Contents

Abstract

Let $(X, \mathcal{T})$ be any T_1 topological space. Given a function $F\colon X \to \mathbb{R}$ and $x \in X$, we define the oscillation of F at x to be $\omega(F, x) = \inf_U \sup_{x_1, x_2 \in U} |F(x_1) - F(x_2)|$, where the infimum is taken over all neighborhoods U of x. It is well known that $\omega(F, \cdot)\colon X \to [0, \infty]$ is upper semicontinuous and vanishes at all isolated points of X.

Suppose an upper semicontinuous function $f\colon X \to [0, \infty]$ vanishing at isolated points of X is given. If there exists a function $F\colon X \to \mathbb{R}$ such that $\omega(F, \cdot) = f$, then we call F an ω-primitive for f. By the 'ω-problem' on a topological space X we mean the problem of the existence of an ω-primitive for a given upper semicontinuous function vanishing at all isolated points of X.

The main topics of the present paper are some results concerning the classical ω-problem and some new generalizations.

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Introduction

The oscillation of a real-valued function is a very well known notion of mathematical analysis. Especially it is very useful in the study of continuity. The problem of characterizing the functions which are the oscillation of some other function is very natural and easily formulated. So, it is surprising that the investigation of such functions has begun only recently. The first attempts at studying the function which is, as we say today, an *ω-primitive*, were probably made by Kostyrko in 1980, [K2].

It is mathematical folklore that the oscillation of any real function is upper semicontinuous and vanishes at isolated points of the domain. Thus the ω-problem is the following question: does every upper semicontinuous real-valued function vanishing at isolated points of its domain the oscillation of some other function? In 2001–2003, the problem of the ω-primitive was studied and positively solved for functions defined on a metric space (see [DGP, EP1, EP2]). Namely, the following theorems were proved:

THEOREM 0.1 ([EP2, Theorem 3]). *Let (X,d) be an arbitrary metric space and $f\colon X \to [0,\infty)$ an upper semicontinuous function which vanishes on $X \setminus X^d$. Then for each lower semicontinuous function $g\colon X \to (0,+\infty)$, there exists a function $F\colon X \to \mathbb{R}$ such that $\omega(F,\cdot) = f$ and $-g < F \leq f$.*

THEOREM 0.2 ([EP2, Theorem 4]). *Let (X,d) be an arbitrary metric space and $f\colon X \to [0,\infty]$ an upper semicontinuous function which vanishes on $X \setminus X^d$. Then there exists a function $F\colon X \to \mathbb{R}$ such that $\omega(F,\cdot) = f$.*

After that, the problem of the existence of an ω-primitive for functions defined on a topological space was investigated. For non-metrizable spaces, a complete solution is still unknown, except for a few partial results found for some particular types of spaces (see, e.g., [EP3], [DV], [Ko1]). Unexpectedly, it turns out that the ω-problem for a topological space is closely connected with the notion of resolvability. A dense in itself topological space is called *resolvable* if it contains two disjoint dense subsets, [H].

The purpose of this paper is to present results concerning the classical ω-problem and some new generalizations.

1. Preliminaries

First, we give the basic definitions and properties which will be used throughout the paper. We will use standard notations. In particular, the set of positive integers, the set of rational numbers, and the set of real numbers are denoted by $\mathbb{N}$, $\mathbb{Q}$, and $\mathbb{R}$, respectively.

$(X, \mathcal{T})$ will always denote a topological space with its family of open subsets. Throughout, we consider only T_1 topological spaces. For each $A \subset X$ we use A^d, $\mathrm{int}(A)$, $\mathrm{cl}(A)$, and $\mathrm{Fr}(A)$ to denote the derived set, interior, closure, and boundary of A, respectively. Capital calligraphic letters usually denote families or classes of sets.

Let $(X, \mathcal{T})$ be any T_1 topological space. Given a function $F \colon X \to \mathbb{R}$ and $x \in X$, we define

$$\omega(F, x) = \inf_U \sup_{x_1, x_2 \in U} |F(x_1) - F(x_2)|,$$
$$M_F(x) = \inf_U \sup_{x_1 \in U} F(x_1),$$
$$m_F(x) = \sup_U \inf_{x_1 \in U} F(x_1)$$

where $\inf_U$ and $\sup_U$ are taken over all neighborhoods U of x. The value $\omega(F, x)$ is called the *oscillation of F at x*, whereas $M_F(x)$ and $m_F(x)$ are called the *upper Baire function of F at x* and the *lower Baire function of F at x*, respectively. It is well known that $\omega(F, \cdot)$ and M_F are upper semicontinuous functions and m_F is lower semicontinuous. Moreover, F is continuous at $x_0 \in X$ if and only if $\omega(F, x_0) = 0$. In general, $\omega(F, \cdot)$ is nonnegative and vanishes at all isolated points of X, and

$$\omega(F, x) = M_F(x) - m_F(x)$$

for each $x \in X$.

Definition 1.1. Let $(X, \mathcal{T})$ be a topological space and $f \colon X \to [0, +\infty]$ an upper semicontinuous function vanishing at all isolated points of X. If there exists a function $F \colon X \to \mathbb{R}$ such that $\omega(F, \cdot) = f$, then we call F an *ω-primitive* for f.

By the 'ω-problem' on a topological space X, we mean the question of whether there always exists an ω-primitive for any given upper semicontinuous function vanishing at isolated points of X. The existence of an ω-primitive has been extensively studied in recent years [DGP], [EP1], [EP2], [EP3], [DV], [Ko1], [Ko2]. It is worth mentioning that one has to distinguish between the cases of f finite and f taking the value ∞, and also between X dense in itself and X having isolated points.

The complete solution of the ω-problem for metrizable spaces was obtained in 2001 [EP2].

As already mentioned, for nonmetrizable spaces, a complete solution is still unknown. Moreover, it turns out that the ω-problem for a topological space is closely connected to the notion of resolvability.

Definition 1.2 ([H]). A dense in itself topological space is called *resolvable* if it contains two disjoint dense subsets.

There are many papers concerning various aspects of resolvability and irresolvability (for example [V], [Pa], [F], [FM], [A], [BMM], [Ce], [El1], [El2], [FL], [G], [GRV]). It is known that, provided X is dense in itself, X metrizable [S], or first countable [H], or locally compact [H], implies X is resolvable. On the other hand, for each dense in itself completely regular topological space $(X, \mathcal{T})$ there exists a topology T^* on X such that $T \subset T^*$ and (X, T^*) is completely regular and not resolvable [H]. In [EP3] it was shown that if X is irresolvable then no upper semicontinuous function $f\colon X \to (0,\infty)$ has an ω-primitive.

In Chapter 2 we investigate properties of the operator ω^{-1}. These properties help solve the ω-problem for continuous functions. It turns out that resolvability is a necessary and sufficient condition for the existence of an ω-primitive for a finite continuous function defined on a dense in itself topological space. Moreover, a full description of the set $\{F\colon X \to \mathbb{R} : \omega(F,\cdot) = f\}$, where $f\colon X \to [0,+\infty)$ is a continuous function, is given. Next, these results are extended to the case of topological spaces with isolated points and to the case of continuous functions taking the value ∞. Moreover, some examples of irresolvable spaces are given in this chapter.

Chapter 3 contains some new sufficient conditions for the solution of the ω-problem. The ω-problem is solved for Baire dense in itself spaces and for separable first countable and completely regular spaces, as well as for some other cases. Moreover, it is proven that each topological space is homeomorphic to a closed subset of a topological space (Y,τ) such that each upper semicontinuous function $f\colon Y \to [0,+\infty]$ has an ω-primitive. If we would like to get similar results for functions taking the value ∞, we need to assume that X is perfectly normal. It is shown that this assumption cannot be omitted. There exists a separable completely regular first countable space $(X,\mathcal{T})$ and an upper semicontinuous function $f\colon X \to [0,+\infty]$ which has no ω-primitive. In the theory of the ω-problem it is useful to define a property of topological spaces which is similar to resolvability. A topological space is *regularly resolvable* if it contains a σ-discrete and F_σ dense subset.

Chapter 4 is devoted to the ω-problem for functions defined on a massive topological space. A topological space is *massive* if no nonempty open set is σ-discrete. In such spaces, an ω-primitive for f has the simple form $F = f\cdot\chi_A$, where χ_A is a characteristic function of some set $A \subset X$. Next, the results are extended to the case of functions taking the value ∞ (certainly, each massive space has to be dense in itself).

In Chapter 5 we discuss the $\omega^\star$-problem. For a given function $F\colon X \to \mathbb{R}$ and $x \in X$ we define

$$\omega^\star(F,x) = \inf_U \sup_{x_1,x_2\in U\setminus\{x\}} |F(x_1) - F(x_2)|,$$

where the infimum is taken over all neighborhoods of x and $(X,\mathcal{T})$ is a dense in itself topological space. Some results concerning the $\omega^\star$-problem were shown in [Ko2]. In the present paper we solve the $\omega^\star$-problem for a metric space. More precisely, if (X,ϱ) is a metric space, then for each upper semicontinuous function $f\colon X \to [0,\infty]$, there exists $F\colon X \to \mathbb{R}$ such that $\omega^\star(F,\cdot) = f$. Moreover, F can be chosen so that $0 \le F \le f$. On the other hand, the $\omega^\star$-problem for nonmetrizable spaces seems to be more difficult. For example, it will be shown that there exists an upper semicontinuous function with finite values defined on the Niemytzki plane which has no $\omega^\star$-primitive.

In the last chapter we study the ω-problem for functions $f\colon (X,\mathcal{T}) \to (Y,\varrho)$, where (Y,ϱ) is a metric space and $\omega(f,x) = \inf_U \sup_{x_1,x_2\in U} \varrho(f(x_1), f(x_2))$. It is proven that if the space Y contains a subset 'similar' to the real line, then the ω-problem can be studied exactly as in the case of real-valued functions.

Proofs are included, as usual, when the result is not known or the proof is simpler than the known one. Otherwise, the reader is referred to the corresponding papers.

2. Properties of the operator ω^{-1}

In this chapter we will describe the set $\omega^{-1}(f) = \{F\colon X \to \mathbb{R} : \omega(F,\cdot) = f\}$ for a continuous function $f\colon X \to [0,+\infty)$. This characterization gives a necessary condition which the topological space $(X,\mathcal{T})$ has to possess for the existence of an ω-primitive for any upper semicontinuous function $f\colon X \to \mathbb{R}$.

Lemma 2.1. *Let $(X,\mathcal{T})$ be a topological space. Suppose that $f\colon X \to [0,\infty]$ is continuous and $F\colon X \to \mathbb{R}$ is an arbitrary function. If there exist functions $g,h\colon X \to \mathbb{R}$ such that $F = g + h$, g is continuous, $0 \le h \le f$, and for each $\varepsilon > 0$ the sets $\{x \in X : h(x) < \varepsilon\}$ and*

$$\{x \in X : (f(x) \neq \infty \wedge h(x) > f(x) - \varepsilon) \vee (f(x) = \infty \wedge h(x) > 1/\varepsilon)\}$$

are dense in X, then $\omega(F,\cdot) = f$.

Proof. Assume that $F = g + h$, g is continuous, $0 \le h \le f$, and for each $\varepsilon > 0$ the sets $\{x \in X : h(x) < \varepsilon\}$ and

$$\{x \in X : (f(x) \neq \infty \wedge h(x) > f(x) - \varepsilon) \vee (f(x) = \infty \wedge h(x) > 1/\varepsilon)\}$$

are dense in X. By the continuity of g, we have $\omega(F,\cdot) = \omega(h,\cdot)$. Since $0 \le h \le f$ and f is continuous, we conclude that $m_h \ge 0$ and $M_h \le f$. Finally, let $x_0 \in X$ be an arbitrary point. Fix any $\varepsilon > 0$ and a neighborhood U of x_0. Since $\{x \in X : h(x) < \varepsilon\}$ is dense, $\inf_{x\in U} h(x) < \varepsilon$. If $f(x_0) \neq +\infty$, then, by the continuity of f at x_0, there exists a neighborhood V of x_0 such that $f(t) < +\infty$ for $t \in V$. Since the set

$$\{x \in X : (f(x) \neq \infty \wedge h(x) > f(x) - \varepsilon) \vee (f(x) = \infty \wedge h(x) > 1/\varepsilon)\}$$

is dense, there exists $y \in U \cap V$ for which $h(y) > f(x_0) - \varepsilon$. Similarly, if $f(x_0) = +\infty$ then, by the continuity of f at x_0, there exists a neighborhood V of x_0 such that $f(x) > 1/\varepsilon$ for all $x \in V$. Since the set

$$\{x \in X : (f(x) \neq \infty \wedge h(x) > f(x) - \varepsilon) \vee (f(x) = \infty \wedge h(x) > 1/\varepsilon)\}$$

is dense, there exists $y \in U \cap V$ for which $f(y) > 1/\varepsilon$. Thus $\sup_{x\in U\cap V} h(x) > f(x_0) - \varepsilon$ if $f(x_0) < +\infty$ and $\sup_{x\in U\cap V} h(x) > 1/\varepsilon$ if $f(x_0) = +\infty$. It follows that $m_h(x_0) \le \varepsilon$, $M_h(x_0) \ge f(x_0) - \varepsilon$ when $f(x_0) \neq +\infty$ and $M_h(x_0) \ge 1/\varepsilon$ when $f(x_0) = +\infty$ for each $\varepsilon > 0$. Therefore $m_h(x_0) \le 0$ and $M_h(x_0) \ge f(x_0)$. Since x_0 is an arbitrary point of X, $m_h \le 0$ and $M_h \ge f$. Therefore $m_h = 0$ and $M_h = f$. Hence $\omega(h,\cdot) = f$. It follows that $\omega(F,\cdot) = f$. ∎

If we consider only functions with finite values, we get the following corollary.

Corollary 2.1. *Let $(X,\mathcal{T})$ be a topological space. Suppose that $f\colon X \to [0,+\infty)$ is continuous and $F\colon X \to \mathbb{R}$ is an arbitrary function. If there exist functions $g,h\colon X \to \mathbb{R}$ such that $F = g + h$, g is continuous, $0 \le h \le f$, and for each $\varepsilon > 0$ the sets $\{x \in X : h(x) < \varepsilon\}$ and $\{x \in X\colon h(x) > f(x) - \varepsilon\}$ are dense in X, then $\omega(F,\cdot) = f$.*

The proof of the next lemma is very similar to the proof of Lemma 2.1, so we omit it.

LEMMA 2.2. *Let $(X, \mathcal{T})$ be a topological space. Suppose that $f\colon X \to [0, +\infty]$ is continuous and $F\colon X \to \mathbb{R}$ is an arbitrary function. If there exist functions $g, h\colon X \to \mathbb{R}$ such that $F = g + h$, g is continuous, $-f \le h \le 0$, and for each $\varepsilon > 0$ the sets $\{x \in X : h(x) > -\varepsilon\}$ and*

$$\{x \in X : (f(x) \neq +\infty \wedge h(x) < -f(x) + \varepsilon) \vee (f(x) = +\infty \wedge h(x) < -1/\varepsilon)\}$$

are dense in X, then $\omega(F, \cdot) = f$.

COROLLARY 2.2. *Let $(X, \mathcal{T})$ be a topological space. Assume that $f\colon X \to [0, +\infty)$ is continuous and $F\colon X \to \mathbb{R}$ is an arbitrary function. If there exist functions $g, h\colon X \to \mathbb{R}$ such that $F = g + h$, g is continuous, $-f \le h \le 0$, and for each $\varepsilon > 0$ the sets $\{x \in X : h(x) > -\varepsilon\}$ and $\{x \in X\colon h(x) < f(x) + \varepsilon\}$ are dense in X, then $\omega(F, \cdot) = f$.*

LEMMA 2.3. *Let $(X, \mathcal{T})$ be a topological space and suppose that $F\colon X \to \mathbb{R}$ is such that m_F is finite and continuous. Put $f = \omega(F, \cdot)$ and $h = F - m_F$. Then $\omega(h, \cdot) = f$, $0 \le h \le f$, and for each $\varepsilon > 0$ the sets $\{x \in X : h(x) < \varepsilon\}$ and*

$$\{x \in X : (f(x) \neq +\infty \wedge h(x) > f(x) - \varepsilon) \vee (f(x) = +\infty \wedge h(x) > 1/\varepsilon)\}$$

are dense in X.

Proof. Since m_F is continuous, $M_h = M_{F-m_F} = M_F - m_F = f$ and $m_h = m_{F-m_F} = m_F - m_F = 0$. It follows that $\omega(h, \cdot) = f - 0 = f$. Moreover, $h \le M_h = f$ and $h \ge m_h = 0$. Hence $0 \le h \le f$.

Fix $\varepsilon > 0$. Let $U \in \mathcal{T}$ be an arbitrary nonempty open set. Since $m_h = 0$, we can find $y_1 \in U$ such that $h(y_1) < \varepsilon$. Moreover, either $f(x) = +\infty$ for all $x \in U$ or there exists an open set $V \subset U$ such that $f(x) \neq \infty$ for all $x \in V$. Take any $x_0 \in U$ in the first case or any $x_0 \in V$ in the second one. First, assume that $f(x_0) < \infty$. Then, by upper semicontinuity of f, there exists a neighborhood W of x_0 such that $f(x) < f(x_0) + \varepsilon$ for all $x \in W$. Moreover, $M_h(x_0) = f(x_0)$. Therefore we can find $y_3 \in U \cap W$ such that $h(y_3) > f(x_0) - \varepsilon/2$.

Finally, assume that $f(x) = \infty$ for all $x \in V$. Then there exists $y_2 \in V$ such that $h(y_2) > 1/\varepsilon$.

Thus we have proven that $U \cap \{x \in X : h(x) < \varepsilon\} \neq \emptyset$ and

$$U \cap \{x\colon (f(x) \neq +\infty \wedge h(x) > f(x) - \varepsilon) \vee (f(x) = +\infty \wedge h(x) > 1/\varepsilon)\} \neq \emptyset.$$

Since U is an arbitrary nonempty open set, $\{x \in X : h(x) < \varepsilon\}$ and

$$\{x \in X : (f(x) \neq +\infty \wedge h(x) > f(x) - \varepsilon) \vee (f(x) = +\infty \wedge h(x) > 1/\varepsilon)\}$$

are dense in X. ∎

The proof of the next lemma is analogous and we omit it.

LEMMA 2.4. *Let $(X, \mathcal{T})$ be a topological space and suppose that $F\colon X \to \mathbb{R}$ is such that M_F is finite and continuous. Put $\omega(F, \cdot) = f$ and $h = F - M_F$. Then $\omega(h, \cdot) = f$, $-f \le h \le 0$, and for each $\varepsilon > 0$ the sets $\{x \in X : h(x) > -\varepsilon\}$ and*

$$\{x \in X : (f(x) \neq +\infty \wedge h(x) < -f(x) + \varepsilon) \vee (f(x) = +\infty \wedge h(x) < -1/\varepsilon)\}$$

are dense in X.

Corollary 2.3. *Let $(X, \mathcal{T})$ be a topological space and $f\colon X \to [0, +\infty)$ an upper semicontinuous function. If there exists $F\colon X \to \mathbb{R}$ for which $\omega(F, \cdot) = f$ and at least one of the functions M_F or m_F is continuous, then there exists $h\colon X \to \mathbb{R}$ such that $\omega(h, \cdot) = f$ and $0 \le h \le f$.*

Proof. If m_F is continuous, we may use Lemma 2.3 and the proof is complete. If M_F is continuous, then by Lemma 2.4 there exists $\widetilde{F}\colon X \to \mathbb{R}$ such that $\omega(\widetilde{F}, \cdot) = f$ and $-f \le \widetilde{F} \le 0$. Then it is sufficient to take $F = -\widetilde{F}$. ■

Lemma 2.5. *Let $(X, \mathcal{T})$ be a topological space and $f\colon X \to [0, +\infty)$ a continuous function. If $F\colon X \to \mathbb{R}$ and $\omega(F, \cdot) = f$, then the functions M_F and m_F are continuous too.*

Proof. Let $\omega(F, \cdot) = f$. Then $M_F = f + m_F$, because f, M_f and m_f are finite. Since M_F is upper semicontinuous and $f + m_f$ is lower semicontinuous, this equality implies that the both M_F and $f + m_f$ are continuous. Therefore m_F and M_F are continuous. ■

The next example shows that Lemma 2.5 does not hold for a continuous function $f\colon X \to [0, +\infty]$ with infinite values, even if it takes the value $+\infty$ only at a single point.

Example 2.1. Define $F\colon X \to \mathbb{R}$ as follows:

$$F(x) = \begin{cases} 1/x & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}, \\ 0 & \text{if } x \in \mathbb{Q}. \end{cases}$$

Then

$$f(x) = \omega(F, x) = \begin{cases} 1/|x| & \text{if } x \neq 0, \\ +\infty & \text{if } x = 0, \end{cases}$$

is continuous. But the functions

$$M_F(x) = \begin{cases} 0 & \text{if } x \in (-\infty, 0), \\ 1/x & \text{if } x \in (0, +\infty), \\ +\infty & \text{if } x = 0, \end{cases} \quad \text{and} \quad m_F(x) = \begin{cases} 1/x & \text{if } x \in (-\infty, 0), \\ 0 & \text{if } x \in (0, +\infty), \\ -\infty & \text{if } x = 0, \end{cases}$$

are discontinuous at 0.

Applying the lemmas we get the following theorem, which characterizes the set $\omega^{-1}(f) = \{F\colon X \to \mathbb{R} : \omega(F, \cdot) = f\}$ for a continuous function $f\colon X \to [0, +\infty)$.

Theorem 2.1. *Let $(X, \mathcal{T})$ be a topological space and $f\colon X \to [0, +\infty)$ a continuous function. For any $F\colon X \to \mathbb{R}$, the following conditions are equivalent:*

(1) *$\omega(F, \cdot) = f$,*
(2) *m_F is continuous and if we put $h = F - m_F$, then $0 \le h \le f$ and for any $\varepsilon > 0$ the sets*

$$\{x \in X : h(x) < \varepsilon\} \quad \textit{and} \quad \{x \in X : h(x) > f(x) - \varepsilon\}$$

are dense in X,

(3) *M_F is continuous and if we put $h = F - M_F$, then $-f \le h \le 0$ and for each $\varepsilon > 0$ the sets*

$$\{x \in X : h(x) > -\varepsilon\} \quad \textit{and} \quad \{x \in X : h(x) < -f(x) + \varepsilon\}$$

are dense in X,

(4) *there exist* $g, h \colon X \to \mathbb{R}$ *such that* $F = g + h$, g *is continuous,* $0 \le h \le f$, *and for each* $\varepsilon > 0$ *the sets*

$$\{x \in X : h(x) < \varepsilon\} \quad \textit{and} \quad \{x \in X : h(x) > f(x) - \varepsilon\}$$

are dense in X.

(5) *there exist* $g, h \colon X \to \mathbb{R}$ *such that* $F = g + h$, g *is continuous,* $-f \le h \le 0$, *and for each* $\varepsilon > 0$ *the sets*

$$\{x \in X : h(x) > -\varepsilon\} \quad \textit{and} \quad \{x \in X : h(x) < -f(x) + \varepsilon\}$$

are dense in X.

Proof. Implication (1)⇒(2) follows from Lemma 2.5 and Lemma 2.3. Similarly, (1)⇒(3) follows from Lemmas 2.5 and 2.4. Implications (2)⇒(4) and (3)⇒(5) are obvious. Finally, (4)⇒(1) follows from Lemma 2.1, and (5)⇒(1) from Lemma 2.2. ■

REMARK 2.1. If $\omega(F, \cdot) = f$ or if for each $\varepsilon > 0$ the sets

$$\{x \in X : h(x) < \varepsilon\} \quad \text{and} \quad \{x \in X : h(x) > f(x) - \varepsilon\}$$

are dense in X, then f vanishes at each isolated point of X.

The next example shows that if f is not continuous and $\omega(F, \cdot) = f$, then the equalities $\omega(F - m_F, \cdot) = \omega(F, \cdot)$ and $\omega(F - M_F, \cdot) = \omega(F, \cdot)$ need not hold.

EXAMPLE 2.2. For any interval $I = (a, b)$, define $g_I \colon I \to [-1, 1]$ by $g_I(x) = \sin \frac{1}{(x-a)(b-x)}$. Then g_I is continuous. Moreover,

$$\limsup_{x \to a^+} g_I(x) = \limsup_{x \to b^-} g_I(x) = 1,$$
$$\liminf_{x \to a^+} g_I(x) = \liminf_{x \to b^-} g_I(x) = -1.$$

Let $A \subset \mathbb{R}$ be any closed, dense in itself, and nowhere dense set. Let $\{I_n = (a_n, b_n) : n \in \mathbb{N}\}$, be the set of all connected components of the complement of A. Let $g \colon A \to [-1, 1]$ be any function. Finally, let us define two more functions $f \colon \mathbb{R} \to [0, +\infty)$, $F \colon \mathbb{R} \to \mathbb{R}$ by putting

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{R} \setminus A, \\ 2 & \text{if } x \in A, \end{cases} \qquad F(x) = \begin{cases} g_{I_n}(x) & \text{if } x \in I_n \text{ and } n \in \mathbb{N}, \\ g(x) & \text{if } x \in A. \end{cases}$$

Then it is easy to check that

$$M_F(x) = \begin{cases} g_{I_n}(x) & \text{if } x \in I_n \text{ and } n \in \mathbb{N}, \\ 1 & \text{if } x \in A, \end{cases} \qquad m_F(x) = \begin{cases} g_{I_n}(x) & \text{if } x \in I_n \text{ and } n \in \mathbb{N}, \\ -1 & \text{if } x \in A. \end{cases}$$

Therefore $\omega(F, \cdot) = f$. On the other hand,

$$(F - M_F)(x) = \begin{cases} 0 & \text{if } x \in \bigcup_{n=1}^{\infty} I_n, \\ g(x) - 1 & \text{if } x \in A, \end{cases}$$

$$(F - m_F)(x) = \begin{cases} 0 & \text{if } x \in \bigcup_{n=1}^{\infty} I_n, \\ g(x) + 1 & \text{if } x \in A. \end{cases}$$

Thus

$$\omega(F - M_F, x) = \begin{cases} 0 & \text{if } x \in \bigcup_{n=1}^{\infty} I_n, \\ 1 - m_g(x) & \text{if } x \in A, \end{cases}$$

$$\omega(F - m_F, x) = \begin{cases} 0 & \text{if } x \in \bigcup_{n=1}^{\infty} I_n, \\ M_g(x) + 1 & \text{if } x \in A. \end{cases}$$

Hence $\omega(F, \cdot) \neq \omega(F - M_F, \cdot)$ and $\omega(F, \cdot) \neq \omega(F - m_F, \cdot)$. Moreover, $\omega(F - M_F, \cdot)$ and $\omega(F - m_F, \cdot)$ depend on an arbitrary function g defined on A, whereas $\omega(F, \cdot) = f$ does not change. This shows that the characterization of an ω-primitive for f as in Theorem 2.1 is no longer true.

