Let $\nu \le \mu \le \kappa$ be infinite cardinal numbers. As usual, every ordinal is considered as the set of all smaller ordinals, and cardinals are defined to be ordinals which cannot be put in one-one correspondence with smaller ordinals. We will study the size of families $\mathscr{F}$ of subsets of κ such that $\mathscr{F}$ is maximal with respect to the following properties: $|A| = \mu$ for all $A \in \mathscr{F}$, and $|A \cap B| < \nu$ for all distinct $A, B \in \mathscr{F}$. We consider the most important cases to be those in which $\nu = \mu = \kappa$, especially for κ regular, the case with $\nu = \mu < \kappa$, and the case $\nu < \mu = \kappa$.

There has been a considerable amount of work done on this subject. Many of the main results are found in Baumgartner [76], where results are given on almost disjoint subsets which have many implications for maximal almost disjoint subsets. Another general source of results is Milner–Prikry [87]. Theorems of a more specialized nature can be found in Blass [93], Erdős–Hechler [75], Wage [79], Kojman–Kubiś–Shelah [∞], and Monk [96b] and [01].

Almost disjoint sets have been studied in many other papers in which the focus is not on the size of maximal almost disjoint families.

The purpose of these notes is to survey all of the results on this topic, giving proofs for many of them. In the course of doing this we indicate some generalizations of consistency results of Baumgartner and of Blass. Some open problems will be mentioned.

The notes are divided into two parts. The first part, entitled *ZFC results*, is mainly a survey of known results, with, however, some new facts and proofs; several problems are mentioned here. The second part, entitled *Consistency results*, begins with a statement of what is known and what is proved in these notes, and then in four sections gives some detailed consistency results. These results are more-or-less straightforward extensions of theorems of Baumgartner and Blass. Section 6 shows that a complete description of possibilities is obtained if one assumes GCH (implicitly this is due to Baumgartner). Sections 7 through 9 give the indicated consistency proofs, described more thoroughly in the introduction to the second part. These proofs are given in rather full detail.

The concepts studied in this paper

Unless otherwise mentioned, throughout these notes κ, ν, μ are infinite cardinals with $\nu \le \mu \le \kappa$. For ordinals $\alpha < \beta$, we write $(\alpha, \beta)_{\text{card}}$ for the collection of all cardinal numbers κ such that $\alpha < \kappa < \beta$; similarly for other intervals. *Denumerable* means countably infinite. For any infinite cardinal κ, the smallest cardinal greater than κ is denoted by κ^+, and if $m \in \omega$ is any infinite cardinal, then κ^{+m} is the mth cardinal successor of κ.

Sets A and B are *ν-almost disjoint* (*ν-ad*) if $|A\cap B|<\nu$. A family $\mathscr{A}$ of sets is *ν-almost disjoint*, for brevity *ν-ad*, if any two distinct members of $\mathscr{A}$ are ν-ad. Now let $\mathscr{F}$ be a family of sets each of size at least ν. We say that $\mathscr{A}$ is *$\mathscr{F},\nu$ almost disjoint* (for brevity *$\mathscr{F},\nu$-ad*) if $\mathscr{A}\subseteq\mathscr{F}$ and it is ν-ad. Furthermore, $\mathscr{A}$ is *$\mathscr{F},\nu$ maximal almost disjoint* (for brevity *$\mathscr{F},\nu$-mad*) if in addition it is maximal among subsets of $\mathscr{F}$ which are ν-ad. Equivalently, $\mathscr{A}$ is $\mathscr{F},\nu$-maximal almost disjoint if $\mathscr{A}\subseteq\mathscr{F}$, it is ν-ad, and for each $X\in\mathscr{F}$ there is a $Y\in\mathscr{A}$ such that $|X\cap Y|\geq\nu$. Instead of *$[\kappa]^\kappa,\kappa$-mad* we say *κ-mad* or *κ-maximal almost disjoint.*

If Γ and Δ are sets of ordinals, we write $\Gamma\preceq\Delta$ if every member of Γ is $\leq$ than some member of Δ. And we write $\Gamma\sqsupseteq\Delta$ if every member of Γ is $\geq$ some member of Δ. Note that this is not quite the same as saying that $\Gamma\succeq\Delta$.

The basic definitions of the concepts we will be working with are as follows. Assume that $\nu\leq\mu\leq\kappa$. Then:

$$\begin{aligned}
\mathrm{AD}(\kappa,\mu,\nu)&=\{|\mathscr{A}|:\mathscr{A}\text{ is }[\kappa]^\mu,\nu\text{-ad}\};\\
\mathrm{AD}(\kappa)&=\mathrm{AD}(\kappa,\kappa,\kappa);\\
\mathrm{MAD}(\kappa,\mu,\nu)&=\{|\mathscr{A}|:\mathscr{A}\text{ is }[\kappa]^\mu,\nu\text{-mad}\};\\
\mathrm{MAD}(\kappa)&=\mathrm{MAD}(\kappa,\kappa,\kappa);\\
\mathrm{MAD}_1(\kappa,\lambda,\mu,\nu)&=\{|\mathscr{A}|:\text{there is a partition }\mathscr{D}\text{ of }\kappa\text{ into }\lambda\text{ sets of size }\mu\\
&\qquad\qquad\text{such that }\mathscr{D}\cap\mathscr{A}=0\text{ and }\mathscr{D}\cup\mathscr{A}\text{ is }[\kappa]^\mu,\nu\text{-mad}\};\\
\mathrm{MAD}_1(\kappa)&=\mathrm{MAD}_1(\kappa,\kappa,\kappa,\kappa);\\
\mathrm{MAD}_2(\kappa,\mu,\nu)&=\{|\mathscr{A}|:\mathscr{A}\subseteq[\kappa\times\mu]^\mu\text{ and }\forall\alpha<\kappa\ (\{\alpha\}\times\mu\notin\mathscr{A})\\
&\qquad\qquad\text{and }\mathscr{A}\cup\{\{\alpha\}\times\mu:\alpha<\kappa\}\text{ is }[\kappa\times\mu]^\mu,\nu\text{-mad}\};\\
\mathfrak{a}_{\kappa\mu\nu}&=\min(\mathrm{MAD}(\kappa,\mu,\nu)\cap[\operatorname{cf}\kappa,\infty));\\
\mathfrak{a}_\kappa&=\mathfrak{a}_{\kappa\kappa\kappa};\\
\mathfrak{a}_{\kappa\lambda\mu\nu1}&=\min(\mathrm{MAD}_1(\kappa,\lambda,\mu,\nu));\\
\mathfrak{a}_{\kappa1}&=\mathfrak{a}_{\kappa\kappa\kappa\kappa1}.
\end{aligned}$$

The last definitions, concerning $\mathfrak{a}$, are valid iff the minimums apply to non-empty sets. MAD_2 turns out to coincide with a special case of MAD_1, so we do not have extensive notation for it. The intersection in the definition of $\mathfrak{a}_{\kappa\mu\nu}$ is there to make the function non-trivial, as we shall see.

Baumgartner's notation $A(\kappa,\lambda,\mu,\nu)$ corresponds to $\lambda\in\mathrm{AD}(\kappa,\mu,\nu)$, and his notation $A(\kappa,\lambda,\mu)$ to $\lambda\in\mathrm{AD}(\kappa,\mu,\mu)$.

In terms of these definitions, we can briefly state some of the main ZFC results in this paper; for consistency results, see the second part. Proposition 2.4: $\mathrm{AD}(\kappa,\mu,\nu)\subseteq[1,\kappa^\nu]$. Proposition 3.2: If $\kappa^+<\lambda$ and $\lambda\in\mathrm{AD}(\kappa^+,\kappa^+,\nu)$, then $\lambda\in\mathrm{AD}(\kappa,\kappa,\nu)$. Example 3.9: There is a singular cardinal κ such that $\operatorname{cf}\kappa=\omega$, $\kappa=\aleph_\kappa$, $\kappa^\omega\in\mathrm{AD}(\kappa,\kappa,\omega)$, and there is no $\varrho\in(\omega,\kappa)_{\mathrm{card}}$ such that $\kappa^\omega\in\mathrm{AD}(\varrho,\varrho,\omega)$. Theorem 4.7: Let μ be a singular cardinal. Suppose that $\nu\in\mathrm{MAD}(\operatorname{cf}\mu)$ and $\varrho\in\mathrm{MAD}(\kappa,\mu,\mu)$. Then $\varrho\cdot\nu\in\mathrm{MAD}(\kappa,\mu,\mu)$. Proposition 4.10: If $\mu<\kappa$, then $\mathrm{MAD}(\kappa,\mu,\mu)\sqsupseteq\mathrm{MAD}(\mu)\cap[\mu,\infty)$. Hence $\mathfrak{a}_\mu\leq\mathfrak{a}_{\kappa\mu\mu}$. Proposition 5.5: $\mathrm{MAD}(\kappa,\mu,\mu)=\mathrm{MAD}_1(\kappa,\kappa,\mu,\mu)$ if κ is regular and $\mu<\kappa$. Proposition 5.6: If κ is regular, then $\mathrm{MAD}(\kappa)\cap[\kappa,\infty)\subseteq\mathrm{MAD}(\kappa^+,\kappa,\kappa)$.

ZFC RESULTS

1. Simple facts

In this section we give very elementary facts about the notions.

Proposition 1.1. *Assume that $\nu \le \mu \le \kappa$. Then*:

(i) $\mathrm{MAD}(\kappa,\mu,\nu)\cap[\kappa,\infty)\neq 0$. *Hence the definition of $\mathfrak{a}_{\kappa\mu\nu}$ always makes sense.*

(ii) *If $\mu<\kappa$, then $\mathrm{MAD}(\kappa,\mu,\nu)\subseteq[\kappa,\infty)$ and so $\mathfrak{a}_{\kappa\mu\nu}\ge\kappa$. Hence the intersection in the definition of $\mathfrak{a}_{\kappa\mu\nu}$ is superfluous if $\mu<\kappa$.*

(iii) *If $\nu<\kappa$, then $[1,\kappa)\subseteq\mathrm{MAD}(\kappa,\kappa,\nu)$.*

(iv) $[1,\operatorname{cf}\kappa)\subseteq\mathrm{MAD}(\kappa)$.

