1. Introduction

The classical Hardy–Littlewood maximal operator M has proved to be a most useful analytical tool ever since its introduction in 1930. Of particular importance are its mapping properties: the celebrated result that M maps L^p space boundedly into itself when $1 < p \leq \infty$ (cf. [HL]), and the corresponding weak $(1,1)$ boundedness when $p = 1$ (cf. [R] and [W]), have had a multitude of applications. For details of some of these and for further historical remarks see [BS] and [Stn1], [Stn2]. This success, together with the increasing sophistication of the questions which arise in applications nowadays, makes it quite natural to try to extend the classical mapping results for M in various ways. One possibility is to replace M by a closely related operator such as the fractional maximal operator; another is to consider scales of spaces more general than L^p. Of course, much progress has already been made in both these directions and is well documented in the research literature. Here we continue this line of work and give mapping theorems which provide a reasonably complete description of the action of the fractional maximal operator (and even a more general operator) between Lorentz spaces of classical and weak type.

For $n \in \mathbb{N}$ and $\gamma \in (0, n)$, the fractional maximal operator M_γ is defined by

$$(M_\gamma f)(x) = \sup_{Q \ni x} |Q|^{\gamma/n-1} \int_Q |f(y)|\, dy, \qquad x \in \mathbb{R}^n, \tag{1.1}$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ with sides parallel to the coordinate axes. The corresponding Riesz potential I_γ, $\gamma \in (0, n)$, is given by

$$(I_\gamma f)(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\gamma}}\, dy, \qquad x \in \mathbb{R}^n. \tag{1.2}$$

It is well known that I_γ satisfies the sharp endpoint estimates

$$I_\gamma : L^1(\mathbb{R}^n) \to L^{n/(n-\gamma),\infty}(\mathbb{R}^n), \tag{1.3}$$

$$I_\gamma : L^{n/\gamma,1}(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n) \tag{1.4}$$

(for (1.3) see, e.g., [Stn1]; (1.4) is equivalent to (1.3) since I_γ is selfadjoint), while (cf. [CKOP])

$$M_\gamma : L^1(\mathbb{R}^n) \to L^{n/(n-\gamma),\infty}(\mathbb{R}^n), \tag{1.5}$$

$$M_\gamma : L^{n/\gamma,\infty}(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n). \tag{1.6}$$

Using estimates (1.3)–(1.6) and the Marcinkiewicz interpolation theorem [BS, Chapter 4, Th. 4.1], we see that both operators I_γ and M_γ are of strong type (p, q), where $1 < p < n/\gamma$ and $1/q = 1/p - \gamma/n$. While, by (1.3) and (1.5), the mapping properties of I_γ and M_γ coincide on L^1, we deduce from (1.4) and (1.6) that the behaviour of I_γ and M_γ is different on spaces which are close to $L^{n/\gamma}$. Since the Riesz potential I_γ is an operator of joint weak type $(1, n/(n-\gamma); n/\gamma, \infty)$ (cf. [BS]) satisfying a convenient lower estimate (cf., e.g., [EOP1, Section 10]), one can describe its precise behaviour on certain spaces close to $L^{n/\gamma}$ (see again [EOP1]; for further results in this direction we refer to the recent papers [CP] and [P]). On the other hand, sufficiently general results describing the behaviour of M_γ on spaces close to $L^{n/\gamma}$ or on spaces which are mapped by M_γ into spaces close to L^∞, e.g. into $\exp L^\beta$, are not mentioned in the literature. To establish such results, we shall look for necessary and sufficient conditions which guarantee that M_γ is bounded between classical (cf. [L]) and weak-type (cf. [CS1], [So]) Lorentz spaces. These scales of spaces are general enough for most purposes and involve many familiar spaces (e.g. spaces of Lebesgue, Lorentz–Zygmund type, and Orlicz spaces with power-logarithmic or exponential Young functions—cf. Section 2).

In fact, in this paper we solve the above-mentioned problem for operators more general than M_γ. We consider fractional maximal operators involving logarithmic terms. Such operators correspond to the potentials with logarithmic smoothness considered in [OT1,2]. Also the maximal operator appearing in [AV] is a local version of the particular case of these operators when $\gamma = 0$. In this limiting situation the maximal operators mentioned above are purely logarithmically fractional and their behaviour differs from that of corresponding potentials even on the space L^1 (cf. [OT2, Remark 3.7(iii)] and Section 10 below). Our methods also enable us to describe mapping properties of these operators on spaces close to L^1 (again cf. Section 10). Although this method gives the best possible result within the chosen scale of spaces, one can sometimes improve it by making use of another approach. However, in such a case the improved result involves spaces which are outside the given scale of spaces. This phenomenon is illustrated in the Appendix of our paper, where the limiting real interpolation is applied to improve a particular result (of Section 10), which involves a local version of the fractional maximal operator M_γ.

Putting $\gamma = 0$ in (1.1), we obtain the classical Hardy–Littlewood maximal operator M, that is, $M = M_0$. Note that the boundedness of M on classical Lorentz spaces $\Lambda^q(w)$ was characterized in [AM]. It was proved that $M : \Lambda^q(w) \to \Lambda^q(w)$, $1 \le q < \infty$, if and only if the weight w belongs to the class B_q. Necessary and sufficient conditions for the boundedness of

$$M : \Lambda^p(v) \to \Lambda^q(w) \tag{1.7}$$

were given in:

- [Sa, Th. 2] if $1 < p, q < \infty$;
- [Stp, Th. 3(a)] if $0 < q < 1 < p < \infty$;
- [Stp, Th. 3(b)] and [CS2, Prop. 2.6(b)] if $0 < p < 1$ and $p \le q < \infty$;
- [SS, Th. 4.1] if $0 < q < p = 1$;
- [CPSS, Th. 4.1(iv)] if $1 = q < p < \infty$.

Sufficient conditions for (1.7) with $0 < q < p < 1$ can be found in [Stp, Prop. 2]. The boundedness of M from the classical Lorentz space $\Lambda^p(v)$ into the weak-type Lorentz space $\Lambda^{q,\infty}(w)$ was characterized in:

- [CAS, Th. 2.3] if $v = w$ and $p = q = 1$;
- [CS3, Th. 3.9] if $1 < p, q < \infty$;
- [CPSS, Th. 4.2] if $0 < p, q < \infty$.

Necessary and sufficient conditions for the boundedness of $M : \Lambda^{p,\infty}(v) \to \Lambda^{q,\infty}(w)$, $0 < p, q < \infty$, were established in [So, Th. 4.1(ii)].

The boundedness of the fractional maximal operator $M_\gamma : \Lambda^p(v) \to \Lambda^q(w)$, $1 < p \le q < \infty$, was characterized in [CKOP] and necessary and sufficient conditions for the boundedness of the power-logarithmic fractional maximal operator $\mathscr{M}_{s,\gamma;\mathbb{A}}$ (see Section 2) from the classical Lorentz space $\Lambda^p(v)$ into $\Lambda^q(w)$ were given without proof in [O1] provided that $0 < p \le q < \infty$.

The results described above on the boundedness of the classical Hardy–Littlewood maximal operator M rely on the estimate $(Mf)^*(t) \approx t^{-1} \int_0^t f^*(\tau)\, d\tau$, $t > 0$, involving the non-increasing rearrangements f^* and $(Mf)^*$ (cf. [BS, Chapter 3, Th. 3.8]) and on weighted inequalities for the averaging operator $(\mathscr{A}g)(t) = t^{-1} \int_0^t g(\tau)\, d\tau$, $t > 0$, considered on the class of all non-negative and non-increasing functions. On the other hand, in the case of fractional maximal operators the role of the averaging operator $\mathscr{A}$ is played by the operator $(\mathscr{T}g)(t) = \sup_{t<\tau<\infty} u(\tau) \int_0^\tau g(\sigma)\, d\sigma$, where $u(\tau)$ is a convenient weight. Therefore, weighted inequalities for the operator $\mathscr{T}$ on the class of all non-negative and non-increasing functions are of basic importance.

The main results of our paper are Theorems 3.1, 4.1, 5.1, 6.1, and 7.1.

2. Notation and preliminaries

Given two quasi-Banach spaces X and Y, we say that X *coincides* with Y (and write $X = Y$) if X and Y are equal in the algebraic and the topological sense (their quasi-norms are equivalent). The symbol $X \hookrightarrow Y$ means that $X \subset Y$ and the natural embedding of X in Y is continuous.

We write $A \lesssim B$ if $A \le cB$ for some constant c independent of appropriate quantities involved in the expressions A and B, and $A \approx B$ if $A \lesssim B$ and $B \lesssim A$. We use the convention $1/\infty = 0$ and $\infty/a = \infty$ for $0 < a < \infty$, and for $0 < q \le \infty$ we define q' by $1/q' + 1/q = 1$ when $q \ne 1$, and $q' = +\infty$ if $q = 1$ (note that $q' < 0$ when $0 < q < 1$).

If $E \subset \mathbb{R}^n$ is a measurable subset (with respect to n-dimensional Lebesgue measure), we denote by $|E|$ its measure, by χ_E its characteristic function, and by $\mathcal{M}(E)$ the set of all measurable functions on E. When $E = (a,b) \subseteq \mathbb{R}$, we write simply $\mathcal{M}(a,b)$. By $\mathcal{M}^+(a,b;\downarrow)$ we mean the subset of $\mathcal{M}(a,b)$ consisting of all non-negative and non-increasing functions on (a,b). The set $\mathscr{W}(a,b)$ of all weights on (a,b) is defined by

$$\mathscr{W}(a,b) := \{w \in \mathcal{M}(a,b);\ 0 < w < \infty \text{ a.e. on } (a,b)\}.$$

If $q \in (0,\infty)$, $\Omega \subset \mathbb{R}^n$ is a domain, and $w \in \mathscr{W}(0,|\Omega|)$, then the *classical Lorentz space*

$\Lambda^q(\Omega;w)$ is the collection of all $f \in \mathcal{M}(\Omega)$ such that the quantity

$$\|f\|_{\Lambda^q(\Omega;w)} := \Big(\int\limits_0^{|\Omega|} [f^*(t)]^q w(t)\, dt\Big)^{1/q}$$

is finite (cf. [L]); here

$$f^*(t) := \inf\{\lambda \geq 0;\ |\{x \in \Omega;\ |f(x)| > \lambda\}| \leq t\}, \qquad t \geq 0,$$

is the non-increasing rearrangement of f. Moreover, a *weak-type* modification of the space $\Lambda^q(\Omega;w)$ is defined by (cf. [CS1], [So])

$$\Lambda^{q,\infty}(\Omega;w) := \Big\{ f \in \mathcal{M}(\Omega);\ \|f\|_{\Lambda^{q,\infty}(\Omega;w)} := \sup_{0<t<|\Omega|} f^*(t) \Big(\int\limits_0^t w(\tau)\, d\tau\Big)^{1/q} < \infty \Big\}.$$

One can easily see that $\Lambda^q(\Omega;w) \hookrightarrow \Lambda^{q,\infty}(\Omega;w)$. If $\Omega = \mathbb{R}^n$, we write simply $\Lambda^q(w)$ and $\Lambda^{q,\infty}(w)$ instead of $\Lambda^q(\mathbb{R}^n;w)$ and $\Lambda^{q,\infty}(\mathbb{R}^n;w)$, respectively.

Recall that classical and weak-type Lorentz spaces include many familiar spaces. In particular:

(i) If $w(t) \equiv 1$, then $\Lambda^q(\Omega;w)$ is the *Lebesgue space* $L^q(\Omega)$ and $\Lambda^{q,\infty}(\Omega;w)$ is the *weak Lebesgue space* $L^{q,\infty}(\Omega)$.

(ii) If $w(t) = t^{q/p-1}$, $t \in (0,|\Omega|)$, $p \in (0,\infty]$, then $\Lambda^q(\Omega;w)$ is the *Lorentz space* $L^{p,q}(\Omega)$ and, moreover, if $p < \infty$, then $\Lambda^{q,\infty}(\Omega;w)$ is the Lorentz space $L^{p,\infty}(\Omega)$.

(iii) If $w(t) = t^{q/p-1}\ell^{q\beta}(t)$, $t \in (0,|\Omega|)$, $p \in (0,\infty]$, $\beta \in \mathbb{R}$ and $\ell(t) := 1 + |\log t|$, then $\Lambda^q(\Omega;w)$ is the *Lorentz–Zygmund space* $L^{p,q}(\log L)^\beta(\Omega)$ introduced in [BR] and, moreover, if $p < \infty$, then $\Lambda^{q,\infty}(\Omega;w)$ is the Lorentz–Zygmund space $L^{p,\infty}(\log L)^\beta(\Omega)$ (see again [BR]).

(iv) If $|\Omega| < \infty$, $\beta < 0$ and $w(t) = t^{-1}\ell^{q\beta-1}(t)$, $t \in (0,|\Omega|)$, then $\Lambda^{q,\infty}(\Omega;w)$ coincides with the *exponential space* $\exp L^{-1/\beta}(\Omega)$ (cf. [BR, Theorem 10.3] or [EOP1, Lemma 2.2(iv)]), which is the Orlicz space $L_\Phi(\Omega)$, where the Young function Φ satisfies $\Phi(t) = \exp t^{-1/\beta}$ for large t. (Note that our space $\exp L^{-1/\beta}(\Omega)$ corresponds to $L^{-\beta}_{\exp}(\Omega)$ of [BS].)

(v) If $w(t) = \ell^{q\beta}(t)$, $t \in (0,|\Omega|)$ and $\beta \in \mathbb{R}$, then $\Lambda^q(\Omega;w)$ is the Lorentz–Zygmund space $L^{q,q}(\log L)^\beta(\Omega)$ and $\Lambda^{q,\infty}(\Omega;w)$ is the Lorentz–Zygmund space $L^{q,\infty}(\log L)^\beta(\Omega)$. Moreover, if $q \in (1,\infty)$, or $q = 1$, $\beta > 0$ and $|\Omega| < \infty$, then $\Lambda^q(\Omega;w)$ coincides with the *Zygmund class* $L^q(\log L)^\beta(\Omega)$, which is the Orlicz space $L_\Phi(\Omega)$, where the Young function Φ satisfies $\Phi(t) \approx t^q\ell^{q\beta}(t)$, $t \in (0,\infty)$ (cf. [OP, Section 8]).

Throughout the paper we denote by $\|\cdot\|_s$, $s \in (0,\infty]$, the usual quasi-norm in the Lebesgue space $L^s(\mathbb{R}^n)$ and by $\|\cdot\|_{s,(a,b)}$, $-\infty \leq a < b \leq +\infty$, the usual L^s-quasi-norm in the Lebesgue space $L^s((a,b))$. Moreover, if $s \in (0,\infty)$ and $\mathbb{A} = (A_0, A_\infty) \in \mathbb{R}^2$, the symbol $\|\cdot\|_{s;\mathbb{A}}$ stands for the quasi-norm in the *generalized Zygmund class* $L^s(\log L)^{\mathbb{A}}(\mathbb{R}^n) := \Lambda^s(\mathbb{R}^n;w) = \Lambda^s(w)$, where

$$w(t) = \ell^{s\mathbb{A}}(t) := \begin{cases} \ell^{sA_0}(t), & t \in (0,1], \\ \ell^{sA_\infty}(t), & t \in (1,\infty). \end{cases}$$

By [OP, Theorem 8.8], the space $L^s(\log L)^{\mathbb{A}}(\mathbb{R}^n)$ coincides with the Orlicz space $L_\Phi(\mathbb{R}^n)$, where

$$\Phi(t) \approx t^s \ell^{s(A_\infty, A_0)}(t), \qquad t > 0,$$

provided that either $s \in (1,\infty)$ and $\mathbb{A} = (A_0, A_\infty) \in \mathbb{R}^2$, or $s = 1$ and $A_\infty \le 0 \le A_0$.

Let $s \in (0,\infty)$, $\gamma \in [0,n)$ and $\mathbb{A} = (A_0, A_\infty) \in \mathbb{R}^2$. The *fractional maximal operator* $\mathscr{M}_{s,\gamma;\mathbb{A}}$ at $f \in \mathcal{M}(\mathbb{R}^n)$ is given by

$$(\mathscr{M}_{s,\gamma;\mathbb{A}} f)(x) = \sup_{Q \ni x} \frac{\|f\chi_Q\|_s}{\|\chi_Q\|_{sn/(n-\gamma);\mathbb{A}}}, \qquad x \in \mathbb{R}^n, \tag{2.1}$$

where the supremum is extended over all cubes Q in $\mathbb{R}^n$ with sides parallel to the coordinate axes.

Since the estimate $\|\chi_Q\|_{sn/(n-\gamma);\mathbb{A}} \approx |Q|^{(1/s)(1-\gamma/n)} \ell^{\mathbb{A}}(|Q|)$ holds for all cubes $Q \subset \mathbb{R}^n$, we have

$$(\mathscr{M}_{s,\gamma;\mathbb{A}} f)(x) \approx \Big[\sup_{Q \ni x} |Q|^{\gamma/n-1} \ell^{-s\mathbb{A}}(|Q|) \int_Q |f(y)|^s \, dy\Big]^{1/s}, \qquad x \in \mathbb{R}^n. \tag{2.2}$$

Hence, if $s = 1$, $\gamma = 0$ and $\mathbb{A} = (0,0)$, then $\mathscr{M}_{s,\gamma;\mathbb{A}}$ is the *classical Hardy–Littlewood maximal operator* M. If $s = 1$, $\gamma \in (0,n)$ and $\mathbb{A} = (0,0)$, then $\mathscr{M}_{s,\gamma;\mathbb{A}}$ is the usual *fractional maximal operator* M_γ from (1.1). Moreover, if $s = 1$, $\gamma \in [0,n)$ and $\mathbb{A} \in \mathbb{R}^2$, then $\mathscr{M}_{s,\gamma;\mathbb{A}}$ is the fractional maximal operator which corresponds to potentials with logarithmic smoothness treated in [OT1] and [OT2]. In particular, if $\gamma = 0$, then $\mathscr{M}_{1,\gamma;\mathbb{A}}$ is the maximal operator of purely logarithmic order. Local versions of this operator were considered in [OT2] and also appeared in [AV]. Finally, the maximal operators considered in [MO] correspond to local versions of $\mathscr{M}_{s,0;\mathbb{A}}$ with $s \in [1,\infty)$ and $\mathbb{A} = (\alpha,\alpha)$, $\alpha \in [0,\infty)$.

Throughout the paper we use the abbreviation LHS($*$) (RHS($*$)) for the left (right) hand side of the relation ($*$).

3. Sharp estimates of $(\mathscr{M}_{s,\gamma;\mathbb{A}} f)^*$

While the results of [AM] and [Sa] rely on the estimate

$$(Mf)^*(t) \approx f^{**}(t) := t^{-1} \int_0^t f^*(\tau)\, d\tau, \qquad t \in (0,\infty), \tag{3.1}$$

in our situation the role of (3.1) is replaced by the following assertion (which is consistent with (3.1) if $s = 1$, $\gamma = 0$ and $\mathbb{A} = (0,0)$).

Theorem 3.1. *Let $s \in (0,\infty)$, $\gamma \in [0,n)$ and $\mathbb{A} = (A_0, A_\infty) \in \mathbb{R}^2$ satisfy*

$$\text{either } \gamma \in (0,n), \text{ or } \gamma = 0 \text{ and } A_0 \ge 0 \ge A_\infty. \tag{3.2}$$

Then there exists a positive constant C depending only on n, s, γ and $\mathbb{A}$ such that for all $f \in \mathcal{M}(\mathbb{R}^n)$ and every $t \in (0,\infty)$,

$$(\mathscr{M}_{s,\gamma;\mathbb{A}} f)^*(t) \le C \Big[\sup_{t<\tau<\infty} \tau^{\gamma/n-1} \ell^{-s\mathbb{A}}(\tau) \int_0^\tau (f^*)^s(\sigma)\, d\sigma\Big]^{1/s}. \tag{3.3}$$

Inequality (3.3) *is sharp in the sense that for every* $\varphi \in \mathcal{M}^+(0,\infty;\downarrow)$ *there exists a function* $f \in \mathcal{M}(\mathbb{R}^n)$ *such that* $f^* = \varphi$ *a.e. on* $(0,\infty)$ *and for all* $t \in (0,\infty)$,

$$(\mathscr{M}_{s,\gamma;\mathbb{A}} f)^*(t) \geq c\Big[\sup_{t<\tau<\infty} \tau^{\gamma/n-1}\ell^{-s\mathbb{A}}(\tau)\int_0^\tau (f^*)^s(\sigma)\,d\sigma\Big]^{1/s}, \tag{3.4}$$

where c *is a positive constant which again depends only on* n, s, γ *and* $\mathbb{A}$.

The proof of Theorem 3.1 will be carried out in several steps. First we establish endpoint estimates for the operator $\mathscr{M}_{1,\gamma;\mathbb{A}}$.

LEMMA 3.2. *Let* $\gamma \in [0,n)$ *and* $\mathbb{A} = (A_0, A_\infty) \in \mathbb{R}^2$ *be such that* (3.2) *holds. Then*

$$\mathscr{M}_{1,\gamma;\mathbb{A}} : L^{n/\gamma,\infty}(\log L)^{-\mathbb{A}}(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n), \tag{3.5}$$

$$\mathscr{M}_{1,\gamma;\mathbb{A}} : L^1(\mathbb{R}^n) \to L^{n/(n-\gamma),\infty}(\log L)^{\mathbb{A}}(\mathbb{R}^n). \tag{3.6}$$

Proof. (i) To prove (3.5), take $f \in L^{n/\gamma,\infty}(\log L)^{-\mathbb{A}}(\mathbb{R}^n)$ and a cube $Q \subset \mathbb{R}^n$. Then the Hardy–Littlewood inequality (cf. [BS, Chapter 2, Theorem 2.2]), the Hölder inequality and the fact that the fundamental functions of the spaces $L^{n/(n-\gamma),1}(\log L)^{\mathbb{A}}(\mathbb{R}^n)$ and $L^{n/(n-\gamma)}(\log L)^{\mathbb{A}}(\mathbb{R}^n)$ are equivalent (cf. [OP, Lemma 3.7]) imply that

$$\begin{aligned}\|f\chi_Q\|_1 &\leq \|f^*\chi_Q^*\|_{1,(0,\infty)} \leq \|\chi_Q\|_{n/(n-\gamma),1;\mathbb{A}}\|f\|_{n/\gamma,\infty;-\mathbb{A}}\\ &\approx \|\chi_Q\|_{n/(n-\gamma);\mathbb{A}}\|f\|_{n/\gamma,\infty;-\mathbb{A}}\end{aligned}$$

and (3.5) follows.

(ii) To prove (3.6), take $f \in L^1(\mathbb{R}^n)$, $\lambda > 0$ and put

$$E(\lambda) = \{x \in \mathbb{R}^n;\ (\mathscr{M}_{1,\gamma;\mathbb{A}} f)(x) > \lambda\}.$$

Then, for any $x \in E(\lambda)$, there is a cube $Q = Q(x)$ containing x such that

$$\lambda\|\chi_{Q(x)}\|_{n/(n-\gamma);\mathbb{A}} \leq \int_{Q(x)} |f(y)|\,dy. \tag{3.7}$$

The collection of all such cubes covers $E(\lambda)$ and we claim that

$$\sup\{\operatorname{diam} Q(x);\ x \in E(\lambda)\} < \infty. \tag{3.8}$$

Indeed, putting $\varepsilon = (1-\gamma/n)/2$, we see that $\varepsilon > 0$ and, since $t^\varepsilon\ell^{\mathbb{A}}(t) \to \infty$ as $t \to \infty$, there is $T = T(\varepsilon,\mathbb{A})$ such that $\ell^{\mathbb{A}}(t) \geq t^{-\varepsilon}$ for all $t > T$. Hence, if $|Q(x)| > T$, then $\ell^{\mathbb{A}}(|Q(x)|) \geq |Q(x)|^{-\varepsilon}$, which in turn yields

$$|Q(x)|^{1-\gamma/n}\ell^{\mathbb{A}}(|Q(x)|) \geq |Q(x)|^\varepsilon. \tag{3.9}$$

Since LHS(3.9) $\approx \|\chi_Q\|_{n/(n-\gamma);\mathbb{A}}$, we infer from (3.9) and (3.7) that

$$|Q(x)|^\varepsilon \lesssim \lambda^{-1}\int_{Q(x)} |f(y)|\,dy \leq \lambda^{-1}\|f\|_1$$

and so

$$\operatorname{diam} Q(x) = |Q(x)|^{1/n}\sqrt{n} \lesssim (\lambda^{-1}\|f\|_1)^{1/(\varepsilon n)}\sqrt{n}.$$

Consequently, for all cubes $Q(x)$, $x \in E(\lambda)$,

$$\operatorname{diam} Q(x) \lesssim \max\{T^{1/n}, (\lambda^{-1}\|f\|_1)^{1/(\varepsilon n)}\}\sqrt{n}$$

and (3.8) follows.

