# Contents

# Abstract

A variety of topological groups is a class of (not necessarily Hausdorff) topological groups closed under the operations of forming subgroups, quotient groups and arbitrary products. The variety of topological groups generated by a class of topological groups is the smallest variety containing the class. In this paper methods are presented to distinguish a number of significant varieties of abelian topological groups, including the varieties generated by (i) the class of all locally compact abelian groups; (ii) the class of all $k_\omega$-groups; (iii) the class of all $\sigma$-compact groups; and (iv) the free abelian topological group on $[0, 1]$. In all cases, hierarchical containments are determined.



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# 1. Introduction

Varieties of groups were introduced in the 1930s by Garrett Birkhoff [6] and B. H. Neumann [65]. A standard reference covering the material that existed until 1967 concerning varieties of groups is [66]. A *variety of groups* is defined as the class of all groups satisfying a certain family of "laws" or "equations"; for example, a group $G$ is abelian if and only if it satisfies the law $w(a_1, a_2) = (a_1)^{-1}(a_2)^{-1}a_1a_2 = 1$. Thus the variety of all abelian groups is the class of all groups that satisfy the law $(a_1)^{-1}(a_2)^{-1}a_1a_2 = 1$.

An equivalent definition for varieties of groups, courtesy of Birkhoff, uses closure under certain operations. A non-empty class of groups is a variety of groups if it is closed under the operations of forming subgroups ($S$), quotient groups ($Q$) and arbitrary Cartesian products ($C$). Birkhoff also proved that if $\Omega$ is any non-empty class of groups and $V(\Omega)$ is the smallest variety of groups containing $\Omega$ then $V(\Omega) = QSC(\Omega)$ [66]. This means that every element of $V(\Omega)$ can be written as a quotient group of a subgroup of some Cartesian product of members of $\Omega$.

For a family of groups $\{G_i : i \in I\}$, we define the *restricted direct product*, denoted $\prod^*_{i\in I} G_i$, to be the subgroup of $\prod_{i\in I} G_i$ consisting of elements $\prod_{i\in I} g_i$ where $g_i = e$ for all but a finite number of $i \in I$. We note that the free abelian group on any set is isomorphic to $\prod^*_{i\in I} \mathbb{Z}_i$, where $\mathbb{Z}_i$ is the additive group of integers, for some index set $I$. Further, every abelian group $G$ is a quotient group of the free abelian group on the underlying set of $G$. Therefore, it is easily seen that the variety of groups generated by $\mathbb{Z}$ is the variety of all abelian groups; that is, the variety of all abelian groups is *singly-generated.* Shortly, when we consider varieties of topological groups, we will see the analogue of this is not true.

In 1970, Ol'shanskiĭ [67] showed that there exist exactly $2^{\aleph_0}$ varieties of groups. Again, when we turn to our discussion of varieties of topological groups, we will see that the situation is very different.

Graham Higman [24], in 1952, suggested a definition for a variety of topological groups, however his work was not followed up. In 1968, Ian D. Macdonald suggested to the second author [57] the now more widely accepted definition that is similar to Birkhoff's definition of a variety of groups. A non-empty class $\mathfrak{V}$ of (not necessarily Hausdorff) topological groups is said to be a *variety of topological groups* [39, 57] if it is closed under the operations of forming subgroups, (not necessarily Hausdorff) quotient topological groups and arbitrary products (with the Tikhonov product topology). For example, the class of all abelian topological groups forms a variety of topological groups. Another example of a class of topological groups that forms a variety of topological groups is the class of all topological groups with a subgroup topology. A topological group $G$ is said to have a *subgroup topology* if a basis for the topology at the identity consists of subgroups, for example, all discrete topological groups have a subgroup topology. However, many classes of 'common' topological groups do not form varieties of topological groups. The class of all compact abelian groups is not closed under subgroups nor is the class of all locally compact abelian groups, nor the class of all separable topological groups. The class of all countable topological groups is not closed under arbitrary products, nor is the class of all discrete topological groups, nor the class of all metrizable topological groups, nor the class of all $\sigma$-compact groups. It is therefore of interest to examine these classes of topological groups in the context of varieties of topological groups. For this, we introduce the concept of a variety of topological groups *generated* by a class of topological groups.

If $\Omega$ is a class of topological groups, then the smallest variety containing $\Omega$ is said to be the *variety generated by* $\Omega$ and is denoted by $\mathfrak{V}(\Omega)$ (cf. [39] and [8]). We use the term *Banach space* to refer to the abelian topological group underlying a Banach (vector) space, or complete normed vector space. In [61] it was shown that the variety of topological groups generated by the class of all Banach spaces is the variety of all abelian topological groups. Contrary to what one may think, the variety of topological groups generated by the class of all locally compact abelian topological groups is quite small. In fact, this variety does not contain any infinite-dimensional Banach spaces.

In this paper, we present methods to distinguish a number of significant varieties of abelian topological groups. In all cases, hierarchical containments are determined.

We investigate the variety generated by $FA[0,1]$, the free abelian topological group on $[0,1]$, and the varieties generated by the following classes of topological groups.

- $\mathcal{A}$, the class of all abelian topological groups.
- $\mathcal{B}$, the class of all topological groups underlying Banach spaces.
- $\mathcal{L}_A$, the class of all locally compact Hausdorff abelian topological groups.
- $\mathcal{D}$, the class of all discrete abelian groups.
- $\mathcal{D}_R$, the class of all discrete abelian groups and $\mathbb{R}$, the additive topological group of all real numbers, with the Euclidean topology.
- $\mathcal{K}_\omega$, the class of all abelian $k_\omega$-groups.
- $\mathcal{C}_\sigma$, the class of all abelian $\sigma$-compact groups.
- $\mathcal{L}_\sigma$, the class of all abelian locally $\sigma$-compact groups.
- $\mathcal{S}$, the class of all abelian separable topological groups.
- $\mathcal{B}_\mathcal{S}$, the class of all topological groups underlying separable Banach spaces.
- $\mathcal{L}_\mathcal{S}$, the class of all abelian locally separable topological groups.
- $\mathcal{L}_m$, the class of all abelian locally-$m$ groups, where $m$ is an infinite cardinal.
- $\mathcal{C}_m$, the class of all abelian topological groups of cardinality less than or equal to $m$, where $m$ is an infinite cardinal.

The terminology here will be explained later in the paper.

Notation. In the following and later theorems, a variety of topological groups $\mathcal{V}$ appears linked by $\Big|$, $\Big\backslash$ or $\Big/$ below a variety $\mathcal{W}$ if and only if $\mathcal{V}$ is a ***proper*** subvariety of $\mathcal{W}$.

The main theorem proved in this paper is the following.

Main Theorem.

$$\mathcal{A} = \mathfrak{V}(\mathcal{B})$$

$$\mathfrak{V}(\mathcal{L}_n) = \mathfrak{V}(\mathcal{C}_n \cup \mathcal{D}), \quad \textit{for all } n > m$$

$$\mathfrak{V}(\mathcal{L}_m) = \mathfrak{V}(\mathcal{C}_m \cup \mathcal{D}), \quad m > \mathfrak{c}$$

$$\mathfrak{V}(\mathcal{L}_{\mathfrak{c}}) = \mathfrak{V}(\mathcal{C}_{\mathfrak{c}} \cup \mathcal{D})$$

$$\mathfrak{V}(\mathcal{L}_{\mathcal{S}}) = \mathfrak{V}(\mathcal{S} \cup \mathcal{D}) = \mathfrak{V}(\mathcal{S} \cup \mathcal{L}_\sigma)$$

$$\mathfrak{V}(\mathcal{C}_\sigma \cup \mathcal{L}_A) = \mathfrak{V}(\mathcal{C}_\sigma \cup \mathcal{D}) = \mathfrak{V}(\mathcal{L}_\sigma)$$

$$\mathfrak{V}(\mathcal{S}) = \mathfrak{V}(\mathcal{B}_{\mathcal{S}}) = \mathfrak{V}(\ell_1)$$

$$\mathfrak{V}(\mathcal{D}_R) = \mathfrak{V}(\mathcal{L}_A)$$

$$\mathfrak{V}(\mathcal{C}_\sigma)$$

$$\mathfrak{V}(\mathcal{D})$$

$$\mathfrak{V}(\mathcal{K}_\omega) = \mathfrak{V}(FA[0,1]) = \mathfrak{V}(FA(X)) \textit{ for } X \textit{ not scattered.}$$

We note that Lydia Außenhofer [3] investigated three varieties of abelian topological groups related to nuclear spaces and nuclear groups (see [4]) and was not only able to distinguish amongst them but to find the containment relationships. In particular, Außenhofer showed that the (Hausdorff) variety of nuclear groups is quite small and hence the variety of topological groups generated by the class of locally compact Hausdorff abelian topological groups is also quite small.

A natural extension of varieties of topological groups is the concept of *wide varieties* of topological groups, which includes continuous homomorphic images as well as arbitrary products and subgroups. We will consider the wide varieties generated by the aforementioned classes of topological groups, and we will find that the situation is very different as many of the wide varieties turn out to be the same (see Theorem E).

Parallel to the theory of varieties of topological groups is the study of varieties of topological vector spaces which has much in common with the theory of varieties of topological groups. In this paper, we present aspects that are specific to varieties of topological groups.

## 2. Preliminaries

For topological groups $G_1$ and $G_2$, we say $G_1$ *is topologically isomorphic to* $G_2$ if there exists a map $f : G_1 \to G_2$ such that $f$ is both an isomorphism of groups and a homeomorphism.

NOTATION. For a topological group $G$, let $|G|$ denote the *group underlying* $G$, that is, the group obtained from $G$ by "forgetting" the topology.

We will often use the symbol $e$ to denote the identity element of a group that we are considering.

LEMMA 2.1 ([39, Lemma 2.7]). *Let $\mathfrak{V}$ be a variety of topological groups and let $G \in \mathfrak{V}$. Then $|G|$ with the indiscrete topology is in $\mathfrak{V}$.*

*Proof.* Consider the countable product $\prod_{i=1}^{\infty} G_i$, where $G_i = G$ for each $i \in \mathbb{N}$. Define the *restricted direct product* (or *weak direct product*), denoted $\prod_{i=1}^{\infty *} G_i$, to be the subgroup of $\prod_{i=1}^{\infty} G_i$ consisting of elements $\prod_{i=1}^{\infty} g_i$ with $g_i = 1_{G_i}$ for all but a finite number of members of $I$, with the topology induced as a subspace of $\prod_{i=1}^{\infty} G_i$.

The restricted direct product $\prod_{i=1}^{\infty *} G_i$ is a dense normal subgroup of $\prod_{i\in I} G_i$. Therefore, the quotient topological group $K = \prod_{i=1}^{\infty} G_i / \prod_{i=1}^{\infty *} G_i$ is indiscrete. Now, let $\rho : \prod_{i=1}^{\infty} G_i \to K$ be the quotient mapping from $\prod_{i=1}^{\infty} G_i$ to $K$ and consider the homomorphism $f : G \to K$ given by $f(g) = \rho(\langle g, g, \ldots \rangle)$ for all $g \in G$. For each $g \in G$, $g \neq e$, $g$ is not contained in the kernel of $f$; that is, the kernel of $f$ is the set $\{e\}$. Therefore, $f$ is a one-to-one homomorphism from $|G|$ to $|K|$ and hence $|G|$ can be embedded in $|K|$. Since $\mathfrak{V}$ is a variety of topological groups and $K \in QC(\mathfrak{V})$, we have $K \in \mathfrak{V}$ and thus $|G|$ with the induced topology from $K$ is also contained in $\mathfrak{V}$; that is, $|G|$ with the indiscrete topology is contained in $\mathfrak{V}$. ■

PROPOSITION 2.2. *Let $\mathfrak{V}$ be a variety of (abelian) topological groups and $\mathbb{T}$ the compact topological group consisting of the multiplicative group of complex numbers of modulus 1 with its usual euclidean topology. If $\mathbb{T}$ is contained in $\mathfrak{V}$, then every indiscrete abelian group appears in $\mathfrak{V}$.*

*Proof.* Note it is well-known that every abelian group is algebraically isomorphic to a subgroup of a product of copies of the divisible group $\mathbb{T}$ and so is contained in $\mathfrak{V}$ with some topological group topology. The result then follows from Lemma 2.1, that whenever a topological group $G$ is in the variety $\mathfrak{V}$, the topological group $|G|$ with the indiscrete topology is also in $\mathfrak{V}$. ■

Corollary 2.3. *Let $\mathfrak{V}$ be a variety of (abelian) topological groups that contains any non-totally disconnected locally compact abelian group. Then every indiscrete abelian group appears in $\mathfrak{V}$.*

*Proof.* Let $G$ be a non-totally disconnected locally compact abelian group contained in $\mathfrak{V}$. Then the identity component $G_0$ of $G$ is a non-trivial connected locally compact abelian group in $\mathfrak{V}$. By Theorem 26 of [56] $G$ is topologically isomorphic to $\mathbb{R}^n \times K$, for $n$ a non-negative integer and $K$ a compact connected abelian group. If $n \geq 1$, then $\mathbb{T}$, as a quotient group of $\mathbb{R}$, is also in $\mathfrak{V}$ and the required result follows from Proposition 2.2. If $K$ is a non-trivial connected compact group, then being a closed subgroup of a product of copies of $\mathbb{T}$ ([56, Corollary 1 to Theorem 14]) it must project onto $\mathbb{T}$ into this product. So $\mathbb{T}$ is a quotient group of $K$ and so it is in $\mathfrak{V}$ from which the required result follows [56]. ∎

Definition 2.4. Let $\Omega$ be a class of (not necessarily Hausdorff) topological groups. Then $S(\Omega)$ is defined to be the class of all topological groups $G$ such that $G$ is isomorphic to a subgroup of a member of $\Omega$. Similarly, the operators $\overline{S}$, $\overline{G}$, $Q$, $\overline{Q}$, $C$ and $P$ denote closed subgroup, quotient group, Hausdorff quotient group, arbitrary cartesian product with the Tikhonov topology and finite product respectively.

The following theorem was shown in [8] (Theorem 1), however, because it is informative, we include it here.

Theorem 2.5. *Let $\Omega$ be a non-empty class of topological groups. Then $\mathfrak{V}(\Omega) = QSC(\Omega)$.*

*Proof.* First, we note that $SS(\Omega) = S(\Omega)$, $QQ(\Omega) = Q(\Omega)$ and $CC(\Omega) = C(\Omega)$.

Further, for $\Omega$ a non-empty class of topological groups, it is routine to establish the following.

(i) $CS(\Omega) \subseteq SC(\Omega)$,
(ii) $CQ(\Omega) \subseteq QC(\Omega)$,
(iii) $SQ(\Omega) \subseteq QS(\Omega)$.

Next we use these results to show that $QSC(\Omega)$ is a variety of topological groups:

$$\begin{aligned} &Q[QSC(\Omega)] = QSC(\Omega),\\ &S[QSC(\Omega)] \subseteq QSSC(\Omega) = QSC(\Omega),\\ &C[QSC(\Omega)] \subseteq QCSC(\Omega) \subseteq QSCC(\Omega) = QSC(\Omega). \end{aligned}$$

Therefore, $QSC(\Omega)$ is a variety of topological groups and so $\mathfrak{V}(\Omega) \subseteq QSC(\Omega)$. Clearly, $QSC(\Omega) \subseteq \mathfrak{V}(\Omega)$ and the proof is complete. ∎

Theorem 2.5 indicates that any topological group contained in the variety generated by a class of topological groups can be obtained by just one application of each of the operators $Q$, $S$ and $C$ to members of the class.

Remark 2.6. Due to the fact that every subgroup of an abelian topological group is normal, if $\Omega$ is a class of *abelian* topological groups, then $QS(\Omega) \subseteq SQ(\Omega)$ and hence $QS(\Omega) = SQ(\Omega)$.

The class of locally compact abelian topological groups and the class of $\sigma$-compact abelian topological groups are both closed under $\bar{S}$ and $\overline{Q}$, but not $C$. However, they are closed under $P$. Therefore, the following result, also shown in [8] (Theorem 2), is powerful. The proof is omitted here as it is more involved than that of Theorem 2.5.

THEOREM 2.7. *Let $\Omega$ be a non-empty class of abelian topological groups. If $G$ is a Hausdorff topological group in $\mathfrak{V}(\Omega)$, then $G \in SC\overline{Q}P(\Omega)$.*

REMARK 2.8. We now see, for example, that if $G$ is a Hausdorff topological group in $\mathfrak{V}(\mathcal{L}_A)$ then $G \in SC(\mathcal{L}_A)$; and for certain groups we can say significantly more. For example, if $B$ is a Banach space contained in $\mathfrak{V}(\mathcal{C}_\sigma)$, then $B \in \bar{S}P(\mathcal{C}_\sigma)$ and hence $B$ is $\sigma$-compact, that is, $B$ is finite-dimensional.

This follows because a Banach space is a special kind of topological group, namely a UFSS-group. A Hausdorff topological group $G$ is defined by Enflo [18] to be *uniformly free from small subgroups*, or a *UFSS-group*, if it has a neighbourhood of the identity, $U$, such that for every neighbourhood of the identity, $V$, there exists a positive integer $n_V$ with the property that if $x \notin V$ then $x^n \notin U$ for some $n \leq n_V$. We note that discrete topological groups and normed vector spaces are UFSS-groups.

Proposition 2.5 and Theorem 3.10 of [63] show that if a UFSS-group $G$ is contained in $\mathfrak{V}(\Omega)$, then $G \in S\overline{Q}\,\bar{S}P(\Omega)$. Again, if $\Omega$ is a class of abelian groups, then $G \in S\overline{Q}P(\Omega)$.

In contrast to Ol'shanskiĭ's result, Morris in [41] proved that there is a proper class of varieties of topological groups. The key to this is the notion of a $T(m)$-group.

DEFINITION 2.9 ([41, §4] and [34]). Let $m$ be any infinite cardinal number. A topological group, $G$, is said to be a *$T(m)$-group* if each neighbourhood of the identity contains a normal subgroup of index in $G$ strictly less than $m$.

Note that a discrete group is a $T(m)$-group if and only if its cardinality is strictly less than $m$. Observe that a variety generated by a class of $T(m)$-groups, for some cardinal $m$, contains only $T(m)$-groups. We say such a variety is a *$T(m)$-variety* and clearly, there is such a variety for every cardinal number and so there is a proper class of varieties of topological groups. (See [41, Theorem 4.2].)