By Theorem 2.1, we get a necessary condition for the existence of an ω-primitive for all upper semicontinuous functions defined on a dense in itself topological space $(X, \mathcal{T})$. To formulate it, we need the notion of a *resolvable space*, introduced in [H].

Definition 2.1 ([H]). A topological space $(X, \mathcal{T})$ is said to be *resolvable* if it is dense in itself and contains two disjoint sets which are dense.

Theorem 2.2. *Suppose that $(X, \mathcal{T})$ is a topological space dense in itself, $\delta > 0$, and $f\colon X \to [\delta, +\infty)$ is a continuous function. If there exists a function $F\colon X \to \mathbb{R}$ such that $\omega(F, \cdot) = f$, then X is resolvable.*

Proof. By the assumptions and by Theorem 2.1, there exists $h\colon X \to \mathbb{R}$ for which $\omega(h, \cdot) = f$ and for any $\varepsilon > 0$ the sets $\{x \in X : h(x) < \varepsilon\}$ and $\{x \in X \colon h(x) > f(x) - \varepsilon\}$ are dense in X. It is sufficient to put $A = \{x \in X : h(x) < \delta/2\}$ and

$$B = \{x \in X : h(x) > f(x) - \delta/2\}.$$

Then A and B are disjoint and dense in X. ■

We have proven that if every upper semicontinuous function $f\colon X \to [0, +\infty)$ defined on a dense in itself topological space X has an ω-primitive, then X is resolvable. Now, we shall show that the resolvability of X is a sufficient condition for the existence of an ω-primitive for each continuous function $f\colon X \to \mathbb{R}$.

Theorem 2.3. *Let $(X, \mathcal{T})$ be a topological space dense in itself. The following conditions are equivalent:*

(1) *X is resolvable,*
(2) *for each continuous function $f\colon X \to [0, +\infty)$ there exists $F\colon X \to \mathbb{R}$ such that $\omega(F, \cdot) = f$,*
(3) *there exist $\delta > 0$ and a continuous function $f\colon X \to [\delta, +\infty)$ that has an ω-primitive.*

Proof. First, assume that X is resolvable. Let A be a dense subset of X such that $X \setminus A$ is dense too. Then, by Lemma 2.1, for each continuous $f\colon X \to \mathbb{R}$, the function $F\colon X \to \mathbb{R}$,

$$F(x) = \begin{cases} 0 & \text{if } x \in A, \\ f(x) & \text{if } x \in X \setminus A, \end{cases}$$

satisfies the equation $\omega(F, \cdot) = f$.

Implication (2)⇒(3) is obvious and (3)⇒(1) follows directly from Theorem 2.2. ■

A similar theorem is true for a space with isolated points.

THEOREM 2.4. *Let $(X,\mathcal{T})$ be a topological space. Assume that any upper semicontinuous function $g\colon X\to[0,+\infty]$ vanishing at all isolated points of X has an ω-primitive. Then there exist A and B, disjoint subsets of X, such that the sets $A\cup(X\setminus X^d)$ and $B\cup(X\setminus X^d)$ are dense in X.*

Proof. If $X^d=\emptyset$ or $X^d=X$, then there is nothing to prove. Assume that $X^d\neq\emptyset$ and $X\setminus X^d\neq\emptyset$. Define a function $f\colon X\to[0,+\infty)$ by putting

$$f(x)=\begin{cases}1 & \text{if } x\in X^d,\\ 0 & \text{if } x\in X\setminus X^d.\end{cases}$$

Since X^d is a closed subset of X, f is upper semicontinuous. By assumption, there exists a function $F\colon X\to\mathbb{R}$ such that $\omega(F,\cdot)=f$. On the other hand, $Y=\operatorname{int}(X^d)$ is an open dense in itself subset of X. Therefore if we put $\tilde f=f_{|Y}$ and $\tilde F=F_{|Y}$, we easily get $\omega(\tilde F,\cdot)=\tilde f$. By the continuity of $\tilde f$ and Theorem 2.1, there exists $h\colon X\to\mathbb{R}$ such that $\omega(h,\cdot)=\tilde f=1$ and the sets $A=\{x\in Y: h(x)<1/2\}$ and $B=\{x\in Y\colon h(x)>1/2\}$ are dense in Y. Since A and B are disjoint and dense in $Y=\operatorname{int}(X^d)$, we deduce that $A\cup(X\setminus X^d)$ and $B\cup(X\setminus X^d)$ are dense in X. ∎

THEOREM 2.5. *Let $(X,\mathcal{T})$ be a topological space. Then the following conditions are equivalent:*

(1) *there exist disjoint subsets A and B of X such that both $A\cup(X\setminus X^d)$ and $B\cup(X\setminus X^d)$ are dense in X,*
(2) *for each continuous function $f\colon X\to[0,+\infty)$ vanishing at all isolated points of X there exists a function $F\colon X\to\mathbb{R}$ such that $\omega(F,\cdot)=f$.*

Proof. Without loss of generality, we may assume that $X^d\neq\emptyset$ and $X\setminus X^d\neq\emptyset$.

First, assume that there exist disjoint sets A and B such that $A\cup(X\setminus X^d)$ and $B\cup(X\setminus X^d)$ are dense in X. For each continuous $f\colon X\to\mathbb{R}$ define $F\colon X\to\mathbb{R}$ by

$$F(x)=\begin{cases}0 & \text{if } x\in A,\\ f(x) & \text{if } x\in X\setminus A,\end{cases}$$

(If $A=\emptyset$, then obviously $F=f$.) Put $C=(X\setminus A)\cup(X\setminus X^d)$. Then $B\cup(X\setminus X^d)\subset C$ and C is a dense subset of X. Since $F_{|C}=f_{|C}$ and f is continuous, we have $M_F=f$. Moreover, $F(x)=0$ for all $x\in A\cup(X\setminus X^d)$ and $A\cup(X\setminus X^d)$ is a dense subset of X. Therefore $m_F=0$. Thus $\omega(F,\cdot)=f$.

The reverse implication follows immediately from Theorem 2.4. ∎

We have shown that in each dense in itself irresolvable topological space, not every upper semicontinuous function has an ω-primitive. Moreover, by Theorem 2.2, in such a space even a constant and positive function has no ω-primitive.

All ‘standard’ topological spaces are resolvable. In [H], some methods of construction of dense in themselves irresolvable topological spaces were given. Based on these methods, we will present some properties of irresolvable spaces. We start from two technical lemmas.

Lemma 2.6 ([H]). *If A is a dense subset of a dense in itself topological space $(X,\mathcal{T})$, then for each nonempty open set $U\in\mathcal{T}$, the intersection $U\cap A$ is an infinite set.*

Proof. Suppose, to the contrary, that for some nonempty open set $U\in\mathcal{T}$ the intersection $U\cap A$ is a finite set, and write $U\cap A=\{x_1,\dots,x_n\}$. Since X is a T_1 space, for each $k\in\{2,3,\dots,n\}$ there exists an open set U_k such that $x_1\in U_k$ and $x_k\notin U_k$. Let $U=U_2\cap U_3\cap\dots\cap U_k$. Then U is a nonempty open set and $U\cap A=\{x_1\}$. Since X is dense in itself, we can find $y\in U\setminus\{x_1\}$. Again, by the T_1 axiom, there exists an open set V such that $y\in V$ and $x_1\notin V$. Hence $V\cap U$ is a nonempty open set and $(V\cap U)\cap A=\emptyset$, which is impossible because A is dense. Therefore our assumption is false and for each nonempty open set $U\in\mathcal{T}$ the intersection $U\cap A$ is an infinite set. ∎

Lemma 2.7. *Let $(X,\mathcal{T})$ be a Tychonoff topological space and let A be any nonempty subset of X. Then the family*

$$\{(U\cap A)\cup(V\setminus A):U,V\in\mathcal{T}\}$$

is a topology τ on the set X such that $\mathcal{T}\subset\tau$ and (X,τ) is also a Tychonoff space.

Moreover, if both A and $X\setminus A$ are dense in $(X,\mathcal{T})$, and if $(X,\mathcal{T})$ is dense in itself, then (X,τ) is dense in itself.

Proof. It follows that $\{(U\cap A)\cup(V\setminus A):U,V\in\mathcal{T}\}$ is a topology on X, since

$$[(U_1\cap A)\cup(V_1\setminus A)]\cap[(U_2\cap A)\cup(V_2\setminus A)]=[(U_1\cap U_2)\cap A]\cup[(V_1\cap V_2)\setminus A]$$

and

$$\bigcup_{s\in S}[(U_s\cap A)\cup(V_s\setminus A)]=\Big(\bigcup_{s\in S}U_s\cap A\Big)\cup\Big(\bigcup_{s\in S}V_s\setminus A\Big)$$

for any $U_1,U_2,U_s,V_s,V_1,V_2\in\mathcal{T}$. Moreover, the definition of τ implies $\mathcal{T}\subset\tau$, because

$$U=(U\cap A)\cup(U\setminus A)\quad\text{ for } U\in\mathcal{T}.$$

Hence (X,τ) is a T_1 space.

Let $f\colon X\to[0,1]$ be any function which is continuous with respect to $\mathcal{T}$. We will show that $\tilde f\colon X\to[0,1]$ defined by

$$\tilde f(x)=\begin{cases}f(x) & \text{if } x\in A,\\ 0 & \text{if } x\in X\setminus A,\end{cases}$$

is continuous with respect to τ. In order to prove this, observe that

$$\{x\in X:\tilde f(x)<a\}=\begin{cases}\emptyset & \text{if } a\le 0,\\ \{x\in X:f(x)<a\}\cup(X\setminus A) & \text{if } 0<a\le 1,\\ X & \text{if } 1<a,\end{cases}$$

$$\{x\in X:\tilde f(x)>a\}=\begin{cases}X & \text{if } a<0,\\ \{x\in X:f(x)>a\}\cap A & \text{if } 0\le a<1,\\ \emptyset & \text{if } 1\le a,\end{cases}$$

By the continuity of f with respect to $\mathcal{T}$ and by the definition of τ, it follows that all preimages of halflines belong to τ, because the sets $A = (X \cap A) \cup (\emptyset \setminus A)$ and $X \setminus A = (\emptyset \cap A) \cup (X \setminus A)$ belong to τ. Therefore $\tilde{f}$ is continuous with respect to τ.

Let $U \in \mathcal{T}$ be any nonempty open set and $x_0 \in U \cap A$. Since $(X, \mathcal{T})$ is a Tychonoff space, there exists a function $f\colon X \to [0,1]$ continuous with respect to $\mathcal{T}$ such that $f(x_0) = 1$ and $f(x) = 0$ for all $x \in X \setminus U$. Let us define $\tilde{f}\colon X \to [0,1]$ by

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in A, \\ 0 & \text{if } x \in X \setminus A. \end{cases}$$

We have proven that $\tilde{f}$ is continuous with respect to τ. Moreover, $\tilde{f}(x_0) = 1$ and $f(x) = 0$ for all $x \in X \setminus (U \cap A)$.

Similarly, if $U \in \mathcal{T}$ is any nonempty open set and $x_0 \in U \setminus A$, then we can construct a function $f\colon X \to [0,1]$ continuous with respect to $\mathcal{T}$ such that $f(x_0) = 1$ and $f(x) = 0$ for all $x \in X \setminus U$. Again, define $\tilde{f}\colon X \to [0,1]$ by

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in X \setminus A, \\ 0 & \text{if } x \in A. \end{cases}$$

Analogously to the previous case, we can show that $\tilde{f}$ is continuous with respect to τ, $\tilde{f}(x_0) = 1$ and $f(x) = 0$ for all $x \in X \setminus (U \setminus A)$.

Since $\{U \cap A : U \in \mathcal{T}\} \cup \{U \setminus A : U \in \mathcal{T}\}$ is a base of τ, we infer that (X, τ) is a Tychonoff space.

Finally, assume that A and $X \setminus A$ are each dense in the space $(X, \mathcal{T})$, which is dense in itself. By Lemma 2.6, the sets $U \cap A$ and $V \setminus A$ for all $U, V \in \mathcal{T}$, $U \neq \emptyset \neq V$ are infinite. Hence all nonempty elements of the base of τ are infinite. This proves that (X, τ) has no isolated point. ∎

Now we can present a construction of a Tychonoff irresolvable topological space. More precisely, we will prove the following:

THEOREM 2.6. *For any dense in itself Tychonoff topological space $(X, \mathcal{T})$, there exists a topology γ on X such that $\mathcal{T} \subset \gamma$ and (X, γ) is a dense in itself Tychonoff irresolvable space.*

Proof. Let $(X, \mathcal{T})$ be any dense in itself Tychonoff topological space. Let $\mathbb{A}$ be the family of all topologies on X such that if $\tau \in \mathbb{A}$, then (X, τ) is a dense in itself Tychonoff space and $\mathcal{T} \subset \tau$. It is obvious that the relation $\leq$ defined by

$$\tau_1 \leq \tau_2 \Leftrightarrow \tau_1 \subset \tau_2$$

is a partial order on $\mathbb{A}$. Let $\{\tau_s : s \in S\}$ be any chain in $(\mathbb{A}, \leq)$. Then $\{U \subset X : U \in \tau_s, s \in S\}$ is a base of some topology τ_0 on X.

For each $s \in S$ and $U \in \tau_s$, if $U \neq \emptyset$, then U is not a singleton. Hence (X, τ_0) has no isolated point. Moreover, for each $s \in S$, each $\emptyset \neq U \in \tau_s$, and each $x_0 \in U$, there exists a function $f\colon X \to [0,1]$ continuous in the topology τ_s such that $f(x_0) = 1$ and $f(x) = 0$ for $x \notin U$. It follows that f is continuous with respect to τ_0. Therefore (X, τ_0) is a dense in itself Tychonoff space. Obviously, $\mathcal{T} \subset \tau_0$. It follows that $\tau_0 \in \mathbb{A}$ and $\tau_s \leq \tau_0$ for $s \in S$.

By the Zorn Lemma, there exists a maximal element γ in the family $\mathbb{A}$. By definition of $(\mathbb{A}, \leq)$, we infer that (X, γ) is a dense in itself Tychonoff space and $\mathcal{T} \subset \gamma$. Moreover, if τ is a topology on X and $\gamma \subsetneq \tau$, then either (X, τ) is not dense in itself or (X, τ) is not a Tychonoff space.

We claim that (X, γ) is not resolvable. Suppose that there exist $A, B \subset X$ such that $A \cap B = \emptyset$ and $\mathrm{cl}_\gamma(A) = X = \mathrm{cl}_\gamma(B)$. Then $\mathrm{cl}_\gamma(X \setminus A) = X$. ($\mathrm{cl}_\gamma(B)$ denotes the closure of $B \subset X$ with respect to the topology γ). Let τ be a topology on X,

$$\tau = \{(U \cap A) \cup (V \setminus A) : U, V \in \gamma\}.$$

By Lemma 2.7, (X, τ) is a dense in itself Tychonoff space and $\gamma \subset \tau$. By definition of γ, it follows that $\gamma = \tau$. In particular, $A = (X \cap A) \cup (\emptyset \setminus A) \in \gamma$. Hence $A = A \cap \mathrm{cl}_\gamma(X \setminus A) = \emptyset$, which is impossible. ∎

Two important corollaries follow directly from Theorem 2.1.

COROLLARY 2.4. *Let X be a topological space. If for some continuous function $f\colon X \to [0, +\infty)$ there exists an ω-primitive $F\colon X \to \mathbb{R}$, then there also exists an ω-primitive $h\colon X \to \mathbb{R}$ for f satisfying the conditions $M_h = f$, $m_h = 0$.*

COROLLARY 2.5. *Let X be a topological space. If for some continuous function $f\colon X \to [0, +\infty)$ there exists an ω-primitive $F\colon X \to \mathbb{R}$, then there exists an ω-primitive $h\colon X \to \mathbb{R}$ for f satisfying the condition $0 \leq h \leq f$.*

The next example shows that for an upper semicontinuous function $f\colon X \to [0, +\infty)$, the last corollary is not generally true.

EXAMPLE 2.3. Let $X = \{(2k+1)/2^n : k \in \mathbb{Z},\ n = 0, 1, 2, \ldots\}$ be endowed with a metric from the natural metric in $\mathbb{R}$. Define $f\colon X \to [0, +\infty)$ by $f((2k+1)/2^n) = 1 + 1/2^n$ for $k \in \mathbb{Z}$ and $n = 0, 1, 2, \ldots$. Obviously, f is upper semicontinuous. Since X is a metric space, there exists a function $F\colon X \to \mathbb{R}$ such that $\omega(F, \cdot) = f$. Suppose that we can find a function $h\colon X \to \mathbb{R}$ for which $\omega(h, \cdot) = f$ and, moreover, $0 \leq h \leq f$. Since $M_h - m_h = f$, $m_h \geq 0$ and $M_h \leq f$, we conclude that $m_h = 0$ and $M_h = f$. Further, $f(x) > 1$ for $x \in X$ and each set $\{t \in X : f(t) \geq \alpha\}$ is closed and discrete for $\alpha > 1$. Therefore $\limsup_{t \to x} h(t) \leq \limsup_{t \to x} f(t) < f(x)$ for $x \in X$. It follows that $M_h(x) = f(x)$ if and only if $h(x) = f(x)$. Hence $h = f$. But it is easy to see that $\omega(f, \cdot) = f - 1$. The contradiction obtained proves that there is no function $h\colon X \to \mathbb{R}$ for which $\omega(h, \cdot) = f$ and $0 \leq h \leq f$.

In the next example we will show that even if for an upper semicontinuous function $f\colon X \to [0, +\infty)$ there exists an ω-primitive F satisfying the conditions $\omega(F, \cdot) = f$ and $0 \leq F \leq f$, the characterization of $\omega^{-1}(f)$ as in Theorem 2.1 need not be true.

EXAMPLE 2.4. Let $f\colon \mathbb{R} \to [0, +\infty)$ be defined by

$$f(x) = \begin{cases} 0 & \text{if } x \neq 0, \\ 2 & \text{if } x = 0. \end{cases}$$

Obviously, f is upper semicontinuous. Moreover, if $F\colon X \to \mathbb{R}$ satisfies $\omega(F, \cdot) = f$ and

$0 \le |F| \le f$, then $F = f$ or $F = -f$. For each $a \in [-1,1]$ we define $F_a\colon \mathbb{R} \to \mathbb{R}$ by

$$F_a(x) = \begin{cases} \sin \frac{1}{x} & \text{if } x \neq 0, \\ a & \text{if } x = 0. \end{cases}$$

Then for $a \in [-1,1]$ we have $\omega(F_a, \cdot) = f$ but F_a cannot be represented as the sum of a continuous function and f or $-f$.

Now let us return to Theorem 2.1 and consider the case of continuous functions taking infinite values. Example 2.1 shows that if $\omega(F,\cdot) = f$ and f has infinite values, then the characterization of the ω-primitive F for f as in Theorem 2.1 is not true. Nevertheless, we can prove quite similar properties of the ω-primitive for functions with infinite values.

Theorem 2.7. *Let $(X, \mathcal{T})$ be a topological space. Suppose that $f\colon X \to [0,+\infty]$ is a continuous function, $A = \{x \in X : f(x) = +\infty\} \neq \emptyset$ and $F\colon X \to \mathbb{R}$. Then the following conditions are equivalent:*

(1) $\omega(F,\cdot) = f$,
(2) *F can be represented in the form $F = g + h$, where $g, h\colon X \to \mathbb{R}$ have the following properties*
   - *the restriction of g to $X \setminus A$ is continuous,*
   - *the restriction of g to* $\operatorname{int}(A)$ *is continuous,*
   - $0 \le h(x) \le f(x)$ *for* $x \in X$,
   - *for each $\varepsilon > 0$, the sets $\{x \in X \setminus A : h(x) < \varepsilon\}$ and $\{x \in X \setminus A : h(x) > f(x) - \varepsilon\}$ are dense in $X \setminus A$ and $\{x \in A : |h(x)| > 1/\varepsilon\}$ is dense in* $\operatorname{int}(A)$.

Proof. (1)$\Rightarrow$(2). Suppose $\omega(F,\cdot) = f$. By the continuity of f, A is closed and $X \setminus A$ is open. Let $\widetilde{f}$ be the restriction of f to $X \setminus A$. By the openness of $X \setminus A$, we infer that $\omega(\widetilde{F},\cdot) = \widetilde{f}$, where $\widetilde{F}$ is the restriction of F to $X \setminus A$. Since $\widetilde{f}$ is continuous and takes only finite values, by Theorem 2.1 there exist $\widetilde{g}\colon X \setminus A \to \mathbb{R}$ and $\widetilde{h}\colon X \setminus A \to \mathbb{R}$ such that: $\widetilde{g}$ is continuous, $0 \le \widetilde{h} \le \widetilde{f}$, and for each $\varepsilon > 0$ the sets $\{x \in X \setminus A : \widetilde{h}(x) < \varepsilon\}$ and $\{x \in X \setminus A : \widetilde{h}(x) > \widetilde{f} - \varepsilon\}$ are dense in $X \setminus A$. Define $f\colon X \to \mathbb{R}$ and $g\colon X \to \mathbb{R}$ by

$$g(x) = \begin{cases} \widetilde{g}(x) & \text{if } x \in X \setminus A, \\ 0 & \text{if } x \in A, \end{cases} \qquad h(x) = \begin{cases} \widetilde{h}(x) & \text{if } x \in X \setminus A, \\ F(x) & \text{if } x \in A. \end{cases}$$

Then $F = g + h$, g is continuous on the set $X \setminus A$, constant on A, and $0 \le h(x) \le f(x)$ for $x \in X$. Fix an $\varepsilon > 0$. It is obvious that the sets

$$\{x \in X \setminus A : h(x) < \varepsilon\} = \{x \in X \setminus A : \widetilde{h}(x) < \varepsilon\}$$

and

$$\{x \in X \setminus A : h(x) > f(x) - \varepsilon\} = \{x \in X \setminus A : \widetilde{h}(x) > \widetilde{f} - \varepsilon\}$$

are dense in $X \setminus A$. Let U be any open set contained in $\operatorname{int}(A)$ and take any $x_0 \in U$. Since $h(x) = F(x)$ for $x \in A$ and $\omega(F, x_0) = f(x_0) = +\infty$, there exists $y \in U$ such that $|h(y)| > 1/\varepsilon$. Therefore $U \cap \{x \in A : h(x) > 1/\varepsilon\} = \emptyset$, so $\{x \in A : |h(x)| > 1/\varepsilon\}$ is dense in $\operatorname{int}(A)$.

(2)$\Rightarrow$(1). Now assume that F can be represented in the form $F = g + h$, where f and g satisfy all the conditions from 2). In particular, $X \setminus A$ is open, g is continuous on $X \setminus A$, and for each $\varepsilon > 0$ the sets $\{x \in X \setminus A : h(x) < \varepsilon\}$ and $\{x \in X \setminus A : h(x) > f(x) - \varepsilon\}$ are dense in $X \setminus A$. By Theorem 2.1, we infer that $\omega(F, x) = f(x)$ for $x \in X \setminus A$. Suppose $x_0 \in A$. Then $f(x_0) = +\infty$. Moreover, $\{x \in A : |h(x)| > 1/\varepsilon\}$ is dense in $\mathrm{int}(A)$ for all $\varepsilon > 0$ and g is continuous on $\mathrm{int}(A)$. It follows that if $x_0 \in \mathrm{int}(A)$, then $\omega(F, x_0) = \omega(h, x_0) = +\infty = f(x_0)$. Finally, consider the case where x_0 belongs to the boundary of A. Fix any real number $M > 0$. Since f is continuous at x_0 and $f(x_0) = +\infty$, there exists a neighborhood V of X such that $f(x) > 2M + 3$ for $x \in V$. Let U be any neighborhood of x_0 and let $y \in U \cap V \cap (X \setminus A)$. Since g is continuous at y, we can find a neighborhood $W \subset X \setminus A$ of y such that $|g(x) - g(y)| < 1$ if $x \in W$. Therefore $|g(u) - g(v)| < 2$ if $u, v \in W$. Moreover, $\{x \in X \setminus A : h(x) < 1\}$ and $\{x \in X \setminus A : h(x) > f(x) - 1\}$ are dense in $X \setminus A$ and $G = U \cap V \cap W \cap (X \setminus A)$ is an open subset of $X \setminus A$. It follows that there exist $x_1, x_2 \in G$ such that $h(x_1) < 1$ and $h(x_2) > f(x_2) - 1$. We get

$$|F(x_1) - F(x_2)| \geq |h(x_2) - h(x_1)| - |g(x_1) - g(x_2)| > f(x_2) - 3 > 2M.$$

Hence $|F(x_1)| > M$ or $|F(x_2)| > M$. Since U is an arbitrary neighborhood of x_0 and $x_1, x_2 \in U$, we have $M_F(x_0) > M$ or $m_F(x_0) < -M$. Thus, since M was arbitrary, we have $M_F = +\infty$ or $m_F = -\infty$. Therefore $\omega(F, x_0) = +\infty$ for any x_0 belonging to the boundary of A. ■

Remark 2.2. In the proof of the implication (2)⇒(1), we have shown the equality $\omega(h, \cdot) = f$.