Condition (3.8) allows us to apply the Besicovitch covering theorem (see [G]) which asserts that one can choose, from among the given cubes $Q(x)$, $x \in E(\lambda)$, a sequence $\{Q_k\}$ (possibly finite) such that

$$E(\lambda) \subset \bigcup_k Q_k, \tag{3.10}$$

$$\sum_k \chi_{Q_k}(x) \le \varrho_n \quad \text{for every } x \in \mathbb{R}^n \tag{3.11}$$

(ϱ_n is a number which depends only on n).

Putting $X := L^{n/(n-\gamma)}(\log L)^{\mathbb{A}}$, we deduce from (3.10) that

$$\|\chi_{E(\lambda)}\|_X \le \|X_{\bigcup_k Q_k}\|_X. \tag{3.12}$$

Since the space X is (equivalent to) a Banach function space (cf. [OP, Theorem 7.1]),

$$\mathrm{RHS}(3.12) \lesssim \sum_k \|\chi_{Q_k}\|_X. \tag{3.13}$$

Now, by (3.7) and (3.11),

$$\mathrm{RHS}(3.13) \le \lambda^{-1} \sum_k \int_{Q_k} |f(y)|\,dy \lesssim \lambda^{-1}\|f\|_1. \tag{3.14}$$

On the other hand, since the fundamental function φ_X of the space X satisfies $\varphi_X(t) \approx t^{1-\gamma/n}\ell^{\mathbb{A}}(t)$, $t \in (0,\infty)$ (cf. [OP, Lemma 3.7(i)]), we see that

$$\mathrm{LHS}(3.12) \approx |E(\lambda)|^{1-\gamma/n}\ell^{\mathbb{A}}(|E(\lambda)|). \tag{3.15}$$

Summarizing estimates (3.12)–(3.15), we arrive at

$$|E(\lambda)|^{1-\gamma/n}\ell^{\mathbb{A}}(|E(\lambda)|)\lambda \lesssim \|f\|_1.$$

Therefore,

$$\|f\|_1 \gtrsim \sup_{\lambda>0} |E(\lambda)|^{1-\gamma/n}\ell^{\mathbb{A}}(|E(\lambda)|)\lambda = \sup_{t>0} t^{1-\gamma/n}\ell^{\mathbb{A}}(t)(\mathscr{M}_{1,\gamma;\mathbb{A}}f)^*(t)$$

and (3.6) follows. ∎

Our next lemma states that estimate (3.3) holds if $s = 1$.

LEMMA 3.3. *Under the assumptions of Lemma* 3.2, *there is a positive constant C depending only on n, γ and $\mathbb{A}$ such that for all $f \in \mathcal{M}(\mathbb{R}^n)$ and every $t \in (0,\infty)$*,

$$(\mathscr{M}_{1,\gamma;\mathbb{A}}f)^*(t) \le C \sup_{t<\tau<\infty} \tau^{\gamma/n}\ell^{-\mathbb{A}}(\tau) f^{**}(\tau). \tag{3.16}$$

Proof. Let $t \in (0,\infty)$ and $f \in \mathcal{M}(\mathbb{R}^n)$. We may assume that

$$\sup_{t<\tau<\infty} \tau^{\gamma/n}\ell^{-\mathbb{A}}(\tau) f^{**}(\tau) < \infty \tag{3.17}$$

(otherwise (3.16) holds trivially). Define functions g_t and h_t on $\mathbb{R}^n$ by

$$g_t(x) = \max\{|f(x)| - f^*(t), 0\} \operatorname{sgn} f(x), \qquad h_t(x) = \min\{|f(x)|, f^*(t)\} \operatorname{sgn} f(x).$$

Then

$$f = g_t + h_t \tag{3.18}$$

and, for all $\tau \in (0,\infty)$,

$$g_t^*(\tau) = \chi_{(0,t)}[f^*(\tau) - f^*(t)], \qquad h_t^*(\tau) = \min\{f^*(\tau), f^*(t)\}. \tag{3.19}$$

Since, by (3.19),

$$\|g_t\|_1 = \int_0^\infty g_t^*(\tau)\,d\tau = \int_0^t [f^*(\tau) - f^*(t)]\,d\tau \le \int_0^t f^*(\tau)\,d\tau, \tag{3.20}$$

we infer from (3.17) that $g_t \in L^1(\mathbb{R}^n)$. Together with (3.6), this implies that

$$\sup_{0<\tau<\infty} \tau^{(n-\gamma)/n}\ell^{\mathbb{A}}(\tau)(\mathscr{M}_{1,\gamma;\mathbb{A}}\, g_t)^*(\tau) \lesssim \|g_t\|_1. \tag{3.21}$$

Analogously, by (3.19) and (3.2),

$$\begin{aligned} \|h_t\|_{n/\gamma,\infty;-\mathbb{A}} &= \sup_{0<\tau<\infty} \tau^{\gamma/n}\ell^{-\mathbb{A}}(\tau)h_t^*(\tau) \\ &= \max\{\sup_{0<\tau<t} \tau^{\gamma/n}\ell^{-\mathbb{A}}(\tau)f^*(t), \sup_{t\le\tau<\infty} \tau^{\gamma/n}\ell^{-\mathbb{A}}(\tau)f^*(\tau)\} \\ &\approx \max\{t^{\gamma/n}\ell^{-\mathbb{A}}(t)f^*(t), \sup_{t\le\tau<\infty} \tau^{\gamma/n}\ell^{-\mathbb{A}}(\tau)f^*(\tau)\} \\ &\le \sup_{t<\tau<\infty} \tau^{\gamma/n}\ell^{-\mathbb{A}}(\tau)f^{**}(\tau). \end{aligned} \tag{3.22}$$

Thus, inequality (3.17) implies that $h_t \in L^{n/\gamma,\infty}(\log L)^{-\mathbb{A}}(\mathbb{R}^n)$. Together with (3.5), this yields

$$\sup_{0<\tau<\infty} (\mathscr{M}_{1,\gamma;A} h_t)^*(\tau) \lesssim \|h_t\|_{n/\gamma,\infty;-\mathbb{A}}. \tag{3.23}$$

Using (3.18) and [BS, Chapter 2, Proposition 1.7], we obtain

$$(\mathscr{M}_{1,\gamma;\mathbb{A}} f)^*(t) \le (\mathscr{M}_{1,\gamma;\mathbb{A}}\, g_t)^*(t/2) + (\mathscr{M}_{1,\gamma;\mathbb{A}} h_t)^*(t/2). \tag{3.24}$$

Thus, combining estimates (3.21), (3.23), (3.20) and (3.22), we arrive at

$$\begin{aligned} (\mathscr{M}_{1,\gamma;\mathbb{A}} f)^*(t) &\lesssim (t/2)^{(\gamma-n)/n}\ell^{-\mathbb{A}}(t/2)\|g_t\|_1 + \|h_t\|_{n/\gamma,\infty;-\mathbb{A}} \\ &\lesssim t^{(\gamma-n)/n}\ell^{-\mathbb{A}}(t)\int_0^t f^*(\tau)\,d\tau + \sup_{t<\tau<\infty} \tau^{\gamma/n}\ell^{-\mathbb{A}}(\tau)f^{**}(\tau) \\ &\lesssim \sup_{t<\tau<\infty} \tau^{\gamma/n}\ell^{-\mathbb{A}}(\tau)f^{**}(\tau) \end{aligned}$$

and (3.16) is proved. ∎

REMARKS 3.4. (i) There is another way of proving Lemma 3.3: First, the endpoint estimates (3.6) and (3.5) imply that

$$\begin{aligned} &K(\mathscr{M}_{1,\gamma;\mathbb{A}} f, t; L^{n/(n-\gamma),\infty}(\log L)^{\mathbb{A}}(\mathbb{R}^n), L^\infty(\mathbb{R}^n)) \\ &\qquad \lesssim K(f, t; L^1(\mathbb{R}^n), L^{n/\gamma,\infty}(\log L)^{-\mathbb{A}}(\mathbb{R}^n)) \end{aligned} \tag{3.25}$$

for all $f \in L^1(\mathbb{R}^n) + L^{n/\gamma,\infty}(\log L)^{-\mathbb{A}}(\mathbb{R}^n)$ and every $t \in (0,\infty)$; here K is the Peetre K-functional.

Second, by [EvO, (8.12)],

$$L^{n/(n-\gamma),\infty}(\log L)^{\mathbb{A}}(\mathbb{R}^n) = (L^{1/2,\infty}(\mathbb{R}^n), L^\infty(\mathbb{R}^n))_{(1/2)(1+\gamma/n),\infty;\mathbb{A}}.$$

(Here, for a compatible couple of quasi-Banach spaces X_0 and X_1, the interpolation space $(X_0, X_1)_{\theta,q;\mathbb{A}}$, $\theta = [0,1]$, $q \in (0,\infty]$ and $\mathbb{A} \in \mathbb{R}^2$, is the set of all $f \in X_0 + X_1$ such that

$$\|f\|_{\theta,q;\mathbb{A}} := \|t^{-\theta-1/q}\ell^{\mathbb{A}}(t)K(f,t;X_0,X_1)\|_{q,(0,\infty)}$$

is finite.)

Hence, applying [EOP2, Theorem 6.6] (with $X_0 = L^{1/2,\infty}(\mathbb{R}^n)$ and $X_1 = L^\infty(\mathbb{R}^n)$), we arrive at

$$\begin{aligned}(3.26)\quad & K(\mathscr{M}_{1,\gamma;\mathbb{A}}f, t^{1-\gamma/n}\ell^{\mathbb{A}}(t);\ L^{n/(n-\gamma),\infty}(\log L)^{\mathbb{A}}(\mathbb{R}^n), L^\infty(\mathbb{R}^n)) \\ &\approx \sup_{0<\tau<t} \tau^{1-\gamma/n}\ell^{\mathbb{A}}(\tau)(\mathscr{M}_{1,\gamma;\mathbb{A}}f)^*(\tau), \qquad t \in (0,\infty).\end{aligned}$$

On the other hand,

$$L^{n/\gamma,\infty}(\log L)^{-\mathbb{A}}(\mathbb{R}^n) = (L^1(\mathbb{R}^n), L^\infty(\mathbb{R}^n))_{1-\gamma/n,\infty;-\mathbb{A}}.$$

Thus, applying [EOP2, Theorem 6.10] or [EOP2, Theorem 6.5], respectively, if $\gamma = 0$ or $\gamma \in (0,n)$, we obtain

$$\begin{aligned}(3.27)\quad & K(f, t^{1-\gamma/n}\ell^{\mathbb{A}}(t); L^1(\mathbb{R}^n), L^{n/\gamma,\infty}(\log L)^{-\mathbb{A}}(\mathbb{R}^n)) \\ &\approx t^{1-\gamma/n}\ell^{\mathbb{A}}(t) \sup_{t<\tau<\infty} \tau^{\gamma/n}\ell^{-\mathbb{A}}(\tau)f^{**}(\tau)\end{aligned}$$

and (3.16) follows from (3.25)–(3.27).

(ii) Since the proof of (3.16) is based on the endpoint estimates (3.5) and (3.6) (cf. part (i)), inequality (3.16) holds not only for the fractional maximal operator $\mathscr{M}_{1,\gamma;\mathbb{A}}$ but for any quasi-linear operator satisfying the same endpoint estimates.

Now, we are going to verify estimate (3.4) provided that $s = 1$.

LEMMA 3.5. *Let the assumptions of Lemma 3.2 be satisfied. Then for every* $\psi \in \mathcal{M}^+(0,\infty;\downarrow)$ *there is* $f \in \mathcal{M}^+(\mathbb{R}^n)$ *such that* $f^* = \psi$ *a.e. on* $(0,\infty)$ *and*

$$(3.28)\qquad (\mathscr{M}_{1,\gamma;\mathbb{A}}f)^*(t) \geq c \sup_{t<\tau<\infty} \tau^{\gamma/n}\ell^{-\mathbb{A}}(\tau)f^{**}(\tau)$$

for all $t \in (0,\infty)$, *where* c *is a positive constant depending only on* n, γ *and* $\mathbb{A}$.

Proof. For $a \in (0,\infty)$ put $Q(a) = \{z \in \mathbb{R}^n;\ |z_i| \leq a,\ i = 1,\dots,n\}$ and $B(a) = \{z \in \mathbb{R}^n;\ |z| \leq a\}$. Let $\omega_n = |B(1)|$. Then

$$(3.29)\qquad \left(\frac{|B(a)|}{|Q(a)|}\right)^{1-\gamma/n} = \left(\frac{\omega_n a^n}{(2a)^n}\right)^{1-\gamma/n} = \omega_n^{1-\gamma/n}2^{\gamma-n} =: c_1 = c_1(n,\gamma).$$

Since the function

$$(3.30)\qquad g(t) := \ell^{\mathbb{A}}(\omega_n t^n)/\ell^{\mathbb{A}}((2t)^n), \qquad t \in (0,\infty),$$

is positive and continuous on $(0,\infty)$ and satisfies $g(t) \to 1$ as $t \to 0+$ or $t \to \infty$, there is a positive constant $c_2 = c_2(n,\mathbb{A})$ such that

$$(3.31)\qquad g(t) \geq c_2 \quad \text{for all } t \in (0,\infty).$$

If $\psi \in \mathcal{M}^+(0,\infty;\downarrow)$ and $f(x) := \psi(\omega_n|x|^n)$, $x \in \mathbb{R}^n \setminus \{0\}$, then $f^* = \psi$ a.e. on $(0,\infty)$. Moreover, we deduce from (3.29)–(3.31) and from the definition of $\mathscr{M}_{1,\gamma;\mathbb{A}}$ that for every

$x, y \in \mathbb{R}^n$ with $|y| > |x|$,

$$(\mathscr{M}_{1,\gamma;\mathbb{A}} f)(x) \gtrsim c_1 c_2 (\omega_n |y|^n)^{\gamma/n-1} \ell^{-\mathbb{A}}(\omega_n |y|^n) \int\limits_{B(|y|)} f(z)\, dz.$$

Since the definition of f and spherical coordinates give

$$\begin{aligned}\int\limits_{B(|y|)} f(z)\, dz &= \int\limits_0^{|y|} \int\limits_{\{|z|=r\}} \psi(\omega_n r^n)\, d\theta\, dr \\ &= \int\limits_0^{|y|} \psi(\omega_n r^n)\, \omega_n n r^{n-1}\, dr = \int\limits_0^{\omega_n |y|^n} \psi(\sigma)\, d\sigma,\end{aligned}$$

we arrive at the estimate

$$(\mathscr{M}_{1,\gamma;\mathbb{A}} f)(x) \ge c_1 c_2 H(\omega_n |y|^n),$$

where $H(\tau) = \tau^{\gamma/n-1} \ell^{-\mathbb{A}}(\tau) \int_0^\tau \psi(\sigma)\, d\sigma$, $\tau \in (0,\infty)$, and (3.28) follows. ■

Proof of Theorem 3.1. Since for all $f \in \mathcal{M}(\mathbb{R}^n)$ and any cube Q in $\mathbb{R}^n$,

$$\|f\chi_Q\|_s = \||f|^s \chi_Q\|_1^{1/s} \quad \text{and} \quad \|\chi_Q\|_{sn/(n-\gamma);\mathbb{A}} = \|\chi_Q\|_{n/(n-\gamma);s\mathbb{A}}^{1/s},$$

we have

$$\mathscr{M}_{s,\gamma;\mathbb{A}} f = [(\mathscr{M}_{1,\gamma;s\mathbb{A}} |f|^s)]^{1/s}. \tag{3.32}$$

Therefore, using Lemma 3.3, for all $t \in (0,\infty)$ we obtain

$$\begin{aligned}(\mathscr{M}_{s,\gamma;\mathbb{A}} f)^*(t) &= [(\mathscr{M}_{1,\gamma;s\mathbb{A}} |f|^s)^*(t)]^{1/s} \\ &\le [C \sup_{t<\tau<\infty} \tau^{\gamma/n} \ell^{-s\mathbb{A}}(\tau) (|f|^s)^{**}(\tau)]^{1/s} \\ &= C^{1/s} \Big[\sup_{t<\tau<\infty} \tau^{\gamma/n-1} \ell^{-s\mathbb{A}}(\tau) \int\limits_0^\tau (f^*)^s(\sigma)\, d\sigma \Big]^{1/s}\end{aligned}$$

(where $C = C(n, \gamma, s\mathbb{A})$) and (3.3) is verified.

To prove (3.4), take $\varphi \in \mathcal{M}^+(0,\infty;\downarrow)$ and put $\psi = \varphi^s$. Then $\psi \in \mathcal{M}^+(0,\infty;\downarrow)$ and, by Lemma 3.5, there is $F \in \mathcal{M}^+(\mathbb{R}^n)$ such that $F^* = \psi$ a.e. in $(0,\infty)$ and, for all $t \in (0,\infty)$,

$$(\mathscr{M}_{1,\gamma;s\mathbb{A}} F)^*(t) \ge c \sup_{t<\tau<\infty} \tau^{\gamma/n} \ell^{-s\mathbb{A}}(\tau) F^{**}(\tau), \tag{3.33}$$

where $c = c(n, \gamma, s\mathbb{A})$. Moreover, on putting

$$f = F^{1/s}, \tag{3.34}$$

we obtain

$$F^* = (f^*)^s \quad \text{and} \quad F^{**}(\tau) = \tau^{-1} \int\limits_0^\tau (f^*)^s(\sigma)\, d\sigma, \qquad \tau > 0. \tag{3.35}$$

Consequently, by (3.34), (3.33) and (3.35),

$$(\mathscr{M}_{1,\gamma;s\mathbb{A}}\, |f|^s)^*(t) = (\mathscr{M}_{1,\gamma;s\mathbb{A}}\, F)^*(t) \ge c \sup_{t<\tau<\infty} \tau^{\gamma/n-1} \ell^{-s\mathbb{A}}(\tau) \int\limits_0^\tau (f^*)^s(\sigma)\, d\sigma \tag{3.36}$$

for all $t \in (0,\infty)$. Estimates (3.36) and (3.32) imply (3.4). Moreover, $f^* = (F^*)^{1/s} = \psi^{1/s} = \varphi$. ■

We conclude this section with some results related to Lemma 3.3. In the proof of Lemma 3.3 (cf. also Remarks 3.4) we have seen that inequality (3.16) is a consequence of the endpoint estimates (3.6) and (3.5). The next assertion shows that the converse holds, that is, inequality (3.16) implies that the endpoint estimates (3.6) and (3.5) are satisfied.

LEMMA 3.6. *Let* $\gamma \in [0,n)$ *and* $\mathbb{A} = (A_0, A_\infty) \in \mathbb{R}^2$ *be such that* (3.2) *holds. Assume that for all* $f \in \mathcal{M}(\mathbb{R}^n)$ *and every* $t \in (0,\infty)$,

$$(\mathscr{M}_{1,\gamma;\mathbb{A}} f)^*(t) \lesssim \sup_{t<\tau<\infty} \tau^{\gamma/n} \ell^{-\mathbb{A}}(\tau) f^{**}(\tau). \tag{3.37}$$

Then

$$\mathscr{M}_{1,\gamma;\mathbb{A}} : L^1(\mathbb{R}^n) \to L^{n/(n-\gamma),\infty}(\log L)^{\mathbb{A}}(\mathbb{R}^n), \tag{3.38}$$

$$\mathscr{M}_{1,\gamma;\mathbb{A}} : L^{n/\gamma,\infty}(\log L)^{-\mathbb{A}}(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n). \tag{3.39}$$

Proof. First, we verify (3.38). Let $f \in \mathcal{M}(\mathbb{R}^n)$. Using estimate (3.37), we obtain

$$\begin{aligned} \|\mathscr{M}_{1,\gamma;\mathbb{A}} f\|_{n/(n-\gamma),\infty;\mathbb{A}} &= \sup_{0<t<\infty} t^{(n-\gamma)/n} \ell^{\mathbb{A}}(t) (\mathscr{M}_{1,\gamma;\mathbb{A}} f)^*(t) \\ &\lesssim \sup_{0<t<\infty} t^{(n-\gamma)/n} \ell^{\mathbb{A}}(t) \sup_{t<\tau<\infty} \tau^{\gamma/n} \ell^{-\mathbb{A}}(\tau) f^{**}(\tau) \\ &= \sup_{0<\tau<\infty} \tau^{\gamma/n} \ell^{-\mathbb{A}}(\tau) f^{**}(\tau) \sup_{0<t<\tau} t^{(n-\gamma)/n} \ell^{\mathbb{A}}(t). \end{aligned} \tag{3.40}$$

Since $\gamma < n$, we see that for all $\tau \in (0,\infty)$,

$$\sup_{0<t<\tau} t^{(n-\gamma)/n} \ell^{\mathbb{A}}(t) \approx \tau^{(n-\gamma)/n} \ell^{\mathbb{A}}(\tau).$$

Thus,

$$\mathrm{RHS}(3.40) \approx \sup_{0<\tau<\infty} \tau f^{**}(\tau) = \int_0^\infty f^*(\sigma)\, d\sigma = \|f\|_1 \tag{3.41}$$

and (3.38) is a consequence of estimates (3.40) and (3.41).

Now, we are going verify (3.39). Taking $f \in \mathcal{M}(\mathbb{R}^n)$, applying (3.37) and the fact that the non-increasing rearrangement of any function is a right-continuous function on $[0,\infty)$, we obtain

$$\begin{aligned} \|\mathscr{M}_{1,\gamma;\mathbb{A}} f\|_\infty &= \sup_{0<t<\infty} (\mathscr{M}_{1,\gamma;\mathbb{A}} f)^*(t) = (\mathscr{M}_{1,\gamma;A} f)^*(0) \\ &\lesssim \sup_{0<\tau<\infty} \tau^{\gamma/n} \ell^{-\mathbb{A}}(\tau) f^{**}(\tau) = \sup_{0<\tau<\infty} \tau^{\gamma/n-1} \ell^{-\mathbb{A}}(\tau) \int_0^\tau f^*(\sigma)\, d\sigma. \end{aligned} \tag{3.42}$$

Since $\gamma < n$,

$$\begin{aligned} \int_0^\tau f^*(\sigma)\, d\sigma &= \int_0^\tau (\sigma^{\gamma/n} \ell^{-\mathbb{A}}(\sigma) f^*(\sigma)) \sigma^{-\gamma/n} \ell^{\mathbb{A}}(\sigma)\, d\sigma \\ &\le \Big(\int_0^\tau \sigma^{-\gamma/n} \ell^{\mathbb{A}}(\sigma)\, d\sigma \Big) \sup_{0<\sigma<\infty} \sigma^{\gamma/n} \ell^{-\mathbb{A}}(\sigma) f^*(\sigma) \\ &\approx \tau^{1-\gamma/n} \ell^{\mathbb{A}}(\tau) \|f\|_{n/\gamma,\infty;-\mathbb{A}} \end{aligned}$$

for all $\tau \in (0,\infty)$. Consequently,

$$\mathrm{RHS}(3.42) \lesssim \|f\|_{n/\gamma,\infty;-\mathbb{A}}$$

and (3.39) follows. ■

Assuming additionally in Lemma 3.3 that $\gamma > 0$, we are able to prove the following variant of (3.16).