(v) *If $\nu\le\varrho\le\mu\le\kappa$, then $\mathrm{MAD}(\kappa,\mu,\nu)\preceq\mathrm{MAD}(\kappa,\mu,\varrho)$.*

(vi) *If $\nu\le\mu\le\kappa\le\varrho$, then $\mathrm{MAD}(\kappa,\mu,\nu)\preceq\mathrm{MAD}(\varrho,\mu,\nu)$.*

Proof. For (i), one can take a partition of size κ of κ into sets of size μ and extend it to a $[\kappa]^\mu,\nu$-mad set.

(ii) is clear.

For (iii), for any $\varrho\in[1,\kappa)$, let $\mathscr{A}$ be a partition of κ into ϱ parts, each of size κ. Then $\mathscr{A}$ is $[\kappa]^\kappa,\nu$-mad, as desired. In fact, suppose that $B\in[\kappa]^\kappa$. If $|B\cap A|<\nu$ for all $A\in\mathscr{A}$, then $B=\bigcup_{A\in\mathscr{A}}(B\cap A)$ would have size at most $\varrho\cdot\nu<\kappa$, contradiction.

For (iv), let $\nu\in[1,\operatorname{cf}\kappa)$. Let $\mathscr{A}$ be a partition of κ into ν sets, each of size κ. Clearly if $B\in[\kappa]^\kappa$ then $|B\cap A|=\kappa$ for some $A\in\mathscr{A}$.

For (v), let $\sigma\in\mathrm{MAD}(\kappa,\mu,\nu)$. Say $\sigma=|\mathscr{A}|$, where $\mathscr{A}$ is $[\kappa]^\mu,\nu$-mad. Then $\mathscr{A}$ is ϱ-ad, and so can be extended to a set which is $[\kappa]^\mu,\varrho$-mad, as desired.

Finally, for (vi), suppose that $\mathscr{A}$ is $[\kappa]^\mu,\nu$-mad. Then $\mathscr{A}\subseteq[\varrho]^\mu$ and it is ν-ad, so it can be extended to a set $\mathscr{B}$ which is $[\varrho]^\mu,\nu$-mad. ∎

The following is the first part of Theorem 2.2(b) of Baumgartner [76].

Proposition 1.2. *Suppose that $\nu\le\mu\le\kappa$, $\nu'\le\mu'\le\kappa'$, $\kappa\le\kappa'$, $\nu\le\nu'$, and $\mu'\le\mu$. Then $\mathrm{AD}(\kappa,\mu,\nu)\subseteq\mathrm{AD}(\kappa',\mu',\nu')$.*

Proof. Suppose that $F\subseteq[\kappa]^\mu$ is ν-ad. For every $X\in F$ choose $Y_X\in[X]^{\mu'}$. Then Y is one-one, and $\{Y_X:X\in F\}\subseteq[\kappa']^{\mu'}$ is ν'-ad. So $|F|\in\mathrm{AD}(\kappa',\mu',\nu')$. ∎

This result is similar to the second part of Theorem 2.2(b) of Baumgartner [76], which says that if $\mu\le\kappa\le\lambda$, $\mu'\le\kappa'\le\lambda'$, $\kappa\le\kappa'$, $\lambda'\le\lambda$, and $\mu'\le\mu$, then from $\lambda\in\mathrm{AD}(\kappa,\mu,\mu)$ it follows that $\lambda'\in\mathrm{AD}(\kappa',\mu',\mu')$. However, this claim is not correct, at least under CH. In fact, take $\kappa=\kappa'=\omega_1$, $\lambda=\lambda'=\omega_2$, $\mu=\omega_1$, and $\mu'=\omega$. Assume CH. By Theorem 2.8 of Baumgartner [76], $\omega_2\in\mathrm{AD}(\omega_1,\omega_1,\omega_1)$. By 2.7 of Baumgartner [76], $\omega_2\in\mathrm{AD}(\omega_1,\omega,\omega)$ would imply that $\omega_2\le\omega_1^\omega$, contradicting CH.

Proposition 1.3. $\mathrm{AD}(\kappa,\mu,\nu)\preceq\mathrm{MAD}(\kappa,\mu,\nu)$. ∎

2. Fundamental results

In this section we give several results which establish some of the fundamental facts about our notions. This will enable us to describe more coherently the special cases above.

First we give a general "decreasing" theorem of Baumgartner, which depends on the following interesting combinatorial lemma (Theorem 3.1 of Baumgartner [76]).

PROPOSITION 2.1. *Suppose that* $\mu \le \kappa \le \lambda$, $F \subseteq [\kappa]^\mu$, $|F| = \lambda$, $\mathrm{cf}\,\mu \ne \mathrm{cf}\,\kappa \ne \mathrm{cf}\,\lambda$. *Then there is an* $\alpha < \kappa$ *such that* $|\{X \in F : |X \cap \alpha| = \mu\}| = \lambda$.

Proof. Since $\mu \le \kappa$ and $\mathrm{cf}\,\mu \ne \mathrm{cf}\,\kappa$, we have $\mu < \kappa$. Next, note:

(1) For every $X \in F$ there is an $\alpha_X < \kappa$ such that $|X \cap \alpha_X| = \mu$.

For, let $\langle \beta_\xi : \xi < \nu\rangle$ enumerate X in increasing order (ν an ordinal). Thus $\nu < \kappa$ since $\mu < \kappa$. If $\mu < \nu$, then β_μ is as desired. If $\nu = \mu$, then note that X is not cofinal in κ, since $\mathrm{cf}\,\mu \ne \mathrm{cf}\,\kappa$. Hence $\sup X$ is as desired.

(2) There is an $\alpha < \kappa$ such that $|\{X \in F : \alpha_X \le \alpha\}| = \lambda$.

For, suppose not. So for all $\alpha < \kappa$, $|\{X \in F : \alpha_X \le \alpha\}| < \lambda$. Note that $F = \bigcup_{\alpha<\kappa}\{X \in F : \alpha_X \le \alpha\}$. Let $\beta_\xi \uparrow \kappa$ for $\xi < \mathrm{cf}\,\kappa$, β continuous, $\beta_0 = 0$. (If κ is regular, we can simply take $\beta_\xi = \xi$ for all $\xi < \kappa$.) Then $F = \bigcup_{\xi<\mathrm{cf}\,\kappa}\{X \in F : \alpha_X \le \beta_\xi\}$. Since $\mathrm{cf}\,\lambda \ne \mathrm{cf}\,\kappa$ and $|F| = \lambda$, we get

(3) $\mathrm{cf}\,\lambda < \mathrm{cf}\,\kappa$.

Now for all $\alpha < \kappa$ there is a $\beta \in (\alpha, \kappa)$ such that $\{X \in F : \alpha_X \le \alpha\} \subset \{X \in F : \alpha_X \le \beta\}$. Define $\xi : \mathrm{cf}\,\kappa \to \mathrm{cf}\,\kappa$ by: $\xi(0) = 0$, $\xi(\gamma + 1)$ minimum such that $\{X \in F : \alpha_X \le \beta_{\xi(\gamma)}\} \subset \{X \in F : \alpha_X \le \beta_{\xi(\gamma+1)}\}$, $\xi(\gamma) = \bigcup_{\delta<\gamma} \xi(\delta)$ for γ limit. Then choose $X_\gamma \in F$ such that $\beta_{\xi(\gamma)} < \alpha_{X_\gamma} \le \beta_{\xi(\gamma+1)}$. Thus X is a one-one function from $\mathrm{cf}\,\kappa$ into F, so $\mathrm{cf}\,\kappa \le \lambda$. Hence λ is singular by (3). Let $\gamma_\xi \uparrow \lambda$ for $\xi < \mathrm{cf}\,\lambda$, the γ_ξ's being cardinals. Then

(4) For every $\xi < \mathrm{cf}\,\lambda$ there is an $\eta_\xi < \mathrm{cf}\,\kappa$ such that $|\{X \in F : \alpha_X \le \beta_{\eta_\xi}\}| \ge \gamma_\xi$.

For, otherwise there is a $\xi < \mathrm{cf}\,\lambda$ such that for all $\eta < \mathrm{cf}\,\kappa$ we have $|\{X \in F : \alpha_X \le \beta_\eta| < \gamma_\xi$. Hence

$$|F| = \Big|\bigcup_{\eta<\mathrm{cf}\,\kappa}\{X \in F : \alpha_X \le \beta_\eta\}\Big| \le \mathrm{cf}\,\kappa \cdot \gamma_\xi < \lambda,$$

contradiction. So (4) holds.

Since $\mathrm{cf}\,\lambda < \mathrm{cf}\,\kappa$, we have $\varrho := \sup_{\xi<\mathrm{cf}\,\lambda} \eta_\xi < \mathrm{cf}\,\kappa$. Hence $|\{X \in F : \alpha_X \le \beta_\varrho\}| = \lambda$, contradicting the "suppose not" for (2).

This contradiction shows that (2) holds. Hence $|\{X \in F : |X \cap \alpha| = \mu\}| = \lambda$, as desired. ∎

COROLLARY 2.2. *Suppose that* $\nu \le \mu < \kappa \le \lambda$, $\mathrm{cf}\,\lambda \ne \mathrm{cf}\,\kappa \ne \mathrm{cf}\,\mu$, *and* $\lambda \in \mathrm{AD}(\kappa, \mu, \nu)$. *Then there is a cardinal* ϱ *with* $\mu \le \varrho < \kappa$ *such that* $\lambda \in \mathrm{AD}(\varrho, \mu, \nu)$. ∎

COROLLARY 2.3. *Suppose that* α *is a limit ordinal*, $m \in \omega$, $\omega \le \nu \le \aleph_\alpha < \aleph_{\alpha+m} < \lambda$, λ *is regular*, *and* $\lambda \in \mathrm{AD}(\aleph_{\alpha+m}, \aleph_\alpha, \nu)$. *Then* $\lambda \in \mathrm{AD}(\aleph_\alpha, \aleph_\alpha, \nu)$. ∎

The following result is due to Tarski; see Baumgartner [76, Theorem 2.7].

PROPOSITION 2.4. $\mathrm{AD}(\kappa,\mu,\nu) \subseteq [1,\kappa^\nu]$. ■

Proof. Suppose that $\mathscr{A} \subseteq [\kappa]^\mu$ is ν-ad. For all $X \in \mathscr{A}$ choose $f(X) \in [X]^\nu$. Then f is a one-one function from $\mathscr{A}$ into $[\kappa]^\nu$, so $|\mathscr{A}| \le \kappa^\nu$. ■

As a corollary, we mention some trivial cases.