Corollary 2.6. *Let $f\colon X \to [0, +\infty]$ be continuous, and suppose that $F\colon X \to \mathbb{R}$ and $\omega(F, \cdot) = f$. Then there exists a function $\varphi\colon X \to \mathbb{R}$ such that $\omega(\varphi, \cdot) = f$ and $0 \leq \varphi \leq f$.*

Proof. Let $A = \{x \in X : f(x) = +\infty\}$. If $A = \emptyset$, then the statement is trivial. Otherwise, by Theorem 2.7, there exist $g, h\colon X \to \mathbb{R}$ satisfying the conditions:

- $F = g + h$,
- the restriction of g to $X \setminus A$ is continuous,
- the restriction of g to $\mathrm{int}(A)$ is continuous,
- $0 \leq h(x) \leq f(x)$ for all $x \in X$,
- for each $\varepsilon > 0$ the sets $\{x \in X \setminus A : h(x) < \varepsilon\}$ and $\{x \in X \setminus A : h(x) > f(x) - \varepsilon\}$ are dense in $X \setminus A$ and $\{x \in A : |h(x)| > 1/\varepsilon\}$ is dense in $\mathrm{int}(A)$.

Let us define $\varphi\colon X \to \mathbb{R}$ in the following way:

$$\varphi(x) = \begin{cases} h(x) & \text{if } x \in X \setminus A, \\ |h(x)| & \text{if } x \in A. \end{cases}$$

It follows directly from the definition of φ and from the assumptions on h that $0 \leq \varphi \leq f$. Repeating the arguments in the proof of Theorem 2.7, we see that $\omega(\varphi, \cdot) = \omega(h, \cdot) = f$. ■

Now, we will prove that the decomposition of F in Theorem 2.1(3) is unique.

Theorem 2.8. *Let $(X, \mathcal{T})$ be a topological space. Assume that $f\colon X \to [0, +\infty)$ is continuous and vanishes at isolated points. Moreover, let $F\colon X \to \mathbb{R}$ be such that $\omega(F, \cdot) = f$. Assume that $F = g_1 + h_1 = g_2 + h_2$, where $g_1, g_2, h_1, h_2\colon X \to \mathbb{R}$, the functions g_1, g_2 are continuous, $0 \leq h_1 \leq f$, $0 \leq h_2 \leq f$, and for each $\varepsilon > 0$ the sets $\{x \in X : h_1(x) < \varepsilon\}$,*

$\{x \in X : h_2(x) < \varepsilon\}$, $\{x \in X : h_1(x) > f(x) - \varepsilon\}$ and $\{x \in X : h_2(x) > f(x) - \varepsilon\}$ are dense in X. Then $g_1 = g_2$ and $h_1 = h_2$.

Proof. Assume that $F = g_1 + h_1 = g_2 + h_2$, where $g_1, g_2, h_1, h_2 \colon X \to \mathbb{R}$ satisfy the assumptions of the theorem. Then $g_1 - g_2 = h_2 - h_1$ and $g = g_1 - g_2$ is continuous. We claim that $g = 0$. Suppose, to the contrary, that there exists $x_0 \in X$ such that $g(x_0) = 2c \neq 0$. Without loss of generality, we may assume that $c > 0$. Since g is continuous, there exists a neighborhood U of x_0 such that $g(x) > c$ for all $x \in U$. By assumption, the set $\{x \in X : h_2(x) < c\}$ is dense. Therefore we can find $x_1 \in U$ for which $h_2(x_1) < c$. Hence

$$c < g(x_1) = g_1(x_1) - g_2(x_1) = h_2(x_1) - h_1(x_1) < c,$$

a contradiction. Hence $g = 0$, so that $g_1 = g_2$ and $h_1 = h_2$. ∎

To end this chapter we study the topological structure of the set $\{F \colon X \to \mathbb{R} : \omega(F, \cdot) = f\}$, where $f \colon X \to [0, +\infty)$.

THEOREM 2.9. *Let $(X, \mathcal{T})$ be a topological space. For each $f \colon X \to [0, +\infty)$, the set $\{F \colon X \to \mathbb{R} : \omega(F, \cdot) = f\}$ is closed in the space $\mathbb{R}^X$ of all functions from X to $\mathbb{R}$ with the metric of uniform convergence*

$$\operatorname{dist}(g, h) = \min\{1, \sup\{|g(x) - h(x)| : x \in X\}\}.$$

Proof. Let $f \colon X \to [0, +\infty)$. We may assume that $\mathbf{B} = \{F \colon X \to \mathbb{R} : \omega(F, \cdot) = f\} \neq \emptyset$. Take any $g \in \operatorname{cl}(\mathbf{B})$ and any $\varepsilon \in (0, 1)$. Moreover, let $F \in \mathbf{B}$ and $|F(x) - g(x)| < \varepsilon$ for $x \in X$. Then $|M_F(x) - M_g(x)| \leq \varepsilon$ and $|m_F(x) - m_g(x)| \leq \varepsilon$ for $x \in X$. Therefore $|\omega(F, x) - \omega(g, x)| = |f(x) - \omega(g, x)| \leq 2\varepsilon$ for all $x \in X$. Since $\varepsilon > 0$ is arbitrary, we have $\omega(g, \cdot) = f$. Hence $g \in \mathbf{B}$ and the proof is complete. ∎

THEOREM 2.10. *Let $f \colon X \to [0, +\infty]$. If there exists $x_0 \in X^d$ such that $f(x_0) < +\infty$, then $\mathbf{B} = \{F \colon X \to \mathbb{R} : \omega(F, \cdot) = f\}$ is a nowhere dense set in the space $\mathbb{R}^X$ with the metric of uniform convergence.*

Proof. If $\mathbf{B}$ is empty, then obviously it is nowhere dense. So, assume that $\mathbf{B} \neq \emptyset$ and take any $F \in \mathbf{B}$. Since $\omega(F, x_0) = f(x_0) < +\infty$, we infer that F is bounded on some neighborhood U of x_0. Take any $\varepsilon > 0$. First, consider the case $f(x_0) > 0$. Then $M_F(x_0) \neq m_F(x_0)$. Let $c = (M_F(x_0) + m_F(x_0))/2$ and define $H \colon X \to \mathbb{R}$ by

$$H(x) = \begin{cases} F(x) + \varepsilon/2 & \text{if } F(x) \geq c, \\ F(x) - \varepsilon/2 & \text{if } F(x) < c. \end{cases}$$

Then $|F(x) - H(x)| \leq \varepsilon/2$ for all $x \in X$, $M_H(x_0) = M_F(x_0) + \varepsilon/2$, and $m_H(x_0) = m_F(x_0) - \varepsilon/2$. Hence $\omega(H, x_0) = \omega(F, x_0) + \varepsilon = f(x_0) + \varepsilon$ and $H \notin \mathbf{B}$.

Now consider the second case where $f(x_0) = 0$. Then $\omega(F, x_0) = f(x_0) = 0$ and F is continuous at x_0. Define $H \colon X \to \mathbb{R}$ by

$$H(x) = \begin{cases} F(x) + \varepsilon/2 & \text{if } x = x_0, \\ F(x) & \text{if } x \neq x_0. \end{cases}$$

Then $|F(x) - H(x)| \le \varepsilon/2$ for all $x \in X$, $M_H(x_0) = M_F(x_0) + \varepsilon/2 = F(x_0) + \varepsilon/2$ and $m_H(x_0) = m_F(x_0) = F(x_0)$. Therefore $\omega(H, x_0) = \varepsilon/2 \ne f(x_0)$ and $H \notin \mathbf{B}$. Hence $\mathbf{B}$ is closed. ∎

THEOREM 2.11. *Let $f\colon X \to [0, +\infty]$ and $f(x) = +\infty$ for $x \in X^d$. Then $\mathbf{B} = \{F\colon X \to \mathbb{R} : \omega(F, \cdot) = f\}$ is open in the space $\mathbb{R}^X$ of all functions from X to $\mathbb{R}$ with the metric of uniform convergence.*

Proof. Assume that $\mathbf{B} \ne \emptyset$ and let $F \in \mathbf{B}$. Then $\omega(F, x) = +\infty$ if $x \in X^d$ and $\omega(F, x) = 0$ if $x \in X \setminus X^d$. Take any $H\colon X \to \mathbb{R}$ such that $\operatorname{dist}(H, F) < 1$. Obviously, $\omega(H, x) = 0$ if $x \in X \setminus X^d$. Take any $x_0 \in X^d$ and suppose U is a neighborhood of x_0. Then for any $x, y \in U$, we have

$$|H(x) - H(y)| \ge |F(x) - F(y)| - |H(x) - F(x)| - |F(y) - H(y)| = |F(x) - F(y)| - 2.$$

Since $\sup_{x,y\in U} |F(x) - F(y)| = +\infty$, we have $\sup_{x,y\in U} |H(x) - H(y)| = +\infty$. Therefore $\omega(H, x_0) = +\infty$ for $x_0 \in X^d$. It follows that $H \in \mathbf{B}$. ∎

3. Basic theorems

In this chapter we will obtain some sufficient conditions for the existence of an ω-primitive in the case of some nonmetrizable spaces.

LEMMA 3.1. *If C is a dense subset of a resolvable space $(X, \mathcal{T})$, then there exists $D \subset C$ such that $\operatorname{cl}(D) = X = \operatorname{cl}(X \setminus D)$.*

Proof. Let A and B be disjoint dense subsets of X. Put $D = (C \setminus \operatorname{int}(C)) \cup (\operatorname{int}(C) \cap A)$. Obviously, $D \subset C$. Moreover, $X \setminus \operatorname{cl}(\operatorname{int}(C)) = \operatorname{cl}(C) \setminus \operatorname{cl}(\operatorname{int}(C)) \subset \operatorname{cl}(C \setminus \operatorname{int}(C))$ and $\operatorname{cl}(\operatorname{int}(C) \cap A) = \operatorname{cl}(\operatorname{int}(C))$ because $A \cap \operatorname{int}(C)$ is a dense subset of $\operatorname{int}(C)$. Therefore

$$\operatorname{cl}(D) = \operatorname{cl}(C \setminus \operatorname{int}(C)) \cup \operatorname{cl}(\operatorname{int}(C) \cap A) \supset (X \setminus \operatorname{cl}(\operatorname{int}(C))) \cup \operatorname{cl}(\operatorname{int}(C)) = X.$$

Similarly, $((X \setminus C) \cup \operatorname{int}(C)) \setminus (\operatorname{int}(C) \cap A) = (X \setminus C) \cup (\operatorname{int}(C) \setminus A)$ and $B \cap \operatorname{int}(C) \subset \operatorname{int}(C) \setminus A$. Hence

$$\begin{aligned}\operatorname{cl}(X \setminus D) &= \operatorname{cl}\big(\big((X \setminus C) \cup \operatorname{int}(C)\big) \cap (X \setminus (\operatorname{int}(C) \cap A))\big)\\ &= \operatorname{cl}((X \setminus C) \cup (\operatorname{int}(C) \setminus A)) \supset (X \setminus \operatorname{int}(C)) \cup \operatorname{cl}(B \cap \operatorname{int}(C))\\ &\supset (X \setminus \operatorname{int}(C)) \cup \operatorname{int}(C) = X. \blacksquare\end{aligned}$$

THEOREM 3.1. *Let $(X, \mathcal{T})$ be a resolvable space. If the set C_f of all continuity points of an upper semicontinuous function $f\colon X \to [0, +\infty)$ is dense in X, then f has an ω-primitive.*

Proof. By Lemma 3.1, we can find $A \subset C_f$ such that $\operatorname{cl}(A) = X = \operatorname{cl}(X \setminus A)$. Define $F\colon X \to \mathbb{R}$ by $F = f \cdot \chi_{X\setminus A}$, where $\chi_{X\setminus A}$ is the characteristic function of $X \setminus A$. We claim that F is an ω-primitive for f.

Since $\operatorname{cl}(A) = X$ and $F \ge 0$, we have $m_F = 0$. Moreover, by the upper semicontinuity of f, we get $M_F(x) = M_f(x) = f(x)$ for $x \in X \setminus A$. If $x \in A$, then f is continuous at x

and

$$\limsup_{t\to x} F(t) = \limsup_{X\setminus A\ni t\to x} F(t) = \limsup_{X\setminus A\ni t\to x} f(t) = f(x) \geq 0.$$

Thus $M_F(x) = f(x)$ for $x \in A$. Finally, $\omega(F, x) = f(x)$ for all $x \in X$. ∎

Later, we will need the following result.

THEOREM 3.2 ([Fo, Theorem 1]). *Let $(X, \mathcal{T})$ be a resolvable space. If $f\colon X \to \mathbb{R}$ is upper or lower semicontinuous, then the set of all discontinuity points of f is a first category set.*

COROLLARY 3.1. *If $(X, \mathcal{T})$ is a Baire space and $f\colon X \to \mathbb{R}$ is upper or lower semicontinuous, then f has a dense set of continuity points.*

THEOREM 3.3. *If $(X, \mathcal{T})$ is a Baire space, then every upper semicontinuous function $f\colon X \to \mathbb{R}$ has an ω-primitive.*

Proof. This follows directly from Theorem 3.1 and Corollary 3.1. ∎

EXAMPLE 3.1. Let $\mathcal{T}_d$ denote the density topology on $\mathbb{R}$, [Wi]. Since each nonempty open subset of $(X, \mathcal{T}_d)$ has positive Lebesgue measure, $(\mathbb{R}, \mathcal{T}_d)$ is dense in itself. Moreover, any Bernstein set [O], and also the complement of any Bernstein set, is dense in the density topology. Thus $(\mathbb{R}, \mathcal{T}_d)$ is a resolvable space. By Theorem 3.3, an upper semicontinuous function $f\colon (\mathbb{R}, \mathcal{T}_d) \to \mathbb{R}$ has an ω-primitive.

THEOREM 3.4. *Let $(X, \mathcal{T})$ be a dense in itself topological space and assume $f\colon X \to \mathbb{R}$ is an upper semicontinuous function. If a subset A of X satisfies the conditions*

(1) $\operatorname{cl}(A) = \operatorname{cl}(X \setminus A) = X$,
(2) $\{x \in A : f(x) - \limsup_{t\to x} f(t) > \varepsilon\}$ *is closed for each* $\varepsilon > 0$,
(3) $\limsup_{X\setminus A\ni t\to x} f(t) = \limsup_{t\to x} f(t)$ *for each* $x \in A$,

then the function $F\colon X \to \mathbb{R}$ defined by

$$F(x) = \begin{cases} f(x) & \text{if } x \in X \setminus A,\\ -\big(f(x) - \limsup_{t\to x} f(t)\big) & \text{if } x \in A,\end{cases}$$

is an ω-primitive for f.

Proof. Let $x_0 \in X \setminus A$. Since $F \leq f$ and f is upper semicontinuous,

$$M_F(x_0) \leq M_f(x_0) = f(x_0) = F(x_0) \leq M_F(x_0).$$

Thus $M_F(x_0) = f(x_0)$. The set A is dense in X and $F(t) \leq 0$ for $t \in A$. Therefore $m_F(x_0) \leq 0$. Fix any $\varepsilon > 0$. From the assumptions, the set $A_\varepsilon = \{x \in A : f(x) - \limsup_{t\to x} f(t) > \varepsilon\}$ is closed. Moreover, $x_0 \notin A_\varepsilon$. Thus there exists a neighborhood U of x_0 such that $A_\varepsilon \cap U = \emptyset$. Hence $m_F(x_0) \geq -\varepsilon$. Since $\varepsilon > 0$ is arbitrary, $m_F(x_0) \geq 0$. Finally, $m_F(x_0) = 0$ and

$$\omega(F, x_0) = f(x_0) - 0 = f(x_0) \quad \text{for } x_0 \in X \setminus A.$$

Now let $x_0 \in A$. Since $F(x_0) \leq 0$ and $X \setminus A$ is dense in X and $F(x) \geq 0$ for $x \in X \setminus A$, we have

$$M_F(x_0) = \limsup_{x\to x_0} F(x) = \limsup_{X\setminus A\ni x\to x_0} F(x).$$

But, it follows from condition (2) that

$$M_F(x_0) = \limsup_{(X\setminus A)\ni x\to x_0} F(x) = \limsup_{(X\setminus A)\ni x\to x_0} f(x) = \limsup_{x\to x_0} f(x).$$

On the other hand, $m_F(x_0) \le F(x_0) \le 0$. Fix any $\varepsilon > -F(x_0)$. By condition (3), the set A_ε is closed. Since $x_0 \notin A_\varepsilon$, we can find a neighborhood U of x_0 such that $U \cap A_\varepsilon = \emptyset$. Hence $m_F(x_0) \ge -\varepsilon$ for $\varepsilon > -F(x_0)$. Thus $m_F(x_0) \ge F(x_0)$ and $m_F(x_0) = F(x_0) = -\big(f(x_0) - \limsup_{x\to x_0} f(x)\big)$. Finally,

$$\omega(F, x_0) = \limsup_{x\to x_0} f(x) + f(x_0) - \limsup_{x\to x_0} f(x) = f(x_0) \quad \text{for each } x_0 \in A. \ \blacksquare$$

LEMMA 3.2. *Let $(X, \mathcal{T})$ be a dense in itself topological space and suppose $f\colon X \to [0, +\infty)$ is upper semicontinuous. Then for any $\varepsilon > 0$,*

$$A_\varepsilon = \Big\{x \in X : f(x) - \limsup_{t\to x} f(t) > \varepsilon\Big\}$$

is nowhere dense in X.

Proof. Fix any $\varepsilon > 0$. Suppose that $\mathrm{cl}(A_\varepsilon)$ contains a nonempty open set $U \in \mathcal{T}$. Let $\alpha = \inf_{x\in U} f(x)$ and pick $x_0 \in U$ for which $f(x_0) < \alpha + \varepsilon/3$. By the upper semicontinuity of f at x_0, there exists a neighborhood $V \subset U$ of x_0 such that $f(x) < f(x_0) + \varepsilon/3 < \alpha + 2\varepsilon/3$ for $x \in V$. Choose any $x_1 \in A_\varepsilon \cap V$. Then $f(x_1) > \limsup_{t\to x_1} f(t) + \varepsilon$. Therefore we can find a neighborhood W of x_1 such that $f(x) + \varepsilon < f(x_1) < \alpha + 2\varepsilon/3$ for $x \in W$. The set $V \cap W$ is a nonempty subset of U. But if $z \in V \cap W$, then $f(z) + \varepsilon < \alpha + 2\varepsilon/3$ and hence $f(z) < \alpha$, which contradicts the definition of α. Thus the set $\mathrm{cl}(A_\varepsilon)$ contains no nonempty open set and for any $\varepsilon > 0$, A_ε is nowhere dense in X. ■

LEMMA 3.3. *Let $(X, \mathcal{T})$ be a Hausdorff first countable dense in itself topological space, $x_0 \in X$, and let $(U_n)_{n\in\mathbb{N}}$ be a decreasing local base at x_0. If $(x_n)_{n\in\mathbb{N}}$ and $(x_{n,k})_{k\in\mathbb{N}}$ for $n \in \mathbb{N}$ are sequences of points from X satisfying the conditions*

(1) $\lim_{n\to+\infty} x_n = x_0$,
(2) $\lim_{k\to+\infty} x_{n,k} = x_n$ *for* $n \in \mathbb{N}$,
(3) $\{x_n\} \cup \{x_{n,k} : k \in \mathbb{N}\} \subset U_n$ *for* $n \in \mathbb{N}$,

then

$$A = \{x_0\} \cup \{x_n : n \ge 1\} \cup \{x_{n,k} : n, k \ge 1\}$$

is closed and nowhere dense in X.

Proof. Fix $y \notin A$. Then $y \ne x_0$ and we can find a neighborhood V of y and a positive integer m such that $y \in V$ and $V \cap U_m = \emptyset$. For each $j = 1, \ldots, m-1$, there exist open sets V_j and Z_j such that $y \in V_j$, $x_j \in Z_j$, and $V_j \cap Z_j = \emptyset$. Thus for each $j = 1, \ldots, m-1$, there is an $i_j \in \mathbb{N}$ such that $x_{j,i} \notin V_j$ for $i \ge i_j$. Hence,

$$(V \cap V_1 \cap \cdots \cap V_{m-1}) \cap A \subset \{x_n : n < m\} \cup \{x_{j,i} : j < m,\ i < i_j\}.$$

Thus, $W = V \cap V_1 \cap \cdots \cap V_{m-1}$ is a neighborhood of y whose intersection with A is finite. For each point of the finite set

$$\{t_1, \ldots, t_k\} = \{x_n : n < m\} \cup \{x_{j,i} : j < m,\ i < i_j\}$$

we can find a neighborhood W_i of y such that $y \notin W_i$ for $i = 1, \dots, k$. Thus, $W \cap W_1 \cap \dots \cap W_k$ is a neighborhood of y which is disjoint from A. This implies that A is closed.

To prove that A is nowhere dense, it is sufficient to show that no nonempty open set is contained in A. Let $U \in \mathcal{T}$, $U \neq \emptyset$. Since X is T_1 and dense in itself, each nonempty open subset is infinite. Let $y \in U$, $y \neq x_0$. We can find a nonempty open set $V \subset U$ and a positive integer $m \in \mathbb{N}$ such that $y \in V$ and $V \cap U_m = \emptyset$. Since V is infinite, there exists $y_1 \in V \setminus \{x_1, \dots, x_{m-1}\}$. For each $j = 1, \dots, m-1$, we can find open sets V_j and W_j such that $y \in V_j$, $x_k \in W_j$, and $V_j \cap W_j = \emptyset$. Hence, for each $j = 1, \dots, m-1$ there exists an $i_j \in \mathbb{N}$ such that $x_{j,i} \notin V_j$ for $i \geq i_j$. Thus,

$$(V \cap V_1 \cap \dots \cap V_{m-1}) \cap A \subset \{x_n : n < m\} \cup \{x_{j,i} : j < m,\ i < i_j\}.$$

It follows that $W = V \cap V_1 \cap \dots \cap V_{m-1}$ is a neighborhood of y contained in U and that $A \cap W$ is finite. This proves that $W \not\subset A$. Thus $U \not\subset A$ and A is a nowhere dense set. ∎

Now we shall present the basic theorem of this chapter.

THEOREM 3.5. *Let $(X, \mathcal{T})$ be a regular dense in itself first countable topological space and let $(U_n(x))_{n\in\mathbb{N}}$ be a decreasing local base at a point $x \in X$. Assume that there exists a countable family $\{F_n : n \geq 1\}$ of nonempty subsets of X satisfying the conditions:*

(a) *each set F_n is discrete and closed,*
(b) *$\bigcup_{n\in\mathbb{N}} F_n$ is dense in X,*
(c) *for each $n \geq 1$ there is a locally finite family $\{G_t^n : t \in F_n\}$ of pairwise disjoint open subsets of X indexed by F_n, such that $t \in G_t^n$ for $t \in F_n$.*

Then for each upper semicontinuous $f\colon X \to [0, +\infty)$ and lower semicontinuous $g\colon X \to (0, +\infty)$, there exists $F\colon X \to \mathbb{R}$ such that $\omega(F, \cdot) = f$ and $-g < F \leq f$. Moreover, if $f(x) = 0$ at some point x, then $F(x) = 0$.

Proof. Let $f\colon X \to [0, +\infty)$ be an upper semicontinuous function and let $g\colon X \to (0, +\infty)$ be lower semicontinuous. In the proof, we will use Theorem 3.4. To this end, we shall construct two families of subsets of X, $\{A_n : n \in \mathbb{N}\}$ and $\{B_n : n \in \mathbb{N}\}$, satisfying

(1) for each $n = 1, 2, \dots$, the sets A_n and B_n are nowhere dense;
(2) $A_n \cap B_k = \emptyset$ for $n, k \in \mathbb{N}$; and for $n \neq k$, both $A_n \cap A_k = \emptyset$ and $B_n \cap B_k = \emptyset$;
(3) $A_n = \{x_{t,k}^n : t \in F_n,\ k \in \mathbb{N}\}$ for $n = 1, 2, \dots$;
(4) $B_n = \{y_{t,k,m}^n : t \in F_n,\ k, m \in \mathbb{N}\}$ for $n = 1, 2, \dots$;
(5) $\lim_{k\to\infty} x_{t,k}^n = t$ for $t \in F_n$, $n = 1, 2, \dots$;
(6) $\lim_{m\to\infty} y_{t,k,m}^n = x_{t,k}^n$ for $t \in F_n$, $k, n = 1, 2, \dots$;
(7) $\limsup_{s\to x_{t,k}^n} f(s) = \lim_{m\to\infty} f(y_{t,k,m}^n)$ for $t \in F_n$, $k, n = 1, 2, \dots$;
(8) $f(x_{t,k}^n) - \limsup_{s\to x_{t,k}^n} f(s) < \min\{g(x_{t,k}^n), 1/n\}$ for $t \in F_n$, $k, n = 1, 2, \dots$.

We shall construct $\{A_n : n \in \mathbb{N}\}$ and $\{B_n : n \in \mathbb{N}\}$ inductively. Let $n = 1$. Choose any $t_0 \in F_1$. Since g is lower semicontinuous, there exist $c_{t_0}^1 \in \mathbb{R}$ and a neighborhood V_{t_0} of t_0 such that $V_{t_0} \subset G_{t_0}^1$ and $g(u) > c_{t_0}$ for $u \in V_{t_0}$. Since X is dense in itself, $\{x \in X : f(x) - \limsup_{s\to x} f(s) \geq 1\}$ is a nowhere dense set, by Lemma 3.2. Therefore, we can find a sequence $(x_{t_0,k}^1)_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} x_{t_0,k}^1 = t_0$, $x_{t_0,k}^1 \in U_k(t_0) \cap V_{t_0}$ and $f(x_{t_0,k}^1) - \limsup_{s\to x_{t_0,k}^1} f(s) < \min\{1, c_{t_0}^1\}$ for $k = 1, 2, \dots$. Again, since X is dense in itself, for each $k = 1, 2, \ldots$ we can find a sequence $(y^1_{t_0,k,m})_{m\in\mathbb{N}}$ with $y^1_{t_0,k,i} \neq y^1_{t_0,k,j}$ for $i \neq j$ satisfying the conditions:

- $\lim_{m\to\infty} f(y^1_{t_0,k,m}) = \limsup_{s\to x^1_{t_0,k}} f(s)$,
- $\{y^1_{t_0,k,m} : k, m \in \mathbb{N}\} \cap \{x^1_{t_0,k} : k \in \mathbb{N}\} = \emptyset$,
- $y^1_{t_0,k,m} \in U_k(t_0) \cap V_{t_0}$ for $m = 1, 2, \ldots$.