Lemma 3.7. *Let $\gamma \in (0,n)$ and $\mathbb{A} = (A_0, A_\infty) \in \mathbb{R}^2$. Then there is a positive constant C depending only on n, γ and $\mathbb{A}$ such that for all $f \in \mathcal{M}(\mathbb{R}^n)$ and every $t \in (0,\infty)$,*

$$(\mathscr{M}_{1,\gamma;\mathbb{A}} f)^{**}(t) \le C \sup_{t<\tau<\infty} \tau^{\gamma/n} \ell^{-\mathbb{A}}(\tau) f^{**}(\tau). \tag{3.43}$$

Proof. For $p,q \in (0,\infty]$ and $\mathbb{A} \in \mathbb{R}^2$ define the quasi-norm $\|\cdot\|_{(p,q;\mathbb{A})}$ on $\mathcal{M}(\mathbb{R}^n)$ by

$$\|f\|_{(p,q;\mathbb{A})} = \|t^{1/p-1/q} \ell^{\mathbb{A}}(t) f^{**}(t)\|_{q,(0,\infty)}$$

and put

$$L_{(p,q;\mathbb{A})}(\mathbb{R}^n) = \{f \in \mathcal{M}(\mathbb{R}^n);\ \|f\|_{(p,q;\mathbb{A})} < \infty\}.$$

By [OP, Theorem 3.8(i)],

$$L^{p,q}(\log L)^{\mathbb{A}}(\mathbb{R}^n) = L_{(p,q;\mathbb{A})}(\mathbb{R}^n)$$

(and the corresponding quasi-norms are equivalent) provided that $p > 1$. Hence, if $\gamma \in (0,n)$, then

$$L^{n/(n-\gamma),\infty}(\log L)^{\mathbb{A}}(\mathbb{R}^n) = L_{(n/(n-\gamma),\infty;\mathbb{A})}(\mathbb{R}^n),$$

which in turn means that the endpoint estimate (3.6) is equivalent to

$$\mathscr{M}_{1,\gamma;\mathbb{A}} : L^1(\mathbb{R}^n) \to L_{(n/(n-\gamma),\infty;\mathbb{A})}(\mathbb{R}^n). \tag{3.6*}$$

For $t \in (0,\infty)$ and $f \in \mathcal{M}(\mathbb{R}^n)$ let g_t and h_t be functions from the proof of Lemma 3.3. As $g_t \in L^1(\mathbb{R}^n)$, estimate (3.6*) implies that

$$\sup_{0<\tau<\infty} \tau^{(n-\gamma)/n} \ell^{\mathbb{A}}(\tau) (\mathscr{M}_{1,\gamma;\mathbb{A}}\, g_t)^{**}(\tau) \lesssim \|g\|_1. \tag{3.21*}$$

Similarly, one can replace estimate (3.23) with

$$\sup_{0<\tau<\infty} (\mathscr{M}_{1,\gamma;\mathbb{A}} h_t)^{**}(\tau) \lesssim \|h_t\|_{n/\gamma,\infty;\mathbb{A}}. \tag{3.23*}$$

Moreover, by (3.18) and [BS, Chapter 2, Theorem 3.4],

$$(\mathscr{M}_{1,\gamma;\mathbb{A}} f)^{**}(t) \le (\mathscr{M}_{1,\gamma;\mathbb{A}}\, g_t)^{**}(t) + (\mathscr{M}_{1,\gamma;\mathbb{A}}\, h_t)^{**}(t). \tag{3.24*}$$

Therefore, (3.24*), (3.21*) and (3.23*) imply that

$$(\mathscr{M}_{1,\gamma;\mathbb{A}} f)^{**}(t) \lesssim t^{(\gamma-n)/n} \ell^{-\mathbb{A}}(t) \|g_t\|_1 + \|h_t\|_{n/\gamma,\infty;\mathbb{A}}.$$

Hence, by (3.20) and (3.22),

$$\begin{aligned}(\mathscr{M}_{1,\gamma;\mathbb{A}} f)^{**}(t) &\lesssim t^{(\gamma-n)/n} \ell^{-\mathbb{A}}(t) \int_0^t f^*(\tau)\, d\tau + \sup_{t<\tau<\infty} \tau^{\gamma/n} \ell^{-\mathbb{A}}(\tau) f^{**}(\tau) \\ &\lesssim \sup_{t<\tau<\infty} \tau^{\gamma/n} \ell^{-\mathbb{A}} f^{**}(\tau),\end{aligned}$$

which is the desired result. ■

One can easily see from the proofs of Lemmas 3.3 and 3.6 that the following assertion holds.

Corollary 3.8. *Let $\gamma \in [0,n)$ and $\mathbb{A} = (A_0, A_\infty) \in \mathbb{R}^2$ be such that (3.2) is satisfied. Let T be a quasi-linear operator on $\mathcal{M}(\mathbb{R}^n)$ with values in $\mathcal{M}(\mathbb{R}^n)$. Then the following statements are equivalent*:

(i) *There is a positive constant C such that for all $f \in \mathcal{M}(\mathbb{R}^n)$ and every $t \in (0,\infty)$,*

$$(Tf)^*(t) \le C \sup_{t<\tau<\infty} \tau^{\gamma/n} \ell^{-\mathbb{A}}(\tau) f^{**}(\tau).$$

(ii) *The mappings*

$$T : L^1(\mathbb{R}^n) \to L^{n/(n-\gamma),\infty}(\log L)^{\mathbb{A}}(\mathbb{R}^n) \quad \textit{and} \quad T : L^{n/\gamma,\infty}(\log L)^{-\mathbb{A}}(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)$$

are bounded.

Similarly, if $\gamma \in (0,n)$, we arrive at the next result.

Corollary 3.9. *Let $\gamma \in (0,n)$ and $\mathbb{A} = (A_0, A_\infty) \in \mathbb{R}^2$. Let T be a quasi-linear operator on $\mathcal{M}(\mathbb{R}^n)$ with values in $\mathcal{M}(\mathbb{R}^n)$. Then the following statements are equivalent*:

(i) *There is a positive constant C such that for all $f \in \mathcal{M}(\mathbb{R}^n)$ and every $t \in (0,\infty)$,*

$$(Tf)^*(t) \le C \sup_{t<\tau<\infty} \tau^{\gamma/n} \ell^{-\mathbb{A}}(\tau) f^{**}(\tau).$$

(ii) *There is a positive constant C such that for all $f \in \mathcal{M}(\mathbb{R}^n)$ and every $t \in (0,\infty)$,*

$$(Tf)^{**}(t) \le C \sup_{t<\tau<\infty} \tau^{\gamma/n} \ell^{-\mathbb{A}}(\tau) f^{**}(\tau).$$

(iii) *The mappings*

$$T : L^1(\mathbb{R}^n) \to L^{n/(n-\gamma),\infty}(\log L)^{\mathbb{A}}(\mathbb{R}^n) \quad \textit{and} \quad T : L^{n/\gamma,\infty}(\log L)^{-\mathbb{A}}(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)$$

are bounded.

4. Boundedness of $\mathscr{M}_{s,\gamma;\mathbb{A}} : \Lambda^p(v) \to \Lambda^q(w),\ 0 < p \le q < \infty$

Using Theorem 3.1 and the definition of the quasi-norms in classical Lorentz spaces, one can see that the operator

$$\mathscr{M}_{s,\gamma;\mathbb{A}} : \Lambda^p(v) \to \Lambda^q(w), \qquad 0 < p, q < \infty, \tag{4.1}$$

is bounded if and only if

$$\Big\{ \int_0^\infty \Big(\sup_{t<\tau<\infty} \tau^{\gamma/n-1} \ell^{-s\mathbb{A}}(\tau) \int_0^\tau [\varphi(\sigma)]^s \, d\sigma \Big)^{q/s} w(t)\, dt \Big\}^{1/q} \lesssim \Big\{ \int_0^\infty [\varphi(t)]^p v(t)\, dt \Big\}^{1/p} \quad \text{for all } \varphi \in \mathcal{M}^+(0,\infty;\downarrow). \tag{4.2}$$

Putting

$$\psi = \varphi^s \quad \text{and} \quad P = p/s, \qquad Q = q/s, \tag{4.3}$$

we can rewrite (4.2) as

$$\Big\{ \int_0^\infty [(\mathscr{T}\psi)(t)]^Q w(t)\,dt \Big\}^{1/Q} \lesssim \Big\{ \int_0^\infty [\psi(t)]^P v(t)\,dt \Big\}^{1/P} \text{ for all } \psi \in \mathcal{M}^+(0,\infty;\downarrow), \tag{4.4}$$

where the operator $\mathscr{T}$ is given on the set $\mathcal{M}^+(0,\infty;\downarrow)$ by

$$(\mathscr{T}\psi)(t) := \sup_{t<\tau<\infty} \tau^{\gamma/n} \ell^{-s\mathbb{A}}(\tau)\psi^{**}(\tau), \qquad t \in (0,\infty). \tag{4.5}$$

Now we claim that for all $\psi \in \mathcal{M}^+(0,\infty;\downarrow)$ and all $t \in (0,\infty)$,

$$(\mathscr{T}\psi)(t) \approx (\mathscr{S}\psi)(t) + (\mathscr{R}\psi)(t), \tag{4.6}$$

where the operators $\mathscr{S}$ and $\mathscr{R}$ are defined on $\mathcal{M}^+(0,\infty;\downarrow)$ by

$$(\mathscr{S}\psi)(t) := t^{\gamma/n}\ell^{-s\mathbb{A}}(t)\psi^{**}(t), \qquad t \in (0,\infty), \tag{4.7}$$

$$(\mathscr{R}\psi)(t) := \sup_{t<\tau<\infty} \tau^{\gamma/n}\ell^{-s\mathbb{A}}(\tau)\psi(\tau), \qquad t \in (0,\infty). \tag{4.8}$$

Consequently, (4.1) is satisfied if and only if both inequalities

$$\Big\{ \int_0^\infty [(\mathscr{S}\psi)(t)]^Q w(t)\,dt \Big\}^{1/Q} \lesssim \Big\{ \int_0^\infty [\psi(t)]^P v(t)\,dt \Big\}^{1/P} \tag{4.9}$$

and

$$\Big\{ \int_0^\infty [(\mathscr{R}\psi)(t)]^Q w(t)\,dt \Big\}^{1/Q} \lesssim \Big\{ \int_0^\infty [\psi(t)]^P v(t)\,dt \Big\}^{1/P} \tag{4.10}$$

hold on $\mathcal{M}^+(0,\infty;\downarrow)$.

To verify (4.6), first note that $\mathscr{T} \gtrsim \mathscr{S} + \mathscr{R}$ on $\mathcal{M}^+(0,\infty;\downarrow)$ since the estimates $\mathscr{T}\psi \ge \mathscr{S}\psi$ and $\mathscr{T}\psi \ge \mathscr{R}\psi$ are evident for any $\psi \in \mathcal{M}^+(0,\infty;\downarrow)$. On the other hand, since for all $\psi \in \mathcal{M}^+(0,\infty;\downarrow)$ and all $t \in (0,\infty)$,

$$(\mathscr{T}\psi)(t) = \sup_{t<\tau<\infty} \tau^{\gamma/n-1}\ell^{-s\mathbb{A}}(\tau)\Big[\int_0^t \psi(\sigma)\,d\sigma + \int_t^\tau \psi(\sigma)\,d\sigma\Big] \tag{4.11}$$

and since $\gamma < n$, we have

$$\sup_{t<\tau<\infty} \tau^{\gamma/n-1}\ell^{-s\mathbb{A}}(\tau)\int_0^t \psi(\sigma)\,d\sigma \approx (\mathscr{S}\psi)(t) \tag{4.12}$$

and

$$\begin{aligned} \sup_{t<\tau<\infty} \tau^{\gamma/n-1}\ell^{-s\mathbb{A}}(\tau)\int_t^\tau \psi(\sigma)\,d\sigma &\le \sup_{t<\tau<\infty} \tau^{\gamma/n-1}\ell^{-s\mathbb{A}}(\tau)\int_t^\tau [\sup_{t<\xi<\infty} \xi^{\gamma/n}\ell^{-s\mathbb{A}}(\xi)\psi(\xi)]\sigma^{-\nu/n}\ell^{s\mathbb{A}}(\sigma)\,d\sigma \\ &\lesssim (\mathscr{R}\psi)(t) \sup_{t<\tau<\infty} \tau^{\gamma/n-1}\ell^{-s\mathbb{A}}(\tau)\int_0^\tau \sigma^{-\gamma/n}\ell^{s\mathbb{A}}(\sigma)\,d\sigma \approx (\mathscr{R}\psi)(t). \end{aligned} \tag{4.13}$$

Thus, the estimate $\mathscr{T}\psi \lesssim \mathscr{S}\psi + \mathscr{R}\psi$ for all $\psi \in \mathcal{M}^+(0,\infty;\downarrow)$ is a consequence of (4.11)–(4.13).

Our main result in this section is the following theorem which provides us with a characterization of (4.1) in the case when $0 < p \le q < \infty$.

THEOREM 4.1. *Let $s \in (0,\infty)$, $n \in \mathbb{N}$, $\gamma \in [0,\infty)$, $\mathbb{A} = (A_0, A_\infty) \in \mathbb{R}^2$ and $v, w \in \mathscr{W}(0,\infty)$. Assume that $0 < p \le q < \infty$ and*

$$\text{either } \gamma \in (0,n), \text{ or } \gamma = 0 \text{ and } A_0 \ge 0 \ge A_\infty. \tag{4.14}$$

Then the following statements are equivalent:

(i) *The operator $\mathscr{M}_{s,\gamma;\mathbb{A}} : \Lambda^p(v) \to \Lambda^q(w)$ is bounded.*

(ii) *For all $\psi \in \mathcal{M}^+(0,\infty;\downarrow)$,*

$$\bigg\{ \int_0^\infty \Big[\sup_{t<\tau<\infty} \tau^{\gamma/n-1} \ell^{-s\mathbb{A}}(\tau) \int_0^\tau \psi(\sigma)\,d\sigma \Big]^{q/s} w(t)\,dt \bigg\}^{1/q} \lesssim \bigg\{ \int_0^\infty [\psi(t)]^{p/s} v(t)\,dt \bigg\}^{1/p}. \tag{4.15}$$

(iii) *For all $r \in (0,\infty)$,*

$$r^{\gamma/(ns)} \ell^{-\mathbb{A}}(r) \Big(\int_0^r w(t)\,dt \Big)^{1/q} \lesssim \Big(\int_0^r v(t)\,dt \Big)^{1/p} \tag{4.16}$$

and either

$$\Big(\int_r^\infty t^{(q/s)(\gamma/n-1)} \ell^{-q\mathbb{A}}(t)\, w(t)\,dt \Big)^{1/q} \times \Big[\int_0^r \Big(t^{-1} \int_0^t v(\tau)\,d\tau \Big)^{p/(s-p)} v(t)\,dt \Big]^{1/s-1/p} \lesssim 1 \quad \text{if } s < p, \tag{4.17}$$

or

$$r^{1/s} \Big(\int_r^\infty t^{(q/s)(\gamma/n-1)} \ell^{-q\mathbb{A}}(t)\, w(t)\,dt \Big)^{1/q} \lesssim \Big(\int_0^r v(t)\,dt \Big)^{1/p} \quad \text{if } p \le s. \tag{4.18}$$

To prove Theorem 4.1, we shall characterize the validity of inequalities (4.9) and (4.10) on $\mathcal{M}^+(0,\infty;\downarrow)$. To find necessary and sufficient conditions under which (4.9) holds on $\mathcal{M}^+(0,\infty;\downarrow)$, we shall use the following assertions.

LEMMA 4.2 (cf. [Sa, Theorem 2]). *Suppose that $\widetilde{v}, \widetilde{w} \in \mathscr{W}(0,\infty)$ and $1 < P \le Q < \infty$. Then the inequality*

$$\bigg\{ \int_0^\infty \Big[t^{-1} \int_0^t \psi(\tau)\,d\tau \Big]^Q \widetilde{w}(t)\,dt \bigg\}^{1/Q} \lesssim \bigg\{ \int_0^\infty [\psi(t)]^P \widetilde{v}(t)\,dt \bigg\}^{1/P} \tag{4.19}$$

holds for all $\psi \in \mathcal{M}^+(0,\infty;\downarrow)$ if and only if, for all $r \in (0,\infty)$,

$$\Big(\int_0^r \widetilde{w}(t)\,dt \Big)^{1/Q} \lesssim \Big(\int_0^r \widetilde{v}(t)\,dt \Big)^{1/P} \tag{4.20}$$

and

$$\Big(\int_r^\infty t^{-Q} \widetilde{w}(t)\,dt \Big)^{1/Q} \Big[\int_0^r \Big(t^{-1} \int_0^t \widetilde{v}(\tau)\,d\tau \Big)^{-P'} \widetilde{v}(t)\,dt \Big]^{1/P'} \lesssim 1. \tag{4.21}$$

Lemma 4.3 (cf. [HM, Theorem 3.2(b)]). *Let* $0 < P \le 1$, $P \le Q < \infty$. *Suppose that* $\widetilde{v}, \widetilde{w} \in \mathscr{W}(0,\infty)$ *and* $\Phi \in \mathcal{M}^+((0,\infty)\times(0,\infty))$. *Then the inequality*

$$\Big\{ \int_0^\infty \Big[\int_0^\infty \Phi(t,\tau)\psi(\tau)\,d\tau \Big]^Q \widetilde{w}(t)\,dt \Big\}^{1/Q} \lesssim \Big\{ \int_0^\infty [\psi(t)]^P \widetilde{v}(t)\,dt \Big\}^{1/P} \tag{4.22}$$

holds for all $\psi \in \mathcal{M}^+(0,\infty;\downarrow)$ *if and only if*

$$\Big\{ \int_0^\infty \Big[\int_0^r \Phi(t,\tau)\,d\tau \Big]^Q \widetilde{w}(t)\,dt \Big\}^{1/Q} \lesssim \Big[\int_0^r \widetilde{v}(t)\,dt \Big]^{1/P} \quad \textit{for all } r \in (0,\infty). \tag{4.23}$$

In the next lemma we present a characterization of inequality (4.9) on $\mathcal{M}^+(0,\infty;\downarrow)$.

Lemma 4.4. *Let all the assumptions of Theorem* 4.1 *be satisfied and* $P = p/s$, $Q = q/s$. *Then inequality* (4.9) *holds on* $\mathcal{M}^+(0,\infty;\downarrow)$ *if and only if either* $s < p$ *and*, *for all* $r \in (0,\infty)$,

$$\Big(\int_0^r t^{\gamma q/(ns)} \ell^{-q\mathbb{A}}(t) w(t)\,dt \Big)^{1/q} \lesssim \Big(\int_0^r v(t)\,dt \Big)^{1/p} \tag{4.24}$$

and

$$\Big(\int_r^\infty t^{(q/s)(\gamma/n-1)} \ell^{-q\mathbb{A}}(t) w(t)\,dt \Big)^{1/q} \Big[\int_0^r \Big(t^{-1} \int_0^t v(\tau)\,d\tau \Big)^{p/(s-p)} v(t)\,dt \Big]^{1/s-1/p} \lesssim 1, \tag{4.25}$$

or $p \le s$ *and*, *for all* $r \in (0,\infty)$, *condition* (4.24) *holds and*

$$r^{1/s} \Big(\int_r^\infty t^{(q/s)(\gamma/n-1)} \ell^{-q\mathbb{A}}(t) w(t)\,dt \Big)^{1/q} \lesssim \Big(\int_0^r v(t)\,dt \Big)^{1/p}. \tag{4.26}$$

Proof. If $0 < s < p$, then $1 < P \le Q < \infty$ and inequality (4.9) can be rewritten as (4.19) with

$$\widetilde{w}(t) = t^{(\gamma/n)Q} \ell^{-sQ\mathbb{A}}(t) w(t), \quad \widetilde{v}(t) = v(t), \quad t \in (0,\infty).$$

Thus, by Lemma 4.2, inequality (4.9) holds on $\mathcal{M}^+(0,\infty;\downarrow)$ if and only if (4.20) and (4.21) are satisfied. However, (4.20) and (4.21), respectively, can be rewritten as (4.24) and (4.25).

If $p \le s$, then $0 < P \le 1$ and $P \le Q < \infty$ and inequality (4.9) can be rewritten as (4.22) with

$$\widetilde{w}(t) = t^{(\gamma/n-1)Q} \ell^{-sQ\mathbb{A}}(t) w(t), \quad \widetilde{v}(t) = v(t), \quad \Phi(t,\tau) = \chi_{(0,t)}(\tau), \quad t,\tau \in (0,\infty).$$

Thus, by Lemma 4.3, inequality (4.9) holds on $\mathcal{M}^+(0,\infty;\downarrow)$ if and only if (4.23) is satisfied. However, for any fixed $r \in (0,\infty)$, the integral $\int_0^\infty \ldots dt$ in (4.23) can be written as $\int_0^r \ldots dt + \int_r^\infty \ldots dt$. This in fact shows that (4.23) is equivalent to two conditions; the first (resp. second) of them is obtained on replacing $\int_0^\infty \ldots dt$ in (4.23) by $\int_0^r \ldots dt$ (resp. $\int_r^\infty \ldots dt$). Finally, it is easy to see that the first (resp. second) condition coincides with (4.24) (resp. (4.26)). ∎

We now turn our attention to (4.10).

LEMMA 4.5. *Let all the assumptions of Theorem* 4.1 *be satisfied and* $P = p/s$, $Q = q/s$. *Then inequality* (4.10) *holds on* $\mathcal{M}^+(0,\infty;\downarrow)$ *if and only if for all* $r \in (0,\infty)$,

$$r^{\gamma/(ns)}\ell^{-\mathbb{A}}(r)\Big(\int_0^r w(t)\,dt\Big)^{1/q} \lesssim \Big(\int_0^r v(t)\,dt\Big)^{1/p}. \tag{4.27}$$

Proof. Necessity. Since, for any $r \in (0,\infty)$,

$$\mathscr{R}\chi_{(0,r)} \approx r^{\gamma/n}\ell^{-s\mathbb{A}}(r)\chi_{(0,r)}, \tag{4.28}$$

the necessity of (4.27) follows by testing (4.10) with $\psi = \chi_{(0,r)}$.