PROPOSITION 2.5. *Suppose that* $\omega \le \varrho \le \nu \le \mu < (2^\nu)^{+m}$, *with* $m \in \omega$. *Then*:

(i) $\mathrm{MAD}((2^\nu)^{+m},\mu,\varrho) = \{(2^\nu)^{+m}\}$.
(ii) $\mathrm{MAD}((2^\nu)^{+m},(2^\nu)^{+m},\varrho) = [1,(2^\nu)^{+m}]$. ■

The following result comes from the proof of 2.8 of Baumgartner [76]:

PROPOSITION 2.6. *If* κ *is regular*, $\mathscr{A} \subseteq [\kappa]^\kappa$, $|\mathscr{A}| = \kappa$, *and* $\mathscr{A}$ *is* κ*-ad*, *then it is not* κ*-mad. So* $\kappa \notin \mathrm{MAD}(\kappa)$. *Hence* $\kappa^+ \in \mathrm{AD}(\kappa)$, $\mathrm{MAD}(\kappa) \cap (\kappa,\infty) \ne 0$, *and* $\kappa < \mathfrak{a}_\kappa$.

Proof. Let $\mathscr{A} = \{A_\alpha : \alpha < \kappa\}$. By induction define

$$x_\alpha \in A_\alpha \setminus \Big(\bigcup_{\beta<\alpha} A_\beta \cup \{x_\beta : \beta < \alpha\}\Big);$$

then $|\{x_\alpha : \alpha < \kappa\} \cap A_\alpha| < \kappa$ for all $\alpha < \kappa$, showing that $\mathscr{A}$ is not maximal. ■

The following is part of 2.2(c) of Baumgartner [76]:

PROPOSITION 2.7. *If* $\lambda_\alpha \in \mathrm{AD}(\kappa,\mu,\nu)$ *for all* $\alpha < \varrho$, *where* $\varrho \le \kappa$, *then* $\sum_{\alpha<\varrho} \lambda_\alpha \in \mathrm{AD}(\kappa,\mu,\nu)$.

Proof. Let $\kappa = \bigcup_{\alpha<\varrho} \Gamma_\alpha$, each Γ_α of size κ and the Γ_α's pairwise disjoint. Let $\mathscr{A}_\alpha \subseteq [\Gamma_\alpha]^\mu$ be of size λ_α and ν-ad. Then $\mathscr{B} := \bigcup_{\alpha<\varrho} \mathscr{A}_\alpha$ is as desired. ■

The following is Theorem 2.10 of Baumgartner [76]:

PROPOSITION 2.8. *Suppose that* $\nu,\sigma,\mu,\kappa,\lambda$ *are cardinals*, *and* $\nu \le \sigma \le \mu \le \kappa \le \lambda$. *Suppose that* $\lambda \in \mathrm{AD}(\kappa,\mu,\nu)$, *and for all* $\alpha < \lambda$, $\lambda_\alpha \in \mathrm{AD}(\mu,\sigma,\nu)$. *Then* $\sum_{\alpha<\lambda} \lambda_\alpha \in \mathrm{AD}(\kappa,\sigma,\nu)$.

Proof. Let $\mathscr{A}$ be $[\kappa]^\mu,\nu$-ad with $|\mathscr{A}| = \lambda$. Let $\langle X_\alpha : \alpha < \lambda\rangle$ enumerate $\mathscr{A}$ without repetitions. For each $\alpha < \lambda$ let $\mathscr{B}_\alpha$ be $[X_\alpha]^\sigma,\nu$-ad, with $|\mathscr{B}_\alpha| = \lambda_\alpha$. Then $\bigcup_{\alpha<\lambda} \mathscr{B}_\alpha$ is $[\kappa]^\sigma,\nu$-ad and has size $\sum_{\alpha<\lambda} \lambda_\alpha$. ■

COROLLARY 2.9. *If* κ *is an infinite cardinal*, $\lambda \in \mathrm{AD}(\kappa)$, *and* $\kappa \le \lambda_\alpha \in \mathrm{AD}(\kappa)$ *for all* $\alpha < \lambda$, *then* $\sum_{\alpha<\lambda} \lambda_\alpha \in \mathrm{AD}(\kappa)$. ■

3. Concerning $\mathrm{AD}(\kappa,\kappa,\nu)$

PROPOSITION 3.1. *Suppose that* $\omega \le \kappa < \lambda$, λ *singular*, *and* $\{\varrho < \lambda : \varrho \in \mathrm{AD}(\kappa,\kappa,\nu)\}$ *is unbounded in* λ. *Then* $\lambda \in \mathrm{AD}(\kappa,\kappa,\nu)$.

Proof. Let $\langle \varrho_\alpha : \alpha < \mathrm{cf}\,\lambda\rangle$ be a strictly increasing sequence of cardinals with limit λ such that $\varrho_\alpha \in \mathrm{AD}(\kappa,\kappa,\nu)$ for all $\alpha < \mathrm{cf}\,\lambda$. Let $\mathscr{A} \subseteq [\kappa]^\kappa$ be ν-ad, $|\mathscr{A}| = \mathrm{cf}\,\lambda$. Let $\langle X_\alpha : \alpha < \varrho\rangle$ be a one-one enumeration of $\mathscr{A}$. For all $\alpha < \varrho$, let $\mathscr{A}_\alpha \subseteq [X_\alpha]^\kappa$ be of size ϱ_α and be ν-ad. Then $\bigcup_{\alpha<\varrho} \mathscr{A}_\alpha$ is as desired. ■

PROPOSITION 3.2. *If* $\nu \le \kappa$, $\kappa^+ < \lambda$, *and* $\lambda \in \mathrm{AD}(\kappa^+,\kappa^+,\nu)$, *then* $\lambda \in \mathrm{AD}(\kappa,\kappa,\nu)$.

Proof. First we prove the special case of the proposition in which $\kappa^+ < \operatorname{cf}\lambda$.

Let $\mathscr{A} \subseteq [\kappa^+]^{\kappa^+}$ be ν-ad, with $|\mathscr{A}| = \lambda$. For every $X \in \mathscr{A}$ there is an $\alpha_X < \kappa^+$ such that $|X \cap \alpha_X| = \kappa$. In fact, one can enumerate X as $\langle x_\alpha : \alpha < \kappa^+\rangle$ in increasing order, and then let $\alpha_X = x_\kappa$. Now

$$\mathscr{A} = \bigcup_{\alpha<\kappa^+} \{X \in \mathscr{A} : \alpha_X = \alpha\},$$

so there exist an $\alpha < \kappa^+$ and an $\mathscr{A}' \in [\mathscr{A}]^\lambda$ such that $\alpha_X = \alpha$ for all $X \in \mathscr{A}'$. Then $\{X \cap \alpha : X \in \mathscr{A}'\}$ is as desired (since $|\alpha| = \kappa$).

This finishes the proof of the special case.

In the general case, we may assume that $\operatorname{cf}\lambda \le \kappa^+$. If μ is regular and $\kappa^+ < \mu < \lambda$, then $\mu \in \mathrm{AD}(\kappa^+, \kappa^+, \nu)$. So by the special case, $\mu \in \mathrm{AD}(\kappa, \kappa, \nu)$. It follows by 3.1 that $\lambda \in \mathrm{AD}(\kappa, \kappa, \nu)$. ∎

Corollary 3.3. *If* $m \in \omega$, $\nu \le \kappa$, $\kappa^{+m} < \lambda$, *and* $\lambda \in \mathrm{AD}(\kappa^{+m}, \kappa^{+m}, \nu)$, *then* $\lambda \in \mathrm{AD}(\kappa, \kappa, \nu)$. ∎

Part of the proof of Theorem 3.7 of Baumgartner [76] can be generalized to give the following.

Proposition 3.4. *Suppose that* $\operatorname{cf}\lambda > \kappa > \nu$, $\operatorname{cf}\kappa > \omega$, κ *is a limit cardinal*, *and* $\lambda \in \mathrm{AD}(\kappa, \kappa, \nu)$. *Then* $\{\varrho : \varrho$ *is a cardinal*, $\varrho \in (\nu, \kappa)$, $\lambda \in \mathrm{AD}(\varrho, \varrho, \nu)\}$ *is unbounded in* κ.

Proof. Suppose that $\delta < \kappa$, δ a cardinal; we want to find a ϱ as above in the interval (δ, κ). Let $\mathscr{A} \subseteq [\kappa]^\kappa$ be ν-ad and of size λ.

(1) For every $X \in \mathscr{A}$ there is an ordinal $\alpha_X \in (\max(\nu, \delta), \kappa)$ such that $|X \cap \alpha_X| = |\alpha_X|$.

For, enumerate X as $\langle \beta_\xi : \xi < \kappa\rangle$ in increasing order. Choose $\alpha_0 < \kappa$ such that $\max(\nu, \delta) < \alpha_0$. Define $\alpha_{i+1} = \beta_{\alpha_i} + 1$ for all $i < \omega$, and let $\alpha_\omega = \sup_{i<\omega} \alpha_i$. Thus $\alpha_\omega < \kappa$ since $\operatorname{cf}\kappa > \omega$. For all $\xi < \alpha_\omega$ let $f(\xi) = \beta_\xi$. For any such ξ choose $i < \omega$ such that $\xi < \alpha_i$. Then $\beta_\xi < \beta_{\alpha_i} < \alpha_{i+1} \le \alpha_\omega$. Thus $f(\xi) \in X \cap \alpha_\omega$. Since f is clearly one-one, it follows that $|\alpha_\omega| \le |X \cap \alpha_\omega|$, and (1) is established.

By (1), $\mathscr{A} = \bigcup_{\alpha\in(\max(\nu,\delta),\kappa)} \{X \in \mathscr{A} : \alpha_X = \alpha\}$, so since $\operatorname{cf}\lambda > \kappa$, there is a $\beta \in (\max(\nu, \delta), \kappa)$ such that $\{X \in \mathscr{A} : \alpha_X = \beta\}$ has size λ. Hence $\{X \cap \beta : X \in \mathscr{A}\} \subseteq [\beta]^{|\beta|}$ is ν-ad and of size λ. ∎

Another part of the proof of Theorem 3.7 in Baumgartner [76] generalizes to give the following.

Proposition 3.5. *Suppose that* κ, ν, λ *are infinite cardinals*, $\operatorname{cf}\lambda > \kappa > \nu$, $\operatorname{cf}\kappa = \omega$, $\kappa \ne \aleph_\kappa$, κ *is a limit cardinal*, *and* $\lambda \in \mathrm{AD}(\kappa, \kappa, \nu)$. *Then* $\{\varrho : \varrho$ *is a cardinal*, $\varrho \in (\nu, \kappa)$, $\lambda \in \mathrm{AD}(\varrho, \varrho, \nu)\}$ *is unbounded in* κ.

