Introduction

Generalizing a characterization of σ-compact spaces Σ and σ due to J. Mogilski ([Mo] and corrections in [CDM]), M. Bestvina and J. Mogilski introduced in [BM] the notion of absorbing set for a class $\mathcal{C}$ of separable metrizable spaces. Though absorbing sets are defined as subsets of s-manifolds (s is the pseudointerior of the Hilbert cube Q), they have a characterization independent of any embedding into an s-manifold, the principal ingredient of this characterization being the condition of strong universality (whose definition is recalled in Section 1).

The notion of absorbing set turns out to be extremely important in infinite-dimensional topology. On the one hand, because of its generality, Bestvina and Mogilski constructed absorbing sets for all Borel classes, but many other classes also admit absorbing sets, for example, the projective classes and the small Borel classes. Moreover, many spaces are strongly universal for the class of their closed subspaces which allows one to prove general theorems like the following [BC_1]: Let X be a regular countable space and let $C_{\mathrm{p}}(X)$ (resp. $C^*_{\mathrm{p}}(X)$) denote the space of continuous (resp. bounded continuous) real functions on X, endowed with the topology of pointwise convergence; then $C^*_{\mathrm{p}}(X)$ is homeomorphic to $C_{\mathrm{p}}(X) \times \sigma$. On the other hand, the characterization of absorbing sets is applicable to spaces that either have no natural completion or are "wildly" embedded in their natural completions (as, for example, the hyperspace of arcs in the plane, see [Ca_3]). A reader who wishes to appreciate the extent and variety of applications of the notion of absorbing set may consult the survey [Ca_7].

If Ω_1, Ω_2 are two $\mathcal{C}$-absorbing sets in s, then for every open cover $\mathcal{U}$ of s there is a homeomorphism between Ω_1 and Ω_2, $\mathcal{U}$-close to the identity map of Ω_1. Let us remark that in general, there may be no autohomeomorphism of s mapping Ω_1 onto Ω_2 (the first example of this sort was given in [Ca_1]). The situation seemed to be different for embeddings of absorbing sets into the Hilbert cube: for every class $\mathcal{C}$ possessing a $\mathcal{C}$-absorbing set, only one (up to homeomorphism) pair (Q, X), where X is a copy of a $\mathcal{C}$-absorbing set such that $Q \setminus X$ is locally homotopy negligible in Q (such subsets X will be called homotopy dense in Q) was known. Moreover, this pair (Q, X) could be characterized by strong universality properties for pairs, as could the different known pairs (s, X), where X is a $\mathcal{C}$-absorbing homotopy dense set in s.

The following questions, which are the motivation for this paper, then appear naturally: 1) What is the number of topologically distinct pairs (M, X), where M is Q or s and X is a homotopy dense copy of a $\mathcal{C}$-absorbing set in M? 2) Could pairs (M, X) of this type always be characterized by properties of strong universality for pairs?

The most complicated part in the proofs of the strong $\mathcal{C}$-universality of a space X is often to show that every space C from the class $\mathcal{C}$ admits a closed embedding into X, i.e., to verify the simple $\mathcal{C}$-universality of X. Suppose X is contained in a space M and C is contained in a compactum K. If there exists an embedding φ of K into M such that $\varphi^{-1}(X) = C$, then the restriction of φ onto C is a closed embedding of C into X. Whenever such a φ exists, its construction is generally much simpler than a direct construction of a closed embedding of C into X.

Given a pair (M, X), a class $\mathcal{C}$ of spaces, and a class $\mathcal{K}$ of compacta, we will show in Section 2 that under some conditions on (M, X), $\mathcal{C}$, and $\mathcal{K}$, the $\mathcal{C}$-universality of X is equivalent to the $(\mathcal{K}, \mathcal{C})$-universality of the pair (M, X) (the latter means that for every pair (K, C), where $K \in \mathcal{K}$ and $C \in \mathcal{C}$, there exists an embedding $\varphi : K \to M$ with $\varphi^{-1}(X) = C$). Applications of this equivalence are numerous, see [Ca_6], [Ca_8], [Ca_9], [BC_1], [BC_2].

Let (M, X) be a pair, where M is an ANR and X is a homotopy dense subset of M. In Section 3 we show that if X has SDAP (the strong discrete approximation property, used by H. Toruńczyk to characterize the s-manifolds [To_2]), then for every pair (K, C) the strong (K, C)-universality of the pair (M, X) implies the strong C-universality of X. As in the case of simple universality, this result has a reciprocal: under some conditions the strong $\mathcal{C}$-universality of X implies the strong $(\mathcal{M}_0 \cap \mathcal{C}, \mathcal{C})$-universality or even the strong $(\mathcal{M}_0, \mathcal{C})$-universality of the pair (M, X) ($\mathcal{M}_0$ is the class of compacta).

These relations between strong universalities of spaces and pairs are used in Section 5 to prove Addition, Deleting, and Negligibility Theorems for absorbing spaces, which resemble well known properties of s- or Σ-manifolds. As an example we mention here the following result that is a particular case of Theorem 5.9: Let Π_{2n} be the absolute retract, absorbing for the projective class $\mathcal{P}_{2n}$ ($n \in \mathbb{N}$). If $A \subset \Pi_{2n}$ is a subset of the class $\mathcal{P}_{2n-1}$, then $\Pi_{2n} \setminus A$ is homeomorphic to Π_{2n}.

We shall need a general technique supplying us with homotopy dense copies of absorbing spaces in s. This technique, developed in Section 6, gives also a new method for constructing absorbing sets. Let C be a space. Embed C as a closed linearly independent subset into a pre-Hilbert space H (this is always possible) and let $L(C)$ denote the linear span of C in H. Then (see [Ca_4]) $L(C)$ is an absorbing set for the smallest topological class containing C and $[0, 1]$ and satisfying the following conditions:

(1) every closed subspace of a space C from $\mathcal{C}$ belongs to the class $\mathcal{C}$;

(2) if $C, C' \in \mathcal{C}$, then $C \times C' \in \mathcal{C}$.

According to [Ca_4], the space $L(C)$ is homeomorphic to its square. Slightly modifying the construction of the space $L(C)$, we assign to each space C an absolute retract $\Omega(C)$, absorbing for the smallest topological class $\mathcal{C}$ containing C and satisfying the conditions (1) and

(3) if $C \in \mathcal{C}$, then $C \times [0, 1] \in \mathcal{C}$.

New phenomena appear here: in general $\Omega(C)$ admits no compatible structure of topological group or of convex set; $\Omega(C)$ may not be homeomorphic to its square. We will give an example of a compactum C such that $\Omega(C)$ does not admit structures of a topological group or a convex set, and all powers $\Omega(C)^n$, $n \in \mathbb{N}$, are topologically distinct.

Let $\mathcal{C}$ be a class of spaces for which there exists a $\mathcal{C}$-absorbing absolute retract Ω. Having developed the necessary tools, we finally come in Section 7 to the problem of topological classification of pairs (s, X) and (Q, X), where X is a homotopy dense copy of Ω in s or Q. If all elements of $\mathcal{C}$ are σ-compact, it is easy to see that for any homotopy dense copy X of Ω in Q (resp. in s), the pair (Q, X) (resp. (s, X)) is strongly $(\mathcal{M}_0, \mathcal{C})$-universal (resp. strongly $(\mathcal{M}_1, \mathcal{C})$-universal ($\mathcal{M}_1$ is the class of Polish spaces)). This implies that the pair (Q, X) (resp. (s, X)) is unique up to homeomorphism. For embeddings into s, the situation changes, whenever $\mathcal{C}$ contains non-sigma-compact spaces. Suppose $\mathcal{C}$ is stable under multiplication by $[0, 1]$ and contains the space ω^ω of irrationals. Then, denoting by $\mathcal{F}_0(M, X)$ the class of pairs $(F, F \cap X)$, where F is a closed subset of M we have:

(1) s contains two homotopy dense copies X_1, X_2 of Ω such that (a) for $i = 1, 2$ the pair (s, X_i) is strongly $\mathcal{F}_0(s, X_i)$-universal, (b) X_1 is contained in a countable union of elements of $\mathcal{M}_0 \cap \mathcal{C}$, but X_2 is not contained in such a union.

We will show in Theorem 7.4 that under some hypotheses on $\mathcal{C}$, there exists up to homeomorphism a unique pair (s, X), where X is a homotopy dense copy of Ω, contained in a σ-compact subset of s.

(2) s contains continuum many homotopy dense copies E_α, $\alpha \in \mathfrak{c}$, of Ω such that (a) for every $\alpha \in \mathfrak{c}$ the pair (s, E_α) is not strongly $\mathcal{F}_0(s, E_\alpha)$-universal; (b) $(s, E_\alpha) \ncong (s, E_\beta)$ if $\alpha \neq \beta$ (here the symbol $\cong$ means "homeomorphic to").

If, in addition, $\mathcal{C}$ contains the class $\mathcal{M}_1$ then

(3) s contains continuum many homotopy dense copies F_α, $\alpha \in \mathfrak{c}$, of Ω such that (a) each (s, F_α) is strongly $\mathcal{F}_0(s, F_\alpha)$-universal; (b) $(s, F_\alpha) \ncong (s, F_\beta)$, provided $\alpha \neq \beta$.

Observe the difference between the families (2) and (3): if $\alpha \neq \beta$, then $\mathcal{F}_0(s, F_\alpha) \neq \mathcal{F}_0(s, F_\beta)$, but the pairs (s, F_α) are homogeneous in the sense that for any $x, y \in F_\alpha$ there is an autohomeomorphism h of (s, F_α) such that $h(x) = y$. On the other hand, $\mathcal{F}_0(s, E_\alpha) = \mathcal{F}_0(s, E_\beta)$ for any α, β, but the pairs (s, E_α) are not homogeneous (though the spaces E_α are).

The case of pairs (Q, X) is much more curious. The situation depends on the intersection $\mathcal{M}_0 \cap \mathcal{C}$. If the class $\mathcal{C}$ satisfies the hypotheses of Theorem 4.1 and contains the class $\mathcal{M}_0$, then there is a unique (up to homeomorphism) pair (Q, X), where X is a homotopy dense copy of Ω. On the other hand, if $\mathcal{C}$ satisfies the hypotheses of Theorem 4.1 but fails to contain $\mathcal{M}_0$, then

(1′) Q contains two homotopy dense copies X_1, X_2 of Ω such that (a) for $i = 1, 2$ the pair (Q, X_i) is strongly $\mathcal{F}_0(Q, X_i)$-universal, (b) X_1 is contained in a countable union of elements of $\mathcal{M}_0 \cap \mathcal{C}$, but X_2 is not contained in such a union.

Here again, the pair (Q, X_1), under some conditions on $\mathcal{C}$, is still unique up to homeomorphism.

(2′) Q contains continuum many homotopy dense copies E_α, $\alpha \in \mathfrak{c}$, of Ω such that (a) (Q, E_α) is not strongly $\mathcal{F}_0(Q, E_\alpha)$-universal for $\alpha \in \mathfrak{c}$; (b) $(Q, E_\alpha) \ncong (Q, E_\beta)$ if $\alpha \neq \beta$.

Finally, if $\mathcal{C}$ contains no strongly infinite-dimensional compactum, then

(3′) Q contains continuum many homotopy dense copies F_α, $\alpha \in \mathfrak{c}$, of Ω such that (a) for every $\alpha \in \mathfrak{c}$ the pair (Q, F_α) is strongly $\mathcal{F}_0(Q, F_\alpha)$-universal; (b) $(Q, F_\alpha) \not\cong (Q, F_\beta)$ if $\alpha \neq \beta$.

1. Preliminaries

All spaces considered in this paper are metrizable and separable, *all maps are continuous.*

Let us recall the main model spaces of infinite-dimensional topology: the Hilbert cube $Q = [-1,1]^\omega$, its pseudointerior $s = (-1,1)^\omega$ which is homeomorphic to the separable Hilbert space l^2 [Mi, §6.6], the radial interior $\Sigma = \{(t_i) \in Q \mid \sup_{i\in\mathbb{N}} |t_i| < 1\}$ of Q, which is known to be homeomorphic to the pseudo-boundary $Q \setminus s$ of Q, and the subspace $\sigma = \{(t_i) \in s \mid t_i = 0 \text{ for almost all } i\} \subset s \subset Q$. As usual, ω is the set of all non-negative integers $\{0,1,2,\ldots\}$, $\mathbb{N} = \omega \setminus \{0\}$; 2^ω is the Cantor set, I stands for the segment [0,1], and ∂I^k is the boundary of the finite-dimensional cube I^k.

By $\mathrm{cov}(X)$ we denote the collection of all open covers of a space X. We say that a cover $\mathcal{U}$ of X refines a cover $\mathcal{V}$ (denoted by $\mathcal{U} \prec \mathcal{V}$) if for every $U \in \mathcal{U}$ there exists $V \in \mathcal{V}$ with $U \subset V$. For a cover $\mathcal{U} \in \mathrm{cov}(X)$ and $A \subset X$ let $\mathrm{St}(A,\mathcal{U}) = \bigcup\{U \in \mathcal{U} \mid A \cap U \neq \emptyset\}$ and $\mathrm{St}\,\mathcal{U} = \{\mathrm{St}(U,\mathcal{U}) \mid U \in \mathcal{U}\}$. Two maps $f, g : Y \to X$ are defined to be $\mathcal{U}$-*close* (denoted by $(f,g) \prec \mathcal{U}$) if for every $y \in Y$ there is $U \in \mathcal{U}$ with $\{f(y), g(y)\} \subset U$. Let X be a space endowed with a metric d. For a cover $\mathcal{U} \in \mathrm{cov}(X)$ let $\mathrm{mesh}(\mathcal{U}) = \sup\{\mathrm{diam}(U) \mid U \in \mathcal{U}\}$. By $O(x,\varepsilon)$ we denote the open ε-ball around $x \in X$.

Further the sentence "a map $f : X \to Y$ can be approximated by a map $\bar{f} : X \to Y$ with a certain property" will mean that for every open cover $\mathcal{U}$ of Y there is a map $\bar{f}$ possessing this property and $\mathcal{U}$-close to f.

We define a subset X of a space Y to be *homotopy dense* if there is a homotopy $h : Y \times [0,1] \to Y$ such that $h(Y \times (0,1]) \subset X$ and $h(y,0) = y$ for all $y \in Y$. A subset $A \subset Y$ is called *homotopy negligible* if $X \setminus A$ is homotopy dense in X. An embedding $e : X \to Y$ is called *homotopy dense* if $e(X)$ is a homotopy dense set in Y.

One can show that if $Z \subset Y$ is a homotopy dense set in Y and $Y \subset X$ is homotopy dense in X then the set $Z \subset X$ is homotopy dense in X. According to [To$_1$, 2.4] a subset A of an ANR-space X is homotopy negligible if and only if it is locally homotopy negligible (i.e. every map $f : I^k \to X$ of a finite-dimensional cube with $f(\partial I^k) \cap A = \emptyset$ can be approximated by a map $\bar{f} : I^k \to X$ such that $\bar{f}|\partial I^k = f|\partial I^k$ and $\bar{f}(I^k) \cap A = \emptyset$). It is well known that every dense convex set $C \subset X$ is homotopy dense, whenever X is a subset of a locally convex linear metric space. We say that a map $f : X \to Y$ is *closed over a subset* $A \subset Y$ if for each $a \in A$ and each neighborhood U of $f^{-1}(a)$ (which may be empty) there exists a neighborhood V of a such that $f^{-1}(V) \subset U$.

A space X is defined to have the *strong discrete approximation property* (briefly SDAP) if any map $f : \bigoplus_{n\in\mathbb{N}} I^n \to X$ can be approximated by a map sending $\{I^n\}_{n\in\mathbb{N}}$ to a discrete collection in X. The strong discrete approximation property plays a crucial role in characterizing manifolds modeled on the pseudo-interior $s = (-1,1)^\omega$ of the Hilbert cube: a space X is an s-manifold if and only if X is a Polish ANR satisfying SDAP [To_2]. On the other hand, X is an ANR satisfying SDAP if and only if X admits a homotopy dense embedding into an s-manifold [Ba_1]. This characterization implies the following fact (cf. [Bo]): if X is an ANR satisfying SDAP then every open subspace in X has SDAP too. Indeed, by [Ba_1], X admits a homotopy dense embedding $X \subset M$ into an s-manifold. Then for every open set $U \subset X$, letting $\widetilde{U} \subset M$ be an open set with $\widetilde{U} \cap X = U$, we see that U is a homotopy dense set in the s-manifold $\widetilde{U}$. Again applying [Ba_1], we conclude that U satisfies SDAP.

The following proposition characterizes SDAP.

1.1. Proposition. *Let X be an ANR. The following conditions are equivalent*:

(1) *X has SDAP*;

(2) *every map $f : \bigoplus_{n\in\mathbb{N}} I^n \to X$ can be approximated by a map sending $\{I^n\}_{n\in\mathbb{N}}$ onto a locally finite family in X*;

(3) *for every space F and every locally finite collection $\{F_i\}_{i\in\mathcal{I}}$ of subsets in F every map $f : F \to X$ can be approximated by a map sending $\{F_i\}_{i\in\mathcal{I}}$ onto a locally finite collection in X.*

Proof. The implication (3)⇒(2) is trivial, and the implication (2)⇒(1) is in [Cu, p. 203] (see also [Ba_1, Lemma 4]). For the proof of (1)⇒(3) we need the following lemma proved in [Ba_1].

1.2. Lemma. *If $\{F_i\}_{i\in\mathcal{I}}$ is a locally finite collection of subsets of a space X then there exists a cover $\mathcal{U} \in \mathrm{cov}(X)$ such that the collection $\{\mathrm{St}(F_i,\mathcal{U})\}_{i\in\mathcal{I}}$ is locally finite in X.*

We proceed to prove (1)⇒(3). Fix $\mathcal{U} \in \mathrm{cov}(X)$ and $f : F \to X$. By [Ba_1], there exists an s-manifold $\widetilde{M}$ containing X as a homotopy dense set. For any $U \in \mathcal{U}$ fix an open set $\widetilde{U} \subset \widetilde{M}$ with $\widetilde{U} \cap X = U$, and consider the open set $M = \bigcup_{U\in\mathcal{U}} \widetilde{U}$ in $\widetilde{M}$ and the cover $\widetilde{\mathcal{U}} = \{\widetilde{U} \mid U \in \mathcal{U}\}$ of M. Notice that $X \subset M \subset \widetilde{M}$ and M is an s-manifold. By Lavrentiev's Theorem [En], there are a Polish space F' containing F and a map $f' : F' \to M$ extending f. Since $\{F_i\}_{i\in\mathcal{I}}$ is locally finite in F, there is an open neighborhood $\widetilde{F}$ of F in F' such that $\{F_i\}_{i\in\mathcal{I}}$ is locally finite in $\widetilde{F}$. Let finally $\widetilde{f} = f'|\widetilde{F} : \widetilde{F} \to M$. Let $\mathcal{V} \in \mathrm{cov}(M)$ be such that $\mathrm{St}\,\mathcal{V} \prec \widetilde{\mathcal{U}}$. According to [$\mathrm{To}_2$], there is a closed embedding $e : \widetilde{F} \to M$ such that $(\widetilde{f}, e) \prec \mathcal{V}$. Then $\{e(F_i)\}_{i\in\mathcal{I}}$ is locally finite in M. By Lemma 1.2, there is $\mathcal{W} \in \mathrm{cov}(M)$ such that $\mathcal{W} \prec \mathcal{V}$ and $\{\mathrm{St}(e(F_i),\mathcal{W})\}_{i\in\mathcal{I}}$ is locally finite in M. Using the homotopy density of X in M, find $\bar{f} : F \to X$ with $(\bar{f}, e|F) \prec \mathcal{W}$. It is easy to verify that $(\bar{f}, f) \prec \mathcal{U}$ and $\{\bar{f}(F_i)\}_{i\in\mathcal{I}}$ is locally finite in X. ∎

A subset $A \subset X$ is defined to be a (*strong*) *Z-set in X* if A is closed in X and if for every cover $\mathcal{U} \in \mathrm{cov}(X)$ there is a map $f : X \to X$ such that $(f, \mathrm{id}) \prec \mathcal{U}$ and $f(X) \cap A = \emptyset$ ($\mathrm{Cl}_X(f(X)) \cap A = \emptyset$ in the case of strong Z-sets). It is well known that a closed subset A of an ANR-space X is a Z-set in X iff A is homotopy negligible and iff every map $f : Q \to X$ of the Hilbert cube can be approximated by maps whose range misses A. One should keep in mind that every strong Z-set is a Z-set, but that the converse is not true (see [BBMW]). However, if $X \in \mathrm{ANR}$ satisfies SDAP then every Z-set in X is a strong Z-set [BM, 1.7]. We will sometimes use the following simple fact: for a homotopy dense subset $X \subset Y$, a closed subset $A \subset Y$ is a Z-set in Y if and only if $A \cap X$ is a Z-set in X. Very often we will use the following result from [BM, 1.1]: if $f : C \to X$ is a map into an ANR such that the restriction $f|B : B \to X$ of f to a closed subset $B \subset C$ is a closed embedding with $f(B)$ being a strong Z-set in X, then for every open cover $\mathcal{U} \in \mathrm{cov}(X)$ there is a map $f' : C \to X$ such that $(f', f) \prec \mathcal{U}$, $f'|B = f|B$, $f'(C \setminus B) \cap f(B) = \emptyset$ and f' is closed over $f(B)$.

