = \beta([b_1, b_2])$, or α and β have no intersections except at p.

(4) γ traverses $\partial B_r(p)$ from $\alpha(a_2)$ to $\beta(b_2)$ in the counter-clockwise direction.

Here, if $\alpha(a_2) = \beta(b_2)$ (or equivalently, if $\alpha([a_1, a_2]) = \beta([b_1, b_2])$), then γ is constant just at the point $\alpha(a_2) = \beta(b_2)$, and S is just the set of all points on the curve α (or equivalently, β). The two curves β and α are called the *start curve* and the *end curve* of S respectively. The *cone* $\mathrm{C}(S)$ of S is defined as

$$\mathrm{C}(S) = \{\mathbf{v} \in S^1 \mid \exists \gamma \in \mathcal{C}^{1:1} : [0,1] \to S \text{ such that } \gamma(0) = p,\ \gamma'(0) = \mathbf{v}\}.$$

S is called *sharp* (resp., *dull*, *flat*) if the center angle of $\mathrm{C}(S)$ is less than π (resp., greater than π, equal to π). If $\alpha([a_1, a_2]) = \beta([b_1, b_2])$, then we call S *degenerate*; otherwise it is *non-degenerate*.

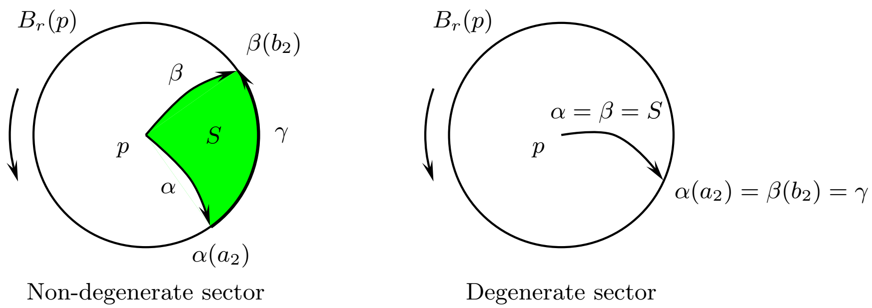


Fig. 7. Sector

Let S_1 and S_2 be two $\mathcal{C}^{1:1}$-sectors with center p and radius r. Then S_1 and S_2 are said to be *non-overlapping* if $S_1 \cap S_2 = \{p\}$. We list some elementary properties of sectors, which follow immediately from Definition 4.1 and Lemma 2.2.

LEMMA 4.1. (1) *Let $\mathcal{C}$ be a class of curves in $\mathcal{C}^{1:1}$ which is closed under restriction, and let S be a $\mathcal{C}$-sector with center p and radius r. Then $B_{r'}(p) \cap S$ is a $\mathcal{C}$-sector with center p and radius r' for every $0 < r' \le r$.*

(2) *Let $\mathcal{M}$ be a Minkowski class, and let S_1 and S_2 be two $\mathcal{M}$-sectors with center p and radius r. Then there exists $0 < r' \le r$ such that, for every $0 < \varrho \le r'$, the set $B_\varrho(p) \cap (S_1 \cup S_2)$ is either $B_\varrho(p)$, or an $\mathcal{M}$-sector with center p and radius ϱ, or a union of two non-overlapping $\mathcal{M}$-sectors with center p and radius ϱ.*

Proof. (1) is obvious, and (2) is immediate from Lemma 2.2. ■

Now we define the domains:

DEFINITION 4.2 (domain). Let $\mathcal{C}$ be a class of curves in $\mathcal{C}^{1:1}$. A subset Ω of $\mathbb{R}^2$ is called a *$\mathcal{C}$-domain* if it satisfies the following conditions:

(1) Ω is connected and compact.

(2) $\partial\Omega$ is a union of a finite number of $\mathcal{C}$-curves, no two of which meet at infinitely many points.

Note that, in view of this definition, the Minkowski sum in Figure 3 is not a $\mathcal{C}^{\omega:\infty}$-domain, though its boundary consists of finitely many $\mathcal{C}^{\omega:\infty}$-curves. In fact, it is not even a $\mathcal{C}^{1:1}$-domain. But the domains in our definition can be of fairly general shape such as the one in Figure 8.

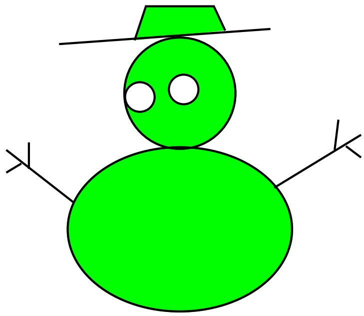


Fig. 8. Example of a "domain" with general shape

Remark 4.1. If $\mathcal{C}$ is $\mathcal{C}^{\omega:\omega}$, $\mathcal{C}_c^{\omega:\omega}$, or a Minkowski class ($\mathcal{W}$, for example), then condition (2) in Definition 4.2 can be omitted.

Now we start to use the local object sector to describe global properties.

Lemma 4.2 (local condition for domain). *Let Ω be a connected compact set in $\mathbb{R}^2$, and let $\mathcal{C}$ be a class of curves in $\mathcal{C}^{1:1}$ which is closed under restriction. Then the following two conditions are equivalent*:

(1) *Ω is a $\mathcal{C}$-domain.*

(2) *For every point p in $\partial\Omega$, there exists $r > 0$ such that $B_r(p) \cap \Omega$ is a union of a finite number of mutually non-overlapping $\mathcal{C}$-sectors with center p and radius r.*

Proof. Suppose Ω is a $\mathcal{C}$-domain. Let $p \in \partial\Omega$. Since $\mathcal{C} \subset \mathcal{C}^{1:1}$, it is easy to see from Definition 4.2(2) that there exist $r > 0$ and $\mathcal{C}$-curves $\gamma_i : [0, a_i] \to B_r(p)$ for $i = 1, \ldots, n$ for some $1 \le n < \infty$ such that $B_r(p) \cap \partial\Omega = \bigcup_{i=1}^n \gamma_i([0, a_i])$ and the function ϱ_i : $[0, a_i] \to [0, r]$ defined by $\varrho_i(t) = |\gamma_i(t) - p|$ is a homeomorphism with $\varrho_i(0) = 0$ for each i. Again by Definition 4.2(2), we can assume γ_i and γ_j do not meet except at p for every $1 \le i \ne j \le n$. Now it is clear that $B_r(p) \cap \Omega$ is a union of a finite number of mutually non-overlapping $\mathcal{C}$-sectors with center p and radius r. Thus (1) implies (2).

Conversely, suppose (2). Then, for every $p \in \partial\Omega$, we can choose $r(p) > 0$ such that $B_{r(p)}(p) \cap \Omega$ is a finite union of mutually non-overlapping $\mathcal{C}$-sectors with center p and radius r, and $B_{r(p)}(p) \cap \partial\Omega$ is a union of a finite number of $\mathcal{C}$-curves, each pair of which have no intersections except at p. Note that $\{B^o_{r(p)}(p) \cap \partial\Omega \mid p \in \partial\Omega\}$ is an open cover of the compact set $\partial\Omega$. So there exist a finite number of points $p_1, \ldots, p_n \in \partial\Omega$ such that $\partial\Omega = \bigcup_{i=1}^n B^o_{r(p_i)}(p_i) \cap \partial\Omega$. Thus $\partial\Omega = \bigcup_{i=1}^n B_{r(p_i)}(p_i) \cap \partial\Omega$ is a union of a finite number of $\mathcal{C}$-curves. From the definition of sector, no pair of these $\mathcal{C}$-curves has infinitely many isolated intersections. Thus Ω is a $\mathcal{C}$-domain. ∎

As can be seen from Definition 4.2, the domains can have quite general shapes. We give a special name for domains with some relatively good geometry.

DEFINITION 4.3 (regular domain). A $\mathcal{C}^{1:1}$-domain is called *regular* if each connected component of $\partial\Omega$ is homeomorphic to S^1, and is not itself a connected component of Ω.

So, the snowman in Figure 8 is not a regular domain. Also, the Minkowski sum in Figure 2 is a $\mathcal{C}^{\omega:\omega}$-domain, but not a regular $\mathcal{C}^{\omega:\omega}$-domain. Note that, for any $\mathcal{C} \subset \mathcal{C}^{1:1}$, the number of connected components of $\partial\Omega$ is finite for a $\mathcal{C}$-domain Ω.

LEMMA 4.3 (local condition for regular domain). *Let Ω be a connected compact set in $\mathbb{R}^2$, and let $\mathcal{C}$ be a class of curves in $\mathcal{C}^{1:1}$ which is closed under restriction. Then the following two conditions are equivalent*:

(1) *Ω is a regular $\mathcal{C}$-domain.*

(2) *For every point p in $\partial\Omega$, there exists $r > 0$ such that $B_r(p) \cap \Omega$ is a non-degenerate $\mathcal{C}$-sector with center p and radius r.*

Proof. Suppose Ω is a regular $\mathcal{C}$-domain. Let $p \in \partial\Omega$. Since $\mathcal{C} \subset \mathcal{C}^{1:1}$, there exists $r > 0$ such that $B_r(p) \cap \partial\Omega$ is a union of two $\mathcal{C}$-curves $\gamma_i : [0, a_i] \to B_r(p)$, $i = 1, 2$, such that $\gamma_1(0) = \gamma_2(0) = p$, γ_1 and γ_2 do not meet except at p, and the function $\varrho_i : [0, a_i] \to [0, r]$ defined by $\varrho_i(t) = |\gamma_i(t) - p|$ is a homeomorphism for $i = 1, 2$. Note that $B_r(p) \cap \Omega \neq B_r(p)$, since $p \in \partial\Omega$. So $B_r(p) \cap \Omega$ is either a non-degenerate $\mathcal{C}$-sector with center p and radius r, or $B_r(p) \cap \Omega = \gamma_1([0, a_1]) \cup \gamma_2([0, a_2])$. Suppose the latter. Then the connected component of $\partial\Omega$ which contains $B_r(p) \cap \Omega$ is itself a connected component of Ω. So we conclude that $B_r(p) \cap \Omega$ is a non-degenerate $\mathcal{C}$-sector with center p and radius r. Thus (1) implies (2).

Conversely, suppose (2). Then $\partial\Omega$ is locally homeomorphic to $\mathbb{R}$ at every point in $\partial\Omega$, and $\partial\Omega$ is a disjoint union of a finite number of 1-dimensional (topological) manifolds embedded in $\mathbb{R}^2$. Since $\partial\Omega$ is bounded, each of these manifolds is homeomorphic to S^1. So $\partial\Omega$ is a disjoint union of a finite number of sets homeomorphic to S^1. Note that each of these sets consists of a finite number of $\mathcal{C}$-curves, since S^1 is compact. From the assumption, none of the connected components of $\partial\Omega$ is itself a connected component of Ω. Thus (2) implies (1). ∎

REMARK 4.2. A subset Ω of $\mathbb{R}^2$ is a regular $\mathcal{C}^{\omega:\omega}$-domain if and only if it satisfies the standing assumptions for domains in [2] and [3]. Note that a domain is a $\mathcal{C}^{\omega:\omega}$-domain if and only if it is a $\mathcal{C}_c^{\omega:\omega}$-domain, since a $\mathcal{C}^{\omega:\omega}$-curve can be cut into a finite number of $\mathcal{C}_c^{\omega:\omega}$-curves.

Finally, we introduce the following terminology.

DEFINITION 4.4 (sharp corner, dull corner and flat point). Let Ω be a regular $\mathcal{C}^{1:1}$-domain. Then a point $p \in \partial\Omega$ is called a *sharp corner* (resp., *dull corner*, *flat point*) if there exists $r > 0$ such that $B_r(p) \cap \Omega$ is a sharp sector (resp., dull sector, flat sector) with center p and radius r.

Note that the above properties are of a local nature of Ω around p, and thus are independent of the choice of r.

5. Virtual boundary

In this section, we introduce the concept of *virtual boundary* for regular domains. This will enable us to treat the regular domains in a more uniform manner, whether they have corners or not.

Let Ω be a *regular* $\mathcal{C}^{1:1}$-domain. By definition, each connected component of $\partial\Omega$ is homeomorphic to S^1. Among them, exactly one is the outer boundary, and the remaining ones are inner boundaries. To each of these components, we give the *standard orientation*, i.e., counter-clockwise orientation for the outer boundary, and clockwise orientation for the inner boundaries. Let C be a connected component of $\partial\Omega$. Fix an orientation-preserving covering map $h : \mathbb{R} \to C$. For any continuous curve $\gamma : [a,b] \to C$, there exists a lifting of γ to $\mathbb{R}$ with respect to h, i.e., a continuous function $\widetilde{\gamma} : [a,b] \to \mathbb{R}$ such that $\gamma(t) = h(\widetilde{\gamma}(t))$ for $t \in [a,b]$. We define

$$O_\Omega(\gamma) = \begin{cases} + & \text{if } \widetilde{\gamma}(b) - \widetilde{\gamma}(a) > 0, \\ 0 & \text{if } \widetilde{\gamma}(b) - \widetilde{\gamma}(a) = 0, \\ - & \text{if } \widetilde{\gamma}(b) - \widetilde{\gamma}(a) < 0. \end{cases}$$

Note that this definition is independent of the choice of h. We say that γ is *in the standard orientation* on Ω if $O_\Omega(\gamma)$ is $+$.

Definition 5.1 (normal cone). Let Ω be a regular $\mathcal{C}^{1:1}$-domain, and let $p \in \partial\Omega$. Let $\gamma_+, \gamma_- : [0,\varepsilon] \to \partial\Omega$ be one-to-one $\mathcal{C}^{1:1}$-curves such that $\gamma_+(0) = \gamma_-(0) = p$ and $O_\Omega(\gamma_\pm) = \pm$. Then the *normal cone* of Ω at p, denoted by $\mathrm{NC}_\Omega(p)$, is defined as follows:

(1) If p is a sharp corner, then $\mathrm{NC}_\Omega(p) = \{\mathbf{n} \in S^1 \mid \mathbf{n}\cdot\mathbf{v}[\gamma_+] \le 0 \text{ and } \mathbf{n}\cdot\mathbf{v}[\gamma_-] \le 0\}$.

(2) If p is a dull corner, then $\mathrm{NC}_\Omega(p) = \{\mathbf{n} \in S^1 \mid \mathbf{n}\cdot\mathbf{v}[\gamma_+] \ge 0 \text{ and } \mathbf{n}\cdot\mathbf{v}[\gamma_-] \ge 0\}$.

(3) If p is a flat point, then $\mathrm{NC}_\Omega(p)$ consists of the (unit) vector obtained by rotating $\mathbf{v}[\gamma_+]$ clockwise through $90°$.

We set $\mathbf{v}^+_\Omega(p) = \mathbf{v}[\gamma_+]$ and $\mathbf{v}^-_\Omega(p) = -\mathbf{v}[\gamma_-]$. Note that these are independent of the choice of $\gamma_\pm$. Also, $\mathbf{v}^+_\Omega(p) = \mathbf{v}^-_\Omega(p)$ if and only if p is a flat point of Ω. In this case, we write $\mathbf{v}_\Omega(p) = \mathbf{v}^+_\Omega(p) = \mathbf{v}^-_\Omega(p)$. We denote by $\mathbf{n}^+_\Omega(p)$ (resp., $\mathbf{n}^-_\Omega(p)$, $\mathbf{n}_\Omega(p)$) the vector obtained by rotating $\mathbf{v}^+_\Omega(p)$ (resp., $\mathbf{v}^-_\Omega(p)$, $\mathbf{v}_\Omega(p)$) clockwise through $90°$. Note that $\mathbf{n}^+_\Omega(p)$ and $\mathbf{n}^-_\Omega(p)$ are the two ends of $\mathrm{NC}_\Omega(p)$.

Definition 5.2 (virtual boundary). Let Ω be a regular $\mathcal{C}^{1:1}$-domain. Then the *virtual boundary* of Ω, denoted by $\partial^v\Omega$, is defined to be

$$\partial^v\Omega = \{(p,\mathbf{n}) \in \partial\Omega \times S^1 \mid \mathbf{n} \in \mathrm{NC}_\Omega(p)\}.$$

Let Ω be a regular $\mathcal{C}^{1:1}$-domain. Then $\partial^v\Omega$ consists of a finite number of connected components, each homeomorphic to S^1, and the connected components of $\partial^v\Omega$ are in one-to-one correspondence with those of $\partial\Omega$. Thus we can also give the *standard orientation* to each of the connected components of $\partial^v\Omega$ in an obvious way. Let $\widehat{C}$ be a connected component of $\partial^v\Omega$. Fix an orientation-preserving covering map $\widehat{h} : \mathbb{R} \to \widehat{C}$. For any continuous map $\phi : [a,b] \to \widehat{C}$, there exists a lifting of ϕ to $\mathbb{R}$ with respect to $\widehat{h}$, i.e., a continuous function $\widetilde{\phi} : [a,b] \to \mathbb{R}$ such that $\phi(t) = \widehat{h}(\widetilde{\phi}(t))$ for $t \in [a,b]$. We define

$$O_\Omega(\phi) = \begin{cases} + & \text{if } \widetilde{\phi}(b) - \widetilde{\phi}(a) > 0, \\ 0 & \text{if } \widetilde{\phi}(b) - \widetilde{\phi}(a) = 0, \\ - & \text{if } \widetilde{\phi}(b) - \widetilde{\phi}(a) < 0. \end{cases}$$

The definition is independent of the choice of $\widehat{h}$. We will say that ϕ is *in the standard orientation* on Ω if $O_\Omega(\phi)$ is $+$.

Let $\phi : [a,b] \to \mathbb{R}^2 \times S^1$, $\phi(t) = (\gamma(t), \mathbf{n}(t))$, be a continuous map. Then there exists a continuous function $\theta : [a,b] \to \mathbb{R}$ such that $\mathbf{n}(t) = \mu(\theta(t))$, where $\mu(s) = (\cos s, \sin s)$ for $s \in \mathbb{R}$. We call θ an *angle function* of ϕ. We define $\Theta(\phi)$, the *total angle* of ϕ, by

$$\Theta(\phi) = \theta(b) - \theta(a).$$

Note that the total angle is independent of the choice of angle functions.

We use the following notations throughout this paper: Let X be a topological space. Let $\gamma : [a,b] \to X$ be a continuous curve. Then the curve $\overline{\gamma} : [a,b] \to X$ is defined by

$$\overline{\gamma}(t) = \gamma(a+b-t)$$

for $t \in [a,b]$. Let $\gamma_i : [a_i, b_i] \to X$, $i = 1,2$, be two continuous curves with $\gamma_1(b_1) = \gamma_2(a_2)$. Then the curve $\gamma = \gamma_1 \cdot \gamma_2 : [a_1, b_1 + b_2 - a_2] \to X$ is defined by

$$\gamma(t) = \begin{cases} \gamma_1(t) & \text{if } t \in [a_1, b_1], \\ \gamma_2(t - b_1 + a_2) & \text{if } t \in [b_1, b_1 + b_2 - a_2]. \end{cases}$$

We denote by $\mathrm{Ind}_\gamma(p)$ the *index* of $p \in \mathbb{R}^2$ with respect to a continuous closed curve $\gamma : [a,b] \to \mathbb{R}^2 \setminus \{p\}$ ($\gamma(a) = \gamma(b)$). It is well known that the index of a point takes integer values and remains the same if we vary the curve homotopically.

The following lemmas are easy consequences of the above definitions.

LEMMA 5.1. *Let Ω be a regular $\mathcal{C}^{1:1}$-domain, and let $\phi : [a,b] \to \partial^v \Omega$, $\phi_i : [a_i, b_i] \to \partial^v \Omega$, $i = 1,2$, be continuous maps such that $\phi_1(b_1) = \phi_2(a_2)$. Then*:

(1) $\Theta(\overline{\phi}) = -\Theta(\phi)$.

(2) $\Theta(\phi_1 \cdot \phi_2) = \Theta(\phi_1) + \Theta(\phi_2)$.

(3) *Suppose $\phi_0 : [a,b] \to \partial^v \Omega$ is a continuous map which is homotopic to ϕ in $\partial^v \Omega$ relative to $\phi(a)$ and $\phi(b)$, i.e., there exists a continuous map $H : [a,b] \times [0,1] \to \partial^v \Omega$ such that $H(t,0) = \phi_0(t)$, $H(t,1) = \phi(t)$ for $t \in [a,b]$, and $H(a,s) = \phi(a)$, $H(b,s) = \phi(b)$ for $s \in [0,1]$. Then $\Theta(\phi_0) = \Theta(\phi)$.*

Proof. (1), (2) are obvious from the definitions. For (3), let $H(t,s) = (\gamma(t,s), \mathbf{n}(t,s))$ for $(t,s) \in [a,b] \times [0,1]$. From the assumption, there exists a continuous map $\theta : [a,b] \times [0,1] \to \mathbb{R}$ such that $(\mu \circ \theta)(t,s) = \mathbf{n}(t,s)$, where $\mu : \mathbb{R} \to S^1$ is defined by $\mu(t) = (\cos t, \sin t)$ for $t \in \mathbb{R}$. Thus $\Theta(\phi_0) = \theta(b,0) - \theta(a,0) = \theta(b,1) - \theta(a,1) = \Theta(\phi)$, since $\mathbf{n}(a,s) = \mathbf{n}(a,0)$ and $\mathbf{n}(b,s) = \mathbf{n}(b,0)$ for every $s \in [0,1]$. ■

LEMMA 5.2. *Let Ω be a regular $\mathcal{C}^{1:1}$-domain, and let $p \in \mathrm{int}\,\Omega$. Let $\phi : [a,b] \to \partial^v \Omega$, $\phi(t) = (\gamma(t), \mathbf{n}(t))$, be a continuous map such that $\phi(a) = \phi(b)$ and $O_\Omega(\phi) = +$. Let C be the connected component of $\partial\Omega$ such that $\gamma([a,b]) \subset C$.*

(1) *If* $\phi|_{[a,b)}$ *is one-to-one, then*

$$\Theta(\phi) = \begin{cases} 2\pi & \text{if } C \text{ is the outer boundary of } \partial\Omega, \\ -2\pi & \text{if } C \text{ is an inner boundary of } \partial\Omega. \end{cases}$$

(2)

$$\mathrm{Ind}_\gamma(p) = \begin{cases} \frac{1}{2\pi}\Theta(\phi) & \text{if } C \text{ is the outer boundary,} \\ 0 & \text{if } C \text{ is an inner boundary.} \end{cases}$$

Proof. (1) This is an easy consequence of the Gauss–Bonnet theorem (see [17, Theorem 8.4]).

(2) It is obvious that $\mathrm{Ind}_\gamma(p) = 0$ if C is an inner boundary. Suppose C is the outer boundary, and let $\widehat{C}$ be the connected component of $\partial^v\Omega$ corresponding to C. Let $\phi_0 : [0,1] \to \widehat{C}$, $\phi_0(t) = (\gamma_0(t), \mathbf{n}_0(t))$, be a continuous map such that $\phi_0(0) = \phi_0(1) = \phi(a) = \phi(b)$, $\phi_0|_{[0,1)}$ is one-to-one, and $O_\Omega(\phi_0) = +$. By (1), we have $\Theta(\phi_0) = 2\pi$. It is easy to see that ϕ is homotopic to $\underbrace{\phi_0 \cdot \ldots \cdot \phi_0}_{\mathrm{Ind}_\gamma(p)}$ if $\mathrm{Ind}_\gamma(p) > 0$, to $\underbrace{\overline{\phi}_0 \cdot \ldots \cdot \overline{\phi}_0}_{-\mathrm{Ind}_\gamma(p)}$ if $\mathrm{Ind}_\gamma(p) < 0$, and to the constant map $\phi(a)$ $(= \phi(b))$ if $\mathrm{Ind}_\gamma(p) = 0$. Now the assertion follows from Lemma 5.1. ∎

Let A be a subset of $\mathbb{R}^2$ and $p \in \mathbb{R}$. Then we define

$$A + p = \{q + p \mid q \in A\} \quad \text{and} \quad -A = \{-q \mid q \in A\}.$$

For later reference, we collect the following elementary facts which can be easily deduced from the definitions.

LEMMA 5.3. *Let* Ω *be a regular* $\mathcal{C}^{1:1}$*-domain, and let* $q \in \mathbb{R}^2$. *Let* $p \in \partial\Omega$, *and let* $\phi : [a,b] \to \partial^v\Omega$, $\phi(t) = (\gamma(t), \mathbf{n}(t))$, *be a continuous map. Then*:

(1) $\mathbf{n}^{\pm}_{\Omega+q}(p+q) = \mathbf{n}^{\pm}_{\Omega}(p)$ *and* $\mathbf{v}^{\pm}_{\Omega+q}(p+q) = \mathbf{v}^{\pm}_{\Omega}(p)$.

(2) $\mathbf{n}^{\pm}_{-\Omega}(-p) = -\mathbf{n}^{\pm}_{\Omega}(p)$ *and* $\mathbf{v}^{\pm}_{-\Omega}(-p) = -\mathbf{v}^{\pm}_{\Omega}(p)$.

(3) $\Theta(\phi + q) = \Theta(\phi)$, *where* $\phi + q : [a,b] \to \partial^v(\Omega + q)$ *is defined by* $(\phi + q)(t) = (\gamma(t) + q, \mathbf{n}(t))$ *for* $t \in [a,b]$.

(4) $\Theta(-\phi) = \Theta(\phi)$, *where* $-\phi : [a,b] \to \partial^v(-\Omega)$ *is defined by* $(-\phi)(t) = (-\gamma(t), -\mathbf{n}(t))$ *for* $t \in [a,b]$.

Now we define the *angle of convexity* of a regular domain. This will be used in defining the *semi-convexity* of domains in Section 7.

DEFINITION 5.3 (angle of convexity). Let Ω be a regular $\mathcal{C}^{1:1}$-domain. The *angle of convexity* of Ω, denoted by $\Theta(\Omega)$, is defined by

$$\Theta(\Omega) = \inf\{\Theta(\phi) : \phi \in S\},$$

where S is the set of all continuous maps from a closed interval to $\partial^v\Omega$ such that $O_\Omega(\phi) = +$.