Proof. Let an ordinal $\delta < \kappa$ be given. From the assumption $\kappa \ne \aleph_\kappa$ we get:

(1) There is an uncountable regular cardinal $\varrho \in (\max(\delta, \nu), \kappa)$ such that $\kappa < \aleph_\varrho$.

Thus by 1.2, $\lambda \in \mathrm{AD}(\kappa, \varrho, \nu)$. Let σ be minimum such that $\varrho \le \sigma$ and $\lambda \in \mathrm{AD}(\sigma, \varrho, \nu)$. Suppose that $\sigma \ne \varrho$. By 2.2, $\operatorname{cf}\sigma = \operatorname{cf}\lambda$ or $\operatorname{cf}\sigma = \varrho$. Since $\operatorname{cf}\lambda > \kappa \ge \sigma$, we have

$\mathrm{cf}\,\sigma \neq \mathrm{cf}\,\lambda$. So $\mathrm{cf}\,\sigma = \varrho > \omega$. Say $\sigma = \aleph_\beta$. Then $\mathrm{cf}\,\sigma = \mathrm{cf}\,\beta = \varrho$, so $\varrho \leq \beta$, and $\aleph_\varrho \leq \aleph_\beta = \sigma \leq \kappa$, contradiction. ∎

Proposition 3.6. *Suppose that* $\mathrm{cf}\,\lambda > \kappa > \nu$ *and* $\lambda \in \mathrm{AD}(\kappa,\kappa,\nu)$. *Then there is a* $\sigma < \kappa$ *such that* $\sigma^\nu \geq \lambda$.

Proof. Choose $\varrho \in [\nu,\kappa)$ such that $\mathrm{cf}\,\varrho \neq \mathrm{cf}\,\kappa$. (If κ is regular, let $\varrho = \nu$. If κ is singular, let $\varrho = \nu^+$ if $\nu^+ \neq \mathrm{cf}\,\kappa$, and $\varrho = \nu^{++}$ otherwise.) Then $\lambda \in \mathrm{AD}(\kappa,\varrho,\nu)$ by 1.2. Let σ be minimum such that $\lambda \in \mathrm{AD}(\sigma,\varrho,\nu)$. Then by 2.2, $\mathrm{cf}\,\sigma = \mathrm{cf}\,\lambda$ or $\mathrm{cf}\,\sigma = \mathrm{cf}\,\varrho$. Since $\mathrm{cf}\,\lambda > \kappa \geq \sigma$, we have $\mathrm{cf}\,\sigma \neq \mathrm{cf}\,\lambda$. Hence $\mathrm{cf}\,\sigma = \mathrm{cf}\,\varrho$. By the choice of ϱ we then have $\sigma < \kappa$. And $\lambda \leq \sigma^\nu$ by 2.4. ∎

Proposition 3.7. *Suppose that* $\lambda > \kappa > \nu$, κ *is a limit cardinal*, $\kappa \neq \aleph_\kappa$, *and* $\lambda \in \mathrm{AD}(\kappa,\kappa,\nu)$. *Then* $\{\varrho : \varrho$ *is a cardinal*, $\varrho \in (\nu,\kappa)$, $\lambda \in \mathrm{AD}(\varrho,\varrho,\nu)\}$ *is unbounded in* κ.

Proof. For λ regular the result follows by 3.4 and 3.5. Now suppose that λ is singular. It suffices to show that $\lambda \in \mathrm{AD}(\varrho,\varrho,\nu)$ for any regular $\varrho \in [\nu,\kappa)$ such that $\kappa < \aleph_\varrho$. By 3.1 it suffices to take any regular $\sigma \in (\kappa,\lambda)$ and show that $\sigma \in \mathrm{AD}(\varrho,\varrho,\nu)$. We have $\sigma \in \mathrm{AD}(\kappa,\varrho,\nu)$ by 1.2. Let τ be minimum such that $\sigma \in \mathrm{AD}(\tau,\varrho,\nu)$. Suppose that $\varrho < \tau$. By 2.2, $\mathrm{cf}\,\tau = \mathrm{cf}\,\sigma$ or $\mathrm{cf}\,\tau = \mathrm{cf}\,\varrho$. Now $\sigma > \kappa \geq \tau$ and σ is regular, so $\mathrm{cf}\,\tau \neq \mathrm{cf}\,\sigma$. Thus $\mathrm{cf}\,\tau = \varrho$. Say $\tau = \aleph_\alpha$. Then $\varrho = \mathrm{cf}\,\tau = \mathrm{cf}\,\alpha$, so $\varrho \leq \alpha$. Hence $\aleph_\varrho \leq \aleph_\alpha = \tau \leq \kappa$, contradiction. ∎

As an application, if $\aleph_{\omega+1} \in \mathrm{AD}(\aleph_\omega,\aleph_\omega,\omega)$, then $\aleph_{\omega+1} \in \mathrm{AD}(\aleph_\alpha,\aleph_\alpha,\omega)$ for some $\alpha < \omega$, and hence by 3.3, $\aleph_{\omega+1} \in \mathrm{AD}(\omega)$.

Another useful fact about singular κ is as follows.

Proposition 3.8. *Suppose that* κ *is singular*, $\langle \mu_\xi : \xi < \mathrm{cf}\,\kappa\rangle$ *is an increasing sequence of infinite cardinals with supremum* κ, *and* λ *is some cardinal* $\geq \kappa$. *Suppose that* ν *is regular*, $\nu \leq \mu_0$, $\nu \leq \mathrm{cf}\,\kappa$. *Assume that* $\lambda \in \mathrm{AD}(\mu_\xi,\mu_\xi,\nu)$ *for all* $\xi < \mathrm{cf}\,\kappa$, *and also* $\lambda \in \mathrm{AD}(\mathrm{cf}\,\kappa,\mathrm{cf}\,\kappa,\nu)$. *Then* $\lambda \in \mathrm{AD}(\kappa,\kappa,\nu)$.

Proof. Write $\kappa = \bigcup_{\xi<\mathrm{cf}\,\kappa} \Gamma_\xi$, the Γ_ξ's pairwise disjoint, $|\Gamma_\xi| = \mu_\xi$. For each $\xi < \mathrm{cf}\,\kappa$ let A^ξ be a one-one function from λ onto a subset of $[\Gamma_\xi]^{\mu_\xi}$ which is ν-ad, and let B be a one-one function from λ onto a subset of $[\mathrm{cf}\,\kappa]^{\mathrm{cf}\,\kappa}$ which is ν-ad. For each $\alpha < \lambda$, let

$$C_\alpha = \bigcup_{\xi \in B_\alpha} A^\xi_\alpha.$$

Clearly $\{C_\alpha : \alpha < \lambda\}$ is the desired family. ∎

The following result shows that Proposition 3.5 cannot be generalized by merely dropping the hypothesis "$\kappa \neq \aleph_\kappa$".

Example 3.9. *There is a singular cardinal* κ *such that* $\mathrm{cf}\,\kappa = \omega$, $\kappa = \aleph_\kappa$, $\kappa^\omega \in \mathrm{AD}(\kappa,\kappa,\omega)$, *and there is no* $\varrho \in (\omega,\kappa)_{\mathrm{card}}$ *such that* $\kappa^\omega \in \mathrm{AD}(\varrho,\varrho,\omega)$.

Proof. We define, by recursion, $\mu_0 = \omega$ and, for any $m \in \omega$,

$$\mu_{m+1} = \aleph_{\mu_m^\omega+1}.$$

Let $\kappa = \sup_{m\in\omega} \mu_m$. Clearly κ is singular with cofinality ω. Suppose that $\kappa < \aleph_\kappa$. Choose $\alpha < \kappa$ such that $\kappa < \aleph_\alpha$. Say $\alpha < \mu_m$. then

$$\kappa < \aleph_\alpha < \aleph_{\mu_m} < \mu_{m+1} < \kappa,$$

contradiction. Thus $\kappa = \aleph_\kappa$.

Now a standard construction yields a family $\langle A_\alpha : \alpha < \kappa^\omega\rangle$ of denumerable subsets of κ such that any two have a finite intersection. Write $\kappa = \bigcup_{\alpha<\kappa} \Gamma_\alpha$, where the Γ_α's are pairwise disjoint and of size κ. For each $\beta < \kappa$ let f_β be a bijection from κ onto Γ_β. Now we set, for each $\alpha < \kappa^\omega$,

$$B_\alpha = \bigcup_{\beta\in A_\alpha} f_\beta[A_\alpha].$$

So each B_α is a subset of κ of size κ. If $\alpha, \gamma < \kappa^\omega$ and $\alpha \neq \gamma$, then

$$B_\alpha \cap B_\gamma = \Big(\bigcup_{\beta\in A_\alpha} f_\beta[A_\alpha]\Big) \cap \Big(\bigcup_{\beta\in A_\gamma} f_\beta[A_\gamma]\Big) = \bigcup_{\beta\in A_\alpha\cap A_\gamma} f_\beta[A_\alpha \cap A_\gamma],$$

and this set is finite, as desired.

Now suppose that $\varrho \in (\omega, \kappa)$ and $\kappa^\omega \in \mathrm{AD}(\varrho, \varrho, \omega)$. Then by Proposition 2.4 we have $\kappa^\omega \le \varrho^\omega$. Since $\varrho < \kappa$, choose $m \in \omega$ such that $\varrho < \mu_m$. Then

$$\varrho^\omega \le \mu_m^\omega \le \aleph_{\mu_m^\omega} < \mu_{m+1} < \kappa \le \varrho^\omega,$$

contradiction. ■

Concerning all these results we mention two problems. 3.7 and 3.9 suggest

Problem 1. *If $\lambda > \kappa > \nu$, κ is a limit cardinal, $\kappa = \aleph_\kappa$, $\lambda \le \varrho^\nu$ for some $\varrho < \kappa$, and $\lambda \in \mathrm{AD}(\kappa, \kappa, \nu)$, is there a $\varrho \in (\nu, \kappa)$ such that $\lambda \in \mathrm{AD}(\varrho, \varrho, \nu)$?*

In turn, 3.1 suggests

Problem 2. *Suppose that $\omega \le \kappa < \lambda$, λ is weakly inaccessible, and $\{\varrho < \lambda : \varrho \in \mathrm{AD}(\kappa, \kappa, \nu)\}$ is unbounded in λ. Does it follow that $\lambda \in \mathrm{AD}(\kappa, \kappa, \nu)$?*

4. On $\mathrm{MAD}(\kappa)$ and $\mathrm{MAD}(\kappa, \mu, \mu)$

Theorem 2.9 of Baumgartner [76] gives:

Proposition 4.1. *Suppose that μ is singular, $\mu \le \kappa$, and $\mathrm{cf}\,\mu \le \mathrm{cf}\,\kappa$. Then:*

(i) *If $\mathscr{A} \subseteq [\kappa]^{\mathrm{cf}\,\mu}$ is $\mathrm{cf}\,\mu$-ad and $|\mathscr{A}| > \kappa$, then there is a $\mathscr{B} \subseteq [\kappa]^\mu$ of size $|\mathscr{A}|$ which is μ-ad.*

(ii) $\mathrm{MAD}(\kappa, \mathrm{cf}\,\mu, \mathrm{cf}\,\mu) \cap (\kappa, \infty) \preceq \mathrm{MAD}(\kappa, \mu, \mu)$.