An embedding $f : A \to X$ is defined to be a *Z-embedding* provided $f(A)$ is a Z-set in X. A (*strong*) *Z_σ-space* is, by definition, a space which is a countable union of its (strong) Z-sets. According to [BM, §1], an ANR is a strong Z_σ-space if and only if it is a Z_σ-space satisfying SDAP.

Now we recall the definition of the absolute Borel classes $\mathcal{M}_\alpha$ and $\mathcal{A}_\alpha$, $\alpha < \omega_1$. For every space X define the classes $\mathcal{A}_\alpha(X)$, $\mathcal{M}_\alpha(X)$, where α is a countable ordinal, as follows. Let $\mathcal{A}_0(X)$ denote the class of all open subsets in X and let $\mathcal{M}_0(X) = \{A \subset X \mid X \setminus A \in \mathcal{A}_0(X)\}$. Assuming that for a countable ordinal α the classes $\mathcal{M}_\xi(X)$ and $\mathcal{A}_\xi(X)$ have been defined for all ordinals $\xi < \alpha$, let $\mathcal{A}_\alpha(X) = \{A \subset X \mid A = \bigcup_{n=1} A_n, A_n \in \bigcup_{\xi<\alpha} \mathcal{M}_\xi(X) \text{ for all } n\}$ and $\mathcal{M}_\alpha(X) = \{A \subset X \mid X \setminus A \in \mathcal{A}_\alpha(X)\}$. A space A is defined to belong to the absolute Borel class $\mathcal{A}_\alpha$ (resp. $\mathcal{M}_\alpha$), provided $e(A) \in \mathcal{A}_\alpha(X)$ (resp. $e(A) \in \mathcal{M}_\alpha(X)$) for every embedding $e : A \to X$. In particular, $\mathcal{M}_0$ is the class of compacta, $\mathcal{M}_1$ is the class of Polish spaces and $\mathcal{A}_1$ is the class of σ-compacta. By $\mathcal{P}_n$, $n \geq 0$, we denote the projective classes. Recall that $\mathcal{P}_0$ is the class of all Borel spaces. The classes $\mathcal{P}_n$ for $n \geq 1$ are defined inductively: $\mathcal{P}_{2n-1}$ is the class of all continuous metrizable images of spaces from $\mathcal{P}_{2n-2}$; and $\mathcal{P}_{2n}$ consists of complements in Polish spaces of spaces from $\mathcal{P}_{2n-1}$.

Let $\mathcal{C}$ be a class of spaces. Define the class $\sigma\mathcal{C}$ as follows. A space X belongs to $\sigma\mathcal{C}$ provided it can be written as a countable union $X = \bigcup_{n\in\mathbb{N}} X_n$ of closed subsets such that for every $n \in \mathbb{N}$, X_n admits a closed embedding into a space $C \in \mathcal{C}$. For a class $\mathcal{D}$ of spaces and a space X let $\mathcal{D}(X) = \{D \subset X \mid \text{there is a compactum } K \supset X \text{ and a subspace } \widetilde{D} \in \mathcal{D} \text{ of } K \text{ such that } \widetilde{D} \cap X = D\}$. Notice that $\mathcal{M}_0(X)$ (resp. $\mathcal{M}_1(X)$) coincides with the family of closed (resp. G_δ-) subsets of X.

A class $\mathcal{C}$ of spaces is called

(1) *topological* if for every $C \in \mathcal{C}$ and every homeomorphism $h : C \to D$ it follows that $D \in \mathcal{C}$;

(2) *local* if a space X belongs to $\mathcal{C}$ when each point $x \in X$ has a neighborhood $U \in \mathcal{C}$;

(3) *compactification-admitting* if for every $C \in \mathcal{C}$ there is a compactum $K \in \mathcal{C}$ containing C;

(4) *T-stable*, where T is a space, if $C \times T \in \mathcal{C}$ for every $C \in \mathcal{C}$;

(5) *$\mathcal{D}$-hereditary*, where $\mathcal{D}$ is a class of spaces, if for every $C \in \mathcal{C}$ we have $\mathcal{D}(C) \subset \mathcal{C}$;

(6) *closed-additive* if $X \in \mathcal{C}$ whenever $X = X_1 \cup X_2$ is the union of two subspaces $X_1, X_2 \in \mathcal{C}$ one of which is closed in X;

(7) *$\mathcal{D}$-additive*, where $\mathcal{D}$ is a class of spaces, if $X \in \mathcal{C}$ whenever $X = C \cup D$, where $C \in \mathcal{C}$ and $D \in \mathcal{D}$;

(8) *weakly $\mathcal{D}$-additive*, where $\mathcal{D}$ is a class of spaces, provided for every compactum $K \in \mathcal{C}$ and subsets $D, C \subset K$, if $D \in \mathcal{D}$ and $C \in \mathcal{C}$ then $D \cup C \in \mathcal{C}$.

For $\mathcal{M}_0$-hereditary classes of spaces we will use the commonly used term "a closed-hereditary class". We distinguish the collection $\mathcal{M}_0(X)$ of closed subsets of a space X and the topological class $\mathcal{F}_0(X)$ of spaces homeomorphic to closed subspaces of X.

Given a class $\mathcal{C}$ of spaces let $\mathcal{C}(\text{c.d.})$ (resp. $\mathcal{C}(\text{s.c.d.})$) denote the subclass of $\mathcal{C}$ consisting of all countable-dimensional (resp. strongly countable-dimensional) spaces from the class $\mathcal{C}$. For a countable ordinal α let also $\mathcal{C}(\alpha)$ (resp. $\mathcal{C}[\alpha]$) be the subclass of $\mathcal{C}$ consisting of all spaces $C \in \mathcal{C}$ with $\operatorname{ind} C < \alpha$ (resp. $\operatorname{ind} C \leq \alpha$).

A space X is defined to be *$\mathcal{C}$-universal*, where $\mathcal{C}$ is a class of spaces, if for every $C \in \mathcal{C}$ there is a closed embedding $e : C \to X$.

Let X, C be two spaces. X is defined to be *strongly C-universal* if for every cover $\mathcal{U} \in \operatorname{cov}(X)$, closed subset $B \subset C$ and map $f : C \to X$ whose restriction $f|B : B \to X$ is a Z-embedding, there is a Z-embedding $\bar{f} : C \to X$ such that $\bar{f}|B = f|B$ and $(\bar{f}, f) \prec \mathcal{U}$. According to [BM, 1.7 and 2.2], an ANR-space X satisfying SDAP is strongly C-universal, provided for open sets $U \subset X$ and $V \subset C$ every map $f : V \to U$ can be approximated by Z-embeddings.

A space X is defined to be *strongly $\mathcal{C}$-universal*, where $\mathcal{C}$ is a class of spaces, if X is strongly C-universal for every $C \in \mathcal{C}$. We define a space X to be *strongly universal* provided X is strongly $\mathcal{F}_0(X)$-universal.

By a pair (X, Y) we will always understand a couple of spaces with $Y \subset X$. For classes $\mathcal{K}, \mathcal{C}$ of spaces by $(\mathcal{K}, \mathcal{C})$ we denote the class of pairs (K, C) with $\mathcal{C} \ni C \subset K \in \mathcal{K}$.

A pair (X, Y) is defined to be *strongly (K, C)-universal* if, given a cover $\mathcal{U} \in \operatorname{cov}(X)$ and a map $f : K \to X$ whose restriction $f|B : B \to X$ onto a given closed subset $B \subset K$ is a Z-embedding with $(f|B)^{-1}(Y) = B \cap C$, there exists a Z-embedding $\bar{f} : K \to X$ such that $(\bar{f}, f) \prec \mathcal{U}$, $\bar{f}|B = f|B$ and $\bar{f}^{-1}(Y) = C$. Notice that a pair (X, Y) is strongly (K, C)-universal if and only if the pair $(X, X \setminus Y)$ is strongly $(K, K \setminus C)$-universal.

For a pair (X, Y), let $\mathcal{M}_0(X, Y) = \{(F, F \cap Y) \mid F \in \mathcal{M}_0(X)\}$ and let $\mathcal{F}_0(X, Y)$ be the class of pairs homeomorphic to couples from the class $\mathcal{M}_0(X, Y)$.

Let $\vec{\mathcal{C}}$ be a class of pairs. A pair (X, Y) is defined to be *$\vec{\mathcal{C}}$-universal* provided for every couple $(K, C) \in \vec{\mathcal{C}}$ there is a closed embedding $e : K \to X$ such that $e^{-1}(Y) = C$. A pair (X, Y) is defined to be *strongly $\vec{\mathcal{C}}$-universal*, provided it is strongly (K, C)-universal for every pair $(K, C) \in \vec{\mathcal{C}}$. We define a pair (X, Y) to be *strongly universal*, provided it is strongly $\mathcal{F}_0(X, Y)$-universal.

Notice that a space X is strongly C-universal if and only if the pair $(X, \emptyset)$ is strongly $(C, \emptyset)$-universal.

Now we establish some important properties of strongly universal spaces and pairs.

1.3. Lemma. *Let $C \subset K$ and $X \subset M$ be two pairs of spaces with M being an ANR. If the pair (M, X) is strongly (K, C)-universal then, for every open subset $U \subset M$ and every closed subset $F \subset K$, the pair $(U, U \cap X)$ is strongly $(F, F \cap C)$-universal.*

Proof. Repeating the arguments from the proof of Proposition 2.1 of [BM], we can show that Lemma 1.3 holds if the open set U is contractible in M (just use the fact that any map from F into U extends to a map from K into M).

Using the fact that any open subset U of the ANR-space M admits a cover by open sets contractible in M and repeating the arguments from the proof of Proposition 2.7 of [BM] show that Lemma 1.3 is valid for any open set U in M. ■

This lemma and [BGM, 6.1] imply

1.4. LEMMA. *Let (M, X), (K, C) be two pairs, where M is an ANR and K is compact, and let Y be a homotopy dense subset of M. If the pair $(Y, Y \cap X)$ is strongly (K, C)-universal then so is the pair (M, X).*

For a space X we denote by $\mathcal{SU}(X)$ the class of all spaces C such that the space X is strongly C-universal. The class $\mathcal{SU}(X)$ has the following properties:

1.5. PROPOSITION. *Let X be an ANR.*

(1) *the class $\mathcal{SU}(X)$ is topological and closed-hereditary*;
(2) *if every Z-set in X is a strong Z-set, then the class $\mathcal{SU}(X)$ is closed-additive*;
(3) *if X satisfies SDAP then the class $\mathcal{SU}(X)$ is local*;
(4) *if X is a strong Z_σ-space then $\mathcal{SU}(X) = \sigma\text{-}\mathcal{SU}(X)$.*

PROOF. The statement (1) follows from Lemma 1.3. The proofs of the second and the third statements depend on the following

1.6. LEMMA. *Let X be a strongly $\mathcal{C}$-universal ANR and C a space that can be expressed as $C = \bigcup_{n\geq 0} C_n$, where $C_0 = \emptyset$ and each $C_n \in \mathcal{C}$ is a closed subset of X with $C_n \subset C_{n+1}$. Suppose $f : C \to X$ is a map such that the collection $\{f(C \setminus C_n)\}_{n\geq 0}$ is locally finite in X. Then for every cover $\mathcal{U} \in \operatorname{cov}(X)$ there exists a Z-embedding $\bar{f} : C \to X$ such that $(\bar{f}, f) \prec \mathcal{U}$.*

PROOF. Fix $\mathcal{U} \in \operatorname{cov}(X)$. By Lemma 1.2, there is $\mathcal{V}_0 \in \operatorname{cov}(X)$ such that the collection $\{\mathrm{St}(f(C \setminus C_n), \mathcal{V}_0)\}_{n\geq 0}$ is locally finite in X. Pick a sequence $\{\mathcal{V}_n\}_{n\geq 1} \subset \operatorname{cov}(X)$ so that $\mathrm{St}\,\mathcal{V}_n \prec \mathcal{V}_{n-1}$ and $\operatorname{mesh}\mathcal{V}_n < 2^{-n}$ for every $n \in \mathbb{N}$. Let $f_0 = f$ and for every $n \geq 1$, using strong C_n-universality of X, construct inductively a map $f_n : C \to X$ such that

$$f_n|C_n : C_n \to X \text{ is a } Z\text{-embedding}, \quad f_n|C_{n-1} = f_{n-1}|C_{n-1}, \quad (f_n, f_{n-1}) \prec \mathcal{V}_n.$$

Then the limit map $\bar{f} = \lim_{n\to\infty} f_n : C \to X$ is $\mathcal{V}_0$-close to $f_0 = f$, and hence, the collection $\{\bar{f}(C \setminus C_n)\}_{n\geq 0}$ is locally finite in X. It follows from the construction that $\bar{f}$ is injective and $\bar{f}|C_n : C_n \to X$ is a Z-embedding for every $n \in \mathbb{N}$. Thus $\bar{f}(C)$ is a local Z-set in X and, being closed, is a Z-set in X. ■

This lemma implies the following useful

1.7. PROPOSITION. *Let $f : C \to X$ be a map of a space C into a strongly C-universal ANR X. For every open set $U \subset X$ and every cover $\mathcal{U} \in \operatorname{cov}(U)$ there is a Z-embedding $g : f^{-1}(U) \to U$ such that $(g, f|f^{-1}(U)) \prec \mathcal{U}$.* ■

Now we prove the second statement of 1.5. Suppose that every Z-set in X is strong and C is a space that can be expressed as $C = C_1 \cup C_2$, where $C_1, C_2 \in \mathcal{SU}(X)$ and C_1 is closed in C. To show that X is strongly C-universal, fix $\mathcal{U} \in \operatorname{cov}(X)$, a closed subset $B \subset C$, and a map $f : C \to X$ that restricts to a Z-embedding on B. According to [BM, 1.1], we may assume that $f(C \setminus B) \cap f(B) = \emptyset$ and f is closed over $f(B)$.

Let $\mathcal{V} \in \operatorname{cov}(X)$ be such that $\operatorname{St}^2 \mathcal{V} \prec \mathcal{U}$. Since X is an ANR, there is $\mathcal{W} \in \operatorname{cov}(X)$ such that every map $f_1 : C_1 \cup B \to X$ with $(f_1, f|C_1 \cup B) \prec \mathcal{W}$ extends to a map $\bar{f}_1 : C \to X$, $\mathcal{V}$-close to f.

By 1.7, there is a Z-embedding $g : C_1 \setminus B \to X \setminus f(B)$ such that $(g, f|C_1 \setminus B) \prec \mathcal{W}$ and $d(g(c), f(c)) < \frac{1}{2} d(f(c), f(B))$ for each $c \in C_1 \setminus B$. Then the map $f_1 : C_1 \cup B \to X$ defined by

$$f_1(c) = \begin{cases} f(c) & \text{if } c \in B, \\ g(c) & \text{if } c \in C_1 \setminus B, \end{cases}$$

is a Z-embedding $\mathcal{W}$-close to $f|C_1 \cup B$. Extend f_1 to a map $\bar{f}_1 : C \to X$ such that $(\bar{f}_1, f) \prec \mathcal{V}$. By [BM, 1.1], we can find a map $\widetilde{f}_1 : C \to X$ such that $(\widetilde{f}_1, \bar{f}_1) \prec \mathcal{V}$, $\widetilde{f}_1|C_1 \cup B = \bar{f}_1|C_1 \cup B$, $\widetilde{f}_1(C \setminus (C_1 \cup B)) \cap \bar{f}_1(C_1 \cup B) = \emptyset$ and $\widetilde{f}_1$ is closed over $\widetilde{f}_1(C_1 \cup B)$.

Using Proposition 1.7, find a Z-embedding $h : C_2 \setminus (C_1 \cup B) \to X \setminus \widetilde{f}_1(C_1 \cup B)$ such that $(h, \widetilde{f}_1|C_2 \setminus (C_1 \cup B)) \prec \mathcal{V}$ and $d(h(c), \widetilde{f}_1(c)) < \frac{1}{2} d(\widetilde{f}_1(c), \widetilde{f}_1(C_1 \cup B))$ for each $c \in C_2 \setminus (C_1 \cup B)$. Then the map $\bar{f} : C \to X$ defined by the formula

$$\bar{f}(c) = \begin{cases} \widetilde{f}_1(c) & \text{if } c \in C_1 \cup B, \\ h(c) & \text{if } c \in C_2 \setminus (C_1 \cup B), \end{cases}$$

is a Z-embedding $\mathcal{U}$-close to f and extending $f|B$. The second statement of the proposition is proved.

To prove the third statement, suppose X has SDAP and C is a space such that each point $c \in C$ has a neighborhood $U \in \mathcal{SU}(X)$. Fix a cover $\mathcal{U} \in \operatorname{cov}(X)$, a closed set $B \subset C$, and a map $f : C \to X$ that restricts to a Z-embedding on B. By [BM, 1.7], each Z-set in X is a strong Z-set. Thus, according to [BM, 1.1], we may assume that $f(C \setminus B) \cap f(B) = \emptyset$ and f is closed over $f(B)$. Let $C' = C \setminus B$ and $\mathcal{V} \in \operatorname{cov}(X \setminus f(B))$ be such that $\operatorname{St} \mathcal{V} \prec \mathcal{U}$ and $\operatorname{St} \mathcal{V} \prec \{B(x, d(x, f(B))/2) \mid x \in X \setminus f(B)\}$.

By the first statement of the theorem, the class $\mathcal{SU}(X)$ is closed-hereditary. Using this and the fact that each point $c \in C$ has a neighborhood $U \in \mathcal{SU}(X)$, pick a countable collection $\{F_n\}_{n \in \mathbb{N}}$ of closed subsets of C such that $C' = \bigcup_{n \in \mathbb{N}} \operatorname{Int} F_n$ and each $F_n \in \mathcal{SU}(X)$. Let $C_n = F_1 \cup \ldots \cup F_n$, $n \in \mathbb{N}$. By the second statement of the theorem, each $C_n \in \mathcal{SU}(X)$. Notice also that the collection $\{C' \setminus C_n\}_{n \in \mathbb{N}}$ is locally finite in C'. Since $X \setminus f(B)$, being an open subspace of X, satisfies SDAP, by 1.1, there is a map $f' : C' \to X \setminus f(B)$ such that $(f', f'|C') \prec \mathcal{V}$ and the collection $\{f'(C' \setminus C_n)\}_{n \in \mathbb{N}}$ is locally finite in $X \setminus f(B)$. Using 1.6, find a Z-embedding $\widetilde{f} : C' \to X \setminus f(B)$ such that $(\widetilde{f}, f') \prec \mathcal{V}$. Then the map $\bar{f} : C \to X$ defined by the formula

$$\bar{f}(c) = \begin{cases} f(c) & \text{if } c \in C, \\ \widetilde{f}(c) & \text{if } c \in C \setminus B, \end{cases}$$

is a Z-embedding $\mathcal{U}$-close to f and extending $f|B$. Thus the third statement of 1.5 is proved.