Finally, we introduce the notion of *contact position* which is important for analyzing Minkowski sum.

Definition 5.4 (contact position). Two regular $\mathcal{C}^{1:1}$-domains Ω_1 and Ω_2 are said to be *in contact position to each other* if they meet at their boundaries only, i.e., $\Omega_1 \cap \Omega_2 = \partial\Omega_1 \cap \partial\Omega_2 \neq \emptyset$.

Let Ω_1 and Ω_2 be two *simply connected* regular $\mathcal{C}^{1:1}$-domains which are in contact position to each other. Let U be the unbounded component of $\mathbb{R}^2 \setminus (\Omega_1 \cup \Omega_2)$. Suppose $p_1 \neq p_2$ are two points in $\partial\Omega_1 \cap \partial\Omega_2$. For $i = 1, 2$, let $\phi_i : [0,1] \to \partial^v \Omega_i$, $\phi_i(t) = (\gamma_i(t), \mathbf{n}_i(t))$, be one-to-one continuous maps such that $\phi_1(0) = (p_1, \mathbf{n}^+_{\Omega_1}(p_1))$, $\phi_1(1) = (p_2, \mathbf{n}^-_{\Omega_1}(p_2))$, $\phi_2(0) = (p_2, \mathbf{n}^+_{\Omega_2}(p_2))$, $\phi_2(1) = (p_1, \mathbf{n}^-_{\Omega_2}(p_1))$, and $O_{\Omega_1}(\phi_1) = O_{\Omega_2}(\phi_2) = +$. Note that, by interchanging p_1 and p_2 if necessary, we can assume $(\gamma_i([0,1]) \setminus \{p_1, p_2\}) \cap \overline{U} = \emptyset$ for $i = 1, 2$. Let α_1 (resp., α_2) be the non-negative angle of counter-clockwise rotation from $-\mathbf{v}^-_{\Omega_2}(p_1)$ to $\mathbf{v}^+_{\Omega_1}(p_1)$ (resp., from $-\mathbf{v}^-_{\Omega_1}(p_2)$ to $\mathbf{v}^+_{\Omega_2}(p_2)$). See Figure 9.

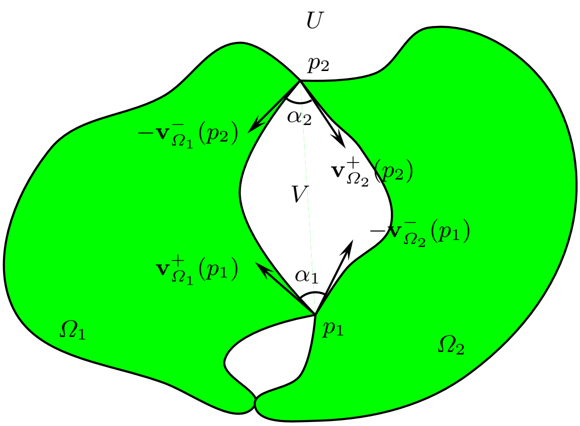


Fig. 9. Contact position

With the above notations, we have the following lemma:

Lemma 5.4. *Let Ω_1 and Ω_2 be simply connected regular $\mathcal{C}^{1:1}$-domains which are in contact position to each other. Suppose $p_1 \neq p_2 \in \partial\Omega_1 \cap \partial\Omega_2$ and α_i, ϕ_i for $i = 1, 2$ are as above. Then*:

(1) $\Theta(\phi_1) + \Theta(\phi_2) + \alpha_1 + \alpha_2 = 0$.

(2) *If $\Theta(\Omega_1), \Theta(\Omega_2) \geq -\Theta$ for some $\Theta \geq 0$, then $-\Theta \leq \Theta(\phi_i) \leq \Theta$ for $i = 1, 2$.*

(3) *There exists a continuous map $H : [0,1] \times [0,1] \to \overline{V}$ such that $H(t,0) = \gamma_1(t)$, $H(t,1) = \overline{\gamma}_2(t)$ for $t \in [0,1]$, and $H(0,s) = p_1$, $H(1,s) = p_2$ for $s \in [0,1]$, where V is the region in $\mathbb{R}^2 \setminus (\Omega_1 \cup \Omega_2)$ bounded by γ_1 and γ_2.*

Proof. (1) This is an easy consequence of Lemma 5.2.

(2) By (1), we have $\Theta(\phi_1) = -\Theta(\phi_2) - \alpha_1 - \alpha_2$. Since $\Theta(\Omega_1), \Theta(\Omega_2) \geq -\Theta$, we have $\Theta(\phi_1), \Theta(\phi_2) \geq -\Theta$. Note that $\alpha_1, \alpha_2 \geq 0$ by definition. Thus $\Theta(\phi_1) \leq \Theta$. We can also see that $\Theta(\phi_2) \leq \Theta$ in the same way.

(3) Obvious. See Figure 9. ∎

6. Minkowski sum of domains

Now we consider the Minkowski sum of domains. For reasonable results, we restrict our analysis to $\mathcal{M}$-domains, where $\mathcal{M}$ is a Minkowski class. After introducing the preliminary facts in Section 6.1, we analyze the behavior of the Minkowski sum of $\mathcal{M}$-sectors in Sections 6.2 and 6.3. Finally, by using these results, we show in Section 6.4 that the set of all $\mathcal{M}$-domains is closed under Minkowski sum for any Minkowski class $\mathcal{M}$.

6.1. *Preliminaries.* Let A and B be two subsets of $\mathbb{R}^2$. We define

$$A+B=\{p+q \mid p\in A,\, q\in B\},$$

and call it the *Minkowski sum* of A and B. The map $M_{A,B}: A\times B\to A+B$ defined by $M_{A,B}(p,q)=p+q$ for $p\in A$, $q\in B$ is called the *Minkowski map* associated to A and B. Note that $M_{A,B}$ is continuous for any $A,B\subset\mathbb{R}^2$. The following are easy consequences of the definition.

Lemma 6.1. *Let $A,B\subset\mathbb{R}^2$. Suppose $A=\bigcup_{i\in I}A_i$ and $B=\bigcup_{j\in J}B_j$. Then*

$$A+B=\bigcup_{i\in I,\, j\in J}(A_i+B_j).$$

Proof. $\supseteq$ is trivial. Suppose $p\in A+B$. Then there exist $p_1\in A$ and $p_2\in B$ such that $p=p_1+p_2$. So there exist $i\in I$ and $j\in J$ such that $p_1\in A_i$ and $p_2\in B_j$. This shows $\subseteq$. ■

Lemma 6.2. *Let $A,B\subset\mathbb{R}^2$, and let $p\in\partial(A+B)$. Then, for any $p_1\in A$ and $p_2\in B$ such that $p=p_1+p_2$, we have $p_1\in\partial A$ and $p_2\in\partial B$. Equivalently, we have*

$$M_{A,B}^{-1}(\partial(A+B))\subset\partial A\times\partial B.$$

Proof. Suppose $p_1\in\operatorname{int}A$. Then we can take a small ball $B_r(p_1)$ around p_1 such that $B_r(p_1)\subset A$. Clearly, $B_r(p)=B_r(p_1)+p_2\subset A+B$, and this implies that $p\in\operatorname{int}(A+B)$. This contradicts the assumption, and we conclude $p_1\in\partial A$. In the same way, we can show that $p_2\in\partial B$. ■

Lemma 6.3. *Let $\Omega_1,\Omega_2\subset\mathbb{R}^2$, and let $\Omega=\Omega_1+\Omega_2$. Let $p\in\mathbb{R}^2$. Then*:

(1) *$p\in\Omega$ if and only if $\Omega_1\cap(-\Omega_2+p)\neq\emptyset$.*

(2) *If $\operatorname{int}\Omega_1\cap(-\Omega_2+p)\neq\emptyset$, then $p\in\operatorname{int}\Omega$.*

(3) *Suppose Ω_1, Ω_2 are regular $\mathcal{C}^{1:1}$-domains. If $p\in\partial\Omega$, then Ω_1 and $-\Omega_2+p$ are in contact position to each other.*

Proof. Suppose $p\in\Omega$. Then there exist $p_1\in\Omega_1$ and $p_2\in\Omega_2$ such that $p_1+p_2=p$. So $\Omega_1\ni p_1=-p_2+p\in-\Omega_2+p$, which means that $\Omega_1\cap(-\Omega_2+p)\neq\emptyset$. Conversely, suppose $\Omega_1\cap(-\Omega_2+p)\neq\emptyset$. Let $p_1\in\Omega_1\cap(-\Omega_2+p)$. Then there exists $p_2\in\Omega_2$ such that $p_1=-p_2+p$. Thus $p=p_1+p_2\in\Omega$. This shows (1).

Suppose $p_1\in\operatorname{int}\Omega_1\cap(-\Omega_2+p)$. Let $p_2=-p_1+p$. Then $p_2\in\Omega_2$, and $p=p_1+p_2\in\Omega$. Since $p_1\in\operatorname{int}\Omega_1$, we have $p\notin\partial\Omega$ by Lemma 6.2. Thus $p\in\operatorname{int}\Omega$. This shows (2).

Suppose $p\in\partial\Omega$. By (1), $\Omega_1\cap(-\Omega_2+p)\neq\emptyset$. Let $p_1\in\Omega_1\cap(-\Omega_2+p)$, and let $p_2=-p_1+p$. Then we have $p_1\in\Omega_1$, $p_2\in\Omega_2$ and $p_1+p_2=p$. By Lemma 6.2, $p_1\in\partial\Omega_1$ and $p_2\in\partial\Omega_2$, and so $p_1=-p_2+p\in\partial(-\Omega_2+p)$. Thus $p_1\in\partial\Omega_1\cap\partial(-\Omega_2+p)$. Since p_1

is taken arbitrarily, it follows that Ω_1 and $-\Omega_2+p$ are in contact position to each other. This shows (3). ∎

REMARK 6.1. The converse of (3) in Lemma 6.3 is false: it is possible that Ω_1 and $-\Omega_2+p$ are in contact position to each other, but still $p \notin \partial\Omega$.

DEFINITION 6.1 (admissible sectors). Two $\mathcal{C}^{1:1}$-sectors S_1 and S_2 with respective centers p_1, p_2 and radius r are said to be *admissible* to each other if they satisfy the following conditions:

(1) $\operatorname{int}(S_1-p_1)\cap(-(S_2-p_2))=\emptyset$ and $\operatorname{int}(S_2-p_2)\cap(-(S_1-p_1))=\emptyset$.

(2) For $i=1,2$, let γ_i be the end curve or start curve of S_i. If the two curves γ_1-p_1 and $-(\gamma_2-p_2)$ (or equivalently, $-(\gamma_1-p_1)$ and γ_2-p_2) meet at a point in $\mathbb{R}^2$ other than 0, then γ_1, γ_2 have the same image.

It is easy to see that if S_1 and S_2 are admissible to each other, then so are $B_{r'}(p_1)\cap S_1$ and $B_{r'}(p_2)\cap S_2$ for every $0<r'\le r$.

LEMMA 6.4. *Let $\mathcal{M}$ be a Minkowski class, and let Ω_1 and Ω_2 be two $\mathcal{M}$-domains. Let $p_1\in\partial\Omega_1$ and $p_2\in\partial\Omega_2$. Suppose $p=p_1+p_2\in\partial\Omega$, where $\Omega=\Omega_1+\Omega_2$. Then for every sufficiently small $r>0$, we have*:

(1) *For $i=1,2$, $B_r(p_i)\cap\Omega_i=\bigcup_{k=1}^{n_i}S_i^k$, where S_i^k is an $\mathcal{M}$-sector with center p_i and radius r for $k=1,\dots,n_i$, and S_i^k's are mutually non-overlapping.*

(2) *S_1^k and S_2^l are admissible to each other for every $k=1,\dots,n_1$ and $l=1,\dots,n_2$.*

Proof. (1) follows from Lemma 4.2. For (2), fix S_1^k and S_2^l. Let α_1, β_1 be the end curve and start curve of $S_1^k-p_1$ respectively, and let α_2, β_2 be the end curve and start curve of $-(S_2^l-p_2)$ respectively. Note that $S_1^k-p_1$ and $-(S_2^l-p_2)$ are $\mathcal{M}$-sectors with center 0 and radius r. Since $\mathcal{M}$ is a Minkowski class, we can assume that any two of α_1, β_1, α_2, β_2 either have the same image, or do not meet except at 0. So, if S_1^k and S_2^l are not admissible, then we would have either $\operatorname{int}S_1^k\cap(-S_2^l+p)\neq\emptyset$ or $\operatorname{int}S_2^l\cap(-S_1^k+p)\neq\emptyset$. Then by Lemma 6.3(2), $p\in\operatorname{int}\Omega$, which is a contradiction. ∎

Let S be a finite union of mutually non-overlapping $\mathcal{C}^{1:1}$-sectors $S_1,\dots,S_n$ with center p and radius $r>0$. Then we define $\mathrm{C}(S)=\bigcup_{k=1}^{n}\mathrm{C}(S_k)$.

LEMMA 6.5. *Let Ω_1 and Ω_2 be two $\mathcal{C}^{1:1}$-domains, and let $\Omega=\Omega_1+\Omega_2$. Let $p_1\in\partial\Omega_1$, $p_2\in\partial\Omega_2$, and choose $r>0$ such that $S_i=B_r(p_i)\cap\Omega_i$ is a finite union of mutually non-overlapping $\mathcal{C}^{1:1}$-sectors with center p_i and radius r for $i=1,2$. Suppose $(p_1,p_2)\in M^{-1}_{\Omega_1,\Omega_2}(\partial\Omega)$ and S_1 is a flat $\mathcal{C}^{1:1}$-sector with center p_1 and radius r. Then $\mathrm{C}(S_2)\subset\mathrm{C}(S_1)$.*

Proof. We can assume $\mathrm{C}(S_1)=\{(x,y)\in S^1\mid y\le 0\}$. Suppose $\mathrm{C}(S_2)\not\subset\mathrm{C}(S_1)$. Then there exists a $\mathcal{C}^{1:1}$-curve $\gamma:[0,\varepsilon]\to S_2$ such that $\gamma(0)=p_2$ and $\mathbf{v}[\gamma]\notin\mathrm{C}(S_1)$. So $\widetilde{\gamma}(0)=p_1$, $\widetilde{\gamma}([0,\varepsilon])\subset -S_2+p$, and $\mathbf{v}[\widetilde{\gamma}]\in\{(x,y)\in S^1\mid y<0\}$, where $p=p_1+p_2$ and the $\mathcal{C}^{1:1}$-curve $\widetilde{\gamma}:[0,\varepsilon]\to\mathbb{R}^2$ is defined by $\widetilde{\gamma}(t)=-\gamma(t)+p$ for $t\in[0,\varepsilon]$. It follows that $\operatorname{int}S_1\cap(-S_2+p)\neq\emptyset$. So by Lemma 6.3(2), $p\in\operatorname{int}(S_1+S_2)\subset\operatorname{int}\Omega$, which is a contradiction. ∎

LEMMA 6.6. *Let $\mathcal{C}$ be a subclass of $\mathcal{C}^{1:1}$ which is closed under restriction, and let Ω_1 and Ω_2 be two $\mathcal{C}$-domains. Let $p\in\partial\Omega$, where $\Omega=\Omega_1+\Omega_2$. Then for any $\varepsilon>0$, there exist*

$0 < r_1, \dots, r_n < \varepsilon$ and $(p_1^1, p_2^1), \dots, (p_1^n, p_2^n)$ in $M^{-1}_{\Omega_1,\Omega_2}(p)$ for some $1 \le n < \infty$ such that each $B_{r_k}(p_i^k) \cap \Omega_i$ is a finite union of mutually non-overlapping $\mathcal{C}$-sectors with center p_i^k and radius r_k, and $M^{-1}_{\Omega_1,\Omega_2}(p) \subset U$, where

$$U = \bigcup_{k=1}^{n} (B^o_{r_k}(p_1^k) \cap \Omega_1) \times (B^o_{r_k}(p_2^k) \cap \Omega_2).$$

Proof. By Lemma 6.2, $M^{-1}_{\Omega_1,\Omega_2}(p) \subset \partial\Omega_1 \times \partial\Omega_2$. So, by Lemma 4.2, we can choose $0 < r(p_1, p_2) < \varepsilon$ for each $(p_1, p_2) \in M^{-1}_{\Omega_1,\Omega_2}(p)$ such that $B_{r(p_1,p_2)}(p_i) \cap \Omega_i$ is a finite union of $\mathcal{C}$-sectors with center p_i and radius $r(p_1, p_2)$ for $i = 1, 2$. Note that

$$\{(B^o_{r(p_1,p_2)}(p_1) \cap \Omega_1) \times (B^o_{r(p_1,p_2)}(p_2) \cap \Omega_2) \mid (p_1, p_2) \in M^{-1}_{\Omega_1,\Omega_2}(p)\}$$

is an open cover of the compact set $M^{-1}_{\Omega_1,\Omega_2}(p)$ in $\Omega_1 \times \Omega_2$. Thus there exists a finite subcover $\{(B^o_{r_k}(p_1^k) \cap \Omega_1) \times (B^o_{r_k}(p_2^k) \cap \Omega_2) \mid 1 \le k \le n\}$, which completes the proof. ∎

6.2. *Minkowski sum of admissible sectors.* Let $\gamma : [a, b] \to \mathbb{R}^2$ be a continuous curve. We define $\widehat{\gamma} : [a, b] \to \mathbb{R}^2$ by

$$\widehat{\gamma}(t) = \gamma(a + b - t) + \gamma(a) - \gamma(b).$$

Note that if we translate the image of γ so that $\gamma(b)$ is moved to $\gamma(a)$, then we get the image of $\widehat{\gamma}$. Note also that $\widehat{\gamma}(a) = \gamma(a)$. See Figure 10.

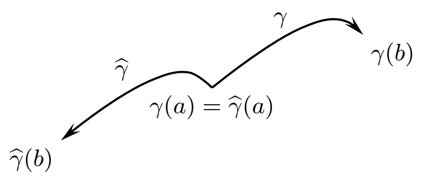


Fig. 10. γ and $\widehat{\gamma}$

Lemma 6.7. *Let $\mathcal{M}$ be a Minkowski class, and let S_1 and S_2 be two admissible $\mathcal{M}$-sectors with center 0 and radius $R > 0$. For some sufficiently small $0 < r \le R$, let $S'_i = B_r(0) \cap S_i$ for $i = 1, 2$, and let $S = S'_1 + S'_2$. Let α_i and β_i be the end curve and start curve of S'_i respectively for $i = 1, 2$. Then, for every sufficiently small $\varrho > 0$, the set $B_\varrho(0) \cap \partial S$ is contained in the union of the images of the following curves*:

(1) α_1, β_1, α_2, β_2.

(2) $\alpha_1 * \alpha_2$, $\alpha_1 * \beta_2$, $\beta_1 * \alpha_2$, $\beta_1 * \beta_2$ (*if defined*).

(3) $\widehat{\alpha}_1$, $\widehat{\beta}_2$ (*if α_1, $-\beta_2$ have the same image*), *and* $\widehat{\beta}_1$, $\widehat{\alpha}_2$ (*if β_1, $-\alpha_2$ have the same image*).

Proof. By abuse of notation, we denote the image of a curve γ also by γ. Set $M = M_{S'_1,S'_2}$. Note that $M^{-1}(\partial S) \subset \partial S'_1 \times \partial S'_2$ by Lemma 6.2. Let $A_i = \partial S'_i \setminus (\alpha_i \cup \beta_i)$ for $i = 1, 2$. Then $\partial S'_1 \times \partial S'_2 = (A_1 \times A_2) \cup (A_1 \times (\alpha_2 \cup \beta_2)) \cup ((\alpha_1 \cup \beta_1) \times A_2) \cup ((\alpha_1 \cup \beta_1) \times (\alpha_2 \cup \beta_2))$. Suppose $M((p_1, p_2)) \in \partial S$ for some $(p_1, p_2) \in A_1 \times A_2$. Let $p = p_1 + p_2$. Lemma 6.5 yields $p_1 = p_2$. So $|p| = |2p_1| = 2r$. This shows that $M(A_1 \times A_2) \cap (B_\varrho(0) \cap \partial S) = \emptyset$ for sufficiently small $\varrho > 0$.

Suppose that $\{(p_1^n, p_2^n)\}$ is a sequence in $(A_1 \times (\alpha_2 \cup \beta_2)) \cup ((\alpha_1 \cup \beta_1) \times A_2)$ such that $M((p_1^n, p_2^n)) = p_1^n + p_2^n \to 0$ as $n \to \infty$. We can assume that $(p_1^n, p_2^n) \in A_1 \times (\alpha_2 \cup \beta_2)$ for every n. Suppose $\alpha_i(0) = \beta_i(0) = 0$ and $|\alpha_i(a_i)| = |\beta_i(b_i)| = r$ for $i = 1, 2$. Since $p_1^n \in A_1$, $p_2^n \in \alpha_2 \cup \beta_2$, and S_1', S_2' are admissible, it is easy to see that there exists a subsequence $\{p_1^{n_k}\}$ such that either $p_1^{n_k} \to \alpha_1(a_1)$ or $p_1^{n_k} \to \beta_1(b_1)$ as $k \to \infty$. Denote this subsequence again by $\{p_1^n\}$; we can assume that $p_1^n \to \alpha_1(a_1)$ as $n \to \infty$. Since S_1' and S_2' are admissible to each other, it follows that $\beta_2(b_2) = -\alpha_1(a_1)$ and $p_2^n \to \beta_2(b_2)$. So we must have $\alpha_1 \approx -\beta_2$. Since we have assumed r to be sufficiently small, we can also assume that β_2 and $\partial B_r(0)$ meet transversally at $\beta_2(b_2)$. So, from Lemma 6.5, $M((p_1^n, p_2^n)) \notin \partial S$ for every sufficiently large n. Thus $M((A_1 \times (\alpha_2 \cup \beta_2)) \cup ((\alpha_1 \cup \beta_1) \times A_2)) \cap (B_\varrho(0) \cap \partial S) = \emptyset$ for sufficiently small $\varrho > 0$.

It follows that $B_\varrho(0) \cap \partial S \subset M((\alpha_1 \cup \beta_1) \times (\alpha_2 \cup \beta_2))$ for sufficiently small $\varrho > 0$. Set $\alpha_i^o = \alpha_i((0, a_i))$ and $\beta_i^o = \beta_i((0, b_i))$ for $i = 1, 2$. We divide $(\alpha_1 \cup \beta_1) \times (\alpha_2 \cup \beta_2)$ into the four parts $\alpha_1^o \times \alpha_2^o$, $\beta_1^o \times \beta_2^o$, $\alpha_1^o \times \beta_2^o$, $\beta_1^o \times \alpha_2^o$, and the twelve parts $\alpha_1 \times \{0\}$, $\beta_1 \times \{0\}$, $\{0\} \times \alpha_2$, $\{0\} \times \beta_2$, $\alpha_1 \times \{\alpha_2(a_2)\}$, $\alpha_1 \times \{\beta_2(b_2)\}$, $\beta_1 \times \{\alpha_2(a_2)\}$, $\beta_1 \times \{\beta_2(b_2)\}$, $\{\alpha_1(a_1)\} \times \alpha_2$, $\{\beta_1(b_1)\} \times \alpha_2$, $\{\alpha_1(a_1)\} \times \beta_2$, $\{\beta_1(b_1)\} \times \beta_2$. Since r is assumed to be small, Lemma 6.5 shows that the intersections of ∂S and the images of the first four parts under M are contained in the union of $\alpha_1 * \alpha_2$, $\beta_1 * \beta_2$, $\alpha_1 * \beta_2$, $\alpha_2 * \beta_1$. The images of the last twelve parts under M are α_1, β_1, α_2, β_2, $\alpha_1 + \alpha_2(a_2)$, $\alpha_1 + \beta_2(b_2)$, $\beta_1 + \alpha_2(a_2)$, $\beta_1 + \beta_2(b_2)$, $\alpha_2 + \alpha_1(a_1)$, $\alpha_2 + \beta_1(b_1)$, $\beta_2 + \alpha_1(a_1)$, $\beta_2 + \beta_1(b_1)$ respectively. It is easy to see that if $0 \in \alpha_1 + \beta_2(b_2)$, then $\beta_2(b_2) = -\alpha_1(a_1)$ and $\alpha_1 + \beta_2(b_2) = \widehat{\alpha}_1$, since $\beta_2(b_2) \in \partial B_r(0)$ and $\alpha_1 \cap \partial B_r(0) = \alpha_1(a_1)$. Also, if $0 \in \alpha_1 + \alpha_2(a_2)$, then $\alpha_2(a_2) = -\alpha_1(a_1)$, which implies $\beta_2(b_2) = -\alpha_1(a_1)$ and $\alpha_1 + \alpha_2(a_2) = \widehat{\alpha}_1$, since S_1' and S_2' are admissible to each other. Applying the same argument to the eight curves $\alpha_1 + \alpha_2(a_2)$, $\alpha_1 + \beta_2(b_2)$, $\beta_1 + \alpha_2(a_2)$, $\beta_1 + \beta_2(b_2)$, $\alpha_2 + \alpha_1(a_1)$, $\alpha_2 + \beta_1(b_1)$, $\beta_2 + \alpha_1(a_1)$, $\beta_2 + \beta_1(b_1)$, we can see that, among these curves, the ones containing 0 are $\widehat{\alpha}_1$, $\widehat{\beta}_2$ (if α_1, $-\beta_2$ have the same image), and $\widehat{\beta}_1$, $\widehat{\alpha}_2$ (if β_1, $-\alpha_2$ have the same image). Now combining the above arguments, we obtain the desired result. ∎

From the above result, we are now able to derive the following theorem:

THEOREM 6.1 (Minkowski sum of admissible sectors). *Let $\mathcal{M}$ be a Minkowski class, and let S_1, S_2 be admissible $\mathcal{M}$-sectors with respective centers p_1, p_2 and radius $R > 0$. Let $S_i' = B_r(p_i) \cap S_i$ for $i = 1, 2$ for some sufficiently small $0 < r < R$, and let $S = S_1' + S_2'$. Then, for every sufficiently small $\varrho > 0$, either $B_\varrho(p) \cap S = B_\varrho(p)$, or $B_\varrho(p) \cap S$ is a finite union of mutually non-overlapping $\mathcal{M}$-sectors with center p and radius ϱ, where $p = p_1 + p_2$.*

Proof. Note that $B_\varrho(p) \cap S = [B_\varrho(0) \cap \{(S_1' - p_1) + (S_2' - p_2)\}] + p$ for every $r > 0$ and $\varrho > 0$. So we can assume that $p_1 = p_2 = 0$. By Lemma 6.7, we can take a finite number of $\mathcal{M}$-curves $\gamma_1, \ldots, \gamma_n : [0, \varepsilon] \to \mathbb{R}^2$ for some $n \geq 1$ such that $\gamma_1(0) = \ldots = \gamma_n(0) = 0$ and $B_\varrho(0) \cap \partial S \subset \bigcup_{k=1}^n \gamma_k([0, \varepsilon]) \subset S$ for every sufficiently small $\varrho > 0$. Since $\mathcal{M}$ is a Minkowski class, we can assume that, for every sufficiently small $\varrho > 0$, $\gamma_k([0, \varepsilon]) \cap \partial B_\varrho(0)$ is a singleton for each k, and $\gamma_i([0, \varepsilon]) \cap \gamma_j([0, \varepsilon]) = \{0\}$ for every $i \neq j$. Since ∂S is compact and ϱ is small, we can assume that either $B_\varrho(0) \cap \partial S = \emptyset$, or there exists $0 < m \le n$ such that $B_\varrho(0) \cap \partial S = \bigcup_{k=1}^m \gamma_k([0,\varepsilon])$. ■

6.3. *Minkowski sum of admissible non-degenerate sectors.* When both S_1 and S_2 are non-degenerate, we have more refined results, which will provide a local building block for dealing with semi-convexity later.