Proof. Without loss of generality, each member of $\mathscr{A}$ has order type $\mathrm{cf}\,\mu$. For all $\alpha \le \kappa$ let $F_\alpha = \{X \in \mathscr{A} : \sup X = \alpha\}$. So $\mathscr{A} = \bigcup_{\alpha\le\kappa} F_\alpha$, and the F_α's are pairwise disjoint. By Proposition 2.7, it is enough to show that $|F_\alpha| \in \mathrm{AD}(\kappa, \mu, \mu)$ for all $\alpha \le \kappa$ with $F_\alpha \neq 0$. Fix α with $F_\alpha \neq 0$. Let $\nu_\xi \uparrow \mu$ for $\xi < \mathrm{cf}\,\mu$, and let $\langle \varrho_\xi : \xi < \mathrm{cf}\,\mu\rangle$ be a continuous strictly increasing sequence of ordinals with supremum α, and with $\varrho_0 = 0$. Let $\langle Y_\eta : \eta < \alpha\rangle$ be pairwise disjoint subsets of κ, with $|Y_\eta| = \nu_\xi$ if $\varrho_\xi \le \eta < \varrho_{\xi+1}$. For each $X \in F_\alpha$ let $X' = \bigcup_{\eta\in X} Y_\eta$. Then $\{X' : X \in F_\alpha\}$ is as desired. ■

The following theorem generalizes a result of Erdős and Hechler; see Milner–Prikry [87, Theorem 3.1].

Theorem 4.2. *Suppose that* $\varrho \le \tau \in \mathrm{MAD}(\kappa,\mu,\mu)$, f *maps* τ *onto* ϱ, *and* $\langle \sigma_\alpha : \alpha < \varrho\rangle \in {}^{\varrho}\mathrm{MAD}(\mu)$. *Then* $\sum_{\alpha<\tau} \sigma_{f(\alpha)} \in \mathrm{MAD}(\kappa,\mu,\mu)$.

Proof. Let $\mathscr{A}$ be $[\kappa]^\mu$, μ-mad with $|\mathscr{A}| = \tau$, and let g be a one-one function from $\mathscr{A}$ onto τ. For each $A \in \mathscr{A}$ let $\mathscr{B}_A$ be $[A]^\mu$, μ-mad with $|\mathscr{B}_A| = \sigma_{f(g(A))}$. Let $\mathscr{C} = \bigcup_{A\in\mathscr{A}} \mathscr{B}_A$. Clearly $|\mathscr{C}| = \sum_{\alpha<\tau} \sigma_{f(\alpha)}$. It is also clear that $\mathscr{C}$ is μ-ad. Now suppose that $X \in [\kappa]^\mu$. Choose $A \in \mathscr{A}$ such that $|A \cap X| = \mu$. Then choose $Y \in \mathscr{B}_A$ such that $|A \cap X \cap Y| = \mu$. So $|X \cap Y| = \mu$. Hence $\mathscr{C}$ is $[\kappa]^\mu$, μ-mad, as desired. ∎

Corollary 4.3. *Suppose that* $\varrho \in \mathrm{MAD}(\kappa,\mu,\mu)$ *and* $\langle \sigma_\alpha : \alpha < \varrho\rangle \in {}^{\varrho}\mathrm{MAD}(\mu)$. *Then* $\sum_{\alpha<\varrho} \sigma_\alpha \in \mathrm{MAD}(\kappa,\mu,\mu)$. ∎

Corollary 4.4. *If* $\varrho \in \mathrm{MAD}(\kappa,\mu,\mu)$ *and* $\sigma \in \mathrm{MAD}(\mu)$, *then* $\varrho\cdot\sigma \in \mathrm{MAD}(\kappa,\mu,\mu)$. ∎

Corollary 4.5. *Suppose that* δ *is singular and is a limit of cardinals in* $\mathrm{MAD}(\mu)$. *Further, assume that* $[\mathrm{cf}\,\delta,\delta) \cap \mathrm{MAD}(\kappa,\mu,\mu) \ne 0$. *Then* $\delta \in \mathrm{MAD}(\kappa,\mu,\mu)$. ∎

Corollary 4.6 (Milner–Prikry [87, Theorem 3.1]). *If* δ *is a singular cardinal which is a limit of members of* $\mathrm{MAD}(\kappa)$, *then* $\delta \in \mathrm{MAD}(\kappa)$. ∎

This corollary naturally leads to the following questions.

Problem 3. *If* δ *is a regular limit cardinal which is a limit of members of* $\mathrm{MAD}(\kappa)$, *is also* $\delta \in \mathrm{MAD}(\kappa)$*?*

Problem 4. *If* δ *is a limit cardinal which is a limit of members of* $\mathrm{MAD}(\kappa,\mu,\nu)$, *is also* $\delta \in \mathrm{MAD}(\kappa,\mu,\nu)$*?*

The following result generalizes Theorem 3.6 in Milner–Prikry [87], also due to Erdős and Hechler.

Theorem 4.7. *Let* μ *be a singular cardinal. Suppose that* $\nu \in \mathrm{MAD}(\mathrm{cf}\,\mu)$ *and* $\varrho \in \mathrm{MAD}(\kappa,\mu,\mu)$. *Then* $\varrho\cdot\nu \in \mathrm{MAD}(\kappa,\mu,\mu)$.

Proof. Let $\mathscr{A}$ be $[\kappa]^\mu$, μ-mad, with $|\mathscr{A}| = \varrho$. For each $A \in \mathscr{A}$, let $\langle S^A_\alpha : \alpha < \mathrm{cf}\,\mu\rangle$ be a partition of A into sets of size less than μ, with $\langle |S^A_\alpha : \alpha < \mathrm{cf}\,\mu\rangle$ strictly increasing. Let $\mathscr{B}$ be $\mathrm{cf}\,\mu$-mad, with $|\mathscr{B}| = \nu$. Then we define

$$\mathscr{C} = \Big\{ \bigcup_{\alpha\in B} S^A_\alpha : A \in \mathscr{A},\ B \in \mathscr{B}\Big\}.$$

Clearly each member of $\mathscr{C}$ has size μ. Suppose that $X, Y \in \mathscr{C}$ with $X \ne Y$. Say $X = \bigcup_{\alpha\in B_0} S^{A_0}_\alpha$ and $Y = \bigcup_{\alpha\in B_1} S^{A_1}_\alpha$, with $B_0, B_1 \in \mathscr{B}$ and $A_0, A_1 \in \mathscr{A}$. If $A_0 \ne A_1$, then $X \cap Y \subseteq A_0 \cap A_1$, and the latter has size less than μ. If $A_0 = A_1$ and $B_0 \ne B_1$, then $X \cap Y = \bigcup_{\alpha\in B_0\cap B_1} S^{A_0}_\alpha$, and $|B_0 \cap B_1| < \mathrm{cf}\,\mu$, and hence $|X \cap Y| < \mu$. Thus $\mathscr{C}$ is μ-ad. Now suppose that $X \in [\kappa]^\mu$. Choose $A \in \mathscr{A}$ such that $|X \cap A| = \mu$. Hence $\sup\{|X \cap S^A_\alpha| : \alpha < \mathrm{cf}\,\mu\} = \mu$, so there is a strictly increasing sequence $\langle \alpha_\xi : \xi < \mathrm{cf}\,\mu\rangle$ of ordinals less than $\mathrm{cf}\,\mu$ such that the sequence $\langle |X \cap S^A_{\alpha_\xi}| : \xi < \mathrm{cf}\,\mu\rangle$ is strictly increasing with supremum μ. Let $Y = \{\alpha_\xi : \xi < \mathrm{cf}\,\mu\}$. Choose $B \in \mathscr{B}$ such that $|Y \cap B| = \mathrm{cf}\,\mu$. Clearly, then, $|X \cap \bigcup_{\alpha\in B} S^A_\alpha| = \mu$. ∎

THEOREM 4.8 (Milner–Prikry [87, Theorem 3.6]). $\mathrm{MAD}(\mathrm{cf}\,\kappa) \subseteq \mathrm{MAD}(\kappa)$. ∎

Recall from 1.1(i) that $\mathrm{MAD}(\kappa) \cap [\kappa, \infty) \neq 0$. So the other inclusion in 4.8 does not hold for singular cardinals, at least in ZFC. Also, $\mathrm{MAD}(\mathrm{cf}\,\kappa) \subseteq [1, \mathrm{cf}\,\kappa^{\mathrm{cf}\kappa}]$ by Proposition 2.4. Thus the following problem is the natural "converse" of 4.8:

PROBLEM 5. *If κ is singular, does the inclusion* $\mathrm{MAD}(\kappa) \cap [1, 2^{\mathrm{cf}\kappa}] \subseteq \mathrm{MAD}(\mathrm{cf}\,\kappa)$ *hold?*

In this connection we mention some related results which will not be proved here. In Kunen [80] it is shown that

$$(\star) \qquad \mathrm{MA} \vdash \mathrm{MAD}(\omega) = \{2^\omega\}.$$

In Kojman–Kubiś–Shelah [∞] the following two results are shown:

$$(\star\star) \qquad \mathrm{MA} + 2^{\aleph_0} > \aleph_\omega \vdash \aleph_\omega \notin \mathrm{MAD}(\aleph_\omega),$$

$(\star\star\star)$ (in ZFC) if μ is singular and $2^{\mathrm{cf}\mu} < \mu$, then $\mu \in \mathrm{MAD}(\mu)$.