REMARK 2.10 (cf. [57, Corollary 6 to Theorem 2]). A variety of topological groups is said to be *singly-generated* if it is generated by a single topological group. A variety of topological groups is singly-generated if and only if it is a $T(m)$-variety for some cardinal number $m$. And so, any subvariety of a singly-generated variety is singly-generated.

## 3. Free abelian topological groups

Graev [20] defined a free abelian topological group as follows.

DEFINITION 3.1 ([20]). Let $X$ be a completely regular [33] (not necessarily Hausdorff) space and $e$ a distinguished point in $X$. An abelian topological group $FA(X)$ is said to be a *free abelian topological group on the space* $X$ if it has the following properties:

(i) $X$ is a subspace of $FA(X)$;
(ii) $X$ generates $FA(X)$ algebraically;
(iii) for any continuous mapping $\phi$ of $X$ into any abelian topological group $G$ which maps the point $e$ onto the identity element of $G$, there exists a continuous homomorphism $\Phi$ of $FA(X)$ into $G$ such that $\Phi(x) = \phi(x)$ on $X$.

Graev [20] showed that $FA(X)$ exists and is unique up to topological isomorphism. In particular, $FA(X)$ does not depend on the choice of the point $e$ in the space $X$. Further, if $X$ is completely regular Hausdorff, $FA(X)$ is also Hausdorff.

We note that $|FA(X)|$, the group underlying $FA(X)$, is the free abelian group on the set $X \setminus \{e\}$ where $e$ is the identity element ([58, Proposition 47, p. 376]).

REMARK 3.2.

(1) Note that each element $w$ of $FA(X)$ can be represented as a product of members of $X \cup X^{-1}$ in an infinite number of ways. Amongst these there is the so-called *reduced representation* (1) which has no occurrences of $e$ (unless $w = e$) and if $x \in X$ appears in the representation then $x^{-1}$ does not appear in the representation.
(2) Any two free abelian topological groups on a given space $X$ are topologically isomorphic; that is, $FA(X)$ is unique up to topological isomorphism. In particular, $FA(X)$ does not depend on the choice of the point $e$ in the space $X$. (See [20].)
(3) The topological group $FA(X)$ has the finest topological group topology on the free abelian group on the set $X \setminus \{e\}$ that satisfies property (i) in Definition 3.1, where $e$ is the identity element of $FA(X)$. In [20], Graev showed the existence of the Graev free abelian topological group, $FA(X)$, on each completely regular Hausdorff space $X$ and gave a description of its topology. We outline Graev's construction of the free abelian topological group topology.

A *continuous* pseudometric is a pseudometric that defines a topology coarser than the given topology on a space $X$. Note that every completely regular space is determined by a family of continuous pseudometrics ([23, Chapter II, §8]). Each continuous pseudometric, $\rho$, on $X$ can be extended to an invariant pseudometric, $\rho'$ on $|FA(X)|$. In summary, Graev [20] defined $\rho'$ as follows.

Let $x, y \in X$ and $x^{-1}, y^{-1} \in X^{-1}$ where $X^{-1} = \{x_i^{-1} : x_i \in X\}$. Then

$$\rho'(x, y) = \rho(x, y); \qquad \rho'(x^{-1}, y^{-1}) = \rho(x, y);$$
$$\rho'(x^{-1}, y) = \rho'(x, y^{-1}) = \rho(x, e) + \rho(y, e).$$

Let $a, b \in |FA(X)|$, $a \neq b$ and let $A = a_1 a_2 \ldots a_s$ be a (not necessarily reduced) representation of $a$, $B = b_1 b_2 \ldots b_s$ a (not necessarily reduced) representation of $b$, in the form of words of equal length such that $a_i, b_i \in X \cup X^{-1}$ for $i = 1, \ldots, s$. Define $R(A, B) = \sum_{i=1}^{s} \rho'(a_i, b_i)$. Then $\rho'(a, b)$ is the infimum of all such $R(A, B)$. Further, this infimum is achieved and a representation $(A, B)$ of $(a, b)$ is said to be an *optimal $\rho$-representation* if $\rho'(a, b) = R(A, B)$. It is readily seen from Graev's proof that if $b = e$, then $\rho'(a, e)$ has the reduced representation of $a$ in any optimal $\rho$-representation.

(1) We use the term *reduced representation* where M. Hall [21] uses the term *reduced word*.

We shall refer to this extension as the *Graev extension* of the pseudometric. The Graev extension determines a topological group topology on the abelian group $|FA(X)|$. The sum of all such topologies on $|FA(X)|$ gives a topological group topology on $|FA(X)|$, and this topological group is the free abelian topological group on $X$ ([58, pp. 378–379]; [62, Proposition 1]). Note that in the non-abelian case, the topological group thus obtained is not necessarily the free topological group. (See [62].)

(4) If $X$ is a completely regular Hausdorff space, then $FA(X)$ is also Hausdorff (see [20]).

(5) As all topological groups are uniform spaces, they are completely regular. So, if $G$ is any topological group then the free abelian topological group on the underlying space of $G$ makes sense. Indeed, if $G$ is an abelian topological group, then $G$ is a quotient group of $FA(G)$ ([39, Theorem 2.12]), as we can extend the identity map $\phi : G \to G$ to an open continuous homomorphism $\Phi$ from $FA(G)$ onto $G$.

LEMMA 3.3. *Let $X$ be a completely regular space whose topology is defined by the family $\{\rho_i : i \in I\}$ of pseudometrics. Then $FA(X)$ can be embedded as a topological subgroup of the product*

$$H = \prod_{i\in I}(|FA(X)|, \rho_i'),$$

*where $|FA(X)|$ is the free abelian group on $X \setminus \{e\}$ and $\rho_i'$ is the Graev extension of $\rho_i$, for each $i \in I$.*

*Proof.* Let $f\colon FA(X) \to H$ be given by $f(w) = \prod_{i\in I} w_i$ where $w \in FA(X)$ and $w_i = w$ for each $i \in I$. The mapping $f$ is clearly a one-to-one homomorphism. A subbasis at $e$ for the topology of $FA(X)$ is given by the family of all open spherical balls about $e$ in $\rho_i$, $i \in I$. Consider the open ball $B_j(e)$ in $\rho_j$ where $B_j(e) = \{w \in FA(X) : \rho_j(w, e) < \varepsilon\}$, for $j \in I$ and some $\varepsilon > 0$. Then

$$f(B_j(e)) = \Big(\prod_{i\in I} U_i\Big) \cap f(FA(X))$$

where $U_j = B_j(e)$ and $U_i = FA(X)$ for each $i \neq j$. Clearly this is open in $f(FA(X))$ and so the corestriction of $f$ is an open mapping. Finally, let $O = \prod_{i\in I} O_i$ be a subbasic open set in $H$, where $O_i$ is open in $(|FA(X)|, \rho_i')$ and $O_i = |FA(X)|$ for all $i \in I \setminus J$, $J \subseteq I$ a finite set. Then

$$f^{-1}(O) = \bigcap_{i\in I} O_i = \bigcap_{j\in J} O_j$$

is open in $FA(X)$. Therefore, $f$ is continuous and the result follows. ∎

The next two results are folklore and their proofs are straightforward.

LEMMA 3.4. *Let $X$ and $Y$ be completely regular spaces such that there exists a quotient mapping $\phi\colon X \to Y$. Then $\phi$ extends to a quotient homomorphism $\Phi\colon FA(X) \to FA(Y)$.*

LEMMA 3.5. *Let $X$ be a completely regular space and $G$ an abelian topological group algebraically generated by $X$ and having the finest topological group topology that induces the given topology on $X$. Then $G$ is a quotient group of $FA(X)$.*

We now recall the concepts of $k_\omega$-space and $k_\omega$-group, the importance of these being that the free abelian topological group on $X$ is easiest to describe when $X$ is a $k_\omega$-space.

Definition 3.6 ([72]). A Hausdorff topological space $X$ is said to be a *$k_\omega$-space* with *$k_\omega$-decomposition* $X = \bigcup_{n=1}^\infty X_n$ if $X$ has compact subspaces $X_n$, for $n = 1, 2, \dots$, such that

(i) $X = \bigcup_{n=1}^\infty X_n$;
(ii) $X_n \subseteq X_{n+1}$ for all $n$;
(iii) a subset $A$ of $X$ is closed in $X$ if and only if $A \cap X_n$ is compact (or closed) for all $n$.

Further, a topological group that is a $k_\omega$-space is said to be a *$k_\omega$-group*.

Of course, every compact Hausdorff space $X$ is a $k_\omega$-space with $k_\omega$-decomposition $X = \bigcup_{n=1}^\infty X_n$ with $X_n = X$, for all $n = 1, 2, \dots$. Every connected locally compact Hausdorff group $G$ is a $k_\omega$-group ([35, §2]) with $k_\omega$-decomposition $G = \bigcup_{n=1}^\infty K^n$ where $K$ is any compact symmetric neighbourhood of the identity in $G$ ([56, Corollaries 1 and 2 to Proposition 8]). For example, the additive topological group of all real numbers, with the Euclidean topology, $\mathbb{R}$, is a $k_\omega$-space with $k_\omega$-decomposition $\mathbb{R} = \bigcup_{n=1}^\infty [-n, n]$. The importance of $k_\omega$-spaces to us is a consequence of the fact that the free abelian topological group on a $k_\omega$-space is a $k_\omega$-space.

Notation. Let $X$ be a completely regular space, $Y$ a subspace of $X$ and let $n \in \mathbb{N}$. We shall denote by $FA_n(Y)$ the set of all words in $FA(X)$ whose reduced representation has length less than or equal to $n$ with respect to $Y$.

Remark 3.7. If $X$ is a $k_\omega$-space with $k_\omega$-decomposition $X = \bigcup_{n=1}^\infty X_n$, then $FA(X)$ is a $k_\omega$-space with $k_\omega$-decomposition $FA(X) = \bigcup_{n=1}^\infty FA_n(X_n)$ ([35, Corollary 1 to Theorem 1]). [Note that in every Hausdorff group topology on $|FA(X)|$ inducing the given topology on $X$, the set $FA_n(X)$ inherits the same compact topology.]

The following lemma uses a now standard application of the Stone–Čech compactification technique introduced in [22].

Lemma 3.8. *Let $X$ be a completely regular Hausdorff space and let $S$ be a subset of $FA(X)$ such that $S \cap FA_n(X)$ is compact for all $n \in \mathbb{N}$. Then $S$ is closed in $FA(X)$.*

*Proof.* Let $\beta X$ be the Stone–Čech compactification of $X$ and let $FA(\beta X)$ be the free abelian topological group on $\beta X$. Then the natural map $\phi\colon X \to \beta X$, where $\beta X \subseteq FA(\beta X)$, can be extended to a continuous, one-to-one homomorphism $\Phi\colon FA(X) \to FA(\beta X)$. Now, clearly $FA(\beta X) = \bigcup_{n=1}^\infty FA_n(\beta X)$, indeed this is the $k_\omega$-decomposition of $FA(\beta X)$.

Now, consider $\Phi(S) \subseteq FA(\beta X)$. We have

$$\Phi(S) \cap FA_n(\beta X) = \Phi(S \cap FA_n(X)).$$

As $S \cap FA_n(X)$ is compact, $\Phi(S) \cap FA_n(\beta X)$ is compact. Therefore, $\Phi(S)$ is closed in $FA(\beta X)$, showing $\Phi^{-1}(\Phi(S))$ is closed in $FA(X)$. The proof is completed by noting that $\Phi^{-1}(\Phi(S)) = S$ as $\Phi$ is one-to-one. ∎

Notation. Let $X$ be a subset of a group $G$. We denote the subset $\bigcup_{i=1}^{n}(X \cup X^{-1})^i$ of $G$ by $\mathrm{gp}_n(X)$.

Corollary 3.9 (cf. Theorem 1.10 of [40]). *Let $X$ be a completely regular Hausdorff space and let $K$ be a compact subspace of $X$ containing the distinguished point $e$ of $X$. Let $G$ be the subgroup of $FA(X)$ algebraically generated by $K$. Then $G$ is topologically isomorphic to $FA(K)$.*

*Proof.* Clearly, $G = \bigcup_{n=1}^{\infty} \mathrm{gp}_n(K)$ and is algebraically the free abelian group generated by $K \setminus \{e\}$. Note that $\mathrm{gp}_n(K)$ is compact, and hence closed in $FA(X)$, for each $n$ as $K$ is compact. Let $A \subseteq G$ be such that $A \cap \mathrm{gp}_n(K)$ is compact for each $n \in \mathbb{N}$. Clearly $A \cap \mathrm{gp}_n(K) = A \cap \mathrm{gp}_n(X)$ and so by Lemma 3.8, $A$ is closed in $FA(X)$. Thus, $A$ is closed in $G$, making $G$ the $k_\omega$-group with $k_\omega$-decomposition $G = \bigcup_{n=1}^{\infty} \mathrm{gp}_n(K)$. So, by Remark 3.7, $G$ is topologically isomorphic to $FA(K)$. ∎

Notation. Let $X$ and $Y$ be disjoint topological spaces. We denote by $X \sqcup Y$ the free union of $X$ and $Y$; that is, $X \sqcup Y$ is the set $X \cup Y$ with the coarsest topology inducing the given topologies on $X$ and $Y$ and having $X$ and $Y$ as open subsets. Further, for each $n = 1, 2, \ldots$, let $Y_n$ be a topological space disjoint from each of $Y_1, \ldots, Y_{n-1}$. We denote by $\bigsqcup_{n=1}^{\infty} Y_n$ the free union of the $Y_n$.

Lemma 3.10. *Let $X = \bigcup_{n=1}^{\infty} X_n$ be a $k_\omega$-decomposition of the $k_\omega$-space $X$. For each $n$, let $Y_n$ be a space homeomorphic to $X_n$ and disjoint from each of $Y_1, \ldots, Y_{n-1}$, and $Y = \bigsqcup_{n=1}^{\infty} Y_n$. Then $FA(X)$ is a quotient group of $FA(Y)$.*

*Proof.* For each $n \in \mathbb{N}$, let $f_n \colon Y_n \to X_n$ be the homeomorphism from $Y_n$ onto $X_n$. Define the mapping $\phi\colon Y \to X$ as follows. For each $y \in Y$, there is a unique $n \in \mathbb{N}$ such that $y \in Y_n$, so let $\phi(y) = f_n(y)$. Clearly $\phi$ is a surjective mapping. We shall show it is also a quotient mapping. Let $O$ be open in $X$. Now, $\phi^{-1}(O) = \bigcup_{n=1}^{\infty}(\phi^{-1}(O) \cap Y_n)$. Further, for each $n \in \mathbb{N}$, $\phi^{-1}(O) \cap Y_n = f_n^{-1}(O)$, which is open in $Y_n$ and hence open in $Y$. Therefore, $\phi^{-1}(O)$ is the union of open sets in $Y$ and hence it is open in $Y$. So $\phi$ is continuous. Now let $U$ be a subset of $X$ such that $\phi^{-1}(U)$ is open in $Y$. Then $\phi^{-1}(U) \cap Y_n$ is open in $Y_n$ for each $n \in \mathbb{N}$. But, $f_n(\phi^{-1}(U) \cap Y_n) = U \cap X_n$ is open in $X_n$ for each $n \in \mathbb{N}$. As $X = \bigcup_{n=1}^{\infty} X_n$ is a $k_\omega$-space, $U$ is open in $X$. Thus $\phi$ is a quotient mapping from $Y$ onto $X$ and hence, by Lemma 3.4, $FA(X)$ is a quotient group of $FA(Y)$. ∎

## 4. The Metrification Mechanism

The Metrification Mechanism proved next will allow us to reduce many problems to the metric case. The Metrification Mechanism is the key to proving the first equality of our main result (see §5).

Note that we call a topological space *pseudometrizable* if its topology is determined by a pseudometric.

Theorem 4.1 (Metrification Mechanism). *Let $(X,\rho)$ be a pseudometrizable topological space. Then $(X,\rho)$ is a subspace of the product of a metrizable space $(Y,d)$ and the set $X$ with the indiscrete topology.*

*Proof.* Define the equivalence relation $\sim$ on $X$ by $x \sim y$ if and only if $\rho(x,y)=0$, for $x,y \in X$, and let $[x]=\{y \in X : \rho(x,y)=0\}$ denote the equivalence class of $x$ under $\sim$. Let $Y$ be the set of all such equivalence classes and define $f : X \to Y$ by $f(x)=[x]$ for all $x \in X$. We note that $f$ is surjective and we put the quotient topology on $Y$ so that $U$ is open in $Y$ if and only if $f^{-1}(U)$ is open in $X$. Consider $O$ open in $X$. Then it is routine to show that $f^{-1}(f(O))=O$, which is open. Therefore, $f(O)$ is open in $Y$ and so $f$ is an open mapping.

We define the metric $d$ on $Y$ by

$$d([x],[y]) = \inf\{\rho(a,b) : a \in [x],\, b \in [y]\}$$

for all $[x],[y] \in Y$. However, we note that for $a \in [x]$ and $b \in [y]$,

$$\rho(a,b) \le \rho(a,x) + \rho(x,y) + \rho(y,b) = \rho(x,y)$$

and

$$\rho(x,y) \le \rho(x,a) + \rho(a,b) + \rho(b,y) = \rho(a,b),$$

giving $\rho(a,b)=\rho(x,y)$. Therefore our definition for $d$ reduces to $d([x],[y])=\rho(x,y)$ for each $[x],[y] \in Y$. From the definition, $d$ is clearly a pseudometric. To see that $d$ is a metric, we take $[x],[y] \in Y$ with $d([x],[y])=\rho(x,y)=0$. Then $x \in [y]$ and so $[x]=[y]$.

Next we need to show that $d$ defines the topology on $Y$. Let $x \in X$ with $[x] \in Y$. Consider $B_\alpha(x,\rho)=\{z \in X : \rho(x,z)<\alpha\}$, the open sphere of radius $\alpha$ about $x \in X$ under $\rho$. Let $B'_\alpha([x],d)=\{[y] \in Y : d([x],[y])<\alpha\}$ denote the open sphere of radius $\alpha$ about $[x] \in Y$ under $d$. Now, $f(B_\alpha(x,\rho))=\{[z] : \rho(x,z)<\alpha\}=B'_\alpha([x],d)$ and so $d$ defines the required topology on $Y$.