Finally, the statement (4) results from [BM, 1.7, 2.3] and the statements (1), (2). ∎

A space X is defined to be *$\mathcal{C}$-absorbing* if $X \in \sigma\mathcal{C}$ is a strongly $\mathcal{C}$-universal ANR satisfying SDAP and X is a Z_σ-space. It is easy to see that every open subspace of a $\mathcal{C}$-absorbing space is $\mathcal{C}$-absorbing too.

Absorbing spaces are of great importance because of the following theorem proven in [BM].

1.8. Theorem (Classification by homotopy type). *Let $\mathcal{C}$ be a class of spaces and let X, Y be two $\mathcal{C}$-absorbing spaces. X and Y are homeomorphic if and only if they are homotopically equivalent. Moreover, if both X and Y are homotopy dense subspaces of an s-manifold M, then for every open cover $\mathcal{U}$ of M there is a homeomorphism $h : X \to Y$, $\mathcal{U}$-close to the identity map of X.*

Notice that by Proposition 1.5, every $\mathcal{C}$-absorbing space is strongly universal. Therefore, we can give a meaning to the term "absorbing space": a space X is defined to be *absorbing* provided X is a strongly universal ANR which is a strong Z_σ-space. Then Theorem 1.8 can be reformulated as follows: two absorbing spaces X and X' are homeomorphic if and only if they are homotopy equivalent and each of them embeds into the other as a closed subset.

Now let us state some important results concerning strongly universal pairs. For a class of pairs $\vec{\mathcal{C}}$ define the class $\sigma\vec{\mathcal{C}}$ as follows. A pair (X, Y) belongs to $\sigma\vec{\mathcal{C}}$ provided $Y \subset X$ and $X = \bigcup_{n\in\mathbb{N}} X_n$, where for every $n \in \mathbb{N}$, X_n is a closed subset in X such that $(X_n, X_n \cap Y) \in \bigcup_{(K,C)\in\vec{\mathcal{C}}} \mathcal{F}_0(K, C)$.

A pair (M, X) is defined to be *$\vec{\mathcal{C}}$-absorbing* provided it is strongly $\vec{\mathcal{C}}$-universal and there is a Z_σ-set $Z \subset M$ such that $(Z, X) \in \sigma\vec{\mathcal{C}}$. We will say that a pair (M, X) is *absorbing* provided it is $\mathcal{F}_0(M, X)$-absorbing.

1.9. Theorem ([Ca$_2$], [DMM]). *Let $\vec{\mathcal{C}}$ be a class of pairs, $M = Q$ or $M = s$, and (M, X), (M, X') two $\vec{\mathcal{C}}$-absorbing pairs. Then for every cover $\mathcal{U} \in \operatorname{cov}(M)$ and every closed set $B \subset M$ with $B \cap X = B \cap X'$ there is a homeomorphism $h : M \to M$ such that $(h, \mathrm{id}) \prec \mathcal{U}$, $h|B = \mathrm{id}\,|B$ and $h^{-1}(X) = X'$.*

The following lemma generalizes [BGM, 9.5] and can be proved by the same technique.

1.10. Lemma. *Let M be an ANR and (M, X) be a strongly $\vec{\mathcal{C}}$-universal pair. Then for every strong Z-set $A \subset M$ and every subset $C \subset A$ the pair $(M, X \cup C)$ is strongly $\vec{\mathcal{C}}$-universal.*

1.11. Proposition. *Let $M = Q$ or $M = s$. Every $\vec{\mathcal{C}}$-absorbing pair (M, X) is $\mathcal{F}_0(M, X)$-absorbing.*

Proof. To prove the proposition, we have to verify the strong $\mathcal{M}_0(M, X)$-universality of (M, X). For this, fix a pair $(F, F \cap X)$, where F is a closed subset in M, a cover $\mathcal{U} \in \operatorname{cov}(M)$, a closed subset $B \subset F$, and a map $f : F \to M$ such that the restriction $f|B : B \to M$ is a Z-embedding with $(f|B)^{-1}(X) = B \cap X$.

By the definition of a $\vec{\mathcal{C}}$-absorbing pair, there is a Z_σ-set $Z \subset M$ such that $(Z, X) \in \sigma\vec{\mathcal{C}}$. Let $\mathcal{V} \in \operatorname{cov}(M)$ be a cover with $\operatorname{St}\mathcal{V} \prec \mathcal{U}$. By [Mi, 6.2.2], there is a Z-embedding $f' : F \to M$ such that $(f', f) \prec \mathcal{V}$, $f'|B = f|B$ and $f'(F \setminus B) \cap Z = \emptyset$. By Lemma 1.10, the pair $(M, X \cup f'(F \cap X))$ is strongly $\vec{\mathcal{C}}$-universal. Moreover, it is easy to see that $Z \cup f'(F \cap Z)$ is a Z_σ-set in M such that $(Z \cup f'(F \cap Z), X \cup f'(F \cap X)) \in \sigma\vec{\mathcal{C}}$. Then by Theorem 1.9, there is a homeomorphism $h : M \to M$ such that $(h, \mathrm{id}) \prec \mathcal{V}$,

$h|f(B) = \mathrm{id}\,|f(B)$ and $h^{-1}(X) = X \cup f'(F \cap X)$. Letting $\bar{f} = h \circ f' : F \to M$ we see that $(\bar{f}, f) \prec \mathrm{St}\,\mathcal{V} \prec \mathcal{U}$, $\bar{f}|B = f|B$ and $\bar{f}^{-1}(X) = F \cap X$, i.e. $\bar{f}$ is the required Z-embedding. ■

The following lemma characterizes the strong universality of pairs and can be proved by arguments of [BM, 2.2].

1.12. Lemma. *Let M be an ANR satisfying SDAP and $X \subset M$. The pair (M, X) is strongly (K, C)-universal if and only if for every open sets $U \subset M$ and $V \subset K$ any map $f : V \to U$ can be approximated by a Z-embedding $\bar{f} : V \to U$ such that $\bar{f}^{-1}(U \cap X) = V \cap C$.*

Next, we have a counterpart of Proposition 2.6 of [BM].

1.13. Lemma. *Suppose Y is an ANR and (M, X) is a strongly (K, C)-universal pair, where M is an ANR such that every Z-set in $M \times Y$ is a strong Z-set. Then the pair $(M \times Y, X \times Y)$ is strongly (K, C)-universal.*

Proof. Fix a cover $\mathcal{U} \in \mathrm{cov}(M \times Y)$, a closed subset $B \subset K$, and a map $f : K \to M \times Y$ such that $f|B : B \to M \times Y$ is a Z-embedding with $(f|B)^{-1}(X \times Y) = B \cap C$. Suppose that each Z-set in $M \times Y$ is a strong Z-set. Then, according to [BM, 1.1], we may assume that $f(K \setminus B) \cap f(B) = \emptyset$ and f is closed over $f(B)$.

Fix metrics d_M and d_Y on M and Y respectively, and consider on $M \times Y$ the metric d defined by

$$d((x, y), (x', y')) = \max\{d_M(x, x'), d_Y(y, y')\}.$$

Let $\varepsilon : M \times Y \to (0, 1]$ be a function such that $\{B(x, \varepsilon(x))\}_{x \in M \times Y} \prec \mathcal{U}$, and define a map $\delta : M \times Y \to [0, 1]$ by letting

$$\delta(x) = \tfrac{1}{2} \min\{\varepsilon(x), d(x, f(B))\}.$$

Let $K_0 = \emptyset$ and $K_n = (\delta \circ f)^{-1}([2^{-n}, 1])$ for $n \geq 1$. Evidently, each K_n is closed in K and $K \setminus B = \bigcup_{n \geq 0} K_n$. Denote by $p_M : M \times Y \to M$, $p_Y : M \times Y \to Y$ the natural projections. Using the strong (K, C)-universality of (M, X), construct inductively a map $g : K \setminus B \to M$ such that for every $n \in \mathbb{N}$ the following conditions are satisfied:

- $g|K_n : K_n \to M$ is a Z-embedding with $(g|K_n)^{-1}(X) = K_n \cap C$ and
- $d_M(g(x), p_M \circ f(x)) < 2^{-n}$ for any $x \in K_n \setminus K_{n-1}$.

Then the map $\bar{f} : K \to M \times Y$ defined by the formula

$$\bar{f}(x) = \begin{cases} f(x) & \text{if } x \in B, \\ (g(x), p_Y \circ f(x)) & \text{if } x \in K \setminus B, \end{cases}$$

is a Z-embedding extending $f|B$ and $\mathcal{U}$-close to f. ■

2. On universal spaces and universal pairs

In this section we will establish some elementary but important properties of universal spaces and universal pairs. Obviously if a pair (M, X) is $(\mathcal{M}_0 \cap \mathcal{C}, \mathcal{C})$-universal, where $\mathcal{C}$ is a compactification-admitting class, then the space X is $\mathcal{C}$-universal. It turns out that in some cases the converse statement is also true.

2.1. Theorem. *Let X be a $\mathcal{C}$-universal space for a 2^ω-stable weakly $\mathcal{A}_1$-additive class $\mathcal{C}$. Then for every Polish space $M \supset X$ the pair (M, X) is $(\mathcal{M}_0 \cap \mathcal{C}, \mathcal{C})$-universal.*

Proof. Fix a pair $(K, C) \in (\mathcal{M}_0 \cap \mathcal{C}, \mathcal{C})$. Let $A \in \mathcal{A}_1 \setminus \mathcal{M}_1$ be any dense subspace in the Cantor set 2^ω, and consider the subset $K \times A \cup C \times 2^\omega \subset K \times 2^\omega$. Since the class $\mathcal{C}$ is 2^ω-stable and weakly $\mathcal{A}_1$-additive, $K \times A \cup C \times 2^\omega \in \mathcal{C}$. Let $f : K \times A \cup C \times 2^\omega \to X$ be a closed embedding. By Lavrentiev's Theorem, f extends to an embedding $\bar{f} : G \to M$ of some G_δ-set $G \subset K \times 2^\omega$ containing $K \times A \cup C \times 2^\omega$. By [En, 3.7.16],

$$\bar{f}^{-1}(X) = K \times A \cup C \times 2^\omega. \tag{1}$$

Notice that the complement $K \times 2^\omega \setminus G$ is σ-compact and its projection B onto 2^ω is a σ-compactum lying in $2^\omega \setminus A$. Since $A \in \mathcal{A}_1 \setminus \mathcal{M}_1$, we have $2^\omega \setminus A \in \mathcal{M}_1 \setminus \mathcal{A}_1$, and hence, $B \neq 2^\omega \setminus A$, i.e., there exists a point

$$t \in 2^\omega \setminus (A \cup B). \tag{2}$$

Then $K \times \{t\} \subset G$. Define the embedding $e : K \to M$ by $e(k) = \bar{f}(k, t)$, $k \in K$, and note that by (1) and (2), $e^{-1}(X) = C$. ∎

For classes $\mathcal{C}$ which are ω^ω-stable and $\mathcal{A}_1$(s.c.d.)-additive we can prove more.

2.2. Theorem. *Let $\mathcal{C}$ be a topological $\mathcal{A}_1$(s.c.d.)-additive ω^ω-stable class of spaces and M be a Polish space. A subspace $X \subset M$ is $\mathcal{C}$-universal if and only if the pair (M, X) is $(\mathcal{M}_0, \mathcal{C})$-universal.*

Proof. The "if" part is trivial. To prove the "only if" part it suffices, for every subset $C \in \mathcal{C}$ of the Hilbert cube $Q = [0,1]^\omega$, to construct an embedding $g : Q \to M$ with $g^{-1}(X) = C$.

For this, consider the following combination of the Cantor cube $2^\omega = \{-1, 1\}^\omega$ and the Hilbert cube Q:

$$K = (\{-1\} \cup [0,1])^\omega.$$

Let $r : K \to 2^\omega$ denote the retraction induced by the natural retraction $\{-1\} \cup [0,1] \to \{-1, 1\}$. In the Cantor set 2^ω, consider the subset $2^\omega_f = \{(t_i) \in 2^\omega : t_i = 1 \text{ for finitely many } i\}$ and let $K_f = r^{-1}(2^\omega_f)$.

For every $t = (t_i)_{i\in\omega} \in K \setminus K_f$ let $n_0(t) < n_1(t) < \dots$ be the enumeration of the infinite set $N(t) = \{i \in \omega : t_i \in [0,1]\}$ and let $h(t) = (h(t)_i)_{i\in\omega}$ be the point in Q defined by $h(t)_i = t_{n_i(t)}$ for $i \in \omega$.

It is easy to see that the map $h : K \setminus K_f \to Q$ is continuous and that moreover, the map

$$H = (r, h) : K \setminus K_f \to (2^\omega \setminus 2^\omega_f) \times Q$$

is a homeomorphism.

Since the class $\mathcal{C}$ is topological and ω^ω-stable, we get $(2^\omega \setminus 2^\omega_f) \times C \in \mathcal{C}$ and $H^{-1}((2^\omega \setminus 2^\omega_f) \times C) \in \mathcal{C}$. Next, by the $\mathcal{A}_1$(s.c.d.)-additivity of $\mathcal{C}$, the set

$$C' = K_f \cup H^{-1}((2^\omega \setminus 2^\omega_f) \times C)$$

belongs to the class $\mathcal{C}$. Since the space X is $\mathcal{C}$-universal, there is a closed embedding $e : C' \to X$. By Lavrentiev's Theorem, it can be extended to an embedding $\bar{e} : G \to M$

of some G_δ-subset $G \subset K$ such that $\overline{e}^{-1}(X) = C'$. The set $K \setminus G$ is σ-compact and so is its image $r(K \setminus G) \subset 2^\omega$. Next, $r(K \setminus G) \cap 2^\omega_f = \emptyset$ because $r^{-1}(2^\omega_f) = K_f \subset G$. By Baire Theorem, there is a $t \in 2^\omega \setminus (2^\omega_f \cup r(K \setminus G))$. Then the map $g : Q \to M$ defined by $g(q) = \overline{e} \circ H^{-1}(t, q)$ is the required embedding of Q into M with $g^{-1}(X) = C$. ■

In the light of Theorems 2.1 and 2.2 it is natural to ask if the Polish space M can be replaced by a space more complex in Borel respect. Let us remark that this is in general not possible since for the class $\mathcal{C} = \mathcal{A}_1$ the space Σ is $\mathcal{A}_1$-universal but the pair (Σ, Σ) is not $(\mathcal{M}_0, \mathcal{A}_1)$-universal. Nevertheless, for more complex classes $\mathcal{C}$ it turns out to be possible to replace the space $M \in \mathcal{M}_1$ in Theorems 2.1 and 2.2 by spaces from the class σ-$\mathcal{M}_1$ or even the class $\mathcal{P}_{2n}$.

We define a space X to be *everywhere C-universal*, where C is a space, if for every non-empty open set $U \subset X$ there exists a closed embedding $e : C \to X$ with $e(C) \subset U$. Recall that a space X is a Baire space provided it contains no open sets of the first Baire category.

2.3. Theorem. *Let $\mathcal{C}$ be a 2^ω-stable weakly $\mathcal{A}_1$-additive class such that for every $C \in \mathcal{C}$ there exists an everywhere C-universal Baire space $\widetilde{C} \in \mathcal{C}$. Then for every embedding $X \subset M$ of a $\mathcal{C}$-universal space X into a space $M \in \sigma$-$\mathcal{M}_1$, the pair (M, X) is $(\mathcal{M}_0 \cap \mathcal{C}, \mathcal{C})$-universal.*

Proof. Fix a pair $(K, C) \in (\mathcal{M}_0 \cap \mathcal{C}, \mathcal{C})$. Let $A \in \mathcal{A}_1 \setminus \mathcal{M}_1$ be any dense subspace in the Cantor set 2^ω, and consider the subset $Y = K \times A \cup C \times 2^\omega \subset K \times 2^\omega$. By our assumptions, there exists an everywhere Y-universal Baire space $\widetilde{Y} \in \mathcal{C}$. Since the space X is $\mathcal{C}$-universal, there is a closed embedding $\widetilde{Y} \subset X$. Write $M = \bigcup_{n \in \mathbb{N}} M_n$, where each M_n is a closed complete-metrizable subset in M. Since $\widetilde{Y}$ is a Baire space, there is an open set $U \subset \widetilde{Y}$ such that $U \subset M_n$ for some $n \in \mathbb{N}$. Let $Y \subset \widetilde{Y}$ be a closed embedding with $Y \subset U$. Proceeding as in the proof of Theorem 2.1, we obtain that the pair $(M_n, M_n \cap X)$ is (K, C)-universal. This implies (K, C)-universality of (M, X). ■

Similar arguments yield

2.4. Theorem. *Let $\mathcal{C}$ be an ω^ω-stable $\mathcal{A}_1(s.c.d.)$-additive topological class such that for every $C \in \mathcal{C}$, there exists an everywhere C-universal Baire space $\widetilde{C} \in \mathcal{C}$. Then for every embedding $X \subset M$ of a $\mathcal{C}$-universal space X into a space $M \in \sigma$-$\mathcal{M}_1$, the pair (M, X) is $(\mathcal{M}_0, \mathcal{C})$-universal.*

Let us remark that for $\alpha \geq 2$ the Borel classes $\mathcal{M}_\alpha$, $\mathcal{A}_\alpha$ as well as the projective classes $\mathcal{P}_n$, $n \geq 0$, satisfy the conditions of Theorems 2.3 and 2.4. This results from the following

2.5. Proposition. *If $\mathcal{C}$ is an $\mathcal{M}_1[0]$-additive class with $\mathcal{C} = \sigma\mathcal{C}$, then for every space $C \in \mathcal{C}$ there exists an everywhere C-universal Baire space $\widetilde{C} \in \mathcal{C}$.*

Proof. Fix any $C \in \mathcal{C}$. It is easy to construct a closed embedding of C into a space X such that C is nowhere dense in X and the complement $N = X \setminus C$ is homeomorphic to the discrete space ω. Fix any point $* \in N$ and let $W(N, *) = \{(y_i)_{i \in \omega} \in N^\omega : y_i = *$ for all but finitely many $i\} \subset X^\omega$. Evidently, the set $W(N, *)$ is countable. For every

$y = (y_i)_{i\in\omega} \in W(N,*)$ let $|y| = \min\{n \in \omega : y_i = * \text{ for all } i \ge n\}$ and

$$C(y) = \{(x_i)_{i\in\omega} \in X \mid x_i = y_i \text{ for } i \ne |y| \text{ and } x_{|y|} \in C\}.$$

Evidently, $C(y)$ is a closed subset of X^ω, homeomorphic to C and thus $C(y) \in \mathcal{C}$. Finally, consider the subspace

$$\widetilde{C} = N^\omega \cup \bigcup_{y\in W(N,*)} C(y).$$

The space $\widetilde{C}$ is Baire since it contains the Polish space N^ω as a dense subset. Since each $C(y) \in \mathcal{C}$ is closed in $\widetilde{C}$, we get $\bigcup_{y\in W(N,*)} C(y) \in \sigma\mathcal{C} = \mathcal{C}$. Because $N^\omega \in \mathcal{M}_1[0]$ and the class $\mathcal{C}$ is $\mathcal{M}_1[0]$-additive, we get $\widetilde{C} \in \mathcal{C}$. It is easy to see that the space $\widetilde{C}$ is everywhere C-universal. ■

Letting $A \subset 2^\omega$ be a dense subspace of the class $\mathcal{P}_{2n-1} \setminus \mathcal{P}_{2n}$ (such a set A exists, according to [Ke, 37.7]), and repeating the arguments of the proof of Theorem 2.1, one can prove

2.6. THEOREM. *Let $n \in \mathbb{N}$, let $\mathcal{C}$ be a 2^ω-stable weakly $\mathcal{P}_{2n-1}$-additive class of spaces, and X be a $\mathcal{C}$-universal space. Then for every space $M \in \mathcal{P}_{2n}$ containing X, the pair (M,X) is $(\mathcal{M}_0 \cap \mathcal{C}, \mathcal{C})$-universal.*

Now we derive some corollaries from the theorems proved above.