In a similar way we may construct sets $\{x^1_{t,k} : k \in \mathbb{N}\}$ and $\{y^1_{t,k,m} : k, m \in \mathbb{N}\}$ for all $t \in F_1$. By Lemma 3.3, $\{x^1_{t,k} : k \in \mathbb{N}\}$ and $\{y^1_{t,k,m} : k, m \in \mathbb{N}\}$ are nowhere dense for each $t \in F_1$. Since the family $\{G^1_t : t \in F_1\}$ is locally finite and $\{x^1_{t,k} : k \in \mathbb{N}\} \cup \{y^1_{t,k,m} : k, m \in \mathbb{N}\} \subset G^1_t$, we infer that $A_1 = \{x^1_{t,k} : t \in F_1,\ k \in \mathbb{N}\}$ and $B_1 = \{y^1_{t,k,m} : t \in F_1,\ k, m \in \mathbb{N}\}$ are nowhere dense. It is easy to verify that conditions 2–8 are fulfilled if $n = 1$.

Assume that $A_1, \ldots, A_{n-1}$ and $B_1, \ldots, B_{n-1}$ satisfying conditions 2–8 have been defined. Then $C_n = A_1 \cup \cdots \cup A_{n-1} \cup B_1 \ldots \cup B_{n-1}$ is a nowhere dense subset of X. Fix any $t_0 \in F_n$. Since $g(t_0) > 0$ and g is lower semicontinuous, there exist $c^n_{t_0} \in \mathbb{R}$ and a neighborhood V_{t_0} of t_0 such that $V_{t_0} \subset G^n_{t_0}$ and $g(u) > c^n_{t_0}$ for $u \in V_{t_0}$. Moreover, X is dense in itself and $\{x \in X : f(x) - \limsup_{s\to x} f(s) \geq 1/n\}$ and C_n are nowhere dense. Therefore there exists a sequence $(x^n_{t_0,k})_{k\in\mathbb{N}}$ such that

- $\lim_{k\to\infty} x^n_{t_0,k} = t_0$,
- $x^n_{t_0,k} \in (U_k(t_0) \cap V_{t_0}) \setminus C_n$,
- $f(x^n_{t_0,k}) - \limsup_{s\to x^n_{t_0,k}} f(s) < \min\{1/n, c^n_{t_0}\}$ for $k = 1, 2, \ldots$.

Next, for each $k = 1, 2, \ldots$ we can find a sequence $(y^n_{t_0,k,m})_{m\in\mathbb{N}}$ satisfying the conditions

- $y^n_{t_0,k,i} \neq y^n_{t_0,k,j}$ for $i \neq j$,
- $\limsup_{s\to x^n_{t_0,k}} f(s) = \lim_{m\to\infty} f(y^n_{t_0,k,m})$,
- $\{y^n_{t_0,k,m} : k, m \in \mathbb{N}\} \cap \{x^n_{t_0,k} : k \in \mathbb{N}\} = \emptyset$,
- $y^n_{t_0,k,m} \in U_k(t_0) \cap V_{t_0}$ for $m = 1, 2, \ldots$.

Similarly, we can find $\{x^n_{t,k} : k \in \mathbb{N}\}$ and $\{y^n_{t,k,m} : k, m \in \mathbb{N}\}$ for each $t \in F_n$. Since $(\{x^n_{t,k} : k \in \mathbb{N}\} \cup \{y^n_{t,k,m} : k, m \in \mathbb{N}\}) \cap C_n = \emptyset$, we have $C_n \cap (A_n \cup B_n) = \emptyset$. It is easily seen that $A_1, \ldots, A_n, B_1, \ldots, B_n$ fulfill all the conditions 1–8.

Thus we have proven by induction that the families of sets $\{A_n : n \in \mathbb{N}\}$ and $\{B_n : n \in \mathbb{N}\}$ satisfying conditions 1–8 can be constructed.

Let $A = \bigcup_{n=1}^{\infty} A_n$ and $B = \bigcup_{n=1}^{\infty} B_n$. We claim that A satisfies all assumptions of Theorem 3.4. Since $F_n \subset \mathrm{cl}(A_n)$ and $A_n \subset \mathrm{cl}(B_n)$ if $n = 1, 2, \ldots$, we obtain

$$\mathrm{cl}(A) = \mathrm{cl}\Big(\bigcup_{n=1}^{\infty} A_n\Big) \supset \bigcup_{n=1}^{\infty} \mathrm{cl}(A_n) \supset \bigcup_{n=1}^{\infty} \mathrm{cl}(F_n) \supset \bigcup_{n=1}^{\infty} F_n,$$

$$\mathrm{cl}(B) = \mathrm{cl}\Big(\bigcup_{n=1}^{\infty} B_n\Big) \supset \bigcup_{n=1}^{\infty} \mathrm{cl}(B_n) \supset \bigcup_{n=1}^{\infty} \mathrm{cl}(A_n) \supset \bigcup_{n=1}^{\infty} F_n.$$

Hence $\mathrm{cl}(A) = X$ and $\mathrm{cl}(B) = X$. But $B \subset X \setminus A$. Thus $\mathrm{cl}(X \setminus A) = X$.

Fix any $\varepsilon > 0$. Let n_0 be a positive integer for which $1/n_0 < \varepsilon$. By (8), we have

$$\Big\{x \in A : f(x) - \limsup_{t\to x} f(t) > \varepsilon\Big\} \subset \bigcup_{n=1}^{n_0-1} A_n.$$

Thus by (1) the set $\{x \in A : f(x) - \limsup_{t\to x} f(t) > \varepsilon\}$ is closed and discrete. By (7), we have $\limsup_{X\setminus A\ni t\to x} f(t) = \limsup_{t\to x} f(t)$ for each $x \in A$, because $B \subset X \setminus A$.

Thus we have proven that the set A satisfies all the assumptions of Theorem 3.4. Therefore the function $F\colon X \to \mathbb{R}$ defined by

$$F(x) = \begin{cases} f(x) & \text{if } x \in X \setminus A, \\ -(f(x) - \limsup_{t\to x} f(t)) & \text{if } x \in A, \end{cases}$$

satisfies $\omega(F,\cdot) = f$.

Directly from the definition of f, we have $F \le f$. Moreover, it follows from (8) that $-g \le F$. If $f(x) = 0$, then $\limsup_{t\to x} f(t) = 0$, because f is upper semicontinuous and nonnegative. Thus $F(x) = 0$ for every x at which $f(x) = 0$. ∎

We find, as a direct corollary from the last theorem, an important class of topological spaces for which every nonnegative and upper semicontinuous real function has an ω-primitive.

THEOREM 3.6. *Let $(X,\mathcal{T})$ be any regular dense in itself and separable topological space. Then for each pair of $f\colon X \to [0,+\infty)$ upper semicontinuous and $g\colon X \to (0,+\infty)$ lower semicontinuous, there exists an $F\colon X \to \mathbb{R}$ such that $\omega(F,\cdot) = f$ and $-g < F \le f$. Moreover, if $f(x) = 0$ for some $x \in X$, then $F(x) = 0$.*

Proof. Let $\{x_n : n = 1, 2, \ldots\}$ be a countable dense subset of X. We may put $F_n = \{x_n\}$ and $G_n = \{X\}$ for $n = 1, 2, \ldots$, and apply Theorem 3.5. ∎

COROLLARY 3.2. *The Niemytzki plane and the Sorgenfrey line are nonmetrizable spaces satisfying the assumptions of the previous theorem. Therefore, if X is one of those spaces, for any $f\colon X \to (0,+\infty)$ upper semicontinuous and $g\colon X \to (0,+\infty)$ lower semicontinuous, there exists an $F\colon X \to \mathbb{R}$ such that $\omega(F,\cdot) = f$ and $-g < F \le f$.*

We shall show that Theorem 3 from [EP2] is also an immediate corollary of Theorem 3.5.

THEOREM 3.7. *Let (X,d) be any dense in itself metric space. Then for any $f\colon X \to [0,+\infty)$ upper semicontinuous and $g\colon X \to (0,+\infty)$ lower semicontinuous, there exists an $F\colon X \to \mathbb{R}$ for which $\omega(F,\cdot) = f$ and $-g < F \le f$.*

Proof. Put

$$\mathcal{A}_n = \{A \subset X : \forall_{\substack{x,y\in A\\ x\neq y}} d(x,y) \ge 1/n\}$$

for each $n \in \mathbb{N}$. By the Teichmüller–Tukey Lemma or by the Zorn Lemma it is easy to see that each family $\mathcal{A}_n$ has a maximal element $\mathbf{A}_n$. Then for any $x \in X$ an open ball $B(x, 1/2n)$ centered at x and with radius $1/2n$ contains at most one element of $\mathbf{A}_n$. Therefore, for any $n\in\mathbb{N}$ the set $\mathbf{A}_n$ is discrete and the family of sets $G_n = \{B(x, 1/4n) : x \in \mathbf{A}_n\}$ is locally finite and is composed of pairwise disjoint sets. Finally, observe that each open ball with radius greater than $2/n$ contains a point from $\mathbf{A}_n$. Hence $\bigcup_{n=1}^{\infty} \mathbf{A}_n$ is a dense subset of X. Therefore all the assumptions of Theorem 3.5 hold. ∎

The next two examples show that Theorem 3.5 can be used in the case of nonseparable nonmetrizable topological spaces.

Example 3.2. Let (Y, τ) be any regular dense in itself separable topological space and let T be any set. It is easy to see that the family

$$\mathcal{B} = \{U \times \{t\} : U \in \tau,\, t \in T\}$$

is a base of a topology $\mathcal{T}$ on $X = Y \times T$. If T is uncountable, then $(X, \mathcal{T})$ is nonseparable. Let $\{y_n : n = 1, 2, \ldots\}$ be any countable dense subset of (Y, τ). For each $n \in \mathbb{N}$, put $F_n = \{(y_n, t) : t \in T\}$ and $G_n = \{Y \times \{t\} : t \in T\}$. Then $(F_n)_{n\in\mathbb{N}}$ and $(G_n)_{n\in\mathbb{N}}$ satisfy all the assumptions of Theorem 3.5. Therefore each upper semicontinuous $f\colon X \to [0, +\infty)$ has an ω-primitive.

Example 3.3. Let (Y, τ) be a regular first countable topological space. Put $X = Y \times [0, 1]$ and define a topology $\mathcal{T}$ on X as follows. Let

$$\{\{y\} \times ((t - \varepsilon, t + \varepsilon) \cap [0, 1]) : \varepsilon > 0\}$$

be a local base at $(y, t) \in X$ where $t \neq 0$, and

$$\{U \times [0, \varepsilon) : y \in U,\, U \in \tau,\, \varepsilon > 0\}$$

a local base at $(y, 0) \in X$. It is easy to check that $Y \times \{0\}$ is a closed subset of $(X, \mathcal{T})$ which is homeomorphic to $(X, \mathcal{T})$.

Let $(q_n)_{n\in\mathbb{N}}$ be a sequence of all the rational numbers from $(0, 1)$. Finally, put

$$F_n = \{(y, q_n) : y \in Y\} \quad \text{and} \quad G_n = \{\{y\} \times (q_n/2, 1) : y \in y\}.$$

It is obvious that the families $(F_n)_{n\in\mathbb{N}}$ and $(G_n)_{n\in\mathbb{N}}$ satisfy all the assumptions of Theorem 3.5. Hence each upper semicontinuous $f\colon X \to [0, +\infty)$ has an ω-primitive.

An important corollary follows directly from the last example.

Corollary 3.3. *For each regular first countable topological space (Y, τ) there exists a topological space $(X, \mathcal{T})$ such that (Y, τ) is homeomorphic to some closed subset of X and for any $f\colon X \to [0, +\infty)$ upper semicontinuous and $g\colon X \to (0, +\infty)$ lower semicontinuous one can find $F\colon X \to \mathbb{R}$ for which $\omega(F, \cdot) = f$ and $-g < F \leq f$.*

Theorem 3.5 motivates the following definition.

Definition 3.1. A topological space $(X, \mathcal{T})$ is said to be *regularly resolvable* if there exists a countable family of sets $\{F_n : n \geq 1\}$ such that

(a) each set F_n is discrete and closed,
(b) $\bigcup_{n\in\mathbb{N}} F_n$ is dense in X.

Assume that a family $\{F_n : n \geq 1\}$ satisfies conditions (a) and (b) of the last definition. Then $X \setminus X^d \subset \bigcup_{n=1}^{\infty} F_n$. On the other hand, if X is dense in itself, then all sets $C_k = \bigcup_{n\geq k} F_n$ are dense in X and $\{C_k : k \geq 1\}$ is a decreasing family of sets. Moreover, if the sets F_n are pairwise disjoint, then $\bigcap_{k=1}^{\infty} C_k = \emptyset$. But any subset of a closed and discrete set is closed and discrete. Thus if we put $F_n' = F_n \setminus \bigcup_{k=1}^{n-1} F_k$ for $n \geq 2$, then the sets F_n' are pairwise disjoint and each of them is closed and discrete. Therefore, without loss of generality, we may assume that if a topological space is regularly resolvable and dense in itself, then there exists a countable family of pairwise disjoint closed and discrete sets whose union is dense.

Certainly each separable topological space is regularly resolvable and each metric space is regularly resolvable. An irresolvable and countable topological space can be considered as an example of a regularly resolvable space which is not resolvable. On the other hand, $\mathbb{R}$ with the density topology [Wi] is a resolvable space which is not regularly resolvable (because each subset of $\mathbb{R}$ which is discrete in the density topology has Lebesgue measure zero).

Theorem 3.8. *If $(X, \mathcal{T})$ is a paracompact Haudorff topological space and F is a closed and discrete subset of X, then there exists a locally finite family of open and pairwise disjoint sets $\{G_t : t \in F\}$ such that each G_t is a neighborhood of t.*

Proof. Since F is discrete, for each $t \in F$ there exists an open set U_t such that $t \in U_t$ and $U_t \cap F = \{t\}$. By the closedness of F, the family $\{U_t : t \in F\} \cup \{X \setminus F\}$ is an open cover of X. Since X is paracompact, there exists a locally finite open cover $\{V_s : s \in S\}$ which is a refinement of $\{U_t : t \in F\} \cup \{X \setminus F\}$. For each $t \in F$, let V_{s_t} be any member of $\{V_s : s \in S\}$ for which $t \in V_{s_t}$. The family $\{V_{s_t} : t \in F\}$ is locally finite (but its members need not be pairwise disjoint). Moreover, $t \in V_{s_t}$ and $V_{s_t} \cap F = \{t\}$ for $t \in F$. Since each paracompact Hausdorff topological space is regular, for each $t \in F$ one can find an open set W_t such that $t \in W_t$ and $\mathrm{cl}(W_t) \subset V_{s_t}$. Then the family $\{W_t \colon t \in F\}$ is locally finite and

$$\mathrm{cl}\Big(\bigcup_{t \in F} W_t\Big) = \bigcup_{t \in F} \mathrm{cl}(W_t) \subset \bigcup_{t \in F} V_{s_t}.$$

Hence $x \notin \mathrm{cl}((\bigcup_{t \in F} W_t) \setminus W_x)$ for $x \in F$. Therefore for each $x \in F$ we can find an open set G_x for which $x \in G_x$, $G_x \subset W_x$, and

$$G_x \cap \Big(\mathrm{cl}\Big(\bigcup_{t \in F} W_t\Big) \setminus W_x\Big) = \emptyset.$$

Obviously, the family $\{G_x : x \in F\}$ has the desired properties. ■

The next corollary follows immediately from Theorems 3.5 and 3.8.

Corollary 3.4. *Let $(X, \mathcal{T})$ be a paracompact regularly resolvable dense in itself and first countable topological space. Then for any $f \colon X \to [0, +\infty)$ upper semicontinuous and $g \colon X \to (0, +\infty)$ lower semicontinuous, there exists $F \colon X \to \mathbb{R}$ such that $\omega(F, \cdot) = f$ and $-g < F \le f$. Moreover, if $f(x) = 0$ for some $x \in X$, then $F(x) = 0$.*

We will discuss one more class of regularly resolvable spaces. To do this we need a few definitions.

Definition 3.2 ([EP3]). Let $(X, \mathcal{T})$ be a first countable topological space and let $\mathcal{N}(x) = \{U_n(x) : n \in \mathbb{N}\}$ be a local base of $\mathcal{T}$ at x for each $x \in X$. We say that a base for the topology $\mathcal{N} = \{\mathcal{N}(x) \colon x \in X\}$:

- satisfies condition $(N1)$ if $\forall_n \forall_{x \in X} \colon \big(U_{n+1}(x) \subset U_n(x)\big)$,
- satisfies condition $(N2)$ if there exists a function $s \colon \mathbb{N} \to \mathbb{N}$ such that

$$\forall_{x,y \in X} \forall_n \colon ((x \in U_{s(n)}(y) \Rightarrow y \in U_n(x)),$$

- satisfies condition $(N3)$ if there exists a function $t\colon \mathbb{N}\to\mathbb{N}$ such that
$$\forall_{x\in X}\,\forall_n\colon \Big(\bigcup_{y\in U_{t(n)}(x)} U_{t(n)}(y)\subset U_n(x)\Big).$$

It was proven in [EP3] that a T_0 topological space $(X,\mathcal{T})$ is metrizable if and only if there exists a family of local bases at each $x\in X$ satisfying conditions $(N1)$–$(N3)$ [EP3, Theorem 1.1]. Moreover, it was proven that the Niemytzki plane has a local base at each $x\in X$ which satisfies $(N1)$ and $(N2)$, but no local base satisfies $(N3)$. Finally, it was shown that the Sorgenfrey line has a local base at each $x\in X$ which satisfies $(N1)$ and $(N3)$, but no local base satisfies $(N2)$.

THEOREM 3.9. *If a first countable topological space $(X,\mathcal{T})$ has a family of local bases $\mathcal{N}=\{\mathcal{N}(x): x\in X\}$, $\mathcal{N}(x)=\{U_n(x): n\in\mathbb{N}\}$ which satisfies conditions $(N1)$ and $(N2)$, then $(X,\mathcal{T})$ is regularly resolvable.*

Proof. Fix $n\in\mathbb{N}$. Let $\mathcal{A}_n$ be the family of all sets $B\subset X$ such that $U_n(x)\cap B=\{x\}$ for each $x\in B$. In other words, $B\in\mathcal{A}_n$ iff $y\notin U_n(x)$ for each $x,y\in B$, $x\neq y$. By the Zorn Lemma, it is easy to see that $\mathcal{A}_n$ has a maximal element $F_n\in\mathcal{A}_n$. Thus if $x\in F_n$, then $U_n(x)\cap F_n=\{x\}$. It follows that F_n is discrete. Let us take any $y\in X\setminus F_n$. Suppose that $y\in\operatorname{cl}(F_n)$. Then y is a limit of a sequence from F_n. Hence $F_n\cap U_{s(n)}(y)$ is an infinite set. Let $x\in F_n\cap U_{s(n)}(y)$. Then by condition $(N2)$, we have $y\in U_n(x)$. Therefore $U_n(x)$ is an open neighborhood of y. Since y is a limit of a sequence from F_n, it follows that $F_n\cap U_n(x)$ is infinite. In particular, there exists a $z\in F_n\cap U_n(x)$ with $z\neq x$. But this contradicts the definition of F_n. Thus we have proven that $y\notin\operatorname{cl}(F_n)$. Therefore $\operatorname{cl}(F_n)\subset F_n$ and F_n is a closed set.

It remains to prove that $\bigcup_{n\in\mathbb{N}}F_n$ is a dense set. Suppose that $U\cap\bigcup_{n\in\mathbb{N}}F_n=\emptyset$ for some nonempty $U\in\mathcal{T}$ and take any $z\in U$. Then we can find $m\in\mathbb{N}$ such that $U_m(z)\subset U$. By $(N1)$, we can find a $k\in\mathbb{N}$ such that $(U_k(z)\cup U_{s(k)}(z))\cap\bigcup_{n\in\mathbb{N}}F_n=\emptyset$. Since $z\notin F_{s(k)}$ and $F_{s(k)}$ is a maximal element of $\mathcal{A}_{s(k)}$, we deduce that $F_{s(k)}\cup\{z\}\notin\mathcal{A}_{s(k)}$. Therefore there exists an $x\in F_{s(k)}$ for which $z\in U_{s(k)}(x)$. By condition $(N2)$, we have $x\in U_k(z)$, whence $x\in U$. Hence $U\cap F_{s(k)}\neq\emptyset$, contrary to $U\cap\bigcup_{n\in\mathbb{N}}F_n=\emptyset$. It follows that $U\cap\bigcup_{n\in\mathbb{N}}F_n\neq\emptyset$, so $\bigcup_{n\in\mathbb{N}}F_n$ is a dense set. ∎

So far, we have discussed the ω-problem for nonnegative real functions defined on a dense in itself topological space and with finite values. Now we will show that the assumption in Theorems 3.4 and 3.5 that $(X,\mathcal{T})$ is dense in itself can be omitted.

THEOREM 3.10. *Let $(X,\mathcal{T})$ be a topological space and let $f\colon X\to[0,+\infty)$ be any upper semicontinuous function vanishing at each isolated point. If $A\subset X^d$ has the properties:*

(1) $\operatorname{cl}(A\cup(X\setminus X^d))=\operatorname{cl}(X\setminus A)=X$,
(2) $\{x\in A: f(x)-\limsup_{t\to x}f(t)>\varepsilon\}$ *is closed for each* $\varepsilon>0$,
(3) $\limsup_{X\setminus A\ni t\to x}f(t)=\limsup_{t\to x}f(t)$ *for all* $x\in A$,

then the function $F\colon X\to\mathbb{R}$ defined by
$$F(x)=\begin{cases} f(x) & \text{if } x\in X\setminus A,\\ -(f(x)-\limsup_{t\to x}f(t)) & \text{if } x\in A,\end{cases}\qquad \textit{satisfies } \omega(F,\cdot)=f.$$

Proof. Let $\widetilde{X} = (X^d \times \{0\}) \cup ((X \setminus X^d) \times [0,1])$. We define a topology $\widetilde{\mathcal{T}}$ on $\widetilde{X}$ in the following way. Let $(x,t) \in \widetilde{X}$. If $t = 0$, then for a local base at (x,t) we take

$$\big\{((U \cap X^d) \times \{0\}) \cup ((U \setminus X^d) \times [0,\varepsilon)) : x \in U \in \mathcal{T},\ \varepsilon \in (0,1)\big\};$$

and if $t \neq 0$ and $x \notin X^d$, then for a local base at (x,t) we take

$$\big\{\{x\} \times (t-\varepsilon, t+\varepsilon) : \varepsilon \in (0,t)\big\}.$$

It is easy to see that $(\widetilde{X}, \widetilde{\mathcal{T}})$ is a topological space dense in itself and $X \times \{0\}$ is a closed subset of $\widetilde{X}$ homeomorphic to X. Define $\widetilde{f} \colon \widetilde{X} \to [0,\infty)$ by

$$\widetilde{f}(x,t) = \begin{cases} f(x) & \text{if } x \in X^d, \\ 0 & \text{if } x \notin X^d. \end{cases}$$

Put

$$\widetilde{A} = (A \times \{0\}) \cup \big((X \setminus X^d) \times (Q \cap (0,1))\big).$$

It is obvious that $\widetilde{A}$ and $\widetilde{f}$ satisfy conditions 1–3. Therefore, $\widetilde{F} \colon \widetilde{X} \to \mathbb{R}$ defined by

$$\widetilde{F}(y) = \begin{cases} \widetilde{f}(y) & \text{if } y \in \widetilde{X} \setminus \widetilde{A}, \\ -(\widetilde{f}(y) - \limsup_{t \to y} \widetilde{f}(t)) & \text{if } x \in \widetilde{A}, \end{cases}$$

has $\omega(\widetilde{F}, \cdot) = \widetilde{f}$. Define $F \colon X \to \mathbb{R}$ by $F(x) = \widetilde{F}((x,0))$ for all $x \in X$. Since $\widetilde{F}(t) = 0$ for $t \in \widetilde{X} \setminus (X \times \{0\})$ and

$$\mathrm{cl}\,\big(\widetilde{X} \setminus (X \times \{0\})\big) \cap \big(X \times \{0\}\big) = \mathrm{cl}\,\big(X \setminus X^d\big) \times \{0\},$$

we deduce that $\omega(F,x) = \omega(\widetilde{F},(x,0)) = \widetilde{f}((x,0)) = f(x)$ for all $x \in X$. ∎

THEOREM 3.11. *Let $(X, \mathcal{T})$ be a regular first countable topological space. Assume that there exists a family $\{F_n : n \geq 1\}$ of nonempty subsets of X satisfying:*

(a) *each F_n is closed and discrete,*
(b) *$\bigcup_{n\in\mathbb{N}} F_n$ is dense in X,*
(c) *for each $n \geq 1$ there exists a locally finite family $\{G_t^n : t \in F_n\}$ of pairwise disjoint open subsets of X indexed by elements F_n such that each G_t^n is a neighborhood of t.*

Then for any $f \colon X \to [0,+\infty)$ upper semicontinuous and $g \colon X \to (0,+\infty)$ lower continuous, there exists $F \colon X \to \mathbb{R}$ for which $\omega(F,\cdot) = f$ and $-g < F \leq f$.

Proof. The proof is similar to the previous one. Let $\widetilde{X} = X^d \times \{0\} \cup (X \setminus X^d) \times [0,1]$. We define a topology $\widetilde{\mathcal{T}}$ on $\widetilde{X}$. If $(x,t) \in \widetilde{X}$ and $t = 0$, then for a local base at (x,t) we take

$$\big\{((U \cap X^d) \times \{0\}) \cup ((U \setminus X^d) \times [0,\varepsilon)) : x \in U \in \mathcal{T},\ \varepsilon \in (0,1)\big\};$$

and if $(x,t) \in \widetilde{X}$ and $t \neq 0$ and $x \notin X^d$, then for a local base at (x,t) we take

$$\big\{\{x\} \times (t-\varepsilon, t+\varepsilon) : \varepsilon \in (0,t)\big\}.$$

It is easy to see that $(\widetilde{X}, \widetilde{\mathcal{T}})$ is a regular first countable and dense in itself topological space. Let

$$\widetilde{F_n} = ((F_n \cap X^d) \times \{0\}) \cup ((X \setminus X^d) \times \{q_n\}),$$
$$\widetilde{G_n} = \{U \times \{0\} : U \in G_n\} \cup \{\{t\} \times (q_n/2, 1)\},$$

where $(q_n)_{n\in\mathbb{N}}$ is a sequence of all the rational numbers from $(0,1)$. Clearly, $(\widetilde{X}, \widetilde{\mathcal{T}})$ and the families $(F_n)_{n\in\mathbb{N}}$ and $(G_n)_{n\in\mathbb{N}}$ satisfy all the assumptions of Theorem 3.5.