Finally, we shall show that $(X,\rho)$ is indeed homeomorphic to a subspace of the product $H=(Y,d)\times X_I$ where $X_I$ is the set $X$ with the indiscrete topology. Consider the mapping $g : (X,\rho) \to H$ given by $g(x)=\langle f(x), x\rangle$ for each $x \in X$. Clearly, $g$ is one-to-one. Let $U$ be an open set in $H$. Then $U=O_1 \times O_2$ where $O_1$ is open in $(Y,d)$ and $O_2$ is either $\emptyset$ or $X$. Now, if $O_2=\emptyset$, then $g^{-1}(U)=\emptyset$, which is open in $(X,\rho)$. On the other hand, if $O_2=X$, then $g^{-1}(U)=f^{-1}(O_1)$, which is open in $(X,\rho)$. Therefore, $g$ is continuous. Finally, if $O$ is an open set in $(X,\rho)$, then $g(O)=(f(O)\times O)\cap g(X)=(f(O)\times X)\cap g(X)$, which is open in $g(X)$. Therefore, $(X,\rho)$ is homeomorphic to $g(X)$, a subspace of $H$. ∎

Notation. If $(X,\rho)$ and $(Y,d)$ are as in the Metrification Mechanism, we refer to $(Y,d)$ as the *metrification of* $(X,\rho)$.

Remark 4.2.

(1) We note that the metrification $(Y,d)$ of a pseudometrizable space $(X,\rho)$ is indeed a quotient space of $(X,\rho)$.

(2) Given a pseudometrizable space $(X,\rho)$, there exists a metrizable space, $Y$, such that $FA(Y)$ is a quotient of $FA(X)$. This is simply an application of the Metrification Mechanism and Lemma 3.4.

(3) It is true that a topological group $G$ is topologically isomorphic to a subgroup of the product of the Hausdorff topological group $G/\overline{\{e\}}$ and $|G|_I$. This can be shown by considering the mapping $h : G \to G/\overline{\{e\}} \times |G|_I$ given by $h(g) = \langle f(g), g\rangle$ where $f : G \to G/\overline{\{e\}}$ is the quotient mapping (cf. [41, proof of Theorem 6.7]). It is routine to show that $h$ is an embedding.
Further, for a pseudometrizable topological group $(G, \rho)$, the quotient group $G/\overline{\{e\}}$ is a metrizable topological group. Indeed, noting that
$$\overline{\{e\}} = \{x : \rho(x, e) = 0\},$$
it is clear that $G/\overline{\{e\}}$ with the metric defined by
$$d(x\overline{\{e\}}, y\overline{\{e\}}) = \rho(x, y),$$
where $x, y \in G$, is the metrification of $(G, \rho)$. Therefore, the metrification of a pseudometrizable topological group is a topological group.

For $X$ a completely regular space, let $\rho$ be a pseudometric on $X$. We wish to identify the metrification of $(|FA(X)|, \rho')$, where $|FA(X)|$ is the group underlying $FA(X)$ and $\rho'$ is the Graev extension of $\rho$. To do this we need the following lemma.

Lemma 4.3. *Let $(X, \rho)$ be a pseudometric space and let $\rho'$ be the Graev extension of $\rho$ to $|FA(X)|$. Let $x = x_1^{\varepsilon_1} \dots x_n^{\varepsilon_n}$, $\varepsilon_i = \pm 1$ for $i = 1, \dots, n$, be the reduced representation of a word in $FA(X)$ such that $\rho'(x, e) = 0$ and for each $i = 1, \dots, n$, $\rho(x_i, e) \neq 0$. Then $x$ can be written the form $x = u_1 \dots u_m$ where each $u_i = z_{i_1} z_{i_2}^{-1}$, $z_{i_1} \in \{x_j : \varepsilon_j = 1\} \subseteq X$ and $z_{i_2} \in \{x_k : \varepsilon_k = -1\} \subseteq X$ and $\rho(z_{i_1}, z_{i_2}) = 0$.*

*Proof.* Let $W_1$ and $W_2$ be the optimal $\rho$-representations for $x$ and $e$ respectively. We note that by Graev's analysis ([20, pp. 313–315]) on $\rho'$, $W_1$ and $W_2$ have length at most $n$; that is, $W_1$ is in fact the reduced representation of $x$. Further, if we write $W_2$ directly under $W_1$, and we apply Graev's analysis on these representations, we can divide the representation $(W_1, W_2)$ into small blocks, each of which has one of the following two forms:

$$\text{Single block: } \begin{cases} x_i^{\varepsilon_i} \\ e \end{cases}$$

$$\text{Double block: } \begin{cases} x_j & x_k^{-1} \\ x_j & x_j^{-1} \end{cases} \text{ where } x_j \text{ and } x_k^{-1} \text{ appear in } W_1, x_j \neq x_k.$$

Note that $\rho'(x, e) = R(W_1, W_2)$. Suppose the single block $\begin{cases} x_i^{\varepsilon_i} \\ e \end{cases}$ appears in the representation $(W_1, W_2)$. Then $\rho(x_i, e)$ contributes to the value of $\rho'(x, e)$. However, $\rho(x_i, e) \neq 0$ and $\rho'(x, e) = 0$, giving a contradiction. Therefore, all blocks must be double blocks and the result follows. ∎

The following proposition applies the Metrification Mechanism to $(X, \rho)$, a pseudometrizable topological space, and $(|FA(X)|, \rho')$, the free abelian group on $X \setminus \{e\}$ with the Graev extension of $\rho$. We see that if $(Y, d)$ is the metrification of $(X, \rho)$ and we take the free abelian group on $Y$ with the Graev extension of $d$, the result is exactly the metrification of $(|FA(X)|, \rho')$.

Proposition 4.4. *Let $(X,\rho)$ be a pseudometric topological space. Let $(Y,d)$ be the metrification of $(X,\rho)$. Let $\rho'$ and $d'$ be the Graev extensions of $\rho$ and $d$ to $F_X$, the group underlying $FA(X,\rho)$, and $F_Y$, the group underlying $FA(Y,d)$, respectively. Then the metrification of $(F_X,\rho')$ is a topological group and is topologically isomorphic to $(F_Y,d')$; indeed, $(F_Y,d')$ is isometrically isomorphic to the metrification of $(F_X,\rho')$.*

*Proof.* The notation for $(X,\rho)$ and $(Y,d)$ will be as in the Metrification Mechanism, Theorem 4.1. Note that $[e]\in Y$ is the identity element in $F_Y$. We shall denote by $\big(F_X/\overline{\{e\}},h\big)$ the metrification of $(F_X,\rho')$ as per Remark 4.2(3).

Let $w\in F_Y$ have reduced representation $w=[x_1]^{\varepsilon_1}\dots[x_n]^{\varepsilon_n}$, where $\varepsilon_i=\pm1$ for each $i=1,\dots,n$. We define the map $f: F_Y\to F_X/\overline{\{e\}}$ by

$$f([x_1]^{\varepsilon_1}\dots[x_n]^{\varepsilon_n})=x_1^{\varepsilon_1}\dots x_n^{\varepsilon_n}\overline{\{e\}}.$$

It is routine to show that $f$ is well-defined. Further, $f$ is clearly a surjective group homomorphism. We shall show that $f$ is a topological group isomorphism from $(F_Y,d')$ onto $(F_X/\overline{\{e\}},h)$.

To show $f$ is one-to-one, let

$$w_1=[x_1]^{\varepsilon_1}\dots[x_n]^{\varepsilon_n},\qquad \varepsilon_i=\pm1 \text{ for each } i=1,\dots,n,$$

and

$$w_2=[y_1]^{\eta_1}\dots[y_m]^{\eta_m},\qquad \eta_i=\pm1 \text{ for each } i=1,\dots,m,$$

be words in $F_Y$, each in its reduced representation. We note that if $[x_i]=[x_j]$, $i\neq j$, then $\varepsilon_i=\varepsilon_j$, and similarly $[y_i]=[y_j]$ implies $\eta_i=\eta_j$. Also, $\rho(x_i,e)\neq0$ for each $i=1,\dots,n$ and $\rho(y_j,e)\neq0$ for each $j=1,\dots,m$. Now, let $f(w_1)=f(w_2)$. Then $x_1^{\varepsilon_1}\dots x_n^{\varepsilon_n}\overline{\{e\}}=y_1^{\eta_1}\dots y_m^{\eta_m}\overline{\{e\}}$ and so $\rho'(x_1^{\varepsilon_1}\dots x_n^{\varepsilon_n},y_1^{\eta_1}\dots y_m^{\eta_m})=0$. By the invariance of $\rho'$, $\rho'(x_1^{\varepsilon_1}\dots x_n^{\varepsilon_n}y_m^{-\eta_m}\dots y_1^{-\eta_1},e)=0$. By Lemma 4.3, the word $x_1^{\varepsilon_1}\dots x_n^{\varepsilon_n}y_m^{-\eta_m}\dots y_1^{-\eta_1}$ can be written in the form $u_1\dots u_q$ where $u_i=z_{2i-1}z_{2i}^{-1}$ such that $\rho(z_{2i-1},z_{2i})=0$ with

$$z_{2i-1}\in\{x_i:\varepsilon_i=1\}\cup\{y_j:\eta_j=-1\}\subseteq X,\qquad z_{2i}\in\{x_i:\varepsilon_i=-1\}\cup\{y_j:\eta_j=1\}\subseteq X.$$

Suppose for some $u_i$, $z_{2i-1}=x_i$ and $z_{2i}=x_j$, $i\neq j$. Clearly $\varepsilon_i\neq\varepsilon_j$. However, as $\rho(z_{2i-1},z_{2i})=0$, $[z_{2i-1}]=[z_{2i}]$, that is, $[x_i]=[x_j]$, implying $\varepsilon_i=\varepsilon_j$. This is contradiction, so for each $u_i$, both $z_{2i-1}$ and $z_{2i}$ cannot be letters from $\{x_1,\dots,x_n\}$. Similarly, for each $u_i$, both $z_{2i-1}$ and $z_{2i}$ cannot be letters from $\{y_1,\dots,y_m\}$. Therefore, for each $u_i$ one letter is from $x_1^{\varepsilon_1}\dots x_n^{\varepsilon_n}$ and the other from $y_m^{-\eta_m}\dots y_1^{-\eta_1}$. This implies that each $x_i^{\varepsilon_i}$ is paired with a $y_j^{-\eta_j}$ such that $\rho(x_i,y_j)=0$, giving $[x_i]=[y_j]$. Further, $m=n$ and using commutativity of $F_Y$, we can form representations of $aw_1$ and $w_2$ by taking one letter from each pair $[x_i]=[y_i]$. This gives $w_1=w_2$ and so $f$ is one-to-one.

To complete the proof, we show that $d'$ and $h$ are isometric. We must show that for two words $w_1$ and $w_2$ in $F_Y$, the equality $d'(w_1,w_2)=h(f(w_1),f(w_2))$ holds. Recall that for $x\overline{\{e\}}$ and $y\overline{\{e\}}$ in $F_X/\overline{\{e\}}$, $h(x\overline{\{e\}},y\overline{\{e\}})=\rho'(x,y)$.

Now, we note that $\rho'(x^\varepsilon,y^\eta)=d'([x]^\varepsilon,[y]^\eta)$, where $x^\varepsilon,y^\eta\in X\cup X^{-1}$ and $[x]^\varepsilon,[y]^\eta\in Y\cup Y^{-1}$. Next we will show that for a word $w\in F_Y$,

$$d'(w,[e])=h(f(w),f([e])).$$

Let

$$w = [x_1]^{\varepsilon_1} \dots [x_n]^{\varepsilon_n}, \qquad \varepsilon_i = \pm 1 \text{ for } i = 1, \dots, n,$$
$$[e] = [a_1]^{\eta_1} \dots [a_n]^{\eta_n}, \qquad \eta_i = \pm 1 \text{ for } i = 1, \dots, n,$$

be an optimal $d$-representation for $(w, [e])$. Then

$$d'(w, [e]) = \sum_{i=1}^{n} d'([x_i]^{\varepsilon_i}, [a_i]^{\eta_i}) = \sum_{i=1}^{n} \rho'(x_i^{\varepsilon_i}, a_i^{\eta_i}) \geq \rho'(x_1^{\varepsilon_1} \dots x_n^{\varepsilon_n}, a_1^{\eta_1} \dots a_n^{\eta_n}).$$

Now, $f(w) = x_1^{\varepsilon_1} \dots x_n^{\varepsilon_n} \overline{\{e\}}$, and $f([e]) = a_1^{\eta_1} \dots a_n^{\eta_n} \overline{\{e\}}$, giving

$$h(f(w), f([e])) = \rho'(x_1^{\varepsilon_1} \dots x_n^{\varepsilon_n}, a_1^{\eta_1} \dots a_n^{\eta_n}),$$

and hence $d'(w, [e]) \geq h(f(w), f([e]))$.

Conversely, take $x = x_1^{\varepsilon_1} \dots x_n^{\varepsilon_n}$. Then $h(f(w), f([e])) = \rho'(x, e)$. Further, let

$$x = b_1^{\xi_1} \dots b_n^{\xi_n}, \qquad \xi_i = \pm 1 \text{ for } i = 1, \dots, n,$$
$$e = c_1^{\gamma_1} \dots c_n^{\gamma_n}, \qquad \gamma_i = \pm 1 \text{ for } i = 1, \dots, n,$$

be an optimal $\rho$-representation for $(x, e)$ (in $F_X$). Then

$$\rho'(x, e) = \sum_{i=1}^{n} \rho'(b_i^{\xi_i}, c_i^{\gamma_i}) = \sum_{i=1}^{n} d'([b_i]^{\xi_i}, [c_i]^{\gamma_i}) \geq d'([b_1]^{\xi_1} \dots [b_n]^{\xi_n}, [c_1]^{\gamma_1} \dots [c_n]^{\gamma_n}).$$

Now as $b_1^{\xi_1} \dots b_n^{\xi_n} = x_1^{\varepsilon_1} \dots x_n^{\varepsilon_n}$, we have

$$f([b_1]^{\xi_1} \dots [b_n]^{\xi_n}) = x_1^{\varepsilon_1} \dots x_n^{\varepsilon_n} \overline{\{e\}} = f([x_1]^{\varepsilon_1} \dots [x_n]^{\varepsilon_n}) = f(w).$$

Therefore, $w = [b_1]^{\xi_1} \dots [b_n]^{\xi_n}$. Similarly, $[e] = [c_1]^{\gamma_1} \dots [c_n]^{\gamma_n}$ and we have $d'(w, [e]) \leq h(f(w), f([e]))$, giving equality.

Finally, let $w_1, w_2 \in FA(Y)$. Then

$$d'(w_1, w_2) = d'(w_1 w_2^{-1}, [e]) = h(f(w_1)[f(w_2)]^{-1}, \overline{\{e\}}) = h(f(w_1), f(w_2)). \ \blacksquare$$

LEMMA 4.5. *Let $X$ be a completely regular space. Then $FA(X)$ is topologically isomorphic to a subgroup of the product of an indiscrete abelian group with a product of free abelian topological groups on metric spaces.*

*Proof.* Let $\rho_i$, $i \in I$, be a family of continuous pseudometrics on the space $X$ which give rise to the given completely regular topology on $X$. For each $i \in I$, let $(Y_i, d_i)$ be the metrification of $(X, \rho_i)$. Let $(|FA(X)|, \rho_i')$ be the free abelian group on $X \setminus \{e\}$ with $\rho_i'$ the Graev extension of the pseudometric $\rho_i$. Further, let $(|FA(Y_i)|, d_i')$ be the free abelian group on $Y_i \setminus \{[e]_i\}$ with $d_i'$ the Graev extension of the metric $d_i$. By Proposition 4.4, $(|FA(Y_i)|, d_i')$ is the metrification of $(|FA(X)|, \rho_i')$. Noting Theorem 4.1, we see that there exists a topological group embedding $g_i : (|FA(X)|, \rho_i') \to (|FA(Y_i)|, d_i') \times K_i$ where $K_i$ is $|FA(X)|$ with the indiscrete topology.

Let $H = \prod_{i \in I} (|FA(X)|, \rho_i')$. By Lemma 3.3, $FA(X)$ is topologically isomorphic to a subgroup of $H$ where the embedding $f : FA(X) \to H$ is given by $f(w) = \prod_{i \in I} w_i$, $w_i = w \in FA(X)$ for each $i \in I$.

Consider $\Phi : FA(X) \to \prod_{i\in I}[(|FA(Y_i)|, d'_i) \times K_i]$ given by $\Phi(w) = \prod_{i\in I} g_i(w)$. Clearly, $\Phi$ is continuous. Also as each $g_i$ is one-to-one and an open mapping onto the image of $(|FA(X)|, \rho'_i)$, the map $\Phi$ is also one-to-one and open onto $\Phi(FA(X))$. Thus $\Phi$ is a topological group embedding and $\Phi|_X$ is a homeomorphism of $X$ onto its image in $\prod_{i\in I}[(Y_i, d_i) \times K_i]$.

For each $i \in I$, let $FA(Y_i, d_i)$ be the free abelian topological group on the metric space $(Y_i, d_i)$. We note that both $(|FA(Y_i)|, d'_i)$ and $FA(Y_i, d_i)$ induce the topology $(Y_i, d_i)$ on $Y_i$, with $FA(Y_i, d_i)$ having the finer topology. Now, let $\Psi : FA(X) \to \prod_{i\in I}(FA(Y_i, d_i) \times K_i)$ be given by $\Psi(w) = \Phi(w)$ for each $w \in FA(X)$. Let $\mathcal{T}'$ be the topology on $|FA(X)|$ for which $\Psi$ is a (topological) embedding. If we denote the topology on $FA(X)$ by $\mathcal{T}$, then clearly $\mathcal{T} \subseteq \mathcal{T}'$. Considering $\Psi(X) = \Phi(X)$, we note $\Psi(X)$ is a subspace of $\prod_{i\in I}((Y_i, d_i) \times K_i)$. Therefore $(|FA(X)|, \mathcal{T}')$ induces the same topology on $X$ as does $FA(X)$ (with $\mathcal{T}$). However, $FA(X)$ has the finest group topology that induces the given topology on $X$, giving $\mathcal{T}' = \mathcal{T}$. Thus $FA(X)$ is topologically isomorphic to a subgroup of $\prod_{i\in I}(FA(Y_i, d_i) \times K_i)$. Further, $\prod_{i\in I}(FA(Y_i, d_i) \times K_i)$ is topologically isomorphic to $\prod_{i\in I} FA(Y_i, d_i) \times K$, where $K = \prod_{i\in I} |FA(X)|_I$, an indiscrete abelian group, and the result follows. ∎

We now present the main theorem for this section, which brings the ideas in this section together and allows us to describe the variety generated by $FA[0, 1]$.