2.7. COROLLARY. *Suppose that a topological class $\mathcal{C}$ of spaces either is 2^ω-stable weakly $\mathcal{A}_1$-additive and compactification-admitting or is ω^ω-stable and $\mathcal{A}_1(s.c.d.)$-additive. A space X is $\mathcal{C}$-universal if and only if X contains a $\mathcal{C}$-universal G_δ-subset.*

PROOF. The "only if" part is trivial. Assume that G is a $\mathcal{C}$-universal G_δ-subspace of X. Let M be any completion of X and $\widetilde{G} \subset M$ be a G_δ-set such that $\widetilde{G} \cap X = G$.

Now consider two cases.

1) The class $\mathcal{C}$ is 2^ω-stable weakly $\mathcal{A}_1$-additive and compactification-admitting. Then by Theorem 2.1, the pair $(\widetilde{G}, G)$ is $(\mathcal{M}_0 \cap \mathcal{C}, \mathcal{C})$-universal. Since $\widetilde{G} \cap X = G$, this yields that the pair (M,X) is $(\mathcal{M}_0 \cap \mathcal{C}, \mathcal{C})$-universal. Since the class $\mathcal{C}$ admits compactifications, we see that X is $\mathcal{C}$-universal.

2) The class $\mathcal{C}$ is ω^ω-stable and $\mathcal{A}_1$(s.c.d.)-additive. Then Theorem 2.2 implies that the pair $(\widetilde{G}, G)$ is $(\mathcal{M}_0, \mathcal{C})$-universal. Since $\widetilde{G} \cap X = G$, this yields that the pair (M,X) is $(\mathcal{M}_0, \mathcal{C})$-universal and the space X is $\mathcal{C}$-universal. ■

Repeating the above arguments and applying Theorems 2.3, 2.4, 2.6, one can prove

2.8. COROLLARY. *Suppose that a topological class $\mathcal{C}$ of spaces is 2^ω-stable weakly $\mathcal{A}_1$-additive and compactification-admitting or $\mathcal{C}$ is ω^ω-stable and $\mathcal{A}_1(s.c.d.)$-additive. Suppose that for every space $C \in \mathcal{C}$ there exists an everywhere C-universal Baire space $\widetilde{C} \in \mathcal{C}$. Then a space X is $\mathcal{C}$-universal if and only if X contains a $\mathcal{C}$-universal subset $G \in \sigma$-$\mathcal{M}_1(X)$.*

2.9. COROLLARY. *Let $n \in \mathbb{N}$, and let $\mathcal{C}$ be a 2^ω-stable weakly $\mathcal{P}_{2n-1}$-additive compactification-admitting class of spaces. A space X is $\mathcal{C}$-universal if and only if X contains a $\mathcal{C}$-universal subspace $Y \in \mathcal{P}_{2n}(X)$.*

2.10. Theorem. *Let X be a $\mathcal{C}$-universal space, where $\mathcal{C}$ is an ω^ω-stable class of spaces. Then for every σ-compact set $A \subset X$ and every $A' \subset A$ the space $X \setminus A'$ is $\mathcal{C}$-universal.*

Proof. Fix a space $X \in \mathcal{C}$ and a σ-compact set $A \subset X$. Since the class $\mathcal{C}$ is ω^ω-stable, $C \times \omega^\omega \in \mathcal{C}$. Let $e : C \times \omega^\omega \to X$ be a closed embedding. Then the set $e^{-1}(A) \subset C \times \omega^\omega$ is σ-compact, and its projection $\mathrm{pr}(e^{-1}(A))$ onto ω^ω is σ-compact. Since $\omega^\omega \notin \mathcal{A}_1$, there is $t \in \omega^\omega \setminus \mathrm{pr}(e^{-1}(A))$. Define $i : C \to X$ by $i(c) = e(c, t)$, $c \in C$. Obviously, the map i is a closed embedding with $i(C) \cap A = \emptyset$. ∎

2.11. Theorem. *Let $\mathcal{C}$ be a 2^ω-stable $\mathcal{M}_1$-hereditary topological class of spaces and X be a $\mathcal{C}$-universal space. Suppose that either $\mathcal{C}$ is $\mathcal{A}_1$(s.c.d.)-additive or $\mathcal{C}$ is weakly $\mathcal{A}_1$-additive and compactification-admitting. Then, for every F_σ-set $A \subset X$ belonging to the class σ-$\mathcal{M}_1$, and every $A' \subset A$, the space $X \setminus A'$ is $\mathcal{C}$-universal.*

Proof. Let X be a $\mathcal{C}$-universal space, $A \in \sigma$-$\mathcal{M}_1$ an F_σ-set in X, and $A' \subset A$. We claim that there is a completion M of X such that A is an F_σ-set in M. Let $\overline{X}$ be any completion of X. Since $A \in \sigma$-$\mathcal{M}_1$ is an F_σ-set in X, it can be written as $A = \bigcup_{n=1}^\infty A_n$, where each $A_n \in \mathcal{M}_1$ is closed in X. For every $n \in \mathbb{N}$ let $\overline{A}_n$ be the closure of A_n in $\overline{X}$. Since A_n is closed in X, $\overline{A}_n \setminus A_n \subset \overline{X} \setminus X$. Moreover, $\overline{A}_n \setminus A_n$ is an F_σ-set in $\overline{X}$, because A_n is a G_δ-set in $\overline{A}_n$. Then $M = \overline{X} \setminus \bigcup_{n=1}^\infty (\overline{A}_n \setminus A_n)$ is a G_δ-set in $\overline{X}$ containing X, and A is an F_σ-set in M.

If $\mathcal{C}$ is $\mathcal{A}_1$(s.c.d.)-additive then by Theorem 2.2, the pair (M, X) is $(\mathcal{M}_0, \mathcal{C})$-universal. Fix any space $C \in \mathcal{C}$. We have to find a closed embedding $C \to X \setminus A'$. Choose any compactum $K \supset C$ and let $P \in \mathcal{M}_1 \setminus \mathcal{A}_1$ be any subset in 2^ω. Since the class $\mathcal{C}$ is 2^ω-stable and $\mathcal{M}_1$-hereditary, $(K \times 2^\omega, C \times P) \in (\mathcal{M}_0, \mathcal{C})$. Then, by $(\mathcal{M}_0, \mathcal{C})$-universality of (M, X), there exists an embedding $e : K \times 2^\omega \to M$ such that $e^{-1}(X) = C \times P$. Since $A \subset X$ is an F_σ-set in M, its preimage $e^{-1}(A) \subset e^{-1}(X) = C \times P$ is σ-compact, and the projection $\mathrm{pr}(e^{-1}(A))$ of A onto 2^ω is a σ-compactum in P. Since $P \notin \mathcal{A}_1$, there is $t \in P \setminus \mathrm{pr}(e^{-1}(A))$. It is easily verified that the map $i : C \to X$ defined by $i(c) = e(c, t)$, $c \in C$, is a closed embedding with $i(C) \cap A' = \emptyset$.

In the case of a weakly $\mathcal{A}_1$-additive compactification-admitting class $\mathcal{C}$ we may apply Theorem 2.1 to see that the pair (M, X) is $(\mathcal{M}_0 \cap \mathcal{C}, \mathcal{C})$-universal. Since the class $\mathcal{C}$ admits compactifications we may find a compactum $K \in \mathcal{C}$ with $K \supset C$. Continuing as in the preceding case, we will produce the required embedding $C \hookrightarrow X$. ∎

Replacing the space $\omega^\omega \in \mathcal{M}_1 \setminus \mathcal{A}_1$ in the proof of 2.10 by any 0-dimensional space $P \in \mathcal{P}_{2n} \setminus \mathcal{P}_{2n-1}$ (such a space exists according to [Ke, 37.7]), one can prove

2.12. Theorem. *Let $n \in \mathbb{N}$, $\mathcal{C}$ be a 2^ω-stable $\mathcal{P}_{2n}$-hereditary class of spaces, and let X be a $\mathcal{C}$-universal space. Then for every subset $A \in \mathcal{P}_{2n-1}$ in X and every $A' \subset A$ the space $X \setminus A'$ is $\mathcal{C}$-universal.*

2.13. Remark. It follows from [LSR] (see also [Ke, 28.19]) that for every countable ordinal $\alpha \geq 2$ and Borel spaces $X \subset M$, if $X \notin \mathcal{A}_\alpha(M)$ then the pair (M, X) is $(\mathcal{M}_0[0], \mathcal{M}_\alpha)$-universal. This implies that for every embedding of a Borel $\mathcal{M}_\alpha[0]$-universal space X into a space $M \in \mathcal{A}_\alpha$, the pair (M, X) is $(\mathcal{M}_0[0], \mathcal{M}_\alpha)$-universal.

In the light of this result it is natural to ask

2.14. QUESTION. Assume that a space X contains an $\mathcal{A}_\alpha$-universal subspace $Y \in \mathcal{M}_\alpha(X)$. Is X $\mathcal{A}_\alpha$-universal? The same question with $\mathcal{A}_\alpha$ and $\mathcal{M}_\alpha$ interchanged.

3. Strong universality for pairs implies strong universality for spaces

3.1. THEOREM. *Let M be an ANR and X be a homotopy dense subspace in M such that X has SDAP. Then for any pair (K,C) the strong (K,C)-universality of the pair (M,X) implies the strong C-universality of the space X.*

PROOF. Suppose that (M,X) is a strongly (K,C)-universal pair. SDAP of X implies that each Z-set in X is a strong Z-set, see [BM, 1.7]. Repeating the arguments of the proof of Proposition 2.2 of [BM], we see that to prove the strong C-universality of X, it suffices for given open subsets $\mathring{C} \subset C$, $\mathring{X} \subset X$, cover $\mathcal{U} \in \operatorname{cov}(\mathring{X})$, and map $f : \mathring{C} \to \mathring{X}$ to find a Z-embedding $\bar{f} : \mathring{C} \to \mathring{X}$ which is $\mathcal{U}$-close to f.

Since X has SDAP, so does its open subspace $\mathring{X}$. Let $\mathcal{V} \in \operatorname{cov}(\mathring{X})$ be a cover with $\operatorname{St}^2 \mathcal{V} = \operatorname{St}(\operatorname{St} \mathcal{V}) \prec \mathcal{U}$. Using Exercise 1 of [Wa], we may find a map $f' : \mathring{K} \to \mathring{X}$ of an open neighborhood $\mathring{K}$ of $\mathring{C}$ in K such that $(f'|\mathring{C}, f) \prec \mathcal{V}$. Write $\mathring{K} = \bigcup_{n=1}^\infty K_n$, where each $K_n \subset \operatorname{int} K_{n+1} \subset \mathring{K}$ is a closed subset in K, and $K_0 = \emptyset$. By Proposition 1.1, there is a map $f_0 : \mathring{K} \to \mathring{X}$ such that $(f_0, f') \prec \mathcal{V}$ and the collection $\{f_0(K_{n+1} \setminus \operatorname{int} K_{n-1})\}_{n\in\mathbb{N}}$ is locally finite in $\mathring{X}$. Using Lemma 1.2, find a cover $\mathcal{W} \in \operatorname{cov}(\mathring{X})$ such that $\operatorname{St} \mathcal{W} \prec \mathcal{V}$ and the collection $\{\operatorname{St}(f_0(K_{n+1} \setminus \operatorname{int} K_{n-1}), \operatorname{St} \mathcal{W})\}_{n\in\mathbb{N}}$ is locally finite in $\mathring{X}$.

For every $W \in \mathcal{W}$ find an open set $\widetilde{W}$ in M such that $\widetilde{W} \cap X = W$ and consider the open set $\mathring{M} = \bigcup_{W\in\mathcal{W}} \widetilde{W}$ in M and the cover $\widetilde{\mathcal{W}} = \{\widetilde{W} \mid W \in \mathcal{W}\}$ of $\mathring{M}$. Notice that $\mathring{M} \cap X = \mathring{X}$ and $\mathring{X}$ is homotopy dense in $\mathring{M}$.

Using Lemma 1.3, construct inductively a sequence of maps $f_n : \mathring{K} \to \mathring{M}$, $n \in \mathbb{N}$, satisfying the following conditions:

$$f_n|K_{n-1} \cup (\mathring{K} \setminus \operatorname{int} K_{n+1}) = f_{n-1}|K_{n-1} \cup (\mathring{K} \setminus \operatorname{int} K_{n+1}), \quad (f_n, f_{n-1}) \prec \widetilde{\mathcal{W}},$$
$$f_n|K_n : K_n \to \mathring{M} \text{ is a } Z\text{-embedding with } (f_n|K_n)^{-1}(X) = K_n \cap \mathring{C}.$$

Let finally $\tilde{f} = \lim_{n\to\infty} f_n : \mathring{K} \to \mathring{M}$ and $\bar{f} = \tilde{f}|\mathring{C} : \mathring{C} \to \mathring{X}$. We claim that the map $\bar{f}$ is a Z-embedding with $(\bar{f}, f) \prec \mathcal{U}$. Indeed, noting that $(\tilde{f}, f_0) \prec \operatorname{St} \widetilde{\mathcal{W}}$, we obtain, for any $n \in \mathbb{N}$, $\bar{f}(\mathring{C} \cap (K_{n+1} \setminus \operatorname{int} K_{n-1})) \subset \operatorname{St}(f_0(K_{n+1} \setminus \operatorname{int} K_{n-1}), \operatorname{St} \mathcal{W})$. Hence, the collection $\{\bar{f}(\mathring{C} \cap (K_{n+1} \setminus \operatorname{int} K_{n-1}))\}_{n\in\mathbb{N}}$ is locally finite in $\mathring{X}$. By construction, for every $n \in \mathbb{N}$, $\tilde{f}|K_n = f_n|K_n$ is a Z-embedding. Since $\mathring{X} \subset \mathring{M}$ is homotopy dense, $\bar{f}(K_n \cap \mathring{C}) = \tilde{f}(K_n) \cap \mathring{X}$ is a Z-set in $\mathring{X}$. Consequently, for every $n \in \mathbb{N}$, the restriction $\bar{f}|(K_{n+1} \setminus \operatorname{int} K_{n-1}) \cap \mathring{C}$ is a Z-embedding. Then $\bar{f} : \mathring{C} \to \mathring{X}$ is a closed embedding and $\bar{f}(\mathring{C})$, being a local Z-set in $\mathring{X}$, is a Z-set in $\mathring{X}$. Thus the map $\bar{f}$ is a Z-embedding.

The second condition, namely $(\bar{f}, f) \prec \mathcal{U}$, easily follows from $(f, f_0) \prec \operatorname{St} \mathcal{V}$, $(\bar{f}, f_0) \prec \operatorname{St} \mathcal{W}$, $\operatorname{St} \mathcal{W} \prec \operatorname{St} \mathcal{V}$, and $\operatorname{St}^2 \mathcal{V} \prec \mathcal{U}$. ∎

4. Strong universality for spaces implies strong universality for pairs

In this section we reverse Theorem 3.1 by proving "strongly universal" counterparts of the results of §2.

4.1. Theorem. *Let M be a Polish ANR and X a homotopy dense subset in M. If the space X is strongly universal for a 2^ω-stable weakly $\mathcal{A}_1$-additive class $\mathcal{C}$ then the pair (M,X) is strongly $(\mathcal{M}_0\cap\mathcal{C},\mathcal{C})$-universal.*

Proof. To prove the strong $(\mathcal{M}_0\cap\mathcal{C},\mathcal{C})$-universality of (M,X) fix a pair $(K,C)\in(\mathcal{M}_0\cap\mathcal{C},\mathcal{C})$, a closed subset $B\subset K$, a cover $\mathcal{U}\in\operatorname{cov}(M)$ and a map $f:K\to M$ whose restriction $f|B:B\to M$ is a Z-embedding with $(f|B)^{-1}(X)=B\cap C$.

Since $f(B)$ is a Z-set in M and X is homotopy dense in M, replacing if necessary, f by a near map,we can assume without loss of generality that $f(K\setminus B)\subset X\setminus f(B)$. Fix any metric d on M and let $\mathcal{V}\in\operatorname{cov}(M\setminus f(B))$ be such that $\mathcal{V}\prec\mathcal{U}$ and $\mathcal{V}\prec\{O(x,d(x,f(B))/2)\mid x\in M\setminus f(B)\}$. Let $A\in\mathcal{A}_1\setminus\mathcal{M}_1$ be any dense set in 2^ω and consider the subspaces $C'=C\times 2^\omega\cup(K\setminus B)\times A$ and $C''=(C\setminus B)\times 2^\omega\cup(K\setminus B)\times A$ in $K\times 2^\omega$. Since the class $\mathcal{C}$ is 2^ω-stable and weakly $\mathcal{A}_1$-additive, $C'\in\mathcal{C}$. Denote by $\operatorname{pr}_K:K\times 2^\omega\to K$ the natural projection and consider the map $f'=f\circ\operatorname{pr}_K|C':C'\to X$. Notice that $(f')^{-1}(X\setminus f(B))=C''$. By Proposition 1.7, there is a Z-embedding $g:C''\to X\setminus f(B)$ such that

$$(g,f'|C'')\prec\mathcal{V}. \tag{1}$$

Since $g(C'')$ is a Z-set in $X\setminus f(B)$, $\operatorname{Cl}_M(g(C''))$ is a Z-set in M. By Lavrentiev's Theorem, the embedding g extends to an embedding $\overline{g}:G\to M\setminus f(B)$ of some G_δ-set $G\subset(K\setminus B)\times 2^\omega$ densely containing C''. By [En, 3.7.16],

$$\overline{g}^{-1}(X\setminus f(B))=C''. \tag{2}$$

Moreover, because of (1), without loss of generality, we can suppose that

$$(\overline{g},f\circ\operatorname{pr}_K|G)\prec\mathcal{V}\prec\mathcal{U}. \tag{3}$$

Note that the complement $((K\setminus B)\times 2^\omega)\setminus G$ is σ-compact and its projection $P=\operatorname{pr}((K\setminus B)\times 2^\omega\setminus G)$ onto 2^ω is a σ-compactum in $2^\omega\setminus A\notin\mathcal{A}_1$. Then there is a point $t\in 2^\omega\setminus(A\cup P)$. Notice that $(K\setminus B)\times\{t\}\subset G$ and define the map $\widetilde{f}:K\setminus B\to M\setminus f(B)$ letting $\widetilde{f}(k)=\overline{g}(k,t)$ for $k\in K\setminus B$. By (2) and (3) we have $\widetilde{f}^{-1}(X\setminus f(B))=C\setminus B$ and $(\widetilde{f},f|K\setminus B)\prec\mathcal{U}$. Letting finally $\bar{f}:K\to M$ be defined by $\bar{f}|B=f|B$ and $\bar{f}|K\setminus B=\widetilde{f}|K\setminus B$, we obtain a closed embedding $\bar{f}$ such that $\bar{f}^{-1}(X)=C$ and $(\bar{f},f)\prec\mathcal{U}$.

To see that $\bar{f}(K)$ is a Z-set in M, notice that $\bar{f}(K)\subset f(B)\cup\operatorname{Cl}_M(g(C''))$ lies in the union of two Z-sets in M. ∎

4.2. Theorem. *Let M be a Polish ANR and X be a homotopy dense subset in M. If the space X is strongly $\mathcal{C}$-universal for a 2^ω-stable $\mathcal{M}_1$-hereditary $\mathcal{A}_1$(s.c.d.)-additive topological class $\mathcal{C}$, then the pair (M,X) is strongly $(\mathcal{M}_0,\mathcal{C})$-universal.*

Proof. To prove the strong $(\mathcal{M}_0,\mathcal{C})$-universality of the pair (M,X) we have to verify its strong (K,C)-universality for each pair $(K,C)\in(\mathcal{M}_0,\mathcal{C})$. So, fix any pair $(K,C)\in$

$(\mathcal{M}_0, \mathcal{C})$. We can assume the compactum K to be a subset of the Hilbert cube Q. According to Lemma 1.3, the strong (K, C)-universality of the pair (M, X) will follow as soon as we show that the pair (M, X) is strongly (Q, C)-universal. To verify this, fix a closed subset $B \subset Q$, $\mathcal{U} \in \mathrm{cov}(M)$, and $f : Q \to M$ such that $f|B : B \to M$ is a Z-embedding with $(f|B)^{-1}(X) = B \cap C$.