PROPOSITION 4.9. *If κ is singular and $\mathscr{A} \subseteq [\kappa]^{\mathrm{cf}\kappa}$ is $\mathrm{cf}\,\kappa$-ad with $|\mathscr{A}| = \kappa$, then it is not mad. Hence $\kappa \notin \mathrm{MAD}(\kappa, \mathrm{cf}\,\kappa, \mathrm{cf}\,\kappa)$, $0 \neq \mathrm{MAD}(\kappa, \mathrm{cf}\,\kappa, \mathrm{cf}\,\kappa) \subseteq (\kappa, \infty)$, and $\kappa < \mathfrak{a}_{\kappa\mathrm{cf}\kappa\mathrm{cf}\kappa}$.*

Proof. Say $\mu_\alpha \uparrow \kappa$ for $\alpha < \mathrm{cf}\,\kappa$. Let $\mathscr{A} = \{A_\alpha : \alpha < \kappa\}$. For each $\alpha < \mathrm{cf}\,\kappa$ choose

$$x_\alpha \in A_\alpha \setminus \Big(\bigcup_{\beta < \mu_\alpha} A_\beta \cup \{x_\beta : \beta < \alpha\} \Big).$$

Let $X = \{x_\alpha : \alpha < \mathrm{cf}\,\kappa\}$. ∎

PROPOSITION 4.10. *If $\mu < \kappa$, then $\mathrm{MAD}(\kappa, \mu, \mu) \sqsupseteq \mathrm{MAD}(\mu) \cap [\mu, \infty)$. Hence $\mathfrak{a}_\mu \leq \mathfrak{a}_{\kappa\mu\mu}$.*

Proof. Note by Proposition 1.1(ii) that $\mathrm{MAD}(\kappa, \mu, \mu) \subseteq [\kappa, \infty)$. Fix a family $\mathscr{A}$ which is $[\kappa]^\mu, \mu$-mad. Fix $\mathscr{A}' \in [\mathscr{A}]^\mu$. Let $\Gamma = \bigcup_{A \in \mathscr{A}'} A$, and let $\mathscr{B} = \{X \cap \Gamma : X \in \mathscr{A}, |X \cap \Gamma| = \mu\}$. Thus $|\Gamma| = \mu$, $\mathscr{B}$ is μ-ad, and $\mu \leq |\mathscr{B}| \leq |\mathscr{A}|$. Hence it suffices to show that $\mathscr{B}$ is $[\Gamma]^\mu, \mu$-mad. Suppose that $Y \in [\Gamma]^\mu$. Since $\mathscr{A}$ is $[\kappa]^\mu, \mu$-mad, there is an $X \in \mathscr{A}$ such that $|X \cap Y| = \mu$. Thus there is a $Z \in \mathscr{B}$ such that $|Y \cap Z| = \mu$, as desired. ∎

For the next result we need a simple set-theoretic lemma.

LEMMA 4.11. *Suppose that $\kappa < \nu$ are infinite cardinals, β is an ordinal, and $\langle \Gamma_\alpha : \alpha < \beta \rangle$ is a sequence of subsets of ν such that if $\alpha < \gamma < \beta$ then $\Gamma_\alpha \subseteq \Gamma_\gamma$. Further, assume that $|\nu \setminus \bigcup_{\alpha<\beta} \Gamma_\alpha| < \nu$. Then there is an $\alpha < \beta$ such that $|\Gamma_\alpha| \geq \kappa$.*

Proof. Suppose not: $\forall \alpha < \beta\ (|\Gamma_\alpha| < \kappa)$. Then

(1) $\forall \alpha < \beta\ \exists \gamma \in (\alpha, \beta)\ [\Gamma_\alpha \subset \Gamma_\gamma]$.

For, otherwise we get an $\alpha < \beta$ such that $\Gamma_\alpha = \bigcup_{\gamma<\beta} \Gamma_\gamma$. Hence $|\bigcup_{\gamma<\beta} \Gamma_\gamma| = |\Gamma_\alpha| < \kappa$ and so $|\nu \setminus \bigcup_{\gamma<\beta} \Gamma_\gamma| = \nu$, contradiction. So (1) holds.

Now define $\alpha \in {}^\beta\beta$ by setting $\alpha_0 = 0$, $\alpha_{\xi+1}$ minimum such that $\Gamma_{\alpha_\xi} \subset \Gamma_{\alpha_{\xi+1}}$, α continuous. For each $\xi < \beta$ let η_ξ be the least element of $\Gamma_{\alpha_{\xi+1}} \setminus \Gamma_{\alpha_\xi}$. Now $|\bigcup_{\gamma<\beta} \Gamma_\gamma| = \nu$ and $|\Gamma_\gamma| < \kappa$ for all $\gamma < \beta$, so $\beta \geq \nu$. Hence $\kappa < \beta$. Now $\eta \restriction \kappa$ is a one-one function mapping into Γ_{α_κ}, so $|\Gamma_{\alpha_\kappa}| \geq \kappa$, contradiction. ∎

PROPOSITION 4.12. *If $\omega \leq \mu < \kappa < \nu$, then $\mathrm{MAD}(\nu, \mu, \mu) \sqsupseteq \mathrm{MAD}(\kappa, \mu, \mu) \cap [\kappa, \infty)$. Hence $\mathfrak{a}_{\kappa\mu\mu} \leq \mathfrak{a}_{\nu\mu\mu}$.*

Proof. Let $\mathscr{A}$ be $[\nu]^\mu, \mu$-mad. By 1.1(ii), $|\mathscr{A}| \geq \nu$.

(1) There is a $\Gamma \in [\nu]^\kappa$ such that $|\{A \in \mathscr{A} : |A \cap \Gamma| = \mu\}| \geq \kappa$.

To see this, let $\langle A_\alpha : \alpha < \tau\rangle$ enumerate $\mathscr{A}$ without repetitions, and for each $\alpha \leq \tau$ let $\Gamma_\alpha = \bigcup_{\beta<\alpha} A_\beta$. Now since $(\nu \setminus \Gamma_\tau) \cap A_\alpha = 0$ for all $\alpha < \tau$ and $\mathscr{A}$ is $[\nu]^\mu, \mu$-mad, we have $|\nu \setminus \Gamma_\tau| < \mu$. Hence by Lemma 4.11, there is an $\alpha \leq \tau$ such that $|\Gamma_\alpha| \geq \kappa$; choose the least such α. If $\alpha = \beta + 1$, then $\Gamma_\alpha = \Gamma_\beta \cup A_\beta$, and $|\Gamma_\alpha| \leq |\Gamma_\beta| + |A_\beta| < \kappa$, contradiction. So α is a limit ordinal. Let $\Delta = \{\xi < \alpha : \Gamma_\xi \subset \Gamma_{\xi+1}\}$. So $\Gamma_\alpha = \bigcup_{\xi\in\Delta}(\Gamma_{\xi+1} \setminus \Gamma_\xi)$, and $|\Gamma_{\xi+1} \setminus \Gamma_\xi| \leq |A_\xi| = \mu$. Hence $|\Delta| \geq \kappa$. If $|\Delta| > \kappa$, then there is a $\xi \in \Delta$ such that $|\{\eta \in \Delta : \eta < \xi\}| = \kappa$. Hence $\Gamma_{\xi+1} = \bigcup_{\eta\in\Delta,\,\eta\leq\xi} \Gamma_{\eta+1} \setminus \Gamma_\eta$, and each $\Gamma_{\eta+1} \setminus \Gamma_\eta \neq 0$, so $|\Gamma_{\xi+1}| \geq \kappa$, contradicting the minimality of α. So $|\Delta| = \kappa$, and hence $|\Gamma_\alpha| = \kappa$ and $|\{A \in \mathscr{A} : |A \cap \Gamma_\alpha| = \mu\}| \geq |\{A_\xi : \xi \in \Delta\}| \geq \kappa$. So (1) holds.

Let $\mathscr{B} = \{A \cap \Gamma : A \in \mathscr{A},\ |A \cap \Gamma| = \mu\}$. So $\kappa \leq |\mathscr{B}| \leq |\mathscr{A}|$. We claim that $\mathscr{B}$ is $[\Gamma]^\mu, \mu$-mad. Clearly it is $[\Gamma]^\mu, \mu$-ad. Suppose that $X \in [\Gamma]^\mu$. Then $X \in [\nu]^\mu$, so choose $A \in \mathscr{A}$ such that $|A \cap X| = \mu$. So $A \cap \Gamma \in \mathscr{B}$ and $|A \cap \Gamma \cap X| = |A \cap X| = \mu$, as desired. ∎

The following is Theorem 3.2 in Milner–Prikry [87].

THEOREM 4.13. *Suppose that κ and ν are infinite cardinals, κ is singular, and $\nu < \kappa$. Then there is a $\delta \in \mathrm{MAD}(\kappa)$ such that $\nu \leq \delta \leq \nu^{\mathrm{cf}\,\kappa}$.*

Proof. Let $\lambda_\alpha \uparrow \kappa$ for $\alpha < \mathrm{cf}\,\kappa$, with $\nu, \mathrm{cf}\,\kappa < \lambda_0$. Let $\langle S_{\alpha\beta} : \alpha < \mathrm{cf}\,\kappa,\ \beta < \nu\rangle$ be a partition of κ such that $|S_{\alpha\beta}| = \lambda_\alpha$ for all α, β. Let

$$\mathscr{F} = \{f : f \text{ is a function, } f \subseteq \mathrm{cf}\,\kappa \times \nu,\ |f| = \mathrm{cf}\,\kappa\}.$$

Let $\mathscr{B}$ be a maximal $\mathrm{cf}\,\kappa$-almost disjoint subset of $\mathscr{F}$.

(1) $\nu \leq |\mathscr{B}| \leq \nu^{\mathrm{cf}\,\kappa}$.

In fact, clearly $|\mathscr{F}| = \nu^{\mathrm{cf}\,\kappa}$, and so $|\mathscr{B}| \leq \nu^{\mathrm{cf}\,\kappa}$. Now suppose that $|\mathscr{B}| < \nu$. For each $\alpha < \mathrm{cf}\,\kappa$ let $f(\alpha)$ be the smallest ordinal β such that $\beta \neq g(\alpha)$ for all $g \in \mathscr{B}$. Thus $f(\alpha) < \nu$ by supposition. So $f \in \mathscr{F}$, and $f \cap g = 0$ for all $g \in \mathscr{B}$, contradiction.