Sufficiency. (i) Assume additionally that

$$\int_0^x v(t)\,dt < \infty \qquad \text{for any } x \in (0,\infty). \tag{4.29}$$

(i-1) First, consider the case when

$$\text{either } \gamma \in (0,n) \text{ and } \mathbb{A} \in \mathbb{R}^2, \text{ or } \gamma = 0 \text{ and } A_0 \ge 0 > A_\infty. \tag{4.30}$$

Then $\lim_{r\to\infty} r^{\gamma/(ns)}\ell^{-\mathbb{A}}(r) = \infty$. Together with (4.27), this implies that

$$\int_0^\infty v(t)\,dt = \infty. \tag{4.31}$$

Then there is an increasing sequence $\{r_k\}_{k\in\mathbb{Z}} \subset (0,\infty)$ such that

$$\int_0^{r_k} v(t)\,dt = 2^k. \tag{4.32}$$

It is clearly sufficient to verify (4.10) for continuous $\psi \in \mathcal{M}^+(0,\infty;\downarrow)$ having compact support in $[0,\infty)$ and $\psi \not\equiv 0$. For such ψ, the set $B \subset \mathbb{Z}$ given by

$$B = \{k \in \mathbb{Z};\ (\mathscr{R}\psi)(r_{k-1}) > (\mathscr{R}\psi)(r_k)\} \tag{4.33}$$

is not empty. Take $k \in B$ and define

$$z_k = \begin{cases} 0 & \text{if } (\mathscr{R}\psi)(t) = (\mathscr{R}\psi)(r_{k-1}),\ t \in (0,r_{k-1}), \\ \min\{r_j;\ (\mathscr{R}\psi)(r_j) = (\mathscr{R}\psi)(r_{k-1})\} & \text{otherwise.} \end{cases} \tag{4.34}$$

Together with the fact that $\mathscr{R}\psi \in \mathcal{M}^+(0,\infty;\downarrow)$, this implies that

$$(\mathscr{R}\psi)(t) \le (\mathscr{R}\psi)(r_{k-1}), \qquad k \in B,\ t \in [z_k, r_k).$$

Moreover, by the definition of B, the supremum appearing in the definition of $(\mathscr{R}\psi)(r_{k-1})$ is attained in $[r_{k-1}, r_k)$. Therefore, for every $k \in B$ and all $t \in [z_k, r_k)$,

$$(\mathscr{R}\psi)(t) \le (\mathscr{R}\psi)(r_{k-1}) = \sup_{r_{k-1}<\tau<r_k} \tau^{\gamma/n}\ell^{-s\mathbb{A}}(\tau)\psi(\tau) \lesssim \psi(r_{k-1})r_k^{\gamma/n}\ell^{-s\mathbb{A}}(r_k). \tag{4.35}$$

Thus, by (4.35), (4.27), (4.32), the monotonicity of ψ and the inequality $Q/P \ge 1$, we obtain

$$\begin{aligned} \int_0^\infty [(\mathscr{R}\psi)(t)]^Q w(t)\,dt &= \sum_{k\in B}\int_{z_k}^{r_k} [(\mathscr{R}\psi)(t)]^Q w(t)\,dt \\ &\lesssim \sum_{k\in B} [\psi(r_{k-1})]^Q r_k^{(\gamma/n)Q}\ell^{-sQ\mathbb{A}}(r_k)\int_{z_k}^{r_k} w(t)\,dt \end{aligned} \tag{4.36}$$

$$\leq \sum_{k\in B} [\psi(r_{k-1})]^Q \Big(\int_0^{r_k} v(t)\,dt \Big)^{Q/P} = 4^{Q/P} \sum_{k\in B} [\psi(r_{k-1})]^Q \Big(\int_{r_{k-2}}^{r_{k-1}} v(t)\,dt \Big)^{Q/P}$$
$$\leq 4^{Q/P} \sum_{k\in B} \Big(\int_{r_{k-2}}^{r_{k-1}} [\psi(t)]^P v(t)\,dt \Big)^{Q/P} \leq 4^{Q/P} \Big(\sum_{k\in B} \int_{r_{k-2}}^{r_{k-1}} [\psi(t)]^P v(t)\,dt \Big)^{Q/P}$$

and (4.10) follows.

(i-2) Second, consider the case when

$$\gamma = 0 \quad \text{and} \quad A_0 \geq 0 = A_\infty. \tag{4.37}$$

Then $\lim_{r\to\infty} r^{\gamma/(ns)} \ell^{-\mathbb{A}}(r) = 1$, and hence, (4.27) does not yield (4.31). If (4.31) holds, then the previous method gives the result. Therefore, we now assume that

$$\int_0^\infty v(t)\,dt < \infty. \tag{4.38}$$

Then there exists $k_0 \in \mathbb{Z}$ such that

$$2^{k_0} \leq \int_0^\infty v(t)\,dt < 2^{k_0+1}. \tag{4.39}$$

Put $\mathbb{Z}_0 = \{k \in \mathbb{Z};\, k \leq k_0\}$ and define the increasing sequence $\{r_k\}_{k\in\mathbb{Z}_0} \subset (0,\infty)$ by (4.32). Moreover, put

$$B_0 = \{k \in \mathbb{Z}_0;\, (\mathscr{R}\psi)(r_{k-1}) > (\mathscr{R}\psi)(r_k)\}. \tag{4.40}$$

Note that now it can happen that $B_0 = \emptyset$.

(i-2.1) Assuming that $B_0 \neq \emptyset$, we assign to any $k \in B_0$ the point z_k just as in (4.34). Let $k_M = \max B_0$. Proceeding as above, we obtain (cf. (4.36))

$$\int_0^{r_{k_M}} [(\mathscr{R}\psi)(t)]^Q w(t)\,dt \lesssim 4^{Q/P} \Big(\sum_{k\in B_0} \int_{r_{k-2}}^{r_{k-1}} [\psi(t)]^P v(t)\,dt \Big)^{Q/P}. \tag{4.41}$$

Put $I = [r_{k_M}, \infty)$. If $\psi \equiv 0$ in I, then $\mathscr{R}\psi \equiv 0$ in I as well and the result follows from (4.41). Therefore we assume that $\psi \not\equiv 0$ in I. We deduce from the definitions of B_0 and k_M that

$$(\mathscr{R}\psi)(t) = (\mathscr{R}\psi)(r_{k_0}) \quad \text{for all } t \in [r_{k_M}, r_{k_0}].$$

Hence, since $\mathscr{R}\psi \in \mathcal{M}^+(0,\infty;\downarrow)$,

$$(\mathscr{R}\psi)(t) \leq (\mathscr{R}\psi)(r_{k_0}) \quad \text{if } t \in I. \tag{4.42}$$

As

$$(\mathscr{R}\psi)(r_{k_0}) = \sup_{r_{k_0}<\tau<\infty} \tau^{\gamma/n} \ell^{-s\mathbb{A}}(\tau)\psi(\tau) \leq \psi(r_{k_0}) \sup_{r_{k_0}<\tau<\infty} \tau^{\gamma/n} \ell^{-s\mathbb{A}}(\tau), \tag{4.43}$$
$$\lesssim \psi(r_{k_0}) \lim_{\tau\to\infty} \tau^{\gamma/n} \ell^{-s\mathbb{A}}(\tau),$$

estimate (4.42) implies that

$$(\mathscr{R}\psi)(t) \lesssim \psi(r_{k_0}) \lim_{\tau\to\infty} \tau^{\gamma/n} \ell^{-s\mathbb{A}}(\tau) \quad \text{for all } t \in I. \tag{4.44}$$

Moreover, by (4.27), (4.39) and (4.32),

$$
\begin{aligned}
(4.45)\qquad \big(\lim_{\tau\to\infty}\tau^{\gamma/n}\ell^{-s\mathbb{A}}(\tau)\big)^Q\int_{r_{k_M}}^{\infty} w(t)\,dt &= \lim_{\tau\to\infty}\tau^{(\gamma/n)Q}\ell^{-sQ\mathbb{A}}(\tau)\Big(\int_{r_{k_M}}^{\tau} w(t)\,dt\Big)\\
&\lesssim \lim_{\tau\to\infty}\Big(\int_0^{\tau} v(t)\,dt\Big)^{Q/P} \le (2^{k_0+1})^{Q/P}\\
&= \Big(8\int_{r_{k_0-2}}^{r_{k_0-1}} v(t)\,dt\Big)^{Q/P}.
\end{aligned}
$$

Applying (4.44), (4.45) and the monotonicity of ψ, we obtain

$$
\begin{aligned}
(4.46)\qquad \int_{r_{k_M}}^{\infty}[(\mathscr{R}\psi)(t)]^Q w(t)\,dt &\le [\psi(r_{k_0})]^Q\big(\lim_{\tau\to\infty}\tau^{\gamma/n}\ell^{-s\mathbb{A}}(\tau)\big)^Q\int_{r_{k_M}}^{\infty} w(t)\,dt\\
&\lesssim 8^{Q/P}\Big(\int_{r_{k_0-2}}^{r_{k_0-1}}[\psi(t)]^P v(t)\,dt\Big)^{Q/P},
\end{aligned}
$$

which, together with (4.41) and the fact that $Q/P \ge 1$, yields the result.

(i-2.2) Assume now that $B_0 = \emptyset$. Then the function $\mathscr{R}\psi$ is constant in $(0, r_{k_0}]$ and thus

$$
(4.47)\qquad (R\psi)(t) \le (\mathscr{R}\psi)(r_{k_0}) \quad \text{for all } t\in(0,\infty).
$$

Together with (4.43), this implies (4.44) with $I = (0,\infty)$. Consequently,

$$
(4.48)\qquad \int_0^{\infty}[(\mathscr{R}\psi)(t)]^Q w(t)\,dt \le [\psi(r_{k_0})]^Q\big(\lim_{\tau\to\infty}\tau^{\gamma/n}\ell^{-s\mathbb{A}}(\tau)\big)^Q\int_0^{\infty} w(t)\,dt.
$$

Proceeding analogously to (4.45) and (4.46), we obtain

$$
\mathrm{RHS}(4.48) \lesssim 8^{Q/P}\Big(\int_{r_{k_0-2}}^{r_{k_0-1}}[\psi(t)]^P v(t)\,dt\Big)^{Q/P}
$$

and the result follows.

(ii) Assume finally that condition (4.29) is violated.

(ii-1) If $\int_0^x v(t)\,dt = \infty$ for any $x\in(0,\infty)$, then $\Lambda^p(v) = \{0\}$. Consequently,

$$
\mathscr{M}_{s,\gamma;\mathbb{A}} : \Lambda^p(v) \to \Lambda^q(w) \text{ is bounded.}
$$

Since also condition (4.27) is satisfied, Lemma 4.5 holds in this case.

(ii-2) Suppose that there exist $x_0\in(0,\infty)$ such that

$$
(4.49)\qquad \int_0^x v(t)\,dt \begin{cases} = \infty & \text{for any } \ x\in(x_0,\infty),\\ < \infty & \text{for any } \ x\in(0,x_0). \end{cases}
$$

Let $\psi\in\mathcal{M}^+(0,\infty;\downarrow)$, $\psi\not\equiv 0$, be a continuous function. If $\int_0^\infty[\psi(t)]^P v(t)\,dt = \infty$, then (4.10) holds trivially. On the other hand, if $\int_0^\infty[\psi(t)]^p v(t)\,dt < \infty$, then, by (4.49), $\operatorname{supp}\psi \subset [0,x_0]$, which implies that $\mathscr{R}\psi\equiv 0$ on (x_0,∞). Now, defining the sequence $\{r_k\}_{k\in\mathbb{Z}}$ by (4.32), we have $\{r_k\}_{k\in\mathbb{Z}}\subset(0,x_0)$ and analogues of the approaches used in (i-2.1) and (i-2.2) (where the role of the interval $(0,\infty)$ is now played by $(0,x_0)$) yield the result. ∎

Proof of Theorem 4.1. The proof that parts (i) and (ii) of Theorem 4.1 are equivalent is clear from what we said at the beginning of this section. This also shows that part (i) is equivalent to the validity of inequalities (4.9) and (4.10) on $\mathcal{M}^+(0,\infty;\downarrow)$. However, by Lemmas 4.4 and 4.5, this is equivalent to the conditions mentioned in part (iii). ∎

Remark 4.6. If $f \in \mathcal{M}(\mathbb{R}^n) \setminus L^s_{\mathrm{loc}}(\mathbb{R}^n)$, then there is a cube $Q_0 \subset \mathbb{R}^n$ such that $f \notin L^s(Q_0)$. Now, if $x \in \mathbb{R}^n$, we take a cube Q such that $x \in Q$ and $Q_0 \subset Q$. Hence $\|f\chi_Q\|_s = \infty$, which implies that $(\mathscr{M}_{s,\gamma;\mathbb{A}}f)(x) = \infty$. Since $x \in \mathbb{R}^n$ was an arbitrary point, we have $\mathscr{M}_{s,\gamma;\mathbb{A}}f \equiv \infty$ on $\mathbb{R}^n$. Consequently, $(\mathscr{M}_{s,\gamma;\mathbb{A}}f)^* \equiv \infty$, and so, there are no $q \in (0,\infty)$ and $w \in \mathscr{W}(0,\infty)$ such that $\mathscr{M}_{s,\gamma;\mathbb{A}}f \in \Lambda^q(w)$. This shows that if $\Lambda^p(v) \not\subset L^s_{\mathrm{loc}}(\mathbb{R}^n)$, then the operator (4.1) is not bounded for any $q \in (0,\infty)$ and any $w \in \mathscr{W}(0,\infty)$.

On the other hand, if the operator (4.1) is bounded and $p \le q$, then it should be somehow hidden in the conditions of Theorem 4.1 that

$$\Lambda^p(v) \subset L^s_{\mathrm{loc}}(\mathbb{R}^n). \tag{4.50}$$

To see it take a cube $Q \subset \mathbb{R}^n$ with $T = |Q|$, and $f \in \Lambda^p(v)$. Substituting $\psi = (f^*)^s$ in (4.15), we get

$$\Big\{\int_0^\infty \Big(\sup_{t<\tau<\infty} \tau^{\gamma/n-1}\ell^{-s\mathbb{A}}(\tau)\int_0^\tau [f^*(\sigma)]^s\,d\sigma\Big)^{q/s} w(t)\,dt\Big\}^{1/q} \lesssim \Big\{\int_0^\infty [f^*(t)]^p v(t)\,dt\Big\}^{1/p}. \tag{4.51}$$

However,

$$\mathrm{LHS}(4.51) \ge \Big(T^{\gamma/n-1}\ell^{-s\mathbb{A}}(T)\int_0^T [f^*(y)]^s\,dy\Big)^{1/s}\Big(\int_0^T w(t)\,dt\Big)^{1/q},$$

which, together with (4.51), implies that

$$\Big\{\int_Q |f(x)|^s\,dx\Big\}^{1/s} C_T \le \Big\{\int_0^T [f^*(y)]^s\,dy\Big\}^{1/s} C_T \lesssim \|f\|_{\Lambda^p(v)}, \tag{4.52}$$

with $C_T = T^{(1/s)(\gamma/n-1)}\ell^{-\mathbb{A}}(t)(\int_0^T w(t)\,dt)^{1/q}$, and (4.50) follows.

We now turn our attention to the conditions of part (iii) of Theorem 4.1. Again, let $Q \subset \mathbb{R}^n$ be a cube with $T = |Q|$.

First consider the case when $p \le s$. Then (4.18) implies that

$$\Big(\int_0^r 1\,dt\Big)^{1/s} \lesssim C_T\Big(\int_0^r v(t)\,dt\Big)^{1/p} \quad \text{for all } r \in (0,T), \tag{4.53}$$

where $C_T = (\int_T^\infty t^{(q/s)(\gamma/n-1)}\ell^{-q\mathbb{A}}(t)w(t)\,dt)^{-1/q}$. However (cf. [Stp, Proposition 1(a)]), (4.53) yields $\Lambda^p(v) \hookrightarrow L^s(Q)$ and (4.50) follows.

Consider now the case when $s < p$ and put $1/r = 1/s - 1/p$. Then

$$\Big[\int_0^T \Big(t^{-1}\int_0^t v(\tau)\,d\tau\Big)^{p/(s-p)} v(t)\,dt\Big]^{1/s-1/p} = \Big[\int_0^T \Big(\int_0^t 1\,d\tau\Big)^{r/s}\Big(\int_0^t v(\tau)\,d\tau\Big)^{-r/s} v(t)\,dt\Big]^{1/r}$$

and, by condition (4.17),

$$(4.54)\qquad \Big[\int_0^T\Big(\int_0^t 1\,d\tau\Big)^{r/s}\Big(\int_0^t v(\tau)\,d\tau\Big)^{-r/s}v(t)\,dt\Big]^{1/r}\lesssim C_T<\infty,$$

where $C_T=(\int_T^\infty t^{(q/s)(\gamma/n-1)}\ell^{-q\mathbb{A}}(t)w(t)\,dt)^{-1/q}$. But, by [Stp, Proposition 1(b)], (4.54) implies that $\Lambda^p(v)\hookrightarrow L^s(Q)$, and (4.50) follows.

5. Boundedness of $\mathscr{M}_{s,\gamma;\mathbb{A}}:\Lambda^p(v)\to\Lambda^{q,\infty}(w),\ 0<p,q<\infty$

Theorem 3.1 and the definitions of the quasi-norms in the classical and weak-type Lorentz spaces imply that the operator

$$(5.1)\qquad \mathscr{M}_{s,\gamma;\mathbb{A}}:\Lambda^p(v)\to\Lambda^{q,\infty}(w),\qquad 0<p,q<\infty,$$

is bounded if and only if

$$(5.2)\qquad \sup_{t>0}\Big\{\sup_{t<\tau<\infty}\tau^{\gamma/n-1}\ell^{-s\mathbb{A}}(\tau)\int_0^\tau[\varphi(\sigma)]^s\,d\sigma\Big\}^{1/s}\Big(\int_0^t w(\tau)\,d\tau\Big)^{1/q}$$
$$\lesssim\Big\{\int_0^\infty[\varphi(t)]^pv(t)\,dt\Big\}^{1/p}\qquad\text{for all }\varphi\in\mathcal{M}^+(0,\infty;\downarrow).$$

On using (4.3) and (4.5), we can rewrite (5.2) as

$$(5.3)\qquad \sup_{t>0}(\mathscr{T}\psi)(t)\Big(\int_0^t w(\tau)\,d\tau\Big)^{1/Q}\lesssim\Big\{\int_0^\infty[\psi(t)]^Pv(t)\,dt\Big\}^{1/P}\qquad\text{for all }\psi\in\mathcal{M}^+(0,\infty;\downarrow).$$

Together with the estimate (4.6), this shows that (5.1) is satisfied if and only if both inequalities

$$(5.4)\qquad \sup_{t>0}(\mathscr{S}\psi)(t)\Big(\int_0^t w(\tau)\,d\tau\Big)^{1/Q}\lesssim\Big\{\int_0^\infty[\psi(t)]^Pv(t)\,dt\Big\}^{1/P}$$

and

$$(5.5)\qquad \sup_{t>0}(\mathscr{R}\psi)(t)\Big(\int_0^t w(\tau)\,d\tau\Big)^{1/Q}\lesssim\Big\{\int_0^\infty[\psi(t)]^Pv(t)\,dt\Big\}^{1/P}$$

hold on $\mathcal{M}^+(0,\infty;\downarrow)$.

Our main result in this section is the following theorem, which provides us with a characterization of (5.1).

THEOREM 5.1. *Let $s\in(0,\infty)$, $n\in\mathbb{N}$, $\gamma\in[0,n)$, $\mathbb{A}=(A_0,A_\infty)\in\mathbb{R}^2$ and let $v,w\in\mathscr{W}(0,\infty)$ satisfy*

$$(5.6)\qquad \int_0^x v(t)\,dt<\infty\quad\textit{and}\quad\int_0^x w(t)\,dt<\infty\qquad\textit{for every }x\in(0,\infty).$$

Assume that $0<p,q<\infty$ and

$$(5.7)\qquad \textit{either }\gamma\in(0,n),\ \textit{or }\gamma=0\ \textit{and}\ A_0\ge0\ge A_\infty.$$

Then the following statements are equivalent:

(i) *The operator* $\mathscr{M}_{s,\gamma;\mathbb{A}} : \Lambda^p(v) \to \Lambda^{q,\infty}(w)$ *is bounded.*

(ii) *For all* $\psi \in \mathcal{M}^+(0,\infty;\downarrow)$,

$$\sup_{t>0}\Big\{\sup_{t<\tau<\infty} \tau^{\gamma/n-1}\ell^{-s\mathbb{A}}(\tau)\int_0^\tau \psi(\sigma)\,d\sigma\Big\}^{1/s}\Big(\int_0^t w(\tau)\,d\tau\Big)^{1/q} \lesssim \Big\{\int_0^\infty [\psi(t)]^{p/s}v(t)\,dt\Big\}^{1/p}.$$

(iii) *For all* $r\in(0,\infty)$,

$$r^{\gamma/(ns)}\ell^{-\mathbb{A}}(r)\Big(\int_0^r w(t)\,dt\Big)^{1/q} \lesssim \Big(\int_0^r v(t)\,dt\Big)^{1/p} \tag{5.8}$$

and either

$$r^{(1/s)(\gamma/n-1)}\ell^{-\mathbb{A}}(r)\Big(\int_0^r w(t)\,dt\Big)^{1/q} \times\Big[\int_0^r\Big(t^{-1}\int_0^t v(\tau)\,d\tau\Big)^{p/(s-p)}v(t)\,dt\Big]^{1/s-1/p} \lesssim 1 \quad \textit{if } s<p, \tag{5.9}$$

or

$$r^{\gamma/(ns)}\ell^{-\mathbb{A}}(r)\sup_{t\in(0,r)}\Big(\frac{t}{r}\Big)^{1/s}\frac{(\int_0^r w(\tau)\,d\tau)^{1/q}}{(\int_0^t v(\tau)\,d\tau)^{1/p}} \lesssim 1 \quad \textit{if } p\le s. \tag{5.10}$$

To prove Theorem 5.1, we shall characterize the validity of inequalities (5.4) and (5.5) on $\mathcal{M}^+(0,\infty;\downarrow)$. To find necessary and sufficient conditions under which (5.4) holds on $\mathcal{M}^+(0,\infty;\downarrow)$, we shall use the following lemma, which is an obvious consequence of [CS2, Theorem 3.3].

LEMMA 5.2. *Suppose that* $0<P,Q<\infty$ *and that* $\widetilde{v},\widetilde{w}\in\mathscr{W}(0,\infty)$ *satisfy* $\int_0^x \widetilde{v}(t)\,dt<\infty$, $\int_0^x \widetilde{w}(t)\,dt<\infty$ *for all* $x>0$. *Let* $\Phi\in\mathcal{M}^+((0,\infty)\times(0,\infty))$ *and let*

$$\int_0^\infty \Phi(t,\tau)\psi(\tau)\,d\tau \in \mathcal{M}^+(0,\infty;\downarrow) \quad \textit{for every } \psi\in\mathcal{M}^+(0,\infty;\downarrow). \tag{5.11}$$

Then the inequality

$$\sup_{t>0}\Big(\int_0^\infty \Phi(t,\tau)\psi(\tau)\,d\tau\Big)\Big(\int_0^t \widetilde{w}(\tau)\,d\tau\Big)^{1/Q} \lesssim \Big\{\int_0^\infty [\psi(t)]^P\widetilde{v}(t)\,dt\Big\}^{1/P} \tag{5.12}$$

holds for all $\psi\in\mathcal{M}^+(0,\infty;\downarrow)$ *if and only if either* $P>1$ *and for all* $r\in(0,\infty)$,

$$\Big[\int_0^\infty\Big(\int_0^t \Phi(r,\tau)\,d\tau\Big)^{P'}\Big(\int_0^t\widetilde{v}(\tau)\,d\tau\Big)^{-P'}\widetilde{v}(t)\,dt\Big]^{1/P'}\Big(\int_0^r\widetilde{w}(\tau)\,d\tau\Big)^{1/Q} \lesssim 1 \tag{5.13}$$

and

$$\Big(\int_0^\infty \Phi(r,\tau)\,d\tau\Big)\Big(\int_0^\infty\widetilde{v}(\tau)\,d\tau\Big)^{-1/P}\Big(\int_0^r\widetilde{v}(\tau)\,d\tau\Big)^{1/Q} \lesssim 1, \tag{5.14}$$

or $P\le 1$ *and for all* $r\in(0,\infty)$,

$$\Big[\sup_{t>0}\Big(\int_0^t \Phi(t,\tau)\,d\tau\Big)\Big(\int_0^t\widetilde{v}(\tau)\,d\tau\Big)^{-1/P}\Big]\Big(\int_0^r\widetilde{w}(\tau)\,d\tau\Big)^{1/Q} \lesssim 1. \tag{5.15}$$

Now, we are able to characterize inequality (5.4).