As in 4.1 we may assume that $f(Q \setminus B) \subset X \setminus f(B)$. Fix any metric d on M and let $\mathcal{V} \in \mathrm{cov}(M \setminus f(B))$ be such that $\mathcal{V} \prec \mathcal{U}$ and $\mathcal{V} \prec \{O(x, d(x, f(B))/2) \mid x \in M \setminus f(B)\}$. To prove the strong (Q, C)-universality of (M, X), it suffices to construct an injective continuous map $\widetilde{f} : Q \setminus B \to M \setminus f(B)$ such that $(\widetilde{f}, f) \prec \mathcal{V}$, $\widetilde{f}^{-1}(X) = C \setminus B$ and $\widetilde{f}(Q \setminus B)$ is a Z-set in $M \setminus f(B)$. Let $\mathcal{V}' \in \mathrm{cov}(M \setminus f(B))$ be a cover with $\mathrm{St}\,\mathcal{V}' \prec \mathcal{V}$.

Next, we follow the notations from the proof of 2.2 where the homeomorphism $H = (r, h) : K \setminus K_f \to (2^\omega \setminus 2^\omega_f) \times Q$ was constructed. Denote by $\mathrm{pr}_Q : (2^\omega \setminus 2^\omega_f) \times Q \to Q$ the natural projection and consider the map $f \circ \mathrm{pr}_Q \circ H : K \setminus K_f \to X \cup f(B)$. Let $Y = H^{-1}((2^\omega \setminus 2^\omega_f) \times (Q \setminus B))$. Evidently, Y is open in $K \setminus K_f$ and $f \circ \mathrm{pr}_Q \circ H(Y) \subset X \setminus f(B)$. By Ex.1 of [Wa], one may find a map $\alpha : U \to X \setminus f(B)$ of an open neighborhood U of Y in K such that $(\alpha|Y, f \circ \mathrm{pr}_Q \circ H|Y) \prec \mathcal{V}$. Now set $C' = (U \cap K_f) \cup H^{-1}((2^\omega \setminus 2^\omega_f) \times (C \setminus B))$. Since the class $\mathcal{C}$ is 2^ω-stable and $\mathcal{M}_1$-hereditary, we see that $(2^\omega \setminus 2^\omega_f) \times (C \setminus B)$, being a G_δ-set in $2^\omega \times C$, belongs to the class $\mathcal{C}$. Next, $U \cap K_f$ is σ-compact and strongly countable-dimensional. Then by $\mathcal{A}_1$(s.c.d.)-additivity of $\mathcal{C}$, we get $C' \in \mathcal{C}$. Since the space $X \setminus f(B)$ is strongly $\mathcal{C}$-universal, see [BM, 2.1], there is a Z-embedding $g : C' \to X \setminus f(B)$ such that $(g, \alpha|C') \prec \mathcal{V}$. Then the closure $\mathrm{Cl}(g(C'))$ of $g(C')$ in $M \setminus f(B)$ is a Z-set in $M \setminus f(B)$ (here we use the homotopy density of $X \setminus f(B)$ in $M \setminus f(B)$). By Lavrentiev's Theorem, g extends to an embedding $\overline{g} : G \to M \setminus f(B)$ of some G_δ-set $G \subset U$ containing C' as a dense subset. Then by [En, 3.7.16], $\overline{g}^{-1}(X \setminus f(B)) = C'$. Moreover, since $(g, \alpha|C') \prec \mathcal{V}$, we may suppose that $(\overline{g}, \alpha|G) \prec \mathcal{V}$. Note that the complement $U \setminus G$ is σ-compact. Then its image $r(U \setminus G)$ under the retraction $r : K \to \{-1, 1\}^\omega$ is σ-compact too. Next, since $(K_f \cap U) \subset G$, we get $U \cap r^{-1}(2^\omega_f) \subset G$ and thus $r(U \setminus G) \cap 2^\omega_f = \emptyset$. By Baire's Theorem, there is a $t_0 \in 2^\omega \setminus (2^\omega_f \cup r(U \setminus G))$. Since $H^{-1}(\{t_0\} \times (Q \setminus B)) \subset Y \subset U$, we get $H^{-1}(\{t_0\} \times (Q \setminus B)) \subset G$. This allows us to consider the map $\widetilde{f} : Q \setminus B \to M \setminus f(B)$ defined for any $q \in Q \setminus B$ by $\widetilde{f}(q) = \overline{g} \circ H^{-1}(t_0, q)$. Analogously to 4.1, it can be proved that $(\widetilde{f}, f) \prec \mathrm{St}(\mathcal{V}') \prec \mathcal{V}$ and $\widetilde{f}^{-1}(X \setminus f(B)) = C \setminus B$. Since $\widetilde{f}(Q \setminus B) \subset \overline{g}(G) \cup \mathrm{Cl}(g(G'))$, we conclude that $\widetilde{f}(Q \setminus B)$ is a Z-set in $M \setminus f(B)$. ■

Replacing in the proof of Theorem 4.1 the set $A \in \mathcal{A}_1 \setminus \mathcal{M}_1$ by any dense subset $A \subset 2^\omega$ of the class $\mathcal{P}_{2n-1} \setminus \mathcal{P}_{2n}$, one can prove

4.3. Theorem. *Let $n \in \mathbb{N}$ and let $\mathcal{C}$ be a 2^ω-stable weakly $\mathcal{P}_{2n-1}$-additive class of spaces. For every absolute neighborhood retract $M \in \mathcal{P}_{2n}$ and every homotopy dense strongly $\mathcal{C}$-universal subspace X of M, the pair (M, X) is strongly $(\mathcal{M}_0 \cap \mathcal{C}, \mathcal{C})$-universal.*

Theorems 4.1, 4.2, 4.3 and 3.1 immediately imply

4.4. Theorem. *Let $\mathcal{C}$ be a 2^ω-stable weakly $\mathcal{A}_1$-additive compactification-admitting class of spaces, M a Polish ANR and $X \subset M$ a homotopy dense subspace satisfying SDAP. The space X is strongly $\mathcal{C}$-universal if and only if the pair (M, X) is strongly $(\mathcal{M}_0 \cap \mathcal{C}, \mathcal{C})$-universal.*

4.5. THEOREM. *Let $\mathcal{C}$ be a 2^ω-stable $\mathcal{M}_1$-hereditary $\mathcal{A}_1$(s.c.d.)-additive class of spaces, M a Polish ANR and $X \subset M$ a homotopy dense subspace satisfying SDAP. The space X is strongly $\mathcal{C}$-universal if and only if the pair (M, X) is strongly $(\mathcal{M}_0, \mathcal{C})$-universal.*

4.6. THEOREM. *Let $n \in \mathbb{N}$ and $\mathcal{C}$ be a 2^ω-stable weakly $\mathcal{P}_{2n-1}$-additive compactification-admitting class of spaces, $M \in \mathcal{P}_{2n}$ an ANR and $X \subset M$ a homotopy dense subspace satisfying SDAP. The space X is strongly $\mathcal{C}$-universal if and only if the pair (M, X) is strongly $(\mathcal{M}_0 \cap \mathcal{C}, \mathcal{C})$-universal.*

5. Enlarging, Deleting, and Strong Negligibility Theorems for strongly universal spaces

In this section we apply the results of §§3,4 to generalize certain well known theorems about Σ- and s-manifolds onto strongly universal and absorbing spaces.

A. Enlarging Theorems

5.1. THEOREM. *Let $\mathcal{C}$ be a 2^ω-stable weakly $\mathcal{A}_1$-additive compactification-admitting class of spaces. An ANR X satisfying SDAP is strongly $\mathcal{C}$-universal if and only if it contains a strongly $\mathcal{C}$-universal homotopy dense G_δ-subspace $G \subset X$.*

5.2. THEOREM. *Let $\mathcal{C}$ be a 2^ω-stable $\mathcal{M}_1$-hereditary $\mathcal{A}_1$(s.c.d.)-additive topological class of spaces. An ANR X satisfying SDAP is strongly $\mathcal{C}$-universal if and only if X contains a strongly $\mathcal{C}$-universal homotopy dense G_δ-subspace $G \subset X$.*

5.3. THEOREM. *Let $n \in \mathbb{N}$ and let $\mathcal{C}$ be a 2^ω-stable weakly $\mathcal{P}_{2n-1}$-additive compactification-admitting class of spaces. An ANR X satisfying SDAP is strongly $\mathcal{C}$-universal if and only if it contains a strongly $\mathcal{C}$-universal homotopy dense subspace $G \in \mathcal{P}_{2n}(X)$.*

PROOF. Because of similarity, we will prove only Theorem 5.3. The "only if" part is trivial. Assume that $G \in \mathcal{P}_{2n}(X)$ is a strongly $\mathcal{C}$-universal homotopy dense subspace in X. Let cX be a compactification of X and $P \in \mathcal{P}_{2n}$ be a subspace in cX such that $G = P \cap X$. According to [To1], there exists a G_δ-set $\widetilde{X} \subset cX$ such that $\widetilde{X} \in \mathrm{ANR}$ and X is homotopy dense in $\widetilde{X}$. Since $\widetilde{G} = P \cap \widetilde{X} \in \mathcal{P}_{2n}$, and G is homotopy dense in $\widetilde{G} \subset \widetilde{X}$, by Theorem 4.6, the pair $(\widetilde{G}, G)$ is strongly $(\mathcal{M}_0 \cap \mathcal{C}, \mathcal{C})$-universal. Since $\widetilde{G}$ is homotopy dense in $\widetilde{X}$ and $\widetilde{G} \cap X = G$, by Lemma 1.4, the pair $(\widetilde{X}, X)$ is strongly $(\mathcal{M}_0 \cap \mathcal{C}, \mathcal{C})$-universal. Applying finally Theorem 3.1, we see that the space X is strongly $\mathcal{C}$-universal. ∎

B. Deleting Theorems. Let us prove firstly the following

5.4. LEMMA. *Let $\mathcal{C}$, $\mathcal{D}$ be two topological closed-hereditary classes of spaces. Suppose that there is a space C such that*

(1) *C is not a countable union of (closed) subspaces belonging to the class $\mathcal{D}$;*

(2) *$I^n \times C \in \mathcal{C}$ for every $n \in \mathbb{N}$;*

(3) *a subset $D \subset C$ belongs to the class $\mathcal{D}$, whenever there exists a (perfect) surjective map $f : D' \to D$, where $D' \in \mathcal{D}$.*

Then for a strongly $\mathcal{C}$-universal ANR X any (F_σ-)subset $F \in \sigma\mathcal{D}$ in X is homotopy negligible.

PROOF. Let $F \in \sigma\mathcal{D}$ be an (F_σ-)set in X. Since X is an ANR, to prove that F is homotopy negligible, it suffices to verify that F is locally homotopy negligible. Let $f : I^n \to X$ be a map with $f(\partial I^n) \cap F = \emptyset$, and $\mathcal{U} \in \mathrm{cov}(X)$ be a cover. Write $I^n \setminus \partial I^n = \bigcup_{k\in\mathbb{N}} A_k$, where $A_1 \subset \mathrm{int}\, A_2 \subset \ldots$ is a tower of compacta. Using the strong $C \times I^n$-universality of X, one can construct a map $f' : I^n \times C \to X$ such that $(f', f \circ \mathrm{pr}_{I^n}) \prec \mathcal{U}$, $f'|\partial I^n \times C = f \circ \mathrm{pr}_{I^n}|\partial I^n \times C$ and for every $k \in \mathbb{N}$ the restriction $f'|A_k \times C : A_k \times C \to X$ is a Z-embedding.

Then $F' = (f')^{-1}(F)$ is an (F_σ-)set in $(I^n \setminus \partial I^n) \times C$ and $F' \in \sigma\mathcal{D}$. Thus F' can be written as $F' = \bigcup_{i=1}^\infty F_i$, where each $F_i \in \mathcal{D}$ is (closed) in $I^n \times C$. Since the projection $\mathrm{pr}_C : I^n \times C \to C$ is a perfect map we see that each $\mathrm{pr}_C(F_i)$ is (closed) in C and by the condition (3) $\mathrm{pr}_C(F_i) \in \mathcal{D}$. By the condition (1), $C \neq \mathrm{pr}_C(F') = \bigcup_{i=1}^\infty \mathrm{pr}_C(F_i)$. Hence, there is a point $c_0 \in C \setminus \mathrm{pr}_C(F')$. It is easily seen that the map $\bar{f} : I^n \to X$ defined by $\bar{f}(t) = f'(t, c_0)$, $t \in I^n$, has the following properties: $\bar{f}|\partial I^n = f|\partial I^n$, $(\bar{f}, f) \prec \mathcal{U}$ and $\bar{f}(I^n) \cap F = \emptyset$. Thus F is homotopy negligible in X. ∎

It is known that if $\mathcal{D}$ is one of the classes $\mathcal{M}_\alpha$, $\mathcal{A}_\alpha$, $\alpha < \omega_1$, or $\mathcal{P}_n$, $n \in \mathbb{N}$, then any perfect image $f(D)$ of a space $D \in \mathcal{D}$ belongs to the class $\mathcal{D}$ [SR]. Hence, the above classes satisfy the "perfect" variant of the condition (3) of the previous lemma.

5.5. THEOREM. *Let X be a strongly $\mathcal{C}$-universal ANR satisfying SDAP, where $\mathcal{C} \supset \{I^n \mid n \in \mathbb{N}\}$ is an ω^ω-stable class of spaces. Then for every σ-compact set $A \subset X$ and every $A' \subset A$,*

(1) *A is homotopy negligible in X;*
(2) *$X \setminus A'$ is strongly $\mathcal{C}$-universal.*

PROOF. Fix a σ-compact set $A \subset X$ and $A' \subset A$. The statement (1) obviously follows from Lemma 5.4 (just let $C = \omega^\omega$ and $\mathcal{D} = \mathcal{M}_0$).

To prove that the space $X \setminus A'$ is strongly $\mathcal{C}$-universal, fix a cover $\mathcal{U} \in \mathrm{cov}(X \setminus A')$, a space $C \in \mathcal{C}$, a closed subset $B \subset C$, and a map $f : C \to X \setminus A'$ whose restriction $f|B : B \to X \setminus A'$ is a Z-embedding.

For every $U \in \mathcal{U}$ find an open set $\widetilde{U} \subset X$ with $\widetilde{U} \cap X \setminus A' = U$. Evidently, $W = \bigcup\{\widetilde{U} \mid U \in \mathcal{U}\}$ is an open set in X, containing $X \setminus A'$, and $\widetilde{\mathcal{U}} = \{\widetilde{U} \mid U \in \mathcal{U}\}$ is an open cover of W. Let $\mathcal{W} \in \mathrm{cov}(W)$ be such that $\mathrm{St}\,\mathcal{W} \prec \widetilde{\mathcal{U}}$. Since the set $X \setminus A'$ is homotopy dense in X and X has SDAP, every Z-set in $X \setminus A'$ is a strong Z-set. By [BM, 1.1], there is a map $f' : C \to X \setminus A'$ such that $(f', f) \prec \mathcal{W}$, $f'|B = f|B$, $f'(C \setminus B) \cap f(B) = \emptyset$ and f' is closed over $f(B)$. Let $\overline{f(B)}$ be the closure of $f(B)$ in W, and let $\mathcal{W}' \in \mathrm{cov}(W \setminus \overline{f(B)})$ be such that $\mathcal{W}' \prec \mathcal{W}$ and $\mathcal{W}' \prec \{O_d(x, d(x, \overline{f(B)})/2) \mid x \in W \setminus \overline{f(B)}\}$ (here d stands for a metric of X).

Denote by $\mathrm{pr}_C : C \times \omega^\omega \to C$ the natural projection and consider the map $g = f' \circ \mathrm{pr}_C : C \times \omega^\omega \to X \setminus A' \subset W$. By Lemma 1.3, the open subspace $W \subset X$ is strongly $C \times \omega^\omega$-universal. Then by Proposition 1.7, there is a Z-embedding $g' : (C \setminus B) \times \omega^\omega \to W \setminus \overline{f(B)}$ such that $(g', g|(C \setminus B) \times \omega^\omega) \prec \mathcal{W}'$. Notice that $(g')^{-1}(A) \subset (C \setminus B) \times \omega^\omega$ is σ-compact and so is its projection $P = \mathrm{pr}_{\omega^\omega}((g')^{-1}(A))$ onto ω^ω. Pick $t \in \omega^\omega \setminus P$ and define

$\bar{f}: C \to X \setminus A'$ by

$$\bar{f}(c) = \begin{cases} f(c) & \text{if } c \in B, \\ g'(c,t) & \text{if } c \in C \setminus B. \end{cases}$$

One can easily verify that $\bar{f}$ is a Z-embedding such that $\bar{f}|B = f|B$ and $(\bar{f}, f) \prec \mathcal{U}$. ■

Replacing the space ω^ω by any zero-dimensional space in $\mathcal{P}_{2n} \setminus \mathcal{P}_{2n-1}$ and repeating the above arguments, one can prove

5.6. THEOREM. *Let $n \in \mathbb{N}$ and let $\mathcal{C} \supset \{I^k \mid k \in \mathbb{N}\}$ be a 2^ω-stable $\mathcal{P}_{2n}$-hereditary class of spaces and X a strongly universal ANR satisfying SDAP. Then for every set $A \in \mathcal{P}_{2n-1}$ in X and every $A' \subset A$,*

(1) *A is homotopy negligible in X;*
(2) *$X \setminus A'$ is strongly $\mathcal{C}$-universal.*

5.7. THEOREM. *Let $\mathcal{C}$ be a 2^ω-stable $\mathcal{A}_1$(s.c.d.)-additive $\mathcal{M}_1$-hereditary topological class of spaces and X a strongly $\mathcal{C}$-universal ANR. For every F_σ-set $A \in \sigma$-$\mathcal{M}_1$ in X and every $A' \subset A$,*

(1) *A is homotopy negligible in X;*
(2) *$X \setminus A'$ is strongly $\mathcal{C}$-universal.*

PROOF. Let $A \in \sigma$-$\mathcal{M}_1$ be an F_σ-subset in X and $A' \subset A$. As in the proof of Theorem 2.11, embed X into a Polish ANR $\widetilde{X}$ so that X is homotopy dense in $\widetilde{X}$ and A is an F_σ-subset in $\widetilde{X}$. By Theorem 4.2, the pair $(\widetilde{X}, X)$ is strongly $(\mathcal{M}_0, \mathcal{C})$-universal.

We are going to show that the pair $(\widetilde{X}, X \setminus A')$ is strongly $(\mathcal{M}_0, \mathcal{C})$-universal. For this, fix a cover $\mathcal{U} \in \operatorname{cov}(\widetilde{X})$, a pair $(K, C) \in (\mathcal{M}_0, \mathcal{C})$, a closed subset $B \subset K$ and a map $f: K \to \widetilde{X}$ whose restriction $f|B : B \to \widetilde{X}$ is a Z-embedding with $(f|B)^{-1}(X \setminus A') = B \cap C$. Since $f(B)$ is a Z-set in $\widetilde{X}$, without loss of generality, $f(K \setminus B) \cap f(B) = \emptyset$. Let $M \in \mathcal{M}_1 \setminus \mathcal{A}_1$ be any dense subset in 2^ω. Since the class $\mathcal{C}$ is 2^ω-stable and $\mathcal{M}_1$-hereditary, $(K \times 2^\omega, C \times M) \in (\mathcal{M}_0, \mathcal{C})$. Denote by $\mathrm{pr}_K : K \times 2^\omega \to K$ the projection and consider the map $g = f \circ \mathrm{pr}_K : K \times 2^\omega \to \widetilde{X}$. Using the strong $(\mathcal{M}_0, \mathcal{C})$-universality of the pair $(\widetilde{X}, X)$ and repeating the arguments of the proof of Lemma 1.6, one can construct a map $g' : K \times 2^\omega \to \widetilde{X}$ such that $g'|B \times 2^\omega = f \circ \mathrm{pr}_K |B \times 2^\omega$, $(g', f \circ \mathrm{pr}_K) \prec \mathcal{U}$, $g'((K \setminus B) \times 2^\omega) \cap g'(B \times 2^\omega) = \emptyset$ and $g'|(K \setminus B) \times 2^\omega : (K \setminus B) \times 2^\omega \to \widetilde{X} \setminus f(B)$ is a Z-embedding with $(g'|(K \setminus B) \times 2^\omega)^{-1}(X) = (C \setminus B) \times M$.