LEMMA 6.8. *Let $\mathcal{M}$ be a Minkowski class, and let S_1, S_2 be non-degenerate $\mathcal{M}$-sectors with center* 0 *and radius $r > 0$ which are admissible to each other. Suppose there exist $r_1, \ldots, r_n > 0$ and $(p_1^1, p_2^1), \ldots, (p_1^n, p_2^n) \in M_{S_1,S_2}^{-1}(0)$ such that $B_{r_k}(p_i^k) \cap S_i$ is an $\mathcal{M}$-sector with center p_i^k and radius r_k for each i and k, and $M_{S_1,S_2}^{-1}(0) \subset U$, where $U = \bigcup_{k=1}^n (B_{r_k}^o(p_1^k) \cap S_1) \times (B_{r_k}^o(p_2^k) \cap S_2)$. Then $M_{S_1,S_2}(U \setminus M_{S_1,S_2}^{-1}(0))$ is connected.*

Proof. Set $M = M_{S_1,S_2}$, and denote the image of a curve γ also by γ. Let $U_k = (B_{r_k}^o(p_1^k) \cap S_1) \times (B_{r_k}^o(p_2^k) \cap S_2)$ for $k = 1, \ldots, n$. Note that $M(U \setminus M^{-1}(0)) = \bigcup_{k=1}^n M(U_k \setminus M^{-1}(0))$. Since S_1 and S_2 are admissible, we must have $(p_1^1, p_2^1), \ldots, (p_1^n, p_2^n) \in (\alpha_1 \cup \beta_1) \times (\alpha_2 \cup \beta_2)$, where $\alpha_i, \beta_i : [0,\varepsilon] \to S_i$ are the end curve and start curve of S_i respectively for $i = 1, 2$. We first show that $U_k \setminus M^{-1}(0)$ is connected for each k. Let $(q_1^1, q_2^1), (q_1^2, q_2^2) \in U_k \setminus M^{-1}(0)$. It is easy to see that $B_{r_k}(p_1^k) \cap S_1$ is a non-degenerate $\mathcal{M}$-sector with center p_1^k and radius r_k. So we can take a continuous curve $\gamma_1 : [0,1] \to B_{r_k}^o(p_1^k) \cap S_1$ such that $\gamma_1(0) = q_1^1$, $\gamma_1(1) = q_1^2$, and $\gamma_1((0,1)) \subset \mathrm{int}(B_{r_k}^o(p_1^k) \cap S_1)$. Take any continuous curve $\gamma_2 : [0,1] \to B_{r_k}^o(p_2^k) \cap S_2$ such that $\gamma_2(0) = q_2^1$, $\gamma_2(1) = q_2^2$. Define $\gamma : [0,1] \to U_k$ by $\gamma(t) = (\gamma_1(t), \gamma_2(t))$. Since $\{p_1 \mid (p_1, p_2) \in M^{-1}(0)$ for some $p_2 \in \Omega_2\} \subset \alpha_1 \cup \beta_1$, it follows that $\mathrm{int}(B_{r_k}^o(p_1^k) \cap S_1) \cap \{p_1 \mid (p_1, p_2) \in M^{-1}(0)$ for some $p_2 \in \Omega_2\} = \emptyset$. Thus $\gamma([0,1]) \in U_k \setminus M^{-1}(0)$, and this shows $U_k \setminus M^{-1}(0)$ is connected.

Now, since S_1, S_2 are admissible to each other, we can assume that $M^{-1}(0)$ is one of $\{(0,0)\}$, $\{(\alpha_1(t), \beta_2(t)) \mid 0 \le t \le \varepsilon\}$, $\{(\beta_1(t), \alpha_2(t)) \mid 0 \le t \le \varepsilon\}$, or $\{(\alpha_1(t), \beta_2(t)) \mid 0 \le t \le \varepsilon\} \cup \{(\beta_1(t), \alpha_2(t)) \mid 0 \le t \le \varepsilon\}$. So we can assume that $(U_k \setminus M^{-1}(0)) \cap (U_{k+1} \setminus M^{-1}(0)) \neq \emptyset$ for $k = 1, \ldots, n-1$, since $M^{-1}(0) \subset U$. Hence the set $U \setminus M^{-1}(0) = \bigcup_{k=1}^n (U_k \setminus M^{-1}(0))$ is connected. Thus $M(U \setminus M^{-1}(0))$ is connected, since M is continuous. ■

THEOREM 6.2 (Minkowski sum of admissible non-degenerate sectors). *Let $\mathcal{M}$ be a Minkowski class, and let S_1, S_2 be non-degenerate $\mathcal{M}$-sectors with respective centers p_1, p_2 and radius $R > 0$, which are admissible to each other. Let $S = S_1' + S_2'$, where $S_i' = B_r(p_i) \cap S_i$ for $i = 1, 2$ for some sufficiently small $0 < r < R$. Let $p = p_1 + p_2$. Suppose $B_\varrho(p) \cap S \neq B_\varrho(p)$ for every $\varrho > 0$. Then, for every sufficiently small $\varrho > 0$, we have*:

1. *$B_\varrho(p) \cap S$ is a non-degenerate $\mathcal{M}$-sector with center p and radius ϱ.*
2. *Let α and β be the end curve and start curve of $B_\varrho(p) \cap S$. Suppose that the image of α (resp., β) is contained in one of the images of $\alpha_1 + p_2$, $\beta_1 + p_2$, $\alpha_2 + p_1$, $\beta_2 + p_1$, $\alpha_1 * \alpha_2$, $\beta_1 * \beta_2$, $\alpha_1 * \beta_2$, $\beta_1 * \alpha_2$, where α_i and β_i are the end curve and start curve of S_i' respectively for $i = 1, 2$. Then there exists a continuous map $\phi_i^\alpha : [0,\varepsilon] \to \partial^v S_i'$, $\phi_i^\alpha(t) = (\gamma_i^\alpha(t), \mathbf{n}_i^\alpha(t))$ (resp., $\phi_i^\beta : [0,\varepsilon] \to \partial^v S_i'$, $\phi_i^\beta(t) = (\gamma_i^\beta(t), \mathbf{n}_i^\beta(t))$), for $i = 1, 2$, with the following properties:*

   (1) *$\gamma_1^\alpha(0) = p_1$ and $\gamma_2^\alpha(0) = p_2$ (resp., $\gamma_1^\beta(0) = p_1$ and $\gamma_2^\beta(0) = p_2$).*

(2) $\alpha(t) = \gamma_1^\alpha(t) + \gamma_2^\alpha(t)$ (*resp.*, $\beta(t) = \gamma_1^\beta(t) + \gamma_2^\beta(t)$) *for every* $t \in [0, \varepsilon]$.

(3) $\mathbf{n}^+_{B_\varrho(p)\cap S}(\alpha(t)) = \mathbf{n}_1^\alpha(t) = \mathbf{n}_2^\alpha(t)$ (*resp.*, $\mathbf{n}^-_{B_\varrho(p)\cap S}(\beta(t)) = \mathbf{n}_1^\beta(t) = \mathbf{n}_2^\beta(t)$) *for every* $t \in [0, \varepsilon]$.

(4) *For* $i = 1, 2$, ϕ_i^α *and* γ_i^α (*resp.*, ϕ_i^β *and* γ_i^β) *are either one-to-one or constant, and if one of* $O_{S_1}(\gamma_1^\alpha)$ *and* $O_{S_2}(\gamma_2^\alpha)$ (*resp.*, $O_{S_1}(\gamma_1^\beta)$ *and* $O_{S_2}(\gamma_2^\beta)$) *is* $-$ (*resp.*, $+$), *then the other is* $+$ (*resp.*, $-$).

3. *Suppose that the image of* α (*resp.*, β) *is not contained in any of the images of* α_1+p_2, β_1+p_2, α_2+p_1, β_2+p_1, $\alpha_1*\alpha_2$, $\beta_1*\beta_2$, $\alpha_1*\beta_2$, $\beta_1*\alpha_2$. *Then* $\alpha\setminus\{p\} \subset \operatorname{int}(S_1+S_2)$ (*resp.*, $\beta \setminus \{p\} \subset \operatorname{int}(S_1 + S_2)$).

Proof. We denote $M_{S_1',S_2'}$ by M. We can assume $p_1 = p_2 = 0$. By Theorem 6.1, $B_\varrho(0) \cap S$ is either $B_\varrho(0)$ or a finite union of mutually non-overlapping $\mathcal{M}$-sectors with center 0 and radius ϱ, for sufficiently small $\varrho > 0$. Note that S_1' and S_2' are non-degenerate. So by Lemma 6.6, there exist $0 < r_1, \dots, r_n < \varrho/2$ and $(p_1^1, p_2^1), \dots, (p_1^n, p_2^n) \in M^{-1}(0)$ such that $B_{r_k}(p_i^k) \cap S_i'$ is a non-degenerate $\mathcal{M}$-sector with center p_i^k and radius r_k for $i = 1, 2$, $k = 1, \dots, n$, and $M^{-1}(0) \subset U$, where $U = \bigcup_{k=1}^n (B^o_{r_k}(p_1^k) \cap S_1') \times (B^o_{r_k}(p_2^k) \cap S_2')$. Note that $M((S_1' \times S_2') \setminus U)$ is compact and does not contain 0. So there exists $0 < \varepsilon < \varrho$ such that $B_\varepsilon(0) \cap M((S_1' \times S_2') \setminus U) = \emptyset$. It follows that $B_\varepsilon(0) \cap S = B_\varepsilon(0) \cap M(U)$. Since $r_k < \varrho/2$ for $k = 1, \dots, n$, it is clear that $M(U \setminus M^{-1}(0)) \subset B_\varrho(0) \cap (S \setminus \{0\})$. By Lemma 6.8, $M(U \setminus M^{-1}(0))$ is connected, since both S_1' and S_2' are non-degenerate. So $M(U \setminus M^{-1}(0))$ is contained in one connected component of $B_\varrho(0) \cap (S \setminus \{0\})$. Since $B_\varepsilon(0) \cap S = B_\varepsilon(0) \cap M(U)$, it follows that $B_\varrho(0) \cap (S \setminus \{0\})$ has exactly one connected component. This implies that $B_\varrho(0) \cap S$ is an $\mathcal{M}$-sector with center 0 and radius ϱ, since we assumed that $B_\varrho(0) \cap S \neq B_\varrho(0)$. Since $0 \in S_1', S_2'$, we have $B_\varrho(0) \cap S_1', B_\varrho(0) \cap S_2' \subset B_\varrho(0) \cap S$. So we conclude that $B_\varrho(0) \cap S$ is a non-degenerate $\mathcal{M}$-sector with center 0 and radius ϱ, since S_1' and S_2' are non-degenerate. Thus we have shown 1.

Suppose the image of α (resp., β) is contained in one of the images of the curves α_1, β_1, α_2, β_2, $\alpha_1 * \alpha_2$, $\beta_1 * \beta_2$, $\alpha_1 * \beta_2$, $\beta_1 * \alpha_2$. We can assume that α_1, β_1, α_2, β_2, $\alpha_1 * \alpha_2$, $\beta_1 * \beta_2$, $\alpha_1 * \beta_2$, $\beta_1 * \alpha_2$ and α, β are parametrized as follows: $\gamma(0) = 0$ for any γ among the above curves, and, for any $*$-admissible γ_1, γ_2 among the above curves, $\mathbf{v}[\gamma_1](t) \,/\!\!/\, \mathbf{v}[\gamma_2](t)$ for every feasible t. Now, depending on in which of the images of the curves α_1, β_1, α_2, β_2, $\alpha_1 * \alpha_2$, $\beta_1 * \beta_2$, $\alpha_1 * \beta_2$, $\beta_1 * \alpha_2$ the image of α (resp., β) is contained, we construct $\phi_i^\alpha : [0, \varepsilon] \to \partial^v S_i'$ (resp., $\phi_i^\beta : [0, \varepsilon] \to \partial^v S_i'$) for $i = 1, 2$ as follows:

α (resp., β)	$\phi_1^\alpha(t)$ (resp., $\phi_1^\beta(t)$)	$\phi_2^\alpha(t)$ (resp., $\phi_2^\beta(t)$)
α_1	$(\alpha_1(t), \mathbf{n}^+_{S_1}(\alpha_1(t)))$	$(0, \mathbf{n}^+_{S_1}(\alpha_1(t)))$
β_1	$(\beta_1(t), \mathbf{n}^-_{S_1}(\beta_1(t)))$	$(0, \mathbf{n}^-_{S_1}(\beta_1(t)))$
α_2	$(0, \mathbf{n}^+_{S_2}(\alpha_2(t)))$	$(\alpha_2(t), \mathbf{n}^+_{S_2}(\alpha_2(t)))$
β_2	$(0, \mathbf{n}^-_{S_2}(\beta_2(t)))$	$(\beta_2(t), \mathbf{n}^-_{S_2}(\beta_2(t)))$
$\alpha_1 * \alpha_2$	$(\alpha_1(t), \mathbf{n}^+_{S_1}(\alpha_1(t)))$	$(\alpha_2(t), \mathbf{n}^+_{S_2}(\alpha_2(t)))$
$\beta_1 * \beta_2$	$(\beta_1(t), \mathbf{n}^-_{S_1}(\beta_1(t)))$	$(\beta_2(t), \mathbf{n}^-_{S_2}(\beta_2(t)))$
$\alpha_1 * \beta_2$	$(\alpha_1(t), \mathbf{n}^+_{S_1}(\alpha_1(t)))$	$(\beta_2(t), \mathbf{n}^-_{S_2}(\beta_2(t)))$
$\beta_1 * \alpha_2$	$(\beta_1(t), \mathbf{n}^-_{S_1}(\beta_1(t)))$	$(\alpha_2(t), \mathbf{n}^+_{S_2}(\alpha_2(t)))$

From the above table, it is easy to check that ϕ_1^α and ϕ_2^α (resp., ϕ_1^β and ϕ_2^β) satisfy (1) and (2) of 2. It is also clear that ϕ_i^α and γ_i^α (resp., ϕ_i^β and γ_i^β) are either one-to-one or constant for $i = 1, 2$. Note that $B_\varrho(0) \cap S$, S_1', S_2' are non-degenerate $\mathcal{M}$-sectors, and $B_\varrho(0) \cap S_i' \subset B_\varrho(0) \cap S$ for $i = 1, 2$. Suppose the image of α is contained in the image of β_1. Since $B_\varrho(0) \cap S_1' \subset B_\varrho(0) \cap S$, it follows that α_1 and β_1 have the same image. But this is impossible, since S_1' is non-degenerate. So the image of α cannot be contained in the image of β_1. In the same way, the image of α cannot be contained in the image of β_2, and the image of β cannot be contained in the images of α_1 or α_2. Suppose the image of α is contained in the image of $\beta_1 * \beta_2$. By Lemma 2.3, $\mathbf{v}[\beta_1 * \beta_2] = \mathbf{v}[\beta_1]$ or $\mathbf{v}[\beta_2]$. With no loss of generality, suppose $\mathbf{v}[\beta_1 * \beta_2] = \mathbf{v}[\beta_1] = (1, 0)$. Clearly, $\mathbf{v}[\alpha] = (1, 0)$. Take non-zero points q_1, q_2, q in the images of β_1, β_2, β respectively such that $q = q_1 + q_2$. Note that these points can be taken arbitrarily close to 0. So there exists a small $\delta > 0$ such that $\{q_1 + u \cdot (0, -1) \mid 0 \le u \le \delta\} \subset S_1'$ and $q_2 + \{q_1 + u \cdot (0, -1) \mid 0 \le u \le \delta\} = \{q + u \cdot (0, -1) \mid 0 \le u \le \delta\} \subset B_\varrho(0) \cap S$. But this contradicts the assumption that α is the end curve of $B_\varrho(0) \cap S$. So the image of α cannot be contained in the image of $\beta_1 * \beta_2$. In the same way, the image of β cannot be contained in the image of $\alpha_1 * \alpha_2$. Now, the above table shows that if one of $O_{S_1}(\gamma_1^\alpha)$ and $O_{S_2}(\gamma_2^\alpha)$ (resp., $O_{S_1}(\gamma_1^\beta)$ and $O_{S_2}(\gamma_2^\beta)$) is $-$ (resp., $+$), then the other is $+$ (resp., $-$). This proves (4) of 2.

Suppose the image of α is contained in $\alpha_1 * \alpha_2$. Then, for every t, either $\mathbf{n}^+_{S_1}(\alpha_1(t)) = \mathbf{n}^+_{S_2}(\alpha_2(t))$ or $\mathbf{n}^+_{S_1}(\alpha_1(t)) = -\mathbf{n}^+_{S_2}(\alpha_2(t))$, since α_1 and α_2 are $*$-admissible to each other. Suppose the latter is true. Since S_1 and S_2 are non-degenerate, we can take t_0 such that, for every sufficiently small $\delta > 0$, $\alpha_1(t_0) - \delta \cdot \mathbf{n}^+_{S_1}(\alpha_1(t)) \in S_1$ and $\alpha_2(t_0) - \delta \cdot \mathbf{n}^+_{S_2}(\alpha_2(t)) \in S_2$. This implies $\alpha(t_0) \pm \delta \cdot \mathbf{n}^+_S(\alpha(t_0)) \in S$, contrary to $\alpha \subset \partial S$. So $\mathbf{n}^+_{S_1}(\alpha_1(t)) = \mathbf{n}^+_{S_2}(\alpha_2(t))$ for every t, and hence, $\mathbf{n}^+_S(\alpha(t)) = \mathbf{n}^+_{S_1}(\alpha_1(t)) = \mathbf{n}^+_{S_2}(\alpha_2(t))$ for every t. We can show that (3) of 2 is true for the remaining cases in a similar way.

Now we show 3. Suppose the image of α (resp., β) is not contained in any of the images of α_1, β_1, α_2, β_2, $\alpha_1 * \alpha_2$, $\beta_1 * \beta_2$, $\alpha_1 * \beta_2$, $\beta_1 * \alpha_2$. Then by Lemma 6.7, the image of α (resp., β) is contained in one of the images of $\widehat{\alpha}_1$, $\widehat{\alpha}_2$, $\widehat{\beta}_1$, $\widehat{\beta}_2$. We first show that the image of α cannot be contained in the images of $\widehat{\alpha}_1$ or $\widehat{\alpha}_2$, and the image of β cannot be contained in the images of $\widehat{\beta}_1$ or $\widehat{\beta}_2$. Suppose the image of α is contained in the image of $\widehat{\alpha}_1$. By Lemma 6.7, α_1, $-\beta_2$ have the same image. We can assume that $\mathbf{v}[\alpha_1] = (-1, 0)$. Since r is small, there exists $f : [-r', 0] \to \mathbb{R}$ whose graph is the image of α_1. Note that $|(-r', f(-r'))| = r$. The graph of $g : [0, r'] \to \mathbb{R}$ defined by $g(x) = f(x - r') - f(-r')$ is the image of $\widehat{\alpha}_1$. Since S_1' is non-degenerate, there exist $\varepsilon, \delta > 0$ such that $\{(-\varepsilon, y) \mid f(-\varepsilon) - \delta \le y \le f(-\varepsilon)\} \subset S_1'$. Since α_1, $-\beta_2$ have the same image, we have $(r', -f(-r')) \in S_2'$. Note that we can take $|r' - \varepsilon|$ and δ as small as desired. So $\{(-\varepsilon + r', y) \mid g(-\varepsilon + r') - \delta \le y \le g(-\varepsilon + r')\} = (r', -f(-r')) + \{(-\varepsilon, y) \mid f(-\varepsilon) - \delta \le y \le f(-\varepsilon)\} \subset B_\varrho(0) \cap S$. This means that $\widehat{\alpha}_1 \cap B_\varrho(0)$ cannot be the end curve of $B_\varrho(0) \cap S$, contrary to the assumption. Thus the image of α cannot be contained in the image of $\widehat{\alpha}_1$. In the same way, the image of α cannot be contained in the image of $\widehat{\alpha}_2$, and the image of β cannot be contained in the images of $\widehat{\beta}_1$ or $\widehat{\beta}_2$.

Suppose the image of α is contained in the image of $\widehat{\beta}_1$. By Lemma 6.7, β_1, $-\alpha_2$ have the same image. Suppose $\sigma(\beta_1) = 0$. Then $\widehat{\beta}_1$, α_2 have the same image. So the image of α is contained in the image of α_2, contrary to the assumption. Suppose $\sigma(\beta_1) = +$. Then the image of $\widehat{\beta}_1$ intersects $\operatorname{int} S_2'$, since S_2' is non-degenerate, $\alpha_2 = -\beta_1$, and r is assumed to be small. So $B_\varrho(0) \cap \widehat{\beta}_1$ cannot be the end curve of $B_\varrho(0) \cap S$, which is a contradiction. Thus we must have $\sigma(\beta_1) = -$ and $\sigma(\widehat{\beta}_1) = +$. We can assume that there exists $\widetilde{r} > r$ such that $T = B_\varrho(0) \cap \{(B_{\widetilde{r}}(0) \cap S_1) + (B_{\widetilde{r}}(0) \cap S_2)\}$ is a non-degenerate $\mathcal{M}$-sector with center 0 and radius ϱ, and $\widetilde{\alpha}$, $\widehat{\widetilde{\beta}}_1$ have the same image, where $\widetilde{\alpha}$ is the end curve of T and $\widetilde{\beta}_1$ is the start curve of $B_{\widetilde{r}}(0) \cap S_1$. Since $\sigma(\widehat{\beta}_1) = +$, $\beta_1 = B_r(0) \cap \widetilde{\beta}_1$, and r is small, it is easy to see that $\alpha \setminus \{0\} \subset \operatorname{int} T$. Thus $\alpha \setminus \{0\}$ is in the interior of $S_1 + S_2$. In the same way, we can show that $\alpha \setminus \{0\} \subset \operatorname{int}(S_1 + S_2)$ if α, $\widehat{\beta}_2$ have the same image, and $\beta \setminus \{0\} \subset \operatorname{int}(S_1 + S_2)$ if β, $\widehat{\alpha}_1$ have the same image or β, $\widehat{\alpha}_2$ have the same image. This shows 3. ∎

6.4. *Closedness of Minkowski sum.* Using the results of Section 6.2, we now analyze the Minkowski sum from a more global point of view, i.e., the Minkowski sum of general domains. It turns out that, for any Minkowski class $\mathcal{M}$, the Minkowski sum of $\mathcal{M}$-domains is also an $\mathcal{M}$-domain, and thus the set of all $\mathcal{M}$-domains is closed under Minkowski sum. Note that this is not true for an arbitrary curve class $\mathcal{C}$ which is closed under restriction. See Figure 3 for an example.

First, we prove a lemma which will also be used later in Section 7:

LEMMA 6.9. *Let $\mathcal{M}$ be a Minkowski class, and let Ω_1 and Ω_2 be two $\mathcal{M}$-domains. Let $\Omega = \Omega_1 + \Omega_2$. Then, for every point $p \in \partial\Omega$ and for every $r > 0$, there exist a finite number of pairs $(p_1^1, p_2^1), \ldots, (p_1^n, p_2^n)$ in $M^{-1}_{\Omega_1,\Omega_2}(p)$, $0 < r_1, \ldots, r_n < r$, such that, for every sufficiently small $\varrho > 0$, the following are satisfied*:

(1) *$S_i^k = B_{r_k}(p_i^k) \cap \Omega_i$ is a finite union of mutually non-overlapping $\mathcal{M}$-sectors with center p_i^k and radius r_k for every $i = 1, 2$ and $k = 1, \ldots, n$.*

(2) *$B_\varrho(p) \cap (S_1^k + S_2^k)$ is a finite union of mutually non-overlapping $\mathcal{M}$-sectors with center p and radius ϱ for $k = 1, \ldots, n$.*

(3) *$B_\varrho(p) \cap \Omega$ is a finite union of mutually non-overlapping $\mathcal{M}$-sectors with center p and radius ϱ, and*

$$B_\varrho(p) \cap \Omega = \bigcup_{k=1}^{n} \{B_\varrho(p) \cap (S_1^k + S_2^k)\}.$$

Proof. Suppose $p \in \partial\Omega$ and $r > 0$. By Lemma 6.6, there exist finitely many pairs $(p_1^1, p_2^1), \ldots, (p_1^n, p_2^n)$ in $M^{-1}_{\Omega_1,\Omega_2}(p)$ and $0 < r_1, \ldots, r_n < r$ such that $S_i^k = B_{r_k}(p_i^k) \cap \Omega_i$ is a finite union of mutually non-overlapping $\mathcal{M}$-sectors with center p_i^k and radius r_k for $i = 1, 2$, $k = 1, \ldots, n$, and $M^{-1}_{\Omega_1,\Omega_2}(p) \subset U$, where $U = \bigcup_{k=1}^n (B^o_{r_k}(p_1^k) \cap \Omega_1) \times (B^o_{r_k}(p_2^k) \cap \Omega_2)$. Thus (1) is satisfied.