LEMMA 5.3. *Let all the assumptions of Theorem* 5.1 *be satisfied and* $P = p/s$, $Q = q/s$. *Then inequality* (5.4) *holds on* $\mathcal{M}^+(0,\infty;\downarrow)$ *if and only if either* $s < p$ *and*, *for all* $r \in (0,\infty)$,

$$r^{\gamma/(ns)}\ell^{-\mathbb{A}}(r)\Big(\int_0^r w(t)\,dt\Big)^{1/q} \lesssim \Big(\int_0^r v(t)\,dt\Big)^{1/p} \tag{5.16}$$

and

$$r^{(1/s)(\gamma/n-1)}\ell^{-\mathbb{A}}(r)\Big(\int_0^r w(t)\,dt\Big)^{1/q}\Big[\int_0^r\Big(t^{-1}\int_0^t v(\tau)\,d\tau\Big)^{p/(s-p)}v(t)\,dt\Big]^{1/s-1/p} \lesssim 1, \tag{5.17}$$

or $p \le s$ *and*, *for all* $r \in (0,\infty)$, *condition* (5.16) *holds and*

$$r^{\gamma/(ns)}\ell^{-\mathbb{A}}(r)\sup_{t\in(0,r)}\Big(\frac{t}{r}\Big)^{1/s}\frac{(\int_0^r w(\tau)\,d\tau)^{1/q}}{(\int_0^t v(\tau)\,d\tau)^{1/p}} \lesssim 1. \tag{5.18}$$

Proof. Our intent is to rewrite (5.4) in the form (5.12) and then apply Lemma 5.2. First, to satisfy the monotonicity demand (5.11), we put $\Phi(t,\tau) = \chi_{(0,t)}(\tau)/t$ for $t, \tau \in (0,\infty)$. Then

$$\mathrm{LHS}(5.4) = \sup_{t>0}\Big(\int_0^\infty \Phi(t,\tau)\psi(\tau)\,d\tau\Big)t^{\gamma/n}\ell^{-s\mathbb{A}}(t)\Big(\int_0^t w(\tau)\,d\tau\Big)^{1/Q}. \tag{5.19}$$

Next, the function g given by $g(0) = 0$ and $g(t) = t^{\gamma/n}\ell^{-s\mathbb{A}}(t)$, $t \in (0,\infty)$, is absolutely continuous and equivalent to a non-decreasing function on $[0,\infty)$. Hence, there is $\widetilde{w} \in \mathscr{W}(0,\infty)$ such that for all $t \in (0,\infty)$,

$$\int_0^t \widetilde{w}(\tau)\,d\tau \approx t^{\gamma/n}\ell^{-s\mathbb{A}}(t)\Big(\int_0^t w(\tau)\,d\tau\Big)^{1/Q}.$$

Together with (5.19), this shows that

$$\mathrm{LHS}(5.4) \approx \sup_{t>0}\Big(\int_0^\infty \Phi(t,\tau)\psi(\tau)\,d\tau\Big)\int_0^t \widetilde{w}(\tau)\,d\tau.$$

Therefore, applying Lemma 5.2 (with $\widetilde{v} = v$ and $Q = 1$), we find that (5.4) holds on $\mathcal{M}^+(0,\infty;\downarrow)$ if and only if either $P > 1$ and, for all $r \in (0,\infty)$, (5.13) and (5.14) are satisfied (with $Q = 1$), or $P \le 1$ and (5.15) holds (again with $Q = 1$) for all $r \in (0,\infty)$. However, after some calculations one can verify that conditions (5.13) and (5.14) are equivalent to (5.16) and (5.17) while condition (5.15) is equivalent to (5.16) and (5.18). ∎

We now turn our attention to inequality (5.5).

LEMMA 5.4. *Let all the assumptions of Theorem* 5.1 *be satisfied and* $P = p/s$, $Q = q/s$. *Then inequality* (5.5) *holds on* $\mathcal{M}^+(0,\infty;\downarrow)$ *if and only if*, *for all* $r \in (0,\infty)$,

$$r^{\gamma/(ns)}\ell^{-\mathbb{A}}(r)\Big(\int_0^r w(t)\,dt\Big)^{1/q} \lesssim \Big(\int_0^r v(t)\,dt\Big)^{1/p}. \tag{5.20}$$

Proof. Necessity. Using (4.28), we obtain (5.20) by testing inequality (5.5) with $\psi = \chi_{(0,r)}$, $r \in (0,\infty)$.

Sufficiency. (Note that the result follows from the embedding $\Lambda^Q(w) \hookrightarrow \Lambda^{Q,\infty}(w)$ and Lemma 4.5 provided that $p \le q$.)

(i) First consider the case (4.30), which (together with (5.20)) implies (4.31). Define $\{r_k\}_{k\in\mathbb{Z}} \subset (0,\infty)$ by (4.32). Again, it is sufficient to verify (5.5) for continuous $\psi \in \mathcal{M}^+(0,\infty;\downarrow)$ having compact support in $[0,\infty)$ and $\psi \not\equiv 0$. Define the set B by (4.33) and the sequence $\{z_k\}_{k\in B}$ by (4.34). Then, by (4.35), (5.20), (4.32) and the monotonicity of ψ,

$$
\begin{aligned}
(5.21)\qquad \sup_{t>0} (\mathscr{R}\psi)(t)\Big(\int_0^t w(\tau)\,d\tau\Big)^{1/Q} &= \sup_{k\in B}\ \sup_{t\in[z_k,r_k)} (\mathscr{R}\psi)(t)\Big(\int_0^t w(\tau)\,d\tau\Big)^{1/Q} \\
&\lesssim \sup_{k\in B}\ \sup_{t\in[z_k,r_k)} \psi(r_{k-1}) r_k^{\gamma/n}\ell^{-s\mathbb{A}}(r_k)\Big(\int_0^t w(\tau)\,d\tau\Big)^{1/Q} \\
&= \sup_{k\in B} \psi(r_{k-1}) r_k^{\gamma/n}\ell^{-s\mathbb{A}}(r_k)\Big(\int_0^{r_k} w(\tau)\,d\tau\Big)^{1/Q} \\
&\lesssim \sup_{k\in B} \psi(r_{k-1})\Big(\int_0^{r_k} v(t)\,dt\Big)^{1/P} \\
&\le 4^{1/P} \sup_{k\in B}\Big(\int_{r_{k-2}}^{r_{k-1}} [\psi(t)]^P v(t)\,dt\Big)^{1/P}
\end{aligned}
$$

and the result follows.

(ii) Second, consider the case (4.37). If (4.31) holds, then the method of part (i) gives the result. Therefore we assume now that (4.38) is satisfied. Put $\mathbb{Z}_0 = \{k \in \mathbb{Z};\ k \le k_0\}$, where k_0 is given by (4.39), define the increasing sequence $\{r_k\}_{k\in\mathbb{Z}_0} \subset (0,\infty)$ by (4.32), and the set B_0 by (4.40).

(ii-1) If $B_0 \ne \emptyset$, we assign to any $k \in B_0$ the point z_k just as in (4.34) and put $k_M = \max B_0$. Then the method of part (i) implies that

$$
(5.22)\qquad \sup_{0<t<r_{k_M}} (\mathscr{R}\psi)(t)\Big(\int_0^t w(\tau)\,d\tau\Big)^{1/Q} \lesssim 4^{1/P} \sup_{k\in B_0}\Big(\int_{r_{k-2}}^{r_{k-1}} [\psi(\tau)]^P v(\tau)\,d\tau\Big)^{1/P}.
$$

Put $I = [r_{k_M},\infty)$. If $\psi \equiv 0$ in I, then $\mathscr{R}\psi \equiv 0$ in I as well and the result follows from (5.22). Thus we assume that $\psi \not\equiv 0$ in I. Since, by (5.20) and (4.32),

$$
\begin{aligned}
(5.23)\qquad \big(\lim_{\tau\to\infty} \tau^{\gamma/n}\ell^{-s\mathbb{A}}(\tau)\big)\Big(\int_0^\infty w(\tau)\,d\tau\Big)^{1/Q} &= \lim_{\tau\to\infty} \tau^{\gamma/n}\ell^{-s\mathbb{A}}(\tau)\Big(\int_0^\tau w(\sigma)\,d\sigma\Big)^{1/Q} \\
&\lesssim \lim_{\tau\to\infty}\Big(\int_0^\tau v(\sigma)\,d\sigma\Big)^{1/P} \le (2^{k_0+1})^{1/P} \\
&= 8^{1/P}\Big(\int_{r_{k_0-2}}^{r_{k_0-1}} v(t)\,dt\Big)^{1/P},
\end{aligned}
$$

applying (4.44), (5.23) and the monotonicity of ψ, we obtain

$$\begin{aligned}(5.24)\qquad \sup_{t\in I}(\mathscr{R}\psi)(t)\Big(\int_0^t w(\tau)\,d\tau\Big)^{1/Q} &\lesssim \sup_{t\in I}\psi(r_{k_0})\big(\lim_{\tau\to\infty}\tau^{\gamma/n}\ell^{-s\mathbb{A}}(\tau)\big)\Big(\int_0^t w(\tau)\,d\tau\Big)^{1/Q}\\ &= \psi(r_{k_0})\big(\lim_{\tau\to\infty}\tau^{\gamma/n}\ell^{-s\mathbb{A}}(\tau)\big)\Big(\int_0^\infty w(\tau)\,d\tau\Big)^{1/Q}\\ &\lesssim \psi(r_{k_0})8^{1/P}\Big(\int_{r_{k_0-2}}^{r_{k_0-1}} v(t)\,dt\Big)^{1/P}\\ &\le 8^{1/P}\Big(\int_{r_{k_0-2}}^{r_{k_0-1}}[\psi(t)]^P v(t)\,dt\Big)^{1/P},\end{aligned}$$

which, together with (5.22), yields the result.

(ii-2) Assume now that $B_0=\emptyset$. Put $I=(0,\infty)$. Estimates (4.47) and (4.43) imply (4.44). Consequently, using also (5.23) and the monotonicity of ψ, we arrive at

$$\begin{aligned}\sup_{t>0}(\mathscr{R}\psi)(t)\Big(\int_0^t w(\tau)\,d\tau\Big)^{1/Q} &\lesssim \sup_{t>0}\psi(r_{k_0})\big(\lim_{\tau\to\infty}\tau^{\gamma/n}\ell^{-s\mathbb{A}}(\tau)\big)\Big(\int_0^t w(\tau)\,d\tau\Big)^{1/Q}\\ &\lesssim \psi(r_{k_0})8^{1/P}\Big(\int_{r_{k_0-2}}^{r_{k_0-1}} v(t)\,dt\Big)^{1/P}\\ &\le 8^{1/P}\Big(\int_{r_{k_0-2}}^{r_{k_0-1}}[\psi(t)]^P v(t)\,dt\Big)^{1/P}\end{aligned}$$

and the result follows. ■

Proof of Theorem 5.1. The proof that parts (i) and (ii) of Theorem 5.1 are equivalent is clear from the remarks at the beginning of this section. This also shows that part (i) is equivalent to the validity of inequalities (5.4) and (5.5) on $\mathcal{M}^+(0,\infty;\downarrow)$. However, by Lemmas 5.3 and 5.4, this is equivalent to the conditions of part (iii). ■

6. Boundedness of $\mathscr{M}_{s,\gamma;\mathbb{A}}:\Lambda^{p,\infty}(v)\to\Lambda^{q,\infty}(w)$, $0<p,q<\infty$

By Theorem 3.1 and the definition of the quasi-norms in weak-type Lorentz spaces, the operator

$$(6.1)\qquad \mathscr{M}_{s,\gamma;\mathbb{A}}:\Lambda^{p,\infty}(v)\to\Lambda^{q,\infty}(w),\qquad 0<p,q<\infty,$$

is bounded if and only if

$$\begin{aligned}(6.2)\qquad \sup_{t>0}\Big\{\sup_{t<\tau<\infty}\tau^{\gamma/n-1}\ell^{-s\mathbb{A}}(\tau)\int_0^\tau[\varphi(\sigma)]^s\,d\sigma\Big\}^{1/s}\Big(\int_0^t w(\tau)\,d\tau\Big)^{1/q}&\\ \lesssim \sup_{t>0}\varphi(t)\Big(\int_0^t v(\tau)\,d\tau\Big)^{1/p}\quad \text{for all } \varphi\in\mathcal{M}^+(0,\infty;\downarrow).&\end{aligned}$$

On using (4.3) and (4.5), inequality (6.2) can be rewritten as

$$(6.3)\qquad \sup_{t>0}(\mathscr{T}\psi)(t)\Big(\int_0^t w(\tau)\,d\tau\Big)^{1/Q} \lesssim \sup_{t>0}\psi(t)\Big(\int_0^t v(\tau)\,d\tau\Big)^{1/P} \quad \text{for all } \psi\in\mathcal{M}^+(0,\infty;\downarrow).$$

Our main result in this section provides a characterization of (6.1) and reads as follows.

THEOREM 6.1. *Let* $s\in(0,\infty)$, $n\in\mathbb{N}$, $\gamma\in[0,n)$, $\mathbb{A}=(A_0,A_\infty)\in\mathbb{R}^2$ *and let* $v,w\in\mathscr{W}(0,\infty)$ *satisfy*

$$(6.4)\qquad \int_0^x v(t)\,dt<\infty \quad\textit{and}\quad \int_0^x w(t)\,dt<\infty \quad \textit{for every } x\in(0,\infty).$$

Assume that $0<p,q<\infty$ *and*

$$(6.5)\qquad \textit{either } \gamma\in(0,n),\ \textit{ or } \gamma=0 \textit{ and } A_0\ge 0\ge A_\infty.$$

Then the following statements are equivalent:

(i) *The operator* $\mathscr{M}_{s,\gamma;\mathbb{A}}:\Lambda^{p,\infty}(v)\to\Lambda^{q,\infty}(w)$ *is bounded.*

(ii) *For all* $\psi\in\mathcal{M}^+(0,\infty;\downarrow)$,

$$\sup_{t>0}\Big\{\sup_{t<\tau<\infty}\tau^{\gamma/n-1}\ell^{-s\mathbb{A}}(\tau)\int_0^\tau\psi(\sigma)\,d\sigma\Big\}^{1/s}\Big(\int_0^t w(\tau)\,d\tau\Big)^{1/q}\lesssim \sup_{t>0}[\psi(t)]^{1/s}\Big(\int_0^t v(\tau)\,d\tau\Big)^{1/p}.$$

(iii) $\sup_{t>0} t^{\gamma/n-1}\ell^{-s\mathbb{A}}(t)(\int_0^t w(\tau)\,d\tau)^{s/q}\int_0^t(\int_0^\tau v(\sigma)\,d\sigma)^{-s/p}\,d\tau<\infty.$

To prove Theorem 6.1, we shall apply the next lemma, where the following notation is used. If $0<p<\infty$ and $v\in\mathscr{W}(0,\infty)$ is such that

$$(6.6)\qquad \int_0^x v(t)\,dt<\infty \quad \text{for every } x\in(0,\infty),$$

we put

$$(6.7)\qquad L^{p,\infty}_{\rm dec}(v):=\Big\{f\in\mathcal{M}^+(0,\infty;\downarrow);\ \|f\|_{L^{p,\infty}_{\rm dec}(v)}:=\sup_{t>0}f(t)\Big(\int_0^t v(\tau)\,d\tau\Big)^{1/p}<\infty\Big\}.$$

LEMMA 6.2 (cf. [So, Proposition 2.7]). *Let* N *be a non-negative functional defined on* $\mathcal{M}(0,\infty)$ *and*

$$(6.8)\qquad X:=\{f\in\mathcal{M}(0,\infty);\ \|f\|_X:=N(f)<\infty\}.$$

Assume that N *satisfies*:

(i) *There exists* $C>0$ *such that* $N(f)\le CN(g)$ *if* $f,g\in X$, $0\le f\le g$.

(ii) *There exists* $C>0$ *such that* $N(\lambda f)\le C\lambda N(f)$ *if* $f\in X$ *and* $\lambda\ge 0$.

Let $0<P<\infty$, *let* $v\in\mathscr{W}(0,\infty)$ *satisfy* (6.6) *and let*

$$\mathscr{L}:L^{P,\infty}_{\rm dec}(v)\to\mathcal{M}^+(0,\infty)$$

be an operator with the property:

(iii) *There exists* $C>0$ *such that* $\mathscr{L}(f)\le C\mathscr{L}(g)$ *if* $0\le f\le g$.

Then $\mathscr{L}:L^{P,\infty}_{\rm dec}(v)\to X$ *is bounded, that is,*

$$(6.9)\qquad \|\mathscr{L}\psi\|_X\lesssim\|\psi\|_{L^{P,\infty}_{\rm dec}(v)} \quad \textit{for all } \psi\in L^{P,\infty}_{\rm dec}(v),$$

if and only if

$$N(\mathscr{L}(V^{-1/P})) < \infty, \tag{6.10}$$

where $V(t) := \int_0^t v(\tau)\,d\tau,\ t \in (0,\infty)$.

Now, we are able to characterize inequality (6.3).

LEMMA 6.3. *Let all the assumptions of Theorem* 6.1 *be satisfied and* $P = p/s$, $Q = q/s$. *Then inequality* (6.3) *holds on* $\mathcal{M}^+(0,\infty;\downarrow)$ *if and only if condition* (iii) *of Theorem* 6.1 *is satisfied.*

Proof. Define the operator $\mathscr{L} : L^{P,\infty}_{\mathrm{dec}}(v) \to \mathcal{M}^+(0,\infty)$ by $\mathscr{L} = \mathscr{T}$, where $\mathscr{T}$ is given by (4.5), and the non-negative functional N on $\mathcal{M}(0,\infty)$ by

$$N(f) := \sup_{t>0} |f(t)| \Big(\int_0^t w(\tau)\,d\tau \Big)^{1/Q}. \tag{6.11}$$

By Lemma 6.2, the operator $\mathscr{L} : L^{P,\infty}_{\mathrm{dec}}(v) \to X$ is bounded, that is, (6.9) holds, if and only if (6.10) is satisfied. But, by (6.8), (6.11), (6.7) and the definition of $\mathscr{L}$, estimate (6.9) coincides with inequality (6.3). Moreover, since

$$\begin{aligned} N(\mathscr{L}(V^{-1/P})) &= \sup_{t>0} (\mathscr{T}(V^{-1/P}))(t) \Big(\int_0^t w(\sigma)\,d\sigma \Big)^{1/Q} \\ &= \sup_{t>0} \Big[\sup_{t<\tau<\infty} \tau^{\gamma/n-1} \ell^{-s\mathbb{A}}(\tau) \int_0^\tau \Big(\int_0^\xi v(\sigma)\,d\sigma \Big)^{-1/P} d\xi \Big] \Big(\int_0^t w(\sigma)\,d\sigma \Big)^{1/Q} \\ &= \sup_{\tau>0} \Big[\sup_{0<t<\tau} \Big(\int_0^t w(\sigma)\,d\sigma \Big)^{1/Q} \Big] \tau^{\gamma/n-1} \ell^{-s\mathbb{A}}(\tau) \int_0^\tau \Big(\int_0^\xi v(\sigma)\,d\sigma \Big)^{-1/P} d\xi \\ &= \sup_{\tau>0} \tau^{\gamma/n-1} \ell^{-s\mathbb{A}}(\tau) \Big(\int_0^\tau w(\sigma)\,d\sigma \Big)^{1/Q} \int_0^\tau \Big(\int_0^\xi v(\sigma)\,d\sigma \Big)^{-1/P} d\xi, \end{aligned}$$

we see that (6.9) and statement (iii) of Theorem 6.1 coincide. ■

Proof of Theorem 6.1. Theorem 6.1 is a consequence of Lemma 6.3 and the facts stated at the beginning of this section. ■

REMARKS 6.4. (i) Note that statement (iii) of Theorem 6.1 is equivalent to the following pair of conditions: For all $r \in (0,\infty)$,

$$r^{\gamma/(ns)} \ell^{-\mathbb{A}}(r) \Big(\int_0^r w(t)\,dt \Big)^{1/q} \lesssim \Big(\int_0^r v(t)\,dt \Big)^{1/p} \tag{6.12}$$

and

$$r^{(1/s)(\gamma/n-1)} \ell^{-\mathbb{A}}(r) \Big(\int_0^r w(t)\,dt \Big)^{1/q} \Big[\int_0^r \Big(\int_0^t v(\sigma)\,d\sigma \Big)^{-s/p} dt \Big]^{1/s} \lesssim 1. \tag{6.13}$$

Indeed, using (4.6), we see that (6.3) is satisfied if and only if both

$$\sup_{t>0} (\mathscr{S}\psi)(t) \Big(\int_0^t w(\tau)\,d\tau \Big)^{1/Q} \lesssim \sup_{t>0} \psi(t) \Big(\int_0^t v(\tau)\,d\tau \Big)^{1/P} \tag{6.14}$$

and

$$\sup_{t>0} (\mathscr{R}\psi)(t)\Big(\int_0^t w(\tau)\,d\tau\Big)^{1/Q} \lesssim \sup_{t>0} \psi(t)\Big(\int_0^t v(\tau)\,d\tau\Big)^{1/P} \tag{6.15}$$

hold on $\mathcal{M}^+(0,\infty;\downarrow)$. Now, applying Lemma 6.2, one can show that (6.15) or (6.14), respectively, holds on $\mathcal{M}^+(0,\infty;\downarrow)$ if and only if (6.12) or (6.13) is satisfied for all $r \in (0,\infty)$.

(ii) To characterize (6.15) on $\mathcal{M}^+(0,\infty;\downarrow)$ one can also proceed similarly to the proof of Lemma 5.4. Indeed, necessity of (6.12) follows by testing (6.15) with $\psi = \chi_{(0,r)}$, $r \in (0,\infty)$. To prove sufficiency of (6.12), consider, for example, the case (4.30). Then, proceeding as in (5.21) without applying the monotonicity of ψ in the last step of this estimate, we arrive at

$$\sup_{t>0} (\mathscr{R}\psi)(t)\Big(\int_0^t w(\tau)\,d\tau\Big)^{1/Q} \lesssim \sup_{k\in B} \psi(r_{k-1})\Big(\int_0^{r_k} v(t)\,dt\Big)^{1/P} \le \sup_{t>0} \psi(t)\Big(\int_0^t v(\tau)\,d\tau\Big)^{1/P}$$

and (6.15) follows. The proof in the case (4.37) is left to the reader.

(iii) Suppose that the assumptions of Lemma 5.4 are satisfied. Then Lemma 5.4 and part (i) of this remark imply that any of inequalities (5.5) and (6.15) holds on $\mathcal{M}^+(0,\infty;\downarrow)$ if and only if (6.12) is satisfied for all $r \in (0,\infty)$. Moreover, by Lemma 4.5, the last condition also characterizes the validity of (4.10) on $\mathcal{M}^+(0,\infty;\downarrow)$ provided that $0 < p \le q < \infty$.

7. Boundedness of $\mathscr{M}_{s,\gamma;\mathbb{A}} : \Lambda^{p,\infty}(v) \to \Lambda^q(w)$, $0 < p, q < \infty$

As in the previous sections, we observe that the operator

$$\mathscr{M}_{s,\gamma;\mathbb{A}} : \Lambda^{p,\infty}(v) \to \Lambda^{q,\infty}(w), \qquad 0 < p, q < \infty, \tag{7.1}$$

is bounded if and only if

$$\Big\{\int_0^\infty \Big(\sup_{t<\tau<\infty} \tau^{\gamma/n-1}\ell^{-s\mathbb{A}}(\tau)\int_0^\tau [\varphi(\sigma)]^s\,d\sigma\Big)^{q/s} w(t)\,dt\Big\}^{1/q} \lesssim \sup_{t>0} \varphi(t)\Big(\int_0^t v(\tau)\,d\tau\Big)^{1/p} \quad \text{for all } \varphi \in \mathcal{M}^+(0,\infty;\downarrow). \tag{7.2}$$

On using the notation (4.3) and (4.5), we see that (7.2) is equivalent to

$$\Big\{\int_0^\infty [(\mathscr{T}\psi)(t)]^Q w(t)\,dt\Big\}^{1/Q} \lesssim \sup_{t>0} \psi(t)\Big(\int_0^t v(\tau)\,d\tau\Big)^{1/P} \quad \text{for all } \psi \in \mathcal{M}^+(0,\infty;\downarrow). \tag{7.3}$$

Our main result in this section is the following theorem.