Theorem 4.6. *Let $X$ be a completely regular Hausdorff space. Then $FA(X)$ is topologically isomorphic to a subgroup of $\prod_{i\in I} FA(Y_i)$, where each $Y_i$ is a metrizable space. If $X$ is also a compact space, then each $Y_i$ is a compact metrizable space and a quotient space of $X$, and each $FA(Y_i)$ is a quotient group of $FA(X)$.*

*Proof.* Let the topology on $X$ be defined by the family of pseudometrics $\{\rho_i : i \in I\}$ and for each $i \in I$, let $(Y_i, d_i)$ be the metrification of $(X, \rho_i)$. By Lemma 4.5, $FA(X)$ is topologically isomorphic to a subgroup of $\prod_{i\in I} FA(Y_i, d_i) \times K$ where $K$ is an indiscrete abelian group. Let $\Phi$ be the topological group isomorphism of $FA(X)$ onto its image in $\prod_{i\in I} FA(Y_i, d_i) \times K$, as in Lemma 4.5. Let $p$ be the projection of $\prod_{i\in I} FA(Y_i, d_i) \times K$ onto $\prod_{i\in I} FA(Y_i, d_i)$ and consider $\gamma = p \circ \Phi$. Clearly, $\gamma$ is a continuous homomorphism of $FA(X)$ into $\prod_{i\in I} FA(Y_i, d_i)$. As $K$ is indiscrete, $\gamma$ is an open mapping of $FA(X)$ onto its image $\gamma(FA(X))$ in $\prod_{i\in I} FA(Y_i, d_i)$.

We now show that $\gamma$ is one-to-one. Let $w \in FA(X)$ such that $w \neq e$. As $FA(X)$ is Hausdorff, there exists $k \in I$ such that $\rho'_k(w, e) \neq 0$. Let $(|FA(Y_k)|, d'_k)$ be the group underlying $FA(Y_k)$ with $d'_k$ the Graev extension of the metric $d_k$. As $(|FA(Y_k)|, d'_k)$ is the metrification of $d'_k([w]_k, [e]_k) \neq 0$, where $[w]_k$ and $[e]_k$ are the respective images of $w$ and $e$ in $FA(Y_k)$. This implies that $[w]_k \neq [e]_k$ for some $k \in I$ and so $\gamma(w)$ is not the identity in $\prod_{i\in I} FA(Y_i, d_i)$. Therefore, $\gamma$ is one-to-one and hence an embedding.

To complete the proof, observe that $(Y_i, d_i)$ is a continuous image of $(X, \rho_i)$, indeed of $X$, and so $(Y_i, d_i)$ is compact. Further, $(Y_i, d_i)$ is a quotient space of $X$ and so $FA(Y_i, d_i)$ is a quotient of $FA(X)$ by Lemma 3.4. ∎

## 5. The variety generated by $FA[0,1]$

In this section, we will commence our proof of the Main Theorem by establishing the following interesting result.

THEOREM A. *The variety of topological groups generated by $FA[0,1]$ is precisely the variety of topological groups generated by the class of all abelian $k_\omega$-groups; that is, $\mathfrak{V}(FA[0,1]) = \mathfrak{V}(\mathcal{K}_\omega)$.*

The variety of topological groups generated by $FA[0,1]$ is trivially contained in $\mathfrak{V}(\mathcal{K}_\omega)$. We must therefore establish that $\mathfrak{V}(\mathcal{K}_\omega)$ is contained in $\mathfrak{V}(FA[0,1])$.

The first two steps in our proof of Theorem A call on Remark 3.2(5), Remark 3.7 and Lemma 3.10 respectively.

STEP 1. The variety of topological groups generated by the class of all abelian $k_\omega$-groups, $\mathfrak{V}(\mathcal{K}_\omega)$, is the same as the variety of topological groups generated by the class of all free abelian topological groups on all $k_\omega$-spaces.

STEP 2. $\mathfrak{V}(\mathcal{K}_\omega)$ equals the variety of topological groups generated by the class of all free abelian topological groups on countable free unions of compact Hausdorff spaces.

In the remainder of this section, we shall see that the free abelian topological group on any compact Hausdorff space is contained in $\mathfrak{V}(FA[0,1])$. This result will then allow us to prove that $\mathfrak{V}(FA[0,1])$ equals $\mathfrak{V}(\mathcal{K}_\omega)$.

As a first step towards establishing that $FA(X)$ is contained in $\mathfrak{V}(FA[0,1])$ for every compact Hausdorff space $X$, we consider the case when $X$ is a compact metrizable space.

PROPOSITION 5.1. *Let $Y$ be a compact metrizable space. Then $FA(Y)$ is topologically isormorphic to a subgroup of a quotient group of $FA[0,1]$, and hence, $FA(Y)$ is contained in $\mathfrak{V}(FA[0,1])$.*

*Proof.* Observe that $Y$ can be embedded in $[0,1]^{\aleph_0}$ ([17, Chapter IX, Section 9, Corollary 9.2]). As $Y$ is compact, by Corollary 3.9, $FA(Y)$ is topologically isomorphic to a closed subgroup of $FA[0,1]^{\aleph_0}$. Now noting that $[0,1]^{\aleph_0}$ is a compact connected locally connected metrizable topological space, by the Hahn–Mazurkiewicz Theorem ([2, Part II, Chapter 1, §1, p. 100]), it is a continuous image of $[0,1]$, and hence a quotient space of $[0,1]$. Thus, by Lemma 3.4, $FA[0,1]^{\aleph_0}$ is a quotient group of $FA[0,1]$ and the result follows. ∎

We now use the Metrification Mechanism and Proposition 5.1 to obtain the desired result for any compact Hausdorff space.

COROLLARY 5.2. *Let $X$ be any compact Hausdorff space. Then $FA(X)$ is contained in $\mathfrak{V}(FA[0,1])$.*

*Proof.* By Theorem 4.6, $FA(X)$ is topologically isomorphic to a subgroup of a product of free abelian topological groups on compact metrizable spaces. From Proposition 5.1, each of these free abelian topological groups is in $\mathfrak{V}(FA[0,1])$ and the result follows. ∎

We now turn our attention back to the problem of establishing that $\mathfrak{V}(FA[0,1])$ is equal to $\mathfrak{V}(\mathcal{K}_\omega)$, where $\mathcal{K}_\omega$ is the class of all abelian $k_\omega$-groups. Using Step 2, it now suffices to show that the free abelian topological group on any countable free union of compact Hausdorff spaces is in $\mathfrak{V}(FA[0,1])$.

NOTATION. We will denote by $\mathrm{gp}(X)$ the algebraic group generated by $X$.

PROPOSITION 5.3. *Let $Y = \bigsqcup_{n=1}^{\infty} Y_n$, where each $Y_n$ is a compact Hausdorff space. If $Y$ is a subspace of a compact Hausdorff space $X$, then $FA(Y)$ is topologically isomorphic to a closed subgroup of $FA(X)$. In particular, $FA(Y)$ is topologically isomorphic to a subgroup of $FA([0,1]^{\aleph})$ for some cardinal number $\aleph$.*

*Proof.* [*Warning.* We are not asserting in the statement that the subgroup of $FA(X)$ algebraically generated by $Y$ is $FA(Y)$.] Without loss of generality, let $e \in Y_1 \subseteq X$. Let $Z$ be the subspace of $FA(X)$ defined by $Z = \bigcup_{n=1}^{\infty} Z_n$, where $Z_n = \{y^n : y \in Y_n\}$. Now, $Z \setminus \{e\}$ freely generates the group $\mathrm{gp}(Z)$. Further, $Z \cap FA_n(X) = \bigcup_{i=1}^{n} Z_i$, which is compact. Therefore, $Z$ is closed in $FA(X)$ by Lemma 3.8. A similar argument shows that for each $n \in \mathbb{N}$, $Z \setminus Z_n$ is closed in $FA(X)$ and so $Z_n$ is open in $Z$. Thus, $Z$ is a free union of the spaces $Z_n$, $n = 1, 2, \ldots.$ So $Z$ is homeomorphic to $Y$. Next, let $Z'_n = Z_1 \cup \cdots \cup Z_n$. Then $Z$ is a $k_\omega$-space with $k_\omega$-decomposition $Z = \bigcup_{n=1}^{\infty} Z'_n$. Now, from the definition of $Z_n$ we see that $\mathrm{gp}(Z) \cap FA_n(X) \subseteq \mathrm{gp}_n(Z'_n)$ and so by Theorem 3 of [35], $\mathrm{gp}(Z)$ is closed in $FA(X)$ and is the free abelian topological group on $Z$, $FA(Z)$. Further, as $Z$ is homeomorphic to $Y$, $FA(Y)$ is topologically isomorphic to $FA(Z)$. Thus, $FA(Y)$ is topologically isomorphic to a closed subgroup of $FA(X)$.

Finally, as $Y$ is completely regular Hausdorff, it can be embedded in $X = [0,1]^{\aleph}$ for some cardinal $\aleph$ ([33, Chapter 4, Theorem 7]). The result follows by noting that $[0,1]$ is compact Hausdorff. ∎

Step 2 and Proposition 5.3 now allow us to deduce the final step in the proof of Theorem A.

STEP 3. $\mathfrak{V}(\mathcal{K}_\omega)$ equals the variety of topological groups generated by the class of all free abelian topological groups on compact spaces.

Theorem A follows from Step 3 and Corollary 5.2.

REMARK 5.4. $\mathfrak{V}(\mathcal{K}_\omega)$ is a singly-generated variety and hence is a $T(m)$-variety, indeed, $\mathfrak{V}(\mathcal{K}_\omega)$ is a $T(\mathfrak{c}^+)$-variety, where $\mathfrak{c}$ is the cardinality of the continuum.

OPEN QUESTIONS. Is the variety of topological groups generated by the class of all $k_\omega$-groups equal to the variety of topological groups generated by the free topological group on $[0,1]$? If not, is the variety of topological groups generated by all $k_\omega$-groups a singly-generated variety? (We note that it can be proved that the variety of topological groups generated by the class of all connected locally compact groups is singly-generated.)

## 6. Locally compact abelian groups and $k_\omega$-groups

In this section, we will prove the following part of the Main Theorem.

THEOREM B.

$$\begin{array}{ccc} & \mathcal{A} = \mathfrak{V}(\mathcal{B}) & \\ & / \quad \backslash & \\ \mathfrak{V}(\mathcal{D}_R) = \mathfrak{V}(\mathcal{L}_A) & & \mathfrak{V}(\mathcal{C}_\sigma) \\ | & & | \\ \mathfrak{V}(\mathcal{D}) & & \mathfrak{V}(\mathcal{K}_\omega) = \mathfrak{V}(FA[0,1]) \end{array}$$

Clearly, the class of all abelian $\sigma$-compact groups, $\mathcal{C}_\sigma$, contains the class of all abelian $k_\omega$ groups, $\mathcal{K}_\omega$. Therefore, we will compare $\mathfrak{V}(\mathcal{K}_\omega)$ with $\mathfrak{V}(\mathcal{C}_\sigma)$.

We will say a variety of topological groups is *closed under completions* if every Hausdorff topological group $G$ contained in the variety has a completion, $\widehat{G}$, in the variety.

PROPOSITION 6.1. *The variety $\mathfrak{V}(\mathcal{K}_\omega)$ is closed under completions.*

*Proof.* Let $G$ be a Hausdorff group in $\mathfrak{V}(\mathcal{K}_\omega)$. We know that $G \in SC\overline{Q}P(\mathcal{K}_\omega)$. Now, $\mathcal{K}_\omega$ is closed under $P$ and $\overline{Q}$ ([19, Results 4, 11]) and so $G$ is topologically isomorphic to a subgroup of $K = \prod_{i\in I} K_i$, a product of $k_\omega$-groups contained in $\overline{Q}P(\mathcal{K}_\omega)$. Now $K$ is complete, since $k_\omega$-groups are complete ([27, Theorem 2]), and so the closure of the group, $\overline{G}$, as a subgroup of $K$ is complete and $\overline{G} = \widehat{G} \in \mathfrak{V}(\mathcal{K}_\omega)$, giving the result. ∎

We will see shortly that $\mathfrak{V}(\mathcal{C}_\sigma)$ is not closed under completions. But first we introduce the concepts of a "miikika" class and a "palirika" (2) class of topological groups. This will allow us to prove results about a number of different classes without dealing with each one individually.

DEFINITION 6.2. Let $\Omega$ be a class of topological groups. Then $\Omega$ is said to be a *miikika* class if it is closed under $\overline{S}$, $\overline{Q}$ and $P$. Further, $\Omega$ is said to be a *palirika* class if it is a miikika class and is also closed under $S$.

EXAMPLE 6.3. The following classes of topological groups are examples of miikika classes that are not palirika classes.

(a) The class of all (abelian) $\sigma$-compact groups.
(b) The class of all locally compact (abelian) Hausdorff groups.
(c) The class of all (abelian) $k_\omega$-groups.
(d) The class of all (abelian) Lie groups.

The following classes of topological groups are examples of palirika classes.

---

(2) *Miikika* (´mi kı ka) and *palirika* (´pa lı rı ka) are the Paakantyi words for "clever" and "nice" respectively. Paakantyi is the language of Aboriginal people living around the Darling River mainly in southwestern New South Wales, Australia [74].

(e) The class of all discrete (abelian) groups.
(f) The class of all (abelian) separable metrizable groups.
(g) The class of all (abelian) topological groups of cardinality less than or equal to $m$ for some infinite cardinal $m$.

The following are examples of classes that are not miikika classes.

(h) The class of all (abelian) pro-Lie groups ([25, Corollary 4.11, p. 179]).
(i) The class of all complete topological groups (see [71]).

LEMMA 6.4.

(i) *Let $\Omega$ be a miikika class. Then any complete UFSS-group in $\mathfrak{V}(\Omega)$ is in $\Omega$. In particular, any discrete group or Banach space in $\mathfrak{V}(\Omega)$ is in $\Omega$.*
(ii) *Let $\Omega$ be a palirika class. Then any UFSS-group in $\mathfrak{V}(\Omega)$ is in $\Omega$.*

*Proof.* To prove part (i), let $G$ be a complete UFSS-group in $\mathfrak{V}(\Omega)$. By Remark 2.8 $G \in S\overline{Q}\,\bar{S}P(\Omega)$. As $G$ is complete, $G \in \bar{S}\,\overline{Q}\,\bar{S}P(\Omega)$. The result follows from Definition 6.2.

The proof of part (ii) is similarly trivial. ∎

PROPOSITION 6.5. *The variety $\mathfrak{V}(\mathcal{C}_\sigma)$ contains no infinite-dimensional Banach spaces. Further, any discrete group contained in $\mathfrak{V}(\mathcal{C}_\sigma)$ is countable.*

*Proof.* Let $B$ be a Banach space and $D$ a discrete group, both contained in $\mathfrak{V}(\mathcal{C}_\sigma)$. Noting that $\mathcal{C}_\sigma$ is a miikika class, both $B$ and $D$ are $\sigma$-compact by Lemma 6.4(i). We see immediately that $D$ is countable.

Suppose $B$ is infinite-dimensional. As $B$ is $\sigma$-compact metrizable, $B$ is separable. Now Theorem 5.2 of Chapter VI in [5] says that any separable Fréchet space, in this case $B$, is homeomorphic to $\mathbb{R}^{\aleph_0}$. However, it is clear that $\mathbb{R}^{\aleph_0}$ is not $\sigma$-compact, giving us a contradiction. Thus $B$ cannot be infinite-dimensional.

(An alternative proof using less powerful machinery goes as follows: A compact subset of an infinite-dimensional Banach space is nowhere dense, otherwise, being closed, it would contain some ball with positive radius, and it is well known that a ball in an infinite-dimensional normed vector space is never compact. It is then enough to apply the Baire Theorem that a complete metric space is not a countable union of nowhere dense subsets. This shows that an infinite-dimensional Banach space is not $\sigma$-compact.) ∎

Note that from Proposition 6.5, we see that $\mathfrak{V}(\mathcal{C}_\sigma)$ is properly contained in $\mathcal{A}$.

PROPOSITION 6.6. *The class $\mathcal{C}_\sigma$ contains all countable-dimensional (real) topological vector spaces.*

*Proof.* Let $N$ be a countable-dimensional (real) topological vector space and let $S$ be countable vector space basis for $N$. Then $N = \operatorname{gp}(\mathbb{R}.S)$ where $\mathbb{R}.S = \bigcup_{s\in S}\{rs : r \in \mathbb{R}\}$. Further, $\{rs : r \in \mathbb{R}\} = \bigcup_{i=1}^{\infty}\{rs : r \in [-i,i]\}$. Clearly, $\{rs : r \in [-i,i]\}$ is compact for each $s \in S$ and so $N$ is a countable union of compact sets and hence is in $\mathcal{C}_\sigma$. ∎

PROPOSITION 6.7. *The variety $\mathfrak{V}(\mathcal{C}_\sigma)$ is not closed under completions.*

*Proof.* Let $N$ be a countably infinite-dimensional normed vector space. By Proposition 6.6, $N \in \mathfrak{V}(\mathcal{C}_\sigma)$. However, $\widehat{N}$, the completion of $N$, is an infinite-dimensional Banach space and by Proposition 6.5 is not in $\mathfrak{V}(\mathcal{C}_\sigma)$, giving the result. ∎

The following corollary follows immediately from Propositions 6.1 and 6.7, and the fact that every $k_\omega$-space is $\sigma$-compact.

Corollary 6.8. *The variety $\mathfrak{V}(\mathcal{K}_\omega)$ is properly contained in $\mathfrak{V}(\mathcal{C}_\sigma)$.*

We now turn our attention to a rich variety of topological groups, $\mathfrak{V}(\mathcal{L}_A)$. First recall the Principal Structure Theorem for locally compact abelian groups.

Theorem 6.9 (Principal Structure Theorem, [23, Theorem 24.30]). *Every locally compact abelian group $G$ is topologically isomorphic to $\mathbb{R}^n \times H$, where $H$ is a locally compact abelian group containing a compact open subgroup and $n$ is a non-negative integer.*

Proposition 6.10. *The variety $\mathfrak{V}(\mathcal{L}_A)$ properly contains $\mathfrak{V}(\mathcal{D})$ and neither is a singly-generated variety of topological groups.*

*Proof.* Clearly, $\mathfrak{V}(\mathcal{D}) \subseteq \mathfrak{V}(\mathcal{L}_A)$.