Since $A \subset X$ is an F_σ-set in $\widetilde{X}$, the set $(g'|(K \setminus B) \times 2^\omega)^{-1}(A) \subset (C \setminus B) \times M$ is σ-compact. Then its projection P onto M is also σ-compact. Since $M \notin \mathcal{A}_1$, there is $t \in M \setminus P$. Define the map $\bar{f} : K \to \widetilde{X}$ by $\bar{f}(k) = g'(k, t)$, $k \in K$, and notice that $\bar{f}$ is a Z-embedding such that $\bar{f}|B = f|B$, $(\bar{f}, f) \prec \mathcal{U}$ and $\bar{f}^{-1}(X \setminus A') = C$. Hence the pair $(\widetilde{X}, X \setminus A')$ is strongly $(\mathcal{M}_0, \mathcal{C})$-universal.

Since the space X is strongly $\mathcal{A}_1$(s.c.d.)-universal, it satisfies SDAP. Next, since $\{I^n \mid n \in \mathbb{N}\} \subset \mathcal{A}_1(\text{s.c.d.}) \subset \mathcal{C}$, the strong $(\mathcal{M}_0, \mathcal{C})$-universality of $(\widetilde{X}, X \setminus A')$ implies that the set $X \setminus A'$ has homotopy negligible complement in $\widetilde{X}$ (see [BGM, 4.3]) and consequently in X. This implies that $X \setminus A'$ satisfies SDAP (recall that X has SDAP). Then, by Theorem 3.1, the space $X \setminus A'$ is strongly $\mathcal{C}$-universal. ■

C. Strong Negligibility Theorems. Following [BP, p. 132] we say that a subset $X \subset M$ is *strongly negligible* if for every open set $U \subset M$ and every cover $\mathcal{U} \in \operatorname{cov}(U)$ there is a homeomorphism $h : U \to U \setminus X$, $\mathcal{U}$-close to the identity.

5.8. THEOREM. *Suppose $\mathcal{C} \supset \{I^n \mid n \in \mathbb{N}\}$ is a 2^ω-stable $\mathcal{M}_1$-hereditary class of spaces and Ω is a $\mathcal{C}$-absorbing space. Then every σ-compact subset $A \subset \Omega$ is strongly negligible in Ω.*

PROOF. Suppose A is a σ-compact subset in Ω. Since every open subspace of Ω is $\mathcal{C}$-absorbing, to prove the theorem, it suffices to show that for every cover $\mathcal{U} \in \operatorname{cov}(\Omega)$ there is a homeomorphism $h : \Omega \to \Omega \setminus A$, $\mathcal{U}$-close to the identity.

Fix a cover $\mathcal{U} \in \operatorname{cov}(\Omega)$. According to [$\mathrm{Ba}_1$], Ω can be embedded into an s-manifold M as a homotopy dense subset. Replacing, if necessary, M by a suitable open neighborhood of Ω in M, we may assume that there is a cover $\widetilde{\mathcal{U}} \in \operatorname{cov}(M)$ such that $\mathcal{U} = \{U \cap \Omega \mid U \in \widetilde{\mathcal{U}}\}$. Using Theorem 5.5 and the $\mathcal{M}_1$-heredity of $\mathcal{C}$, one can prove that $\Omega \setminus A$ is a $\mathcal{C}$-absorbing homotopy dense subspace of M. Then Theorem 1.8 supplies us with a homeomorphism $h : \Omega \to \Omega \setminus A$, $\mathcal{U}$-close to $\mathrm{id}\,|\Omega$. ∎

The following two theorems can be proved analogously, using Theorems 5.6 and 5.7.

5.9. THEOREM. *Let $n \in \mathbb{N}$ and let Ω be a $\mathcal{C}$-absorbing space for a 2^ω-stable $\mathcal{P}_{2n}$-hereditary class $\mathcal{C} \supset \{I^n \mid n \in \mathbb{N}\}$. Then every subset $A \subset \Omega$ of class $\mathcal{P}_{2n-1}$ is strongly negligible in Ω.*

5.10. THEOREM. *Let $\mathcal{C}$ be a 2^ω-stable $\mathcal{A}_1(s.c.d.)$-additive $\mathcal{M}_1$-hereditary class of spaces and Ω a $\mathcal{C}$-absorbing space. Every F_σ-subset $A \in \sigma\text{-}\mathcal{M}_1$ in Ω is strongly negligible in Ω.*

In light of Theorem 5.9 the following question arises naturally:

5.11. QUESTION. Let Ω be an absorbing space for the Borel class $\mathcal{M}_\alpha$ (resp. $\mathcal{A}_\alpha$), $\alpha < \omega_1$. Is every subset $A \subset \Omega$ of class $\mathcal{A}_\alpha$ (resp. $\mathcal{M}_\alpha$) strongly negligible in Ω?

6. Existence of absorbing spaces

The main result of this section states that for a closed-hereditary $[0,1]$-stable class $\mathcal{C}$, a $\mathcal{C}$-absorbing AR exists if and only if the class $\sigma\mathcal{C}$ contains a $\mathcal{C}$-universal space. Till now this result was known under an additional assumption that the class $\mathcal{C}$ is *multiplicative*, i.e. $X \times Y \in \mathcal{C}$ whenever $X, Y \in \mathcal{C}$. The arguments are as follows. Given a space C, we may consider C as a subset of a linearly independent compactum in l^2. Then by [Ca_4], the linear hull $L(C)$ of C in l^2 is an absorbing space for the class $\bigcup_{n\in\mathbb{N}} \mathcal{F}_0(C^n \times I^n)$, the smallest $[0,1]$-stable closed-hereditary multiplicative class containing C. In the particular case when $C \in \sigma\mathcal{C}$ is a $\mathcal{C}$-universal space, where $\mathcal{C}$ is a $[0,1]$-stable multiplicative class, the construction of $L(C)$ just supplies us with a $\mathcal{C}$-absorbing AR.

Now we consider a minor modification of this construction that will allow us to build absorbing AR's for $[0,1]$-stable classes which are not necessarily multiplicative.

Given a space C, fix a space $\widetilde{C}$ containing C as a closed subset such that $D = \widetilde{C} \setminus C$ is a countable dense set in $\widetilde{C}$. Embed $\widetilde{C}$ into a linearly independent compact subset of l^2

and consider the space

$$\Omega(C) = \{tx + y \mid t \in [0,1],\ x \in C,\ y \in \operatorname{span} D\}$$

which is dense in $L(\widetilde{C})$.

6.1. THEOREM. *$\Omega(C)$ is an absorbing AR for the class $\bigcup_{n\in\mathbb{N}} \mathcal{F}_0(C \times I^n)$.*

The class $\bigcup_{n\in\mathbb{N}} \mathcal{F}_0(C \times I^n)$ is the smallest $[0,1]$-stable closed-hereditary class containing C. As we will see later, the powers $\Omega(C)^n$ are absorbing AR's too, but they can be topologically distinct. We will construct a compactum C such that the powers $\Omega(C)^n$ are pairwise not homeomorphic. For this compactum, the absorbing space $\Omega(C)$ supports neither a topological group structure nor the structure of a convex set in a topological vector space, and hence $\Omega(C)$ is a counterexample to Question 4.9 of [DM]. Note that the first example of an absorbing space supporting no topological group structure, the absorbing AR for the class $\mathcal{M}_1[\omega_0]$ of Polish spaces of transfinite dimension $\leq \omega_0$, was constructed by M. Zarichnyi [Za] by a completely different method.

Theorem 6.1 immediately implies the following characterization.

6.2. THEOREM. *Let $\mathcal{C}$ be a $[0,1]$-stable class of spaces. There exists a $\mathcal{C}$-absorbing AR if and only if the class $\sigma\mathcal{C}$ contains a $\mathcal{C}$-universal space.*

For the proof of Theorem 6.1 we need to recall some notions. Given a subset X of a linear space L, we set $\operatorname{Ker}(X) = \{x \in X \mid [x,y] \subset X \text{ for every } y \in X\}$ denote the kernel of X. It is well known that $\operatorname{Ker}(X)$ is a convex (possibly empty) set in L.

Let L be a linear space and $A, B \subset L$ be two subsets in L. Denote by $\operatorname{span} A$ the linear span of A and by $\pi : L \to L/\operatorname{span}(A)$ the corresponding quotient map. We say that the set A has *infinite codimension* in B if the set $\pi(B)$ is algebraically infinite-dimensional (i.e. there is an infinite linearly independent subset $C \subset \pi(B)$). Recall that a subset B of a linear topological space L is called *bounded* if for every neighborhood $U \subset L$ of the origin there is $n \in \mathbb{N}$ with $B \subset n \cdot U$.

6.3. PROPOSITION. *Let L be a locally convex linear metric space, L_0 a linear subspace in L, and let $X \subset L$ be a set such that $\operatorname{Ker}(X) \cap L_0$ is dense in X. Suppose $A \subset X$ is a bounded closed set in L of infinite codimension in $\operatorname{Ker}(X) \cap L_0$. Then the pair $(X, X \cap L_0)$ is strongly $(A, A \cap L_0)$-universal.*

PROOF. To verify that the pair $(X, X \cap L_0)$ is strongly $(A, A \cap L_0)$-universal, fix a cover $\mathcal{U} \in \operatorname{cov}(X)$, a closed subset $B \subset A$, and a map $f : A \to X$ such that $f|B : B \to X$ is a Z-embedding with $(f|B)^{-1}(L_0) = B \cap L_0$.

Let $\overline{L}$ denote the completion of L with respect to any invariant metric d and let $\overline{X}$ be the closure of X in $\overline{L}$. Since the convex set $\operatorname{Ker}(X)$ is dense in X, the closure $\overline{X}$ of X is convex and X is homotopy dense in $\overline{X}$. According to [DT], $\overline{X}$ is either a Q- or an s-manifold, hence every Z-set in $\overline{X}$ is a strong Z-set. The same is true for X because of the homotopical density of X in $\overline{X}$. Thus $f(B)$ is a strong Z-set in X. Then by [BM, 1.1], without loss of generality, we can assume that $f(A \setminus B) \cap f(B) = \emptyset$ and f is closed over $f(B)$.

Let $A' = A \setminus B$, $X' = X \setminus f(B)$, and $\mathcal{U}' \in \operatorname{cov}(X')$ be such that $\operatorname{St}\mathcal{U}' \prec \mathcal{U}$ and $\operatorname{St}\mathcal{U}' \prec \{O(x, d(x, f(B))/2) \mid x \in X'\}$. Since X' is homotopy dense in $\overline{X} \setminus \overline{f(B)}$ which is a

Q- or an s-manifold, there exists $p : X' \to X'$ such that $(p, \mathrm{id}) \prec \mathcal{U}'$ and $F = \mathrm{Cl}_{X'}(p(X'))$ is locally compact. Indeed, in case X' is a Q-manifold that is trivial (just take $p = \mathrm{id}$). If X' is an s-manifold then we may find a countable locally finite simplicial complex K and two maps $X' \xrightarrow{\alpha} K \xrightarrow{\beta} X'$ such that $(\beta \circ \alpha, \mathrm{id}) \prec \mathcal{V}$ for some $\mathcal{V} \in \mathrm{cov}(X')$ with $\mathrm{St}(\mathcal{V}) \prec \mathcal{U}'$ (this follows from the fact that $X' \in \mathrm{ANR}$). Next, applying the strong $\mathcal{M}_1$-universality of the s-manifold X', we may approximate the map β by a closed embedding $\beta' : K \to X'$ such that $(\beta', \beta) \prec \mathcal{V}$. Then $\beta'(K)$ is a closed locally compact subset in X' and thus the map $p = \beta' \circ \alpha : X' \to X'$ is as required.

Using the infinite codimension of A in $\mathrm{Ker}(X) \cap L_0$, find a countable dense subset $S \subset \mathrm{Ker}(X) \cap L_0$ such that $\mathrm{span}(A) \cap \mathrm{span}(S) = \{0\}$ and $\mathrm{span}(S \cup A)$ has infinite codimension in $\mathrm{Ker}(X) \cap L_0$. Let $x_0 \in \mathrm{Ker}(X) \cap L_0 \setminus \mathrm{span}(S \cup A)$ be any point. The set $\mathrm{conv}\, S$, being convex and dense, is homotopy dense in X.

Replacing, if necessary, p by a near map, without loss of generality, we may assume that $F \subset \mathrm{conv}\, S \subset \mathrm{Ker}(X)$. Since $F \subset X'$ is closed and locally compact, there is a locally finite cover $\mathcal{W} \in \mathrm{cov}(X')$, $\mathcal{W} \prec \mathcal{U}'$, such that for every $W \in \mathcal{W}$ the intersection $\overline{W} \cap F$ is compact.

Using the continuity of linear operations and the boundedness of A find a continuous function $\varepsilon : F \to (0, 1]$ such that every $x \in F \subset \mathrm{Ker}(X)$ has a neighborhood $W \in \mathcal{W}$ such that $(1 - \varepsilon(x))x + \frac{\varepsilon(x)}{2} x_0 + \frac{\varepsilon(x)}{2} A \subset W$.

Define a map $f' : A' \to X'$ by the formula

$$f'(a) = (1 - \varepsilon \circ p \circ f(a))\, p \circ f(a) + \frac{\varepsilon \circ p \circ f(a)}{2} x_0 + \frac{\varepsilon \circ p \circ f(a)}{2} a, \quad a \in A'. \tag{1}$$

Let us show that f' is a Z-embedding such that $(f', p \circ f|A') \prec \mathcal{W}$ and $(f')^{-1}(L_0) = A' \cap L_0$. In fact, the last two properties easily follow from the definition of f', and our task now is to show that f' is a Z-embedding. By the choice of x_0 and S, the equality $f'(a) = f'(a')$, where $a, a' \in A'$, implies $\varepsilon \circ p \circ f(a) = \varepsilon \circ p \circ f(a')$ and $a = a'$, i.e., the map f' is injective.

To show that f' is a closed embedding, it now suffices to verify that $f' : A' \to X'$ is perfect. Fix a compactum $K \subset X'$. To show that $K^- = (f')^{-1}(K) \subset A'$ is compact, notice first that the set K^- is closed not only in A' but also in A (this follows from $(f|A', f') \prec \mathcal{U}'$ and the closedness of f over $f(B)$). Since $(f', p \circ f|A') \prec \mathcal{W}$, we have $p \circ f(K^-) \subset \mathrm{St}(K, \mathcal{W})$. By the choice of $\mathcal{W}$, the set $M = \mathrm{Cl}_X(\mathrm{St}(K, \mathcal{W})) \cap F$ is compact. Let $\varepsilon_0 = \min\{\varepsilon(x) \mid x \in M\}$ and observe that the set $D = [1, 2/\varepsilon_0](K - [0,1]M - [0,1]x_0) \subset L$ is compact.

It follows from (1) that for every $a \in K^-$,

$$a = \frac{2}{\varepsilon \circ p \circ f(a)} \left(f'(a) - (1 - \varepsilon \circ p \circ f(a))\, p \circ f(a) - \frac{\varepsilon \circ p \circ f(a)}{2} x_0 \right) \in D.$$

Thus $K^- \subset D$ is compact.

By the choice of S and x_0, and by the definition of f', the set $f'(A') \subset \mathrm{span}(A \cup S \cup \{x_0\})$ has infinite codimension in $\mathrm{Ker}(X)$. By standard arguments (see e.g. [Ba$_2$]), one can show that $f'(A')$ is a Z-set in X'.

Letting finally $\bar{f}|B = f|B$ and $\bar{f}|A \setminus B = f'$, we define a Z-embedding $\bar{f} : A \to X$

such that $\bar{f}|B = f|B$, $(\bar{f}, f) \prec \mathcal{U}$, and $\bar{f}^{-1}(L_0) = A \cap L_0$. Thus $(X, X \cap L_0)$ is strongly $(A, A \cap L_0)$-universal. ∎

Proof of 6.1. Recall that C is closed in $\widetilde{C} \subset l^2$, $D = \widetilde{C} \setminus C$ is a countable dense set in $\widetilde{C}$, and the closure $\widetilde{K}$ of $\widetilde{C}$ in l^2 is a linearly independent compactum. Let $K = \overline{C} \subset \widetilde{K}$ be the closure of C, and notice that $K \cap D = \emptyset$.

We will show firstly that $\Omega(C) \in \sigma\mathcal{C}$, where $\mathcal{C} = \bigcup_{n\in\mathbb{N}} \mathcal{F}_0(C \times I^n)$. Write $D = \{d_n\}_{n\in\mathbb{N}}$ and remark that $\operatorname{span} D = \bigcup_{n\in\mathbb{N}} D_n$, where $D_n = n \cdot \operatorname{conv}\{\pm d_1, \ldots, \pm d_n\}$ is homeomorphic to I^n for each $n \in \mathbb{N}$.

By the definition of $\Omega(C)$, we have

$$\Omega(C) = \bigcup_{n\in\mathbb{N}} D_n \cup \bigcup_{n\in\mathbb{N}} ([1/n, 1] \cdot C + D_n). \tag{2}$$

Since $\widetilde{K} \supset K \cup D$ is a linearly independent compactum, the map $i_n : K \times [1/n, 1] \times D_n \to l^2$, $i_n : (k, t, d) \mapsto t \cdot k + d$, is injective, and consequently, it is an embedding. Notice that $([1/n, 1]\cdot K + D_n) \cap \Omega(C) = [1/n, 1]\cdot C + D_n = i_n(C \times [1/n, 1] \times D_n)$. Hence, $[1/n, 1]\cdot C + D_n$ is a closed subset in $\Omega(C)$, homeomorphic to $C \times [1/n, 1] \times D_n \cong C \times I^{n+1}$. Since $D_n \cong I^n$ are closed in $\Omega(C)$, this and (2) yield $\Omega(C) \in \sigma\mathcal{C}$.

Since $\Omega(C)$ is homotopy dense in the σ-compact pre-Hilbert space $\operatorname{span} \widetilde{K}$, $\Omega(C)$ is a strong Z_σ-space and $\Omega(C)$ is an AR.

To show that the space $\Omega(C)$ is strongly $\mathcal{C}$-universal, we will apply Proposition 6.3. Notice that for every $n \in \mathbb{N}$ the set $C + D_n$ is closed and bounded in $L = \operatorname{span} \widetilde{C}$, and has infinite codimension in $\operatorname{span} D$. Since the kernel of $\Omega(C)$ contains the dense in L set $\operatorname{span} D$, Proposition 6.3 is applicable and thus the space $\Omega(C)$ is strongly $(C + D_n)$-universal. As $C + D_n$ is homeomorphic to $C \times I^n$, by 1.5 we get $\Omega(C)$ is strongly $\mathcal{F}_0(C \times I^n)$-universal for every $n \in \mathbb{N}$. Thus $\Omega(C)$ is a $\mathcal{C}$-absorbing AR. ∎

6.4. Theorem. *If $C_1 \cong C_1 \times [0, 1]$ and $C_2 \cong C_2 \times [0, 1]$ are two absorbing spaces then their product $C_1 \times C_2$ is an absorbing space.*

Proof. We first consider the case when both C_1 and C_2 are AR's. Fix $i \in \{1, 2\}$. Since $C_i \cong C_i \times [0, 1]$, the class $\mathcal{F}_0(C_i)$ is $[0, 1]$-stable. Then the space $\Omega(C_i)$ is $\mathcal{F}_0(C_i)$-absorbing and thus homeomorphic to C_i according to Theorem 1.8. Applying Proposition 6.3, we may prove that the product $\Omega(C_1) \times \Omega(C_2)$ is strongly $C_1 \times C_2$-universal. Using this fact, we may easily deduce that $C_1 \times C_2$ is an absorbing space.