Define $\widetilde{f}\colon \widetilde{X} \to [0,\infty)$ and $\widetilde{g}\colon \widetilde{X} \to (0,\infty)$ by

$$\widetilde{f}(x,t) = \begin{cases} f(x) & \text{if } x \in X^d, \\ 0 & \text{if } x \notin X^d, \end{cases}$$

and $\widetilde{g}(x,t) = g(x)$ for $(x,t) \in \widetilde{X}$. Obviously, $\widetilde{f}$ is upper semicontinuous and $\widetilde{g}$ is lower semicontinuous. By Theorem 3.5, there exists $\widetilde{F}\colon \widetilde{X} \to \mathbb{R}$ such that $\omega(\widetilde{F},\cdot) = \widetilde{f}$ and $-\widetilde{g} < \widetilde{F} \le \widetilde{f}$. Moreover, $\widetilde{F}(x,t) = 0$ for $x \notin X^d$, $t \in [0,1]$. Define $F\colon X \to \mathbb{R}$ by $F(x) = \widetilde{F}(x,0)$. Then $\omega(F,\cdot) = f$ and $-g < F \le f$. ∎

Corollary 3.5. *Let $(X,\mathcal{T})$ be a regular first countable and separable topological space. Then for any $f\colon X \to [0,+\infty)$ upper semicontinuous and $g\colon X \to (0,+\infty)$ lower semicontinuous, there exists $F\colon X \to \mathbb{R}$ such that $\omega(F,\cdot) = f$ and $-g < F \le f$.*

If we want any upper semicontinuous functions with an infinite value to have an ω-primitive, then we need the additional assumption that X is perfectly normal.

Theorem 3.12. *Let $(X,\mathcal{T})$ be a regularly resolvable perfectly normal topological space. Moreover, assume that for each nonempty open set $U \subset X$ and for each upper semicontinuous function $g\colon U \to [0,+\infty)$ vanishing at all isolated points of X there exists a set $A_{U,g} \subset U$ such that the function $G\colon U \to \mathbb{R}$ defined by*

$$G(x) = \begin{cases} g(x) & \text{if } x \in U \setminus A_{U,g}, \\ -(g(x) - \limsup_{t\to x} g(t)) & \text{if } x \in A_{U,g}, \end{cases}$$

satisfies $\omega(G,\cdot) = g$. Then each upper semicontinuous $f\colon X \to [0,+\infty]$ (possibly taking infinite values) vanishing at all isolated points of X has an ω-primitive.

Proof. Clearly, we may assume that the set $B_f = \{x \in X : f(x) = +\infty\}$ is nonempty (otherwise the proof is trivial). Since f is upper semicontinuous, B_f is closed. Moreover, $B_f \subset X^d$, because f vanishes at isolated points. By the assumptions, there exists $\widetilde{F}\colon X \setminus B_f \to \mathbb{R}$ such that $\omega(\widetilde{F},x) = f(x)$ for $x \in X \setminus B_f$ and $\{x \in X \setminus B_f : \widetilde{F}(x) \ge 0\}$ is dense in $X \setminus B_f$. By the perfect normality of X, we can find a continuous function $h\colon X \to [0,1]$ such that $h(x) = 0$ for $x \in B_f$ and $h(x) > 0$ for $x \in X \setminus B_f$. Since X is regularly resolvable, there exists a decreasing family $(C_n)_{n\in\mathbb{N}}$ of dense subsets of X for which $X^d \cap \bigcap_{n\in\mathbb{N}} C_n = \emptyset$. Now define $F\colon X \to \mathbb{R}$ by

$$F(x) = \begin{cases} \widetilde{F}(x) + 1/h(x) & \text{if } x \in X \setminus B_f, \\ n & \text{if } x \in B_f \cap (C_n \setminus C_{n-1}),\ n = 2,3,\ldots, \\ 0 & \text{if } x \in B_f \setminus C_1. \end{cases}$$

Since $X \setminus B_f$ is open and $1/h\colon X \setminus B_f \to (0,+\infty)$ is continuous,

$$\omega(F,x) = \omega(\widetilde{F} + 1/h, x) = \omega(\widetilde{F}, x) = f(x)$$

for $x \in X \setminus B_f$. Moreover,

$$\limsup_{t\to x} F(x) = +\infty \qquad \text{for } x \in \bigcap_{n\in\mathbb{N}} \mathrm{cl}(B_f \cap C_n).$$

Since the C_n are dense, we deduce that $\mathrm{cl}(\mathrm{int}(B_f)) \subset \mathrm{cl}(B_f \cap C_n)$ for each $n \ge 1$. Therefore $\omega(F,x) = +\infty = f(x)$ for $x \in \mathrm{cl}(\mathrm{int}(B_f))$.

By the definition of h, it follows that $\lim_{t\to x} 1/h(t) = +\infty$ if $x \in \mathrm{cl}(X \setminus B_f) \cap B_f$. Since $\widetilde{F}$ is nonnegative on a dense subset of $X \setminus B_f$, we have $\limsup_{t\to x} F(t) = +\infty$ for $x \in \mathrm{cl}(X \setminus B_f) \cap B_f$. Hence $\omega(F,x) = +\infty = f(x)$ for $x \in \mathrm{cl}(X \setminus B_f) \cap B_f$. It follows that $\omega(F,x) = f(x)$ for all x in $(X \setminus B_f) \cup \mathrm{cl}(\mathrm{int}(B_f)) \cup (\mathrm{cl}(X \setminus B_f) \cap B_f) = X$. ■

THEOREM 3.13. *Let $(X,\mathcal{T})$ be a first countable and perfectly normal topological space. Assume that there exists a family $\{F_n : n \ge 1\}$ of pairwise disjoint subsets of X satisfying:*

(a) *each F_n is closed and discrete,*
(b) *$\bigcup_{n\in\mathbb{N}} F_n$ is dense in X,*
(c) *for each $n \ge 1$ there exists a locally finite family $\{G^n_t : t \in F_n\}$ of open subsets of X such that each G^n_t is a neighborhood of t.*

Then for any $f\colon X \to [0,+\infty]$ upper semicontinuous vanishing at all isolated points of X and $g\colon X \to (0,+\infty)$ lower semicontinuous, there exists $F\colon X \to \mathbb{R}$ such that $\omega(F,\cdot) = f$ and $-g < F$.

Proof. Without loss of generality, we may assume that g is bounded. Put $Y = \{x \in X : f(x) \neq \infty\}$. Since Y is an open subset of X, $(Y,\mathcal{T}_Y)$ is a perfectly normal topological space satisfying all the assumptions of Theorem 3.11. Moreover, $\tilde{f} = f_{|Y}$ is upper semicontinuous and $\tilde{g} = g_{|Y}$ is lower semicontinuous. By Theorem 3.11, there exists a function $\widetilde{F}\colon Y \to \mathbb{R}$ such that $\omega(\widetilde{F},\cdot) = \tilde{f}$ and $-\tilde{g} < \widetilde{F} \le \tilde{f}$. Let $Z = \mathrm{int}(X \setminus Y) = \mathrm{int}(\{x \in X : f(x) = \infty\})$. Then Z is an open subset of X without isolated points, because $f_{|(X\setminus X^d)} = 0$. In [H] it was proven that each dense in itself first countable topological space is resolvable. Therefore we can find $Z_1 \subset Z$ and $W_1 \subset Z$ such that $Z_1 \cap W_1 = \emptyset$, $Z \subset \mathrm{cl}(Z_1)$ and $Z \subset \mathrm{cl}(W_1)$. On the other hand, W_1 is a dense in itself first countable topological space too. Hence we can find $Z_2 \subset W_1$ and $W_2 \subset W_1$ for which $Z_2 \cap W_2 = \emptyset$, $W_1 \subset \mathrm{cl}(Z_2)$ and $W_1 \subset \mathrm{cl}(Y_2)$. It follows that $Z \subset \mathrm{cl}(W_1) \subset \mathrm{cl}(Z_2)$ and $Z \subset \mathrm{cl}(W_1) \subset \mathrm{cl}(W_2)$. Repeating the above construction, we can inductively define a sequence $(Z_n)_{n\in\mathbb{N}}$ of pairwise disjoint subsets of Z such that $Z \subset \mathrm{cl}(Z_n)$ for $n = 1,2,\ldots.$ Since X is perfectly normal, there exists a continuous function $h\colon X \to [0,1]$ such that $h^{-1}(0) = X \setminus Y$. Therefore

$$\lim_{t\to x} h(t) = 0 \qquad \text{for } x \in \mathrm{Fr}(X \setminus Y).$$

We now define $F\colon X \to \mathbb{R}$ by

$$F(x) = \begin{cases} \widetilde{F}(x) + 1/h(x) & \text{if } x \in Y, \\ 0 & \text{if } x \in Z \setminus \bigcup_{n=1}^{\infty} Z_n, \\ n & \text{if } x \in Z_n. \end{cases}$$

Since Y is an open subset of X and the function $1/n$ is continuous on Y, we have $\omega(F,x) = \omega(\widetilde{F},x) = \widetilde{f}(x) = f(x)$ for $x \in Y$. Moreover, $M_F(x) = +\infty$ and $\omega(F,x) = +\infty = f(x)$ for $x \in \mathrm{int}(Z)$. Since $-g < F$ and g is bounded, we deduce that $m_F(x)$ is finite for all $x \in X$. Moreover, applying the fact that $\lim_{t\to x} h(t) = 0$ for $x \in \mathrm{Fr}(X\setminus Y)$, we have $M_F(x) = +\infty$ and $\omega(F,x) = +\infty = f(x)$ for $x \in \mathrm{Fr}(X \setminus Y)$. Finally $\omega(F,\cdot) = f$. The inequality $-g < F$ follows directly from the construction of F and from the properties of $\widetilde{F}$. ■

The following theorem is an easy corollary of the previous theorem.

Theorem 3.14. *Let $(X,\mathcal{T})$ be a perfectly normal first countable separable topological space. Then for any $f\colon X \to [0,+\infty]$ upper semicontinuous and $g\colon X \to (0,+\infty)$ lower semicontinuous, there exists an $F\colon X \to \mathbb{R}$ such that $\omega(F,\cdot) = f$ and $-g < F$.*

Example 3.4. Define $f\colon \mathbb{R} \to [0,\infty]$ by

$$f(x) = \begin{cases} 0 & \text{if } x \neq 0, \\ \infty & \text{if } x = 0, \end{cases}$$

and $g\colon \mathbb{R} \to (0,\infty)$ by $g(x) = 1$ for $x \in \mathbb{R}$. Then f is upper semicontinuous and g is lower semicontinuous. If $F\colon \mathbb{R} \to \mathbb{R}$ is any function for which $-g < F \leq f$, then F is bounded by $\max\{1, |f(0)|\}$. In particular, $\omega(F,0) \neq \infty$. Hence $\omega(F,\cdot) \neq f$. Since $\mathbb{R}$ is a metric space (with the natural metric), we know that f has an ω-primitive. On the other hand, there is no ω-primitive for f lying between $-g$ and f.

The next example shows that the assumption in Theorem 3.13 that $(X,\mathcal{T})$ is perfectly normal cannot be omitted.

Example 3.5. Let $(X,\mathcal{T})$ be the Niemytzki plane. Then $(X,\mathcal{T})$ is completely regular separable and first countable, but $(X,\mathcal{T})$ is not perfectly normal. By Corollary 3.2, each nonnegative upper semicontinuous function with finite values has an ω-primitive. We will show that this is not true for a function with infinite values. Put $Y \subset X$, $Y = \mathbb{Q} \times \{0\}$. Define $f\colon X \to [0,+\infty]$ by

$$f(z) = \begin{cases} 0 & \text{if } z \in X \setminus Y, \\ +\infty & \text{if } z \in Y. \end{cases}$$

Since Y is a closed subset of X, we infer that f is upper semicontinuous. Assume that there exists $F\colon X \to \mathbb{R}$ such that $\omega(F,\cdot) = f$. Then $f(z) = \omega(F,z) = 0$ and F is continuous at each $z \in X\setminus Y$. Therefore for each $x \in \mathbb{R}\setminus\mathbb{Q}$ there exists a positive integer k_x such that $F(x,y) < k_x$ for $y \in (0,1/k_x)$. Let $A_n = \{x \in \mathbb{R}\setminus\mathbb{Q} : k_x = n\}$. By the completeness of $\mathbb{R}$ with the natural metric and by the countability of $\mathbb{Q}$, we can find an open interval (a,b) and a positive integer n_0 for which $(a,b) \subset \mathrm{cl}(A_{n_0})$. It follows that

$$F(x,y) < n_0 \quad \text{for } (x,y) \in ((a,b) \cap (\mathbb{R}\setminus\mathbb{Q})) \times (0,1/n_0).$$

Since

$$(a,b) \times (0,1/n_0) \subset \mathrm{cl}((a,b) \cap (\mathbb{R}\setminus\mathbb{Q}))$$

and F is continuous on $(a,b) \times (0,1/n_0)$, we deduce that $F(x,y) \leq n_0$ for $(x,y) \in (a,b) \times (0,1/n_0)$. Finally, if $x \in \mathbb{Q} \cap (a,b)$, then $\omega(F,(x,0)) < +\infty$ and $\omega(F,(x,0)) \neq f((x,0))$, which contradicts the assumption that $\omega(F,\cdot) = f$. This proves that there is no $F\colon X \to \mathbb{R}$ for which $\omega(F,\cdot) = f$.

4. The ω-problem for a massive space

In the theory of the ω-problem for metric spaces, an important role is played by so-called massive spaces. Each upper semicontinuous function f defined on a massive metric space has an ω-primitive of the very simple form $F = f \cdot \chi_A$, where χ_A is the characteristic function of some set A of type F_σ. In this chapter, we study the problem of the existence of a set $A \subset X$ for which $\omega(f \cdot \chi_A, \cdot) = f$ for a function defined on a massive topological space.

Definition 4.1 ([DGP]). Let $(X, \mathcal{T})$ be a topological space. We say that the space X is

- *σ-discrete at $x \in X$* if there exists a neighborhood U of x which is a σ-discrete set,
- *massive at $x \in X$* if it is not σ-discrete at x, which means that no neighborhood of x is a σ-discrete set,
- *massive* if it is massive at each point $x \in X$.

Obviously, a massive topological space is dense in itself. Let $(X, \mathcal{T})$ be a topological space. We denote by $\mathcal{T}_\mathbb{R}$ the topology on $X \times \mathbb{R}$ that is the Cartesian product of $\mathcal{T}$ and the natural topology on $\mathbb{R}$. Let $f\colon X \to \mathbb{R}$ be any function. We will consider the graph $\operatorname{Gr} f = \{(x, f(x)) : x \in X\}$ as a topological space endowed with the topology from $(X \times \mathbb{R}, \mathcal{T}_\mathbb{R})$.

First, we will find necessary and sufficient conditions on a set A for a function of the form $f \cdot \chi_A$ to be an ω-primitive of an upper semicontinuous function $f\colon X \to [0, \infty)$.

Theorem 4.1. *Suppose $(X, \mathcal{T})$ is a topological space, A is a nonempty subset of X, and $f\colon X \to [0, +\infty)$ is an upper semicontinuous function vanishing at isolated points of X. Then the following conditions are equivalent:*

(1) $\omega(f \cdot \chi_A, \cdot) = f$,
(2) *the graph of f restricted to the set $B = A \cup \{x \in X : f(x) = 0\}$ is dense in the graph of f, and for $\varepsilon > 0$ the set*

$$(X \setminus A) \cup \{x \in A : f(x) < \varepsilon\}$$

is dense in X.

Proof. First, assume that $\omega(f \cdot \chi_A, \cdot) = f$. Since $0 \le f \cdot \chi_A \le f$ and f is upper semicontinuous, $M_{f\cdot\chi_A} \le M_f = f$ and $m_{f\cdot\chi_A} \ge 0$. Hence $M_{f\cdot\chi_A} = f$ and $m_{f\cdot\chi_A} = 0$. From $m_{f\cdot\chi_A} = 0$ it follows that for each $\varepsilon > 0$ the set $(X \setminus A) \cup \{x \in A : f(x) < \varepsilon\}$ is dense in X.

Fix any $(x, f(x)) \in \operatorname{Gr} f$. If $f(x) = 0$, then $(x, f(x)) \in \operatorname{Gr} f_{|B}$. So, we may assume that $f(x) > 0$. Let U be any neighborhood of x and take any $\varepsilon \in (0, f(x))$. Since $M_{f\cdot\chi_A}(x) = f(x)$, there exists a $y \in U \cap A$ such that $|(f \cdot \chi_A)(y) - f(x)| < \varepsilon$. Since U and ε were arbitrary, we have $(x, f(x)) \in \operatorname{cl}(\operatorname{Gr} f_{|B})$. Hence $\operatorname{Gr} f_{|B}$ is dense in $\operatorname{Gr} f$.

Now, assume that (2) holds. Then $m_{f\cdot\chi_A} \le 0$. On the other hand, $f \cdot \chi_A \ge 0$. Hence $m_{f\cdot\chi_A} \ge 0$. Thus $m_{f\cdot\chi_A} = 0$. Since $f \cdot \chi_A \le f$ and f is upper semicontinuous, we deduce that $M_{f\cdot\chi_A} \le M_f = f$. Let $x \in X$. If $f(x) = 0$, then $0 \le M_{f\cdot\chi_A}(x) \le M_f(x) = f(x) = 0$ because f is nonnegative and upper semicontinuous. This implies that

$M_{f\cdot\chi_A}(x) = 0 = f(x)$. Finally, assume that $f(x) > 0$. Let U be any neighborhood of x and fix any $\varepsilon \in (0, f(x))$. By (2), there exists $y \in A$ such that $y \in U$ and $|f(y) - f(x)| < \varepsilon$. Hence $|(f \cdot \chi_A)(y) - f(x)| < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, we have $M_{f\cdot\chi_A}(x) \geq f(x)$. Thus we have proven that $M_{f\cdot\chi_A}(x) = f(x)$ for all $x \in X$. Therefore $M_{f\cdot\chi_A} = f$ and finally

$$\omega(f \cdot \chi_A, \cdot) = M_{f\cdot\chi_A} - m_{f\cdot\chi_A} = f - 0 = f. \ \blacksquare$$

Corollary 4.1. *Let $(X, \mathcal{T})$ be any dense in itself topological space and let $f\colon X \to [0, +\infty)$ be any upper semicontinuous function. If a set $A \subset X$ satisfies*

$$\mathrm{cl}(A) = \mathrm{cl}(X \setminus A) = X \quad \textit{and} \quad \mathrm{Gr}\, f_{|A} \textit{ is dense in } \mathrm{Gr}\, f,$$

then $\omega(f \cdot \chi_A, \cdot) = f$.

Corollary 4.2. *Let $(X, \mathcal{T})$ be a topological space and let $f\colon X \to [0, +\infty)$ be an upper semicontinuous function such that $\inf\{f(x) : x \in X\} > 0$. If $\omega(f \cdot \chi_A, \cdot) = f$ for a set $A \subset X$, then $\mathrm{cl}(A) = \mathrm{cl}(X \setminus A) = X$ and $\mathrm{Gr}\, f_{|A}$ is dense in $\mathrm{Gr}\, f$.*

We will need the following definition.

Definition 4.2. We say that a topological space $(X, \mathcal{T})$ is *weakly regularly resolvable* if it contains a dense subset which is σ-discrete. We say that a subset $A \subset X$ is *weakly regularly resolvable* if there exists a σ-discrete set $B \subset A$ such that $A \subset \mathrm{cl}(B)$.

Obviously, each regularly resolvable topological space is weakly regularly resolvable. The next example shows that the converse is not true.

Example 4.1. Let $X = [0, \omega_1]$, where ω_1 is the first uncountable ordinal number. Define a topology $\mathcal{T}$ on X in the following way: let a local base of a point $\eta \in X$, $\eta \neq \omega_1$, be $\{\{\eta\}\}$, and a local base of ω_1 be $\{(\xi, \omega_1] : \xi < \omega_1\}$. Then $A = [0, \omega_1)$ is a discrete subset of X and $\mathrm{cl}(A) = X$. Therefore $(X, \mathcal{T})$ is weakly regularly resolvable.

We will prove that $(X, \mathcal{T})$ is not regularly resolvable. Let $\{H_n : n \in \mathbb{N}\}$ be any countable family of discrete and closed subsets of X. Fix $n \in \mathbb{N}$. If $\omega_1 \in H_n$, then there exists $\eta_n < \omega_1$ such that $(\eta_n, \omega_1) \cap H_n = \emptyset$, since H_n is discrete. On the other hand, if $\omega_1 \notin H_n$, then there exists $\eta_n < \omega_1$ such that $(\eta_n, \omega_1] \cap H_n = \emptyset$, since H_n is closed. Therefore for each $n \in \mathbb{N}$ there exists $\eta_n < \omega_1$ for which $(\eta_n, \omega_1) \cap H_n = \emptyset$. Put $\eta = \sup\{\eta_n : n \geq 1\} < \omega_1$. Then $(\eta, \omega_1) \cap \bigcup_{n\in\mathbb{N}} H_n = \emptyset$ and (η, ω_1) is an open and nonempty subset of X. Hence $\mathrm{cl}(\bigcup_{n\in\mathbb{N}} H_n) \neq X$. This shows that $(X, \mathcal{T})$ is not a regularly resolvable space.

It turns out that in perfect topological spaces the notions of regular resolvability and weakly regular resolvability are equivalent.

Lemma 4.1. *If H is a discrete subset of a perfect topological space $(X, \mathcal{T})$, then H can be represented in the form $H = \bigcup_{n=1}^{\infty} H_n$, where each H_n is discrete and closed.*

Proof. By the assumptions, $H^d \cap H = \emptyset$. Since $(X, \mathcal{T})$ is a perfect space and H^d is a closed subset of H, there exists a countable family $\{U_n : n \in \mathbb{N}\}$ of open subsets of H such that $\bigcap_{n\in\mathbb{N}} U_n = H^d$. Let $H_n = H \setminus U_n$ for $n \in \mathbb{N}$. Obviously, each H_n is discrete and

$$\bigcup_{n\in\mathbb{N}} H_n = \bigcup_{n\in\mathbb{N}} (H \setminus U_n) = H \setminus \bigcap_{n\in\mathbb{N}} U_n = H \setminus H^d = H.$$

Moreover, $\mathrm{cl}(H_n) \subset X \setminus U_n$. Therefore

$$\mathrm{cl}(H_n) \cap H^d = \emptyset \quad \text{and} \quad \mathrm{cl}(H_n) = H_n \cup H_n^d \subset H_n.$$

Hence $\mathrm{cl}(H_n) = H_n$. ∎

Corollary 4.3. *In a perfect topological space, each σ-discrete set is an F_σ set.*

Corollary 4.4. *If a perfect topological space $(X, \mathcal{T})$ is weakly regularly resolvable, then it is regularly resolvable.*

Theorem 4.2. *Let $\{H_n : n \in \mathbb{N}\}$ be a family of nonempty closed and discrete subsets of a dense in itself topological space $(X, \mathcal{T})$. Then there exists an upper semicontinuous function $f\colon X \to [0, +\infty)$ such that*

$$\Big\{x \in X : f(x) > \limsup_{t \to x} f(t)\Big\} = \bigcup_{n \in \mathbb{N}} H_n.$$

Proof. Put $\widetilde{H}_1 = H_1$ and $\widetilde{H}_n = H_n \setminus \bigcup_{k<n} H_k$ for $n \geq 2$. Then each $\widetilde{H}_n$ is closed and discrete. Moreover, $\bigcup_{n \in \mathbb{N}} H_n = \bigcup_{n \in \mathbb{N}} \widetilde{H}_n$ and $\widetilde{H}_n \cap \widetilde{H}_k = \emptyset$ if $n \neq k$. Thus, without loss of generality, we may assume that the H_n are pairwise disjoint. Define $f\colon X \to [0, +\infty)$ by

$$f(x) = \begin{cases} 1/n & \text{if } x \in H_n,\ n = 1, 2, \ldots, \\ 0 & \text{if } x \in X \setminus \bigcup_{n \in \mathbb{N}} H_n. \end{cases}$$

It is easy to see that f is upper semicontinuous and $\limsup_{t \to x} f(t) = 0$ for $x \in X$. It follows that $f(x) - \limsup_{t \to x} f(t) = 1/n > 0$ if $x \in H_n$, $n = 1, 2, \ldots$, and $f(x) - \limsup_{t \to x} f(t) = 0$ if $x \in X \setminus \bigcup_{n \in \mathbb{N}} H_n$. ∎

Corollary 4.5. *Let $(X, \mathcal{T})$ be a dense in itself perfect topological space. If H is a nonempty σ-discrete subset of X, then there exists an upper semicontinuous function $f\colon X \to [0, +\infty)$ such that*

$$\Big\{x \in X : f(x) > \limsup_{t \to x} f(t)\Big\} = H.$$

Observe that for any $f\colon X \to [0, +\infty)$, any point $(x_0, f(x_0))$ where $x_0 \in \{x \in X : f(x) - \limsup_{t \to x} f(t) > 0\}$ is an isolated point of $\mathrm{Gr}\, f$. Therefore, if $A \subset X$ and $\mathrm{Gr}\, f_{|A}$ is dense in $\mathrm{Gr}\, f$, then

$$\Big\{x \in X : f(x) - \limsup_{t \to x} f(t) > 0\Big\} \subset A.$$

Thus, by Theorem 4.1, we have

Lemma 4.2. *If $(X, \mathcal{T})$ is a topological space, $f\colon X \to \mathbb{R}$ is an upper semicontinuous function, and A is a subset of X for which $\omega(f \cdot \chi_A, \cdot) = f$, then*

$$\Big\{x \in X : f(x) > \limsup_{t \to x} f(t)\Big\} \subset A.$$

Theorem 4.3. *Let $(X, \mathcal{T})$ be a dense in itself perfect topological space. If for each upper semicontinuous function $f\colon X \to [0, +\infty)$ there exists an $A \subset X$ such that $\omega(f \cdot \chi_A, \cdot) = f$, then X is a massive space.*

Proof. Let H be any σ-discrete subset of X. By Lemma 4.1 and Corollary 4.5, there exists an upper semicontinuous function $f\colon X \to [0, +\infty)$ such that

$$\Big\{x \in X : f(x) > \limsup_{t\to x} f(t)\Big\} = H.$$

By the assumptions, we can find an $A \subset X$ for which $\omega(f \cdot \chi_A, \cdot) = f$. Then $H \subset A$ by Lemma 4.2. Finally, by Corollary 4.2, $X \setminus A$ is dense in X. Hence $X \setminus H$ is dense. Therefore H is a boundary set. It follows that X is a massive space. ∎

THEOREM 4.4. *Assume that a topological space $(X, \mathcal{T})$ satisfies the following conditions:*

(1) *X is massive,*
(2) *each set $H \subset X$ of the form $H = F \cap G$, where F is closed and G open, is weakly regularly resolvable.*

Then for each upper semicontinuous function $f\colon X \to [0, +\infty)$ there exists a set $A \subset X$ such that

$$\mathrm{cl}(A) = \mathrm{cl}(X \setminus A) = X \quad \textit{and} \quad \mathrm{cl}(\mathrm{Gr}\, f_{|A}) = \mathrm{Gr}\, f.$$

Moreover, if each set $H \subset X$ of the form $H = F \cap G$, where F is closed and G open, is regularly resolvable, then we can choose A to be an F_σ.