For each $f \in \mathscr{B}$ let

$$X_f = \bigcup_{(\alpha,\beta)\in f} S_{\alpha\beta},$$

and let $\mathscr{A} = \{X_f : f \in \mathscr{B}\}$. We claim that $\mathscr{A}$ is κ-mad, and X is one-one. Clearly $|X_f| = \kappa$ for all $f \in \mathscr{B}$. Suppose that $f, g \in \mathscr{B}$ and $f \neq g$. Then

$$X_f \cap X_g = \bigcup_{(\alpha,\beta)\in f\cap g} S_{\alpha\beta},$$

and this set has fewer than κ elements.

Finally, suppose that $Y \in [\kappa]^\kappa$. Then

(2) $\forall\alpha < \mathrm{cf}\,\kappa\ \exists\beta \in [\alpha, \mathrm{cf}\,\kappa)\ \exists\gamma < \nu\ [|Y \cap S_{\beta\gamma}| \geq \lambda_\alpha]$.

In fact, otherwise we get $\alpha < \mathrm{cf}\,\kappa$ such that $\forall\beta \in [\alpha, \mathrm{cf}\,\kappa)\ \forall\gamma < \nu\ [|Y \cap S_{\beta\gamma}| < \lambda_\alpha]$. So

$$|Y| = \Big|\bigcup_{\beta<\mathrm{cf}\kappa}\bigcup_{\gamma<\nu}(Y \cap S_{\beta\gamma})\Big| = \Big|\bigcup_{\beta<\alpha}\Big(\bigcup_{\gamma<\nu} Y \cap S_{\beta\gamma}\Big)\Big| + \Big|\bigcup_{\alpha\leq\beta<\mathrm{cf}\kappa}\Big(\bigcup_{\gamma<\nu} Y \cap S_{\beta\gamma}\Big)\Big| \leq \lambda_\alpha,$$

contradiction. So (2) holds.

By (2), let $\langle \alpha_\varrho : \varrho < \mathrm{cf}\,\kappa \rangle$ and $\langle \beta_\varrho : \varrho < \mathrm{cf}\,\kappa \rangle$ be such that $\langle \alpha_\varrho : \varrho < \mathrm{cf}\,\kappa \rangle$ is strictly increasing, and $|Y \cap S_{\alpha_\varrho \beta_\varrho}| \geq \lambda_\varrho$. Let $f = \{(\alpha_\varrho, \beta_\varrho) : \varrho < \mathrm{cf}\,\kappa\}$. Thus $f \in \mathscr{F}$. Choose $g \in \mathscr{B}$ such that $|f \cap g| = \mathrm{cf}\,\kappa$. Then $|Y \cap X_g| = \kappa$, as desired. ■

Corollary 4.14. *If κ is singular and $\forall \nu < \kappa\ (\nu^{\mathrm{cf}\kappa} < \kappa)$, then $\kappa \in \mathrm{MAD}(\kappa)$.*

Proof. By 4.13 and 4.6. ■

Corollary 4.15. *If κ is strong limit singular, then $\kappa \in \mathrm{MAD}(\kappa)$.* ■

In connection with these corollaries, see the results of Kojman–Kubiś–Shelah [∞] mentioned above.

Here is a generalization of Proposition 4.9.

Proposition 4.16. *Suppose that κ is singular, $\mu < \kappa$, and $\mathrm{cf}\,\kappa = \mathrm{cf}\,\mu$. Suppose that $\mathscr{A} \subseteq [\kappa]^\mu$ is μ-ad and $|\mathscr{A}| = \kappa$. Then it is not mad. So $\kappa \notin \mathrm{MAD}(\kappa, \mu, \mu)$; hence $0 \neq \mathrm{MAD}(\kappa, \mu, \mu) \subseteq (\kappa, \infty)$. Moreover, $\mathfrak{a}_{\kappa\mu\mu} > \kappa$.*

Proof. Assume the hypothesis. Say $\mathscr{A} = \{X_\alpha : \alpha < \kappa\}$. By Proposition 4.9 we may assume that μ is singular. Let $\nu_\alpha \uparrow \mu$ for $\alpha < \mathrm{cf}\,\mu$ and $\varrho_\alpha \uparrow \kappa$ for $\alpha < \mathrm{cf}\,\kappa$. We now define subsets Y_α of κ of size at most μ for each $\alpha < \mathrm{cf}\,\mu$. Suppose that we have done this for all $\beta < \alpha$. Then $\bigcup_{\beta < \varrho_\alpha} X_\beta \cup \bigcup_{\beta < \alpha} Y_\beta$ has size less than κ, so we can choose

$$Y_\alpha \subseteq \kappa \setminus \Big(\bigcup_{\beta < \varrho_\alpha} X_\beta \cup \bigcup_{\beta < \alpha} Y_\beta \Big)$$

of size ν_α. Let $Z = \bigcup_{\alpha < \mathrm{cf}\mu} Y_\alpha$. Then $|Z| = \mu$ and $|Z \cap X_\beta| < \mu$ for all $\beta < \kappa$. ■

As an special case of Proposition 4.16 we have $\aleph_{\omega+\omega} \notin \mathrm{MAD}(\aleph_{\omega+\omega}, \aleph_\omega, \aleph_\omega)$.

Corollary 4.17. *If κ is singular, then $\mathrm{MAD}(\kappa) \cap (\kappa, \infty) \neq 0$.*

Proof. By Propositions 1.1(i) and 4.9, $\mathrm{MAD}(\kappa, \mathrm{cf}\,\kappa, \mathrm{cf}\,\kappa) \cap (\kappa, \infty) \neq 0$, so the corollary follows by 4.1. ■

The next result is in Milner–Prikry [87, p. 165].

Proposition 4.18. *Suppose that κ is singular, $\mathscr{A} \subseteq [\kappa]^\kappa$ is κ-ad, and $|\mathscr{A}| = \mathrm{cf}\,\kappa$. Then $\mathscr{A}$ is not mad. Hence $\mathrm{cf}\,\kappa \notin \mathrm{MAD}(\kappa)$, and so $\mathfrak{a}_\kappa > \mathrm{cf}\,\kappa$.*

Proof. Assume the hypothesis; say $\mathscr{A} = \{A_\xi : \xi < \mathrm{cf}\,\kappa\}$. Let $\kappa_\xi \uparrow \kappa$ for $\xi < \mathrm{cf}\,\kappa$, with $\kappa_0 = \omega$. By induction, for each $\xi < \mathrm{cf}\,\kappa$ choose

$$B_\xi \subseteq A_\xi \setminus \Big(\bigcup_{\eta < \xi} A_\eta \cup \bigcup_{\eta < \xi} B_\eta \Big)$$

of size κ_ξ. Clearly $\bigcup_{\xi < \mathrm{cf}\,\kappa} B_\xi$ has size κ and its intersection with each A_ξ is of size less than κ, for each $\xi < \mathrm{cf}\,\kappa$. ■

The following is due to Tarski; see Baumgartner [76, Theorem 2.3].

Proposition 4.19. *Suppose that $1 < \nu \leq \kappa \geq \omega$, and let μ be minimum such that $\kappa < \nu^\mu$. Then $\mathrm{MAD}(\kappa, \mu, \mu)$ and $\mathrm{MAD}(\kappa, \mathrm{cf}\,\mu, \mathrm{cf}\,\mu)$ both have members $\geq \nu^\mu$.*

Proof. First statement: Let $T=\bigcup_{\alpha<\mu}{}^{\alpha}\nu$. Thus $|T|\le\kappa$. It suffices to exhibit a family $\mathscr{F}\subseteq[T]^{\mu}$ which is μ-ad and of size ν^{μ}. Let

$$\mathscr{F}=\{\{f\restriction\alpha:\alpha<\mu\}:f\in{}^{\mu}\nu\}.$$

Second statement: this follows from the first if μ is regular. Suppose that μ is singular, and let $\langle\nu_\alpha:\alpha<\operatorname{cf}\mu\rangle$ be a strictly increasing sequence of cardinals with supremum μ. Let $T'=\bigcup_{\alpha<\operatorname{cf}\mu}{}^{\nu_\alpha}\nu$, and let

$$\mathscr{F}=\{\{f\restriction\nu_\alpha:\alpha<\operatorname{cf}\mu\}:f\in{}^{\mu}\nu\}.\ \blacksquare$$

The following is Corollary 2.11 of Baumgartner [76].

Corollary 4.20. *If κ is an infinite cardinal and λ is the least cardinal such that $\lambda\ge\kappa$ and $\lambda\notin\mathrm{AD}(\kappa)$, then λ is regular.*

Proof. Obviously $\kappa\in\mathrm{AD}(\kappa)$. So the result follows by Corollary 2.9. $\blacksquare$

Corollary 4.21. *Suppose that $\kappa,\lambda,\lambda',\mu$ are cardinals, with κ,λ',μ infinite. Also suppose that $\operatorname{cf}\mu\le\kappa$, $\kappa=\aleph_\alpha$, $\lambda^{<\mu}<\aleph_{\alpha+\operatorname{cf}\mu}$, $\lambda^{<\mu}<\lambda^{\mu}$, $\lambda'\le\lambda^{\mu}$, and $\operatorname{cf}\lambda'>\lambda^{<\mu}$. Then $\lambda'\in\mathrm{AD}(\kappa,\operatorname{cf}\mu,\operatorname{cf}\mu)$.*

Proof. Apply 4.19 with ν and κ replaced by λ and $\lambda^{<\mu}$ respectively. Then the μ of 4.19 is our μ as well, and so $\lambda^{\mu}\in\mathrm{AD}(\lambda^{<\mu},\operatorname{cf}\mu,\operatorname{cf}\mu)$. Hence $\lambda'\in\mathrm{AD}(\lambda^{<\mu},\operatorname{cf}\mu,\operatorname{cf}\mu)$. Let ϱ be minimum such that $\lambda'\in\mathrm{AD}(\varrho,\operatorname{cf}\mu,\operatorname{cf}\mu)$. So $\varrho\le\lambda^{<\mu}$. Now $\operatorname{cf}\mu\le\varrho\le\lambda'$, so by 2.2, $\operatorname{cf}\lambda'=\operatorname{cf}\varrho$ or $\operatorname{cf}\varrho=\operatorname{cf}\mu$. Since $\operatorname{cf}\lambda'>\lambda^{<\mu}\ge\varrho$, we have $\operatorname{cf}\lambda'\ne\operatorname{cf}\varrho$, so $\operatorname{cf}\varrho=\operatorname{cf}\mu$. But $\varrho\le\lambda^{<\mu}<\aleph_{\alpha+\operatorname{cf}\mu}$, so $\varrho\le\kappa$. Hence $\lambda'\in\mathrm{AD}(\kappa,\operatorname{cf}\mu,\operatorname{cf}\mu)$. $\blacksquare$

5. The notion MAD_1

Proposition 5.1.