Theorem 7.1. *Let $s \in (0,\infty)$, $n \in \mathbb{N}$, $\gamma \in [0,n)$, $\mathbb{A} = (A_0, A_\infty) \in \mathbb{R}^2$, $v, w \in \mathscr{W}(0,\infty)$ and let*

$$\int_0^x v(t)\,dt < \infty \qquad \textit{for every } x \in (0,\infty). \tag{7.4}$$

Assume that $0 < p, q < \infty$ *and*

$$\text{either } \gamma \in (0, n), \text{ or } \gamma = 0 \text{ and } A_0 \geq 0 \geq A_\infty. \tag{7.5}$$

Then the following statements are equivalent:

(i) *The operator* $\mathscr{M}_{s,\gamma;\mathbb{A}} : \Lambda^{p,\infty}(v) \to \Lambda^q(w)$ *is bounded.*

(ii) *For all* $\psi \in \mathcal{M}^+(0, \infty; \downarrow)$,

$$\Big\{ \int_0^\infty \Big[\sup_{t<\tau<\infty} \tau^{\gamma/n-1} \ell^{-s\mathbb{A}}(\tau) \int_0^\tau \psi(\sigma)\, d\sigma \Big]^{q/s} w(t)\, dt \Big\}^{1/q} \lesssim \sup_{t>0} [\psi(t)]^{1/s} \Big(\int_0^t v(\tau)\, d\tau \Big)^{1/p}.$$

(iii) $\int_0^\infty [\sup_{t<\tau<\infty} \tau^{(q/s)(\gamma/n-1)} \ell^{-q\mathbb{A}}(\tau) (\int_0^\tau (\int_0^\sigma v(\xi)\, d\xi)^{-s/p}\, d\sigma)^{q/s}] w(t)\, dt < \infty.$

To prove Theorem 7.1, we shall characterize the validity of inequality (7.3) on $\mathcal{M}^+(0, \infty; \downarrow)$.

Lemma 7.2. *Let all the assumptions of Theorem* 7.1 *be satisfied and* $P = p/s$, $Q = q/s$. *Then inequality* (7.3) *holds on* $\mathcal{M}^+(0, \infty; \downarrow)$ *if and only if condition* (iii) *of Theorem* 7.1 *is satisfied.*

Proof. We apply Lemma 6.2 with

$$N(f) = \Big\{ \int_0^\infty |f(x)|^Q w(t)\, dt \Big\}^{1/Q}$$

and $\mathscr{L} = \mathscr{T}$ (recall that $\mathscr{T}$ is given by (4.5)). First we observe that

$$\mathscr{L} : L^{P,\infty}_{\mathrm{dec}}(v) \to X \text{ is bounded,} \tag{7.6}$$

that is,

$$N(\mathscr{L}\psi) \lesssim \|\psi\|_{L^{P,\infty}_{\mathrm{dec}}(v)} \quad \text{for all } \psi \in \mathcal{M}^+(0, \infty; \downarrow),$$

if and only if (7.3) holds. Second, by Lemma 6.2, statement (7.6) is equivalent to (6.10). Since

$$\begin{aligned} N(\mathscr{L}(V^{-1/P})) &= \Big\{ \int_0^\infty [(\mathscr{T}(V^{-1/P}))(t)]^Q w(t)\, dt \Big\}^{1/Q} \\ &= \Big\{ \int_0^\infty \Big[\sup_{t<\tau<\infty} \tau^{\gamma/n-1} \ell^{-s\mathbb{A}}(\tau) \int_0^\tau \Big(\int_0^\sigma v(\xi)\, d\xi \Big)^{-1/P} d\sigma \Big]^Q w(t)\, dt \Big\}^{1/Q}, \end{aligned}$$

we see that (6.10) and statement (iii) of Theorem 7.1 coincide. ∎

Proof of Theorem 7.1. Theorem 7.1 is a consequence of Lemma 7.2 and the facts stated at the beginning of this section. ∎

Remark 7.3. Just as in Remark 6.4(i), we can see that statement (iii) of Theorem 7.1 is equivalent to the following pair of conditions:

$$\int_0^\infty \Big[\sup_{t<\tau<\infty} \tau^{\gamma q/(ns)} \ell^{-q\mathbb{A}}(\tau) \Big(\int_0^\tau v(\sigma)\, d\sigma \Big)^{-q/p} \Big] w(t)\, dt < \infty \tag{7.7}$$

and

$$\int_0^\infty t^{(q/s)(\gamma/n-1)} \ell^{-q\mathbb{A}}(t) \Big[\int_0^t \Big(\int_0^\tau v(\sigma)\, d\sigma \Big)^{-s/p} d\tau \Big]^{q/s} w(t)\, dt < \infty. \tag{7.8}$$

Note that (7.7) characterizes inequality (7.3) with $\mathscr{T}$ replaced by $\mathscr{R}$ while (7.8) is equivalent to (7.3) with $\mathscr{T}$ replaced by $\mathscr{S}$.

8. Boundedness of $\mathscr{M}_{s,\gamma;\mathbb{A}} : \Lambda^p(v) \to \Lambda^q(w)$, $0 < q < p < \infty$

We have already seen in Section 4 that (4.1) is satisfied if and only if inequality (4.4) holds on $\mathcal{M}^+(0,\infty;\downarrow)$ and that the latter statement is satisfied if both (4.9) and (4.10) hold on $\mathcal{M}^+(0,\infty;\downarrow)$. In contrast to Theorem 4.1, we now consider the case when $0 < q < p < \infty$. In such a case characterization of the validity of inequality (4.10) on $\mathcal{M}^+(0,\infty;\downarrow)$ is a more difficult problem than that with $0 < p \le q < \infty$. Nevertheless, in [GOP] an even more general result was proved. The next lemma is a particular case of [GOP, Theorem 3.4].

LEMMA 8.1. *Let $v, w \in \mathscr{W}(0,\infty)$ and suppose $\int_0^x v(t)\,dt < \infty$ for every $x \in (0,\infty)$. Assume that $0 < Q < P < \infty$ and $1/R = 1/Q - 1/P$. Then inequality* (4.10) *holds on $\mathcal{M}^+(0,\infty;\downarrow)$ if and only if*

$$\Big\{ \int_0^\infty \Big(\int_0^x w(t)\,dt \Big)^{R/P} w(x) \Big(x^{\gamma/n} \ell^{-s\mathbb{A}}(x) \Big)^R \Big(\int_0^x v(t)\,dt \Big)^{-R/P} dx \Big\}^{1/R} < \infty. \tag{8.1}$$

Using known results on weighted inequalities for the averaging operator

$$(\mathscr{A}\psi)(t) := t^{-1} \int_0^x \psi(\tau)\,d\tau \tag{8.2}$$

on $\mathcal{M}^+(0,\infty;\downarrow)$, one can find characterizations of (4.9) on $\mathcal{M}^+(0,\infty;\downarrow)$ when $0 < Q < P < \infty$ with the exception of the case $0 < Q < P < 1$. Combining these results with Lemma 8.1, one can obtain a characterization of (4.4) on $\mathcal{M}^+(0,\infty;\downarrow)$ when $0 < Q < P < \infty$ with the exception of the case $0 < Q < P < 1$. We omit details since this is a particular case of [GOP, Theorem 3.5 and Remark 3.6]. In such a way we arrive at the following theorem.

THEOREM 8.2. *Let $s \in (0,n)$, $n \in \mathbb{N}$, $\gamma \in [0,n)$, $\mathbb{A} = (A_0, A_\infty) \in \mathbb{R}^2$ and let either $\gamma \in (0,n)$, or $\gamma = 0$ and $A_0 \ge 0 \ge A_\infty$. Assume that $v, w \in \mathscr{W}(0,\infty)$ and*

$$\int_0^x v(t)\,dt < \infty \quad \text{and} \quad \int_0^x w(t)\,dt < \infty \quad \text{for every } x \in (0,\infty).$$

Let $0 < q < p < \infty$, $P = p/s$, $Q = q/s$ and $1/R = 1/Q - 1/P$. Then the following statements are equivalent:

(i) *The operator $\mathscr{M}_{s,\gamma;\mathbb{A}} : \Lambda^p(v) \to \Lambda^q(w)$ is bounded.*
(ii) *For all $\psi \in \mathcal{M}^+(0,\infty;\downarrow)$,*

$$\Big\{ \int_0^\infty \Big[\sup_{t<\tau<\infty} \tau^{\gamma/n-1} \ell^{-s\mathbb{A}}(\tau) \int_0^\tau \psi(\sigma)\,d\sigma \Big]^{q/s} w(t)\,dt \Big\}^{1/q} \lesssim \Big\{ \int_0^\infty [\psi(t)]^{p/s} v(t)\,dt \Big\}^{1/p}. \tag{8.3}$$

Moreover, if $P \ge 1$, then the statement (ii) *is equivalent to*:

$$\Big\{ \int_0^\infty \Big(\int_0^x w(t)\,dt \Big)^{R/P} w(x) (x^{\gamma/n} \ell^{-s\mathbb{A}}(x))^R \Big(\int_0^x v(t)\,dt \Big)^{-R/P} dx \Big\}^{1/R} < \infty \tag{8.4}$$

and either

$$(8.5)\qquad \Big\{\int_0^\infty \Big(\int_x^\infty t^{(\gamma/n-1)Q}\ell^{-sQ\mathbb{A}}(t)w(t)\,dt\Big)^{R/Q}\Big[\int_0^x \Big(t^{-1}\int_0^t v(\tau)\,d\tau\Big)^{-P'} v(t)\,dt\Big]^{R/Q'} \times \Big(x^{-1}\int_0^x v(t)\,dt\Big)^{-P'} v(x)\,dx\Big\}^{1/R} < \infty \quad \text{if } Q\neq 1 < P,$$

or

$$(8.6)\qquad \Big\{\int_0^\infty \Big(\int_0^x v(t)\,dt\Big)^{1-P'}\Big(\int_0^x t^{(\gamma/n)Q}\ell^{-sQ\mathbb{A}}(t)w(t)\,dt + x\int_x^\infty t^{(\gamma/n)Q-1}\ell^{-sQ\mathbb{A}}(t)w(t)\,dt\Big)^{P'-1} \times \Big(\int_x^\infty t^{(\gamma/n)Q-1}\ell^{-sQ\mathbb{A}}(t)w(t)\,dt\Big)\,dx\Big\}^{1/P'} < \infty \quad \text{if } Q = 1,$$

or

$$(8.7)\qquad \Big\{\int_0^\infty \Big(\int_x^\infty t^{(\gamma/n-1)Q}\ell^{-sQ\mathbb{A}}(t)\,w(t)\,dt\Big)^{-Q'} \times \Big(\sup_{0<t<x} t^{-1}\int_0^t v(\tau)\,d\tau\Big)^{Q'} x^{(\gamma/n-1)Q}\ell^{-sQ\mathbb{A}}(x)w(x)\,dx\Big\}^{-1/Q'} < \infty \quad \text{if } P = 1.$$

REMARK 8.3. There is a gap in Theorem 8.2 since no characterization of (ii) is given if $0 < Q < P < 1$. This corresponds to the fact that a characterization of the weighted inequality, involving the averaging operator $\mathscr{A}$ from (8.2),

$$(8.8)\qquad \Big\{\int_0^\infty [(\mathscr{A}\psi)(t)]^Q\overline{w}(t)\,dt\Big\}^{1/Q} \lesssim \Big\{\int_0^\infty [\psi(t)]^P\overline{v}(t)\,dt\Big\}^{1/P}$$

on $\mathcal{M}^+(0,\infty;\downarrow)$ is not available in the literature when $0 < Q < P < 1$. In this case only sufficient conditions for the validity of (8.8) on $\mathcal{M}^+(0,\infty;\downarrow)$ are known (cf. [Stp, Proposition 2]). Using this result, one can obtain conditions that guarantee the validity of (4.9) and, at the end, (8.3) on $\mathcal{M}^+(0,\infty;\downarrow)$. In such a way one can prove that statement (i) of Theorem 8.2 holds if $0 < Q < P < 1$ and both condition (8.4) and

$$(8.9)\qquad \Big\{\int_0^\infty \Big(\int_x^\infty t^{(\gamma/n-1)Q}\ell^{-sQ\mathbb{A}}(t)\,w(t)\,dt\Big)^{R/Q}\Big(\int_0^x v(t)\,dt\Big)^{-R/P} x^{R-1}\,dx\Big\}^{1/R} < \infty$$

are satisfied.

9. Local results

Now we indicate the changes which result from the assumption that we consider the local version of the fractional maximal operator $\mathscr{M}_{s,\gamma;\mathbb{A}}$ given by

$$(9.1)\qquad (\mathscr{M}_{s,\gamma;\alpha}f)(x) = \sup_{\substack{Q\ni x\\ Q\subset\Omega}} \frac{\|f\chi_Q\|_s}{\|\chi_Q\|_{sn/(n-\gamma);\alpha}}, \qquad f\in\mathcal{M}(\Omega),\ x\in\Omega.$$

Here $s\in(0,\infty)$, $\gamma\in[0,n)$, $\alpha\in\mathbb{R}$ and the supremum is extended over all cubes Q (with sides parallel to the coordinate axes) contained in a domain $\Omega\subset\mathbb{R}^n$ with $|\Omega|<\infty$, and

$\|\cdot\|_{q;\alpha} = \|\cdot\|_{q;\alpha,\Omega}$, $0 < q < \infty$, stands for the quasi-norm in the space

$$L^q(\log L)^\alpha(\Omega) := \Lambda^q(\Omega; u), \quad u \in \mathscr{W}(0, |\Omega|),\ u(t) = \ell^{\alpha q}(t),\ t \in (0, |\Omega|).$$

If $\Omega \subset \mathbb{R}^n$ is a domain, $|\Omega| < \infty$, we denote by $\mathscr{W}_0(0, |\Omega|)$ the subset of all weights $w \in \mathscr{W}(0, |\Omega|)$ which have singularities or degeneracies only at the origin, that is, $w \in \mathscr{W}(0, |\Omega|)$ belongs to $\mathscr{W}_0(0, |\Omega|)$ if and only if for any $a \in (0, |\Omega|)$ there is a positive constant $c = c(a, w)$ such that $c^{-1} < w(x) < c$ for all $x \in (a, |\Omega|)$.

We can see that our global results of Sections 4–8 can be easily adapted to give analogous local results provided that the weights v and w belong to $\mathscr{W}_0(0, |\Omega|)$. This essentially involves replacing the intervals $(0, \infty)$, (r, ∞), (x, ∞) by $(0, 1)$, $(r, 1)$, $(x, 1)$ respectively, the vector exponents by their first components, and omitting all the assumptions on their second components. For example, the local result corresponding to Theorem 4.1 reads as follows.

THEOREM 4.1*. *Let $s \in (0, \infty)$, $n \in \mathbb{N}$, $\gamma \in [0, n)$ and $\alpha \in \mathbb{R}$. Let Ω be a domain in $\mathbb{R}^n$, $|\Omega| < \infty$, and $v, w \in \mathscr{W}_0(0, |\Omega|)$. Assume that $0 < p \le q < \infty$ and*

$$\text{(4.14*)} \qquad \text{either } \gamma \in (0, n), \text{ or } \gamma = 0 \text{ and } \alpha \ge 0.$$

Then the following statements are equivalent:

(i) *The operator $\mathscr{M}_{s,\gamma;\alpha} : \Lambda^p(\Omega; v) \to \Lambda^q(\Omega; w)$ is bounded.*

(ii) *For all $\psi \in \mathcal{M}^+(0, 1; \downarrow)$,*

$$\text{(4.15*)} \quad \Big\{\int_0^1 \Big[\sup_{t<\tau<1} \tau^{\gamma/n-1} \ell^{-s\alpha}(\tau) \int_0^\tau \psi(\sigma)\, d\sigma\Big]^{q/s} w(t)\, dt\Big\}^{1/q} \lesssim \Big\{\int_0^1 [\psi(t)]^{p/s} v(t)\, dt\Big\}^{1/p}.$$

(iii) *For all $r \in (0, 1)$,*

$$\text{(4.16*)} \qquad r^{\gamma/(ns)} \ell^{-\alpha}(r) \Big(\int_0^r w(t)\, dt\Big)^{1/q} \lesssim \Big(\int_0^r v(t)\, dt\Big)^{1/p}$$

and either

$$\text{(4.17*)} \quad \Big(\int_r^1 t^{(q/s)(\gamma/n-1)} \ell^{-q\alpha}(t)\, w(t)\, dt\Big)^{1/q} \times \Big[\int_0^r \Big(t^{-1} \int_0^t v(\tau)\, d\tau\Big)^{p/(s-p)} v(t)\, dt\Big]^{1/s-1/p} \lesssim 1 \quad \text{if } s < p,$$

or

$$\text{(4.18*)} \qquad r^{1/s} \Big(\int_r^1 t^{(q/s)(\gamma/n-1)} \ell^{-q\alpha}(t) w(t)\, dt\Big)^{1/q} \lesssim \Big(\int_0^r v(t)\, dt\Big)^{1/p} \quad \text{if } p \le s.$$

In the case of general weights $v, w \in \mathscr{W}(0, |\Omega|)$ the above-mentioned replacement yields only sufficient conditions for the boundedness of the operator $\mathscr{M}_{s,\gamma;\alpha}$ since an analogue of the lower estimate (3.4) is available only for small t.

10. Applications

In this section we apply our general results to describe mapping properties of some maximal operators in a limiting situation. For simplicity, we restrict ourselves to a local case.

Our first theorem deals with a local version of the usual fractional maximal operator M_γ from (1.1); we use the same notation for this local operator.

THEOREM 10.1. *Let Ω be a domain in $\mathbb{R}^n$, $|\Omega| < \infty$, $\gamma \in (0,n)$ and $\varrho \in [1,\infty)$.*

(i) *Let $\delta, \theta \in \mathbb{R}$. Then the operator*

$$M_\gamma : L^{n/\gamma,\varrho}(\log L)^\delta(\Omega) \to L^{\infty,\varrho}(\log L)^{\theta-1/\varrho}(\Omega) \tag{10.1}$$

is bounded if and only if $\theta < 0$ and $\delta \geq \theta$.

(ii) *Let $\delta \in \mathbb{R}$ and $\theta < 0$. Then the operator*

$$M_\gamma : L^{n/\gamma,\varrho}(\log L)^\delta(\Omega) \to \exp L^{-1/\theta}(\Omega) \tag{10.2}$$

is bounded if and only if $\delta \geq \theta$.

(iii) *Let $\delta \in \mathbb{R}$ and $\theta < 0$. Then the operator*

$$M_\gamma : L^{n/\gamma,\infty}(\log L)^\delta(\Omega) \to \exp L^{-1/\theta}(\Omega) \tag{10.3}$$

is bounded if and only if $\delta \geq \theta$.

(iv) *Let $\beta, \delta \in \mathbb{R}$. Then the operator*

$$M_\gamma : L^{n/\gamma,\infty}(\log L)^\delta(\Omega) \to L^{\infty,\varrho}(\log L)^\beta(\Omega) \tag{10.4}$$

is bounded if and only if either $\delta < 0$ and $\beta < \delta - 1/\varrho$, or $\delta \geq 0$ and $\beta < -1/\varrho$.

COROLLARY 10.2. *Let Ω be a domain in $\mathbb{R}^n$, $|\Omega| < \infty$, $\gamma \in (0,n)$, $\varrho \in [1,\infty)$, $\theta < 0$ and $\beta < \theta - 1/\varrho$. Then the following operators are bounded*:

$$M_\gamma : L^{n/\gamma,\varrho}(\log L)^\theta(\Omega) \to L^{\infty,\varrho}(\log L)^{\theta-1/\varrho}(\Omega), \tag{10.5}$$

$$M_\gamma : L^{n/\gamma,\varrho}(\log L)^\theta(\Omega) \to \exp L^{-1/\theta}(\Omega), \tag{10.6}$$

$$M_\gamma : L^{n/\gamma,\infty}(\log L)^\theta(\Omega) \to \exp L^{-1/\theta}(\Omega), \tag{10.7}$$

$$M_\gamma : L^{n/\gamma,\infty}(\log L)^\theta(\Omega) \to L^{\infty,\varrho}(\log L)^\beta(\Omega). \tag{10.8}$$

Proof of Theorem 10.1. (i) Since

$$L^{n/\gamma,\varrho}(\log L)^\delta(\Omega) = \Lambda^\varrho(\Omega; v) \qquad \text{with } v(t) = t^{\varrho\gamma/n-1}\ell^{\varrho\delta}(t),\ t \in (0,|\Omega|), \tag{10.9}$$

$$L^{\infty,\varrho}(\log L)^{\theta-1/\varrho}(\Omega) = \Lambda^\varrho(\Omega; w) \qquad \text{with } w(t) = t^{-1}\ell^{\varrho(\theta-1/\varrho)}(t),\ t \in (0,|\Omega|), \tag{10.10}$$

and

$$M_\gamma = \mathscr{M}_{1,\gamma;(0,0)}, \tag{10.11}$$

the result follows on applying (a local version of) Theorem 4.1 (cf. Section 9) with $p = q = \varrho$, $s = 1$, $\mathbb{A} = (0,0)$ and the weights v and w from (10.9) and (10.10), respectively.

(ii) Since

$$\exp L^{-1/\theta}(\Omega) = \Lambda^{1,\infty}(\Omega; w) \qquad \text{with } w(t) = t^{-1}\ell^{\theta-1}(t),\ t \in (0,|\Omega|), \tag{10.12}$$

the result follows on applying (a local version of) Theorem 5.1 with $p = \varrho$, $q = 1$, $s = 1$, $\mathbb{A} = (0,0)$ and the weights v and w from (10.9) and (10.12), respectively.

(iii) Since

$$L^{n/\gamma,\infty}(\log L)^{\delta}(\Omega) = \Lambda^{1,\infty}(\Omega; v) \quad \text{with } v(t) = t^{\gamma/n-1}\ell^{\delta}(t),\ t \in (0,|\Omega|), \tag{10.13}$$

the result follows on using (a local version of) Theorem 6.1 (cf. Section 9) with $p = q = 1$, $s = 1$, $\mathbb{A} = (0,0)$ and the weights v and w from (10.13) and (10.12), respectively.