Proof. Let $f\colon X \to [0, +\infty)$ be any upper semicontinuous function. Put

$$A_{n,k} = \{x \in X : (k-1)/n \le f(x) < k/n\}$$

for each $n, k \in \mathbb{N}$. Since f is upper semicontinuous,

$$A_{n,k} = \{x \in X : f(x) < k/n\} \cap \{x \in X : f(x) \ge (k-1)/n\}$$

is the intersection of an open and a closed set. By the assumptions, for each $A_{n,k}$ there exists a countable family $\{H_{n,k,m} : m \in \mathbb{N}\}$ of discrete sets such that

$$\bigcup_{m\in\mathbb{N}} H_{n,k,m} \subset A_{n,k} \quad \text{and} \quad A_{n,k} \subset \mathrm{cl}\Big(\bigcup_{m\in\mathbb{N}} H_{n,k,m}\Big).$$

Put

$$A = \bigcup_{n,k,m\in\mathbb{N}} H_{n,k,m}.$$

Then A is a σ-discrete set. Since X is massive, we have $\mathrm{cl}(X \setminus A) = X$. By the equality $X = \bigcup_{k\in\mathbb{N}} A_{1,k}$, we obtain

$$\mathrm{cl}(A) \supset \mathrm{cl}\Big(\bigcup_{k,m\in\mathbb{N}} H_{1,k,m}\Big) \supset \bigcup_{k\in\mathbb{N}} A_{1,k} = X.$$

Take any $(x, f(x)) \in \mathrm{Gr}\, f$. Let U be any neighborhood of x and let $\varepsilon > 0$. There exist natural numbers n and k such that $1/n < \varepsilon$ and $|f(x) - (2k-1)/2n| < 1/2n$. Hence $x \in A_{n,k}$. Since $\bigcup_{m\in\mathbb{N}} H_{n,k,m}$ is dense in $A_{n,k}$, we can find $m \in \mathbb{N}$ and $y \in H_{n,k,m}$ such that $y \in U$. Certainly, $y \in A_{n,k}$. Therefore $|f(y) - f(x)| \le 1/n < \varepsilon$. We have shown that

$$(U \times (f(x) - \varepsilon, f(x) + \varepsilon)) \cap \mathrm{Gr}\, f_{|A} \ne \emptyset.$$

Since U and $\varepsilon > 0$ were arbitrary, we obtain $(x, f(x)) \in \mathrm{cl}(\mathrm{Gr}\, f_{|A})$.

If we assume that if each set $H \subset X$ of the form $H = F \cap G$, where F is closed and G open, is regularly resolvable, then we can choose $H_{n,k,m}$ to be closed sets. Then the set A is of type F_σ. ■

Combining Corollary 4.1 and Theorem 4.4, we obtain the following theorem.

THEOREM 4.5. *Let $(X, \mathcal{T})$ be a massive topological space such that each subset which is the intersection of an open set and a closed set is weakly regularly resolvable. Then for any upper semicontinuous function $f\colon X \to [0, +\infty)$, there exists a σ-discrete set $A \subset X$ such that $\omega(f \cdot \chi_A, \cdot) = f$. Moreover, if all the $H \subset X$ of the form $H = F \cap G$ where F is closed and G is open, are regularly resolvable, then we can find an A which is an F_σ set.*

COROLLARY 4.6.

(1) *If a metric space (X, d) is massive, then for any upper semicontinuous function $f\colon X \to [0, +\infty)$, there exists an F_σ set $A \subset X$ such that $\omega(f \cdot \chi_A, \cdot) = f$.*
(2) *If a massive topological space $(X, \mathcal{T})$ is first countable and satisfies conditions $(N1)$ and $(N2)$ (page 27) then for each upper semicontinuous function $f\colon X \to [0, +\infty)$, there exists an $A \subset X$ of type F_σ such that $\omega(f \cdot \chi_A, \cdot) = f$.*
(3) *If a massive topological space $(X, \mathcal{T})$ is second countable, then for each upper semicontinuous function $f\colon X \to [0, +\infty)$ there exists an $A \subset X$ of type F_σ such that $\omega(f \cdot \chi_A, \cdot) = f$.*

Proof. It is well known that each metric space is regularly resolvable, and each subset of a metric space is itself a metric space and therefore is regularly resolvable. Similarly, each first countable topological space satisfying conditions $(N1)$ and $(N2)$ is regularly resolvable, and each subset of such a space is first countable and satisfies $(N1)$ and $(N2)$ and therefore is regularly resolvable. Finally, each subset of a second countable topological space is separable and therefore is regularly resolvable. The remaining part of the proof follows from the last theorem. ■

THEOREM 4.6. *Let $(X, \mathcal{T})$ be a dense in itself topological space such that for every upper semicontinuous function $f\colon X \to [0, +\infty)$ there exists a σ-discrete set $A \subset X$ for which $\omega(f \cdot \chi_A, \cdot) = f$. Then each $H \subset X$ that is the intersection of an open and a closed set is weakly regularly resolvable.*

Proof. Let $P = H \cap U$, where H is closed and U is open. Without loss of generality, we may assume that $P \neq \emptyset$. Define $f\colon X \to [0, +\infty)$ by

$$f(x) = \begin{cases} 1 & \text{if } x \in U \setminus H, \\ 2 & \text{if } x \in U \cap H, \\ 5 & \text{if } x \in X \setminus U. \end{cases}$$

Since

$$\{x \in X : f(x) < a\} = \begin{cases} \emptyset & \text{if } a \leq 1, \\ U \setminus H & \text{if } 1 < a \leq 2, \\ U & \text{if } 2 < a \leq 5, \\ X & \text{if } 5 < a, \end{cases}$$

f is upper semicontinuous. By the assumptions, there exists a σ-discrete set A contained in X such that $\omega(f\cdot\chi_A,\cdot)=f$. In particular, $\omega(f\cdot\chi_A,x)=2$ for $x\in H\cap U$. The function $f\cdot\chi_A$ takes at most four values 0, 1, 2, 5. It follows that $M_{f\cdot\chi_A}$ and $m_{f\cdot\chi_A}$ take at most these four values too. Therefore $M_{f\cdot\chi_A}(x)=2$ and $m_{f\cdot\chi_A}(x)=0$ for $x\in H\cap U$. Thus

$$P\subset \operatorname{cl}(\{x\in X: f\cdot\chi_A(x)=2\}).$$

On the other hand,

$$\{x\in X: f\cdot\chi_A(x)=2\}=P\cap A$$

and $P\cap A$ is σ-discrete as a subset of a σ-discrete set A. It follows that P is weakly regularly resolvable. ∎

Corollary 4.7. *Let $(X,\mathcal{T})$ be a massive topological space. Then the following conditions are equivalent:*

(1) *each subset of X, which is the intersection of an open and a closed set is weakly regularly resolvable,*
(2) *for each upper semicontinuous function $f\colon X\to[0,+\infty)$ there exists a σ-discrete set $A\subset X$ such that $\omega(f\cdot\chi_A,\cdot)=f$.*

Theorem 4.7. *Suppose that $(X,\mathcal{T})$ is a topological space, $f\colon X\to[0,+\infty)$ is an upper semicontinuous function, and $A\subset X$ is a set such that $\omega(f\cdot\chi_A,\cdot)=f$. If $A\subset B$ and $\operatorname{cl}(X\setminus B)=X$, then $\omega(f\cdot\chi_B,\cdot)=f$.*

Proof. Since $f\cdot\chi_A\le f\cdot\chi_B\le f$, we have $f=M_{f\cdot\chi_A}\le M_{f\cdot\chi_B}\le f$. Therefore $M_{f\cdot\chi_B}=f$.

On the other hand, $m_{f\cdot\chi_B}=0$, because $f\cdot\chi_B\ge 0$ and $\operatorname{cl}(X\setminus B)=X$. Thus $\omega(f\cdot\chi_B,\cdot)=f$. ∎

Each massive topological space is dense in itself. Nevertheless, similar theorems are true for spaces with isolated points.

Theorem 4.8. *Let $(X,\mathcal{T})$ be a topological space such that X^d is a massive space and each subset of X^d is weakly regularly resolvable. Then for any upper semicontinuous function $f\colon X\to[0,+\infty)$ vanishing at all isolated points there exists a σ-discrete set A contained in X such that $\omega(f\cdot\chi_A,\cdot)=f$.*

Proof. The proof of the theorem is similar to the proof of Theorem 3.11. Put $\widetilde{X}=(X^d\times\{0\})\cup((X\setminus X^d)\times[0,1])$. Define a topology $\widetilde{\mathcal{T}}$ on $\widetilde{X}$ in the following way. A local base at a point $(x,t)\in\widetilde{X}$ is

$$\big\{((U\cap X^d)\times\{0\})\cup((U\setminus X^d)\times[0,\varepsilon)) : U\in\mathcal{T}(x),\ \varepsilon\in(0,1)\big\}$$

if $t=0$ and

$$\big\{\{x\}\times(t-\varepsilon,t+\varepsilon):\varepsilon\in(0,t)\cap(0,1-t)\big\},$$

if $t>0$ and $x\notin X^d$. It is easily seen that $X^d\times\{0\}$ is homeomorphic to X^d and $(\widetilde{X},\widetilde{\mathcal{T}})$ is a massive space. Moreover, the subspace $(X\setminus X^d)\times[0,1]\subset\widetilde{X}$ is metrizable by the metric

$$\varrho((x_1,t_1),(x_2,t_2))=\begin{cases}1 & \text{if } x_1\ne x_2,\\ |t_1-t_2| & \text{if } x_1=x_2.\end{cases}$$

Therefore each subset of $(X \setminus X^d) \times [0,1]$ is weakly regularly resolvable. Moreover, if $Z \subset (X \setminus X^d) \times [0,1]$ is discrete in $(X \setminus X^d) \times [0,1]$, then all the sets $Z_0 = Z \cap (X \setminus X^d) \times \{0\}$ and $Z_n = Z \cap (X \setminus X^d) \times [1/(n+1), 1/n]$ for $n \geq 1$ are discrete in $(\widetilde{X}, \widetilde{\mathcal{T}})$ and $Z = Z_0 \cup \bigcup_{n=1}^{\infty} Z_n$. Hence every subset of $(\widetilde{X}, \widetilde{\mathcal{T}})$ is weakly regularly resolvable.

Let $f\colon X \to [0, +\infty)$ be any upper semicontinuous function vanishing at isolated points of X. Define $\widetilde{f}\colon \widetilde{X} \to [0, \infty)$ by

$$\widetilde{f}(x,t) = \begin{cases} f(x) & \text{if } x \in X^d, \\ 0 & \text{if } x \notin X^d. \end{cases}$$

Obviously, $\widetilde{f}$ is upper semicontinuous. By Theorem 4.5, there exists a σ-discrete set $\widetilde{A} \subset \widetilde{X}$ such that $\omega(\widetilde{f} \cdot \chi_{\widetilde{A}}, \cdot) = \widetilde{f}$. Put

$$A = \{x \in X : \exists_{t \in [0,1]}\, (x,t) \in \widetilde{A}\}.$$

Since $A \cap X^d$ is homeomorphic to $\widetilde{A} \cap (X^d \times \{0\})$ and $A \cap (X \setminus X^d)$ is a discrete set, we infer that A is σ-discrete. Moreover, $\widetilde{f} \geq 0$, $\omega(\widetilde{f} \cdot \chi_{\widetilde{A}}, \cdot) = \widetilde{f}$, and $\widetilde{f}(x,t) = 0$ for $x \in X \setminus X^d$. It follows that $\omega(f \cdot \chi_A, \cdot) = f$. ∎

So far, looking for a set A such that $\omega(f \cdot \chi_A, \cdot) = f$ for a given function f, we considered only functions with finite values. Now, we will briefly study the case where the functions may take infinite values. Assume that for some set A the function $f \cdot \chi_A$ is an ω-primitive for f. Then $f \cdot \chi_A$ must take only finite values. That is why we must assume $0 \cdot \infty = 0$. Thus $f \cdot \chi_A(x) = 0$ if $x \notin A$ and $f(x) = \infty$. It is easy to see that a necessary condition for the equality $\omega(f \cdot \chi_A, \cdot) = f$ is $A \subset \{x \in X : f(x) < +\infty\}$. Therefore if $\{x \in X : f(x) < +\infty\}$ is not dense, then there is no set $A \subset X$ for which $f \cdot \chi_A$ is a function with finite values and $\omega(f \cdot \chi_A, \cdot) = f$. It turns out that there is a stronger necessary condition for the existence of a set $A \subset X$ with the required properties.

THEOREM 4.9. *Let $(X, \mathcal{T})$ be a topological space and let $f\colon X \to [0, +\infty]$ be any upper semicontinuous function. Put*

$$A_\infty = \{x \in X : f(x) = \infty\} \quad \textit{and} \quad \widetilde{f} = f \cdot \chi_{|(X \setminus A_\infty)}.$$

If $M_{\widetilde{f}}(x) < +\infty$ at some $x \in A_\infty$, then there exists no set A for which $f \cdot \chi_A$ is finite and $\omega(f \cdot \chi_A, \cdot) = f$.

Proof. Let $x \in A_\infty$ with $M_{\widetilde{f}}(x) < +\infty$. Then there exists a real number K and a neighborhood U of x such that either $f(t) = +\infty$ or $f(t) < K$ for $t \in U$. Let A be any subset of $X \setminus A_\infty$. Then $f \cdot \chi_A(t) = 0$ for $t \in U \cap A_\infty$ and $f \cdot \chi_A(t) < K$ for $t \in U \setminus A_\infty$. It follows that $M_{f \cdot \chi_A}(x) \leq K \neq +\infty = f(x)$. ∎

REMARK 4.1. Observe that, in the notations of the last theorem, if $x \in \operatorname{int}(A_\infty)$, then $M_{\widetilde{f}}(x) = 0$.

THEOREM 4.10. *Let $(X, \mathcal{T})$ be a massive topological space such that each subset of X which is the intersection of an open and a closed set is weakly regularly resolvable. Suppose $f\colon X \to [0, +\infty]$ is an upper semicontinuous function. Put*

$$A_\infty = \{x \in X : f(x) = +\infty\} \quad \textit{and} \quad \widetilde{f} = f \cdot \chi_{(X \setminus A_\infty)}.$$

If

$$A_\infty \subset \{x \in X : M_{\widetilde{f}}(x) = +\infty\},$$

then there exists a σ-discrete set $A \subset \{x \in X : f(x) < +\infty\}$ for which $\omega(f \cdot \chi_A, \cdot) = f$.

Proof. Consider the function $g\colon X \setminus A_\infty \to [0, +\infty)$, $g = f_{|(X\setminus A_\infty)}$. Obviously, g is upper semicontinuous and $X \setminus A_\infty$, being an open subset of X, is a massive space. Let $B \subset X \setminus A_\infty$ and $B = H \cap G$, where H is a closed and G is an open subset of $X \setminus A_\infty$. Then there exists a closed set C such that $H = C \cap (X \setminus A_\infty)$. Since $X \setminus A_\infty$ is an open subset of X, the set $B = (G \cap (X \setminus A_\infty)) \cap C$ is the intersection of an open and a closed set. By the assumptions, B is weakly regularly resolvable. By Theorem 4.5, one can find a σ-discrete subset A of $X \setminus A_\infty$ such that $\omega(g \cdot \chi_A, \cdot) = g$.

Clearly, A is σ-discrete in X. Therefore $X \setminus A$ is dense. We claim that $\omega(f \cdot \chi_A, \cdot) = f$. By the openness of $X \setminus A_\infty$, this equality is obvious for $x \in X \setminus A_\infty$. Fix any $x \in A_\infty$. Then $f(x) = +\infty$ and $f \cdot \chi_A(x) = 0$. Let K be any positive number and let U be any neighborhood of x. By the assumptions, $M_{\widetilde{f}}(x) = +\infty$. Therefore we can find a $y \in U$ such that $\widetilde{f}(y) = f \cdot \chi_{X\setminus A_\infty}(y) > K$. Hence $y \in X \setminus A_\infty$ and $f(y) > K$. We know that $\omega(f \cdot \chi_A, y) = f(y)$. Since $\{x \in X : f \cdot \chi_A(x) = 0\}$ is dense in X and $f \cdot \chi_A \geq 0$, we have $m_{f\cdot\chi_A}(y) = 0$. Thus $M_{f\cdot\chi_A}(y) = f(y) > K$. By the last inequality and from the fact that U is a neighborhood of y, there exists a point $z \in A \cap U$ for which $f(z) > K$. Since U was an arbitrary neighborhood of x, we obtain $M_{f\cdot\chi_A}(x) > K$ for all $K > 0$. It follows that $M_{f\cdot\chi_A}(x) = +\infty$ and $\omega(f \cdot \chi_A, x) = +\infty = f(x)$. Thus we have shown that $\omega(f \cdot \chi_A, x) = f(x)$ for all $x \in X$. ∎

Corollary 4.8. *Let $(X, \mathcal{T})$ be a massive topological space such that each subset of X which is the intersection of an open and a closed set is weakly regularly resolvable. Moreover, suppose $f\colon X \to [0, +\infty]$ is an upper semicontinuous function, $A_\infty = \{x \in X : f(x) = +\infty\}$, and put $\widetilde{f} = f \cdot \chi_{X\setminus A_\infty}$. Then the following condition are equivalent:*

(1) *there exists a σ-discrete set $A \subset \{x \in X : f(x) < +\infty\}$ such that $\omega(f\chi_A, \cdot) = f$,*
(2) *$A_\infty \subset \{x \in X : M_{\widetilde{f}}(x) = +\infty\}$,*
(3) *if $f(x) = \infty$, then for all $K > 0$ and for all neighborhoods U of x, there exists an $y \in U$ for which $K < f(y) < \infty$.*

Proof. The equivalence (1)⇔(2) was proven in Theorems 4.9 and 4.10.

The equivalence (2)⇔(3) follows directly from the definition of $\widetilde{f}$. ∎

5. The ω^*-problem

Let (X, ϱ) be a dense in itself metric space. Given $F\colon X \to \mathbb{R}$, we can define two functions $f\colon X \to [-\infty, +\infty]$ and $g\colon X \to [-\infty, +\infty]$ by

$$f(x) = \limsup_{t\to x} F(t) \quad \text{and} \quad g(x) = \liminf_{t\to x} F(t) \qquad \text{for } x \in X.$$

It is clear that f is upper semicontinuous and g is lower semicontinuous. Moreover, we may define another function $\omega^*(F,\cdot)\colon X\to[0,\infty]$ by

$$\omega^*(F,x)=\lim_{r\to 0^+}\sup\{|f(y)-f(z)| : 0<\varrho(x,y)<r,\ 0<\varrho(x,z)<r\},$$

which is also upper semicontinuous. If

$$x_0\notin\{x\in X : f(x)=+\infty=g(x)\}\cup\{x\in X : f(x)=-\infty=g(x)\}$$

then

$$\omega^*(F,x_0)=\limsup_{x\to x_0}F(x)-\liminf_{x\to x_0}F(x).$$

On the other hand, it is easy to see that if $f(x)=+\infty=g(x)$ or $f(x)=-\infty=g(x)$, then $\omega^*(F,x)=\infty$.

Although the definitions of $\omega(f,\cdot)$ and $\omega^*(f,\cdot)$ are similar, their properties may be quite different.

EXAMPLE 5.1. Let $X=\{(2k-1)/2^n : k=1,\dots,2^{n-1}, n\in\mathbb{N}\cup\{0\}\}\subset\mathbb{R}$ and define $f\colon X\to\mathbb{R}$ by $f((2k-1)/2^n)=1/2^n$ for $(2k-1)/2^n\in X$.

It is easily seen that $\omega(f,\cdot)=f$ and $\omega^*(f,\cdot)=0$. Hence $\omega(f,x)\neq\omega^*(f,x)$ for $x\in X$.

We shall show that if (X,ϱ) is a dense in itself metric space, then for any upper semicontinuous function $h\colon X\to[0,+\infty]$, there exists an $F\colon X\to\mathbb{R}$ such that $h=\omega^*(F,\cdot)$. We shall prove even more. Under some weak additional conditions, for any pair of functions $f\colon X\to[-\infty,+\infty]$ and $g\colon X\to[-\infty,+\infty]$ such that f is upper semicontinuous, g is lower semicontinuous, and $g\le f$, there exists a function $F\colon X\to\mathbb{R}$ for which

$$\limsup_{t\to x}F(t)=f(x)\quad\text{and}\quad\liminf_{t\to x}F(t)=g(x)$$

for $x\in X$. Example 2.3 shows that there exist a metric space, an upper semicontinuous function $f\colon X\to\mathbb{R}$, and a lower semicontinuous function $g\colon X\to\mathbb{R}$ such that $g\le f$ and there are no functions $F\colon X\to\mathbb{R}$ which fulfill the conditions

$$M_F(x)=f(x)\quad\text{and}\quad m_F(x)=g(x)\qquad\text{for } x\in X.$$

First, we consider the case of functions f and g which have only finite values. Let $\varrho(x,A)$ denote the distance of a point x from a nonempty set A in a metric space (X,ϱ), and let

$$B(A,\varepsilon)=\bigcup_{x\in A}B(x,\varepsilon)=\bigcup_{x\in A}\{t\in X : \varrho(x,t)<\varepsilon\}$$

for $\emptyset\neq A\subset X$ and $\varepsilon>0$.

LEMMA 5.1. *Let (X,ϱ) be a dense in itself metric space. Given a dense subset M of X, a nonempty $A\subset X$, and an $\varepsilon>0$, there exists a set $T_{M,A,\varepsilon}\subset M$ such that*

(1) $\varrho(z_1,z_2)\ge\varepsilon$ *if* $z_1,z_2\in T_{M,A,\varepsilon}$, $z_1\neq z_2$,
(2) $\varrho(z,A)<\varepsilon$ *if* $z\in T_{M,A,\varepsilon}$,
(3) $\varrho(x,T_{M,A,\varepsilon})<2\varepsilon$ *if* $x\in A$.

Proof. First, observe that another way of stating (2) is $T_{M,A,\varepsilon}\subset B(A,\varepsilon)$, and an equivalent formulation of (3) is $A\subset B(T_{M,A,\varepsilon},2\varepsilon)$.

Fix $\varepsilon > 0$. Since M is a dense subset of X, $M \cap B(A,\varepsilon) \neq \emptyset$. Let $\mathfrak{B}$ be the set of all subsets B of X satisfying

(a) $B \subset M \cap B(A,\varepsilon)$,
(b) $\varrho(z_1,z_2) \geq \varepsilon$ for each $z_1,z_2 \in B$.

The family $\mathfrak{B}$ is nonempty because it contains each singleton $\{x\}$ where $x \in M \cap B(A,\varepsilon)$. Furthermore, $\mathfrak{B}$ is partially ordered by inclusion. It is easy to check that if $\{B_s : s \in S\}$ is a linear chain in $\mathfrak{B}$, then the set $B = \bigcup_{s\in S} B_s$ belongs to $\mathfrak{B}$ and B is an upper bound of $\{B_s : s \in S\}$. Hence, by the Zorn Lemma, the family $\mathfrak{B}$ has a maximal element, which we denote by $T_{M,A,\varepsilon}$.

We will show that $T_{M,A,\varepsilon}$ fulfils all the required conditions. By (a) it is clear that $T_{M,A,\varepsilon} \subset M$. Next, $\varrho(z_1,z_2) \geq \varepsilon$ for $z_1,z_2 \in T_{M,A,\varepsilon}$ from (b). Again applying (a), we get $T_{M,A,\varepsilon} \subset B(A,\varepsilon)$. Thus $\varrho(z,A) < \varepsilon$ for every $z \in T_{M,A,\varepsilon}$.

Suppose that $\varrho(x_0,T_{M,A,\varepsilon}) \geq 2\varepsilon$ for some $x_0 \in A$. Since M is a dense subset of X, there exists a $z_0 \in M$ such that $\varrho(x_0,z_0) < \varepsilon$. Hence

$$\varrho(t,z_0) \geq \varrho(t,x_0) - \varrho(x_0,z_0) > \varrho(x_0,T_{M,A,\varepsilon}) - \varepsilon \geq \varepsilon$$

provided $t \in T_{M,A,\varepsilon}$. It follows that $T_{M,A,\varepsilon} \cup \{z_0\} \in \mathfrak{B}$, which contradicts the fact that $T_{M,A,\varepsilon}$ is a maximal element of $\mathfrak{B}$. Therefore $\varrho(x,T_{M,A,\varepsilon}) < 2\varepsilon$ for every $x \in A$ and the set $T_{M,A,\varepsilon}$ satisfies all the conditions (1)–(3). ∎

Remark 5.1. It follows from condition (1) of the previous lemma that $T_{M,A,\varepsilon}$ is a closed and discrete set.