Let $S_i^k = \bigcup_{j=1}^{n_i^k} S_i^{k,j}$, where $S_i^{k,j}$'s are mutually non-overlapping $\mathcal{M}$-sectors with center p_i^k and radius r_k. Note that r_k's can be taken to be arbitrarily small. So by Lemma 6.4, we can assume that $S_1^{k,j}$ and $S_2^{k,j'}$ are admissible to each other for every $k = 1, \ldots, n$ and $1 \le j \le n_1^k$, $1 \le j' \le n_2^k$. By Theorem 6.1, the set $B_\varrho(p) \cap (S_1^{k,j} + S_2^{k,j'})$ is either $B_\varrho(p)$, or a finite union of mutually non-overlapping $\mathcal{M}$-sectors with center p and radius ϱ for sufficiently small $\varrho > 0$. So by Lemmas 6.1 and 4.1(2), the set $B_\varrho(p) \cap (S_1^k + S_2^k)$ is either $B_\varrho(p)$, or a finite union of mutually non-overlapping $\mathcal{M}$-sectors with center p and radius ϱ for sufficiently small $\varrho > 0$. It follows that $B_\varrho(p) \cap (S_1^k + S_2^k)$ is a finite union of mutually non-overlapping $\mathcal{M}$-sectors with center p and radius ϱ, since $p \in \partial\Omega$ and $S_1^k + S_2^k \subset \Omega$. Thus (2) is satisfied.

Lemma 4.1(2) shows that $\bigcup_{k=1}^n \{B_\varrho(p) \cap (S_1^k + S_2^k)\}$ is a finite union of mutually non-overlapping $\mathcal{M}$-sectors with center p and radius ϱ for sufficiently small $\varrho > 0$, since $p \in \partial\Omega$ and $\bigcup_{k=1}^n (S_1^k + S_2^k) \subset \Omega$. Note that the set $M_{\Omega_1,\Omega_2}((\Omega_1 \times \Omega_2) \setminus U)$ in Ω is compact, and does not contain p, since $M^{-1}_{\Omega_1,\Omega_2}(p) \subset U$. So, for sufficiently small $\varrho > 0$, we have $B_\varrho(p) \cap M_{\Omega_1,\Omega_2}((\Omega_1 \times \Omega_2) \setminus U) = \emptyset$. This implies that $B_\varrho(p) \cap \Omega = B_\varrho(p) \cap M_{\Omega_1,\Omega_2}(\overline{U})$. Thus (3) is satisfied, since $M_{\Omega_1,\Omega_2}(\overline{U}) = \bigcup_{k=1}^n (S_1^k + S_2^k)$. ∎

It is now easy to prove the following result:

Theorem 6.3 (closedness under Minkowski sum). *Let M be a Minkowski class, and let Ω_1 and Ω_2 be two $\mathcal{M}$-domains. Then their Minkowski sum $\Omega = \Omega_1 + \Omega_2$ is an $\mathcal{M}$-domain.*

Proof. First, note that Ω is compact and connected, since it is the image of the compact connected set $\Omega_1 \times \Omega_2$ under the continuous Minkowski map M_{Ω_1,Ω_2}. By Lemma 6.9, there exist $r > 0$ such that $B_r(p) \cap \Omega$ is a finite union of mutually non-overlapping $\mathcal{M}$-sectors with center p and radius r for every $p \in \partial\Omega$. Thus Ω is an $\mathcal{M}$-domain by Lemma 4.2. ∎

7. Minkowski sum of semi-convex domains

Let us first define the *semi-convexity*:

Definition 7.1 (semi-convex domain). A regular $\mathcal{C}^{1:1}$-domain Ω is called *semi-convex* if $\Theta(\Omega) \geq -\pi$.

Remark 7.1. In fact, if $\Theta(\Omega) > -2\pi$ for a regular $\mathcal{C}^{1:1}$-domain Ω, then Ω must be simply connected. So a semi-convex domain is automatically simply connected. It is also easy to see that a regular $\mathcal{C}^{1:1}$-domain Ω is *convex* if and only if $\Theta(\Omega) = 0$.

The domains in Figures 11 and 12 are examples of regular $\mathcal{C}^{1:1}$-domains which are semi-convex and not semi-convex respectively.

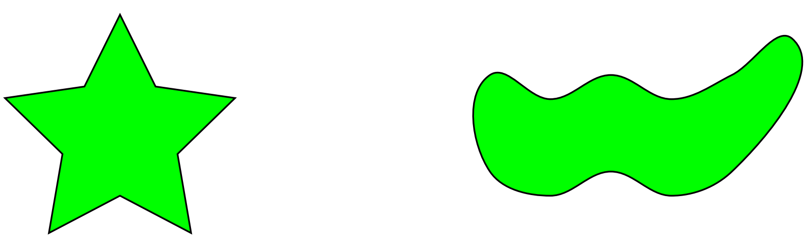


Fig. 11. Examples of semi-convex domains

In this section, we will show that the Minkowski sum of two semi-convex $\mathcal{M}$-domains is homeomorphic to the unit disk in $\mathbb{R}^2$ for any Minkowski class $\mathcal{M}$. This answers Problem 1 posed in Section 1 within the category of $\mathcal{M}$-domains.

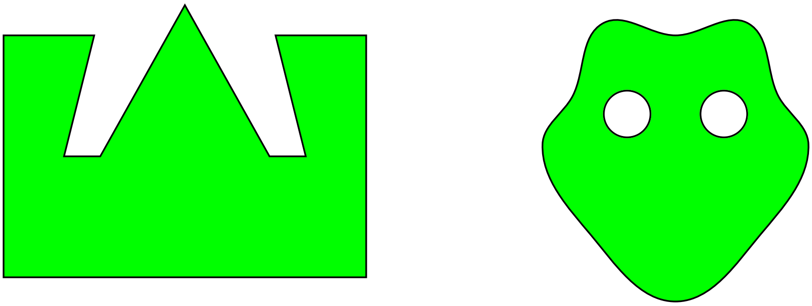


Fig. 12. Examples of regular domains which are not semi-convex

Let $\mathcal{M}$ be a Minkowski class, and let Ω_1 and Ω_2 be two semi-convex $\mathcal{M}$-domains. Let $\Omega = \Omega_1 + \Omega_2$ be their Minkowski sum. The proof is divided into two major steps: First, we show that Ω is regular in Section 7.1, and then we show that Ω is simply connected in Section 7.2. The result will finally follow, since a domain is homeomorphic to the unit disk if and only if it is regular and simply connected.

7.1. *Regularity*

Lemma 7.1. *Let $\mathcal{M}$ be a Minkowski class, and let Ω_1, Ω_2 be regular $\mathcal{M}$-domains. Let $\Omega = \Omega_1 + \Omega_2$, and let $p \in \partial\Omega$. Suppose $B_r(p) \cap \Omega = \bigcup_{k=1}^n S^k$, where S^k's are mutually non-overlapping $\mathcal{M}$-sectors with center p and radius r. Then there exist $\varrho > 0$ and $(p_1^1, p_2^1), \ldots, (p_1^n, p_2^n)$ in $M^{-1}_{\Omega_1,\Omega_2}(p)$ such that $S_i^k = B_\varrho(p_i^k) \cap \Omega_i$ is a non-degenerate $\mathcal{M}$-sector with center p_i^k and radius ϱ, and $S_i^k - p_i^k \subset (S_1^k + S_2^k) - p \subset S^k - p$ for each $i = 1, 2$ and $k = 1, \ldots, n$.*

Proof. By Lemma 6.9, there exist $r_1, \ldots, r_m > 0$, $(q_1^1, q_2^1), \ldots, (q_1^m, q_2^m)$ in $M^{-1}_{\Omega_1,\Omega_2}(p)$, and $0 < \varrho < \min\{r/2, r_1, \ldots, r_m\}$ such that $T_i^j = B_{r_j}(q_i^j) \cap \Omega_i$ is a finite union of mutually non-overlapping $\mathcal{M}$-sectors with center q_i^j and radius r_j for $i = 1, 2$ and $j = 1, \ldots, m$, $B_{2\varrho}(p) \cap (T_1^j + T_2^j)$ is a finite union of mutually non-overlapping $\mathcal{M}$-sectors with center p and radius 2ϱ for each j, and $B_{2\varrho}(p) \cap \Omega = \bigcup_{j=1}^m B_{2\varrho}(p) \cap (T_1^j + T_2^j)$. Since Ω_1 and Ω_2 are regular, each T_i^j is a non-degenerate $\mathcal{M}$-sector. Since r_j's can be taken arbitrarily small, we can assume that T_1^j and T_2^j are admissible to each other for each j. So by Theorem 6.2, $B_{2\varrho}(p) \cap (T_1^j + T_2^j)$ is a non-degenerate $\mathcal{M}$-sector with center p and radius 2ϱ for each j. Note that $B_{2\varrho}(p) \cap S^k$'s are mutually non-overlapping $\mathcal{M}$-sectors with center p and radius 2ϱ, and $B_{2\varrho}(p) \cap \Omega = \bigcup_{k=1}^n B_{2\varrho}(p) \cap S^k$. So it is easy to see that there exists $1 \le j_k \le m$ such that $B_{2\varrho}(p) \cap (T_1^{j_k} + T_2^{j_k}) \subset B_{2\varrho}(p) \cap S^k$ for each k. Let $p_i^k = q_i^{j_k}$, and let $S_i^k = B_\varrho(p_i^k) \cap \Omega_i = B_\varrho(q_i^{j_k}) \cap T_i^{j_k}$ for $i = 1, 2$ and $k = 1, \ldots, n$. Then $S_1^k + S_2^k = (B_\varrho(q_1^{j_k}) \cap T_1^{j_k}) + (B_\varrho(q_2^{j_k}) \cap T_2^{j_k}) \subset B_{2\varrho}(p) \cap (T_1^{j_k} + T_2^{j_k}) \subset B_{2\varrho}(p) \cap S^k \subset S^k$ for each k. Clearly, $S_i^k - p_i^k \subset (S_1^k + S_2^k) - p$ for $i = 1, 2$ and $k = 1, \ldots, n$. ∎

Lemma 7.2. *Let $\mathcal{M}$ be a Minkowski class. Let $\gamma_i : [0, a_i] \to \mathbb{R}^2$, $i = 1, 2$, be two $\mathcal{M}$-curves such that $\gamma_1(0) = \gamma_2(0) = 0$ and their images $S_1 = \gamma_1([0, a_1])$, $S_2 = \gamma_2([0, a_2])$ are degenerate $\mathcal{M}$-sectors with center 0 and radius $r > 0$. Suppose S_1, S_2 are admissible to each other, and $S_1 \ne -S_2$. Take $0 < r' \le r$ such that either $B_{r'}(0) \cap S_1 = B_{r'}(0) \cap S_2$,* *or* $B_{r'}(0) \cap S_1$, $B_{r'}(0) \cap S_2$ *do not meet except at* 0. *Let* S *be the* $\mathcal{M}$*-sector with center* 0 *and radius* r' *which is uniquely determined by the following conditions*:

(1) *S is bounded by* $B_{r'}(0) \cap S_1$, $B_{r'}(0) \cap S_2$ *and an arc in* $\partial B_{r'}(0)$.

(2) *S is a sharp sector if* $\mathbf{v}[\gamma_1] \neq -\mathbf{v}[\gamma_2]$.

(3) *When* $\mathbf{v}[\gamma_1] = -\mathbf{v}[\gamma_2]$, *the start curve of* S *is* $B_{r'}(0) \cap S_1$ (*resp.*, $B_{r'}(0) \cap S_2$), *and the end curve of* S *is* $B_{r'}(0) \cap S_2$ (*resp.*, $B_{r'}(0) \cap S_1$) *if* $\gamma_1 \lhd \gamma_2$ (*resp.*, $\gamma_1 \rhd \gamma_2$).

Then $B_\varrho(0) \cap S \subset S_1 + S_2$ *for every sufficiently small* $\varrho > 0$.

Proof. We can assume $\mathbf{v}[\gamma_1] = (\cos\theta, \sin\theta)$, $\mathbf{v}[\gamma_2] = (\cos(\pi - \theta), \sin(\pi - \theta))$ for some $0 \leq \theta \leq \pi/2$. In case $\theta = 0$, we can also assume that $\sigma(\gamma_1) = +$, $\sigma(\gamma_2) = 0$ or $+$, and $\gamma_1 \rhd \gamma_2$. Then $(0, \varrho) \in S_1 + S_2$ for every sufficiently small $\varrho > 0$ when $\theta \neq \pi/2$. Note that $(0, \varrho) \in \operatorname{int} S$ for sufficiently small $\varrho > 0$ when $\theta \neq \pi/2$. In case $\theta = \pi/2$, it is also easy to see that there exists a point in $B_\varrho(0) \cap \operatorname{int} S$ (in $B_\varrho(0) \cap S$ if S has no interior) which is contained in $S_1 + S_2$, for every sufficiently small $\varrho > 0$. By Lemma 6.7 and Theorem 6.2(3), there exist $0 < r'' < r'$ and $\varrho > 0$ such that the set $B_\varrho(0) \cap \partial((B_{r''}(0) \cap S_1) + (B_{r''}(0) \cap S_2))$ is contained in the union of the images of γ_1, γ_2 and $\gamma_1 * \gamma_2$ (if defined). Hence, $B_\varrho(0) \cap S \subset S_1 + S_2$ for every sufficiently small $\varrho > 0$. ∎

THEOREM 7.1 (regularity of Minkowski sum of semi-convex domains). *Let* $\mathcal{M}$ *be a Minkowski class, and let* Ω_1 *and* Ω_2 *be semi-convex* $\mathcal{M}$*-domains. Then their Minkowski sum* $\Omega = \Omega_1 + \Omega_2$ *is a regular* $\mathcal{M}$*-domain.*

Proof. By Theorem 6.3, Ω is an $\mathcal{M}$-domain. Suppose Ω is not regular. Then there exists a point $p \in \partial\Omega$ and $r > 0$ such that $B_r(p) \cap \Omega = \bigcup_{k=1}^n S^k$, where S^k's are mutually non-overlapping $\mathcal{M}$-sectors with center p and radius r and $n \geq 2$. By Lemma 7.1, there exist $(p_1^1, p_2^1), (p_1^2, p_2^2)$ in $M^{-1}_{\Omega_1,\Omega_2}(p)$ and $\varrho > 0$ such that, for each $k = 1, 2$ and $i = 1, 2$, $S_i^k = B_\varrho(p_i^k) \cap \Omega_i$ is a non-degenerate $\mathcal{M}$-sector with center p_i^k and radius ϱ, and $S_i^k - p_i^k \subset (S_1^k + S_2^k) - p \subset S^k - p$. Let $\widetilde{\Omega}_2 = -\Omega_2 + p$. Then by Lemma 6.3(3), Ω_1 and $\widetilde{\Omega}_2$ are in contact position to each other, and meet at p_1^1 and p_1^2. Since S^1 and S^2 are non-overlapping, it is easy to see that $p_1^1 \neq p_1^2$. For $i = 1, 2$, let $\phi_i : [0, 1] \to \partial^v \Omega_i$, $\phi_i(t) = (\gamma_i(t), \mathbf{n}_i(t))$, be a one-to-one continuous map such that $\phi_1(0) = (p_1^1, \mathbf{n}^+_{\Omega_1}(p_1^1))$, $\phi_1(1) = (p_1^2, \mathbf{n}^-_{\Omega_1}(p_1^2))$, $\phi_2(0) = (p_2^2, \mathbf{n}^+_{\Omega_2}(p_2^2))$, $\phi_2(1) = (p_2^1, \mathbf{n}^-_{\Omega_2}(p_2^1))$. By interchanging (p_1^1, p_2^1) and (p_1^2, p_2^2) if necessary, we can assume that $O_{\Omega_1}(\phi_1) = O_{\Omega_2}(\phi_2) = +$, and $(\gamma_1([0, 1]) \setminus \{p_1^1, p_1^2\}) \cap \overline{U} = \emptyset$, where U is the unbounded component of $\mathbb{R}^2 \setminus (\Omega_1 \cup \widetilde{\Omega}_2)$. Define $\widetilde{\phi}_2 : [0, 1] \to \partial^v \widetilde{\Omega}_2$, $\widetilde{\phi}_2(t) = (\widetilde{\gamma}_2(t), \widetilde{\mathbf{n}}_2(t))$, by $\widetilde{\phi}_2(t) = (-\gamma_2(t) + p, -\mathbf{n}_2(t))$ for $t \in [0, 1]$. Then $(\widetilde{\gamma}_2([0, 1]) \setminus \{p_1^1, p_1^2\}) \cap \overline{U} = \emptyset$. Since Ω_1 and $\widetilde{\Omega}_2$ are semi-convex, we have $-\pi \leq \Theta(\phi_1), \Theta(\widetilde{\phi}_2) \leq \pi$ by Lemma 5.4(2). So $-\pi \leq \Theta(\phi_i) \leq \pi$ for $i = 1, 2$, since $\Theta(\widetilde{\phi}_2) = \Theta(\phi_2)$ by Lemma 5.3.

We will show that, in fact, $-\pi < \Theta(\phi_i) < \pi$ for $i = 1, 2$. Suppose $\Theta(\phi_1) = -\pi$. We can assume $\mathbf{n}_1(0) = (-1, 0)$. Let α_i^k and β_i^k be the end curve and start curve of the $\mathcal{M}$-sector S_i^k respectively. Then $\mathbf{v}[\alpha_1^1] = \mathbf{v}[\beta_1^2] = (0, -1)$. Suppose $\sigma(\alpha_1^1) = +$. Then there exists $t_0 \in (0, 1)$ such that $\Theta(\phi_1|_{[0,t_0]}) > 0$. So by Lemma 5.1, we have $\Theta(\phi_1|_{[t_0,1]}) = \Theta(\phi_1) - \Theta(\phi_1|_{[0,t_0]}) < -\pi$, which is impossible since Ω_1 is semi-convex. Thus $\sigma(\alpha_1^1) = 0$ or $-$. In the same way, $\sigma(\beta_1^2) = 0$ or $+$. Since $S_1^1 - p_1^1 \subset S^1 - p$, $S_1^2 - p_1^2 \subset S^2 - p$, and $S^1 - p$

and $S^2 - p$ are non-overlapping, it follows that $\mathbf{v}[\beta_1^1] = \mathbf{v}[\alpha_1^2] = \mathbf{v}[\beta^1] = \mathbf{v}[\alpha^2] = (0, -1)$, and either $\sigma(\beta^1) = -$ or $\sigma(\alpha^2) = +$, where α^k and β^k are the end curve and start curve of S^k for $k = 1, 2$. Let α_1 be the non-negative angle of counter-clockwise rotation from $-\mathbf{v}^-_{\tilde{\Omega}_2}(p_1^1) = -\mathbf{v}[\beta_2^1]$ to $\mathbf{v}^+_{\Omega_1}(p_1^1) = \mathbf{v}[\alpha_1^1]$, and let α_2 be the non-negative angle of counter-clockwise rotation from $-\mathbf{v}^-_{\Omega_1}(p_1^2) = \mathbf{v}[\beta_1^2]$ to $\mathbf{v}^+_{\tilde{\Omega}_2}(p_1^2) = -\mathbf{v}[\alpha_2^2]$. Suppose $\alpha_1 < \pi$. Then by Lemma 7.2, the Minkowski sum of β_1^1 and β_2^1, which is contained in S^1, must intersect S^2. But this is impossible, since S^1 and S^2 are non-overlapping. So $\alpha_1 \geq \pi$. In the same way, $\alpha_2 \geq \pi$. By Lemma 5.4(1), we have $\Theta(\phi_1) + \Theta(\widetilde{\phi_2}) + \alpha_1 + \alpha_2 = 0$, and hence $\Theta(\phi_1) + \Theta(\phi_2) + \alpha_1 + \alpha_2 = 0$. Since $-\pi \leq \Theta(\phi_2) \leq \pi$ and $\alpha_1 + \alpha_2 \geq 2\pi$, we must have $\alpha_1 = \alpha_2 = \pi$ and $\Theta(\phi_2) = -\pi$. So $\mathbf{v}[\beta_2^1] = \mathbf{v}[\alpha_2^2] = (0, -1)$. Remember that either $\sigma(\beta^1) = -$ or $\sigma(\alpha^2) = +$. Suppose $\sigma(\beta^1) = -$. Then $\sigma(\beta_2^1)$ is also $-$. So there exists $t_0 \in (0, 1)$ such that $\Theta(\phi_2|_{[t_0,1]}) > 0$. By Lemma 5.1, we have $\Theta(\phi_2|_{[0,t_0]}) = \Theta(\phi_2) - \Theta(\phi_2|_{[t_0,1]}) < -\pi$, which is impossible since Ω_2 is semi-convex. In the same way, we get a contradiction if $\sigma(\alpha^2) = -$. Thus we conclude that $\Theta(\phi_1) \neq -\pi$. By a symmetric argument, $\Theta(\phi_2) \neq -\pi$. It follows from Lemma 5.4(1) that $\Theta(\phi_i) \neq \pi$ for $i = 1, 2$. Thus $-\pi < \Theta(\phi_i) < \pi$ for $i = 1, 2$.

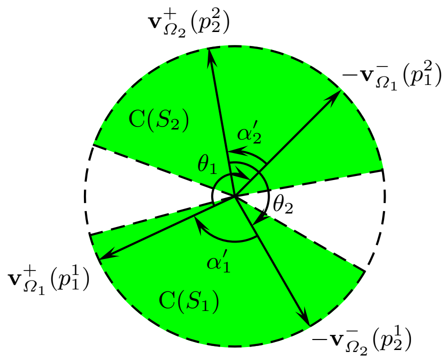


Fig. 13. $\mathrm{C}(S_1)$ and $\mathrm{C}(S_2)$

Now let θ_1 (≤ 0) be the angle of clockwise rotation from $\mathbf{v}^+_{\Omega_1}(p_1^1)$ to $-\mathbf{v}^-_{\Omega_1}(p_1^2)$, and θ_2 (≤ 0) be the angle of clockwise rotation from $\mathbf{v}^+_{\Omega_2}(p_2^2)$ to $-\mathbf{v}^-_{\Omega_2}(p_2^1)$. Note that $\theta_i = \Theta(\phi_i) - \pi + 2\pi n_i$ for some $n_i \in \mathbb{Z}$ for $i = 1, 2$. Since S^1 and S^2 are non-overlapping, we have $-2\pi \leq \theta_i \leq 0$ (see Figure 13). So $n_i = 0$ for $i = 1, 2$, since $-\pi < \Theta(\phi_i) < \pi$. Thus $\theta_i = \Theta(\phi_i) - \pi$ for $i = 1, 2$. Let α_1' be the angle of rotation *in* $\mathrm{C}(S^1)$ from $-\mathbf{v}^-_{\Omega_2}(p_2^1)$ to $\mathbf{v}^+_{\Omega_1}(p_1^1)$, and α_2' be the angle of rotation *in* $\mathrm{C}(S^2)$ from $-\mathbf{v}^-_{\Omega_1}(p_1^2)$ to $\mathbf{v}^+_{\Omega_2}(p_2^2)$. We understand α_i' to be positive if the rotation is counter-clockwise, and negative if the rotation is clockwise. Note that $-\mathbf{v}^-_{\tilde{\Omega}_2}(p_1^1) = \mathbf{v}^-_{\Omega_2}(p_2^1)$, and $-\mathbf{v}^+_{\tilde{\Omega}_2}(p_1^2) = \mathbf{v}^+_{\Omega_2}(p_2^2)$. So clearly $\alpha_i' = \alpha_i - \pi$ for $i = 1, 2$ by Lemma 7.2.

From the definitions, it is obvious that $\theta_1 + \theta_2 + \alpha_1' + \alpha_2' = -2\pi$ (see Figure 13). So from the above relations between θ_i's and $\Theta(\phi_i)$'s, and α_i''s and α_i's, it follows that $\Theta(\phi_1) + \Theta(\phi_2) + \alpha_1 + \alpha_2 = 2\pi$, which contradicts Lemma 5.4(1). ∎

7.2. *Simple connectedness.* In this section, we show that the Minkowski sum of two semi-convex $\mathcal{M}$-domains is simply connected for any Minkowski class $\mathcal{M}$.

Let $\mathcal{M}$ be a Minkowski class, and let Ω be a simply connected regular $\mathcal{M}$-domain. For each $q \in \Omega$, we fix a homotopy $H_{\Omega;q} : \Omega \times [0,1] \to \Omega$ such that $H_{\Omega;q}(p,0) = p$ and $H_{\Omega;q}(p,1) = q$ for every $p \in \Omega$. For each $q \in \mathbb{R}^2$, we define $I_q : \mathbb{R}^2 \to \mathbb{R}^2$ by $I_q(p) = -p+q$ for $p \in \mathbb{R}^2$. Note that $I_q \circ I_q$ is the identity map.