In the general case (C_i may be non-contractible), C_i is homeomorphic to an open subset $U_i \subset \Omega(C_i)$. This can be proved as follows. The space C_i, being an ANR, is homotopy equivalent to a locally finite simplicial complex K. By the Open Embedding Theorem for s-manifolds, $K \times s$, being an s-manifolds, is homeomorphic to an open subset $U \subset s$. By the construction, $\Omega(C_i)$ is a homotopy dense set in s. Then $U \cap \Omega(C_i)$ is an $\mathcal{F}_0(C_i)$-absorbing space homotopy equivalent to C_i. Finally, applying Theorem 1.8, we conclude that C_i is homeomorphic to the open subset $U \cap \Omega(C_i)$ in $\Omega(C_i)$. Since $\Omega(C_i) \cong \Omega(C_i) \times [0, 1]$, $i = 1, 2$, are absorbing AR's, it follows from the first case that $\Omega(C_1) \times \Omega(C_2)$ is an absorbing space. Then $C_1 \times C_2$, being homeomorphic to the open subspace $U_1 \times U_2$ of $\Omega(C_1) \times \Omega(C_2)$, is an absorbing space too. ∎

6.5. Corollary. *If a space* $\Omega\cong\Omega\times[0,1]$ *is absorbing, then so is* Ω^n *for every* $n\in\mathbb{N}$.

6.6. Question. Is Corollary 6.5 valid without the assumption $\Omega\cong\Omega\times[0,1]$?

We shall say that a map $*:X\times Y\to Z$ is a *cancellative operation* if for every $x,x'\in X$ and $y,y'\in Y$, $x*y=x*y'$ implies $y=y'$ and $x*y=x'*y$ implies $x=x'$. Notice that spaces X supporting group or convex structures admit a continuous cancellative operation $*:X\times X\to X$ (in the convex case set $x*y=\frac{1}{2}x+\frac{1}{2}y$).

6.7. Theorem. *There exists a compactum* C *such that*

(1) $\Omega(C)$ *does not admit any continuous cancellative operation* $\Omega(C)\times\Omega(C)\to\Omega(C)$ (*hence* $\Omega(C)$ *does not support neither convex nor topological group structures*);

(2) *for every* $n\neq m$ $\Omega(C)^n$ *is not homeomorphic to* $\Omega(C)^m$.

Proof. We will make use of Cook's continuum [Co]. This is a hereditarily indecomposable continuum M such that every map $A\to M$ of a subcontinuum $A\subset M$ is either the identity or constant.

Let C_i, $i\in\mathbb{N}$, be pairwise disjoint subcontinua of M, and let $C=\prod_{i\in\mathbb{N}}C_i$. Let us show that the absorbing space $\Omega(C)$ satisfies the conditions (1) and (2).

To prove (1) assume that $\Omega(C)$ admits a continuous cancellative operation $\Omega(C)\times\Omega(C)\to\Omega(C)$. Since C embeds into $\Omega(C)$, this implies the existence of a continuous map $\varphi:C\times C\to\Omega(C)$ such that for every $c\in C$ the restrictions $\varphi|C\times\{c\}$ and $\varphi|\{c\}\times C$ are injective. Since $\Omega(C)\in\sigma\mathcal{C}$, where $\mathcal{C}=\bigcup_{n\in\mathbb{N}}\mathcal{F}_0(C\times I^n)$, Baire's Theorem implies the existence of an open set $U\subset C\times C$ such that $\varphi(U)$ embeds into $C\times I^n$ for some $n\in\mathbb{N}$. Denote by $\pi_1:C\times I^n\to C$ and $\pi_2:C\times I^n\to I^n$ the natural projections.

Fix $(c,c')\in U\subset C\times C$ such that $c_i\neq c_i'$ for every $i\in\mathbb{N}$, where $c=(c_i)_{i=1}^\infty$ and $c'=(c_i')_{i=1}^\infty$. For every $i\in\mathbb{N}$ pick disjoint continua $B_i,B_i'\subset C_i$ such that $c_i\in B_i$, $c_i'\in B_i'$ and $\prod_{i=1}^\infty B_i\times\prod_{i=1}^\infty B_i'\subset U$. Let $Y=\prod_{i=1}^\infty B_i$, $Y_i=\prod_{j\neq i}B_j$ and identify Y with $B_i\times Y_i$. For $y\in Y_i$ define $\varphi_y^i:B_i\to C_i$ by $\varphi_y^i(b)=p_i\circ\pi_1\circ\varphi((b,y),c')$, $b\in B_i$ (here $p_i:C\to C_i$ denote the coordinate projections).

Claim 1. *The map* φ_y^i *does not depend on the choice of* y *and is either a constant or the identity embedding* $\mathrm{id}\,|B_i$.

Indeed, because of the choice of the continuum M, φ_y^i is either a constant or the identity embedding $\mathrm{id}\,|B_i$. Since Y_i is connected, this yields that either $\varphi_y^i=\mathrm{id}\,|B_i$ for all $y\in Y_i$, or φ_y^i is a constant for all y. In the second case, if φ_y^i depended on y, one could find two points $y,y'\in Y_i$ differing in one coordinate only (say i_0) such that $\varphi_y^i\neq\varphi_{y'}^i$. Represent Y_i as the product $Y_i=B_{i_0}\times\prod_{j\neq i,i_0}B_j$ and write $y=(y_{i_0},\overline{y})$ and $y'=(y_{i_0}',\overline{y})$. Fix any $d\in B_i$. The map $\psi:B_{i_0}\to C_i$ defined by $\psi(b)=p_i\circ\pi_1\circ\varphi((d,b,\overline{y}),c')$, $b\in B_{i_0}$, is not constant because $\psi(y_{i_0})=\varphi_y^i(d)\neq\varphi_{y'}^i(d)=\psi(y_{i_0}')$. On the other hand, since $i_0\neq i$, $B_{i_0}\cap C_i=\emptyset$, therefore every map $B_{i_0}\to C_i\subset M$ must be constant.

Our claim shows that the map $\pi_1\circ\varphi|Y\times\{c'\}$ can be written as $\pi_1\circ\varphi(y,c')=(\varphi_i(y_i))_{i\in\mathbb{N}}$, where $\varphi_i=\varphi_y^i$ is either the identity or a constant. Let $\mathcal{I}_0$ denote the set of indices i for which φ_i are constant. The set $\mathcal{I}_0$ is finite. Indeed, fixing any $b\in\prod_{i\notin\mathcal{I}_0}B_i$ we see that $\pi_1\circ\varphi|\prod_{i\in\mathcal{I}_0}B_i\times\{b\}\times\{c'\}$ is constant. Since $\varphi|C\times\{c'\}$ is injective, $\pi_2\circ$

$\varphi|\prod_{i\in\mathcal{I}_0} B_i\times\{b\}\times\{c'\}$ must be injective. Hence the dimension of $\prod_{i\in\mathcal{I}_0} B_i$ is not greater than n, and consequently, $|\mathcal{I}_0|\le n$.

Therefore, $p_i\circ\pi_1\circ\varphi(c,c')=c_i$ for all but finitely many i. By a symmetric reasoning, we can prove that $p_i\circ\pi_1\circ\varphi(c,c')=c_i'$ for almost all i. Hence $c_i=c_i'$ for almost all i's, but this contradicts the choice of $c=(c_i)$ and $c'=(c_i')$.

For the proof of the property (2), it suffices to show that there is no continuous injection $C^M\to\Omega(C)^N$ if $M>N$. Assume the converse. Since $\Omega(C)\in\sigma\mathcal{C}$, Baire's Theorem yields an open set $U\subset C^M$ such that U embeds into $C^N\times I^q$ for some $q\in\mathbb{N}$. The set U being open in $C^M=(\prod_{i=1}^\infty C_i)^M$ contains a topological copy of the product $(\prod_{j=r}^\infty C^j)^M$ for some $r\in\mathbb{N}$. Write $(\prod_{j=r}^\infty C_j)^M=\prod_{m=1}^M\prod_{j=r}^\infty C_{m,j}$, where $C_{m,j}=C_j$, and $C^N=\prod_{n=1}^N\prod_{i=1}^\infty C_{n,i}$, where $C_{n,i}=C_i$, and let φ denote the embedding of $(\prod_{j=r}^\infty C_j)^M$ into $C^N\times I^q=(\prod_{n=1}^N\prod_{i=1}^\infty C_{n,i})\times I^q$.

Denote by $\pi_1:C^N\times I^q\to C^N$, $\pi_2:C^N\times I^q\to I^q$, $p_{n,i}:\prod_{n=1}^N\prod_{i=1}^\infty C_{n,i}\to C_{n,i}=C_i$ and $\varrho_{m,j}:\prod_{m=1}^M\prod_{j=r}^\infty C_{m,j}\to C_{m,j}=C_j$ the corresponding projections.

Claim 2. *For every $(n,i)\in\{1,\dots,N\}\times\mathbb{N}$ the map $p_{n,i}\circ\pi_1\circ\varphi$ is either constant or coincides with the projection $\varrho_{m,i}$ for some $m\in\{1,\dots,M\}$.*

Assume that $p_{n,i}\circ\pi_1\circ\varphi$ is not constant. Then we can find two points $y,y'\in\prod_{m=1}^M\prod_{j=r}^\infty C_{m,j}$ differing by one coordinate only (say by the (m,j)-coordinate) such that $p_{n,i}\circ\pi_1\circ\varphi(y)\ne p_{n,i}\circ\pi_1\circ\varphi(y')$. Write $\prod_{m=1}^M\prod_{j=r}^\infty C_{m,j}=C_{m,j}\times Z$. For every $z\in Z$ define the map $\psi_z:C_{m,j}=C_j\to C_i=C_{n,i}$ by $\psi_z(c)=p_{n,i}\circ\pi_1\circ\varphi(c,z)$. By the properties of Cook's continuum M and by the connectedness of Z either ψ_z is constant for all $z\in Z$ or $\psi_z=\mathrm{id}\,|C_j$ for all $z\in Z$. Notice that the first case is impossible since writing $y=(c,z), y'=(c',z)$ we have $\psi_z(c)=p_{n,i}\circ\pi_1\circ\varphi(y)\ne p_{n,i}\circ\pi_1\circ\varphi(y')=\psi_z(c')$. Therefore $p_{n,i}\circ\pi_1\circ\varphi=\varrho_{m,j}$. Notice also that in this case necessarily $j=i$.

Denote by $\mathcal{I}_0$ the set of all pairs (m,j) for which there is an $n\in\{1,\dots,N\}$ with $p_{n,j}\circ\pi_1\circ\varphi=\varrho_{m,j}$. Since $N<M$, the set $\mathcal{I}_1=(\{r,r+1,\dots\}\times\{1,\dots,M\})\setminus\mathcal{I}_0$ is infinite. Fixing any point $z\in\prod_{(m,j)\in\mathcal{I}_0}C_{m,j}$ we find that the restriction of $\pi_1\circ\varphi$ to $B=\prod_{(m,j)\in\mathcal{I}_1}C_{m,j}\times\{z\}$ is constant. Thus $\pi_2\circ\varphi|B:B\to I^q$ is injective, which is impossible because B is infinite-dimensional. ∎

7. On embeddings of absorbing spaces

Throughout the section, $\mathcal{C}$ is a closed-hereditary $[0,1]$-stable topological class of spaces and Ω is a $\mathcal{C}$-absorbing absolute retract. We are concerned with the topological classification of pairs (M,X), where $M=Q$ or $M=s$, and X is a homotopy dense topological copy of Ω in M. The case $\mathcal{C}\subset\mathcal{A}_1$ is not so interesting because, as one can easily see, there is a unique (up to homeomorphism) such pair (M,X) (this fact holds for all subclasses $\mathcal{C}$ of $\mathcal{A}_1$, not necessarily $[0,1]$-stable). To begin, let us remark that there always exists a homotopy dense embedding of Ω into s with image in a σ-compact subset of s.

7.1. Theorem. *There exists a homotopy dense embedding $\Omega \subset s$ such that Ω lies in a σ-compact subset $A \subset s$ and the pairs (s, Ω) and (Q, Ω) are strongly universal. Moreover, if the class $\mathcal{C}$ admits compactifications then we can assume that $A \in (\mathcal{M}_0 \cap \mathcal{C})_\sigma$.*

We will derive this theorem from the following

7.2. Lemma. *Let K be a Polish space and $C \subset K$. Let $\vec{\mathcal{C}} = \bigcup_{n\in\mathbb{N}} \mathcal{F}_0(K \times I^n, C \times I^n)$. There is a Z_σ-subset $A \subset l^2$ with dense kernel, and a subset $\Omega \subset A$ such that $(A, \Omega) \in \vec{\mathcal{C}}_\sigma$ and the pairs (A, Ω) and (l^2, Ω) are strongly $\vec{\mathcal{C}}$-universal.*

Proof. We will repeat arguments of the proof of Theorem 6.1. Let $\widetilde{K}$ be a Polish space containing K as a closed subset with $D = \widetilde{K} \setminus K$ being a countable discrete dense subset in $\widetilde{K}$. According to [BP], there is an embedding $\widetilde{K} \subset l^2$ such that $\widetilde{K}$ is a closed bounded linearly independent set with dense $\operatorname{span} \widetilde{K}$ in l^2. Consider the sets

$$A = \{tx + y \mid t \in [0,1],\ x \in K,\ y \in \operatorname{span} D\},$$
$$\Omega = \{tx + y \mid t \in [0,1],\ x \in C,\ y \in \operatorname{span} D\}.$$

Let $L = l^2$ and $L_0 = \operatorname{span}(C \cup D)$. It is easy to see that $\operatorname{Ker}(A) \cap L_0 \supset \operatorname{span} D$ is dense in A, and $\Omega = A \cap L_0$.

Write $D = \{d_n\}_{n\in\mathbb{N}}$ and fix an $n \in \mathbb{N}$. Since the set $\widetilde{K} \supset D$ is linearly independent, the set $D_n = \operatorname{conv}\{\pm d_1, \dots, \pm d_n\} \subset l^2$ is homeomorphic to I^n, and the map $h_n : K \times D_n \to K + D_n$, $h_n : (k, d) \mapsto k + d$, is injective. Moreover, using the closedness of K in l^2, one can show that the map h_n is a homeomorphism onto the closed set $K + D_n \subset l^2$. Notice that $(K + D_n) \cap L_0 = C + D_n$. Since $K + D_n$ has infinite codimension in $\operatorname{span} D$, by 6.3, the pair (A, Ω) is strongly $(K + D_n, C + D_n)$-universal. By the same reason, the pair (l^2, Ω) is strongly $(K + D_n, C + D_n)$-universal. Since the pair $(K + D_n, C + D_n)$ is homeomorphic to $(K \times I^n, C \times I^n)$, we see that the pairs (A, Ω) and (l^2, Ω) are strongly $\vec{\mathcal{C}}$-universal.

The rest of the statements of the lemma ($(A, \Omega) \in \sigma\vec{\mathcal{C}}$ and A is a Z_σ-subset in l^2) can be easily derived from the definition of A and Ω and the known fact stating that for a closed subset $F \subset l^2$ and compacta $G \subset l^2$ and $H \subset \mathbb{R} \setminus \{0\}$ the map $F \times G \times H \to l^2$ sending a triple (f, g, h) onto $h \cdot f + g \in l^2$ is perfect. ■

Proof of Theorem 7.1. Let Ω be a $\mathcal{C}$-absorbing AR for a $[0,1]$-stable class $\mathcal{C}$. Write $\Omega = \bigcup_{n\in\mathbb{N}} C_n$, where $C_n \in \mathcal{C}$, $n \in \mathbb{N}$, are closed subsets in Ω. If the class $\mathcal{C}$ admits compactification, find for every $n \in \mathbb{N}$ a compactum $K_n \in \mathcal{C}$ with $C_n \subset K_n$, in the other case let K_n be any compactum with $C_n \subset K_n$. Let $K = \sqcup_{n\in\mathbb{N}} K_n$, $C = \sqcup_{n\in\mathbb{N}} C_n$ be topological sums of K_n's and C_n's, respectively, and $\vec{\mathcal{C}} = \bigcup_{n\in\mathbb{N}} \mathcal{F}_0(K \times I^n, C \times I^n)$. By Lemma 7.2, there is a Z_σ-set $A \subset l^2$ and a subset $\Omega' \subset A$ such that $(A, \Omega') \in \sigma\vec{\mathcal{C}}$ and the pair (l^2, Ω') is strongly $\vec{\mathcal{C}}$-universal. It is easily seen that $\Omega' \in \sigma\mathcal{C}$ (just use $[0,1]$-stability of $\mathcal{C}$). Since $(K_n, C_n) \in \vec{\mathcal{C}}$, the strong $\vec{\mathcal{C}}$-universality of (l^2, Ω') and Theorem 3.1 imply the strong $\mathcal{C}$-universality of Ω'. Since Ω' is a homotopy dense subset contained in a σ-compact set $A \subset l^2$, Ω' is a Z_σ-space, and $\Omega' \in AR$. Then Proposition 1.5 implies that Ω' is strongly $\bigcup_{n\in\mathbb{N}} C_n$-universal. Since every space from $\mathcal{C}$ embeds as a closed subset into $\bigcup_{n\in\mathbb{N}} C_n = \Omega$, we see that the space Ω' is strongly $\mathcal{C}$-universal. Hence Ω' is a $\mathcal{C}$-absorbing AR which, by Theorem 1.8, is homeomorphic to Ω. Identifying

Ω with Ω' and l^2 with $s \subset Q$, we obtain the required homotopy dense embedding $\Omega \subset s$. Notice that Lemma 1.4, and the strong $\vec{\mathcal{C}}$-universality of the pair (s, Ω) imply the strong $\vec{\mathcal{C}}_0$-universality of (Q, Ω), where $\vec{\mathcal{C}}_0 = \{(B, D) \in \vec{\mathcal{C}} \mid B \in \mathcal{M}_0\}$. Observe also that because of the equality $\sigma\text{-}\vec{\mathcal{C}}_0 = \sigma\text{-}\vec{\mathcal{C}}$, we have $(A, \Omega) \in \sigma\text{-}\vec{\mathcal{C}}_0$. By Proposition 1.11, the pairs (Q, Ω) and (s, Ω) are strongly universal.

If the class $\mathcal{C}$ admits compactifications, by the choice of the compacta K_n, $A \in (\mathcal{M}_0 \cap \mathcal{C})_\sigma$. ∎

If $\mathcal{C}$ admits compactifications and is weakly $\mathcal{A}_1$-additive, then for $M = Q$ or $M = s$ there is a unique (up to homeomorphism) pair (M, X), where X is a homotopy dense copy of Ω contained in a subset $A \in (\mathcal{M}_0 \cap \mathcal{C})_\sigma$.

7.3. Theorem. *Suppose that the class $\mathcal{C}$ is weakly $\mathcal{A}_1$-additive and admits compactifications. Let M be either a Q-manifold or an s-manifold. For $i = 1, 2$, let $X_i \subset M$ be a $\mathcal{C}$-absorbing homotopy dense subspace of M such that $X_i \subset A_i \subset M$ for some $A_i \in (\mathcal{M}_0 \cap \mathcal{C})_\sigma$. Then $(M, X_1) \cong (M, X_2)$.*

A similar result holds also if the class $\mathcal{C}$ is $\mathcal{M}_1$-hereditary and $\mathcal{A}_1$(s.c.d.)-additive.

7.4. Theorem. *Suppose the class $\mathcal{C}$ is $\mathcal{M}_1$-hereditary and $\mathcal{A}_1$(s.c.d.)-additive. Let M be either a Q-manifold or an s-manifold. For $i = 1, 2$, let $X_i \subset M$ be a $\mathcal{C}$-absorbing homotopy dense subspace of M such that $X_i \subset A_i \subset M$ for some σ-compact A_i. Then $(M, X_1) \cong (M, X_2)$.*

These theorems follow in an obvious way from Theorems 4.1, 4.2 and 1.9. Note also that the condition of $[0, 1]$-stability of the class $\mathcal{C}$ in Theorem 7.3, 7.4 can be replaced by the 2^ω-stability of $\mathcal{C}$.

If the class $\mathcal{C}$ contains the space ω^ω of irrationals, then there are topologically distinct homotopy dense embeddings of Ω into s.

7.5. Theorem. (1) *If $\omega^\omega \in \mathcal{C}$, then s contains two homotopy dense copies E_0, E_1 of Ω such that* (a) *each (s, E_i) is strongly universal, and* (b) $(s, E_0) \not\cong (s, E_1)$.

(2) *If $\mathcal{M}_1 \subset \mathcal{C}$ then s contains continuum many homotopy dense copies E_α, $\alpha \in \mathfrak{c}$, of Ω such that* (a) *each (s, E_α) is strongly universal and* (b) $(s, E_\alpha) \not\cong (s, E_\beta)$ *if $\alpha \neq \beta$.*

Proof. Let K be any compactification of Ω.