Theorem 5.1. *Suppose (X,ϱ) is a dense in itself metric space and Y is a dense subset of X. Suppose further that $f\colon X \to \mathbb{R}$ and $g\colon X \to \mathbb{R}$ are such that f is upper semicontinuous, g is lower semicontinuous, and $g \leq f$. Then there exists a function $F\colon X \to \mathbb{R}$ for which*

(1) $\limsup_{t\to x} F(t) = f(x)$ *and* $\liminf_{t\to x} F(t) = g(x)$ *for* $x \in X$,
(2) $F(x) = g(x)$ *if* $x \in X \setminus Y$.

Proof. Put

$$K = \{(n,k) \in \mathbb{N} \times \mathbb{Z} : -n^2 \leq k < n^2\}$$

and let $\preceq$ be the relation in K defined as follows:

$$(n_1,k_1) \preceq (n_2,k_2) \ \Leftrightarrow\ n_1 < n_2 \vee (n_1 = n_2 \wedge k_1 \leq k_2).$$

It is easily seen that K is well ordered by $\preceq$. Next, define

$$A_{n,k} = \{x \in X : k/n \leq f(x) < (k+1)/n\},$$
$$B_{n,k} = \{x \in X : k/n \leq g(x) < (k+1)/n\}$$

for $(n,k) \in K$. We shall construct by induction two families $\{R_{n,k} : (n,k) \in K\}$ and $\{S_{n,k} : (n,k) \in K\}$ of closed and discrete subsets of X which satisfy the following conditions:

(a) $R_{n_1,k_1} \cap R_{n_2,k_2} = \emptyset = S_{n_1,k_1} \cap S_{n_2,k_2}$ if $(n_1,k_1),(n_2,k_2) \in K$, $(n_1,k_1) \neq (n_2,k_2)$ and $R_{n,k} \cap S_{i,j} = \emptyset$ if $(n,k),(i,j) \in K$,

(b) $\bigcup_{(n,k)\in K}(R_{n,k}\cup S_{n,k})\subset Y$,
(c) $R_{n,k}\subset B(A_{n,k},1/n)$ and $S_{n,k}\subset B(B_{n,k},1/n)$ for $(n,k)\in K$,
(d) $\varrho(x,R_{n,k})<2/n$ for $x\in A_{n,k}$, $(n,k)\in K$ and $\varrho(x,S_{n,k})<2/n$ for $x\in B_{n,k}$, $(n,k)\in K$.

If $A_{n,k}=\emptyset$, then we set $R_{n,k}=\emptyset$, and if $B_{n,k}=\emptyset$, then we set $S_{n,k}=\emptyset$. We have to construct $R_{n,k}$ if $A_{n,k}\neq\emptyset$ and $S_{n,k}$ if $B_{n,k}\neq\emptyset$. Let $R_{1,-1}=T_{Y,A_{1,-1},1}$, where $T_{Y,A_{1,-1},1}$ is the set from Lemma 5.1 for $M=Y$, $A=A_{1,-1}$ and $\varepsilon=1$. Since $R_{1,-1}$ is a closed and discrete subset of X and X is dense in itself, $Y\setminus R_{1,-1}$ is dense in X. Thus we may set $S_{1,-1}=T_{Y\setminus R_{1,-1},B_{1,-1},1}$. Next, let

$$\widetilde{Y}_{1,0}=Y\setminus(R_{1,-1}\cup S_{1,-1}),\qquad R_{1,0}=T_{\widetilde{Y}_{1,0},A_{1,0},1},\qquad S_{1,0}=T_{\widetilde{Y}_{1,0}\setminus R_{1,0},B_{1,0},1}.$$

Fix $(n,k)\in K$. Assume that the closed and discrete sets $R_{i,j}$ and $S_{i,j}$ have already been chosen for $(i,j)\prec(n,k)$, and let

$$\widetilde{Y}_{n,k}=Y\setminus\bigcup_{(i,j)<(n,k)}(R_{i,j}\cup S_{i,j}).$$

Set

$$R_{n,k}=T_{\widetilde{Y}_{n,k},A_{n,k},1/n}\quad\text{and}\quad S_{n,k}=T_{\widetilde{Y}_{n,k}\setminus R_{n,k},B_{n,k},1/n}.$$

By Lemma 5.1, it is obvious that the families

$$\{R_{n,k}:(n,k)\in K\}\quad\text{and}\quad\{S_{n,k}:(n,k)\in K\}$$

constructed inductively satisfy the conditions (a)–(d). Define $F\colon X\to\mathbb{R}$ by

$$F(x)=\begin{cases}k/n & \text{if } x\in R_{n,k},\ (n,k)\in K,\\ (k+1)/n & \text{if } x\in S_{n,k},\ (n,k)\in K,\\ g(x) & \text{if } x\in X\setminus\bigcup_{(n,k)\in K}(R_{n,k}\cup S_{n,k}).\end{cases}$$

We shall show that (1) and (2) hold. Fix $x_0\in X$ and $\varepsilon>0$. There exists $n_0\in\mathbb{N}$ such that $1/n_0<\varepsilon$ and $f(x_0)<n_0+1$. For every $n\geq n_0$ we can find $k_n\in\mathbb{Z}$ such that $-n^2\leq k_n<n^2$ and $k_n/n\leq f(x_0)<(k_n+1)/n$. Thus $x_0\in A_{n,k_n}$. By condition (d), for every $n\geq n_0$ there exists $y_n\in R_{n,k_n}$ for which $d(x_0,y_n)<2/n$. Hence $\lim_{n\to\infty}y_n=x_0$. From this we obtain

$$F(y_n)=k_n/n\quad\text{and}\quad 0<f(x_0)-F(y_n)<1/n.$$

This gives $\lim_{n\to\infty}F(y_n)=f(x_0)$. We have proven that $\limsup_{x\to x_0}f(x)\geq f(x_0)$. In the same manner we can prove that $\liminf_{x\to x_0}f(x)\leq g(x_0)$.

Let $(x_m)_{m\in\mathbb{N}}$ be a sequence of elements of X converging to x_0 such that $x_m\neq x_0$ for $n\in\mathbb{N}$ and $\lim_{m\to\infty}F(x_m)=\alpha$, where $\alpha\in\mathbb{R}\cup\{-\infty,+\infty\}$. Without loss of generality, we can assume that all elements of the sequence belong to one of the three sets

$$\bigcup_{(n,k)\in K}R_{n,k},\qquad\bigcup_{(n,k)\in K}S_{n,k}\quad\text{or}\quad X\setminus\bigcup_{(n,k)\in K}(R_{n,k}\cup S_{n,k}).$$

First, suppose that $x_m\in\bigcup_{(n,k)\in K}R_{n,k}$ for $m\geq 1$. Then for every $m\in\mathbb{N}$ we can find $(n_m,k_m)\in K$ such that $x_m\in R_{n_m,k_m}$. The sets $R_{n,k}$ are closed and discrete and for fixed $n\in\mathbb{N}$ there are finitely many integers $k\in\mathbb{Z}$ for which $(n,k)\in K$. Moreover, $(x_m)_{m\in\mathbb{N}}$

is convergent and is not constant. Hence $\lim_{m\to\infty} n_m = +\infty$. From (c), for every $m \in \mathbb{N}$ there exists $z_m \in A_{n_m,k_m}$ such that $d(x_m, z_m) < 2/m$. Moreover,

$$F(x_m) = k_m/n_m \quad \text{and} \quad k_m/n_m \le f(z_m) < (k_m+1)/n_m.$$

Since the function f is upper semicontinuous,

$$\alpha = \lim_{m\to\infty} F(x_m) = \lim_{m\to\infty} f(z_m) \le f(x_0).$$

Now, let $x_m \in \bigcup_{(n,k)\in K} S_{n,k}$ for $m \ge 1$. Then for every $m \in \mathbb{N}$ we can find $(n_m, k_m) \in K$ such that $x_m \in S_{n_m,k_m}$. As before, we can see that $\lim_{m\to\infty} n_m = +\infty$. From (c), for every $m \in \mathbb{N}$ there exists $z_m \in B_{n_m,k_m}$ such that $d(x_m, z_m) < 2/n$. Moreover,

$$F(x_m) = (k_m+1)/n_m \quad \text{and} \quad k_m/n_m \le f(z_m) < (k_m+1)/n_m.$$

It follows that

$$\alpha = \lim_{m\to\infty} F(x_m) = \lim_{m\to\infty} g(z_m) \le \lim_{m\to\infty} f(z_m) \le f(x_0).$$

Finally, if $x_m \in X \setminus \bigcup_{(n,k)\in K}(R_{n,k} \cup S_{n,k})$, then $F(x_m) = g(x_m)$ for $m \in \mathbb{N}$. Therefore

$$\alpha = \lim_{m\to\infty} F(x_m) = \lim_{m\to\infty} g(x_m) \le \lim_{m\to\infty} f(x_m) \le f(x_0).$$

Thus we have proven that $\alpha \le f(x_0)$. Since α was an arbitrary limit number of f at x_0, $\limsup_{x\to x_0} F(x) \le f(x_0)$. Finally,

$$\limsup_{x\to x_0} F(x) = f(x_0)$$

for every $x_0 \in X$.

Applying the lower semicontinuity of g, we can show similarly that $\liminf_{t\to x} F(t) = g(x)$ for $x \in X$. The equality $F(x) = g(x)$ for $x \in X \setminus Y$ is clear, since $\bigcup_{(n,k)\in K}(R_{n,k} \cup S_{n,k}) \subset Y$ and $F(x) = g(x)$ for $x \in \bigcup_{(n,k)\in K}(R_{n,k} \cup S_{n,k})$. ∎

Remark 5.2. The previous theorem was proven in [Ko2]; we have repeated the proof here for completeness.

Remark 5.3. Using the notation from the proof of the previous theorem, define $\widetilde{F}\colon X \to \mathbb{R}$ by

$$\widetilde{F}(x) = \begin{cases} k/n & \text{if } x \in R_{n,k},\ (n,k) \in K, \\ (k+1)/n & \text{if } x \in S_{n,k},\ (n,k) \in K, \\ f(x) & \text{if } x \in X \setminus \bigcup_{(n,k)\in K}(R_{n,k} \cup S_{n,k}), \end{cases}$$

Then it is easily seen that

$$\limsup_{t\to x} \widetilde{F}(t) = f(x) \quad \text{and} \quad \liminf_{t\to x} \widetilde{F}(t) = g(x) \qquad \text{for } x \in X.$$

Thus we get the following theorem, analogous to Theorem 5.1.

Theorem 5.2. *Let (X, ϱ) be a dense in itself metric space and let Y be a dense subset of X. Let $f\colon X \to \mathbb{R}$ and $g\colon X \to \mathbb{R}$ be such that f is upper semicontinuous, g is lower semicontinuous, and $g \le f$. Then there exists $F\colon X \to \mathbb{R}$ for which*

(1) $\limsup_{t\to x} F(t) = f(x)$ *and* $\liminf_{t\to x} F(t) = g(x)$ *for* $x \in X$,
(2) $F(x) = f(x)$ *for* $x \in X \setminus Y$.

Now we will extend Theorem 5.1 to the case of functions with infinite values. But first we have to prove a few technical lemmas.

LEMMA 5.2. *Let $(X,\mathcal{T})$ be a dense in itself topological space. Let $F\colon X\to\mathbb{R}$ and $F\colon X\to[-\infty,+\infty]$ be such that $\limsup_{t\to x}F(t)=f(x)$ for all $x\in X$. If the set $A=\{x\in X: f(x)=-\infty\}$ is nonempty, then it can be represented as a countable union of sets which are closed in A and discrete.*

Proof. Let

$$A_n=\{x\in A:|F(x)|\le n\}\qquad\text{for } n\in\mathbb{N}.$$

Then $A=\bigcup_{n\in\mathbb{N}}A_n$. We shall show that the A_n are closed in A and discrete. Fix any $n_0\in\mathbb{N}$. It is sufficient to prove that for any convergent sequence $(x_k)_{k\in\mathbb{N}}\subset A_{n_0}$ such that $x_k\ne x_m$ for $k\ne m$, its limit does not belong to A.

Since $|F(x_k)|\le n_0$ for $k\in\mathbb{N}$, we can find a subsequence $(F(x_{k_m}))_{m\in\mathbb{N}}$ convergent to some real number α such that $|\alpha|\le n_0$. Hence

$$\limsup_{t\to x_0}F(t)\ge\alpha\ge -n_0>-\infty.$$

It follows that $x_0\notin A$. Thus every set A_n is closed in A and discrete. ∎

Exactly in the same way, we can prove a similar lemma for the lower limit.

LEMMA 5.3. *Suppose that $(X,\mathcal{T})$ is a dense in itself topological space, $f\colon X\to[-\infty,+\infty]$, $F\colon X\to\mathbb{R}$, and $\liminf_{t\to x}F(t)=f(x)$ for all $x\in X$. If the set $A=\{x\in X: f(x)=+\infty\}$ is nonempty, then it can be represented as a countable union of sets which are closed in A and discrete.*

LEMMA 5.4. *Let (X,ϱ) be a metric space, M a dense subset of X, and A a closed and nonempty subset of X. Then there exists a family $\{A_n:n\in\mathbb{N}\}$ of pairwise disjoint sets such that*

(1) $\bigcup_{n\ge1}A_n\subset M$,
(2) *each set A_n is closed in X and discrete,*
(3) $(\bigcup_{n\ge1}A_n)^d=A$.

Proof. In the proof we apply Lemma 5.1 (and also the notations from that lemma). Define

$$A_1=T_{M,A,1},\qquad A_2=T_{M\setminus A_1,A,1/2},\qquad A_3=T_{M\setminus(A_1\cup A_2),A,1/3}$$

and, in general,

$$A_{n+1}=T_{M\setminus(A_1\cup\dots\cup A_n),A,1/(n+1)}.$$

It is obvious that every A_n is closed in X and discrete, $A_n\cap A_m=\emptyset$ for $n\ne m$ and $\bigcup_{n\ge1}A_n\subset M$. Since $d(z,A)<1/n$ for $z\in A_n$, $(\bigcup_{n\ge1}A_n)^d\subset\mathrm{cl}(A)=A$. On the other hand, since $d(x,A_n)<2/n$ for $x\in A$, we have $A\subset(\bigcup_{n\ge1}A_n)^d$. Finally, $(\bigcup_{n\ge1}A_n)^d=A$. ∎

THEOREM 5.3. *Let (X,ϱ) be a dense in itself metric space and let Y be a dense subset of X. Let $f\colon X\to\mathbb{R}$ and $g\colon X\to\mathbb{R}$ be such that f is upper semicontinuous, g is lower* *semicontinuous, $g \le f$ and the sets*

$$C = \{x \in X : f(x) = -\infty\} \quad \textit{and} \quad D = \{x \in X : g(x) = +\infty\}$$

can be represented as countable unions of sets which are discrete and closed in C and D, respectively. Then there exists $F\colon X \to \mathbb{R}$ for which

$$\limsup_{t\to x} F(t) = f(x) \quad \textit{and} \quad \liminf_{t\to x} F(t) = g(x) \quad \textit{for}\, x \in X.$$

Proof. Since every dense in itself metric space is resolvable, we can find four pairwise disjoint subsets Y_1, Y_2, Y_3, Y_4 of X, each dense in X. Let

$$A = \{x \in X : f(x) = +\infty\} \quad \text{and} \quad B = \{x \in X : g(x) = -\infty\}.$$

Then $D \subset A$ and $C \subset B$. Since f is upper semicontinuous and g is lower semicontinuous, A and B are closed subsets of X. It follows from our assumption that $C = \bigcup_{n\in\mathbb{N}} C_n$ and $D = \bigcup_{n\in\mathbb{N}} D_n$, where the C_n are closed in C and discrete and the D_n are closed in D and discrete for each $n \in \mathbb{N}$. Hence $(C_n)^d \cap C = \emptyset$ and $(D_n)^d \cap D = \emptyset$ for $n \in \mathbb{N}$.

The next part of the proof is analogous to the proof of Theorem 5.1. Let

$$K = \{(n,k) \in \mathbb{N} \times \mathbb{Z} : -n^2 \le k < n^2\}.$$

Let $\preceq$ be a well ordering of K defined by

$$(n_1,k_1) \preceq (n_2,k_2) \;\Leftrightarrow\; n_1 < n_2 \vee (n_1 = n_2 \wedge k_1 \le k_2).$$

Put

$$A_{n,k} = \{x \in X : k/n \le f(x) < (k+1)/n\},$$
$$B_{n,k} = \{x \in X : k/n \le g(x) < (k+1)/n\}.$$

As in the proof of Theorem 5.1, we can construct by induction two families $\{R_{n,k} : (n,k) \in K\}$ and $\{S_{n,k} : (n,k) \in K\}$ of closed and discrete subsets of X satisfying:

(a) $R_{n_1,k_1} \cap R_{n_2,k_2} = \emptyset = S_{n_1,k_1} \cap S_{n_2,k_2}$ if $(n_1,k_1),(n_2,k_2) \in K$, $(n_1,k_1) \ne (n_2,k_2)$ and $R_{n,k} \cap S_{i,j} = \emptyset$ if $(n,k),(i,j) \in K$,
(b) $\bigcup_{(n,k)\in K} R_{n,k} \subset Y_1$ and $S_{n,k} \subset Y_2$,
(c) $R_{n,k} \subset B(A_{n,k}, 1/n)$, $S_{n,k} \subset B(B_{n,k}, 1/n)$ if $(n,k) \in K$,
(d) $\varrho(x, R_{n,k}) < 2/n$ if $x \in A_{n,k}$, $(n,k) \in K$, and $\varrho(x, S_{n,k}) < 2/n$ if $x \in B_{n,k}$, $(n,k) \in K$.

By Lemma 5.4, there exists a family $\{P_n : n \in \mathbb{N}\}$ of discrete and closed (in X) subsets of Y_3 such that $\left(\bigcup_{n\in\mathbb{N}} P_n\right)^d = A$. Similarly, there exists a sequence $\{Q_n : n \in \mathbb{N}\}$ of discrete and closed in X subsets of Y_4 such that $(\bigcup_{n\in\mathbb{N}} Q_n)^d = B$. Observe that all $R_{n,k}$ and $S_{n,k}$ for $(n,k) \in K$ and all P_n and Q_n for $n \in \mathbb{N}$ are pairwise disjoint. Put

$$T = \bigcup_{n\in\mathbb{N}} \Big(\bigcup_{(n,k)\in K} (R_{n,k} \cup S_{n,k}) \cup P_n \cup Q_n \Big).$$

Define $F\colon X \to \mathbb{R}$ by

$$F(x) = \begin{cases} k/n & \text{if } x \in R_{n,k},\ (n,k) \in K, \\ (k+1)/n & \text{if } x \in S_{n,k},\ (n,k) \in K, \\ n & \text{if } x \in P_n \cup (D_n \setminus T),\ n \in \mathbb{N}, \\ -n & \text{if } x \in Q_n \cup (C_n \setminus T),\ n \in \mathbb{N}, \\ 0 & \text{if } x \in (A \cap B) \setminus T, \\ f(x) & \text{if } x \in B \setminus (A \cup C \cup T), \\ g(x) & \text{if } x \in X \setminus (B \cup D \cup T). \end{cases}$$

We have to show that

$$\limsup_{t\to x} F(t) = f(x) \quad \text{and} \quad \liminf_{t\to x} F(t) = g(x)$$

for $x \in X$.

Let $x \in A = \mathrm{cl}(\bigcup_{n\in\mathbb{N}} P_n)$. Then we can find a sequence $(x_k)_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} x_k = x$, $x_k \neq x_m$ for $k \neq m$ and $x_k \in P_{n_k}$ for $k \in \mathbb{N}$. Since each set P_n is closed and discrete, $\lim_{k\to\infty} n_k = +\infty$. Thus $\lim_{k\to\infty} F(x_k) = \lim_{k\to\infty} n_k = +\infty$. It follows that

$$\limsup_{t\to x} F(x) = +\infty = f(x) \quad \text{if } x \in A. \tag{5.1}$$

Now, let $x \in X \setminus (A \cup C)$. Then $|f(x)| \neq +\infty$. Since $x \notin ((\bigcup_{n\in\mathbb{N}} P_n) \cup (A \cap B))^d$, by repeating the arguments from the proof of Theorem 5.1, we can show that

$$\limsup_{t\to x} F(x) = f(x) \quad \text{if } x \in X \setminus (A \cup C). \tag{5.2}$$

Finally, let $x \in C$. Then $f(x) = -\infty$ and $x \in B \setminus A$. The function f is continuous at x, since it is upper semicontinuous and $f(x) = -\infty$. Thus, if a sequence $(t_n)_{n\in\mathbb{N}}$ converges to x, then $\lim_{n\to\infty} f(t_n) = -\infty$. But $g \le f$. Therefore $\lim_{n\to\infty} g(t_n) = -\infty$ too. Let $\lim_{m\to\infty} x_m = x$ and $x_m \neq x$ for $m \in \mathbb{N}$. Without loss of generality, we can assume that all elements of the sequence $(x_m)_{m\in\mathbb{N}}$ belong to one of the following sets: $\bigcup_{(n,k)\in K} R_{n,k}$, $\bigcup_{(n,k)\in K} S_{n,k}$, $\bigcup_{n\in\mathbb{N}} Q_n$, $\bigcup_{n\in\mathbb{N}} C_n \setminus T$, $B \setminus (A \cup C \cup T)$.

First consider the case where $x_m \in \bigcup_{(n,k)\in K} R_{n,k}$ for all $m \in \mathbb{N}$. Then for each $m \in \mathbb{N}$ there exists $(n_m, k_m) \in K$ such that $x_m \in R_{n_m,k_m}$. For each $(n,k) \in K$ the set $R_{n,k}$ contains only finitely many elements of $(x_m)_{m\in\mathbb{N}}$, since $R_{n,k}$ is closed in X and discrete. For each $n \in \mathbb{N}$, the set $\{k \in \mathbb{Z} : (n,k) \in K\}$ is finite. Hence $\lim_{m\to\infty} n_m = +\infty$. From (d), for every $m \in \mathbb{N}$ we can find $z_m \in X$ such that

$$|x_m - z_m| < 2/n_m \quad \text{and} \quad |F(x_m) - f(z_m)| < 1/n_m.$$

Hence

$$\lim_{m\to\infty} z_m = \lim_{m\to\infty} x_m = x.$$

By the continuity of f at x, we obtain

$$\lim_{m\to\infty} F(x_m) = \lim_{m\to\infty} f(z_m) = -\infty.$$

In the case where $x_m \in \bigcup_{(n,k)\in K} S_{n,k}$ if $m \in \mathbb{N}$, we argue similarly. For $m \in \mathbb{N}$ there exists $(n_m, k_m) \in K$ such that $x_m \in S_{n_m,k_m}$ and $\lim_{m\to\infty} n_m = +\infty$. From (d), for every

$m \in \mathbb{N}$ there exists a $z_m \in X$ for which

$$|x_m - z_m| < 2/n_m \quad \text{and} \quad |F(x_m) - g(z_m)| < 1/n_m.$$

Hence

$$\lim_{m\to\infty} z_m = \lim_{m\to\infty} x_m = x.$$

By the continuity of g at x, we obtain

$$\lim_{m\to\infty} F(x_m) = \lim_{m\to\infty} g(z_m) = -\infty.$$

Now, let $x_m \in \bigcup_{n\in\mathbb{N}} Q_n$ for $m \in \mathbb{N}$. Then for each $m \in \mathbb{N}$ there exists an $n_m \in \mathbb{N}$ such that $x_m \in Q_{n_m}$. Every Q_n is closed and discrete. Therefore $\lim_{m\to\infty} n_m = +\infty$. Hence

$$\lim_{m\to\infty} F(x_m) = \lim_{m\to\infty} (-n_m) = -\infty.$$

The next case, where $x_m \in \bigcup_{n\in\mathbb{N}} C_n \setminus T$ for $m \in \mathbb{N}$, is very similar. For each $m \in \mathbb{N}$ there exists an $n_m \in \mathbb{N}$ such that $x_m \in C_{n_m}$. Each Q_n is closed and discrete. Therefore $\lim_{m\to\infty} n_m = +\infty$. Hence

$$\lim_{m\to\infty} F(x_m) = \lim_{m\to\infty} (-n_m) = -\infty.$$

Finally, if $x_m \in B \setminus (A \cup C \cup T)$ for $m \in \mathbb{N}$, then

$$\lim_{m\to\infty} F(x_m) = \lim_{m\to\infty} f(x_m) = -\infty.$$

Thus we have proven that for a sequence $(x_m)_{m\in\mathbb{N}}$, $x_m \neq x$, converging to x, the equality $\lim_{m\to\infty} F(x_m) = -\infty$ holds. Thus

$$\limsup_{t\to x} F(t) = -\infty = f(x) \quad \text{for } x \in X \setminus (A \cup C). \tag{5.3}$$

Combining (5.1)–(5.3), we have

$$\limsup_{t\to x} F(t) = f(x) \quad \text{for } x \in X.$$

The proof of $\liminf_{t\to x} F(t) = g(x)$ is very similar and we omit it. ∎

If we set $g = 0$ in Theorem 5.3 (observe that then $C = \emptyset = D$), we obtain the following theorem.

THEOREM 5.4. *Let (X, d) be a metric space. For any upper semicontinuous function $f\colon X \to [0, \infty]$, there exists an $F\colon X \to \mathbb{R}$ such that $\omega^*(F, x) = f(x)$ for $x \in X$.*

We have shown that the ω^*-problem has a positive solution for a metric space. The next example proves that the case of a nonmetrizable space is quite different.