Suppose $\mathbb{R} \in \mathfrak{V}(\mathcal{D})$. As $\mathcal{D}$ is a miikika class and $\mathbb{R}$ is a complete UFSS-group, by Lemma 6.4(i), $\mathbb{R}$ must be discrete. This is clearly a contradiction. So $\mathbb{R} \notin \mathfrak{V}(\mathcal{D})$ and $\mathfrak{V}(\mathcal{D})$ is a proper subvariety of $\mathfrak{V}(\mathcal{L}_A)$.

Now, suppose $\mathfrak{V}(\mathcal{D})$ is singly-generated. Then it is a $T(m)$-variety for some infinite cardinal $m$ (Remark 2.10). However, there exists a discrete group of cardinal $m$ which is not a $T(m)$-group and we have a contradiction. Therefore, $\mathfrak{V}(\mathcal{D})$ is not singly-generated and the result follows from Remark 2.10. ∎

The next proposition shows that it suffices to add the locally compact abelian group $\mathbb{R}$ to the class of all discrete abelian groups to obtain the variety generated by $\mathcal{L}_A$. We use the following lemma, which is folklore.

Lemma 6.11 ([16, Lemma 3.1]). *Let $G$ be a locally compact abelian group containing an open compact subgroup. Then $G$ is topologically isomorphic to a closed subgroup of a product $\mathbb{T}^m \times D$, where $m$ is a cardinal and $D$ is a discrete abelian group.*

*Proof.* Let $K$ be an open compact subgroup of $G$. Then there exists an embedding $\phi : K \to \mathbb{T}^m$ for some cardinal $m$. As $\mathbb{T}^m$ is divisible, $\phi$ can be extended to a homomorphism $\Phi : G \to \mathbb{T}^m$. Now, $\Phi$ is continuous as $K$ is open in $G$ and the restriction $\phi$ to $K$ is continuous. Let $h : G \to G/K$ be the canonical homomorphism to the quotient group $D = G/K$. Note that as $K$ is open, $D$ is discrete. Consider the mapping $f : G \to \mathbb{T}^m \times D$ given by $f(g) = \langle \Phi(g), h(g) \rangle$. Clearly, $f$ is a continuous homomorphism. Further, the kernel of $f$ is the identity of $G$ as $\phi$ is one-to-one on $K$. Since the restriction of $f$ to the open subgroup $K$ coincides with the embedding $\phi : K \to \mathbb{T}^m$, $f$ is also an open mapping, and hence an embedding, which completes the proof. ∎

Proposition 6.12. *The variety $\mathfrak{V}(\mathcal{L}_A)$ equals $\mathfrak{V}(\mathcal{D}_R)$, the variety generated by the class of all discrete abelian groups and $\mathbb{R}$.*

*Proof.* The result follows from Proposition 6.10, the Principal Structure Theorem, and Lemma 6.11. ∎

To establish that $\mathfrak{V}(\mathcal{L}_A)$ is not the variety of all abelian topological groups, we consider connected abelian topological groups that are contained in $\mathfrak{V}(\mathcal{L}_A)$.

PROPOSITION 6.13. *Let $G$ be a complete Hausdorff topological group contained in $\mathfrak{V}(\mathcal{L}_A)$. Then the connected component of the identity of $G$ is topologically isomorphic to $\mathbb{R}^{\aleph} \times K$, where $\aleph$ is some cardinal number and $K$ is a connected compact Hausdorff abelian topological group.*

*Proof.* Let $G_0$ denote the connected component of the identity of $G$. By Theorem 2.7, $G \in SC\overline{Q}P(\mathcal{L}_A)$ and noting that $G_0$ is a closed subgroup of the complete $G$, we have $G_0 \in \bar{S}C\overline{Q}P(\mathcal{L}_A) = \bar{S}C(\mathcal{L}_A)$. Now by [54] (Theorem on p.123), a connected closed subgroup of a product of locally compact Hausdorff abelian groups is topologically isomorphic to $\mathbb{R}^{\aleph} \times K$, for some cardinal number $\aleph$ and some compact connected Hausdorff abelian group $K$, giving the result. ∎

LEMMA 6.14 ([8, Corollary 1 to Theorem 2]). *Let $\Omega$ be a family of locally compact abelian Hausdorff groups. Then every Hausdorff topological group $G$ in $\mathfrak{V}(\Omega)$ has a completion $\widehat{G}$ in $\mathfrak{V}(\Omega)$.*

*Proof.* By Theorem 2.7, $G \in SC\overline{Q}P(\Omega)$. Now, the class of all locally compact abelian Hausdorff groups is closed under $\overline{Q}$, and $P$ and so $G$ is topologically isomorphic to a subgroup of a product, $H$, of locally compact abelian Hausdorff groups contained in $\overline{Q}P(\Omega)$. As every product of locally compact abelian Hausdorff groups is complete and every closed subgroup of a complete group is complete, the completion, $\widehat{G}$, of $G$ satisfies $\widehat{G} = \overline{G}$ as a subgroup of $H$. Thus, $\widehat{G} \in \mathfrak{V}(\Omega)$. ∎

REMARK 6.15. By Example 5.1 of [25] (p. 212) every locally compact abelian group is an abelian pro-Lie group. Further, by Definition C of [25] (p. 161) every pro-Lie group is a closed subgroup of a product of (finite-dimensional real) Lie groups. So, $\mathfrak{V}(\mathcal{L}_A) = \mathfrak{V}(\mathcal{A}_{\mathcal{L}})$, where $\mathcal{A}_{\mathcal{L}}$ is the class of all abelian Lie groups. If a Hausdorff group $G$ is contained in $\mathfrak{V}(\mathcal{A}_{\mathcal{L}})$ then $G \in SC\overline{Q}P(\mathcal{A}_{\mathcal{L}})$. As $\mathcal{A}_{\mathcal{L}}$ is closed under $\overline{Q}$ and $P$, $G \in SC(\mathcal{A}_{\mathcal{L}})$ and as the product of pro-Lie groups is a pro-Lie group, $G$ is a subgroup of a pro-Lie group. If $G$ is a complete Hausdorff topological group in $\mathfrak{V}(\mathcal{L}_A)$, then $G$ is a pro-Lie group as, by Theorem 3.35 of [25] (p. 158), every closed subgroup of a pro-Lie group is a pro-Lie group. By Lemma 5.12 of [25] (p. 221), every connected pro-Lie group is topologically isomorphic to $\mathbb{R}^{\aleph} \times K$, for some cardinal number $\aleph$ and some compact connected Hausdorff abelian group $K$. This remark reproves Lemma 6.14 but extends the result to show that every complete Hausdorff topological group in $\mathfrak{V}(\mathcal{L}_A)$ is a pro-Lie group.

We now show the connected abelian topological groups contained in $\mathfrak{V}(\mathcal{L}_A)$ are in the variety of topological groups generated by $\mathbb{R}$.

PROPOSITION 6.16. *Let $G$ be a Hausdorff connected abelian topological group contained in $\mathfrak{V}(\mathcal{L}_A)$. Then $G \in \mathfrak{V}(\mathbb{R})$.*

*Proof.* By Lemma 6.14, the completion, $\widehat{G}$, of $G$ is in $\mathfrak{V}(\mathcal{L}_A)$. Then, by Proposition 6.13, $\widehat{G}$ is topologically isomorphic to $\mathbb{R}^{\aleph} \times K$, where $K$ is compact connected Hausdorff abelian and $\aleph$ is some cardinal number. Therefore $G \in \mathfrak{V}(\mathbb{R})$ since every compact Hausdorff abelian topological group is topologically isomorphic to a subgroup of a product of copies of $\mathbb{T}$ ([56, Corollary 1 to Theorem 14]) and is thus contained in $\mathfrak{V}(\mathbb{R})$. Therefore, $G \in \mathfrak{V}(\mathbb{R})$. ∎

THEOREM 6.17. *A normed vector space contained in $\mathfrak{V}(\mathcal{L}_A)$ is finite-dimensional.*

*Proof.* By Lemma 6.14, the completion, $\widehat{G}$, of $G$ (a Banach space) is contained in $\mathfrak{V}(\mathcal{L}_A)$. As $\mathcal{L}_A$ is a miikika class, $\widehat{G} \in \mathcal{L}_A$. Therefore, $\widehat{G}$, indeed $G$, is finite-dimensional. ∎

From Proposition 6.16 and Theorem 6.17, we see that $\mathfrak{V}(\mathcal{L}_A)$ is not equal to $\mathcal{A}$. Indeed, $\mathfrak{V}(\mathcal{L}_A)$ is relatively small, despite not being singly-generated.

To complete the proof of Theorem B, we establish that there are no further containment relationships between $\mathfrak{V}(\mathcal{K}_\omega)$, $\mathfrak{V}(\mathcal{C}_\sigma)$, $\mathfrak{V}(\mathcal{D})$ and $\mathfrak{V}(\mathcal{L}_A)$.

LEMMA 6.18. *Let $\mathbb{Z}$ be the discrete additive group of integers. Then $FA(\mathbb{Z})$ is topologically isomorphic to a subgroup of $FA[0,1]$.*

*Proof.* By Theorem 1 of [32], $FA(0,1)$ is topologically isomorphic to a subgroup of $FA[0,1]$. Now $\mathbb{Z}$ is homeomorphic to a closed subspace of $(0,1)$, a $k_\omega$-space. Thus, by Theorem 3 of [35], the group algebraically generated by $\mathbb{Z}$ in $FA(0,1)$ is topologically isomorphic to $FA(\mathbb{Z})$. Therefore, $FA(\mathbb{Z})$ is topologically isomorphic to a subgroup of $FA[0,1]$. ∎

LEMMA 6.19 (cf. [43, Lemma 2]). *Let $D$ be a discrete group in $\mathfrak{V}(\mathbb{R})$. Then $D$ is finitely generated.*

*Proof.* By Remark 2.8, $D \in S\overline{Q}P(\mathbb{R})$. Therefore, $D$ is a subgroup of $H$, where $H$ is a compactly generated locally compact Hausdorff abelian topological group. By Theorem 9.14 of [23], $H$ is topologically isomorphic to $\mathbb{R}^a \times \mathbb{Z}^b \times F$, where $F$ is a compact group and $a$ and $b$ are non-negative integers.

Let $Y$ be the subgroup of $D$ consisting of all elements of finite order and $X$ be the complement of $Y$ in $D$. We will show that both $Y$ and the group generated by $X$, $\mathrm{gp}(X)$, are finitely generated, and hence so too is $D$.

Let $S$ be any finite subset of $X$. Then the group generated by $S$ is $\mathbb{Z}^r$ for some non-negative integer $r$. By Theorem 9.12 of [23], $r \le a+b$. That is, any finitely generated subgroup of $\mathrm{gp}(X)$ is generated by $a+b$ elements. Thus $\mathrm{gp}(X)$ is finitely generated.

Let $p_1$, $p_2$ and $p_3$ be the natural projection mappings of $H$ onto $\mathbb{R}^a$, $\mathbb{Z}^b$ and $F$, respectively. Let $y$ be any element in $Y$. Then, since $y$ is of finite order, $p_1(y) = e_1$ and $p_2(y) = e_2$, where $e_1$ and $e_2$ denote identity elements. Thus $p_3(Y)$ is topologically isomorphic to $Y$. That is, $p_3(Y)$ is a discrete subgroup of $F$. Since $F$ is compact, this implies $p_3(Y)$, and hence $Y$, is a finite set. The proof is complete. ∎

PROPOSITION 6.20. *The topological group $FA[0,1]$ is not in $\mathfrak{V}(\mathcal{L}_A)$.*

*Proof.* Suppose $FA[0,1] \in \mathfrak{V}(\mathcal{L}_A)$. We note that any group algebraically generated by a connected set containing the identity is connected. Thus as $[0,1]$ is connected Hausdorff, $FA[0,1]$ is also connected Hausdorff and so from Proposition 6.16, $FA[0,1] \in \mathfrak{V}(\mathbb{R})$. Now any discrete subgroup $D$ of $FA[0,1]$ is also in $\mathfrak{V}(\mathbb{R})$ and hence by Lemma 6.19, $D$ is finitely generated. However, by Lemma 6.18, $FA(\mathbb{Z})$ is a discrete subgroup of $FA[0,1]$ which is not finitely generated. Thus, we have a contradiction and so $FA[0,1]$ is not contained in $\mathfrak{V}(\mathcal{L}_A)$. ∎

REMARK 6.21. Proposition 6.20 tells us that $\mathfrak{V}(\mathcal{C}_\sigma)$ and $\mathfrak{V}(\mathcal{K}_\omega)$ are not subvarieties of $\mathfrak{V}(\mathcal{L}_A)$. Further, as $\mathfrak{V}(\mathcal{L}_A)$ and $\mathfrak{V}(\mathcal{D})$ contain uncountable discrete groups which are not in $\mathfrak{V}(\mathcal{C}_\sigma)$ we see that $\mathfrak{V}(\mathcal{D})$ and $\mathfrak{V}(\mathcal{L}_A)$ are not contained in $\mathfrak{V}(\mathcal{C}_\sigma)$.

This completes the proof of Theorem B.

## 7. Marrang properties

DEFINITION 7.1. Let $\mathcal{P}$ be a property of topological spaces. A topological space $X$ is said to be *locally* $\mathcal{P}$ if each neighbourhood of each point of $X$ contains a neighbourhood of that point with property $\mathcal{P}$.

REMARK 7.2. Clearly a topological group is locally $\mathcal{P}$ if each neighbourhood of the identity contains a neighbourhood of the identity with property $\mathcal{P}$.

Local properties we consider in this paper share a number of useful characteristics. Therefore, we introduce the concept of a "marrang"(3) property, which has these characteristics. We are then able to establish a number of general results which can be applied later.

DEFINITION 7.3. Let $\mathcal{P}$ be a topological property. Then $\mathcal{P}$ is said to be a *marrang property* if

- (i) $\mathcal{P}$ is preserved under finite products;
- (ii) $\mathcal{P}$ is preserved under quotients;
- (iii) every singleton space $\{x\}$ has property $\mathcal{P}$;
- (iv) any topological group that has property $\mathcal{P}$ is also locally $\mathcal{P}$;
- (v) any topological group $G$ algebraically generated by a subspace $X$ with property $\mathcal{P}$, also has property $\mathcal{P}$.

LEMMA 7.4. *Let $\mathcal{P}$ be a marrang property. Then a topological group $G$ is locally $\mathcal{P}$ if and only if $G$ has an open subgroup with the property $\mathcal{P}$.*

*Proof.* Let $G$ have an open subgroup $H$ with the property $\mathcal{P}$. By Definition 7.3(iv), $H$ is locally $\mathcal{P}$. Let $U$ be a neighbourhood of the identity $e$ of $G$. Then $U \cap H$ is clearly a neighbourhood of $e$ in $H$ and so contains a neighbourhood of $e$ with the property $\mathcal{P}$. Therefore, $U$ contains a neighbourhood of $e$ with property $\mathcal{P}$, implying $G$ is locally $\mathcal{P}$.

(3) *Marrang* (´ma [r]ang –rolled r) is the Wiradjuri word for "good" or "friend". Wiradjuri is one of the largest Aboriginal language groupings in New South Wales, Australia [74].

Conversely, if $G$ is a locally $\mathcal{P}$ group, then there exists a neighbourhood $U$ of the identity with property $\mathcal{P}$. From Definition 7.3(v), the open subgroup $\mathrm{gp}(U)$ has property $\mathcal{P}$ and the result follows. ∎

The previous lemma gives an alternative definition for a locally $\mathcal{P}$ *group* when $\mathcal{P}$ is a marrang property. We shall use it in all cases without reference.

Remark 7.5.

(1) Let $\mathcal{P}$ be a marrang property. Then a connected locally $\mathcal{P}$ group has the property $\mathcal{P}$.
(2) Let $\mathcal{P}$ be a marrang property and $X$ a completely regular space with property $\mathcal{P}$. Then $FA(X)$ has property $\mathcal{P}$ from Definition 7.3(v).
(3) It follows from Definition 7.3(iii) that if $\mathcal{P}$ is a marrang property, every discrete space is locally $\mathcal{P}$.

Example 7.6. For an infinite cardinal $m$, a *locally-$m$ group* is a topological group with a neighbourhood of the identity of cardinality less than or equal to $m$. In other words, a group $G$ is locally-$m$ if and only if it is locally $\mathcal{P}_1$, where $\mathcal{P}_1$ is the property of having cardinality less than or equal to $m$.

*The property $\mathcal{P}_1$ of having cardinality less than or equal to $m$ is a marrang property.*

Note that $\sigma$-compactness and separability will be seen to be marrang properties.

Lemma 7.7. *If $\mathcal{P}$ is a marrang property, then the class of all abelian locally $\mathcal{P}$ groups is closed under $Q$ and $P$.*

*Proof.* Let $G$ be a locally $\mathcal{P}$ group with open subgroup $H$ that has property $\mathcal{P}$. Clearly, if $F$ is a quotient topological group of $G$ with quotient homomorphism $f : G \to F$, then $f(H)$ is an open subgroup of $F$ with property $\mathcal{P}$ and so $F$ is locally $\mathcal{P}$. Further, let $G_i$ be a locally $\mathcal{P}$ group for each $i = 1, \dots, n$, and let $H_i$ be an open subgroup of $G_i$ with property $\mathcal{P}$. Clearly $\prod_{i=1}^n H_i$ is an open subgroup of $\prod_{i=1}^n G_i$ that has property $\mathcal{P}$ and so $\prod_{i=1}^n G_i$ is locally $\mathcal{P}$. ∎

Notation. Let $\mathcal{P}$ be a marrang property. We shall denote by $\mathfrak{P}$ the class of all abelian topological groups with property $\mathcal{P}$ and by $\mathcal{L}_{\mathcal{P}}$ the class of all abelian locally $\mathcal{P}$ topological groups.

Proposition 7.8. *Let $\mathcal{P}$ be a marrang property and $G$ a connected Hausdorff abelian topological group. Then $G \in \mathfrak{V}(\mathcal{L}_{\mathcal{P}})$ if and only if $G \in \mathfrak{V}(\mathfrak{P})$.*

*Proof.* Clearly, if $G \in \mathfrak{V}(\mathfrak{P})$, then $G \in \mathfrak{V}(\mathcal{L}_{\mathcal{P}})$ (see Definition 7.3(iv)).