(iv) Since

$$L^{\infty,\varrho}(\log L)^{\beta}(\Omega) = \Lambda^{\varrho}(\Omega; w) \quad \text{with } w(t) = t^{-1}\ell^{\varrho\beta}(t),\ t \in (0,|\Omega|), \tag{10.14}$$

the result follows on applying (a local version of) Theorem 7.1 (cf. Section 9) with $p = 1$, $q = \varrho$, $s = 1$, $\mathbb{A} = (0,0)$ and the weights v and w from (10.13) and (10.14), respectively. ■

Compare Corollary 10.2 with the next sharp results on the Riesz potential I_γ, $\gamma \in (0,n)$, defined by

$$(I_\gamma f)(x) = \int_\Omega \frac{f(y)}{|x-y|^{n-\gamma}}\,dy, \qquad x \in \Omega.$$

THEOREM 10.3. *Let Ω be a domain in $\mathbb{R}^n$, $|\Omega| < \infty$, $\gamma \in (0,n)$, $\varrho \in [1,\infty)$, $\theta < 0$ and $\beta < \theta - 1/\varrho$. Then the following operators are bounded*:

$$I_\gamma : L^{n/\gamma,\varrho}(\log L)^{\theta+1/\varrho'}(\Omega) \to L^{\infty,\varrho}(\log L)^{\theta-1/\varrho}(\Omega), \tag{10.15}$$

$$I_\gamma : L^{n/\gamma,\varrho}(\log L)^{\theta+1/\varrho'}(\Omega) \to \exp L^{-1/\theta}(\Omega), \tag{10.16}$$

$$I_\gamma : L^{n/\gamma,\infty}(\log L)^{\theta+1}(\Omega) \to \exp L^{-1/\theta}(\Omega), \tag{10.17}$$

$$I_\gamma : L^{n/\gamma,\infty}(\log L)^{\theta+1}(\Omega) \to L^{\infty,\varrho}(\log L)^{\beta}(\Omega). \tag{10.18}$$

Proof. Assertion (10.15) follows from [EOP1, (12.2.1)] with $p = n/(n-\gamma)$, $r = \varrho$ and $\beta = 0$. Similarly, (10.17) is a consequence of [EOP1, (12.2.1)] with $p = n/(n-\gamma)$, $r = \infty$ and $\beta = 0$ since

$$\exp L^{-1/\theta}(\Omega) = L^{\infty,\infty}(\log L)^{\theta}(\Omega) \tag{10.19}$$

(cf. [EOP1, Lemma 2.2(iv)]). Assertion (10.16) follows from [EOP1, Theorem 4.5(ii)] with $p_1 = 1$, $q_1 = n/(n-\gamma)$, $p_2 = n/\gamma$, $q_2 = \infty$, $r = \varrho$, $s = \infty$ and $\beta = 0$. Finally, (10.18) is a consequence of [EOP1, Theorem 4.2(ii)] with $p_1 = 1$, $q_1 = n/(n-\gamma)$, $p_2 = n/\gamma$, $q_2 = \infty$, $r = \infty$, $s = \varrho$, $\delta = \theta + 1$ and $\gamma = \beta$. ■

REMARK 10.4. Note that (10.6) is a consequence of (10.5) and the embedding

$$L^{\infty,\varrho}(\log L)^{\theta-1/\varrho}(\Omega) \hookrightarrow \exp L^{-1/\theta}(\Omega)$$

(cf. [EOP1, Theorem 6.3]). Similarly, (10.8) follows from (10.7). On the other hand, (10.6) follows from (10.7) since $L^{n/\gamma,\varrho}(\log L)^{\theta}(\Omega) \hookrightarrow L^{n/\gamma,\infty}(\log L)^{\theta}(\Omega)$. Finally, using (10.19), we see that (10.7) (and hence (10.8)) remains true if $\theta = 0$ since then statement (10.7) coincides with the endpoint estimate $M_\gamma : L^{n/\gamma,\infty}(\Omega) \to L^{\infty}(\Omega)$ (cf. (1.6)).

In our next result we establish the behaviour of the local maximal operator $M^{0,\alpha} := \mathscr{M}_{1,0;1+\alpha}$, $\alpha > 0$ (cf. (9.1)), on spaces close to $L^1(\Omega)$, where Ω is a domain in $\mathbb{R}^n$ with $|\Omega| < \infty$. This operator satisfies the following endpoint estimates (cf. Lemma 3.2):

$$M^{0,\alpha} : L^1(\Omega) \to L^{1,\infty}(\log L)^{1+\alpha}(\Omega), \tag{10.20}$$

$$M^{0,\alpha} : L^{\infty}(\log L)^{-1-\alpha}(\Omega) \to L^{\infty}(\Omega). \tag{10.21}$$

It is worthwhile to compare (10.20) and (10.21) with the sharp endpoint estimates for the corresponding Riesz-type potential operator $I^{0,\alpha}$, $\alpha > 0$, defined by

$$(I^{0,\alpha} f)(x) = \int_{\Omega} \frac{f(y)}{|x-y|^n \,\ell^{1+\alpha}(|x-y|)}\, dy, \qquad x \in \Omega. \tag{10.22}$$

By [OT2, (3.34)], for any domain Ω in $\mathbb{R}^n$ with $|\Omega| < \infty$,

$$I^{0,\alpha} : L^1(\Omega) \to L_{(1,\infty;\alpha)}(\Omega), \tag{10.23}$$

$$I^{0,\alpha} : L^{\infty,1}(\log L)^{-1-\alpha}(\Omega) \to L^{\infty}(\Omega). \tag{10.24}$$

Here the symbol $L_{(p,q;\beta)}(\Omega)$, $p, q \in (0,\infty]$, $\beta \in \mathbb{R}$, stands for the space given by

$$L_{(p,q;\beta)}(\Omega) = \{f \in \mathcal{M}(\Omega);\ \|f\|_{(p,q;\beta),\Omega} := \|t^{1/p-1/q}\ell^{\beta}(t) f^{**}(t)\|_{q,(0,|\Omega|)} < \infty\}$$

(recall that $f^{**}(t) := t^{-1}\int_0^t f^*(\tau)\, d\tau$).

While the operators I_γ and M_γ, $\gamma \in (0,n)$, have the same behaviour on the space L^1 (cf. (1.3) and (1.5)), in the limiting case when $\gamma = 0$ the behaviour of $I^{0,\alpha}$ and $M^{0,\alpha}$ on the space $L^1(\Omega)$ is different since, by [OP, Theorem 3.16(iii)],

$$L^{1,\infty}(\log L)^{1+\alpha}(\Omega) \subsetneqq L_{(1,\infty;\alpha)}(\Omega).$$

THEOREM 10.5. *Let Ω be a domain in $\mathbb{R}^n$, $|\Omega| < \infty$, $\alpha > 0$, $\beta, \delta, \theta \in \mathbb{R}$ and $\varrho \in [1,\infty)$.*

(i) *The operator*

$$M^{0,\alpha} : L^{1,\varrho}(\log L)^{\delta+1/\varrho'}(\Omega) \to L^{1,\varrho}(\log L)^{\alpha+\theta+1/\varrho'}(\Omega) \tag{10.25}$$

is bounded if and only if $\delta > 0$ and $\theta \le \delta$, or $\varrho = 1$, $\delta = 0$ and $\theta < 0$.

(ii) *The operator*

$$M^{0,\alpha} : L^{1,\varrho}(\log L)^{\delta+1/\varrho'}(\Omega) \to L^{1,\infty}(\log L)^{\alpha+\theta+1}(\Omega) \tag{10.26}$$

is bounded if and only if $\delta > 0$ and $\theta \le \delta$, or $\varrho = 1$, $\delta = 0$ and $\theta \le 0$.

(iii) *The operator*

$$M^{0,\alpha} : L^{1,\infty}(\log L)^{\delta}(\Omega) \to L^{1,\infty}(\log L)^{\alpha+\theta}(\Omega) \tag{10.27}$$

is bounded if and only if $\delta > 1$ and $\theta \le \delta$.

(iv) *The operator*

$$M^{0,\alpha} : L^{1,\infty}(\log L)^{\delta}(\Omega) \to L^{1,\varrho}(\log L)^{\beta}(\Omega) \tag{10.28}$$

is bounded if and only if $\delta > 1$ and $\beta < \alpha + \delta - 1/\varrho$.

COROLLARY 10.6. *Let Ω be a domain in $\mathbb{R}^n$, $|\Omega| < \infty$, $\alpha > 0$, $\varrho \in [1,\infty)$, $\theta > 0$, $\delta > 1$ and $\beta < \alpha + \delta - 1/\varrho$. Then the following operators are bounded*:

$$M^{0,\alpha} : L^{1,\varrho}(\log L)^{\theta+1/\varrho'}(\Omega) \to L^{1,\varrho}(\log L)^{\alpha+\theta+1/\varrho'}(\Omega), \tag{10.29}$$

$$M^{0,\alpha} : L^{1,\varrho}(\log L)^{\theta+1/\varrho'}(\Omega) \to L^{1,\infty}(\log L)^{\alpha+\theta+1}(\Omega), \tag{10.30}$$

$$M^{0,\alpha} : L^{1,\infty}(\log L)^{\delta}(\Omega) \to L^{1,\infty}(\log L)^{\alpha+\delta}(\Omega), \tag{10.31}$$

$$M^{0,\alpha} : L^{1,\infty}(\log L)^{\delta}(\Omega) \to L^{1,\varrho}(\log L)^{\beta}(\Omega). \tag{10.32}$$

Proof of Theorem 10.5. (i) Since

$$L^{1,\varrho}(\log L)^{\delta+1/\varrho'}(\Omega) = \Lambda^{\varrho}(\Omega; v) \quad \text{with } v(t) = t^{\varrho-1}\ell^{\varrho(\delta+1/\varrho')}(t),\ t \in (0,|\Omega|), \tag{10.33}$$

$$(10.34)\qquad L^{1,\varrho}(\log L)^{\alpha+\theta+1/\varrho'}(\Omega)=\Lambda^{\varrho}(\Omega;w)\quad \text{with } w(t)=t^{\varrho-1}\ell^{\varrho(\alpha+\theta+1/\varrho')}(t),\ t\in(0,|\Omega|),$$

$$(10.35)\qquad M^{0,\alpha}=\mathscr{M}_{1,0;1+\alpha},$$

the result follows on applying (a local version of) Theorem 4.1 (cf. Section 9) with $p=q=\varrho$, $s=1$, $\gamma=0$, $\mathbb{A}=(1+\alpha,1+\alpha)$ and the weights v and w from (10.33) and (10.34), respectively.

(ii) Since

$$(10.36)\qquad L^{1,\infty}(\log L)^{\alpha+\theta+1}(\Omega)=\Lambda^{1,\infty}(\Omega;w)\quad \text{with } w(t)=\ell^{\alpha+\theta+1}(t),\ t\in(0,|\Omega|),$$

the result follows on applying (a local version of) Theorem 5.1 with $p=\varrho$, $q=1$, $s=1$, $\gamma=0$, $\mathbb{A}=(1+\alpha,1+\alpha)$ and the weights v and w from (10.33) and (10.36), respectively.

(iii) Since

$$(10.37)\qquad L^{1,\infty}(\log L)^{\delta}(\Omega)=\Lambda^{1,\infty}(\Omega;v)\qquad \text{with } v(t)=\ell^{\delta}(t),\ t\in(0,|\Omega|),$$

$$(10.38)\qquad L^{1,\infty}(\log L)^{\alpha+\theta}(\Omega)=\Lambda^{1,\infty}(\Omega;w)\quad \text{with } w(t)=\ell^{\alpha+\theta}(t),\ t\in(0,|\Omega|),$$

the result follows on using (a local version of) Theorem 6.1 with $p=q=1$, $s=1$, $\gamma=0$, $\mathbb{A}=(1+\alpha,1+\alpha)$ and the weights v and w from (10.37) and (10.38), respectively.

(iv) Since

$$(10.39)\qquad L^{1,\varrho}(\log L)^{\beta}(\Omega)=\Lambda^{\varrho}(\Omega;w)\quad \text{with } w(t)=t^{\varrho-1}\ell^{\varrho\beta}(t),\ t\in(0,|\Omega|),$$

the result follows on applying (a local version of) Theorem 7.1 with $p=1$, $q=\varrho$, $s=1$, $\gamma=0$, $\mathbb{A}=(1+\alpha,1+\alpha)$ and the weights v and w from (10.37) and (10.39), respectively. ∎

For the Riesz-type potential $I^{0,\alpha}$ we have the following sharp result (see [OT2, Theorem 6.8 (a)]).

THEOREM 10.7. *Let Ω be a domain in $\mathbb{R}^n$, $|\Omega|<\infty$, $\alpha>0$, $\varrho\in[1,\infty]$, and $\theta>0$. Then the operator*

$$(10.40)\qquad I^{0,\alpha}: L^{1,\varrho}(\log L)^{\theta+1/\varrho'}(\Omega)\to L_{(1,\varrho;\alpha+\theta-1/\varrho)}(\Omega)$$

is bounded.

REMARKS 10.8. (i) By [EOP1, Theorem 6.3], there is no embedding between the target spaces in (10.29) and (10.30). Consequently, under the assumption of Corollary 10.6,

$$M^{0,\alpha}: L^{1,\varrho}(\log L)^{\theta+1/\varrho'}(\Omega)\to L^{1,\varrho}(\log L)^{\alpha+\theta+1/\varrho'}(\Omega)\cap L^{1,\infty}(\log L)^{\alpha+\theta+1}(\Omega).$$

In particular,

$$(10.41)\qquad M^{0,\alpha}: L^{1}(\log L)^{\theta}(\Omega)\to L^{1}(\log L)^{\alpha+\theta}(\Omega)\cap L^{1,\infty}(\log L)^{\alpha+\theta+1}(\Omega).$$

This should be compared with [EOP1, (12.3.6)] which asserts that the local version M_Ω of the classical Hardy–Littlewood maximal operator M satisfies

$$(10.42)\qquad M_\Omega: L^{1}(\log L)^{\theta}(\Omega)\to L^{1}(\log L)^{\theta-1}(\Omega)\cap L^{1,\infty}(\log L)^{\theta}(\Omega)\quad \text{if } \theta>0.$$

(ii) By [EOP1, Theorem 6.3],

$$L^{1,\infty}(\log L)^{\alpha+\delta}(\Omega)\hookrightarrow L^{1,\varrho}(\log L)^{\beta}(\Omega),\qquad \varrho\in[1,\infty),$$

if and only if $\beta<\alpha+\delta-1/\varrho$. Consequently, (10.32) follows from (10.31).

(iii) Since, by [OP, Theorem 3.16(ii)],

$$L_{(1,1;\alpha+\theta-1)}(\Omega) = L^1(\log L)^{\alpha+\theta}(\Omega),$$

the source and target spaces in (10.40) and (10.29) coincide if $\varrho = 1$. On the other hand, the target spaces in (10.40) and (10.29) (resp. in (10.40) and (10.30)) are different if $\varrho \in (1,\infty)$ (resp. if $\varrho = \infty$) since, by [OP, Theorem 3.16(iii)],

$$L^{1,\varrho}(\log L)^{\alpha+\theta+1/\varrho'}(\Omega) \subsetneqq L_{(1,\varrho;\alpha+\theta-1/\varrho)}(\Omega) \quad \text{if } \varrho \in (1,\infty].$$

(iv) Compare (10.31) (and (10.27)) with the following result involving the local version M_Ω of the classical Hardy–Littlewood maximal operator M:

Let Ω be a domain in $\mathbb{R}^n$, $|\Omega| < \infty$, and let $\beta, \delta \in \mathbb{R}$. Then the operator

$$M_\Omega : L^{1,\infty}(\log L)^{\delta}(\Omega) \to L^{1,\infty}(\log L)^{\beta}(\Omega) \tag{10.43}$$

is bounded if and only if $\delta > 1$ and $\beta \le \delta - 1$. In particular, the operator

$$M_\Omega : L^{1,\infty}(\log L)^{\delta}(\Omega) \to L^{1,\infty}(\log L)^{\delta-1}(\Omega)$$

is bounded if and only if $\delta > 1$.

Note that (10.43) is a consequence of the next assertion.

LEMMA 10.9. *Let Ω be a domain in $\mathbb{R}^n$, $|\Omega| < \infty$, let $s \in (0,\infty)$ and $\beta, \delta \in \mathbb{R}$. Put*

$$(M_{s,\Omega} f)(x) = \sup_{\substack{Q \ni x\\ Q \subset \Omega}} \frac{\|f\chi_Q\|_s}{\|\chi_Q\|_s}, \qquad f \in \mathcal{M}(\Omega),\ x \in \Omega.$$

Then the operator

$$M_{s,\Omega} : L^{s,\infty}(\log L)^{\delta}(\Omega) \to L^{s,\infty}(\log L)^{\beta}(\Omega)$$

is bounded if and only if $\delta > 1/s$ and $\beta \le \delta - 1/s$.

Proof. Since

$$L^{s,\infty}(\log L)^{\delta}(\Omega) = \Lambda^s(\Omega; v) \qquad \text{with } v(t) = \ell^{s\delta}(t),\ t \in (0,|\Omega|), \tag{10.44}$$

$$L^{s,\infty}(\log L)^{\beta}(\Omega) = \Lambda^s(\Omega; w) \qquad \text{with } w(t) = \ell^{s\beta}(t),\ t \in (0,|\Omega|), \tag{10.45}$$

the result follows on applying (a local version of) Theorem 6.1 with $\gamma = 0$, $\mathbb{A} = (0,0)$, $p = q = s$ and with the weights v and w from (10.44) and (10.45), respectively. ∎

REMARK 10.10. Let Ω be a domain in $\mathbb{R}^n$, $|\Omega| < \infty$, and consider the operator $\mathscr{M}_{1,0;1(\Omega)} := \mathscr{M}_{1,0;1}$ given by (9.1). (Note that it coincides with the operator $M^{0,\alpha}$ from Theorem 10.5 provided that $\alpha = 0$.) Applying Theorem 4.1, one can prove that

$$\mathscr{M}_{1,0;1(\Omega)} : L^2(\Omega) \to L^2(\log L)^{\beta}(\Omega)$$

is bounded if and only if $\beta \le 1$. In particular,

$$\mathscr{M}_{1,0;1(\Omega)} : L^2(\Omega) \to L^2(\Omega) \tag{10.46}$$

is bounded.

Now, let $G = \Omega \times \Omega$. Using (10.46) and Fubini's theorem, one can show that the operator

$$\mathscr{M}_{1,0;1(\Omega)} : L^2(G) \to L^2(G)$$

is bounded. Consequently, for any $f \in L^2(G)$, the function $\mathscr{M}_{1,0;1(\Omega)} f \in L^2(G)$. Note that this result corresponds to that of [AV, p. 1189].

11. Appendix

In the previous sections we have found necessary and sufficient conditions for the boundedness of fractional maximal operators between classical and weak-type Lorentz spaces. Therefore, our results are the best possible within these scales of spaces, which are general enough for most applications.

The aim of this section is to show that some of our results can be improved if we consider spaces which are outside the scales mentioned above. For simplicity, we shall consider the local version of the usual fractional maximal operator M_γ from (1.1) (cf. also (9.1)); we use the same notation for this local operator.

Putting $q = \varrho$ and $\theta = \alpha + 1/q$ in (10.5)–(10.7), we see that the operators

$$M_\gamma : L^{n/\gamma,q}(\log L)^{\alpha+1/q}(\Omega) \to L^{\infty,q}(\log L)^{\alpha}(\Omega), \qquad q \in [1,\infty), \tag{11.1}$$

$$M_\gamma : L^{n/\gamma,q}(\log L)^{\alpha+1/q}(\Omega) \to L^{\infty,\infty}(\log L)^{\alpha+1/q}(\Omega), \quad q \in [1,\infty], \tag{11.2}$$

are bounded provided that $\Omega \subset \mathbb{R}^n$ is a domain with $|\Omega| < \infty$, $\gamma \in (0,n)$ and $\alpha + 1/q < 0$.

On the other hand, making use of the endpoint estimates (cf. (1.5) and (1.6))

$$M_\gamma : L^1(\Omega) \to L^{n/(n-\gamma),\infty}(\Omega), \tag{11.3}$$

$$M_\gamma : L^{n/\gamma,\infty}(\Omega) \to L^{\infty}(\Omega), \tag{11.4}$$

and the limiting real interpolation involving logarithmic functors, we arrive at

$$M_\gamma : (L^1(\Omega), L^{n/\gamma,\infty}(\Omega))_{1,q;\alpha} \to (L^{n/(n-\gamma),\infty}(\Omega), L^{\infty}(\Omega))_{1,q;\alpha} \tag{11.5}$$

for all $q \in [1,\infty]$ and $\alpha \in \mathbb{R}$. Here, for a compatible couple of quasi-Banach spaces X_0 and X_1 satisfying $X_1 \hookrightarrow X_0$, the interpolation space $(X_0, X_1)_{\theta,q;\alpha}$, $\theta \in [0,1]$, $q \in [1,\infty]$, $\alpha \in \mathbb{R}$, is the set of all $f \in X_0$ such that

$$\|f\|_{\theta,q;\alpha} := \|t^{-\theta-1/q}\ell^{\alpha}(t) K(f,t;X_0,X_1)\|_{q,(0,1)} < \infty, \tag{11.6}$$

where K is the Peetre K-functional. Note that, by [EOP2, Sect. 9, Th. 2.2*], $(X_0, X_1)_{1,q;\alpha} \neq \{0\}$ if $\alpha + 1/q < 0$, or $q = \infty$ and $\alpha = 0$.

Since

$$L^{n/(n-\gamma),\infty}(\Omega) = (L^1(\Omega), L^{\infty}(\Omega))_{\gamma/n,\infty;0},$$

we deduce from [EOP2, Th. 7.1(v) and Sect. 9] (cf. also [D]) that

$$\begin{aligned}(L^{n/(n-\gamma),\infty}(\Omega), L^{\infty}(\Omega))_{1,q;\alpha} &= ((L^1(\Omega), L^{\infty}(\Omega))_{\gamma/n,\infty;0}, L^{\infty}(\Omega))_{1,q;\alpha} \\ &= (L^1(\Omega), L^{\infty}(\Omega))_{1,q;\alpha}\end{aligned}$$

if $\alpha + 1/q < 0$, or $q = \infty$ and $\alpha = 0$. Moreover, one can easily prove that

$$(L^1(\Omega), L^{\infty}(\Omega))_{1,q;\alpha} = L^{\infty,q}(\log L)^{\alpha}(\Omega) \tag{11.7}$$

if $\alpha + 1/q < 0$, or $q = \infty$ and $\alpha = 0$. Consequently, (11.7) implies that the target space in (11.5) coincides with the space $L^{\infty,q}(\log L)^{\alpha}(\Omega)$, which is a member of the scales of classical and weak-type Lorentz spaces.

Now, we are going to identify the source space in (11.5). Since

$$L^{n/\gamma,\infty}(\Omega) = (L^1(\Omega), L^\infty(\Omega))_{(n-\gamma)/n,\infty;0},$$

we infer from [EvO, Th. 5.9* (page 950), Lemma 8.6 and Section 7] (cf. also [D]) that

$$(L^1(\Omega), L^{n/\gamma,\infty}(\Omega))_{1,q;\alpha} = X_2, \tag{11.8}$$

where

$$X_2 = \{f \in L^1(\Omega);\ \|f\|_{X_2} := \|t^{-1/q}\ell^\alpha(t)\|\tau^{\gamma/n} f^{**}(\tau)\|_{\infty,(t,1)}\|_{q,(0,1)} < \infty\}. \tag{11.9}$$

Thus, by (11.5), (11.7)–(11.9), the operator

$$M_\gamma : X_2 \to L^{\infty,q}(\log L)^\alpha(\Omega) \tag{11.10}$$

is bounded provided that $\Omega \subset \mathbb{R}^n$ is a domain with $|\Omega| < \infty$ and the numbers $q \in [1,\infty]$ and $\alpha \in \mathbb{R}$ satisfy

$$\alpha + 1/q < 0, \quad \text{or} \quad q = \infty \text{ and } \alpha = 0.$$

On the other hand, the source space in (11.1) and (11.2) is

$$X_1 := L^{n/\gamma,q}(\log L)^{\alpha+1/q}(\Omega). \tag{11.11}$$

Consequently, a natural question arises: What is the relationship between the spaces X_1 and X_2?

First, by [EvO, Lemma 4.9, (6.3) and Sect. 9], we observe that

$$X_1 = X_2 \quad \text{if} \quad q = \infty \text{ and } \alpha \le 0. \tag{11.12}$$

Thus, in this case (11.10) reads

$$M_\gamma : L^{n/\gamma,\infty}(\log L)^\alpha(\Omega) \to L^{\infty,\infty}(\log L)^\alpha(\Omega) \quad \text{if } \alpha \le 0, \tag{11.13}$$

which is also a consequence of (11.2) and (11.4).

Second, by [EvO, Th. 4.7(i), (6.3) and Sect. 9], we have

$$X_1 \hookrightarrow X_2 \quad \text{if } q \in [1,\infty) \text{ and } \alpha + 1/q < 0. \tag{11.14}$$

We would like to know whether $X_1 \subsetneqq X_2$ in (11.14). To this end, first we compare the fundamental functions φ_{X_i} of the spaces X_i, $i = 1, 2$ (for the definition of this notion we refer to [BS, p. 65]). By [OP, Lemma 3.14(i)], $\varphi_{X_1}(t) \approx t^{\gamma/n}\ell^{\alpha+1/q}(t)$ for all $t \in [0, |\Omega|]$. Moreover, after some calculations one can arrive at

$$\varphi_{X_2} \approx \varphi_{X_1}. \tag{11.15}$$

Thus, we still do not know whether

$$X_1 \subsetneqq X_2 \quad \text{if } q \in [1,\infty) \text{ and } \alpha + 1/q < 0. \tag{11.16}$$

But (11.15) shows that the spaces X_1 and X_2 are very close to each other.