LEMMA 7.3. *Let $\mathcal{M}$ be a Minkowski class, and let Ω_1 and Ω_2 be two semi-convex $\mathcal{M}$-domains with $0 \in \Omega_1, \Omega_2$. Let $\Omega = \Omega_1 + \Omega_2$, and let $p \in \partial\Omega$. Then there exist one-to-one continuous maps $\phi^+, \phi^- : [0,1] \to \partial^v\Omega$, $\phi^\pm(t) = (\gamma^\pm(t), \mathbf{n}^\pm(t))$, and continuous maps $\phi_i^+, \phi_i^- : [0,1] \to \partial^v\Omega_i$, $\phi_i^\pm(t) = (\gamma_i^\pm(t), \mathbf{n}_i^\pm(t))$ for $i = 1,2$, which satisfy the following conditions:*

(1) *$\phi^\pm(0) = (p, \mathbf{n}_\Omega^\pm(p))$, $O_\Omega(\phi^\pm) = \pm$, and $\gamma^\pm(t)$ is a flat point for every $t \in (0,1]$.*

(2) *Each of $\phi_i^\pm$'s and $\gamma_i^\pm$'s is either one-to-one or constant, and if one of $O_{\Omega_1}(\gamma_1^\pm)$ and $O_{\Omega_2}(\gamma_2^\pm)$ is $\mp$, then the other is $\pm$.*

(3) *$\gamma^\pm(t) = \gamma_1^\pm(t) + \gamma_2^\pm(t)$ and $\mathbf{n}^\pm(t) = \mathbf{n}_1^\pm(t) = \mathbf{n}_2^\pm(t)$ for $t \in [0,1]$.*

(4) *$\Theta(\phi^\pm) = \Theta(\phi_1^\pm) = \Theta(\phi_2^\pm)$ and, for $i = 1,2$, $\gamma_i^\pm$ is homotopic to $\gamma^\pm$ in $\mathbb{R}^2 \setminus \operatorname{int}\Omega_i$ via the homotopy $H_i^\pm : [0,1] \times [0,1] \to \mathbb{R}^2 \setminus \operatorname{int}\Omega_i$ defined by*

$$H_1^\pm(t,s) = I_{\gamma^\pm(t)}(H_{\Omega_2;0}(\gamma_2^\pm(t),s)), \qquad H_2^\pm(t,s) = I_{\gamma^\pm(t)}(H_{\Omega_1;0}(\gamma_1^\pm(t),s)),$$

for $(t,s) \in [0,1] \times [0,1]$.

Proof. By Theorem 7.1, Ω is a regular $\mathcal{M}$-domain. Let $p \in \partial\Omega$. By Lemma 4.3, there exists $r > 0$ such that $B_r(p) \cap \Omega$ is a non-degenerate $\mathcal{M}$-sector with center p and radius r. By Lemma 6.9, there exist $0 < r_1, \ldots, r_n < r$, $0 < \varrho < r$ and $(p_1^1, p_2^1), \ldots, (p_1^n, p_2^n)$ in $M_{\Omega_1,\Omega_2}^{-1}(p)$ such that $S_i^k = B_{r_k}(p_i^k) \cap \Omega_i$ is a finite union of mutually non-overlapping $\mathcal{M}$-sectors with center p_i^k and radius r_k for $i = 1,2$ and $k = 1, \ldots, n$, and $S^k = B_\varrho(p) \cap (S_1^k + S_2^k)$ is a finite union of mutually non-overlapping sectors with center p and radius ϱ, and $S = B_\varrho(p) \cap \Omega = \bigcup_{k=1}^n S^k$. Since r can be taken arbitrarily small and Ω_1, Ω_2 are regular, we can assume that $B_{r_k}(p_i^k) \cap \Omega_i$ is a non-degenerate $\mathcal{M}$-sector with center p_i^k and radius r_k for each i and k. By Theorem 6.2, we can also assume each S^k is a non-degenerate $\mathcal{M}$-sector with center p and radius ϱ, since r_k's can be taken arbitrarily small. Note that S is a non-degenerate $\mathcal{M}$-sector with center p and radius ϱ. Let $\gamma^+, \gamma^- : [0,1] \to S$ be the end curve and start curve of S respectively. Since $S = \bigcup S^k$, there exist $1 \le k^+, k^- \le n$ such that γ^+ and γ^- are the end curve of S^{k^+} and the start curve of S^{k^-} respectively. Since γ^+ and γ^- are in the boundary of Ω, they are in the boundary of $S_1^{k^+} + S_2^{k^+}$ and $S_1^{k^-} + S_2^{k^-}$ respectively. So by Theorem 6.2, there exist $0 < \varepsilon < 1$ and continuous maps $\phi_i^+ : [0,\varepsilon] \to \partial^v S_i^{k^+}$, $\phi_i^+(t) = (\gamma_i^+(t), \mathbf{n}_i^+(t))$, and $\phi_i^- : [0,\varepsilon] \to \partial^v S_i^{k^-}$, $\phi_i^-(t) = (\gamma_i^-(t), \mathbf{n}_i^-(t))$, for $i = 1,2$, such that $\phi_i^\pm(0) = p_i^{k^\pm}$, $\gamma^\pm(t) = \gamma_1^\pm(t) + \gamma_2^\pm(t)$ and $\mathbf{n}_{S^{k^\pm}}^\pm(\gamma^\pm(t)) = \mathbf{n}_1^\pm(t) = \mathbf{n}_2^\pm(t)$ for $t \in [0,\varepsilon]$, each $\phi_i^\pm$ and each $\gamma_i^\pm$ is either one-to-one or constant, and if one of $O_{S_1^{k^\pm}}(\gamma_1^\pm)$ and $O_{S_2^{k^\pm}}(\gamma_2^\pm)$ is $\mp$, then the other is $\pm$. Define $\phi^+, \phi^- : [0,\varepsilon] \to \partial^v S$, $\phi^\pm(t) = (\gamma^\pm(t), \mathbf{n}^\pm(t))$, by $\phi^\pm(t) = (\gamma^\pm(t), \mathbf{n}_S^\pm(\gamma^\pm(t)))$ for $t \in [0,\varepsilon]$. Note that $\phi_1^\pm$, $\phi_2^\pm$ and $\phi^\pm$ are in $\partial^v\Omega_1$, $\partial^v\Omega_2$ and $\partial^v\Omega$ respectively. Thus, by reparametrizing them on the interval $[0,1]$, (1)–(3) are checked easily.

Now we show (4). First, $\Theta(\phi^\pm) = \Theta(\phi_1^\pm) = \Theta(\phi_2^\pm)$, since $\mathbf{n}^\pm(t) = \mathbf{n}_1^\pm(t) = \mathbf{n}_2^\pm(t)$ for every $t \in [0,1]$. Note that, for $i = 1,2$, $H_i^\pm(t,0) = \gamma_i^\pm(t)$ and $H_i^\pm(t,1) = \gamma^\pm(t)$ for every

$t \in [0,1]$. By the definition of $H_{\Omega_i;0}$'s, we have $H_{\Omega_2;0}(\gamma_2^{\pm}(t),s) \in \Omega_2$ and $H_{\Omega_1;0}(\gamma_1^{\pm}(t),s) \in \Omega_1$ for all $t,s \in [0,1]$. So $H_1^{\pm}(t,s) \in -\Omega_2 + \gamma^{\pm}(t)$ and $H_2^{\pm}(t,s) \in -\Omega_1 + \gamma^{\pm}(t)$ for all t,s. By Lemma 6.3(3), Ω_1 and $-\Omega_2 + \gamma^{\pm}(t)$ are in contact position to each other, and Ω_2 and $-\Omega_1 + \gamma^{\pm}(t)$ are in contact position to each other for every t. So $-\Omega_2 + \gamma^{\pm}(t) \subset \mathbb{R}^2 \setminus \operatorname{int}\Omega_1$ and $-\Omega_1 + \gamma^{\pm}(t) \subset \mathbb{R}^2 \setminus \operatorname{int}\Omega_2$ for every t. Thus $H_1^{\pm}(t,s) \in \mathbb{R}^2 \setminus \operatorname{int}\Omega_1$ and $H_2^{\pm}(t,s) \in \mathbb{R}^2 \setminus \operatorname{int}\Omega_2$ for all t,s. This shows (4), and the proof is complete. ∎

Let us introduce the following useful notations: Let $F_1 : [a_1,b_1] \times [c,d] \to \mathbb{R}^2$ and $F_2 : [a_2,b_2] \times [c,d] \to \mathbb{R}^2$ be two homotopies such that $F_1(b_1,s) = F_2(a_2,s)$ for every $s \in [c,d]$. Then we define $F_1 \cdot F_2 : [a_1, b_1+b_2-a_2] \times [c,d] \to \mathbb{R}^2$ by

$$(F_1 \cdot F_2)(t,s) = \begin{cases} F_1(t,s) & \text{if } (t,s) \in [a_1,b_1] \times [c,d], \\ F_2(t-b_1+a_2,s) & \text{if } (t,s) \in [b_1, b_1+b_2-a_2] \times [c,d]. \end{cases}$$

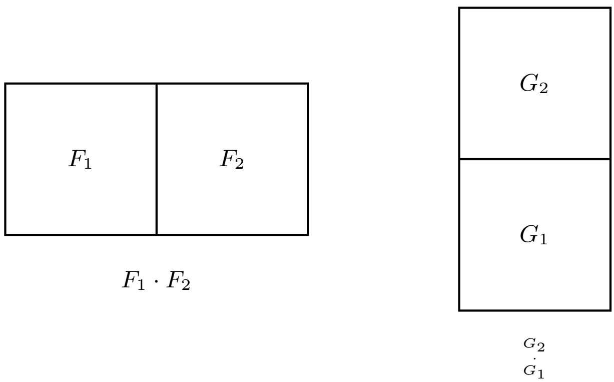


Fig. 14. The homotopies $F_1 \cdot F_2$ and $\substack{G_2 \\ \cdot \\ G_1}$

Let $G_1 : [a,b] \times [c_1,d_1] \to \mathbb{R}^2$ and $G_2 : [a,b] \times [c_2,d_2] \to \mathbb{R}^2$ be two homotopies such that $G_1(t,d_1) = G_2(t,c_2)$ for every $t \in [a,b]$. Then we define $\substack{G_2 \\ \cdot \\ G_1} : [a,b] \times [c_1, d_1+d_2-c_2] \to \mathbb{R}^2$ by

$$\begin{pmatrix} G_2 \\ \cdot \\ G_1 \end{pmatrix}(t,s) = \begin{cases} G_1(t,s) & \text{if } (t,s) \in [a,b] \times [c_1,d_1], \\ G_2(t,s-d_1+c_2) & \text{if } (t,s) \in [a,b] \times [d_1, d_1+d_2-c_2]. \end{cases}$$

It is clear that $F_1 \cdot F_2$ and $\substack{G_2 \\ \cdot \\ G_1}$ are well defined and continuous. See Figure 14.

Let $F_{ij} : [a_i,b_i] \times [c_j,d_j] \to \mathbb{R}^2$ be a homotopy for $i,j = 1,2$. Suppose that $F_{1j}(b_1,s) = F_{2j}(a_2,s)$ for every $s \in [c_j,d_j]$ and $j = 1,2$, and $F_{i1}(t,d_1) = F_{i2}(t,c_2)$ for every $t \in [a_i,b_i]$ and $i = 1,2$. Then we define $\substack{F_{12} \cdot F_{22} \\ \cdot \quad \cdot \\ F_{11} \cdot F_{21}} : [a_1, b_1+b_2-a_2] \times [c_1, d_1+d_2-c_2] \to \mathbb{R}^2$ by

$$\begin{matrix} F_{12} \cdot F_{22} \\ \cdot \qquad \cdot \\ F_{11} \cdot F_{21} \end{matrix} = \begin{pmatrix} F_{12} \\ \cdot \\ F_{11} \end{pmatrix} \cdot \begin{pmatrix} F_{22} \\ \cdot \\ F_{21} \end{pmatrix} = \begin{matrix} (F_{12} \cdot F_{22}) \\ \cdot \\ (F_{11} \cdot F_{21}) \end{matrix}.$$

See Figure 15.

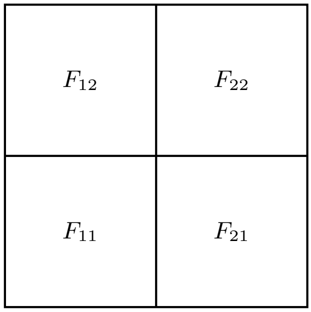


Fig. 15. The homotopy $\begin{smallmatrix} F_{12} & \cdot & F_{22} \\ \cdot & & \cdot \\ F_{11} & \cdot & F_{21} \end{smallmatrix}$

For any $m, n \geq 1$, we define in an obvious way the appropriate homotopy, when given the homotopies F_{ij}, $i = 1, \dots, m$, $j = 1, \dots, n$ with the continuity conditions on the common boundaries.

Now, let $\mathcal{M}$ be a Minkowski class, and let Ω_1 and Ω_2 be semi-convex $\mathcal{M}$-domains with $0 \in \Omega_1, \Omega_2$. Let $\Omega = \Omega_1 + \Omega_2$. Suppose $\phi^k : [0,1] \to \partial^v \Omega$, $\phi^k(t) = (\gamma^k(t), \mathbf{n}^k(t))$, and $\phi_i^k : [0,1] \to \partial^v \Omega_i$, $\phi_i^k(t) = (\gamma_i^k(t), \mathbf{n}_i^k(t))$, are continuous maps for $k, i = 1, 2$, which satisfy the following conditions:

(1) γ^k is one-to-one and $O_\Omega(\gamma^k) = +$, and $\gamma^1(1) = \gamma^2(0)$.

(2) Each of ϕ_i^k's and γ_i^k's is either one-to-one or constant, and if one of $O_{\Omega_1}(\gamma_1^k)$, $O_{\Omega_2}(\gamma_2^k)$ is $-$, then the other is $+$.

(3) $\gamma^k(t) = \gamma_1^k(t) + \gamma_2^k(t)$ and $\mathbf{n}^k(t) = \mathbf{n}_1^k(t) = \mathbf{n}_2^k(t)$ for every $t \in [0,1]$.

(4) $\Theta(\phi^k) = \Theta(\phi_1^k) = \Theta(\phi_2^k)$, and for $i, k = 1, 2$, γ_i^k is homotopic to γ^k in $\mathbb{R}^2 \setminus \operatorname{int} \Omega_i$ via the homotopy $H_i^k : [0,1] \times [0,1] \to \mathbb{R}^2 \setminus \operatorname{int} \Omega_i$, where

$$H_1^k(t,s) = I_{\gamma^k(t)}(H_{\Omega_2;0}(\gamma_2^k(t), s)), \qquad H_2^k(t,s) = I_{\gamma^k(t)}(H_{\Omega_1;0}(\gamma_1^k(t), s)),$$

for $(t,s) \in [0,1] \times [0,1]$ and $k = 1, 2$.

Let $p = \gamma^1(1) = \gamma^2(0)$. From the assumptions on ϕ^k's, it is obvious that $\phi^1(1) = (p, \mathbf{n}_\Omega^-(p))$ and $\phi^2(0) = (p, \mathbf{n}_\Omega^+(p))$. Let $\psi : [0,1] \to p \times \mathrm{NC}_\Omega(p) \subset \partial^v \Omega$, $\psi(t) = (\eta(t), \mathbf{m}(t))$, be a continuous map which is either one-to-one or constant and $\mathbf{m}(0) = \mathbf{n}_\Omega^-(p)$, $\mathbf{m}(1) = \mathbf{n}_\Omega^+(p)$. Note that $\psi(0) = \phi^1(1)$, $\psi(1) = \phi^2(0)$, and $\eta(t) = p$ for $t \in [0,1]$. Note also that ψ, $\mathbf{m}$ are one-to-one if p is a corner point, and constant if p is a flat point.

Let $p_i^1 = \gamma_i^1(1)$ and $p_i^2 = \gamma_i^2(0)$ for $i = 1, 2$. Note that $p = p_1^1 + p_2^1 = p_1^2 + p_2^2$, i.e., (p_1^1, p_2^1) and (p_1^2, p_2^2) are in $M_{\Omega_1, \Omega_2}^{-1}(p)$. Let $\psi_i : [0,1] \to \partial^v \Omega_i$, $\psi_i(t) = (\eta_i(t), \mathbf{m}_i(t))$, be a continuous map which is either one-to-one or constant, and $\psi_i(0) = (p_i^1, \mathbf{n}_i^1(1)) = \phi_i^1(1)$, $\psi_i(1) = (p_i^2, \mathbf{n}_i^2(0)) = \phi_i^2(0)$. Let $\widetilde{\Omega}_i = -\Omega_i + p$. Note that Ω_1 and $\widetilde{\Omega}_2$ are in contact position to each other, and $p_1^1, p_1^2 \in \Omega_1 \cap \widetilde{\Omega}_2$. Also, Ω_2 and $\widetilde{\Omega}_1$ are in contact position to each other and $p_2^1, p_2^2 \in \Omega_2 \cap \widetilde{\Omega}_1$. We assume that $(\eta_i([0,1]) \setminus \{p_i^1, p_i^2\}) \cap \overline{U}_i = \emptyset$, where U_1 is the unbounded component of $\mathbb{R}^2 \setminus (\Omega_1 \cup \widetilde{\Omega}_2)$, and U_2 is the unbounded component of $\mathbb{R}^2 \setminus (\Omega_2 \cup \widetilde{\Omega}_1)$.

Note that $p_1^1 = p_1^2$ if and only if $p_2^1 = p_2^2$. Suppose first $p_1^1 = p_1^2$. Then clearly η_1, η_2 are constant, and ψ_1, ψ_2 are either one-to-one or constant. Let $p_1 = p_1^1 = p_1^2$ and

$p_2 = p_2^1 = p_2^2$. Take $r > 0$ such that $S_i = B_r(p_i) \cap \Omega_i$ is a non-degenerate $\mathcal{M}$-sector with center p_i and radius r for $i = 1, 2$, and $S = B_{2r}(p) \cap \Omega$ is a non-degenerate $\mathcal{M}$-sector with center p and radius $2r$. We can assume that S_1, S_2 are admissible to each other by Lemma 6.4. Note that $S_i - p_i \subset (S_1 + S_2) - p \subset S - p$. Since $\mathbf{m}(0) = \mathbf{m}_1(0) = \mathbf{m}_2(0)$ and $\mathbf{m}(1) = \mathbf{m}_1(1) = \mathbf{m}_2(1)$, we have $\Theta(\psi_i) = \Theta(\psi) + 2n_i\pi$ for some $n_i \in \mathbb{Z}$. Note that $-\pi \le \Theta(\psi), \Theta(\psi_1), \Theta(\psi_2) \le \pi$, since $\mathbf{m}$, $\mathbf{m}_1$, $\mathbf{m}_2$ rotate in $\mathrm{NC}_\Omega(p)$, $\mathrm{NC}_{\Omega_1}(p_1)$, $\mathrm{NC}_{\Omega_2}(p_2)$ respectively. So, if $-\pi < \Theta(\psi) < \pi$, we get $\Theta(\psi) = \Theta(\psi_1) = \Theta(\psi_2)$.

Suppose $\Theta(\psi) = \pi$. Then S becomes a sharp sector, and $\mathrm{C}(S)$ contains only one element. We can assume that $\mathrm{C}(S) = \{(0, -1)\}$. Since $S_i - p_i \subset S - p$ for $i = 1, 2$, S_1, S_2 are also sharp sectors, and $\mathrm{C}(S_1) = \mathrm{C}(S_2) = \{(0, -1)\}$. So we must have $\Theta(\psi_1) = \Theta(\psi_2) = \pi$. Thus $\Theta(\psi) = \Theta(\psi_1) = \Theta(\psi_2)$. Suppose $\Theta(\psi) = -\pi$. Let α and β be the end curve and start curve of S respectively, and let α_i and β_i be the end curve and start curve of S_i respectively. In this case, S becomes a dull sector, and $\mathbf{v}[\alpha] = \mathbf{v}[\beta]$. With no loss of generality, assume $\mathbf{v}[\alpha] = \mathbf{v}[\beta] = (0, 1)$. Note that $\Theta(\psi_1)$, $\Theta(\psi_2)$ are π or $-\pi$. Since $\Theta(\psi_1), \Theta(\psi_2) \ne 0$, S_1 and S_2 cannot be flat sectors. Since S_1, S_2 are admissible to each other, they cannot be dull sectors simultaneously. Suppose both S_1 and S_2 are sharp sectors. Then it is easy to see that $\mathrm{C}(S_i) = \{(0, 1)\}$ or $\{(0, -1)\}$ for $i = 1, 2$. So, from Lemma 7.2, we can see that at least one of α and β is not contained in $S_1 + S_2$, which contradicts the assumption that $\gamma^1 = \gamma_1^1 + \gamma_2^1$ and $\gamma^2 = \gamma_1^2 + \gamma_2^2$. So S_1, S_2 cannot be sharp sectors simultaneously. It follows that one of S_1 and S_2, say S_1, is a sharp sector and the other is a dull sector. Then it is easy to see that $\mathbf{v}[\alpha_1] = \mathbf{v}[\beta_1] = (0, -1)$, $\mathbf{v}[\alpha_2] = \mathbf{v}[\beta_2] = (0, 1)$, and so $\Theta(\psi_1) = \Theta(\psi_2) = -\pi$. Thus we conclude that $\Theta(\psi) = \Theta(\psi_1) = \Theta(\psi_2)$ if $p_1^1 = p_1^2$ (or equivalently, $p_2^1 = p_2^2$).

Suppose now $p_1^1 \ne p_1^2$. Then it is easy to see that one of $O_{\Omega_1}(\eta_1)$, $O_{\Omega_2}(\eta_2)$ is $+$ and the other is $-$. Moreover, $O_{\Omega_i}(\eta_i|_{[a,b]})$ cannot be $-O_{\Omega_i}(\eta_i)$ for any $[a, b] \subset [0, 1]$, for $i = 1, 2$. We will also show that $\Theta(\psi) = \Theta(\psi_1) = \Theta(\psi_2)$ in this case. First, it is easy to see that $\Theta(\psi_1) = \Theta(\psi_2)$ since $\mathbf{m}_1(0) = \mathbf{m}_2(0)$, $\mathbf{m}_1(1) = \mathbf{m}_2(1)$, and Ω_1, $\widetilde{\Omega}_2$ are in contact position to each other. Since $\mathbf{m}(0) = \mathbf{m}_1(0)$ and $\mathbf{m}(1) = \mathbf{m}_1(1)$, we have $\Theta(\psi_1) = \Theta(\psi) + 2n\pi$ for some $n \in \mathbb{Z}$. We have seen that one of $O_{\Omega_1}(\eta_1)$, $O_{\Omega_2}(\eta_2)$ is $+$ and the other is $-$. So, one of $O_{\Omega_1}(\psi_1)$, $O_{\Omega_2}(\psi_2)$, say $O_{\Omega_1}(\psi_1)$, is $+$ and the other is $-$. Since Ω_1, Ω_2 are semi-convex, we have $\Theta(\psi_1) \ge -\pi$ and $\Theta(\psi_2) \le \pi$. So $-\pi \le \Theta(\psi_1) = \Theta(\psi_2) \le \pi$. Thus, $\Theta(\psi) = \Theta(\psi_1) = \Theta(\psi_2)$ if $-\pi < \Theta(\psi) < \pi$.

It remains to consider the cases when $\Theta(\psi) = \pi$ or $-\pi$. Take $r > 0$ such that $S_i^k = B_r(p_i^k) \cap \Omega_i$ is a non-degenerate $\mathcal{M}$-sector with center p_i^k and radius r for $i, k = 1, 2$, and $S = B_{2r}(p) \cap \Omega$ is a non-degenerate $\mathcal{M}$-sector with center p and radius $2r$. We can assume S_1^k and S_2^k are admissible to each other. Note that $S_i^k - p_i^k \subset (S_1^k + S_2^k) - p \subset S - p$. Let α and β be the end curve and start curve of S respectively, and let α_i^k and β_i^k be the end curve and start curve of S_i^k respectively.

Suppose $\Theta(\psi) = \pi$. Then S is a sharp sector, and $\mathrm{C}(S)$ contains only one element, which we assume to be $(0, -1)$. Since $S_i^k - p_i^k \subset S - p$, we have $\mathrm{C}(S_i^k) = \{(0, -1)\}$. So $\mathbf{v}[\alpha_i^k] = \mathbf{v}[\beta_i^k] = (0, -1)$. We can assume that $O_{\Omega_1}(\psi_1) = +$ and $O_{\Omega_2}(\psi_2) = -$. Suppose $\Theta(\psi_1) = \Theta(\psi_2) = -\pi$. Note that $\mathbf{m}(0) = (1, 0)$ and $\mathbf{m}(1) = (-1, 0)$. So $-\pi = \Theta(\psi_1) = \pi + \Theta(\psi_1') + \pi$, where $\psi_1' : [0, 1] \to \partial^v \Omega_1$ is a one-to-one continuous map such that $\psi_1'(0) = (p_1^1, \mathbf{n}_{\Omega_1}^+(p_1^1))$, $\psi_1'(1) = (p_1^2, \mathbf{n}_{\Omega_1}^-(p_1^2))$, and $O_{\Omega_1}(\psi_1') = +$. Now $\Theta(\psi_1') = -3\pi$, which is a contradiction since Ω_1 is semi-convex. So we must have $\Theta(\psi_1) = \Theta(\psi_2) = \pi$, and hence $\Theta(\psi) = \Theta(\psi_1) = \Theta(\psi_2)$ if $\Theta(\psi) = \pi$.