Let $G \in \mathfrak{V}(\mathcal{L}_{\mathcal{P}})$, then $G \in SC\overline{Q}P(\mathcal{L}_{\mathcal{P}})$ (see Theorem 2.7). By Lemma 7.7, $\mathcal{L}_{\mathcal{P}}$ is closed under $Q$ and $P$ and so $G \in SC(\mathcal{L}_{\mathcal{P}})$. Therefore, $G$ is topologically isomorphic to a subgroup of $\prod_{i\in I} L_i$ for some index set $I$, where each $L_i$ has an open (and closed) subgroup, $H_i$, with property $\mathcal{P}$. For $j \in I$, let $p_j$ be the projection map from $\prod_{i\in I} L_i$ onto $L_j$. As $p_j(G)$ is connected, $p_j(G)$ is a subgroup of $H_j$. Therefore, $G$ is topologically isomorphic to a subgroup of $\prod_{i\in I} H_i$, where each $H_i$ has property $\mathcal{P}$. Thus, $G \in \mathfrak{V}(\mathfrak{P})$, giving the result. ∎

As an immediate consequence of Proposition 7.8, we have the following result concerning normed vector spaces.

COROLLARY 7.9. *Let $\mathcal{P}$ be a marrang property and $N$ a normed vector space. Then $N \in \mathfrak{V}(\mathcal{L}_{\mathcal{P}})$ if and only if $N \in \mathfrak{V}(\mathfrak{P})$.*

THEOREM 7.10. *Let $\mathcal{P}$ be a marrang property. Then $\mathfrak{V}(\mathcal{L}_{\mathcal{P}}) = \mathfrak{V}(\mathfrak{P} \cup \mathcal{D})$.*

*Proof.* As all abelian topological groups with property $\mathcal{P}$ and all discrete abelian groups are locally $\mathcal{P}$ (see Remark 7.5(3)), $\mathfrak{V}(\mathfrak{P} \cup \mathcal{D}) \subseteq \mathfrak{V}(\mathcal{L}_{\mathcal{P}})$.

Let $G \in \mathcal{L}_{\mathcal{P}}$. Then $G$ has an open subgroup $H$ which has property $\mathcal{P}$. Note that $G$ has the finest group topology which induces the given topology on $H$. Choose one element out of each coset of $H$ different from $H$ and form the set $D$ which is clearly discrete and disjoint from $H$. Thus, $H \sqcup D = H \cup D$ is a subspace of $G$, and $G$ has the finest topology which induces the given topology on $H \sqcup D$, for if there were a finer topology on $G$ that induces the given topology on $H \sqcup D$, it would also induce the given topology on $H$, which is a contradiction. Further, $G = \mathrm{gp}(H \cup D)$ and so by Lemma 3.5, $G$ is a quotient group of $FA(H \sqcup D)$. By Theorem 6 of [48] we find that $FA(H \sqcup D)$ is topologically isomorphic to $FA(H) \times FA(D)$. By Remark 7.5(2), $FA(H)$ has property $\mathcal{P}$ and we know $FA(D)$ is discrete. So $FA(H) \times FA(D) \in \mathfrak{V}(\mathfrak{P} \cup \mathcal{D})$. Therefore, $G \in \mathfrak{V}(\mathfrak{P} \cup \mathcal{D})$ and the result follows. ■

## 8. Locally $\sigma$-compact groups

In this section, we extend Theorem B to the following.

THEOREM C.

$$\begin{array}{ccc}
 & \mathcal{A} = \mathfrak{V}(\mathcal{B}) & \\
 & | & \\
 & \mathfrak{V}(\mathcal{C}_\sigma \cup \mathcal{L}_A) = \mathfrak{V}(\mathcal{C}_\sigma \cup \mathcal{D}) = \mathfrak{V}(\mathcal{L}_\sigma) & \\
 / & & \backslash \\
\mathfrak{V}(\mathcal{D}_R) = \mathfrak{V}(\mathcal{L}_A) & & \mathfrak{V}(\mathcal{C}_\sigma) \\
| & & | \\
\mathfrak{V}(\mathcal{D}) & & \mathfrak{V}(\mathcal{K}_\omega) = \mathfrak{V}(FA[0,1])
\end{array}$$

DEFINITION 8.1. A topological space $G$ is said to be *locally $\sigma$-compact* if each neighbourhood of each point in $G$ contains a $\sigma$-compact neighbourhood of that point.

PROPOSITION 8.2. *The property of $\sigma$-compactness is a marrang property.*

*Proof.* Clearly, $\sigma$-compactness satisfies Definition 7.3(i)–(iii).

Let $G$ be a $\sigma$-compact group and $N$ an open neighbourhood of the identity $e \in G$. Then, as $G$ is a regular space, $N$ contains a closed neighbourhood $C$ of $e$. So, $C$ is $\sigma$-compact. Therefore, $G$ is locally $\sigma$-compact.

Let $X$ be a $\sigma$-compact space and $H = \mathrm{gp}(X)$. Then $H = \bigcup_{n=1}^{\infty}(X \cup X^{-1})^n$ and so is obviously $\sigma$-compact, thereby completing the proof. ∎

From Remark 7.5(1), we see that a connected locally $\sigma$-compact group $G$ is $\sigma$-compact. Further, the class of all abelian locally $\sigma$-compact groups, $\mathcal{L}_\sigma$, is closed under $Q$ and $P$ (see Lemma 7.7).

Example 8.3. Every locally compact Hausdorff abelian group $G$ is locally $\sigma$-compact as by the Principal Structure Theorem ([56, Theorem 25]) $G$ has an open subgroup topologically isomorphic to the $\sigma$-compact group $\mathbb{R}^n \times K$, $K$ a compact abelian group and $n$ a non-negative integer.

Remark 8.4. It is of interest to know also that $\bar{S}$ preserves the property of locally $\sigma$-compact. This can be seen by taking a locally $\sigma$-compact group $G$ with open $\sigma$-compact subgroup $H$ and $K$ a closed subgroup of $G$. Then $K \cap H$ is a closed subgroup of $H$ and is therefore $\sigma$-compact. So $K$ has $K \cap H$ as an open $\sigma$-compact subgroup and so is locally $\sigma$-compact.

Proposition 8.5. *The variety $\mathfrak{V}(\mathcal{L}_\sigma)$ contains no infinite-dimensional Banach spaces.*

*Proof.* Let $B$ be a Banach space in $\mathfrak{V}(\mathcal{L}_\sigma)$. By Corollary 7.9, $B \in \mathfrak{V}(\mathcal{C}_\sigma)$. The result follows from Proposition 6.5. ∎

Proposition 8.5 shows that $\mathfrak{V}(\mathcal{L}_\sigma)$ is a proper subvariety of $\mathcal{A}$.

In Proposition 6.7 we saw that the variety generated by the class of all abelian $\sigma$-compact groups is not closed under completions. Using exactly the same argument, we have the following result.

Proposition 8.6. *The variety $\mathfrak{V}(\mathcal{L}_\sigma)$ is not closed under completions.*

Proposition 8.7. *The variety $\mathfrak{V}(\mathcal{L}_\sigma)$ properly contains both $\mathfrak{V}(\mathcal{L}_A)$ and $\mathfrak{V}(\mathcal{C}_\sigma)$.*

*Proof.* From Example 8.3 it is clear that $\mathfrak{V}(\mathcal{L}_A)$ is contained in $\mathfrak{V}(\mathcal{L}_\sigma)$. Note that $\mathfrak{V}(\mathcal{L}_A)$ is closed under completions (see Lemma 6.14) while $\mathfrak{V}(\mathcal{L}_\sigma)$ is not, hence establishing proper containment.

As $\sigma$-compactness is a marrang property, $\mathfrak{V}(\mathcal{C}_\sigma)$ is contained in $\mathfrak{V}(\mathcal{L}_\sigma)$. Noting that $\mathfrak{V}(\mathcal{L}_\sigma)$ contains all discrete abelian groups (indeed, all groups in $\mathcal{L}_A$) while $\mathfrak{V}(\mathcal{C}_\sigma)$ contains no uncountable discrete abelian groups (Proposition 6.5), we again have proper containment. ∎

Theorem 8.8. *Consider the variety $\mathfrak{V}(\mathcal{L}_\sigma)$. Then*

$$\mathfrak{V}(\mathcal{L}_\sigma) = \mathfrak{V}(\mathcal{C}_\sigma \cup \mathcal{D}) = \mathfrak{V}(\mathcal{C}_\sigma \cup \mathcal{L}_A).$$

*Proof.* By Proposition 8.2, $\sigma$-compactness is a marrang property. Therefore, by Theorem 7.10, $\mathfrak{V}(\mathcal{L}_\sigma) = \mathfrak{V}(\mathcal{C}_\sigma \cup \mathcal{D})$. As all abelian $\sigma$-compact groups and all locally compact Hausdorff abelian groups are locally $\sigma$-compact (see Example 8.3), we see that $\mathfrak{V}(\mathcal{C}_\sigma \cup \mathcal{D}) \subseteq \mathfrak{V}(\mathcal{C}_\sigma \cup \mathcal{L}_A) \subseteq \mathfrak{V}(\mathcal{L}_\sigma)$ and the result follows. ■

This completes the proof of Theorem C.

## 9. Separable and locally separable groups

In this section, we extend Theorem C to the following.

THEOREM D.

$$\mathcal{A} = \mathfrak{V}(\mathcal{B})$$

$$\mathfrak{V}(\mathcal{L}_\mathcal{S}) = \mathfrak{V}(\mathcal{S} \cup \mathcal{D}) = \mathfrak{V}(\mathcal{S} \cup \mathcal{L}_\sigma)$$

$$\mathfrak{V}(\mathcal{C}_\sigma \cup \mathcal{L}_A) = \mathfrak{V}(\mathcal{C}_\sigma \cup \mathcal{D}) = \mathfrak{V}(\mathcal{L}_\sigma) \qquad \mathfrak{V}(\mathcal{S}) = \mathfrak{V}(\mathcal{B}_\mathcal{S}) = \mathfrak{V}(\ell_1)$$

$$\mathfrak{V}(\mathcal{D}_R) = \mathfrak{V}(\mathcal{L}_A) \qquad \mathfrak{V}(\mathcal{C}_\sigma)$$

$$\mathfrak{V}(\mathcal{D}) \qquad \mathfrak{V}(\mathcal{K}_\omega) = \mathfrak{V}(FA[0,1])$$

We will make use of the following well-known result, the proof of which is straightforward.

LEMMA 9.1. *Let $G$ be an abelian topological group algebraically generated by a separable subspace $X$. Then $G$ is separable.*

The next separability result concerning $FA(X)$ follows immediately from Lemma 9.1

COROLLARY 9.2. *Let $(X, \rho)$ be a separable pseudometrizable (hence completely regular) topological space. Let $F$ be the free abelian group on $X \setminus \{e\}$ for some $e \in X$, and let $\rho'$ be the Graev extension of $\rho$ onto $F$. Then $(F, \rho')$ is separable.*

REMARK 9.3. We note that open subspaces of separable spaces are separable and all subspaces of separable metrizable spaces are separable.

Recall the fact that every separable Banach space is a quotient group of the separable Banach space $\ell_1$ [15]. Thus we have the following result.

Proposition 9.4. *The variety of topological groups, $\mathfrak{V}(\ell_1)$, generated by the topological group underlying the separable Banach space $\ell_1$ is precisely $\mathfrak{V}(\mathcal{B}_{\mathcal{S}})$.*

Remark 9.5. Let $(X, d)$ be a metric space and let $F$ be the free abelian group on the set $X \setminus \{e\}$, for $e \in X$. Let $d'$ be the Graev extension of $d$ to $F$. In Corollary 3.4 of [38] it was proved that $(F, d')$ is a topological group which is topologically isomorphic to a subgroup of a Banach space.

Proposition 9.6. *Let $X$ be a separable completely regular topological space. Then $FA(X)$ is contained in $\mathfrak{V}(\ell_1)$.*

*Proof.* Let $\{\rho_i : i \in I\}$ be the family of all continuous pseudometrics on $X$. Clearly each $(X, \rho_i)$ is a separable space. Let $(Y_i, d_i)$ be the metrification of $(X, \rho_i)$ for each $i \in I$. Then $(Y_i, d_i)$ is separable metrizable. Further, let $F_i$ be the free abelian group on $Y_i \setminus \{e_i\}$ and let $d'_i$ be the Graev extension of $d_i$ onto $F_i$. From Corollary 9.2, $(F_i, d'_i)$ is separable and by Remark 9.5, $(F_i, d'_i)$ is a topological group which is topologically isomorphic to a subgroup of a Banach space, indeed, a separable Banach space. Therefore, $(F_i, d'_i) \in \mathfrak{V}(\mathcal{B}_{\mathcal{S}}) = \mathfrak{V}(\ell_1)$. Now $FA(X)$ is topologically isomorphic to a subgroup of the product $H = \prod_{i\in I}(|FA(X)|, \rho'_i)$, where $|FA(X)|$ is the group underlying $FA(X)$ and $\rho'_i$ is the Graev extension of $\rho_i$ for each $i \in I$ (see Lemma 3.3). By Proposition 4.4, $(F_i, d'_i)$ is the metrification of $(|FA(X)|, \rho'_i)$ and so $H$ can be embedded in the product $\prod_{i\in I}(F_i, d'_i) \times L$, where $L$ is an indiscrete abelian group. Now, $\mathbb{R}$ is a separable Banach space, so $\mathbb{R}, \mathbb{T} \in \mathfrak{V}(\ell_1)$. Therefore, by Proposition 2.2, $L$ is contained in $\mathfrak{V}(\ell_1)$, thus, $FA(X)$ can be embedded as a topological group in a product of topological groups contained in $\mathfrak{V}(\ell_1)$ giving $FA(X) \in \mathfrak{V}(\ell_1)$. ■

Theorem 9.7. *$\mathfrak{V}(\ell_1) = \mathfrak{V}(\mathcal{B}_{\mathcal{S}}) = \mathfrak{V}(\mathcal{S})$.*

*Proof.* We have already established that $\mathfrak{V}(\ell_1) = \mathfrak{V}(\mathcal{B}_{\mathcal{S}})$.

Clearly, $\mathfrak{V}(\ell_1) \subseteq \mathfrak{V}(\mathcal{S})$. Let $G$ be an abelian separable topological group. By Proposition 9.6, $FA(G) \in \mathfrak{V}(\ell_1)$. Now $G$ is a quotient group of $FA(G)$ and so $G \in \mathfrak{V}(\ell_1)$. Therefore, $\mathcal{S} \subseteq \mathfrak{V}(\ell_1)$ and thus $\mathfrak{V}(\mathcal{S}) = \mathfrak{V}(\ell_1)$. ■

Although $\mathfrak{V}(\ell_1)$ contains all separable topological groups, it does not contain any non-separable normed vector spaces.

Lemma 9.8. *Any normed vector space contained in $\mathfrak{V}(\mathcal{S})$ is separable.*

*Proof.* Firstly, by Theorem 9.7, $\mathfrak{V}(\mathcal{S}) = \mathfrak{V}(\ell_1)$. Let $N$ be a normed vector space in $\mathfrak{V}(\ell_1)$. We note that $\ell_1$ is a separable metric topological group and the class of all separable metrizable groups is a palirika class. So by Lemma 6.4(ii), $N$ is separable. ■

Proposition 9.9. *Let $X$ be a $\sigma$-compact completely regular topological space. Then $FA(X)$ is contained in $\mathfrak{V}(\mathcal{S})$.*

*Proof.* Let $\{\rho_i : i \in I\}$ be the family of all continuous pseudometrics on $X$. Clearly each $(X, \rho_i)$ is a $\sigma$-compact pseudometrizable space and is therefore separable. We know that $FA(X)$ is topologically isomorphic to a subgroup of the product $\prod_{i\in I}(|FA(X)|, \rho'_i)$, where $|FA(X)|$ is the group underlying $FA(X)$ and $\rho_i'$ is the Graev extension of $\rho_i$ for each $i \in I$. By Corollary 9.2, $(|FA(X)|, \rho_i')$ is separable for each $i \in I$, and therefore contained in $\mathfrak{V}(\mathcal{S})$. Hence, $FA(X) \in \mathfrak{V}(\mathcal{S})$. ∎

THEOREM 9.10. *The variety $\mathfrak{V}(\mathcal{S})$ properly contains $\mathfrak{V}(\mathcal{C}_\sigma)$.*

*Proof.* Let $G$ be an abelian $\sigma$-compact group. By Proposition 9.9, $FA(G)$ is contained in $\mathfrak{V}(\mathcal{S})$. Further $G$ is a quotient group of $FA(G)$ and so contained in $\mathfrak{V}(\mathcal{S})$. Therefore, $\mathfrak{V}(\mathcal{C}_\sigma) \subseteq \mathfrak{V}(\mathcal{S})$. By Proposition 6.5, $\mathfrak{V}(\mathcal{C}_\sigma)$ only contains finite-dimensional Banach spaces. However, $\mathfrak{V}(\mathcal{S})$ contains $\ell_1$, and so $\mathfrak{V}(\mathcal{C}_\sigma)$ is a proper subvariety of $\mathfrak{V}(\mathcal{S})$. ∎

PROPOSITION 9.11. *The variety $\mathfrak{V}(\mathcal{S})$ is not contained in, nor does it contain, $\mathfrak{V}(\mathcal{L}_\sigma)$.*

*Proof.* By Proposition 8.5, $\mathfrak{V}(\mathcal{L}_\sigma)$ contains no infinite-dimensional Banach spaces and so $\mathfrak{V}(\mathcal{S})$ is not contained in $\mathfrak{V}(\mathcal{L}_\sigma)$. On the other hand, $\ell_1$ is a separable metrizable topological group and so has cardinality $\mathfrak{c}$ ($\ell_1$ is a subspace of $[0,1]^{\aleph_0}$; see [33, Chapter 4, Theorem 17]). Thus, $\ell_1$ is a $T(\mathfrak{c}^+)$-group, where $\mathfrak{c}^+$ is the smallest cardinal strictly greater than $\mathfrak{c}$, and so every topological group contained in $\mathfrak{V}(\mathcal{S}) = \mathfrak{V}(\ell_1)$ is a $T(\mathfrak{c}^+)$-group; that is, every discrete group contained in $\mathfrak{V}(\mathcal{S})$ has cardinality strictly less than $\mathfrak{c}^+$. However, $\mathfrak{V}(\mathcal{L}_\sigma)$ contains every discrete group and so is not contained in $\mathfrak{V}(\mathcal{S})$. ∎

We see now that although $\mathfrak{V}(\mathcal{S})$ contains $\mathfrak{V}(\mathcal{C}_\sigma)$, it does not contain $\mathfrak{V}(\mathcal{L}_A)$. Thus we introduce the concept of locally separable topological groups, to find a variety that contains all locally $\sigma$-compact groups and all separable abelian topological groups.