If we really can prove that (11.16) holds, then (11.10) gives a better result than (11.1). Since (11.1) is the best possible result within the scale of Lorentz–Zygmund spaces, this means that the space X_2 is outside that scale.

To verify (11.16), it is sufficient to show (cf. (11.14)) that $X_2 \setminus X_1 \neq \emptyset$. To this end, we shall need a very careful analysis. We start with the following assertion.

Lemma 11.1. *Let $n \in \mathbb{N}$, $\gamma \in (0,n)$, $q \in [1,\infty)$ and $\alpha + 1/q < 0$. Assume that $\{a_j\}_{j=1}^{\infty}$ and $\{b_j\}_{j=1}^{\infty}$ are two strictly increasing sequences of positive numbers with $b_1 = 1$. Put $A_j = e^{a_j}$, $t_j = e^{-b_j}$, $j \in \mathbb{N}$, and define the function g by*

$$g(t) = \sum_{j=1}^{\infty} A_j \chi_{[t_{j+1},t_j)}(t), \qquad t \in [0,\infty). \tag{11.17}$$

Moreover, let

$$\|g\|_1 = \|t^{\gamma/n-1/q}\ell^{\alpha+1/q}(t) g^*(t)\|_{q,(0,1)}, \tag{11.18}$$

$$\|g\|_2 = \big\|t^{-1/q}\ell^{\alpha}(t) \|\tau^{\gamma/n} g^{**}(\tau)\big\|_{\infty,(t,1)}\|_{q,(0,1)}. \tag{11.19}$$

If $c_j = e^{a_j-(\gamma/n)b_j}$, $j \in \mathbb{N}$, then

$$\|g\|_1^q \approx \sum_{j=1}^{\infty} (c_j b_j^{\alpha})^q\, b_j \left[1 - e^{-(\gamma/n)q(b_{j+1}-b_j)} \left(\frac{b_j}{b_{j+1}}\right)^{-\alpha q-1}\right] \tag{11.20}$$

and

$$\|g\|_2 \approx V(g) + W(g), \tag{11.21}$$

where

$$[V(g)]^q = \sum_{j=1}^{\infty} (c_j b_j^{\alpha})^q \left[1 - e^{-(\gamma/n)q(b_{j+1}-b_j)} \left(\frac{b_j}{b_{j+1}}\right)^{-\alpha q}\right], \tag{11.22}$$

$$W(g) = \|t^{-1/q}\ell^{\alpha}(t) \sup\{c_j;\ j \in \mathbb{N}\ \&\ t_j > t\}\|_{q,(0,1)}. \tag{11.23}$$

Remark 11.2. Let all the assumptions of Lemma 11.1 be satisfied. Then, since $b_j/b_{j+1} < 1$ and $\alpha + 1/q < 0$, estimate (11.20) implies that

$$\|g\|_1^q \gtrsim \sum_{j=1}^{\infty} (c_j b_j^{\alpha})^q b_j [1 - e^{-(\gamma/n)q(b_{j+1}-b_j)}]. \tag{11.24}$$

Similarly, from (11.22) we obtain

$$[V(g)]^q \le \sum_{j=1}^{\infty} (c_j b_j^{\alpha})^q. \tag{11.25}$$

Proof of Lemma 11.1. It is clear that $g^* = g$. Together with (11.17), this yields

$$\|g\|_1^q = \sum_{j=1}^{\infty} A_j^q \int_{t_{j+1}}^{t_j} t^{(\gamma/n)q-1}\ell^{\alpha q+1}(t)\,dt. \tag{11.26}$$

Moreover, for all $j \in \mathbb{N}$,

$$\begin{aligned}\int_{t_{j+1}}^{t_j} t^{(\gamma/n)q-1}\ell^{\alpha q+1}(t)\,dt &= \int_0^{t_j} \ldots\,dt - \int_0^{t_{j+1}} \ldots\,dt \\ &\approx (t_j)^{(\gamma/n)q}\ell^{\alpha q+1}(t_j) - (t_{j+1})^{(\gamma/n)q}\ell^{\alpha q+1}(t_{j+1}) \\ &= (t_j)^{(\gamma/n)q}\ell^{\alpha q+1}(t_j)[1 - (t_{j+1}/t_j)^{(\gamma/n)q}(\ell(t_{j+1})/\ell(t_j))^{\alpha q+1}],\end{aligned}$$

$$A_j(t_j)^{\gamma/n} = e^{a_j} e^{-(\gamma/n)b_j} = c_j, \quad \ell(t_j) = 1 - \ln t_j = 1 + b_j \approx b_j,$$
$$(t_{j+1}/t_j)^{(\gamma/n)q} = e^{-(\gamma/n)q(b_{j+1}-b_j)},$$

and (11.20) easily follows.

To verify (11.21), we define operators $\mathscr{T}, \mathscr{S}$ and $\mathscr{R}$ on the set $\mathcal{M}^+(0,1;\downarrow)$ by (cf. (4.5), (4.7) and (4.8))

$$(\mathscr{T}\varphi)(t) = \sup_{t<\tau<1} \tau^{\gamma/n}\varphi^{**}(\tau), \tag{11.27}$$

$$(\mathscr{S}\varphi)(t) = t^{\gamma/n}\varphi^{**}(t), \tag{11.28}$$

$$(\mathscr{R}\varphi)(t) = \sup_{t<\tau<1} \tau^{\gamma/n}\varphi^{*}(\tau). \tag{11.29}$$

Then (cf. (4.6))

$$\mathscr{T} \approx \mathscr{S} + \mathscr{R} \quad \text{on } \mathcal{M}^+(0,1;\downarrow).$$

Since

$$\|g\|_2 = \|t^{-1/q}\ell^{\alpha}(t)(\mathscr{T}g)(t)\|_{q,(0,1)},$$

we see that

$$\|g\|_2 \approx \|t^{-1/q}\ell^{\alpha}(t)(\mathscr{S}g)(t)\|_{q,(0,1)} + \|t^{-1/q}\ell^{\alpha}(t)(\mathscr{R}g)(t)\|_{q,(0,1)} =: \mathrm{I} + \mathrm{II}. \tag{11.30}$$

By (11.28),

$$\mathrm{I} = \|t^{-1/q}\ell^{\alpha}(t)t^{\gamma/n}g^{**}(t)\|_{q,(0,1)},$$

which, together with the inequality $n/\gamma > 1$, implies that (cf. [OP, Th. 3.16(i)])

$$\mathrm{I} \approx \|t^{\gamma/n-1/q}\ell^{\alpha}(t)g^{*}(t)\|_{q,(0,1)}.$$

Moreover, using the fact that $g^* = g$ and (11.17), we arrive at

$$\mathrm{I} \approx \sum_{j=1}^{\infty} A_j^q \int_{t_{j+1}}^{t_j} t^{(\gamma/n)q-1}\ell^{\alpha q}(t)\,dt, \tag{11.31}$$

and the same arguments as those used to calculate $\|g\|_1$ (cf. (11.31) and (11.26)) show that

$$\mathrm{I} \approx V(g). \tag{11.32}$$

If we prove that

$$\mathrm{II} = W(g), \tag{11.33}$$

then (11.21) is a consequence of (11.30) and (11.32).

To verify (11.33), we use (11.29) and (11.17), which imply that for all $t \in (0,1)$,

$$(\mathscr{R}g)(t) = \sup_{j\in\mathbb{N}} \sup_{\tau\in(t,1)\cap[t_{j+1},t_j)} \tau^{\gamma/n}A_j. \tag{11.34}$$

Let $t \in (0,1)$ and $\mathbb{N}(t) = \{j \in \mathbb{N};\, t_j > t\}$. Then

$$(t,1)\cap[t_{j+1},t_j) \neq \emptyset \quad \text{if and only if} \quad j \in \mathbb{N}(t).$$

Together with (11.34), this yields

$$(\mathscr{R}g)(t) = \sup_{j\in\mathbb{N}(t)} A_j \sup_{\tau\in(\max\{t,t_{j+1}\},t_j)} \tau^{\gamma/n} = \sup_{j\in\mathbb{N}(t)} A_j(t_j)^{\gamma/n}.$$

Hence,

$$\mathrm{II} = \|t^{-1/q}\ell^\alpha(t) \sup_{j\in\mathbb{N}(t)} A_j(t_j)^{\gamma/n}\|_{q,(0,1)},$$

and (11.33) follows since $A_j t_j^{\gamma/n} = c_j$ for all $j \in \mathbb{N}$. ∎

Now, we construct a function g given by (11.17) and such that

$$\|g\|_1 = \infty \quad \text{and} \quad \|g\|_2 < \infty. \tag{11.35}$$

LEMMA 11.3. *Let $n \in \mathbb{N}$, $\gamma \in (0, n)$, $q \in [1, \infty)$ and $\alpha + 1/q < 0$. Take $\varepsilon \in (0, \min\{1/q, -(\alpha + 1/q)\})$, put*

$$b_j = j, \quad a_j = (\gamma/n)j + (-\alpha - 1/q - \varepsilon)\ln j, \quad c_j = j^{-(\alpha+1/q+\varepsilon)}, \quad j \in \mathbb{N}, \tag{11.36}$$

and define g by (11.17). Then (11.35) is satisfied.

Proof. It is easy see that the sequences $\{a_j\}_{j=1}^\infty$, $\{b_j\}_{j=1}^\infty$ and $\{c_j\}_{j=1}^\infty$ given by (11.36) satisfy the assumptions of Lemma 11.1. Moreover, $c_j\, b_j^\alpha = j^{-(1/q+\varepsilon)}$, $j \in \mathbb{N}$, which, together with the fact that $1 + \varepsilon q > 1$, implies

$$\sum_{j=1}^\infty (c_j b_j^\alpha)^q = \sum_{j=1}^\infty j^{-(1+\varepsilon q)} < \infty.$$

This estimate and (11.25) show that

$$V(g) < \infty. \tag{11.37}$$

Furthermore, $(c_j b_j^\alpha)^q b_j = j^{-\varepsilon q}$, $j \in \mathbb{N}$, and, since $\varepsilon q < 1$, we obtain

$$\sum_{j=1}^\infty (c_j b_j^\alpha)^q\, b_j = \infty. \tag{11.38}$$

Our choice of b_j, $j \in \mathbb{N}$, implies that

$$1 - e^{-(\gamma/n)q(b_{j+1}-b_j)} = 1 - e^{-\gamma/n} > 0 \quad \text{for all } j \in \mathbb{N},$$

which, together with (11.38) and (11.24), shows that $\|g\|_1 = \infty$.

Let $t \in (0,1)$ and $\mathbb{N}(t) = \{j \in \mathbb{N};\, t_j > t\}$. Since

$$t_j > t \;\Leftrightarrow\; j < \ln\frac{1}{t},$$

we have

$$\sup\{c_j;\, j \in \mathbb{N}(t)\} = \sup\left\{c_j;\, j \in \mathbb{N} \;\&\; j < \ln\frac{1}{t}\right\}.$$

Thus, if $j \in \mathbb{N}(t)$, then

$$c_j = j^{-(\alpha+1/q+\varepsilon)} < \left(\ln\frac{1}{t}\right)^{-(\alpha+1/q+\varepsilon)} < [\ell(t)]^{-(\alpha+1/q+\varepsilon)}.$$

This implies that

$$W(g) = \|t^{-1/q}\ell^\alpha(t)\sup\{c_j;\, j \in \mathbb{N}(t)\}\|_{q,(0,1)} \le \|t^{-1/q}\ell^{-\varepsilon-1/q}(t)\|_{q,(0,1)} < \infty \tag{11.39}$$

since $\varepsilon > 0$. The inequality $\|g\|_2 < \infty$ now follows from (11.37), (11.39) and (11.21). ∎

Finally, we are able to verify that

$$X_2 \setminus X_1 \neq \emptyset. \tag{11.40}$$

Indeed, let g be the function from Lemma 11.3. Without loss of generality, we can assume that $|\Omega| = 1$. By [BS, Chapter 2, Corollary 7.8], there is a function $f \in \mathcal{M}(\Omega)$ such that $f^* = g$. This implies that $\|f\|_{X_1} = \|g\|_1$ and $\|f\|_{X_2} = \|g\|_2$. Therefore, (11.40) is a consequence of (11.35).

As mentioned above, (11.10) gives a better result than (11.1) since (11.16) holds. The next theorem provides an interesting characterization of the space X_2. In the proof of this assertion we shall use the following notation: Let $f \in \mathcal{M}(\mathbb{R}^n)$ and let $\Omega \subset \mathbb{R}^n$ be a domain. Then the symbol f^* stands for the non-increasing rearrangement of f while f^*_Ω is used to denote the non-increasing rearrangement of the restriction f_Ω of the function f to Ω.

THEOREM 11.4. *Let $n \in \mathbb{N}$, $\gamma \in (0, n)$ and let $\Omega \subset \mathbb{R}^n$ be a domain with $|\Omega| < \infty$. Assume that the numbers $q \in [1, \infty]$ and $\alpha \in \mathbb{R}$ satisfy $\alpha + 1/q < 0$, or $q = \infty$ and $\alpha = 0$. Put $Y(\Omega) = L^{\infty,q}(\log L)^\alpha(\Omega)$ and let $X_2 = X_2(\Omega)$ be the space from* (11.9). *Then X_2 is the largest rearrangement-invariant Banach function space* (*abbreviation r.i. B.f.s.*) *which is mapped by M_γ into the space $Y(\Omega)$.*

Proof. First, it is clear that X_2 is an r.i. B.f.s. Moreover, by (11.10), X_2 is mapped by M_γ into $Y(\Omega)$. Thus, it remains to show that X_2 is the largest r.i. B.f.s. with such a property. To this end, consider another r.i. B.f.s. $X = X(\Omega)$ which is mapped by M_γ into the space $Y(\Omega)$. Let $f \in X$. Put $\varphi = f^*$, choose some point $x_0 \in \Omega$ and define the function g by

$$g(x) = \varphi(\omega_n |x - x_0|^n), \qquad x \in \mathbb{R}^n, \tag{11.41}$$

where $\omega_n := |\{x \in \mathbb{R}^n;\ |x| \le 1\}|$. Then $g^*_\Omega \le g^* = f^*$, which implies that $\|g_\Omega\|_{X(\Omega)} \le \|f\|_{X(\Omega)} < \infty$. Consequently, $g_\Omega \in X(\Omega)$ and, since M_γ maps $X(\Omega)$ into $Y(\Omega)$, we have $\|M_\gamma g_\Omega\|_{Y(\Omega)} < \infty$. In other words,

$$\|t^{-1/q}\ell^\alpha(t)(M_\gamma g_\Omega)^*(t)\|_{q,(0,1)} < \infty. \tag{11.42}$$

(Note that $M_\gamma g_\Omega$ is defined only on Ω; cf. (9.1).) On the other hand, since Ω is open, (11.41) and the fact that $\varphi = f^*$ imply that there is $\varepsilon \in (0, |\Omega|)$ with

$$g^*_\Omega(t) = f^*(t) \qquad \text{for all } t \in (0, \varepsilon).$$

Making use of the ideas of the proof of Lemma 3.5, one can show that there is $\delta \in (0, \varepsilon)$ such that for all $t \in (0, \delta)$,

$$(M_\gamma g_\Omega)^*(t) \gtrsim \sup_{t<\tau<\delta} \tau^{\gamma/n-1} \int_0^\tau g^*_\Omega(\sigma)\, d\sigma = \sup_{t<\tau<\delta} \tau^{\gamma/n} f^{**}(\tau) \gtrsim \sup_{t<\tau<1} \tau^{\gamma/n} f^{**}(\tau).$$

Together with (11.42), this yields

$$\infty > \|t^{-1/q}\ell^\alpha(t) \sup_{t<\tau<1} \tau^{\gamma/n} f^{**}(\tau)\|_{q,(0,\delta)} \gtrsim \|f\|_{X_2}.$$

Since f was an arbitrary element of $X(\Omega)$, we have proved that $X(\Omega) \subset X_2(\Omega)$ and the proof is complete. ■

We conclude this section with the following remark.

REMARK 11.5. We have found the space $X_2 = X_2(\Omega)$ by means of limiting real interpolation. On the other hand, the proof of Theorem 11.4 clarifies the structure of the space $X_2(\Omega)$ (that is, the structure of the largest r.i. B.f.s., which is mapped by M_γ into $Y(\Omega)$) and shows that the expression defining the norm $\|\cdot\|_{X_2(\Omega)}$ (and involving the sharp estimate of the non-increasing rearrangement of $M_\gamma f$) is quite natural. We have (cf. (11.9))

$$\|f\|_{X_2(\Omega)} = \|t^{-1/q}\ell^\alpha(t) \sup_{t<\tau<1} \tau^{\gamma/n} f^{**}(\tau)\|_{q,(0,1)}, \tag{11.43}$$

and so (recall that $|\Omega| < \infty$)

$$\|f\|_{X_2(\Omega)} \approx \| \sup_{t<\tau<\infty} \tau^{\gamma/n} f^{**}(\tau)\|_{\overline{Y}(\Omega)}, \tag{11.44}$$

where $\overline{Y}(\Omega)$ stands for the (one-dimensional) representation space of the space $Y(\Omega)$ (cf. [BS, pp. 62–64]).

Having in mind formula (11.44) (and its analogues for other operators of harmonic analysis), the characterization of X_2 given in Theorem 11.4 (and its analogues for convenient spaces $Y(\Omega)$) and the well known mapping properties of M_γ on some classical spaces (e.g., on the Lebesgue spaces $L^p(\Omega)$ or the Lorentz spaces $L^{p,q}(\Omega)$), one can discover equivalent (quasi-)norms on a variety of familiar spaces simply by considering convenient spaces $Y(\Omega)$. For example, such an approach has led to interesting results mentioned in [EdO], [O2] and [O3].

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References

[AV] D. R. Adams and R. F. Vanderhouten, *Stability for polyharmonic obstacle problems with varying obstacles*, Comm. Partial Differential Equations 25 (2000), 1171–1200.

[AM] M. Ariño and B. Muckenhoupt, *Maximal functions on classical Lorentz spaces and Hardy's inequality with weights for nonincreasing functions*, Trans. Amer. Math. Soc. 320 (1990), 727–735.

[BR] C. Bennett and K. Rudnick, *On Lorentz–Zygmund spaces*, Dissertationes Math. 175 (1980).

[BS] C. Bennett and R. Sharpley, *Interpolation of Operators*, Pure Appl. Math. 129, Academic Press, New York, 1988.

[CAS] M. Carro, A. García del Amo and J. Soria, *Weak-type weights and normable Lorentz spaces*, Proc. Amer. Math. Soc. 124 (1996), 849–857.

[CPSS] M. J. Carro, L. Pick, J. Soria and V. D. Stepanov, *On embeddings between classical Lorentz spaces*, Math. Inequal. Appl. 4 (2001), 397–428

[CS1] M. J. Carro and J. Soria, *Weighted Lorentz spaces and the Hardy operator*, J. Funct. Anal. 112 (1993), 480–494.

[CS2] —, —, *Boundedness of some integral operators*, Canad. J. Math. 45 (1993), 1155–1166.

[CS3] —, —, *The Hardy–Littlewood maximal function and weighted Lorentz spaces*, J. London Math. Soc. 55 (1997), 146–158.

[CKOP] A. Cianchi, R. Kerman, B. Opic and L. Pick, *Sharp rearrangement inequality for the fractional maximal operator*, Studia Math. 138 (2000), 277–284.

[CP] M. Cwikel and E. Pustylnik, *Weak type interpolation near "endpoint" spaces*, J. Funct. Anal. 171 (2000), 235–277.

[D] R. Ya. Doktorskiĭ, *Reiteration relations of the real interpolation method*, Soviet Math. Dokl. 44 (1992), 665–669.

[EdO] D. E. Edmunds and B. Opic, *Equivalent quasi-norms on Lorentz spaces*, Proc. Amer. Math. Soc., to appear.

[EvO] W. D. Evans and B. Opic, *Real interpolation with logarithmic functors and reiteration*, Canad. J. Math. 52 (2000), 920—960.

[EOP1] W. D. Evans, B. Opic and L. Pick, *Interpolation of operators on scales of generalized Lorentz–Zygmund spaces*, Math. Nachr. 182 (1996), 127–181.

[EOP2] —, —, —, *Real interpolation with logarithmic functors*, J. Inequal. Appl., 7 (2002), 187–269.

[GOP] A. Gogatishvili, B. Opic and L. Pick, *Weighted inequalities for Hardy-type operators involving suprema*, submitted.

[G] M. de Guzmán, *Differentiation of Integrals in* $\mathbb{R}^n$, Lecture Notes in Math. 481, Springer, Berlin, 1975.

[HL] G. H. Hardy and J. E. Littlewood, *A maximal theorem with function-theoretic applications*, Acta Math. 54 (1930), 81–116.

[HM] H. P. Heinig and L. Maligranda, *Weighted inequalities for monotone and concave functions*, Studia Math. 116 (1995), 133–165.

[L] G. Lorentz, *On the theory of spaces* Λ, Pacific J. Math. 1 (1951), 411–429.

[MO] M. Milman and B. Opic, *Real interpolation and two variants of Gehring's lemma*, J. London Math. Soc. 58 (1998), 394–412.

[O1] B. Opic, *On boundedness of fractional maximal operators between classical Lorentz spaces*, in: Function Spaces, Differential Operators and Nonlinear Analysis Vol. 4 (Syöte, 1999), V. Mustonen and J. Rákosník (eds.), Math. Inst. AS CR, Prague, 2000, 187–196.

[O2] —, *New characterization of Lorentz spaces*, Proc. Roy. Soc. Edinburgh Sect. A, to appear.

[O3] —, *On equivalent quasi-norms on Lorentz spaces*, preprint, Prague, 2001, 11 pp. (submitted).

[OP] B. Opic and L. Pick, *On generalized Lorentz–Zygmund spaces*, Math. Inequal. Appl. 2 (1999), 391–467.

[OT1] B. Opic and W. Trebels, *Bessel potentials with logarithmic components and Sobolev-type embeddings*, Anal. Math. 26 (2000), 299—319.

[OT2] —, —, *Sharp embeddings of Bessel potential spaces with logarithmic smoothness*, Math. Proc. Cambridge Philos. Soc., to appear.

[P] E. Pustylnik, *Optimal interpolation in spaces of Lorentz–Zygmund type*, J. Anal. Math. 79 (1999), 113–158.

[R] F. Riesz, *Sur un théorème de maximum de MM. Hardy et Littlewood*, J. London Math. Soc. 7 (1932), 10–13.

[Sa] E. Sawyer, *Boundedness of classical operators on classical Lorentz spaces*, Studia Math. 96 (1990), 145–158.

[SS] G. Sinnamon and V. D. Stepanov, *The weighted Hardy inequality: new proofs and the case* $p = 1$, J. London Math. Soc. 54 (1996), 89–101.

[So] J. Soria, *Lorentz spaces of weak-type*, Quart. J. Math. Oxford 49 (1998), 93–103.

[Stn1] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Univ. Press, Princeton, NJ, 1970.

[Stn2] E. M. Stein, *Harmonic Analysis*: *Real-Variable Methods*, *Orthogonality*, *and Oscillatory Integrals*, Princeton Univ. Press, Princeton, NJ, 1993.
[Stp] V. D. Stepanov, *The weighted Hardy's inequality for nonincreasing functions*, Trans. Amer. Math. Soc. 338 (1993), 173–186.
[W] N. Wiener, *The ergodic theorem*, Duke Math. J. 5 (1939), 1–18.