Suppose $\Theta(\psi) = -\pi$. Then S is a dull sector, and $\mathbf{v}[\alpha] = \mathbf{v}[\beta]$, which we assume to be $(0, 1)$. Let $S' = B_{2r}(p) \setminus S$. Suppose $\Theta(\psi_1) = \Theta(\psi_2) = \pi$. We can assume that $O_{\Omega_1}(\psi_1) = +$ and $O_{\Omega_2}(\psi_2) = -$. Note that $\mathbf{m}(0) = (-1, 0)$, $\mathbf{m}(1) = (1, 0)$, and $S_i^k - p_i^k \subset S_1^k + S_2^k - p \subset S - p$. Let $\mathbf{v}[\beta_2^1] = (\cos\theta, \sin\theta)$. If $\pi/2 < \theta < 3\pi/2$, then $\mathbf{m}_1(0) = (-1, 0)$ cannot be in $\mathrm{NC}_{\Omega_2}(p_2^1)$. If $-\pi/2 \leq \theta < \pi/2$, then there exists $t_0 \in (0, 1)$ such that $\Theta(\psi_2|_{[0,t_0]}) < 0$. So $\Theta(\psi_2|_{[t_0,1]}) = \Theta(\psi_2) - \Theta(\psi_2|_{[0,t_0]}) > \pi$, which is impossible since Ω_2 is semi-convex and $O_{\Omega_2}(\psi_2) = -$. Thus we must have $\mathbf{v}[\beta_2^1] = (0, 1)$. Note also that $\sigma(\beta_2^1) \neq -$ for the same reason. In the same way, we can see that $\mathbf{v}[a_2^2] = (0, 1)$ and $\sigma(\alpha_2^2) \neq +$. Let $\mathbf{v}[\alpha_1^1] = (\cos\theta_1, \sin\theta_1)$ and $\mathbf{v}[\beta_1^2] = (\cos\theta_2, \sin\theta_2)$. Suppose $\pi/2 < \theta_1 < 3\pi/2$. Then we should have $\mathbf{v}[\beta_1^1] = (0, 1)$ in order for $\mathbf{m}(0) = (-1, 0)$ to be in $\mathrm{NC}_{\Omega_1}(p_1^1)$. Since $\mathbf{v}[\beta_2^1] = (0, 1)$, it follows from Lemma 7.2 that $B_\varrho(p) \subset S_1^1 + S_2^1 \subset S$ for sufficiently small $\varrho > 0$, which is impossible. Suppose $\theta_1 = \pi/2$. Then, in order for $\mathbf{m}(0)$ to be in $\mathrm{NC}_{\Omega_1}(p_1^1)$, we must have $\mathbf{v}[\beta_1^1] = (0, 1)$ again, and $\mathrm{C}(S_1^1) = \{(0, 1)\}$ or $\partial B_1(0)$. If $\mathrm{C}(S_1^1) = \partial B_1(0)$, then we would also have the same contradiction $B_\varrho(p) \subset S_1^1 + S_2^1$ by Lemma 7.2. So $\mathrm{C}(S_1^1) = \{(0, 1)\}$. Let W_1 be the sharp sector with center 0 and radius $2r$, whose start curve and end curve are $\beta - p$ and $\{(x, 0) \mid 0 \leq x \leq 2r\}$ respectively. Let W_2 be the sharp sector with center 0 and radius $2r$, whose start curve and end curve are $\{(x, 0) \mid -2r \leq x \leq 0\}$ and $\alpha - p$ respectively. Note that $\alpha_1^1 - p_1^1, \beta_1^1 - p_1^1 \subset W_i$ and $\beta_2^1 - p_2^1 \subset W_j$ for some $i, j = 1, 2$. If $i \neq j$, then $B_\varrho(p) \cap S' \subset S_1^1 + S_2^1 \subset S$ by Lemma 7.2, which is a contradiction. So $i = j$. Suppose $i = j = 2$. Since $S_2^1 - p_2^1 \subset S - p$, we also have $\alpha_2^1 - p_2^1 \subset W_2$ and $\mathbf{v}[\alpha_2^1] = (0, 1)$. Now from Lemma 2.3, we can see that $\beta - p$ cannot be any of $\alpha_1^1 - p_1^1$, $\beta_1^1 - p_1^1$, $\alpha_2^1 - p_2^1$, $\beta_2^1 - p_2^1$, or their convolutions. From Lemma 6.7 and Theorem 6.2(3), we see that this contradicts the assumption that $\gamma^1 = \gamma_1^1 + \gamma_2^1$. Suppose $i = j = 1$. Since $\sigma(\beta_2^1) \neq -$, we must have $\sigma(\beta) \neq -$ and $\sigma(\alpha) = +$. So, $\alpha_2^2 - p_2^2, \beta_2^2 - p_2^2 \subset W_1$ and $\mathbf{v}[\beta_2^2] = (0, 1)$, since $\mathbf{v}[\alpha_2^2] = (0, 1)$, $\sigma(\alpha_2^2) \neq +$ and $S_2^2 - p_2^2 \subset S - p$. Since $\gamma^2 = \gamma_1^2 + \gamma_2^2$, $\alpha - p$ should be one of $\alpha_1^2 - p_1^2$, $\beta_1^2 - p_1^2$, $\alpha_2^2 - p_2^2$, $\beta_2^2 - p_2^2$, or their convolutions by Lemma 6.7 and Theorem 6.2(3). There are only two cases to make this possible: Either $\alpha - p$ is one of $\alpha_1^2 - p_1^2$, $\beta_1^2 - p_1^2$, or $\sigma(\beta_2^2) = +$ and $\alpha = \beta_2^2 * \gamma$ for $\gamma = \alpha_1^2$ or β_1^2. But it is easy to see from Lemma 7.2 that, in both cases, $S_1^2 + S_2^2$ would intersect $B_\varrho(p) \cap S'$ for sufficiently small $\varrho > 0$, which is a contradiction. Thus $\mathbf{v}[\alpha_1^1] \neq (0, 1)$. So we have $-\pi/2 \leq \theta_1 < \pi/2$. Similarly, $\pi/2 < \theta_2 \leq 3\pi/2$.

Suppose $\theta_1 = -\pi/2$, i.e., $\mathbf{v}[\alpha_1^1] = (0, -1)$. Let α_1 be the non-negative angle of counter-clockwise rotation from $-\mathbf{v}[\beta_2^1]$ to $\mathbf{v}[\alpha_1^1]$ *in* V, where V is the region bounded by η_1 and $-\eta_2 + p$. Let α_2 be the non-negative angle of counter-clockwise rotation from $\mathbf{v}[\beta_1^2]$ to $-\mathbf{v}[\alpha_2^2]$ *in* V. Suppose either $\sigma(\alpha_1^1) = -$ or $\alpha_1^1 \lhd \beta_2^1$. Then $\alpha_1 = 2\pi$, since S_1^1, S_2^1 are admissible to each other. For $i = 1, 2$, we can choose $[a_i, b_i] \subset [0, 1]$ such that $\psi_1(a_1) = (p_1^1, \mathbf{n}_{\Omega_1}^+(p_1^1)$, $\psi_1(b_1) = (p_1^2, \mathbf{n}_{\Omega_1}^-(p_1^2))$, and $\psi_2(a_2) = (p_2^1, \mathbf{n}_{\Omega_2}^-(p_2^1)$, $\psi_2(b_2) = (p_2^2, \mathbf{n}_{\Omega_2}^+(p_2^2))$. Note that $a_1 = a_2 = 0$, since $\mathbf{v}[\alpha_1^1] = -\mathbf{v}[\beta_2^1] = (0, -1)$. By Lemma 5.4(1), $\Theta(\psi_1|_{[0,b_1]}) - \Theta(\psi_2|_{[0,b_2]}) + 2\pi + \alpha_2 = 0$. Since Ω_1, Ω_2 are semi-convex, we must have $\Theta(\psi_1|_{[0,b_1]}) = -\pi$, $\Theta(\psi_2|_{[0,b_2]}) = \pi$ and $\alpha_2 = 0$. Since $\alpha_2 = 0$ and $\mathbf{m}(1) = \mathbf{m}_1(1) = \mathbf{m}_2(1) = (1, 0)$, it follows that $\Theta(\psi_1) = -\pi$, which contradicts the assumption. Thus if $\mathbf{v}[\alpha_1^1] = (0, -1)$ then we must have $\sigma(\alpha_1^1) \neq -$ and either $\alpha_1^1 \rhd \beta_2^1$ or $\alpha_1^1 \sim \beta_2^1$. Similarly, if $\mathbf{v}[\beta_1^2] = (0, -1)$ then $\sigma(\beta_1^2) \neq +$ and either $\beta_1^2 \rhd \alpha_2^2$ or $\beta_1^2 \sim \alpha_2^2$. Now it is easy to see from Lemma 7.2 that $(S_1^1 + S_2^1) \cup (S_1^2 + S_2^2)$ contains $B_\varrho(p) \cap S'$ for sufficiently small $\varrho > 0$. This is a contradiction, since $S_1^1 + S_2^1, S_1^2 + S_2^2 \subset S$. Thus we must have $\Theta(\psi_1) = \Theta(\psi_2) = -\pi$.

Summarizing the above arguments, we conclude that $\Theta(\psi) = \Theta(\psi_1) = \Theta(\psi_2)$ in any case.

For $i = 1, 2$, define $\widetilde{\eta}_i : [0,1] \to \mathbb{R}^2$ by $\widetilde{\eta}_i(t) = -\eta_i(t) + p$. It is easy to see that $\widetilde{\eta}_i$ is in $\partial \widetilde{\Omega}_i$ and $\widetilde{\eta}_1(0) = p_2^1$, $\widetilde{\eta}_1(1) = p_2^2$, $\widetilde{\eta}_2(0) = p_1^1$, $\widetilde{\eta}_2(1) = p_1^2$. Let V_1 be the region enclosed by η_1 and $\widetilde{\eta}_2$, and V_2 be the region enclosed by η_2 and $\widetilde{\eta}_1$. By Lemma 5.4(3), there exists a homotopy $A_i : [0,1] \times [0,1] \to \overline{V_i}$ such that $A_1(t, 0) = \eta_1(t)$, $A_1(t, 1) = \widetilde{\eta}_2(t)$, $A_1(0, s) = p_1^1$, $A_1(1, s) = p_1^2$, and $A_2(t, 0) = \eta_2(t)$, $A_2(t, 1) = \widetilde{\eta}_1(t)$, $A_2(0, s) = p_2^1$, $A_2(1, s) = p_2^2$ for every $(t, s) \in [0,1] \times [0,1]$. For $i = 1, 2$, let $B_i : [0,1] \times [0,1] \to \mathbb{R}^2$ be the homotopies defined by

$$B_1(t, s) = I_p(H_{\Omega_2;0}(\eta_2(t), s)), \qquad B_2(t, s) = I_p(H_{\Omega_1;0}(\eta_1(t), s)),$$

for $(t, s) \in [0,1] \times [0,1]$. Then $B_1(t, 0) = \widetilde{\eta}_2(t)$, $B_2(t, 0) = \widetilde{\eta}_1(t)$, and $B_1(t, 1) = B_2(t, 1) = p$ for $t \in [0, 1]$. It is also easy to see that $B_i([0,1] \times [0,1]) \subset \mathbb{R}^2 \setminus \operatorname{int} \Omega_i$. For $i, k = 1, 2$ we define $E_i^k : [0,1] \times [0,1] \to \mathbb{R}^2$ by

$$E_i^k(t, s) = \gamma_i^k(t)$$

for $(t, s) \in [0,1] \times [0,1]$.

Now we can see that the homotopy $G_i = \begin{array}{ccccc} H_i^1 & \cdot & B_i & \cdot & H_i^2 \\ \cdot & & & & \cdot \\ E_i^1 & \cdot & A_i & \cdot & E_i^2 \end{array}$ is well defined, where H_i's are defined as in Lemma 7.3, and $G_i([0,3] \times [0,2]) \subset \mathbb{R}^2 \setminus \operatorname{int} \Omega_i$ for $i = 1, 2$. See Figure 16. Note that $\gamma_i^1 \cdot \eta_i \cdot \gamma_i^2$ is homotopic to $\gamma^1 \cdot \eta \cdot \gamma^2$ in $\mathbb{R}^2 \setminus \operatorname{int} \Omega_i$ via G_i.

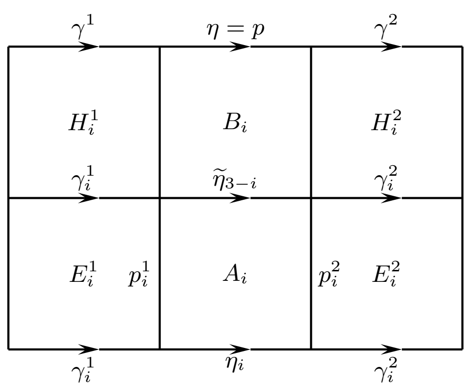


Fig. 16. The homotopy $G_i = \begin{array}{ccccc} H_i^1 & \cdot & B_i & \cdot & H_i^2 \\ \cdot & & & & \cdot \\ E_i^1 & \cdot & A_i & \cdot & E_i^2 \end{array}$

Let $\widetilde{\phi} : [0,1] \to \partial^v \Omega$, $\widetilde{\phi}(t) = (\widetilde{\gamma}(t), \widetilde{\mathbf{n}}(t))$, be a locally one-to-one, continuous map such that $O_\Omega(\widetilde{\phi}) = +$ and $\widetilde{\gamma}(0)$, $\widetilde{\gamma}(1)$ are flat points. Since $[0, 1]$ is compact, Lemma 7.3 shows that there exist $\widetilde{\gamma}(0) = p^0, \ldots, p^n = \widetilde{\gamma}(1) \in \partial \Omega$ and continuous maps $\phi^k : [0,1] \to \partial^v \Omega$,

$\phi^k(t) = (\gamma^k(t), \mathbf{n}^k(t))$, and $\phi_i^k : [0,1] \to \partial^v \Omega_i$, $\phi_i^k(t) = (\gamma_i^k(t), \mathbf{n}_i^k(t))$, for $i = 1,2$ and $k = 1, \ldots, n$, such that:

(1) $\gamma^k(0) = p^{k-1}$, $\gamma^k(1) = p^k$, γ^k is one-to-one, and $O_\Omega(\gamma^k) = +$.

(2) Each of ϕ_i^k's and γ_i^k's is either one-to-one or constant, and, if one of $O_{\Omega_1}(\gamma_1^k)$, $O_{\Omega_2}(\gamma_2^k)$ is $-$, then the other is $+$.

(3) $\gamma^k(t) = \gamma_1^k(t) + \gamma_2^k(t)$ and $\mathbf{n}^k(t) = \mathbf{n}_1^k(t) = \mathbf{n}_2^k(t)$ for $t \in [0,1]$.

(4) $\Theta(\phi^k) = \Theta(\phi_1^k) = \Theta(\phi_2^k)$, and, for $i = 1, 2$, γ_i^k is homotopic to γ^k in $\mathbb{R}^2 \setminus \operatorname{int} \Omega_i$ via the homotopy $H_i^k : [0,1] \times [0,1] \to \mathbb{R}^2 \setminus \operatorname{int} \Omega_i$, where

$$H_1^k(t,s) = I_{\gamma^k(t)}(H_{\Omega_2;0}(\gamma_2^k(t), s)), \qquad H_2^k(t,s) = I_{\gamma^k(t)}(H_{\Omega_1;0}(\gamma_1^k(t), s)),$$

for $(t,s) \in [0,1] \times [0,1]$.

(5) There exists a continuous, onto, non-decreasing function $\widetilde{h} : [0,1] \to [0,n]$ such that $\widetilde{\gamma}(t) = (\gamma^1 \cdot \ \ldots \ \cdot \gamma^n)(\widetilde{h}(t))$ for $t \in [0,1]$.

From the above arguments, there exist continuous maps $\psi^k : [0,1] \to \partial^v \Omega$, $\psi^k(t) = (\eta^k(t), \mathbf{m}^k(t))$, and $\psi_i^k : [0,1] \to \partial^v \Omega_i$, $\psi_i^k(t) = (\eta_i^k(t), \mathbf{m}_i^k(t))$, and a homotopy $A_i^k : [0,1] \times [0,1] \to \overline{V_i^k}$ for $i = 1,2$ and $k = 1, \ldots, n-1$ ($k = 1, \ldots, n$ if $\widetilde{\phi}(0) = \widetilde{\phi}(1)$), where $V_i^k \subset \mathbb{R}^2 \setminus (\Omega_i \cup (-\Omega_{3-i} + p^k))$ is the region bounded by η_i^k and $\widetilde{\eta}_{3-i}^k = -\eta_{3-i}^k + p^k$, such that $\Theta(\psi^k) = \Theta(\psi_1^k) = \Theta(\psi_2^k)$, and $\gamma_i^k \cdot \eta_i^k \cdot \gamma_i^{k+1}$ is homotopic to $\gamma^k \cdot \eta^k \cdot \gamma^{k+1}$ in $\mathbb{R}^2 \setminus \operatorname{int} \Omega_i$ via $\begin{smallmatrix} H_i^k & \cdot & B_i^k & \cdot & H_i^{k+1} \\ \cdot & & \cdot & & \cdot \\ E_i^k & \cdot & A_i^k & \cdot & E_i^{k+1} \end{smallmatrix}$ for $i = 1,2$ and $k = 1, \ldots, n-1$ ($k = 1, \ldots, n$ if $\widetilde{\phi}(0) = \widetilde{\phi}(1)$). Here, we let $\phi^{n+1} = \phi^1$, $\gamma^{n+1} = \gamma^1$, and $\phi_i^{n+1} = \phi_i^1$, $\gamma_i^{n+1} = \gamma_i^1$, $H_i^{n+1} = H_i^1$, $E_i^{n+1} = E_i^1$ for $i = 1, 2$. For $i = 1,2$ and $k = 1, \ldots, n-1$ (or n), define $E_i^k(t,s) = \gamma_i^k(t)$ for $(t,s) \in [0,1] \times [0,1]$, and

$$B_1^k(t,s) = I_{p^k}(H_{\Omega_2;0}(\eta_2^k(t), s)), \qquad B_2^k(t,s) = I_{p^k}(H_{\Omega_1;0}(\eta_1^k(t), s)),$$

for $(t,s) \in [0,1] \times [0,1]$.

For $i = 1, 2$, let

$$\begin{aligned}
\phi &= \phi^1 \cdot \psi^1 \cdot \phi^2 \cdot \ \ldots \ \cdot \psi^{n-1} \cdot \phi^n, \\
\phi_i &= \phi_i^1 \cdot \psi_i^1 \cdot \phi_i^2 \cdot \ \ldots \ \cdot \psi_i^{n-1} \cdot \phi_i^n, \\
\gamma &= \gamma^1 \cdot \eta^1 \cdot \gamma^2 \cdot \ \ldots \ \cdot \eta^{n-1} \cdot \gamma^n, \\
\gamma_i &= \gamma_i^1 \cdot \eta_i^1 \cdot \gamma_i^2 \cdot \ \ldots \ \cdot \eta_i^{n-1} \cdot \gamma_i^n, \\
\widetilde{\gamma}_i &= \gamma_i^1 \cdot \widetilde{\eta}_{3-i}^1 \cdot \gamma_i^2 \cdot \ \ldots \ \cdot \widetilde{\eta}_{3-i}^{n-1} \cdot \gamma_i^n, \\
P_i &= E_i^1 \cdot A_i^1 \cdot E_i^2 \cdot \ \ldots \ \cdot A_i^{n-1} \cdot E_i^n, \\
Q_i &= H_i^1 \cdot B_i^1 \cdot H_i^2 \cdot \ \ldots \ \cdot B_i^{n-1} \cdot H_i^n.
\end{aligned}$$

When $\widetilde{\phi}(0) = \widetilde{\phi}(1)$, we let

$$\begin{aligned}
\phi &= \phi^1 \cdot \psi^1 \cdot \phi^2 \cdot \ \ldots \ \cdot \psi^{n-1} \cdot \phi^n \cdot \psi^n, \\
\phi_i &= \phi_i^1 \cdot \psi_i^1 \cdot \phi_i^2 \cdot \ \ldots \ \cdot \psi_i^{n-1} \cdot \phi_i^n \cdot \psi_i^n, \\
\gamma &= \gamma^1 \cdot \eta^1 \cdot \gamma^2 \cdot \ \ldots \ \cdot \eta^{n-1} \cdot \gamma^n \cdot \eta^n, \\
\gamma_i &= \gamma_i^1 \cdot \eta_i^1 \cdot \gamma_i^2 \cdot \ \ldots \ \cdot \eta_i^{n-1} \cdot \gamma_i^n \cdot \eta_i^n,
\end{aligned}$$

$$\widetilde{\gamma}_i = \gamma_i^1 \cdot \widetilde{\eta}_{3-i}^1 \cdot \gamma_i^2 \cdot \ \dots \ \cdot \widetilde{\eta}_{3-i}^{n-1} \cdot \gamma_i^n \cdot \widetilde{\eta}_{3-i}^n,$$
$$P_i = E_i^1 \cdot A_i^1 \cdot E_i^2 \cdot \ \dots \ \cdot A_i^{n-1} \cdot E_i^n \cdot A_i^n,$$
$$Q_i = H_i^1 \cdot B_i^1 \cdot H_i^2 \cdot \ \dots \ \cdot B_i^{n-1} \cdot H_i^n \cdot B_i^n,$$

for $i = 1, 2$.

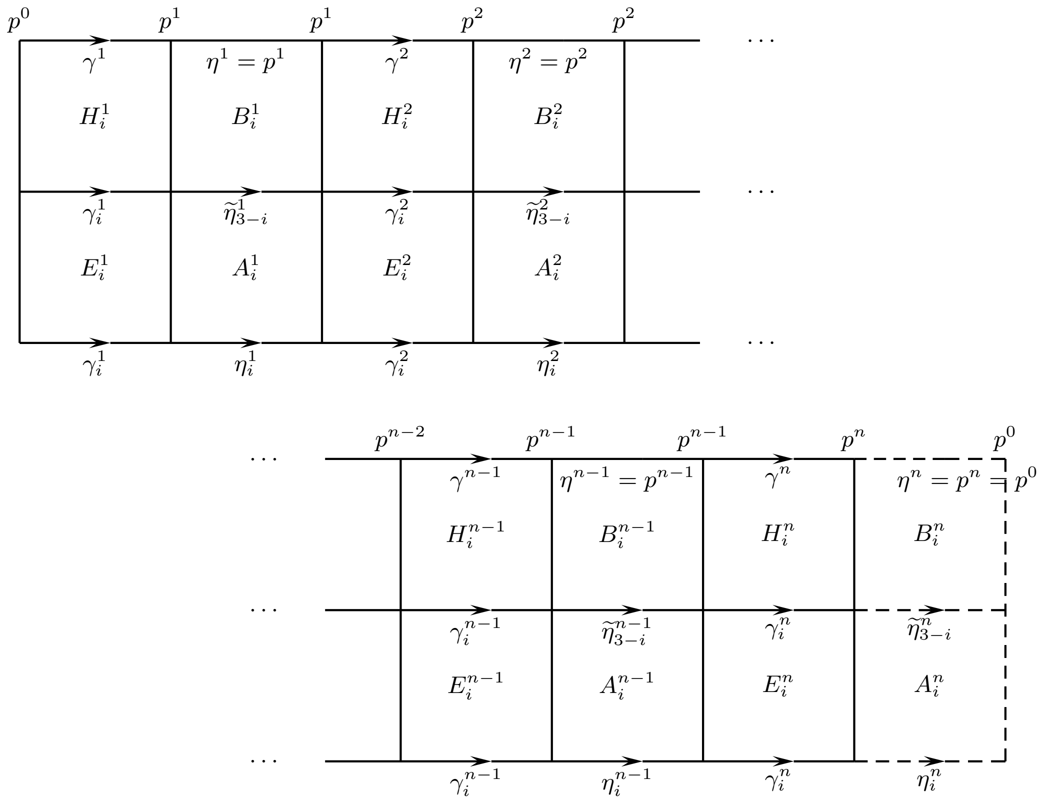


Fig. 17. The homotopy $H_i = \overset{Q_i}{\underset{P_i}{\cdot}}$

Note that $Q_i(t,s) = I_{\gamma(t)}(H_{\Omega_{3-i};0}(I_{\gamma(t)}(\widetilde{\gamma}_i(t)), s))$ for $(t,s) \in [0, 2n-1] \times [0,1]$ (for $(t,s) \in [0,2n] \times [0,1]$ if $\widetilde{\phi}(0) = \widetilde{\phi}(1)$), for $i = 1, 2$. Let $H_i = \overset{Q_i}{\underset{P_i}{\cdot}}$. See Figure 17. Now it is easy to see that γ_i is homotopic to $\widetilde{\gamma}_i$ in $\mathbb{R}^2 \setminus \operatorname{int} \Omega_i$ via P_i, and $\widetilde{\gamma}_i$ is homotopic to γ in $\mathbb{R}^2 \setminus \operatorname{int} \Omega_i$ via Q_i. So γ_i is homotopic to γ in $\mathbb{R}^2 \setminus \operatorname{int} \Omega_i$ via H_i. Furthermore, if $\widetilde{\phi}(0) = \widetilde{\phi}(1)$, then $H_i(0,s) = H_i(2n,s)$ for $s \in [0,2]$. It is also easy to see that $\Theta(\widetilde{\phi}) = \Theta(\phi) = \Theta(\phi_1) = \Theta(\phi_2)$.

Finally, we obtain the following theorem by using the above arguments:

Theorem 7.2 (simple connectedness of Minkowski sum of semi-convex domains). *Let $\mathcal{M}$ be a Minkowski class, and let Ω_1 and Ω_2 be semi-convex $\mathcal{M}$-domains. Then their Minkowski sum $\Omega = \Omega_1 + \Omega_2$ is a simply connected regular $\mathcal{M}$-domain.*

Proof. From Theorem 7.1, Ω is a regular $\mathcal{M}$-domain. We can assume $0 \in \operatorname{int}\Omega_1, \operatorname{int}\Omega_2$. Clearly, this implies $0 \in \operatorname{int}\Omega$. Suppose Ω is not simply connected. Then there exists an inner boundary C of Ω. Let $\widetilde{C}$ be the connected component of $\partial^v\Omega$ corresponding to C, and let $\widetilde{\phi} : [0,1] \to \widetilde{C}$, $\widetilde{\phi}(t) = (\widetilde{\gamma}(t), \widetilde{\mathbf{n}}(t))$, be a continuous map such that $\widetilde{\phi}(0) = \widetilde{\phi}(1)$, $\widetilde{\gamma}(0) = \widetilde{\gamma}(1)$ is a flat point, $\widetilde{\phi}|_{[0,1)}$ is one-to-one, and $O_\Omega(\widetilde{\phi}) = +$. (That is, $\widetilde{\phi}$ traverses $\widetilde{C}$ exactly once in the standard orientation.) Then $\Theta(\widetilde{\phi}) = -2\pi$ by Lemma 5.2(1).