EXAMPLE 5.2. Let $(X, \mathcal{T})$, $X = \mathbb{R} \times [0, +\infty)$ be the Niemytzki plane. Then X is a 'nice' nonmetrizabe topological space which is separable Tychonoff and Baire. Define $f\colon X \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \times \{0\}, \\ 0 & \text{if } x \notin \mathbb{Q} \times \{0\}. \end{cases}$$

We shall show that an ω^*-primitive for f does **not** exist. Let $F\colon X \to \mathbb{R}$ be any function such that $\omega^*(F, x) = f$ for each $x \in X \setminus (\mathbb{Q} \times \{0\})$. Then the function F has a limit at

$(x, 0)$ for each $x \in \mathbb{R} \setminus \mathbb{Q}$. Let

$A_{n,k} = \{x \in \mathbb{R} \setminus \mathbb{Q} : F(v) \in (k/4 - 1/4, k/4 + 1/4)$ for $v \in (x - 1/n, x + 1/n) \times (0, 1/n)\}$

for each $n, k \in \mathbb{N}$. Then $\mathbb{R} \setminus \mathbb{Q} = \bigcup_{n,k\in\mathbb{N}} A_{n,k}$ and by the Baire Theorem there exist $n_0, k_0 \in \mathbb{N}$ and an open interval (a, b) such that A_{n_0,k_0} is dense in (a, b). Hence $f(v) \in (k_0/4-1/4, k_0/4+1/4)$ for all $v \in (a, b)\times(0, 1/n_0)$. But then for any $x_0 \in (a, b)\cap\mathbb{Q}$ there exists an open neighborhood U of $(x_0, 0) \in X$ such that $\sup_{u,v\in U\setminus\{(x_0,0)\}} |F(u) - F(v)| \leq 1/2$. Therefore $\omega^*(F, x_0) \leq 1/2$. Thus $\omega^*(F, x_0) \neq f(x_0) = 1$ and $\omega^*(F, \cdot) \neq f$. We have proven that an ω^*-primitive for f does not exist.

6. Functions with values in a metric space

In this chapter we investigate the problem of the existence of ω-primitives in the following situation. Let $(X, \mathcal{T})$ be a topological space and let (Y, ϱ) be a metric space. For $F\colon X \to Y$ and $x \in X$ we define

$$\omega(F, x) = \inf_U \sup_{x_1,x_2\in U} \varrho(F(x_1), F(x_2)),$$

where the infimum is taken over all neighborhoods U of x. Does every upper semicontinuous function $f\colon X \to [0, +\infty]$ possess a function $F\colon X \to Y$ such that $\omega(F, \cdot) = f$? For such a function F, it is impossible to define upper and lower Baire functions. We shall prove that the ω-problem can be solved for metric spaces which contain a subset similar in some sense to the real line.

Definition 6.1. Let (Y, ϱ) be a metric space. We say that a function $\varphi\colon \mathbb{R} \to Y$ is *monotonically continuous* if it is continuous and the following two conditions are fulfilled.

(c1) $\varrho(\varphi(x), \varphi(z)) > \max\{\varrho(\varphi(x), \varphi(y)), \varrho(\varphi(y), \varphi(z))\}$ for $x < y < z$,
(c2) $\lim_{r\to+\infty} \varrho(\varphi(0), \varphi(r)) = \lim_{r\to-\infty} \varrho(\varphi(0), \varphi(r)) = +\infty$.

Lemma 6.1. *If $\varphi\colon \mathbb{R} \to Y$ is monotonically continuous, then φ is a homeomorphism between $\mathbb{R}$ and $\varphi(\mathbb{R})$.*

Proof. It is easily seen from condition (c1) that φ is an injection, because if $x < y$, then

$$\varrho(\varphi(x), \varphi(y)) > \max(\varrho(\varphi(x), \varphi(t)), \varrho(\varphi(t), \varphi(y))) \geq 0$$

where $t = (x + y)/2$. Let $y \in \varphi(\mathbb{R})$ and let $(y_n)_{n\in\mathbb{N}}$ be a sequence from $\varphi(\mathbb{R})$ converging to y. Let $x_n = \varphi^{-1}(y_n)$ for $n \in \mathbb{N}$ and $x = \varphi^{-1}(y)$. Then $\varphi(x_n) = y_n$ for $n \in \mathbb{N}$ and $\varphi(x) = y$. Fix $\varepsilon > 0$. Let $\delta_1 = \varrho(y, \varphi(x + \varepsilon)) > 0$ and $\delta_2 = \varrho(y, \varphi(x - \varepsilon)) > 0$. It follows from (c1) that if $t \geq x + \varepsilon$, then $\varrho(y, \varphi(t)) \geq \delta_1$ and if $t \leq x - \varepsilon$, then $\varrho(y, \varphi(t)) \geq \delta_2$. Let $\delta = \min\{\delta_1, \delta_2\}$. We can find a positive integer n_0 such that $\varrho(y, y_n) < \delta$ for $n \geq n_0$. Hence $x_n \in (x - \varepsilon, x + \varepsilon)$ for $n \geq n_0$. Thus $\lim_{n\to\infty} x_n = x$ and φ^{-1} is continuous at y. ■

Definition 6.2. We say that a subset K of a metric space (Y, ϱ) is *monotonically homeomorphic to the real line* if there exists a monotonically continuous function $\varphi\colon \mathbb{R} \to Y$ such that $\varphi(\mathbb{R}) = K$.

Example 6.1. Let $f\colon \mathbb{R} \to (Y, \varrho)$ be a function satisfying a Lipschitz condition with the constant $L < 1$. Define a metric $\tilde{\varrho}$ in the set $\mathbb{R} \times Y$ by

$$\tilde{\varrho}((x_1, y_1), (x_2, y_2)) = \sqrt{|x_1 - x_2|^2 + (\varrho(y_1, y_2))^2}.$$

We shall show that the function $\varphi\colon \mathbb{R} \to (\mathbb{R} \times Y, \tilde{\varrho})$, $\varphi(x) = (x, f(x))$ for $x \in \mathbb{R}$ is monotonically continuous. It is obvious that φ is a continuous injection. Let x, y, z be real numbers, $x < y < z$. Then

$$\begin{aligned}\varrho^2(f(x), f(z)) &\geq \varrho^2(f(x), f(y)) - 2\varrho(f(x), f(y))\varrho(f(y), f(z)) + \varrho^2(f(y), f(z))\\ &\geq \varrho^2(f(x), f(y)) - 2L^2|x - y| \cdot |y - z| + \varrho^2(f(y), f(z)).\end{aligned}$$

Thus

$$\begin{aligned}\tilde{\varrho}^2(\varphi(x), \varphi(z)) &> |x - z|^2 + \varrho^2(f(x), f(y)) - 2|x - y| \cdot |y - z|\\ &\quad + \varrho^2(f(y), f(z)) = |x - y|^2 + |y - z|^2 + \varrho^2(f(x), f(y)) + \varrho^2(f(y), f(z))\\ &= \tilde{\varrho}^2(\varphi(x), \varphi(y)) + \tilde{\varrho}^2(\varphi(y), \varphi(z)).\end{aligned}$$

Hence $\tilde{\varrho}(\varphi(x), \varphi(z)) > \max\{\tilde{\varrho}(\varphi(x), \varphi(y)), \tilde{\varrho}(\varphi(y), \varphi(z))\}$ and condition ($c1$) holds. Since $\tilde{\varrho}(\varphi(x), \varphi(y)) \geq |x - y|$ for any real x and y, condition (c2) is obviously fulfilled.

Corollary 6.1. *If $f\colon \mathbb{R} \to (Y, \varrho)$ is a function satisfying a Lipschitz condition with constant $L < 1$, then the graph of f is monotonically homeomorphic to the real line.*

Example 6.2. Let $\varphi\colon \mathbb{R} \to \mathbb{R}$ be any strictly monotonic function and let d_φ be the metric in $\mathbb{R}$ defined by $d_\varphi(x, y) = |\varphi(x) - \varphi(y)|$. It is easy to verify that the identity function $\mathrm{Id}\colon (\mathbb{R}, d_n) \to (\mathbb{R}, d_\varphi)$, where d_n is the natural metric in $\mathbb{R}$, is monotonically continuous and $(\mathbb{R}, d_\varphi)$ is monotonically homeomorphic to the real line.

Let $\varphi\colon \mathbb{R} \to (Y, d)$ be a monotonically continuous function and $K = \varphi(\mathbb{R})$. Since $\varphi\colon \mathbb{R} \to K$ is a bijection, we can define a linear order on the set K.

Definition 6.3. For $\varphi\colon \mathbb{R} \to (Y, \varrho)$ monotonically continuous and $K = \varphi(\mathbb{R})$, a linear order $\leq_\varphi$ on K is defined by

$$x \leq_\varphi y \;\Leftrightarrow\; \varphi^{-1}(x) \leq \varphi^{-1}(y) \quad \text{for } x, y \in K.$$

It follows from condition (c1) that if x, y, u, v belong to K and $x \leq_\varphi y <_\varphi u \leq_\varphi v$, then $\varrho(x, v) \geq \varrho(y, u)$.

Since the order $\leq_\varphi$ on K is isomorphic to the natural order on $\mathbb{R}$, we can define in the standard way the supremum and infimum of a subset of K and we can define the upper limit (φ-$\limsup_{t\to x} f(t)$), lower limit (φ-$\liminf_{t\to x} f(t)$), an upper Baire function ($M_f^\varphi(x)$), and a lower Baire function ($m_f^\varphi(x)$) at any point $x \in X$ for a function $f\colon (X, \mathcal{T}) \to K$. Moreover, the following equalities hold:

- $\sup A = \varphi(\sup \varphi^{-1}(A))$ for $A \subset K$,
- $\inf A = \varphi(\inf \varphi^{-1}(A))$ for $A \subset K$,
- φ-$\limsup_{t\to x} f(t) = \varphi(\limsup_{t\to x}(\varphi^{-1} \circ f)(t))$,
- φ-$\liminf_{t\to x} f(t) = \varphi(\liminf_{t\to x}(\varphi^{-1} \circ f)(t))$,
- $M_f^\varphi(x) = \varphi(M_{\varphi^{-1}\circ f}(x))$,
- $m_f^\varphi(x) = \varphi(m_{\varphi^{-1}\circ f}(x))$.

Let $\varphi\colon \mathbb{R} \to (Y,d)$ be a monotonically continuous function and $K = \varphi(\mathbb{R})$. Fix any $y_0 \in K$. Define

$$\alpha : \{y \in K : y_0 \leq_\varphi y\} \to [0,+\infty) \quad \text{and} \quad \beta : \{z \in K : z \leq_\varphi y_0\} \to [0,+\infty)$$

by $\alpha(y) = \varrho(y,y_0)$ and $\beta(z) = \varrho(z,y_0)$. Obviously, α and β are continuous. It follows from condition (c1) and the definition of $\leq_\varphi$ that α is increasing and β is decreasing. Hence α and β are injections. The space K is homeomorphic to the real line, so it is connected. Hence by the continuity of the metric ϱ and by condition (c2), the functions α and β are bijections. Thus we have proven the following theorem.

Theorem 6.1. *Let $\varphi\colon \mathbb{R} \to (Y,d)$ be a monotonically continuous function, $K = \varphi(\mathbb{R})$ and $y_0 \in K$. For every nonnegative λ there is a unique $z_1 \in K$ such that $y_0 \leq_\varphi z_1$ and $\varrho(y_0,z_1) = \lambda$. There is also a unique $z_2 \in K$ such that $z_2 \leq_\varphi y_0$ and $\varrho(y_0,z_2) = \lambda$.*

Definition 6.4. We say that a function $\varphi\colon \mathbb{R} \to (Y,\varrho)$ is a *conformable homeomorphism* if it is monotonically continuous and

(c3) $\varrho(\varphi(x),\varphi(0)) = |x|$

for every real x.

Let us remark that condition (c3) implies (c2) from Definition 6.1.

Theorem 6.2. *If a subset K of a metric space (Y,ϱ) is monotonically homeomorphic to the real line, then for every $y_0 \in K$ there exists a conformable homeomorphism $\varphi\colon \mathbb{R} \to (Y,\varrho)$ such that $\varphi(\mathbb{R}) = K$ and $\varphi(0) = y_0$.*

Proof. Let $\phi\colon \mathbb{R} \to (Y,d)$ be any monotonically continuous function such that $K = \phi(\mathbb{R})$ and let y_0 be any point of K. There exists $x_0 \in \mathbb{R}$ such that $\phi(x_0) = y_0$. It is easy to verify that $\phi_{y_0} : \mathbb{R} \to Y$, $\phi_{y_0}(x) = \phi(x - x_0)$, is also monotonically continuous. Thus we may assume that $\phi(0) = y_0$. We shall define $\varphi\colon \mathbb{R} \to Y$ applying Theorem 6.1. For $x \geq 0$, let $\varphi(x)$ be the unique element of K such that $\phi(0) \leq_\phi \varphi(x)$ and $\varrho(\phi(0),\varphi(x)) = x$. Similarly, for $x \leq 0$, let $\varphi(x)$ be the unique element of K such that $\varphi(z) \leq_\phi \phi(0)$ and $\varrho(\phi(0),z) = -x$. It is easily seen that φ is an injection and that condition (c3) holds.

We claim that φ is an increasing function (with respect to the natural order on $\mathbb{R}$ and the order $\leq_\phi$ in K). Let $x,y \in \mathbb{R}$ and $x < y$. If $x \leq 0 \leq y$, then $\varphi(x) \leq_\phi \varphi(0) \leq_\phi \varphi(y)$. So, let $0 \leq x < y$. There exist $x_1 \geq 0$ and $y_1 \geq 0$ such that $\phi(x_1) = \varphi(x)$ and $\phi(y_1) = \varphi(y)$. Suppose $y_1 < x_1$. Then

$$\varrho(\phi(0),\phi(x_1)) > \max\{\varrho(\phi(0),\phi(y_1)),\varrho(\phi(y_1),\phi(x_1))\},$$

by condition (c1). But

$$\varrho(\phi(0),\phi(x_1)) = \varrho(\phi(0),\varphi(x)) = x,$$
$$\varrho(\phi(0),\phi(y_1)) = \varrho(\phi(0),\varphi(y)) = y > x.$$

This is a contradiction. Therefore $x_1 < y_1$ and $\varphi(x) = \phi(x_1) <_\phi \phi(y_1) = \varphi(y)$. Similarly, we can show $\varphi(x) <_\phi \varphi(y)$ in the case $x < y \leq 0$.

Let $x < y < z$. Since φ is an increasing function, $\varphi(x) <_\phi \varphi(y) <_\phi \varphi(z)$. Hence

$$\varrho(\varphi(x),\varphi(z)) > \max\{\varrho(\varphi(x),\varphi(y)),\varrho(\varphi(y),\varphi(z))\},$$

because ϕ satisfies condition (c1). Thus φ satisfies condition (c1).

Suppose $x \in \mathbb{R}$ and $\varepsilon > 0$. Let $a = \varphi(x)$. There exist $b, c \in K$ such that $b <_\phi a <_\phi c$ and $\varrho(a, b) = \varrho(a, c) = \varepsilon$. Let $y = \varphi^{-1}(b)$, $z = \varphi^{-1}(c)$ and $\delta = \min\{|x - y|, |x - z|\}$. If $t \in (x - \delta, x + \delta)$, then $b <_\phi \varphi(t) <_\phi c$. Hence $b <_\phi \varphi(t) \leq_\phi a$ or $a \leq_\phi \varphi(t) <_\phi c$. By condition (c1), we get $\varrho(\varphi(t), a) < \varrho(a, b) = \varepsilon$ in the first case and $\varrho(\varphi(t), a) < \varrho(a, c) = \varepsilon$ in the second. Thus $\varrho(\varphi(t), a) < \varepsilon$. This proves the continuity of φ at x. Since x was arbitrary, φ is a continuous conformable homeomorphism. ∎

The next theorem gives a connection between the oscillation of a function $f\colon X \to K$ and upper and lower Baire functions.

THEOREM 6.3. *Let $\varphi\colon \mathbb{R} \to (Y, \varrho)$ be a monotonically continuous function, $K = \varphi(\mathbb{R})$ and let $(X, \mathcal{T})$ be a topological space. For every $f\colon X \to K$ and every $x \in X$, if f is bounded in some neighborhood of x, then*

$$\omega(f, x) = \varrho(M_f^\varphi(x), m_f^\varphi(x)).$$

Proof. Let $x \in X$. Fix any $\varepsilon > 0$. Since f is bounded in some neighborhood of x, $M_f^\varphi(x)$ and $m_f^\varphi(x)$ exist and belong to K. Let z_1 be the unique element of K such that $M_f^\varphi(x) <_\varphi z_1$ and $\varrho(M_f^\varphi(x), z_1) = \varepsilon$ and let z_2 be the unique element of K such that $z_2 <_\varphi m_f^\varphi(x)$ and $\varrho(m_f^\varphi(x), z_2) = \varepsilon$. There exist neigborhoods U_1 and U_2 of x such that $f(t) <_\varphi z_1$ for $t \in U_1$ and $z_2 <_\varphi f(t)$ for $t \in U_2$. Hence for $x_1, x_2 \in U = U_1 \cap U_2$, we have $\varrho(f(x_1), f(x_2)) \leq \varrho(z_1, z_2) \leq \varrho(z_1, M_f^\varphi(x)) + \varrho(M_f^\varphi(x), m_f^\varphi(x)) + \varrho(m_f^\varphi(x), z_2) < \varrho(M_f^\varphi(x), m_f^\varphi(x)) + 2\varepsilon$. Thus

$$\sup_{x_1, x_2 \in U} \varrho(f(x_1), f(x_2)) \leq \varrho(M_f^\varphi(x), m_f^\varphi(x)) + 2\varepsilon$$

and

$$\omega(f, x) \leq \varrho(M_f^\varphi(x), m_f^\varphi(x)) + 2\varepsilon.$$

Since ε was arbitrary, $\omega(f, x) \leq \varrho(M_f^\varphi(x), m_f^\varphi(x))$.

Again, fix any $\varepsilon > 0$. Let $z_1 \in K$ be unique such that $z_1 <_\varphi M_f^\varphi(x)$ and $\varrho(M_f^\varphi(x), z_1) = \varepsilon$, and let $z_2 \in K$ be unique such that $m_f^\varphi(x) <_\varphi z_2$ and $\varrho(m_f^\varphi(x), z_2) = \varepsilon$. Take any neighborhood V of x. There exist $x_1, x_2 \in U$ for which $z_1 <_\varphi f(x_1)$ and $f(x_2) <_\varphi z_2$. Thus

$$\begin{aligned}\varrho(f(x_1), f(x_2)) \geq \varrho(z_1, z_2) &\geq \varrho(M_f^\varphi(x), m_f^\varphi(x)) - \varrho(z_1, M_f^\varphi(x)) - \varrho(m_f^\varphi(x), z_2) \\ &\geq \varrho(M_f^\varphi(x), m_f^\varphi(x)) - 2\varepsilon.\end{aligned}$$

Hence

$$\sup_{u, v \in U} \varrho(f(u), f(v)) \geq \varrho(f(x_1), f(x_2)) \geq \varrho(M_f^\varphi(x), m_f^\varphi(x)) - 2\varepsilon$$

and

$$\omega(f, x) \geq \varrho(M_f^\varphi(x), m_f^\varphi(x)) - 2\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we have $\omega(f, x) \geq \varrho(M_f^\varphi(x), m_f^\varphi(x))$. ∎

Now we may formulate the main theorem of this chapter.

THEOREM 6.4. *Suppose a metric space (Y, ϱ) contains a subset K which is monotonically homeomorphic to the real line. Let $(X, \mathcal{T})$ be a topological space dense in itself and let $f\colon X \to \mathbb{R}$ be upper semicontinuous. If there exists a set $A \subset X$ satisfying:*

(1) $\mathrm{cl}(A) = \mathrm{cl}(X \setminus A) = X$,
(2) *the set* $\{x \in A : f(x) - \limsup_{t\to x} f(t) > \varepsilon\}$ *is closed and discrete for every* $\varepsilon > 0$,
(3) $\limsup_{(X\setminus A)\ni t\to x} f(t) = \limsup_{t\to x} f(t)$ *for every* $x \in A$,

then there exists a function $F\colon X \to Y$ *for which* $\omega(F,\cdot) = f$.

Proof. Let $\varphi\colon \mathbb{R} \to Y$ be a conformable homeomorphism such that $\varphi(\mathbb{R}) = K$. Repeating the arguments from the proof of Theorem 6.1, we can find for every $x \in X$ a unique element $\alpha(x) \in K$ such that $\varrho(\varphi(\limsup_{t\to x} f(t)), \alpha(x)) = f(x)$ and $\alpha(x) \leq_\varphi \varphi(0)$. Define $F\colon X \to Y$ by

$$F(x) = \begin{cases} \varphi(f(x)) & \text{if } x \in X \setminus A, \\ \alpha(x) & \text{if } x \in A. \end{cases}$$

We have $F \leq_\varphi \varphi \circ f$. Furthermore, f is upper semicontinuous and $M^\varphi_{\varphi\circ f}(x) = \varphi(f(x))$. Hence

$$F(x) \leq_\varphi M^\varphi_F(x) \leq_\varphi M^\varphi_{\varphi\circ f}(x) = \varphi(f(x)) = F(x)$$

for $x \in X \setminus A$. Thus $M^\varphi_F(x) = \varphi(f(x))$ for $x \in X \setminus A$. Applying the facts that $X \setminus A$ is dense, $F(x) \leq_\varphi \varphi(0)$ for $x \in A$, and $\varphi(0) \leq_\varphi F(x)$ for $x \in X \setminus A$, we have $M^\varphi_F(x) = \varphi\text{-}\limsup_{t\to x} F(t)$ for $x \in A$. By 3), we deduce that

$$\begin{aligned} M^\varphi_F(x) &= \varphi\text{-}\limsup_{t\to x} F(t) = \varphi\text{-} \limsup_{X\setminus A\ni t\to x} (\varphi \circ f)(t) \\ &= \varphi\Big(\limsup_{X\setminus A\ni t\to x} f(t)\Big) = \varphi\Big(\limsup_{t\to x} f(t)\Big) \end{aligned}$$

for $x \in A$. Since A is dense and $F(t) \leq_\varphi \varphi(0)$ for $t \in A$, we obtain $m^\varphi_F(x) \leq_\varphi \varphi(0)$ for $x \in X$. Furthermore, $m^\varphi_F(x) \leq_\varphi \alpha(x)$ for $x \in A$. Now we will prove the following property:

(c4) for every $\varepsilon > 0$ and for every $c \in K$ satisfying $\varphi(0) \leq_\varphi c$, there exists $\delta > 0$ such that if $x, y, z \in K$, $\varphi(0) \leq_\varphi x \leq_\varphi y \leq_\varphi c$, $z \leq_\varphi \varphi(0)$, $\varrho(\varphi(0), y) = \varrho(z, x)$, and $\varrho(x, y) < \delta$, then $\varrho(z, \varphi(0)) < \varepsilon$.

Fix $\varepsilon > 0$ and $c \in K$ such that $\varphi(0) \leq_\varphi c$. Suppose that $(c4)$ does not hold. Then we can find sequences $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ and $(z_n)_{n\in\mathbb{N}}$ in K such that

- $\varphi(0) \leq_\varphi x_n \leq_\varphi y_n \leq_\varphi c$, $z_n \leq_\varphi \varphi(0)$,
- $\varrho(\varphi(0), y_n) = \varrho(z_n, x_n)$, $\varrho(x_n, y_n) < 1/n$,
- $\varrho(z_n, \varphi(0)) \geq \varepsilon$

for $n \in \mathbb{N}$. Since $H = \{x \in K : \varphi(0) \leq_\varphi x \leq_\varphi\leq c\}$ is a compact metric space (because H is homeomorphic to the closed interval $[0, \varphi^{-1}(c)]$) and $x_n \in H$ for $n \in \mathbb{N}$, we can find a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ converging to some $x_0 \in H$. From the inequality $\varrho(x_n, y_n) < 1/n$ for $n \in \mathbb{N}$, it follows that the sequence $(y_{n_k})_{k\in\mathbb{N}}$ also converges to x_0. Then

$$\varrho(\varphi(0), x_0) = \lim_{k\to\infty} \varrho(\varphi(0), y_{n_k}) = \lim_{k\to\infty} \varrho(x_{n_k}, z_{n_k}) = \lim_{k\to\infty} \varrho(x_0, z_{n_k}).$$

But this implies that $\lim_{k\to\infty} z_{n_k} = \varphi(0)$, which contradicts the inequality $\varrho(z_n, \varphi(0)) \geq \varepsilon$ for $n \in \mathbb{N}$. Thus we have proven property (c4).

Fix any $x \in X$ and $\varepsilon > 0$. Since f is upper semicontinuous, it is locally bounded. Let V be a neighborhood of x and let $P > 0$ be such that $f(t) \le P$ for $t \in V$. Choose $\delta > 0$ from (c4) for ε and $c = \varphi(P)$. Since φ is continuous, it is uniformly continuous on $[0, P]$. Hence there exists $\eta > 0$ such that if $0 \le x \le y \le P$ and $|x - y| < \eta$, then $\varrho(\varphi(x), \varphi(y)) < \delta$. By (2), there exists a neighborhood U of x such that

$$f(t) - \limsup_{s \to t} f(s) < \eta \quad \text{for } t \in U \setminus \{x\}.$$

Let $t \in A \cap (U \setminus \{x\})$. Then

$$f(t) - \limsup_{s \to t} f(s) < \eta \quad \text{and} \quad \varrho\Big(\varphi(M_f(t)), \varphi\Big(\limsup_{s \to t} f(s)\Big)\Big) < \delta.$$

So, by property (c4) we get $\varrho(\varphi(0), \alpha(t)) < \varepsilon$. Thus

$$\inf_{t \in A \cap (U \setminus \{x\})} F(t) \ge_{\varphi} \varphi(-\varepsilon).$$

Since $\varepsilon > 0$ was arbitrary, we get $m_F^{\varphi}(x) = \varphi(0)$ for $x \in X \setminus A$ and $m_F^{\varphi}(x) = \alpha(x)$ for $x \in A$. Since $F \le \varphi \circ f$ and $\varphi \circ f$ is upper semicontinuous, F is locally bounded. Thus $\omega(F, \cdot) = f$ follows immediately from Theorem 6.3 ■

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