DEFINITION 9.12. A topological space $G$ is said to be *locally separable* if each neighbourhood of each point contains a separable neighbourhood of that point.

PROPOSITION 9.13. *The property of separability is a marrang property.*

*Proof.* Clearly, separability satisfies Definition 7.3(i)–(iii).

Let $G$ be a separable group and $H$ an open neighbourhood of $G$. As open subspaces of separable spaces are separable, $H$ is a separable neighbourhood of $e$. Therefore, $G$ is locally separable. Lemma 9.1 completes the proof. ∎

As a direct consequence of Proposition 9.13, $\mathcal{S} \subseteq \mathfrak{V}(\mathcal{L}_\mathcal{S})$. Also a connected locally separable group $G$ is separable. Further, the class of all abelian locally separable groups, $\mathcal{L}_\mathcal{S}$, is closed under $Q$ and $P$.

COROLLARY 9.14. *The variety $\mathfrak{V}(\mathcal{L}_\mathcal{S})$ properly contains $\mathfrak{V}(\mathcal{S})$.*

*Proof.* As separability is a marrang property, $\mathfrak{V}(\mathcal{S})$ is contained in $\mathfrak{V}(\mathcal{L}_\mathcal{S})$. Proper containment follows by noting that $\mathfrak{V}(\mathcal{L}_\mathcal{S})$ contains all discrete groups (Remark 7.5(3)) while $\mathfrak{V}(\mathcal{S})$ contains only discrete groups of cardinality strictly less than $\mathfrak{c}^+$ (see the proof of Proposition 9.11). ∎

THEOREM 9.15. *Consider the variety $\mathfrak{V}(\mathcal{L}_\mathcal{S})$. Then*

$$\mathfrak{V}(\mathcal{L}_\mathcal{S}) = \mathfrak{V}(\mathcal{S} \cup \mathcal{D}) = \mathfrak{V}(\mathcal{S} \cup \mathcal{L}_\sigma).$$

*Proof.* By Proposition 9.13, separability is a marrang property. Therefore, by Theorem 7.10, $\mathfrak{V}(\mathcal{L}_\mathcal{S}) = \mathfrak{V}(\mathcal{S} \cup \mathcal{D})$. Theorem 9.10 clearly implies that $\mathcal{C}_\sigma \subseteq \mathfrak{V}(\mathcal{S})$, and so $\mathfrak{V}(\mathcal{C}_\sigma \cup \mathcal{D}) \subseteq \mathfrak{V}(\mathcal{S} \cup \mathcal{D})$. Further, by Theorem 8.8, $\mathfrak{V}(\mathcal{C}_\sigma \cup \mathcal{D}) = \mathfrak{V}(\mathcal{L}_\sigma)$, giving $\mathcal{L}_\sigma \subseteq \mathfrak{V}(\mathcal{S} \cup \mathcal{D}) = \mathfrak{V}(\mathcal{L}_\mathcal{S})$. Thus, as $\mathcal{D} \subseteq \mathcal{L}_\sigma$, $\mathfrak{V}(\mathcal{S} \cup \mathcal{D}) \subseteq \mathfrak{V}(\mathcal{S} \cup \mathcal{L}_\sigma) \subseteq \mathfrak{V}(\mathcal{L}_\mathcal{S})$ and the result follows. ∎

Corollary 9.16. *The variety $\mathfrak{V}(\mathcal{L}_\sigma)$ is properly contained in $\mathfrak{V}(\mathcal{L}_\mathcal{S})$.*

*Proof.* From Theorem 9.15, $\mathfrak{V}(\mathcal{L}_\sigma) \subseteq \mathfrak{V}(\mathcal{L}_\mathcal{S})$. Recall that $\mathfrak{V}(\mathcal{L}_\sigma)$ contains no infinite-dimensional Banach spaces (Proposition 8.5). Now all separable Banach spaces are locally separable and contained in $\mathfrak{V}(\mathcal{L}_\mathcal{S})$. Therefore, $\mathfrak{V}(\mathcal{L}_\sigma)$ is properly contained in $\mathfrak{V}(\mathcal{L}_\mathcal{S})$. ∎

Remark 9.17. As separability is a marrang property, we can apply Corollary 7.9 to a normed vector space $N$ contained in $\mathfrak{V}(\mathcal{L}_\mathcal{S})$ and see that $N \in \mathfrak{V}(\mathcal{S})$. However, by Lemma 9.8, $N$ is separable. Therefore, $\mathfrak{V}(\mathcal{L}_\mathcal{S})$ clearly does not contain all normed vector spaces and so is properly contained in $\mathcal{A}$.

This completes the proof of Theorem D.

## 10. Locally-$m$ groups

In this section, we complete the proof of the Main Theorem. The "missing link" is a chain of varieties generated by classes of locally-$m$ groups, $m$ an infinite cardinal (see Example 7.6).

Proposition 10.1. *Let $m$ be an infinite cardinal. A normed vector space $N$ is in $\mathfrak{V}(\mathcal{L}_m)$ if and only if the cardinality of $N$ is less than or equal to $m$.*

*Proof.* Note that having cardinality less than or equal to $m$ is a marrang property, and the class, $\mathcal{M}_m$, of all abelian topological groups of cardinality less than or equal to $m$ is a palirika class. Let $N$ be a normed vector space. By Corollary 7.9, $N$ is in $\mathfrak{V}(\mathcal{L}_m)$ if and only if $N$ is in $\mathfrak{V}(\mathcal{M}_m)$. Finally, if $N \in \mathfrak{V}(\mathcal{M}_m)$ then, by Lemma 6.4(ii), $N \in \mathcal{M}_m$ and the result follows. ∎

Remark 10.2. Let $m$ be an infinite cardinal number greater than or equal to $\mathfrak{c}$. Observe that there exists a normed vector space $N$ of cardinality $m$. If $m = \mathfrak{c}$ choose $N = \mathbb{R}$. For $m > \mathfrak{c}$, let $I$ be an index set of cardinality $m$, $X = \prod_{i \in I} X_i$, where each $X_i = [0, 1]$, and let $\mathbf{C}(X)$ be the Banach space of all continuous functions $X \to \mathbb{R}$. $\mathbf{C}(X)$ has dimension greater than or equal to $m$. So choose $m$ linearly independent vectors in $\mathbf{C}(X)$ and let $N$ be the normed vector space spanned by these vectors. Then $N$ has cardinality $m$.

Proposition 10.3. *Let $m$ and $n$ be infinite cardinals greater than or equal to $\mathfrak{c}$. Then $\mathfrak{V}(\mathcal{L}_m)$ is a proper subvariety of $\mathfrak{V}(\mathcal{L}_n)$ if and only if $m < n$.*

*Proof.* If $m < n$ then, clearly, $\mathcal{L}_m \subseteq \mathcal{L}_n$ and so $\mathfrak{V}(\mathcal{L}_m) \subseteq \mathfrak{V}(\mathcal{L}_n)$. Now, there exists a normed vector space $N$ of cardinality $n$ (Remark 10.2) and $N \in \mathfrak{V}(\mathcal{L}_n)$. However, by Proposition 10.1, $N \notin \mathfrak{V}(\mathcal{L}_m)$, giving $\mathfrak{V}(\mathcal{L}_m) \subsetneq \mathfrak{V}(\mathcal{L}_n)$. ∎

The following theorem immediately follows.

THEOREM 10.4. *The class $\mathfrak{C}$ of varieties of topological groups, $\mathfrak{V}(\mathcal{L}_m)$, ranging over all infinite cardinals $m$, is a proper class such that the smallest variety in the class is $\mathfrak{V}(\mathcal{L}_\mathfrak{c})$, and if $\mathcal{V}_1, \mathcal{V}_2 \in \mathfrak{C}$, then $\mathcal{V}_1$ is properly contained in $\mathcal{V}_2$ or $\mathcal{V}_2$ is properly contained in $\mathcal{V}_1$.*

PROPOSITION 10.5. *Let $G$ be an abelian topological group with a neighbourhood of the identity which is separable. Then $G$ is contained in $\mathfrak{V}(\mathcal{L}_\mathfrak{c})$.*

*Proof.* Let $\{\rho_i' : i \in I\}$ be the family of all continuous pseudometrics on $G$. Let $U$ be a neighbourhood of the identity in $G$ which is separable and let $\rho$ be a continuous pseudometric on $G$ in which $U$ is a neighbourhood of the identity. We shall consider $\{\rho_i : \rho_i = \rho_i' + \rho,\, i \in I\}$, a family of continuous pseudometrics on $G$ that also defines the topology on $G$. Then $G$ is topologically isomorphic to a subgroup of the product $\prod_{i\in I}(G, \rho_i)$ and we note that $U$ is a separable neighbourhood of the identity in each $(G, \rho_i)$. Let $G_i = (G/\overline{\{e\}}^{\,i}, d_i)$ be the metrification of $(G, \rho_i)$ for each $i \in I$, with $f_i : G \to G_i$ the quotient homomorphism from $G$ onto $G_i$. Theorem 4.1 and Remark 4.2(3) clearly imply that $G$ is topologically isomorphic to a subgroup of the product $\prod_{i\in I} G_i \times H$, where $H$ is an abelian indiscrete group. We note that each $G_i$ has a separable neighbourhood of the identity in $G_i$, namely $f_i(U)$. Further, $f_i(U)$ is metrizable and we know that separable metrizable topological spaces have cardinality less than or equal to $\mathfrak{c}$. Therefore for each $i \in I$, $G_i$ is an abelian locally-$\mathfrak{c}$ group and so is in $\mathfrak{V}(\mathcal{L}_\mathfrak{c})$. Since $\mathbb{T}$ is a locally-$\mathfrak{c}$ group, by Proposition 2.2 every indiscrete abelian group is in $\mathfrak{V}(\mathcal{L}_\mathfrak{c})$. Therefore, $G$ is in $\mathfrak{V}(\mathcal{L}_\mathfrak{c})$, as required. ∎

THEOREM 10.6. *The variety $\mathfrak{V}(\mathcal{L}_\mathfrak{c})$ properly contains $\mathfrak{V}(\mathcal{L}_\mathcal{S})$.*

*Proof.* Since every locally separable topological group has at least one separable neighbourhood of the identity, Proposition 10.5 implies that $\mathfrak{V}(\mathcal{L}_\mathcal{S}) \subseteq \mathfrak{V}(\mathcal{L}_\mathfrak{c})$.

Let $N$ be a normed vector space of cardinality $\mathfrak{c}$ that is not separable (for example, $\ell_\infty$; see [70]). From Remark 9.17 we see that $\mathfrak{V}(\mathcal{L}_\mathcal{S})$ contains no non-separable normed vector spaces, and so $N \notin \mathfrak{V}(\mathcal{L}_\mathcal{S})$. However, $N \in \mathfrak{V}(\mathcal{L}_\mathfrak{c})$ and the result follows. ∎

THEOREM 10.7. *Let $m$ be an infinite cardinal number. Then $\mathfrak{V}(\mathcal{L}_m) = \mathfrak{V}(\mathcal{C}_m \cup \mathcal{D})$.*

*Proof.* The result follows from Example 7.6 and Theorem 7.10. ∎

This completes the proof of our Main Theorem.

OPEN QUESTION. What is the analogue of the Main Theorem for non-abelian topological groups?

We make a few observations.

The variety generated by the class of all locally compact groups, $\mathcal{L}_C$, is not the class of all topological groups. Let $B$ be a Banach space contained in $\mathfrak{V}(\mathcal{L}_C)$ then it is contained in $SC\overline{Q}\bar{S}P(\mathcal{L}_C)$. Now, $\overline{Q}\bar{S}P(\mathcal{L}_C) = \mathcal{L}_C$ and so $B \in SC(\mathcal{L}_C)$. As $B$ is abelian, it is contained in $SC(\mathcal{L}_A)$ and by Theorem 6.17, is finite-dimensional. Using a similar argument, we can show that a Banach space contained in the variety generated by all $\sigma$-compact groups is finite-dimensional. Therefore, once again, the variety generated by the class of all $\sigma$-compact groups is not the class of all topological groups. Indeed, Proposition 10.1 applies to the non-abelian case and therefore for any infinite cardinal $m$, $\mathfrak{V}(\mathcal{L}_m)$ is not the class of all topological groups. This is true as there are normed vector spaces of arbitrary size.

It is by no means obvious that it is trivial to convert the proofs we have presented here for abelian groups to non-abelian groups. The key difficulty is that we have relied heavily on the fact that the free abelian topological group topology is equal to the Graev topology, namely that obtained by extending pseudometrics. Except in exceptional cases, the free topological group topology is not equal to the Graev topology (see, for example, [62]).

## 11. Wide varieties of topological groups

The concept of a variety of topological groups is a natural extension of the concept of variety of groups. An equally natural extension is that of a wide variety of topological groups where instead of saying the class is closed under quotients, we say the class of topological groups is closed under continuous homomorphic images, namely the operator $H$, rather than (or as well as) $Q$. Indeed, the alternative definition of variety of groups using laws extends more satisfactorily to the case of wide varieties. To see this, see [73, 57, 34, 64].

Definition 11.1. Let $\Omega$ be a class of topological groups. The operator $H$ is defined on $\Omega$ to give the class of topological groups as follows. The topological group $G \in H(\Omega)$ if $G$ is a continuous homomorphic image of a topological group in $\Omega$.

Definition 11.2. A class of topological groups is said to be a *wide variety of topological groups* if it is closed under $S$, $H$ and $C$ [73]. Further, if $\Omega$ is a class of topological groups, the smallest wide variety containing $\Omega$, denoted by $\mathfrak{W}(\Omega)$, is the *wide variety generated by* $\Omega$.

Example 11.3.

(a) It is clear that the class of all abelian topological groups is a wide variety of topological groups as any continuous homomorphic image of an abelian topological group is also abelian.
(b) Let $m$ be any cardinal number. The class of all $T(m)$-groups is a wide variety of topological groups [41]. This is seen by considering a $T(m)$-group $G$ with continuous homomorphism $f : G \to K$ from $G$ onto $K$. Clearly, for $U$ an open neighbourhood of the identity in $K$, $f^{-1}(U)$ is an open neighbourhood of the identity in $G$ and thus contains a normal subgroup, $N$ of index strictly less than $m$. The image $f(N)$ in $K$ is also a normal subgroup of index strictly less than $m$ and $f(N) \subseteq U$. Thus, $K$ is a $T(m)$-group.
(c) Recall that the class $\Omega$ of all topological groups that have the subgroup topology is a variety of topological groups. Note also that all discrete groups are contained in $\Omega$, but $\mathbb{R}$ is not. Therefore, $\Omega$ is not the class of all abelian topological groups. We shall soon see that any class that contains all discrete groups generates as its wide variety the class of all abelian topological groups (see Remark 11.11). Therefore, $\mathfrak{W}(\Omega) \neq \Omega$,

that is, $\Omega$ is an example of a class of topological groups that forms a variety, but not a wide variety of topological groups.

REMARK 11.4.

(1) In a manner analogous to that for varieties generated by topological groups, we know that $\mathfrak{W}(\Omega) = HSC(\Omega)$ [73]. It also indicates that just as with varieties generated by classes of topological groups, any topological group contained in the smallest wide variety can be obtained by just one application of each of the operators $H$, $S$ and $C$.
(2) Let $\Omega$ be a class of topological groups. Note that a quotient group of a topological group $G$ is a continuous image of $G$ and so it is clear that the variety generated by $\Omega$ is contained in the wide variety generated by $\Omega$.

PROPOSITION 11.5. *Let $\Omega$ be a class of topological groups and let $G \in \mathfrak{W}(\Omega)$. Then $G$ is a continuous one-to-one homomorphic image of $G'$ where $G' \in \mathfrak{V}(\Omega)$.*

*Proof.* By the definition of $\mathfrak{W}(\Omega)$, $G \in HSC(\Omega)$. Therefore, $G$ is a continuous homomorphic image of $K \in SC(\Omega)$. Let $f : K \to G$ be the continuous homomorphism of $K$ onto $G$ and let $\mathcal{T}$ be the given topology on $G$. Further, let $\mathcal{T}'$ be the quotient topology induced on $|G|$ by $f$. Clearly, $(|G|, \mathcal{T}') = G'$ is a quotient group of $K$ and so is contained in $\mathfrak{V}(\Omega)$. Finally, we note that $\mathcal{T}'$ is finer than $\mathcal{T}$ and so the identity mapping $i : G' \to G$ is a continuous, one-to-one homomorphism, giving the result. ■

We present a number of useful corollaries to Proposition 11.5.

COROLLARY 11.6. *Let $\Omega$ be a class of topological groups. A discrete group $D$ is contained in $\mathfrak{W}(\Omega)$ if and only if $D \in \mathfrak{V}(\Omega)$.*

*Proof.* Clearly, if $D \in \mathfrak{V}(\Omega)$, then $D \in \mathfrak{W}(\Omega)$. Let $D \in \mathfrak{W}(\Omega)$. By Proposition 11.5, $D$ is a continuous one-to-one homomorphic image of a group $G \in \mathfrak{V}(\Omega)$. As the topology on $G$ is finer than the topology on $D$, $G$ must be a discrete group. Therefore, $D$ is topologically isomorphic to $G$ and so $D \in \mathfrak{V}(\Omega)$. ■

COROLLARY 11.7. *Let $E$ be a locally convex Hausdorff topological vector space contained in $\mathfrak{W}(\mathbb{R})$, the wide variety generated by $\mathbb{R}$. Then $E \in \mathfrak{V}(\mathbb{R})$.*

*Proof.* By Proposition 11.5, there exists a topological group $G \in \mathfrak{V}(\mathbb{R})$ such that $E$ is a continuous, one-to-one homomorphic image of $G$; that is, there exists $f : G \to E$ such that $f$ is a continuous isomorphism. Let $D$ be a discrete subgroup of $E$, then $f^{-1}(D)$ is a discrete subgroup of $G$. By Lemma 6.19 $f^{-1}(D)$ is finitely generated and hence $D$ is finitely generated. Therefore, by the main Theorem of [44], $E$ has the weak topology and so is contained in $\mathfrak{V}(\mathbb{R})$. ■

NOTATION. We denote by $I$ the operation of formation of continuous, algebraically isomorphic images, so that $H = IQ$.