Now take ϕ, ϕ_i, and H_i for $i = 1, 2$ as in the above arguments. We have $\Theta(\widetilde{\phi}) = \Theta(\phi) = \Theta(\phi_1) = \Theta(\phi_2)$, and γ_i is homotopic to γ in $\mathbb{R}^2 \setminus \operatorname{int}\Omega_i$ via H_i for $i = 1, 2$. Also, γ and $\widetilde{\gamma}$ are homotopic in $\partial\Omega$. So $\operatorname{Ind}_{\widetilde{\gamma}}(0) = \operatorname{Ind}_\gamma(0) = \operatorname{Ind}_{\gamma_1}(0) = \operatorname{Ind}_{\gamma_2}(0)$, since $0 \in \operatorname{int}\Omega, \operatorname{int}\Omega_1, \operatorname{int}\Omega_2$. Since C is an inner boundary of Ω and $0 \in \operatorname{int}\Omega$, we have $\operatorname{Ind}_\gamma(0) = 0$ by Lemma 5.2(2). So $\operatorname{Ind}_{\gamma_i}(0) = 0$ for $i = 1, 2$. It follows that $\Theta(\phi_1) = \Theta(\phi_2) = 0$ again by Lemma 5.2(2), since Ω_1 and Ω_2 are simply connected and hence have no inner boundaries. So $\Theta(\widetilde{\phi}) = 0$, which is a contradiction. ∎

8. Maximality of semi-convexity

Let $\mathcal{C}$ be a subclass of $\mathcal{C}^{1:1}_c$ which is closed under restriction. In this section, we show that for any regular $\mathcal{C}$-domain which is not semi-convex, there exists a semi-convex $\mathcal{C}$-domain so that their Minkowski sum is not simply connected. Combined with Theorem 7.2, this answers Problem 2 posed in Section 1 within the category of $\mathcal{M}$-domains for any Minkowski class $\mathcal{M}$. In fact, it is shown that we can choose this domain among a special kind of semi-convex $\mathcal{C}$-domains, which we call *flag domains.* Note that $\mathcal{C}$ need not be a Minkowski class.

First, we observe the following easy fact:

LEMMA 8.1. *Let Ω be a regular $\mathcal{C}^{1:1}_c$-domain which is not semi-convex. Then there exists a one-to-one continuous map $\phi : [-\varepsilon, 1+\varepsilon] \to \partial^v\Omega$, $\phi(t) = (\gamma(t), \mathbf{n}(t))$, for some $\varepsilon > 0$, which satisfies the following conditions:*

(1) *$O_\Omega(\phi) = +$ and $\Theta(\phi|_{[0,1]}) = -\pi$.*

(2) *$-\pi < \Theta(\phi|_{[s,t]}) < \pi$ for every proper subinterval $[s,t]$ of $[0,1]$.*

(3) *Let $\theta : [-\varepsilon, 1+\varepsilon] \to \mathbb{R}$ be an angle function of ϕ. Then θ is strictly decreasing on $[-\varepsilon, \varepsilon]$ and $[1-\varepsilon, 1+\varepsilon]$.*

Proof. Since Ω is not semi-convex, there exists a one-to-one continuous map $\widetilde{\phi} : [a,b] \to \partial^v\Omega$ such that $O_\Omega(\widetilde{\phi}) = +$ and $\Theta(\phi) < -\pi$. Let $\widetilde{\theta} : [a,b] \to \mathbb{R}$ be an angle function of $\widetilde{\phi}$. Since Ω is a $\mathcal{C}^{1:1}_c$-domain, we can divide $[a,b]$ into a finite number of subintervals on which $\widetilde{\theta}$ is either strictly increasing or strictly decreasing or constant. It follows that the number of critical values of $\widetilde{\theta}$ is finite. So we can take $a < a' < b' < b$ such that $\Theta(\widetilde{\phi}|_{[a',b']}) = -\pi$ and $\widetilde{\theta}$ is strictly monotone near every $t \in [a,b]$ such that $\widetilde{\theta}(t) = \widetilde{\theta}(a')$ or $\widetilde{\theta}(b')$. Now there exist $a' \le a'' < b'' \le b'$ such that $\widetilde{\theta}(a'') = \widetilde{\theta}(a')$, $\widetilde{\theta}(b'') = \widetilde{\theta}(b')$, $\widetilde{\theta}$ is strictly decreasing near a'' and b'', and $\widetilde{\theta}(b'') < \widetilde{\theta}(t) < \widetilde{\theta}(a'')$ for every $t \in (a'', b'')$. So we can take a strictly increasing continuous function $h : [-\varepsilon, 1+\varepsilon] \to [a,b]$ for some $\varepsilon > 0$ such that $h(0) = a''$, $h(1) = b''$ and $\widetilde{\theta}$ is strictly decreasing on $h([-\varepsilon, \varepsilon])$ and $h([1-\varepsilon, 1+\varepsilon])$.

Taking $\phi(t) = \widetilde{\phi}(h(t))$ for $t \in [-\varepsilon, 1+\varepsilon]$, we can easily check that ϕ satisfies conditions (1)–(3). ■

For any $p \in \mathbb{R}^2$, we denote the x-coordinate of p by p_x, and the y-coordinate of p by p_y.

THEOREM 8.1 (maximality of semi-convexity). *Let $\mathcal{C} \subset \mathcal{C}_c^{1:1}$ be closed under restriction, and let Ω_1 be a regular $\mathcal{C}$-domain which is not semi-convex. Then there exists a semi-convex $\mathcal{C}$-domain Ω_2 such that $\Omega = \Omega_1 + \Omega_2$ is not simply connected.*

Proof. By Lemma 8.1, we can take a one-to-one continuous map $\phi : [-\varepsilon', 1+\varepsilon'] \to \partial^v \Omega_1$, $\phi(t) = (\gamma(t), \mathbf{n}(t))$, for some $\varepsilon' > 0$, such that $O_{\Omega_1}(\phi) = +$, $\Theta(\phi|_{[0,1]}) = -\pi$, $-\pi < \Theta(\phi|_{[s,t]}) < \pi$ for every proper subinterval $[s,t]$ of $[0,1]$, and θ is strictly decreasing on $[-\varepsilon', \varepsilon']$ and $[1-\varepsilon', 1+\varepsilon']$, where θ is an angle function of ϕ. We can assume $\mathbf{n}(0) = (-1,0)$. Let $a = \gamma(1)_x$ and $b = \gamma(0)_x$. Let $\mathbf{v}(t)$ be the unit vector obtained by rotating $\mathbf{n}(t)$ counter-clockwise through $90°$ for $t \in [-\varepsilon', 1+\varepsilon']$. Suppose $\gamma(t_0)_x \ge b$ for some $t_0 \in (0,1)$ such that $\gamma(t_0) \ne \gamma(0)$. Then there exists $t_1 \in (0, t_0)$ such that $\mathbf{v}(t_1)_x \ge 0$. So $\mathbf{n}(t_1)_y \le 0$, and hence $\Theta(\phi|_{[0,t_1]}) \ge 0$, since $-\pi < \Theta(\phi|_{[0,t_1]}) < \pi$. It follows that $\Theta(\phi|_{[t_1,1]}) = \Theta(\phi|_{[0,1]}) - \Theta(\phi|_{[0,t_1]}) \le -\pi$, which is impossible. Thus $\gamma(t)_x < b$ for every $t \in (0,1)$ such that $\gamma(t) \ne \gamma(0)$. Analogously, $\gamma(t)_x > a$ for every $t \in (0,1)$ such that $\gamma(t) \ne \gamma(1)$. Thus $a < \gamma(t)_x < b$ for every $t \in (0,1)$ such that $\gamma(t) \ne \gamma(0), \gamma(1)$. Suppose $\gamma(t_1)_x = \gamma(t_2)_x$ for some $0 < t_1 < t_2 < 1$ such that $\gamma(t_1) \ne \gamma(t_2)$. Then there exists $t_3 \in (t_1, t_2)$ such that $\mathbf{v}(t_3)_x = 0$. So $\mathbf{n}(t_3)_y = 0$, which implies $\mathbf{n}(t_3) = (1,0)$ or $(-1,0)$. It follows that either $|\Theta(\phi|_{[0,t_3]})| \ge \pi$ or $|\Theta(\phi|_{[t_3,1]})| \ge \pi$. But this contradicts the assumption that $-\pi < \Theta(\phi|_{[0,t_3]}), \Theta(\phi|_{[t_3,1]}) < \pi$. Thus $\gamma(t_1)_x \ne \gamma(t_2)_x$ for every $t_1, t_2 \in (0,1)$ such that $\gamma(t_1) \ne \gamma(t_2)$. From these observations, it is clear that there exists a continuous function $f : [a,b] \to \mathbb{R}$ whose graph is $\gamma([0,1])$.

Let $\gamma^+ : [0,1] \to \partial\Omega_1$ be a one-to-one continuous curve such that $\gamma^+(0) = \gamma(1)$ and $O_{\Omega_1}(\gamma^+) = +$. Note that if $\gamma^+((0,\varepsilon'')) \not\subset \{(x,y) \in \mathbb{R}^2 \mid a < x < b,\ y > f(x)\}$ for every small $\varepsilon'' > 0$, then θ cannot be strictly decreasing on $[1-\varepsilon', 1+\varepsilon']$. So we can take a continuous function $g : [a, a+\varepsilon] \to \mathbb{R}$, for some small $\varepsilon > 0$, such that the graph of g is contained in $\partial\Omega_1$, $g(a) = f(a)$, and $f(x) < g(x)$ for every $x \in (a, a+\varepsilon]$. In the same way, we can take a continuous function $h : [b-\varepsilon, b] \to \mathbb{R}$ such that the graph of h is contained in $\partial\Omega_1$, $h(b) = f(b)$, and $f(x) < h(x)$ for every $x \in [b-\varepsilon, b)$. See Figure 18.

We can assume $a + \varepsilon < 0 < b - \varepsilon$ and $f(0) = 0$. Let F, G and H be the graphs of f, g and h respectively. Let $A = \overline{\partial\Omega_1 \setminus (F \cup G \cup H)}$. Since F and A are compact and $F \cap A = \emptyset$, we can take $\delta > 0$ such that

$$2\delta < \min\{d(F,A), g(a+\varepsilon) - f(a+\varepsilon), h(b-\varepsilon) - f(b-\varepsilon)\}.$$

Let F_δ, $F_{2\delta}$ be the graphs of $f+\delta$, $f+2\delta$ respectively. Since $2\delta < g(a+\varepsilon) - f(a+\varepsilon)$ and $2\delta < h(b-\varepsilon) - f(b-\varepsilon)$, $F_{2\delta}$ must meet both G and H. Let $a_1 = \max\{p_x : p \in F_{2\delta} \cap G\}$ and $b_1 = \min\{p_x : p \in F_{2\delta} \cap H\}$. Then the set

$$\Omega_2' = \{(x,y) \in \mathbb{R}^2 \mid a_1 \le x \le b_1,\ f(x) + \delta \le y \le f(x) + 2\delta\}$$

is a simply connected regular $\mathcal{C}$-domain, and Ω_1 and Ω_2' are in contact position to each other. It is also easy to see that Ω_2' is semi-convex.

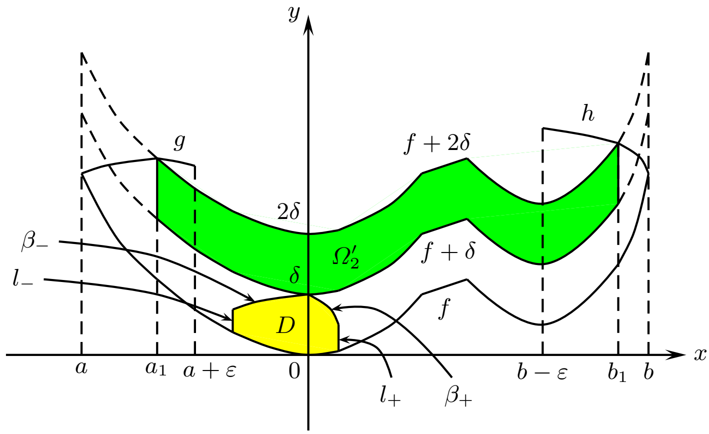


Fig. 18. Ω_2'

Let $\Omega_2 = -\Omega_2' + (0,\delta)$ and $\Omega = \Omega_1 + \Omega_2$. Clearly, Ω_2 is also semi-convex. Define $\beta_- : [a, a_1] \to \mathbb{R}^2$ and $\beta_+ : [b_1, b] \to \mathbb{R}^2$ by

$$\beta_-(t) = (t, g(t)) - (a_1, g(a_1)) + (0,\delta), \qquad t \in [a, a_1],$$
$$\beta_+(t) = (t, h(t)) - (b_1, h(b_1)) + (0,\delta), \qquad t \in [b_1, b].$$

If β_- does not meet F, then let l_- be the line segment that starts from $\beta_-(a)$ and goes in the direction of $(0,-1)$ until it meets F. Also, if β_+ does not meet F, then let l_+ be the line segment that starts from $\beta_+(b)$ and goes in the direction of $(0,-1)$ until it meets F. Note that $a + \varepsilon < 0 < b - \varepsilon$. Let D be the simply connected regular $\mathcal{C}$-domain which is enclosed by the curves F, β_-, β_+ (and l_-, l_+ if needed), and let $\beta : [0,1] \to \partial D$ be a closed curve which traverses ∂D once in the standard orientation of D. Now note that $(-\Omega_2 + p) \cap \Omega_1 \neq \emptyset$ for every $p \in \partial D$. So $\beta(t) \in \Omega$ for every $t \in [0,1]$ by Lemma 6.3(1). On the other hand, note that $-\Omega_2 + (0,\delta/2) = \Omega_2' - (0,\delta/2)$ has no intersections with Ω_1. So $(0,\delta/2) \notin \Omega$ again by Lemma 6.3(1). Since $(0,\delta/2) \in \operatorname{int} D$, we have $\operatorname{Ind}_\beta((0,\delta/2)) = 1$. Now suppose Ω is simply connected. Then $\operatorname{Ind}_{\widetilde{\beta}}(p) = 0$ for every $p \notin \Omega$ and every closed curve $\widetilde{\beta}$ in Ω. So we have $\operatorname{Ind}_\beta((0,\delta/2)) = 0$. This is a contradiction. ∎

Remark 8.1. Theorem 8.1 does not guarantee that for every regular non-semi-convex domain, there exists a *convex* domain such that their Minkowski sum is not simply connected. In fact, this is false: Let Ω be the domain depicted in Figure 12 (left). The Minkowski sum of Ω and any convex domain is simply connected. This can be easily seen from the fact that there should be a "trapping region" in order for a Minkowski sum to be non-simply connected.

Note that the domain Ω_2 in the proof of Theorem 9.2 is of a special shape, which is not always shared by every semi-convex domain. Since these domains play an important role in Section 9, we give a name to them:

Definition 8.1 (flag domain). A simply connected regular $\mathcal{C}^{1:1}$-domain Ω is called a *flag domain* if there exists a piecewise C^1 function $f : [a, b] \to \mathbb{R}$ such that:

(1) $-\infty < f'(x+), f'(x-) < \infty$ for every $x \in [a, b]$.

(2) For some rigid motion in $\mathbb{R}^2$,

$$\Omega = \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b,\ f(x) \leq y \leq f(x) + d\}$$

for some $d > 0$.

See Figure 20 for an example of a flag domain. It is easy to see that a flag domain is semi-convex, but not vice versa. Note that the domain Ω_2 in Theorem 9.2 is a flag domain. Thus we have the following statement which is stronger than Theorem 9.2:

Theorem 8.2. *Let $\mathcal{C} \subset \mathcal{C}_c^{1:1}$ be closed under restriction, and let Ω_1 be a regular $\mathcal{C}$-domain which is not semi-convex. Then there exists a flag $\mathcal{C}$-domain Ω_2 such that the Minkowski sum $\Omega = \Omega_1 + \Omega_2$ is not simply connected.*

9. Closedness of semi-convexity

In this section, we show that the Minkowski sum of two semi-convex $\mathcal{M}$-domains is again a semi-convex $\mathcal{M}$-domain for any Minkowski class $\mathcal{M}$. Thus the set of all semi-convex $\mathcal{M}$-domains is closed under Minkowski sum.

We start with some basic observations:

Lemma 9.1. *Let Ω_1 and Ω_2 be two simply connected regular $\mathcal{C}^{1:1}$-domains such that $\Omega_1 \subset \Omega_2$, and let $p \in \partial\Omega_1 \cap \partial\Omega_2$, $q_i \in \partial\Omega_i$ for $i = 1, 2$ with $q_2 \neq p$. For $i = 1, 2$, let $\gamma_i : [0, 1] \to \partial\Omega_i$ be continuous maps such that $\gamma_i(0) = p$, $\gamma_i(1) = q_i$, and let $\beta : [0, 1] \to \Omega_2 \setminus \operatorname{int} \Omega_1$ be a continuous map such that $\beta(0) = q_1$, $\beta(1) = q_2$, and either β is constant or $\beta((0, 1]) \subset \Omega_2 \setminus \Omega_1$. Suppose there exists a homotopy $H : [0, 1] \times [0, 1] \to \mathbb{R}^2 \setminus \operatorname{int} \Omega_1$ such that $H(t, 0) = \gamma_1(t)$, $H(t, 1) = \gamma_2(t)$ for $t \in [0, 1]$, and $H(0, s) = p$, $H(1, s) = \beta(s)$ for $s \in [0, 1]$. Then $O_{\Omega_1}(\gamma_1) \cdot O_{\Omega_2}(\gamma_2) \neq -$.*

Proof. Let $\widetilde{\gamma}_2 : [0, 1] \to \partial\Omega_2$ be a one-to-one continuous map such that $\widetilde{\gamma}_2(0) = p$, $\widetilde{\gamma}_2(1) = q_2$, and $O_{\Omega_2}(\widetilde{\gamma}_2) = +$. Let $\widetilde{\gamma}_1 : [0, 1] \to \partial\Omega_1$ be a continuous map such that $\widetilde{\gamma}_1(0) = p$, $\widetilde{\gamma}_1(1) = q_1$, $O_{\Omega_1}(\widetilde{\gamma}_1) \neq -$, $\widetilde{\gamma}_1|_{[0,1)}$ is either one-to-one or constant. Clearly, we can find a homotopy $\widetilde{H} : [0, 1] \times [0, 1] \to \mathbb{R}^2 \setminus \operatorname{int} \Omega_1$ such that $\widetilde{H}(t, 0) = \widetilde{\gamma}_1(t)$, $\widetilde{H}(t, 1) = \widetilde{\gamma}_2(t)$ for every $t \in [0, 1]$, and $\widetilde{H}(0, s) = p$, $\widetilde{H}(1, s) = \beta(s)$ for every $s \in [0, 1]$. For $i = 1, 2$, let $\nu_i : [0, 2] \to \mathbb{R}$ be the continuous function such that $\nu_i(0) = 0$ and $\mu_i(\nu_i(t)) = (\widetilde{\gamma}_i \cdot \overline{\gamma_i})(t)$ for $t \in [0, 2]$, where $\mu_i : \mathbb{R} \to \partial\Omega_i$ are covering maps in the standard orientation of $\partial\Omega_i$ with period 1 such that $\mu_i(0) = p$. See Section 5 for the definition of $\overline{\gamma}$ for a curve γ.

Clearly, $0 \leq \nu_1(1) \leq 1$ and $0 < \nu_2(1) < 1$. Note also that the two closed curves $\widetilde{\gamma}_1 \cdot \overline{\gamma}_1$ and $\widetilde{\gamma}_2 \cdot \overline{\gamma}_2$ are homotopic in $\mathbb{R}^2 \setminus \operatorname{int} \Omega_1$. So $\operatorname{Ind}_{\widetilde{\gamma}_1 \cdot \overline{\gamma}_1}(0) = \operatorname{Ind}_{\widetilde{\gamma}_2 \cdot \overline{\gamma}_2}(0)$, where we have assumed $0 \in \operatorname{int} \Omega_1$. Note that $\operatorname{Ind}_{\widetilde{\gamma}_i \cdot \overline{\gamma}_i}(0) = \nu_i(2)$ for $i = 1, 2$. So $\nu_1(2) = \nu_2(2) \in \mathbb{Z}$. Thus the assertion follows, since $O_{\Omega_i}(\gamma_i)$ is the sign of $\nu_i(1) - \nu_i(2)$. ∎

Lemma 9.2. *Let Ω be a simply connected regular $\mathcal{C}^{1:1}$-domain. Let $(p_1, \mathbf{n}_1)$ and $(p_2, \mathbf{n}_2)$ be two points in $\partial^v\Omega$ such that $\mathbf{n}_1 = -\mathbf{n}_2$. Suppose*

$$\Omega \cap (\{p_1 + t \cdot \mathbf{n}_1 \mid t > 0\} \cup \{p_2 + t \cdot \mathbf{n}_2 \mid t > 0\}) = \emptyset.$$

Then $\Theta(\phi) = \pi$ for any one-to-one continuous map $\phi : [0,1] \to \partial^v \Omega$, $\phi(t) = (\gamma(t), \mathbf{n}(t))$, such that $\phi(0) = (p_1, \mathbf{n}_1)$, $\phi(1) = (p_2, \mathbf{n}_2)$ and $O_\Omega(\phi) = +$.

Proof. We can assume that $\mathbf{n}_1 = (1,0)$. Since Ω is bounded, there exist $a_1 < a_2$ and $b_1 < b_2$ such that $\Omega \subset \{(x,y) \in \mathbb{R}^2 \mid a_1 < x < a_2,\ b_1 < y < b_2\}$. See Figure 19.

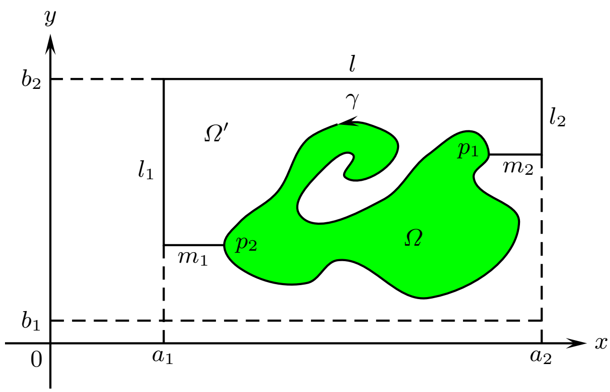


Fig. 19. Figure for Lemma 9.2

Let $l_1 = \{(x,y) \in \mathbb{R}^2 \mid x = a_1, (p_2)_y \le y \le b_2\}$ and $l_2 = \{(x,y) \in \mathbb{R}^2 \mid x = a_2, (p_1)_y \le y \le b_2\}$. Let $m_1 = \{(x,y) \in \mathbb{R}^2 \mid a_1 \le x \le (p_2)_x,\ y = (p_2)_y\}$ and $m_2 = \{(x,y) \in \mathbb{R}^2 \mid (p_1)_x \le x \le a_2,\ y = (p_1)_y\}$. Let $l = \{(x,y) \in \mathbb{R}^2 \mid a_1 \le x \le a_2,\ y = b_2\}$. By the assumptions, the curve γ and the line segments m_2, l_2, l, l_1, m_1 constitute the boundary of a simply connected regular $\mathcal{C}^{1:1}$-domain, which we call Ω'. Let $\psi : [0,1] \to \partial^v \Omega'$ be a one-to-one continuous map such that $\psi(0) = (p_1, (0,-1))$, $\psi(1) = (p_2, (0,-1))$ and $O_{\Omega'}(\psi) = +$. It is easy to see that $\Theta(\psi) = 2\pi$. By Lemma 5.2(1), we have $\Theta(\psi) + \pi/2 - \Theta(\phi) + \pi/2 = 2\pi$. Thus $\Theta(\phi) = \pi$. ∎

Lemma 9.3. *Let Ω_1 be a flag $\mathcal{C}^{1:1}$-domain and Ω_2 a semi-convex $\mathcal{C}^{1:1}$-domain. Suppose that Ω_1 and Ω_2 are in contact position to each other, and that V is a bounded connected component of $\mathbb{R}^2 \setminus (\Omega_1 \cup \Omega_2)$. Then for any $p_1 \in \partial V \setminus \partial\Omega_2$, there exist $p_2 \in \partial V \setminus \partial\Omega_1$ and a continuous curve $\beta : [0,1] \to \overline{V}$ such that $\beta(0) = p_1$, $\beta(1) = p_2$, $\beta((0,1)) \subset V$, and $(\Omega_1 + \beta(u) - \beta(0)) \cap \Omega_2 \ne \emptyset$ for every $u \in [0,1]$.*

Proof. We can assume that

$$\Omega_1 = \{(x,y) \mid f(x) \le y \le f(x) + d,\ |x| \le 1\},$$

for some piecewise C^1 function $f : [-1,1] \to \mathbb{R}$. Let F and F_d be the graphs of f and $f+d$ respectively, and let l_-, l_+ be the line segments (without end points) joining $(-1, f(-1))$ to $(-1, f(-1)+d)$ and $(1, f(1))$ to $(1, f(1)+d)$ respectively. See Figure 20. Note that $\partial\Omega_1 = F \cup F_d \cup l_- \cup l_+$. If $p_1 \in F$ (resp., $p_1 \in F_d$), then take $p_2 \in \partial V \setminus \partial\Omega_1$ such that $(p_2)_x = (p_1)_x$ and $(p_2)_y = \max\{p_y : p \in \partial V \setminus \partial\Omega_1,\ p_x = (p_1)_x,\ p_y < (p_1)_y\}$ (resp., $(p_2)_y = \min\{p_y : p \in \partial V \setminus \partial\Omega_1,\ p_x = (p_1)_x,\ p_y > (p_1)_y\}$). If $p_1 \in l_-$ (resp., $p_1 \in l_+$), we take $p_2 \in \partial V \setminus \partial\Omega_1$ such that $(p_2)_y = (p_1)_y$ and $(p_2)_x = \max\{p_x : p \in \partial V \setminus \partial\Omega_1,\ p_y = (p_1)_y,\ p_x < (p_1)_x\}$ (resp., $(p_2)_x = \min\{p_x : p \in \partial V \setminus \partial\Omega_1,\ p_y = (p_1)_y,\ p_x > (p_1)_x\}$). In

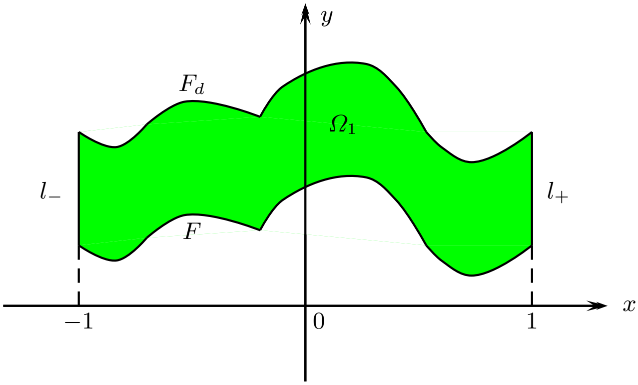


Fig. 20. Flag domain

any case, we define $\beta(u) = (1-u)p_1 + up_2$ for $u \in [0,1]$. It is clear that $\beta(0) = p_1$, $\beta(1) = p_2$, $\beta([0,1]) \subset \overline{V}$, and $\beta((0,1)) \subset V$.