(1) Applying Lemma 7.2 to the pairs (K, Ω) and $(K \sqcup \omega^\omega, \Omega \sqcup \omega^\omega)$ we construct strongly universal pairs (l^2, E_0) and (l^2, E_1) such that E_0 is contained in a σ-compact subset of s, while E_1 contains a copy of ω^ω closed in l^2 and hence E_1 is contained in no σ-compact subset of l^2. Thus $(s, E_0) \not\cong (s, E_1)$.

(2) According to [Ma], Cook's continuum contains a family $\{M_\alpha \mid \alpha \in \mathfrak{c}\}$ of pairwise disjoint nondegenerate continua. For every $\alpha \in \mathfrak{c}$ fix a point $a_\alpha \in M_\alpha$ and let $X_\alpha = M_\alpha \setminus \{a_\alpha\}$. Then every continuous function of X_α into X_β is constant, whenever $\alpha \neq \beta$. Applying Lemma 7.2 to the Polish space $K_\alpha = K \sqcup X_\alpha^\omega$ and its subspace $C_\alpha = \Omega \sqcup X_\alpha^\omega$ we get a strongly universal pair (l^2, E_α) such that $E_\alpha \subset A_\alpha$ for some Z_σ-set $A_\alpha \subset l^2$ with $(A_\alpha, E_\alpha) \in \sigma\text{-}\vec{\mathcal{C}}_\alpha$, where $\vec{\mathcal{C}}_\alpha = \bigcup_{n=1}^\infty \mathcal{F}_0(K_\alpha \times I^n, C_\alpha \times I^n)$. It follows from the proof of Theorem 7.1 that E_α is homeomorphic to Ω.

Let $\alpha \neq \beta$. To show $(l^2, E_\alpha) \not\cong (l^2, E_\beta)$ suppose the converse. Then E_β contains a copy of X_α^ω closed in l^2 and hence also in A_β. Since X_α^ω is complete, we may use Baire's Theorem to conclude that there are an $n \in \mathbb{N}$ and an open set $U \subset X_\alpha^\omega$ such that $\overline{U}$ is homeomorphic to a closed subset of $K_\beta \times I^n = (K \times I^n) \sqcup (X_\beta^\omega \times I^n)$. But $\overline{U}$ contains a closed copy of X_α^ω which, being connected, is homeomorphic to a closed subset of $K \times I^n$ or of $X_\beta^\omega \times I^n$. The first case is impossible because $K \times I^n$ is compact but X_α^ω is not, and the second case also leads to the absurd conclusion that the infinite-dimensional space X_α^ω embeds into I^n. ∎

Under the hypotheses of Theorem 7.5(1), there also exist homotopy dense embeddings $\Omega \subset s$ which are not strongly universal.

7.6. Theorem. *Suppose that s contains two copies E_0, E_1 of Ω such that* (a) *each (s, E_i) is strongly universal and* (b) $(s, E_0) \not\cong (s, E_1)$. *Then s contains continuum many homotopy dense copies of F_α, $\alpha \in \mathfrak{c}$, of Ω such that*

(i) *each (s, F_α) is not strongly universal*;
(ii) $(s, F_\alpha) \not\cong (s, F_\beta)$ *if $\alpha \neq \beta$.*

Proof. Since $\mathcal{C}$ is $[0,1]$-stable, Ω can be obtained by the construction of Theorem 6.1, hence it contains a homotopy dense copy of σ (the set $\operatorname{span} D$). According to [BP], if σ_1, σ_2 are two homotopy dense copies of σ in s, then $(s, \sigma_1) \cong (s, \sigma_2)$ (see also Theorem 7.3) which allows us to suppose that $\sigma \subset E_0 \cap E_1$. Let

$$T = [0, \infty) \times \{0\} \cup \Big(\bigcup_{n \in \mathbb{N}} \{n\} \times [0,1] \Big) \subset \mathbb{R}^2.$$

Since T is a Polish AR, the product $T \times s$ is homeomorphic to s and we will construct the F_α's as subsets of $T \times s$.

For every function $\alpha : \mathbb{N} \to \{0,1\}$ consider the following subset of $T \times s$:

$$F_\alpha = \bigcup_{n=1}^{\infty} (2n-2, 2n-1) \times \{0\} \times E_0 \cup (2n-1, 2n) \times \{0\} \times E_1 \cup \{n\} \times [0,1] \times E_{\alpha(n)}.$$

Then F_α contains the homotopy dense subset $(T \setminus \{(0,0)\}) \times \sigma$ of $T \times s$. Consequently, F_α is an AR, and to see that $F_\alpha \cong \Omega$, it is enough to show that each point of F_α has a neighborhood homeomorphic to Ω. The points for which that is not obvious have a neighborhood homeomorphic to a subset of the type

$$U = (-1,1) \times \{0\} \times E_i \cup \{0\} \times (0,1) \times E_j,$$

where $i = 0$ or 1 and $j = 1 - i$. Since $\{(0,0)\} \times E_i$ is a strong Z-set in $\{(0,0)\} \times E_i \cup \{0\} \times (0,1) \times E_j$, this union is a $\mathcal{C}$-absorbing AR. Then according to [BM, 3.2], there is a homeomorphism h of $\{(0,0)\} \times E_i \cup \{0\} \times (0,1) \times E_j$ onto $\{0\} \times [0,1) \times E_i$ such that $h|\{(0,0)\} \times E_i = \mathrm{id}$. Then $\mathrm{id} \cup h$ is a homeomorphism of U onto $\big((-1,1) \times \{0\} \cup \{0\} \times [0,1)\big) \times E_i \cong \Omega$.

Let $\alpha, \beta : \mathbb{N} \to \{0,1\}$ and let h be a homeomorphism of $(T \times s, F_\alpha)$ onto $(T \times s, F_\beta)$. Using the fact that every point of s has a neighborhood U such that $(U, U \cap E_i) \cong (s, E_i)$ $(i = 0,1)$ it is easy to see that the points of $\mathbb{N} \times \{0\} \times s$ are the only ones having no neighborhood V such that $(V, V \cap F_\alpha)$ is homeomorphic to one of the pairs (s, E_i), $i = 0,1$.

Consequently, $h(\mathbb{N} \times \{0\} \times s) = \mathbb{N} \times \{0\} \times s$. Since $(T \setminus \{(n,0)\}) \times s$ has 3 components containing respectively 0, $n-1$, and an infinity of components of $\mathbb{N} \times \{0\} \times s$, we get $h(\{(n,0)\} \times s) = \{(n,0)\} \times s$ for each $n \in \mathbb{N}$. Examining the component (the components if $n = 1$) of $(T \setminus \{(n,0)\}) \times s$ disjoint from $\mathbb{N} \times \{0\} \times s$, we conclude $\alpha(n) = \beta(n)$ for each $n \in \mathbb{N}$. Thus $\alpha = \beta$.

If $(T \times s, F_\alpha)$ were strongly universal then for every $x, y \in F_\alpha$ there would exist an autohomeomorphism of $(T \times s, F_\alpha)$ sending x onto y (for the construction of such an autohomeomorphism, see the proof of Theorem 2.1 in [Ca$_2$]). The foregoing arguments show us that this is impossible. ∎

The case of embeddings into Q is more curious. First, let us note the following corollary of Theorems 7.3 and 7.4.

7.7. Theorem. *Suppose that either $\mathcal{C}$ is $\mathcal{A}_1$-additive and $\mathcal{C} \supset \mathcal{M}_0$ or that $\mathcal{C}$ is $\mathcal{A}_1(s.c.d.)$-additive and $\mathcal{M}_1$-hereditary. If X_1, X_2 are two homotopy dense copies of Ω in Q then the pairs (Q, X_1) and (Q, X_2) are homeomorphic.*

The following two theorems show us the necessity of the conditions on $\mathcal{C}$.

7.8. Theorem. *Suppose that $\mathcal{C}$ admits compactifications and contains the space ω^ω.*

(1) *If $Q \notin \mathcal{C}$, then Q contains two homotopy dense copies E_0, E_1 of Ω such that* (a) *(Q, E_i) is strongly universal*, (b) *$Q \setminus E_0 \ncong Q \setminus E_1$.*

(2) *If $\mathcal{C}$ contains no strongly infinite-dimensional compactum then Q contains continuum many homotopy dense copies $\{E_\alpha \mid \alpha \in A\}$ of Ω such that* (a) *each (Q, E_α) is strongly universal*, (b) *$Q \setminus E_\alpha \ncong Q \setminus E_\beta$ if $\alpha \neq \beta$.*

Proof. Write $\Omega = \bigcup_{n=1}^\infty C_n$, where $C_n \in \mathcal{C}$ is a closed subset of Ω. For every $n \in \mathbb{N}$, fix a compactification $K_n \in \mathcal{C}$ of C_n. To each continuum X let us assign a copy $E(X) \subset Q$ of Ω as follows. Let $\widetilde{X}$ be a compactum containing X so that $N(X) = \widetilde{X} \setminus X$ is countable, discrete, and dense in $\widetilde{X}$. Applying Lemma 7.2 to the Polish space $\widetilde{M}(X) = \widetilde{X}^\omega \sqcup \left(\bigsqcup_{n\in\mathbb{N}} K_n\right)$ and its subspace $C(X) = N(X)^\omega \sqcup \left(\bigsqcup_{n\in\mathbb{N}} C_n\right)$, we obtain a pair $(A(X), E(X))$ with $E(X) \subset A(X) \subset s \subset Q$. It follows from 7.2, 1.4, and 1.11 that $(Q, E(X))$ is strongly universal. Since $N(X)^\omega \cong \omega^\omega \in \mathcal{C}$, the arguments of Theorem 7.1 show that $E(X) \cong \Omega$.

Claim. *If $Q \setminus E(X) \cong Q \setminus E(X')$ then either $X^\omega \in \mathcal{C}$ or $X^\omega \in \mathcal{F}_0((X')^\omega \times I^n)$ for some n.*

Suppose that h is a homeomorphism of $Q \setminus E(X)$ onto $Q \setminus E(X')$. According to Lavrentiev's Theorem, h extends to a homeomorphism $\overline{h} : G \to G'$ between G_δ-sets of Q. According to the construction of $E(X)$, Q contains a copy of $\widetilde{X}^\omega$ such that $\widetilde{X}^\omega \cap E(X) = N(X)^\omega$. Then $\widetilde{X}^\omega \setminus G$ is a σ-compact subspace of $N(X)^\omega$. Let us show that G contains a compactum H such that $(H, H \setminus E(X)) \cong (\widetilde{X}^\omega, \widetilde{X}^\omega \setminus N(X)^\omega)$. For that, fix a homeomorphism $\psi : \widetilde{X}^\omega \to (\widetilde{X}^\omega)^2$ such that $\psi^{-1}((N(X)^\omega)^2) = N(X)^\omega$. Since $\psi(\widetilde{X}^\omega \setminus G)$ is σ-compact and $N(X)^\omega$ is not, there is $t \in N(X)^\omega$ such that $\{t\} \times \widetilde{X}^\omega \subset \psi(\widetilde{X}^\omega \cap G)$. One can readily verify that $H = \psi^{-1}(\{t\} \times \widetilde{X}^\omega) \subset \widetilde{X}^\omega \cap G$ has the desired properties. Then for $\overline{h}|H : H \to G'$ we have $(\overline{h}|H)^{-1}(E(X')) = N(X)^\omega$, where $\widetilde{X}^\omega$ is identified with H. Since $E(X') \subset A(X')$, Baire's Theorem helps us to find an integer n and an open set

$U \subset N(X)^\omega$ such that $\overline{h}(\overline{U})$ is contained in $K_m \times I^n$ $(m \in \mathbb{N})$ or in $(\widetilde{X}')^\omega \times I^n$. Since $\overline{U}$ has non-empty interior in $\widetilde{X}^\omega$, $\overline{U}$ contains a copy of X^ω which embeds in $K_m \times I^n \in \mathcal{C}$ or in $(\widetilde{X}')^\omega \times I^n$. In the latter case, because of the connectedness of X, X^ω embeds into $\{\mathrm{pt}\} \times I^n \in \mathcal{C}$ or into $(X')^\omega \times I^n$.

Now to prove the statement (1) of Theorem 7.8, it suffices to take $E_0 = E(\{\mathrm{pt}\})$ and $E_1 = E(Q)$ (remark that E_0 is just the copy of Ω supplied by the second part of Theorem 7.1).

To prove (2), it suffices to set $E_\alpha = E(X_\alpha)$, where $\{X_\alpha \mid \alpha \in \mathfrak{c}\}$ is a family of continuum many pairwise disjoint nondegenerate subcontinua of the Cook continuum. ∎

The simplest example of a class $\mathcal{C}$ containing $\mathcal{M}_0$ and such that $\sigma\mathcal{C}$ is not $\mathcal{A}_1$-additive, is the class $\mathcal{M}_1$ of all Polish spaces.

7.9. Theorem. *Q contains a family $\{E_\alpha \mid \alpha \in \mathfrak{c}\}$ of continuum many homotopy dense copies of the $\mathcal{M}_1$-absorbing spaces $s \times \sigma$ such that*

(a) *each (Q, E_α) is strongly universal*;
(b) *$Q \setminus E_\alpha \not\cong Q \setminus E_\beta$ if $\alpha \neq \beta$.*

Proof. Let us consider any family $\{\Omega_\alpha \mid \alpha \in \mathfrak{c}\}$ consisting of continuum many pairwise non-homeomorphic $\mathcal{C}_\alpha$-absorbing absolute retracts Ω_α, where $\mathcal{C}_\alpha \subset \mathcal{A}_1$ are $[0,1]$-stable classes (one can take the family of pre-Hilbert σ-compact spaces constructed in [Ca$_4$]). We may assume each Ω_α to be a homotopy dense subset of Q. Then by [Cu], $Q \setminus \Omega_\alpha$ is homeomorphic to s and $E_\alpha = (Q \setminus \Omega_\alpha) \times \sigma$ is a copy of $s \times \sigma$ in $Q \times Q \cong Q$. Since Ω_α is absorbing for the class $\mathcal{C}_\alpha \subset \mathcal{A}_1$, the pair (Q, Ω_α) is strongly $(\mathcal{M}_0, \sigma\mathcal{C}_\alpha)$-universal and thus $(Q, Q \setminus \Omega_\alpha)$ is strongly universal for the class of pairs $\vec{\mathcal{C}_\alpha} = \{(K, C) \mid (K, K \setminus C) \in (\mathcal{M}_0, \sigma\mathcal{C}_\alpha)\}$, and so is the pair $(Q \times \sigma, (Q \setminus \Omega_\alpha) \times \sigma)$. Since $(Q \times \sigma, (Q \setminus \Omega_\alpha) \times \sigma) \in \sigma\vec{\mathcal{C}_\alpha}$, Proposition 1.11 implies that $(Q \times Q, (Q \setminus \Omega_\alpha) \times \sigma)$ is strongly universal.

Finally, (b) results from the following lemma. ∎

7.10. Lemma. *Let $\mathcal{C}'$, $\mathcal{C}''$ be closed-hereditary $[0,1]$-stable classes and let $\Omega', \Omega'' \subset Q$ be $\mathcal{C}'$- and $\mathcal{C}''$-absorbing spaces, respectively. If the complements of $(Q \setminus \Omega') \times \sigma$ and of $(Q \setminus \Omega'') \times \sigma$ in $Q \times Q$ are homeomorphic, then $\Omega' \cong \Omega''$.*

Proof. It suffices to prove that if the complements are homeomorphic then each of the spaces Ω', Ω'' admits a closed embedding into the other. Since $\Omega' \in \mathcal{F}_0(Q \times Q \setminus (Q \setminus \Omega') \times \sigma)$, there is a closed embedding $f : \Omega' \to Q \times Q \setminus (Q \setminus \Omega'') \times \sigma$. By Lavrentiev's Theorem, there is a complete space $X \supset \Omega'$ and a continuous function $\bar{f} : X \to Q \times Q$ extending f. We may assume Ω' to be dense in X. Then $\bar{f}(X \setminus \Omega') \subset (Q \setminus \Omega'') \times \sigma$. Write $\sigma = \bigcup_{n=1}^\infty I^n$. Since Ω' is of the first Baire category, $X \setminus \Omega'$ is a Baire space, dense in X. Consequently, there exists $n \in \mathbb{N}$ such that the interior U of $\bar{f}^{-1}((Q \setminus \Omega'') \times I^n)$ relatively $X \setminus \Omega'$ is not empty. Since $X \setminus \Omega'$ is dense in X, the interior V of $\overline{U}$ in X is not empty. The strong universality of Ω' guarantees that $V \cap \Omega'$ contains a copy A of Ω' closed in Ω'. Then $f|A$ is a closed embedding of A into $Q \times Q \setminus (Q \setminus \Omega'') \times \sigma$. By continuity, $f(A) \subset \bar{f}(\overline{U}) \subset Q \times I^n$. Because $Q \times I^n \setminus ((Q \setminus \Omega'') \times \sigma) = \Omega'' \times I^n$, we see that $\Omega' \cong A$ is homeomorphic to a closed subspace of $\Omega'' \times I^n$. Since $\mathcal{C}_2$ is $[0,1]$-stable, $\Omega'' \times I^n \cong \Omega''$, and thus $\Omega' \in \mathcal{F}_0(\Omega'')$. Analogously, $\Omega'' \in \mathcal{F}_0(\Omega')$. ∎

Theorem 7.6 has its counterpart for embeddings into Q.

7.11. THEOREM. *Suppose* Q *contains two copies* E_0, E_1 *of* Ω *such that* (a) *each* (Q, E_i) *is strongly universal and* (b) $(Q, E_0) \not\cong (Q, E_1)$. *Then* Q *contains continuum many homotopy dense copies* $\{F_\alpha,\ \alpha \in \mathfrak{c}\}$ *of* Ω *such that*

(i) (Q, F_α) *is not strongly universal*;
(ii) $(Q, F_\alpha) \not\cong (Q, F_\beta)$ *if* $\alpha \neq \beta$.

For the proof it suffices to repeat the proof of Theorem 7.6 replacing T by its one-point compactification $\widehat{T}$ which is an AR and thus $\widehat{T} \times Q \cong Q$.

It is interesting to notice that in spite of the fact that the pairs (s, E_α), (s, F_α) constructed in 7.5 and 7.6 are pairwise distinct, it may happen that all the complements $s \setminus E_\alpha$, $s \setminus F_\alpha$ are nevertheless homeomorphic.

7.12. THEOREM. *Suppose* $\mathcal{C}$ *is a* 2^ω*-stable* $\mathcal{A}_1$*-additive class of spaces*, Ω *a* $\mathcal{C}$*-absorbing homotopy dense subspace in* Q *and* G *a homotopy dense* G_δ*-subset in* Q. *Then for every homotopy dense embedding* $e : \Omega \to G$ *the complement* $G \setminus e(\Omega)$ *is homeomorphic to* $Q \setminus \Omega$.

PROOF. By Theorem 4.1, the pair $(G, e(\Omega))$ is strongly $(\mathcal{M}_0, \mathcal{C})$-universal, and by Lemma 1.4, so is the pair $(Q, (Q \setminus G) \cup e(\Omega))$. Since the class $\mathcal{C}$ is $\mathcal{A}_1$-additive, $(Q \setminus G) \cup e(\Omega) \in \sigma\mathcal{C}$. Notice also that $(Q \setminus G) \cup e(\Omega)$ is contained in a Z_σ-set in Q. Thus the pair $(Q, (Q \setminus G) \cup e(\Omega))$ is $(\mathcal{M}_0, \mathcal{C})$-absorbing. The same arguments show us that the pair (Q, Ω) is $(\mathcal{M}_0, \mathcal{C})$-absorbing too. By Theorem 1.9, these pairs are homeomorphic and consequently, $Q \setminus ((Q \setminus G) \cup e(\Omega)) = G \setminus e(\Omega)$ is homeomorphic to $Q \setminus \Omega$. ∎

Finally let us state an open problem connected with the results of this section.

7.13. PROBLEM. Let X be a homotopy dense F_σ-subset of s homeomorphic to $s \times \sigma$. Is the pair (s, X) homeomorphic to $(s \times s, s \times \sigma)$?

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