COROLLARY 11.8. *Let $\Omega$ be a non-empty class of abelian topological groups. If $G$ is a Hausdorff topological group in $\mathfrak{W}(\Omega)$, then $G \in ISC\overline{Q}P(\Omega)$.*

*Proof.* By Proposition 11.5, $G$ is a continuous, one-to-one image of $K \in \mathfrak{V}(\Omega)$. As $G$ is Hausdorff, $K$ must be Hausdorff, so $K \in SC\overline{Q}P(\Omega)$ and so $G \in ISC\overline{Q}P(\Omega)$. ∎

We have presented here a very brief overview of wide varieties, giving only those results that are needed in this section. More information on wide varieties can be found in the work of Taylor [73] and that of Kopperman, Mislove, Morris, Nickolas, Pestov and Svetlichny in [34] and that of Morris, Nickolas and Pestov in [64].

The first two wide varieties of topological groups we consider are the wide variety generated by the class of all Banach spaces and the wide variety generated by the class of all locally compact abelian groups. For both classes, we easily characterize the wide varieties they generate.

Remark 11.9. We know that $\mathfrak{V}(\mathcal{B})$, the variety generated by the class of all topological groups underlying Banach spaces, is exactly the variety of all abelian topological groups. Thus, $\mathfrak{W}(\mathcal{B}) = \mathfrak{V}(\mathcal{B}) = \mathcal{A}$, where $\mathcal{A}$ is the variety of all abelian topological groups.

Proposition 11.10. *The wide variety of topological groups generated by the class of all locally compact abelian groups is precisely the variety of all abelian topological groups.*

*Proof.* Clearly, $\mathfrak{W}(\mathcal{L}_A) \subseteq \mathcal{A}$, where $\mathcal{L}_A$ is the class of all locally compact abelian groups and $\mathcal{A}$ is the variety of all abelian topological groups. Let $G$ be an abelian topological group and let $|G|_D$ be the group underlying $G$ equipped with the discrete topology. Clearly, $|G|_D$ is contained in $\mathcal{L}_A$ and the identity homomorphism $i : |G|_D \to G$ is continuous. Therefore, $G \in \mathfrak{W}(\mathcal{L}_A)$ and so $\mathcal{A} \subseteq \mathfrak{W}(\mathcal{L}_A)$, giving the result. ∎

Note it follows immediately from Remark 11.4(2), Theorem 6.17 and Proposition 11.10 that $\mathfrak{V}(\mathcal{L}_A) \subsetneq \mathfrak{W}(\mathcal{L}_A)$.

Remark 11.11. We note that if a class $\Omega$ of topological groups contains $\mathcal{L}_A$—even $\mathcal{D}$, the class of all discrete abelian groups—then $\mathfrak{W}(\Omega)$ is the variety of all abelian topological groups. Therefore, considering the diagram in our Main Theorem, we see that the following wide varieties of topological groups are all equal to the variety of all abelian topological groups.

(i) $\mathfrak{W}(\mathcal{B})$, generated by all Banach spaces;
(ii) $\mathfrak{W}(\mathcal{L}_m)$, generated by all locally-$m$ groups for $m \geq c$;
(iii) $\mathfrak{W}(\mathcal{L}_S)$, generated by all locally separable groups;
(iv) $\mathfrak{W}(\mathcal{L}_\sigma)$, generated by all locally $\sigma$-compact groups;
(v) $\mathfrak{W}(\mathcal{L}_A)$, generated by all locally compact abelian topological groups;
(vi) $\mathfrak{W}(\mathcal{D})$, generated by all discrete abelian topological groups.

The wide variety generated by $FA[0,1]$ turns out to be the most interesting of the wide varieties in question. From Theorem A it follows trivially that $\mathfrak{W}(FA[0,1]) = \mathfrak{W}(\mathcal{K}_\omega)$.

Proposition 11.12. *The wide variety of topological groups generated by $FA[0,1]$ contains all countable abelian topological groups.*

*Proof.* Clearly, every countable discrete abelian topological group is a $k_\omega$-group and by Theorem A is contained in $\mathfrak{V}(FA[0,1])$. Every countable abelian topological group $G$

is a continuous homomorphic image of $|G|_D$, the group underlying $G$ with the discrete topology, and as $|G|_D \in \mathfrak{V}(FA[0,1]) \subseteq \mathfrak{W}(FA[0,1])$, the result follows. ■

We will shortly show that $\mathfrak{W}(\mathcal{C}_\sigma) = \mathfrak{W}(FA[0,1])$. To do this, we need the following two lemmas.

LEMMA 11.13. *Let $X$ be a $\sigma$-compact space. Then there exists a $k_\omega$-space $Y$ such that $X$ is the continuous image of $Y$.*

*Proof.* Let $X = \bigcup_{n=1}^{\infty} X_n$ where each $X_n$ is compact. Let $H_n$ be a homeomorphic copy of $X_n$, disjoint from $H_1, \ldots, H_{n-1}$, with $f_n : H_n \to X_n$ the corresponding homeomorphism. Let $Y = \bigsqcup_{n=1}^{\infty} H_n$ be the free union of the $H_n$. Now, let $Y_n = H_1 \sqcup \cdots \sqcup H_n$, and note that each $Y_n$ is compact. Clearly, $Y = \bigcup_{n=1}^{\infty} Y_n$ and we will show that with this decomposition, $Y$ is a $k_\omega$-space. Let $A$ be a subset of $Y$ such that $A \cap Y_n$ is compact for each $n \in \mathbb{N}$. Clearly, for each $i \in \mathbb{N}$, $A \cap H_i$ is compact and so $H_i \setminus A$ is open in $H_i$, indeed, $H_i \setminus A$ is open in $Y$. Noting that $Y \setminus A = \bigcup_{i=1}^{\infty} (H_i \setminus A)$, $A$ is closed in $Y$. So $Y = \bigcup_{n=1}^{\infty} Y_n$ is a $k_\omega$-decomposition of $Y$. Finally, let $f : Y \to X$ be the mapping defined as follows. If $y \in Y$ then $y \in H_n$ for some $n \in \mathbb{N}$ and so define $f(y) = f_n(y)$. The mapping $f$ is clearly onto, and for an open set $U$ in $X$, $f^{-1}(U) = \bigcup_{n=1}^{\infty} f_n^{-1}(U)$, which is open in $Y$. Therefore, $X$ is the continuous image of $Y$, a $k_\omega$-space. ■

The following result is analogous to Lemma 3.4:

LEMMA 11.14. *Let $X$ and $Y$ be completely regular spaces such that there exists a continuous mapping $\phi : X \to Y$ from $X$ onto $Y$. Then there exists a continuous homomorphism $\Phi : FA(X) \to FA(Y)$ from $FA(X)$ onto $FA(Y)$.*

*Proof.* Let $\Phi$ be the continuous homomorphism from $FA(X)$ into $FA(Y)$ that extends naturally from $\phi$, according to the definition of a free abelian topological group. To show that $\Phi$ is an onto homomorphism, take $w \in FA(Y)$ such that $w = y_1^{\varepsilon_1} \ldots y_n^{\varepsilon_n}$, $y_i \in Y$ and $\varepsilon_i = \pm 1$ for each $i = 1, \ldots, n$. For each $y_i$, there exists an $x_i \in X$ such that $\phi(x_i) = \Phi(x_i) = y_i$. Further,

$$\Phi(x_1^{\varepsilon_1} \ldots x_n^{\varepsilon_n}) = \Phi(x_1)^{\varepsilon_1} \ldots \Phi(x_n)^{\varepsilon_n} = y_1^{\varepsilon_1} \ldots y_n^{\varepsilon_n} = w.$$

Therefore, $\Phi$ is an onto continuous homomorphism. ■

THEOREM 11.15. *The wide variety of topological groups generated by $FA[0,1]$ is equal to the wide variety generated by $\mathcal{C}_\sigma$, the class of all abelian $\sigma$-compact groups.*

*Proof.* As $FA[0,1]$ is a $k_\omega$-group, it is $\sigma$-compact and so $FA[0,1] \in \mathfrak{W}(\mathcal{C}_\sigma)$ giving $\mathfrak{W}(FA[0,1]) \subseteq \mathfrak{W}(\mathcal{C}_\sigma)$. Now, let $G$ be an abelian $\sigma$-compact group then by Lemma 11.13, there exists a $k_\omega$-space $X$ such that $G$ (as a topological space) is a continuous image of $X$. Further, $FA(G)$ is a continuous homomorphic image of $FA(X)$, by Lemma 11.14. We know that $FA(X)$ is a $k_\omega$-group (see [35, Corollary 1]) and therefore is contained in $\mathfrak{V}(FA[0,1])$ and hence also in $\mathfrak{W}(FA[0,1])$. Thus, $FA(G) \in \mathfrak{W}(FA[0,1])$ and as $G$ is a quotient group of $FA(G)$ (Remark 3.2(5)), $G \in \mathfrak{W}(FA[0,1])$ also. Hence, $\mathfrak{W}(\mathcal{C}_\sigma) \subseteq \mathfrak{W}(FA[0,1])$ and the result follows. ■

From Theorems 11.15 and B it is clear that $\mathfrak{V}(FA[0,1]) \subsetneq \mathfrak{W}(FA[0,1])$.

Definition 11.16 ([15]). Let $X$ be a completely regular space. A topological group $FLCS(X)$ is said to be a *free locally convex topological vector space on the space* $X$ if it has the following properties:

(1) $X$ is a subspace of $FLCS(X)$;
(2) $X$ is a (vector space) basis for $FLCS(X)$;
(3) for any continuous mapping $\phi$ of $X$ into any locally convex topological vector space $V$, there exists a continuous linear transformation $\Phi$ of $FLCS(X)$ into $V$ such that $\Phi(x) = \phi(x)$ on $X$. ∎

The following theorem is not an obvious result. It shows that for a completely regular space $X$, $FA(X)$ is a subgroup of $FLCS(X)$.

Theorem 11.17 ([75, Theorem 3]; [76]). *Let $X$ be a completely regular Hausdorff space and let $FLCS(X)$ be the free locally convex topological vector space on $X$. Then the subgroup of $FLCS(X)$ that is algebraically generated by $X$ is (with the induced topology) topologically isomorphic to the free abelian topological group on $X$.*

Theorem 11.18. $\mathfrak{W}(FLCS[0,1]) = \mathfrak{W}(FA[0,1])$.

*Proof.* In $FLCS[0,1]$, for $n \in \mathbb{N}$ define the set $A_n = \{\lambda t : \lambda \in [-n,n],\, t \in [0,1]\}$. Further, for $m \in \mathbb{N}$, define $mA_n = A_n + \cdots + A_n$ ($m$ terms). Note that each $mA_n$ is compact. Further,

$$FLCS[0,1] = \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} mA_n$$

Therefore, $FLCS[0,1]$ is $\sigma$-compact. Thus, $\mathfrak{W}(FLCS[0,1]) \subseteq \mathfrak{W}(\mathcal{C}_\sigma) = \mathfrak{W}(FA[0,1])$. To complete the proof, we note that $[0,1]$ is completely regular and so by Theorem 11.17, $FA[0,1] \in \mathfrak{W}(FLCS[0,1])$. ∎

In summary, the following wide varieties are all equal:

(i) $\mathfrak{W}(FA[0,1])$, the wide variety generated by $FA[0,1]$;
(ii) $\mathfrak{W}(\mathcal{K}_\omega)$, the wide variety generated by the class of all abelian $k_\omega$-groups;
(iii) $\mathfrak{W}(\mathcal{C}_\sigma)$, the wide variety generated by the class of all abelian $\sigma$-compact groups.
(iv) $\mathfrak{W}(FLCS[0,1])$, the wide variety generated by $FLCS[0,1]$.

Earlier, we considered the variety of topological groups generated by $\mathcal{S}$, the class of all separable abelian groups. We now examine $\mathfrak{W}(\mathcal{S})$.

Recall that a *Polish space* is a separable complete metrizable space [7]. For example, $\mathbb{P}$, the set of irrationals with the topology induced from $\mathbb{R}$, is a Polish space. A topological space is said to be a *Suslin space* if it is metrizable and is a continuous image of a Polish space. Note that $\mathbb{R}^{\aleph_0}$, indeed every separable Banach space, is a Polish space and hence a Suslin space.

Remark 11.19. It can be shown (cf. [7, p. 261, 6(a)]) that if $X$ is a Suslin space, then $X$ is a continuous image of $\mathbb{N}^{\aleph_0}$, indeed of $\mathbb{P}$, where $\mathbb{P}$ is the set of irrationals with the induced topology from $\mathbb{R}$.

Proposition 11.20. *The following wide varieties of topological groups are equal:*

(i) $\mathfrak{W}(\ell_1)$;
(ii) $\mathfrak{W}(\mathcal{B}_\mathcal{S})$, *where* $\mathcal{B}_\mathcal{S}$ *is the class of all separable Banach spaces;*
(iii) $\mathfrak{W}(\mathcal{S})$, *where* $\mathcal{S}$ *is the class of all separable abelian topological groups;*
(iv) $\mathfrak{W}(FA(\mathbb{R}^{\aleph_0}))$, *where* $FA(\mathbb{R}^{\aleph_0})$ *is the free abelian topological group on* $\mathbb{R}^{\aleph_0}$;
(v) $\mathfrak{W}(FA(\mathbb{P}))$, *where* $FA(\mathbb{P})$ *is the free abelian topological group on* $\mathbb{P}$.

*Proof.* From Theorem 9.7, the first three wide varieties are clearly equal.

As $\mathbb{R}^{\aleph_0}$ is a Polish space, Proposition 9.6 and Remark 11.4(2) imply that $FA(\mathbb{R}^{\aleph_0}) \in \mathfrak{V}(\ell_1)$. We also know that $FA(\ell_1)$ is topologically isomorphic to $FA(\mathbb{R}^{\aleph_0})$ as $\ell_1$ is homeomorphic to $\mathbb{R}^{\aleph_0}$ (see [29, Part I, Section 2]). Therefore, $\ell_1 \in \mathfrak{V}(FA(\mathbb{R}^{\aleph_0}))$, giving $\mathfrak{V}(\ell_1) = \mathfrak{V}(FA(\mathbb{R}^{\aleph_0}))$, indeed $\mathfrak{W}(\ell_1) = \mathfrak{W}(FA(\mathbb{R}^{\aleph_0}))$.

We now note that as $\ell_1$ is a separable Banach space, it is a Suslin space. So there exists a continuous surjective mapping, $f : \mathbb{P} \to \ell_1$, which extends to a continuous homomorphism $F$ of $FA(\mathbb{P})$ onto $\ell_1$. Hence, $\ell_1 \in \mathfrak{W}(FA(\mathbb{P}))$ and $\mathfrak{W}(\ell_1) \subseteq \mathfrak{W}(FA(\mathbb{P}))$. Equality follows from Proposition 9.6 and Remark 11.4(2). ∎

We note now that $\mathfrak{W}(\mathcal{S}) = \mathfrak{W}(\ell_1)$ is not the wide variety of all abelian topological groups.

Proposition 11.21. *The wide variety generated by* $\ell_1$ *is properly contained in the wide variety of all abelian topological groups.*

*Proof.* Recall that $\ell_1$ has cardinality $c$ and so every topological group contained in $\mathfrak{V}(\ell_1)$ is a $T(c^+)$-group. By Corollary 11.6, a discrete group contained in $\mathfrak{W}(\ell_1)$ must also be contained in $\mathfrak{V}(\ell_1)$; that is, every discrete group contained in $\mathfrak{W}(\ell_1)$ has cardinality strictly less than $c^+$. Therefore, $\mathfrak{W}(\ell_1)$ does not contain all discrete abelian topological groups and is therefore properly contained in the wide variety of all abelian topological groups. ∎

We now summarize our wide variety results in the following theorem.

Theorem E.

$$\mathcal{A} = \mathfrak{W}(\mathcal{B}) = \mathfrak{W}(\mathcal{D}) = \mathfrak{W}(\mathcal{L}_A) = \mathfrak{W}(\mathcal{L}_\sigma) = \mathfrak{W}(\mathcal{L}_\mathcal{S}) = \mathfrak{W}(\mathcal{L}_m), \text{ for all } m \geq c$$

/ \

$$\mathfrak{W}(FA[0,1]) = \mathfrak{W}(\mathcal{K}_\omega) = \mathfrak{W}(\mathcal{C}_\sigma) = \mathfrak{W}(FLCS[0,1])$$

$$\mathfrak{W}(\mathcal{S}) = \mathfrak{W}(\mathcal{B}_\mathcal{S}) = \mathfrak{W}(\ell_1) = \mathfrak{W}(FA(\mathbb{P})) = \mathfrak{W}(FA(\mathbb{R}^{\aleph_0}))$$

Concerning the previous theorem, note that although we know that $\mathfrak{W}(\mathcal{C}_\sigma) \subseteq \mathfrak{W}(\mathcal{S})$, deduced from the main theorem of this paper, we do not know the full relationship between $\mathfrak{W}(\mathcal{C}_\sigma)$ and $\mathfrak{W}(\mathcal{S})$. We leave this as an open question.

Open Question. In the structure given in the Main Theorem, we saw that $\mathfrak{V}(\mathcal{C}_\sigma)$ is properly contained in $\mathfrak{V}(\mathcal{S})$, the variety generated by the class of all separable abelian topological groups. Thus, it is clear that $\mathfrak{W}(\mathcal{C}_\sigma) \subseteq \mathfrak{W}(\mathcal{S})$. However, the question remains whether these two wide varieties are equal.

Note that we already know that $\mathfrak{W}(\mathcal{C}_\sigma) = \mathfrak{W}(FA[0,1])$ and $\mathfrak{W}(\mathcal{S}) = \mathfrak{W}(FA(\mathbb{R}^{\aleph_0}))$. Therefore our question of whether $\mathfrak{W}(\mathcal{C}_\sigma)$ equals $\mathfrak{W}(\mathcal{S})$ can be reduced to whether $\mathfrak{W}(FA[0,1])$ equals $\mathfrak{W}(FA(\mathbb{R}^{\aleph_0}))$, that is, whether $FA(\mathbb{R}^{\aleph_0}) \in \mathfrak{W}(FA[0,1])$.

Recall that $\mathfrak{V}(FA[0,1])$ is closed under completions, but $\mathfrak{V}(\mathcal{C}_\sigma)$ is not. An alternative way of considering this open question is to determine whether $\mathfrak{W}(FA[0,1]) = \mathfrak{W}(\mathcal{C}_\sigma)$ is closed under completions or not.

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