Now we only have to show that $(\Omega_1 + \beta(u) - \beta(0)) \cap \Omega_2 \neq \emptyset$ for every $u \in [0,1]$. For $i = 1,2$, let $\phi_i : [0,1] \to \partial^v \Omega_i$, $\phi_i(t) = (\gamma_i(t), \mathbf{n}_i(t))$, be a one-to-one continuous map such that $O_{\Omega_i}(\phi_i) = +$, $\gamma_i([0,1]) = \partial V \cap \partial \Omega_i$, $\phi_i(0) = (\gamma_i(0), \mathbf{n}^+_{\Omega_i}(\gamma_i(0)))$, $\phi_i(1) = (\gamma_i(1), \mathbf{n}^-_{\Omega_i}(\gamma_i(1)))$. Note that Ω_1 is semi-convex, since a flag domain is semi-convex. So by Lemma 5.4(2), we have $-\pi \leq \Theta(\phi_i) \leq \pi$ for $i = 1,2$. From this, it is easy to see that at least one of F, F_d, l_-, l_+ has no intersections with $\gamma_1([0,1])$, and if $\gamma_1([0,1])$ intersects one of l_+, l_-, then it does not intersect the other. Thus, by symmetry, it is sufficient to consider the following four cases when $\gamma_1([0,1])$ intersects only (1) F, (2) l_-, (3) F and l_-, (4) F, l_- and F_d. See Figure 21.

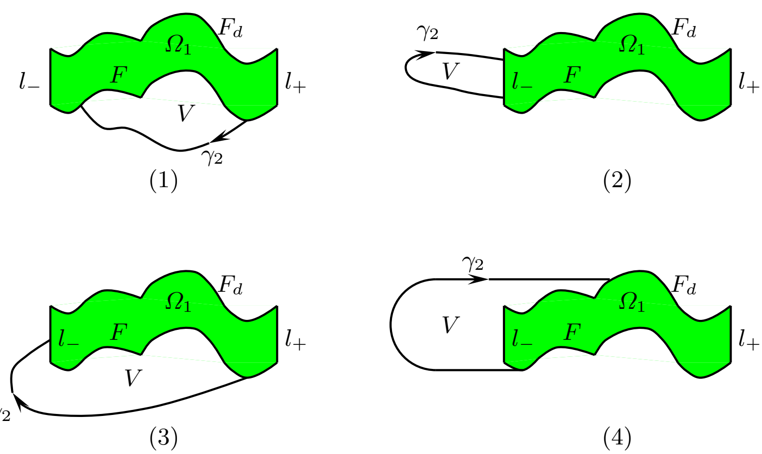


Fig. 21. Four cases of contact positions

First consider case (1). Let $U = \{(x,y) \in \mathbb{R}^2 \mid \gamma_1(0)_x \leq x \leq \gamma_1(1)_x,\ y \leq f(x)\}$. Suppose there does not exist t_1' nor t_2' in $[0,1]$ such that $\gamma_2([0,t_1']) \subset U$, $\gamma_2(t_1')_x =$

$\gamma_1(0)_x$, and $\gamma_2([t_2',1]) \subset U$, $\gamma_2(t_2')_x = \gamma_1(1)_x$. Then there exist $0 < t_1 < t_2 < 1$ and $\varepsilon > 0$ such that $\mathbf{n}_2(t_1) = -\mathbf{n}_2(t_2) = (-1,0)$, $\Theta(\phi_1|_{[t_1-\varepsilon,t_1]}), \Theta(\phi_1|_{[t_2,t_2+\varepsilon]}) < 0$, and $\overline{V} \cap \{\gamma_2(t_i) - u\cdot \mathbf{n}_2(t_i) \mid u > 0\} = \emptyset$ for $i = 1,2$. By applying Lemma 9.2 to $\overline{V}$, we have $\Theta(\phi_2|_{[t_1,t_2]}) = -\pi$. So $\Theta(\phi_2|_{[t_1-\varepsilon,t_2+\varepsilon]}) < -\pi$, which is impossible since Ω_2 is semi-convex. Thus at least one of t_1', t_2' above should exist. Then we can see easily that $(\Omega_1 + \beta(u) - \beta(0)) \cap \Omega_2 \neq \emptyset$ for every $u \in [0,1]$.

Case (2) can be treated with the same argument as in (1). Consider case (3). Note that the case when $\gamma_1(1) = (-1, f(-1))$ can be treated by the same method as for case (2). So we assume $\gamma_1(1) \neq (-1, f(-1))$. Suppose $p_1 \in F$. Let $U = \{(x,y) \in \mathbb{R}^2 \mid -1 \le x \le \gamma_1(1)_x,\ y \le f(x)\}$. Suppose there does not exist $t_1' \in [0,1]$ such that $\gamma_2([0,t_1']) \subset U$, $\gamma_2(t_1')_x = -1$. Then there exist $0 < t_1 < t_2 < 1$ and $\varepsilon > 0$ such that $\mathbf{n}_2(t_1) = -\mathbf{n}_2(t_2) = (-1,0)$, $\Theta(\phi_2|_{[t_1-\varepsilon,t_1]}), \Theta(\phi_2|_{[t_2,t_2+\varepsilon]}) < 0$, and $\overline{V} \cap \{\gamma_2(t_i) - u\cdot \mathbf{n}_2(t_i) \mid u > 0\} = \emptyset$ for $i = 1,2$. By Lemma 9.2, we have $\Theta(\phi_2|_{[t_1,t_2]}) = -\pi$. So $\Theta(\phi_2|_{[t_1-\varepsilon,t_2+\varepsilon]}) < -\pi$, which is impossible since Ω_2 is semi-convex. Thus there exists $t_1' \in [0,1]$ as above, and it follows easily that $(\Omega_1 + \beta(u) - \beta(0)) \cap \Omega_2 \neq \emptyset$ for every $u \in [0,1]$.

Suppose $p_1 \in l_-$. Let U_1 be the (closed) region bounded by $\{\gamma_1(0) + u\cdot(-1,0) \mid u \ge 0\}$, $\{\gamma_1(1) + u\cdot(0,-1) \mid u \ge 0\}$ and γ_1, which does not contain Ω_1. Let $U_2 = \{(x,y) \in \mathbb{R}^2 \mid x \le -1,\ f(-1) \le y \le \gamma_1(0)_y\}$. Suppose there do not exist t_1', t_2' in $[0,1]$ such that $\gamma_2([0,t_1']) \subset U_1$, $\gamma_2(t)_y \ge \gamma_1(1)_y$ for every $t \in [0,t_1']$, $\gamma_2(t_1')_y = \gamma_1(0)_y$, and $\gamma_2([t_2',1]) \subset U_2$, $\gamma_2(t_2')_y = f(-1)$. Then there exist $0 < t_1 < t_2 < 1$ and $\varepsilon > 0$ such that $\mathbf{n}_2(t_1) = -\mathbf{n}_2(t_2) = (0,1)$, $\Theta(\phi_2|_{[t_1-\varepsilon,t_1]}), \Theta(\phi_2|_{[t_2,t_2+\varepsilon]}) < 0$, and $\overline{V} \cap \{\gamma_2(t_i) - u\cdot \mathbf{n}_2(t_i) \mid u > 0\} = \emptyset$ for $i = 1,2$. By Lemma 9.2, we have $\Theta(\phi_2|_{[t_1,t_2]}) = -\pi$. It follows that $\Theta(\phi_2|_{[t_1-\varepsilon,t_2+\varepsilon]}) < -\pi$, which is impossible since Ω_2 is semi-convex. So at least one of t_1', t_2' exists. Now it is easy to see that $(\Omega_1 + \beta(u) - \beta(0)) \cap \Omega_2 \neq \emptyset$ for every $u \in [0,1]$.

Finally, consider case (4). Note that the cases when $\gamma_1(0) = (-1, f(-1)+d)$ or $\gamma_1(1) = (-1, f(-1))$ can be treated with the same methods as for cases (2) and (3). So assume $\gamma_1(0) \neq (-1, f(-1)+d)$ and $\gamma_1(1) \neq (-1, f(-1))$. By using the same argument as for case (3) when $p_1 \in F$, we can see that $(\Omega_1 + \beta(u) - \beta(0)) \cap \Omega_2 \neq \emptyset$ for every $u \in [0,1]$ if $p \in F \cup F_d$. Suppose $p \in l_-$. Let U_3 be the (closed) region bounded by $\{\gamma_1(1) + u\cdot(0,-1) \mid u \ge 0\}$, $\{(-1, f(-1)+d) + u\cdot(-1,0) \mid u \ge 0\}$ and $\gamma_1([0,1]) \cap (F \cup l_-)$, which does not contain Ω_1. Let U_4 be the (closed) region bounded by $\{\gamma_1(0) + u\cdot(0,1) \mid u \ge 0\}$, $\{(-1, f(-1)) + u\cdot(-1,0) \mid u \ge 0\}$ and $\gamma_1([0,1]) \cap (F_d \cup l_-)$ which does not contain Ω_1. Suppose there do not exist t_1', t_2' in $[0,1]$ such that $\gamma_2([0,t_1']) \subset U_3$, $\gamma_2(t)_y \ge \gamma_1(1)_y$ for every $t \in [0,t_1']$, $\gamma_2(t_1')_y = f(-1)+d$, and $\gamma_2([t_2',1]) \subset U_4$, $\gamma_2(t)_y \le \gamma_1(0)_y$ for every $t \in [t_2',1]$, $\gamma_2(t_2')_y = f(-1)$. Then there exist $0 < t_1 < t_2 < 1$ and $\varepsilon > 0$ such that $\mathbf{n}_2(t_1) = -\mathbf{n}_2(t_2) = (0,1)$, $\Theta(\phi_2|_{[t_1-\varepsilon,t_1]}), \Theta(\phi_2|_{[t_2,t_2+\varepsilon]}) < 0$, and $\overline{V} \cap \{\gamma_2(t_i) - u\cdot \mathbf{n}_2(t_i) \mid u > 0\} = \emptyset$ for $i = 1,2$. By Lemma 9.2, we have $\Theta(\phi_2|_{[t_1,t_2]}) = -\pi$, and so $\Theta(\phi_2|_{[t_1-\varepsilon,t_2+\varepsilon]}) < -\pi$. But this is impossible since Ω_2 is semi-convex. So at least one of t_1', t_2' exists. Now clearly $(\Omega_1 + \beta(u) - \beta(0)) \cap \Omega_2 \neq \emptyset$ for every $u \in [0,1]$. ∎

THEOREM 9.1 (semi-convex + flag $\Rightarrow$ semi-convex). *For any Minkowski class $\mathcal{M}$, the Minkowski sum of a semi-convex $\mathcal{M}$-domain and a flag $\mathcal{M}$-domain is a semi-convex $\mathcal{M}$-domain.*

Proof. Let $\mathcal{M}$ be a Minkowski class. Let Ω_1 be a flag $\mathcal{M}$-domain, and let Ω_2 be a semi-convex $\mathcal{M}$-domain. We can assume that $0 \in \Omega_1, \Omega_2$. Since a flag domain is semi-convex, the Minkowski sum $\Omega = \Omega_1 + \Omega_2$ is a simply connected regular $\mathcal{M}$-domain by Theorem 7.2. Suppose Ω is not semi-convex. Then we can take a one-to-one continuous map $\widetilde{\phi} : [0,1] \to \partial^v \Omega$, $\widetilde{\phi}(t) = (\widetilde{\gamma}(t), \widetilde{\mathbf{n}}(t))$, such that $O_\Omega(\widetilde{\phi}) = +$ and $\Theta(\widetilde{\phi}) < -\pi$. We can assume that $\widetilde{\gamma}(0)$ and $\widetilde{\gamma}(1)$ are flat points. Now we can take the maps ϕ, ϕ_i, ϕ^k, ϕ_i^k, ψ^k, ψ_i^k associated to $\widetilde{\phi}$ as in Section 7.2. We also use all the related notations therein.

Let $\mu : \mathbb{R} \to \partial\Omega$ and $\mu_i : \mathbb{R} \to \partial\Omega_i$ for $i = 1, 2$ be covering maps in the standard orientations of $\partial\Omega$ and $\partial\Omega_i$ respectively with period 1, such that $\mu(0) = \gamma(0)$ and $\mu_i(0) = \gamma_i(0)$. Then there exist continuous functions $\nu, \nu_1, \nu_2 : [0, 2n-1] \to \mathbb{R}$ such that $\nu(0) = \nu_1(0) = \nu_2(0) = 0$ and $\gamma = \mu \circ \nu$, $\gamma_i = \mu_i \circ \nu_i$. Note that such ν and ν_i's are unique, and $O_\Omega(\gamma|_{[a,b]})$ and $O_{\Omega_i}(\gamma_i|_{[a,b]})$ are the signs of $\nu(b) - \nu(a)$ and $\nu_i(b) - \nu_i(a)$ respectively, for any $[a, b] \subset [0, 2n-1]$.

Note that $\Theta(\widetilde{\phi}) = \Theta(\phi) = \Theta(\phi_1) = \Theta(\phi_2)$. Since Ω_1 is semi-convex and $\Theta(\phi_1) < -\pi$, we have $O_{\Omega_1}(\phi_1) = -$. It follows that $O_{\Omega_1}(\gamma_1) = -$, since $\Theta(\phi_1) < -\pi$. So $\nu_1(2n-1) < 0$. Note that ν_1 is either non-decreasing or non-increasing on the interval $[k-1, k]$ for $k = 1, \ldots, 2n-1$. So there exist $0 = a_0 \le b_0 < a_1 < \ldots < b_{m-1} < a_m \le b_m = 2n-1$ such that ν_1 is non-increasing on $[a_j, b_j]$ for $j = 0, \ldots, m$, and $\nu_1(a_{j+1}) - \nu_1(b_j) = 0$ for $j = 0, \ldots, m-1$. Note that, from the constructions in Section 7, we can assume that $O_\Omega(\gamma|_{[b_j, a_{j+1}]}) = +$, and either $\gamma(b_j) = \gamma_1(b_j) + \gamma_2(b_j)$ or $\gamma(a_{j+1}) = \gamma_1(a_{j+1}) + \gamma_2(a_{j+1})$ for $j = 0, \ldots, m-1$. We can also assume that ν_2 is non-decreasing on $[a_j, b_j]$ for $j = 0, \ldots, m$.

Suppose $\gamma(c) \ne \gamma_1(c) + \gamma_2(c)$ for some $c = a_0, b_0, \ldots, a_m, b_m$. Note that Ω_2 and $-\Omega_1 + \gamma(c)$ are in contact position to each other by Lemma 6.3(3). Since $\gamma(c) \ne \gamma_1(c) + \gamma_2(c)$, it follows that, for some k, $\gamma_i(c)$ is on η_i^k for $i = 1, 2$. Let V be the connected component of the set $\mathbb{R}^2 \setminus (\Omega_2 \cup (-\Omega_1 + \gamma(c)))$ such that $-\gamma_1(c) + \gamma(c) \in \overline{V}$. Note that V is bounded by η_2^k and $\widetilde{\eta}_1^k$. By applying Lemma 9.3 to $-\Omega_1 + \gamma(c)$ and Ω_2, we have a continuous curve $\beta_c : [0,1] \to \overline{V}$ such that $\beta_c(0) = -\gamma_1(c) + \gamma(c)$, $\beta_c(1) \in \partial V \setminus \partial(-\Omega_1 + \gamma(c))$, $\beta_c((0,1)) \subset V$, and $(-\Omega_1 + \gamma(c) + \beta_c(u) - \beta_c(0)) \cap \Omega_2 \ne \emptyset$ for every $u \in [0,1]$. Now from the constructions in Section 7, it is easy to see that we can take γ_2 (more exactly, η_2^k's) and A_2^k's such that:

(1) if $\gamma(c) = \gamma_1(c) + \gamma_2(c)$, then $P_2(c, s) = \gamma_2(c)$ for $s \in [0,1]$,
(2) if $\gamma(c) \ne \gamma_1(c) + \gamma_2(c)$, then $P_2(c, s) = \overline{\beta}_c(s)$ for $s \in [0,1]$,

for each $c = a_0, b_0, \ldots, a_m, b_m$.

Now we show $\nu_2(a_{j+1}) - \nu_2(b_j) \ge 0$ for $j = 0, \ldots, m-1$. Fix j, and let $b = b_j$, $a_{j+1} = a$. Note that $b < a$, $O_\Omega(\gamma|_{[b,a]}) = +$, and either $\gamma(b) = \gamma_1(b) + \gamma_2(b)$ or $\gamma(a) = \gamma_1(a) + \gamma_2(a)$.

Suppose $\gamma(b) = \gamma_1(b) + \gamma_2(b)$. Let $\breve{\Omega}_1 = \Omega_1 - \gamma_1(b)$, and let $\breve{\Omega} = \Omega - \gamma_1(b)$. Then $\breve{\Omega} = \breve{\Omega}_1 + \Omega_2$, and $\Omega_2 \subset \breve{\Omega}$ since $0 \in \breve{\Omega}_1$. Define $\breve{\gamma}_1(t) = \gamma_1(t) - \gamma_1(b)$ and $\breve{\gamma}(t) = \gamma(t) - \gamma_1(b)$ for $t \in [0, 2n-1]$. Then, clearly, $O_{\breve{\Omega}}(\breve{\gamma}|_{[b,a]}) = O_\Omega(\gamma|_{[b,a]}) = +$, $O_{\breve{\Omega}_1}(\breve{\gamma}_1|_{[b,a]}) = O_{\Omega_1}(\gamma_1|_{[b,a]}) = 0$, and $\gamma_2(b) = \breve{\gamma}(b)$. Define $\breve{Q}_2(t, s) = I_{\breve{\gamma}(t)}(H_{\breve{\Omega}_1;0}(I_{\breve{\gamma}(t)}(P_2(t,1)), s))$ for $(t, s) \in [0, 2n-1] \times [0,1]$ and $\breve{H}_2 = \overset{\breve{Q}_2}{\underset{P_2}{\cdot}}$. Then $\breve{H}_2$ is well defined and continuous, $\gamma_2|_{[b,a]}$ is homotopic to $\breve{\gamma}|_{[b,a]}$ in $\mathbb{R}^2 \setminus \operatorname{int} \Omega_2$ via $\breve{H}_2|_{[b,a] \times [0,2]}$, and $\breve{H}_2(b, s) = \gamma_2(b)$ for $s \in [0, 2]$.

Suppose $\gamma(a) = \gamma_1(a) + \gamma_2(a)$. Then also $\breve{H}_2(a,s) = \gamma_2(a)$ for $s \in [0,2]$. So by Lemma 9.1, we have $O_{\Omega_2}(\gamma_2|_{[b,a]}) \neq -$, which implies $\nu_2(a) - \nu_2(b) \geq 0$. Suppose $\gamma(a) \neq \gamma_1(a) + \gamma_2(a)$. Then $P_2(a,s) = \overline{\beta}_a(s)$ and $\breve{Q}_2(a,s) = \overline{\beta}_a(1) = -\gamma_1(a) + \gamma(a)$ for $s \in [0,1]$. So $\breve{H}_2(a,(0,2]) \subset \mathbb{R}^2 \setminus \Omega_2$. Since $\nu_1(b) = \nu_1(a)$ (hence $\gamma_1(b) = \gamma_1(a)$) and $\beta_a(0) = -\gamma_1(a) + \gamma(a)$, we have $-\Omega_1 + \gamma(a) + \beta_a(u) - \beta_a(0) = -\breve{\Omega}_1 + \beta_a(u)$ for $u \in [0,1]$. So $(-\breve{\Omega}_1 + \beta_a(u)) \cap \Omega_2 \neq \emptyset$ for $u \in [0,1]$, and hence $\beta_a([0,1]) \subset \breve{\Omega}$ by Lemma 6.3(1). So $\breve{H}_2(a,(0,2]) \subset \breve{\Omega} \setminus \Omega_2$. Thus by applying Lemma 9.1 again, we have $O_{\Omega_2}(\gamma_2|_{[b,a]}) \neq -$, which implies that $\nu_2(a) - \nu_2(b) \geq 0$. In the same way, $\nu_2(a) - \nu_2(b) \geq 0$, when $\gamma(b) \neq \gamma_1(b) + \gamma_2(b)$ and $\gamma(a) = \gamma_1(a) + \gamma_2(a)$. Thus $\nu_2(a_{j+1}) - \nu_2(b_j) \geq 0$ for $j = 0, \ldots, m-1$.

Now since ν_2 is non-decreasing on $[a_j, b_j]$ for $j = 0, \ldots, m$, we have $\nu_2(2n-1) \geq 0$, and hence $O_{\Omega_2}(\gamma_2) \neq -$. Note that $O_{\Omega_2}(\gamma_2) \neq 0$, since $\Theta(\phi_2) < -\pi$. So $O_{\Omega_2}(\gamma_2) = +$. But this is impossible, since Ω_2 is semi-convex and $\Theta(\phi_2) < -\pi$. ∎

Finally, we prove the main theorem of this section:

THEOREM 9.2 (semi-convex + semi-convex ⇒ semi-convex). *For any Minkowski class $\mathcal{M}$, the Minkowski sum of two semi-convex $\mathcal{M}$-domains is also a semi-convex $\mathcal{M}$-domain.*

Proof. Let Ω_1 and Ω_2 be two semi-convex $\mathcal{M}$-domains, and let $\Omega = \Omega_1 + \Omega_2$ be their Minkowski sum. We know from Theorem 7.2 that Ω is a simply connected regular $\mathcal{M}$-domain. Suppose Ω is not semi-convex. Then, by Theorem 8.2, there exists a flag $\mathcal{M}$-domain Ω_3 such that $\Omega + \Omega_3$ is not simply connected. By Theorem 9.1, $\Omega_2 + \Omega_3$ is a semi-convex $\mathcal{M}$-domain. So $\Omega + \Omega_3 = \Omega_1 + (\Omega_2 + \Omega_3)$ is simply connected by Theorem 7.2. This is a contradiction. ∎

10. Conclusion

Here we briefly summarize the important results in this paper, and mention some further research directions. Let $\mathcal{M}$ be a Minkowski class. We denote the major classes of domains in this paper as follows:

M = The set of all $\mathcal{M}$-domains.
D = The set of all $\mathcal{M}$-domains homeomorphic to the unit disk.
S = The set of all semi-convex $\mathcal{M}$-domains.
F = The set of all flag $\mathcal{M}$-domains.
C = The set of all convex $\mathcal{M}$-domains homeomorphic to the unit disk.

The inclusion relations between them are shown in Figure 22.

The inclusions $\mathbf{C} \subset \mathbf{S} \subset \mathbf{D} \subset \mathbf{M}$ and $\mathbf{F} \subset \mathbf{S}$ are all proper. By Theorem 6.3, **M** is closed under Minkowski sum. Let $\mathcal{D}$ be the class of all subsets $\mathbf{X}$ of $\mathbf{D}$ such that $A + B \in \mathbf{D}$ for every $A, B \in \mathbf{X}$. By Theorem 7.2, **S** is in $\mathcal{D}$, and is maximal in $\mathcal{D}$ with respect to the inclusion by Theorem 8.1. In fact, **S** is the unique maximal element in $\mathcal{D}_{\mathbf{F}}$ by Theorem 8.2, where $\mathcal{D}_{\mathbf{F}} = \{\mathbf{X} \in \mathcal{D} \mid \mathbf{F} \subset \mathbf{X}\}$. Finally, **S** is closed under Minkowski sum by Theorem 9.2.

Now let us mention some further research directions in the subject of semi-convexity. First, note that the semi-convexity is amenable to the algorithmic setting in that only the rotation of normal vectors needs to be checked. Also, it is a natural generalization

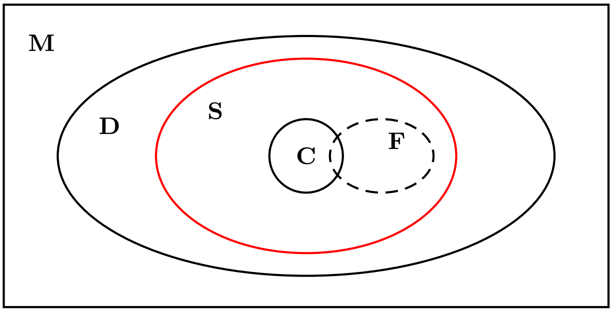


Fig. 22. Relations between classes of domains

of the usual convexity. Usually, the computation of the Minkowski sum of general shapes can essentially be divided into a few steps:

1. Decompose the shapes into unions of simpler shapes, which are usually convex.
2. Select the simple parts which can contribute to the boundary of the Minkowski sum.
3. Do Minkowski sum operations on these selected parts.
4. Integrate the results to form the Minkowski sum boundary, and hence the Minkowski sum itself.

The most important reason for using convex shapes in Step 1 is that they are closed under Minkowski sum. But in general, the number of the decomposed parts will be large since the convexity is very restrictive, and this results in the slow-down of the algorithms. So if we can use semi-convex shapes instead of convex ones, it would be possible to compute the Minkowski sum in a significantly more efficient way.

An immediate further research direction is to generalize the semi-convexity to 3 or higher dimensions, which would be most needed in various applications. Also, note that the current definition of semi-convexity requires some differentiability of the boundary, i.e., $C^{1:1}$. Compared to the fact that convexity has no such *a priori* requirements, this may be considered as a severe restriction. So an important next step would be to remove the regularity requirements from the definition of semi-convexity, which will be dealt with in [4] along with relationships of semi-convexity with other notions such as visibility.

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