(i) *If* $\nu=1$, *then* $\mathrm{MAD}_1(\kappa,\lambda,\mu,\nu)\subseteq\{0\}$.
(ii) *If* $\kappa>\mu$ *and* $\kappa>\lambda$, *then* $\mathrm{MAD}_1(\kappa,\lambda,\mu,\nu)=0$.
(iii) *If* $\lambda<\operatorname{cf}\kappa$, *then* $\mathrm{MAD}_1(\kappa,\lambda,\kappa,\nu)=\{0\}$.
(iv) *If* $\operatorname{cf}\kappa\le\lambda<\kappa$ *and* $\nu<\kappa$, *then* $\mathrm{MAD}_1(\kappa,\lambda,\kappa,\nu)=\{0\}$.
(v) *If* $\operatorname{cf}\kappa\le\lambda<\kappa$, *then* $\mathrm{MAD}_1(\kappa,\lambda,\kappa,\kappa)\subseteq[(\operatorname{cf}\kappa)^+,\infty)$.
(vi) *If* $\operatorname{cf}\kappa\le\lambda<\kappa$, *then* $\mathrm{MAD}_1(\kappa,\lambda,\kappa,\kappa)\cap[\kappa,\infty)\ne0$.
(vii) *If* $\mu<\kappa$ *and* $1<\nu$, *then* $\mathrm{MAD}_1(\kappa,\kappa,\mu,\nu)\subseteq[\kappa,\infty)$.
(viii) *If* $1<\nu<\kappa$, *then* $\mathrm{MAD}_1(\kappa,\kappa,\kappa,\nu)\subseteq[\kappa,\infty)$.
(ix) *If* κ *is regular*, *then* $\mathrm{MAD}_1(\kappa)\subseteq[\kappa^+,\infty)$.

Proof. (i) is clear.

Under the assumptions of (ii), there is no partition of κ into λ sets, each of power μ. For (iii), if $\mathscr{D}$ is a partition of κ into λ sets each of power κ, then $\mathscr{D}$ is $[\kappa]^{\kappa},\nu$-mad. The same is true under the assumptions of (iv). In fact, suppose that $\Gamma\in[\kappa]^{\kappa}$ and $|\Gamma\cap X|<\nu$ for all $X\in\mathscr{D}$. Then $\Gamma=\bigcup_{X\in\mathscr{D}}(\Gamma\cap X)$, which has size at most $\lambda\cdot\nu<\kappa$, contradiction.

We turn to (v). Assume that $\operatorname{cf}\kappa\le\lambda<\kappa$. Suppose that $\varrho\in\mathrm{MAD}_1(\kappa,\lambda,\kappa,\kappa)$. Accordingly, let $\mathscr{D}$ be a partition of κ into λ sets, each of size κ; say that $\mathscr{D}=\{D_\alpha:\alpha<\lambda\}$, without repetitions, and let $\mathscr{A}$ be such that $|\mathscr{A}|=\varrho$, $\mathscr{D}\cap\mathscr{A}=0$, $\mathscr{D}\cup\mathscr{A}$ is κ-mad. Say $\mathscr{A}=\{A_\xi:\xi<\varrho\}$ without repetitions. Say $\kappa_\xi\uparrow\kappa$ for $\xi<\operatorname{cf}\kappa$.

First suppose that $\varrho<\operatorname{cf}\kappa$. Now $|A\cap D_\alpha|<\kappa$ for all $A\in\mathscr{A}$ and $\alpha<\lambda$, so $|D_\alpha\cap\bigcup_{A\in\mathscr{A}}A|<\kappa$ for all $\alpha<\lambda$. For each $\xi<\operatorname{cf}\kappa$ choose $C_\xi\subseteq D_\xi\setminus\bigcup_{A\in\mathscr{A}}A$ such that $|C_\xi|=\kappa_\xi$. Let $B=\bigcup_{\xi<\operatorname{cf}\kappa}C_\xi$. Then $B\in[\kappa]^\kappa$, $B\cap A=0$ for all $A\in\mathscr{A}$, and $|B\cap D_\xi|<\kappa$ for all $\alpha<\lambda$, contradiction.

Second, suppose that $\varrho=\operatorname{cf}\kappa$. We claim

(1) $\forall\xi,\eta,\theta<\operatorname{cf}\kappa\ \exists\alpha\in\lambda\setminus\max\{\xi+1,\eta+1,\theta+1\}\ [|A_\xi\cap D_\alpha|\ge\kappa_\eta]$.

For, assume otherwise; choose $\xi,\eta,\theta<\operatorname{cf}\kappa$ such that for all $\alpha\in\lambda\setminus\max\{\xi+1,\eta+1,\theta+1\}$ we have $|A_\xi\cap D_\alpha|<\kappa_\eta$. Hence

$$\Big|\bigcup\{A_\xi\cap D_\alpha:\alpha\in\lambda\setminus\max\{\xi+1,\eta+1,\theta+1\}\}\Big|\le\lambda\cdot\kappa_\eta<\kappa.$$

So

$$\Big|\bigcup\{A_\xi\cap D_\alpha:\alpha<\max\{\xi+1,\eta+1,\theta+1)\}\}\Big|=\kappa.$$

But $|A_\xi\cap D_\alpha|<\kappa$ for all $\alpha<\max\{\xi+1,\eta+1,\theta+1\}$, and $\max\{\xi+1,\eta+1,\theta+1\}<\operatorname{cf}\kappa$, contradiction. So (1) holds.

Now we define $B_\xi\subseteq\kappa$ and $\alpha_\xi<\operatorname{cf}\kappa$ for $\xi<\operatorname{cf}\kappa$ so that always $|B_\xi|=\kappa_\xi$. Suppose defined for all $\eta<\xi$. Then

$$E_\xi:=\bigcup_{\eta<\xi}(A_\xi\cap A_\eta)\cup\bigcup_{\eta<\xi}B_\eta\cup\kappa_\xi^+$$

has size less than κ; say that its size is less than κ_τ, where $\xi<\tau<\operatorname{cf}\kappa$. By (1), choose $\alpha_\xi\in\lambda\setminus\max\{\eta+1,\sup\{\alpha_\eta:\eta<\xi\}+1\}$ so that $|A_\xi\cap D_{\alpha_\xi}|\ge\kappa_\tau$. Choose $B_\xi\subseteq A_\xi\cap D_{\alpha_\xi}$ so that $|B_\xi|=\kappa_\xi$ and $B_\xi\cap E_\xi=0$.

Let $B=\bigcup_{\xi<\operatorname{cf}\kappa}B_\xi$. Then $|B|=\kappa$, $|B\cap A_\xi|<\kappa$ for all $\xi<\operatorname{cf}\kappa$ (since for $\eta>\xi$ we have $B_\eta\cap A_\xi=B_\eta\cap A_\xi\cap A_\eta\subseteq B_\eta\cap E_\eta=0$), $B\cap D_{\alpha_\xi}=B_\xi$, which has size less than κ, for each $\xi<\operatorname{cf}\kappa$, and $B\cap D_\beta=0$ for all $\beta\in\lambda\setminus\{\alpha_\xi:\xi<\operatorname{cf}\kappa\}$. Thus $\mathscr{D}\cup\mathscr{A}$ is not κ-mad, contradiction. So (v) holds.

Next, we take (vi); assume that $\operatorname{cf}\kappa\le\lambda<\kappa$. Let $\omega\le\kappa_\alpha\uparrow\kappa$ for $\alpha<\operatorname{cf}\kappa$. Let $\mathscr{D}$ be a partition of κ into λ sets each of size κ; say $\mathscr{D}=\{D_\alpha:\alpha<\lambda\}$. For each $\alpha<\operatorname{cf}\kappa$, let $\langle E_{\alpha\beta}:\beta<\kappa\rangle$ be a partition of D_α into sets of size κ. For all $\beta<\kappa$ let A_β be defined by requiring that $A_\beta\cap D_\alpha$ is a subset of $E_{\alpha\beta}$ of size κ_α for each $\alpha<\operatorname{cf}\kappa$, while $A_\beta\cap D_\alpha=0$ if $\operatorname{cf}\kappa\le\alpha<\lambda$. Thus $|A_\beta|=\kappa$, the A_β's are pairwise disjoint, and $|A_\beta\cap D_\alpha|<\kappa$ for all $\beta<\kappa$ and $\alpha<\operatorname{cf}\kappa$, as desired.

For (vii), assume that $\mathscr{D}$ is a partition of κ into κ sets each of size $\mu<\kappa$, $\mathscr{A}\subseteq[\kappa]^\mu$, $\mathscr{D}\cap\mathscr{A}=0$, $\mathscr{D}\cup\mathscr{A}$ is ν-ad, and $|\mathscr{A}|<\kappa$. For each $X\in\mathscr{A}$ let $M_X=\{Y\in\mathscr{D}:X\cap Y\ne0\}$. Clearly $|M_X|\le\mu$ for each $X\in\mathscr{A}$. Hence $\bigcup_{X\in\mathscr{A}}M_X$ has fewer than κ elements. So there is a subset Z of $\kappa\setminus\bigcup\bigcup_{X\in\mathscr{A}}M_X$ of size μ which has at most one element in common with each member of $\mathscr{D}$, and is disjoint from each member of $\mathscr{A}$. Thus $\mathscr{D}\cup\mathscr{A}$ is not $[\kappa]^\mu,\nu$-mad.

For (viii), assume that $\mathscr{D}$ is a partition of κ into κ sets each of size κ, $\mathscr{A}\subseteq[\kappa]^\kappa$, $\mathscr{D}\cap\mathscr{A}=0$, $\mathscr{D}\cup\mathscr{A}$ is ν-ad, and $|\mathscr{A}|<\kappa$. For each $D\in\mathscr{D}$ the set $\{D\cap A:A\in\mathscr{A}\}$ has size less than κ, and each set $D\cap A$ has size less than ν, so $\bigcup_{A\in\mathscr{A}}(D\cap A)$ has fewer than

κ elements; so choose $a_D \in D \setminus \bigcup_{A\in\mathscr{A}}(D\cap A)$. Let $E = \{a_D : D \in \mathscr{D}\}$. Now $|\mathscr{D}| = \kappa$, so $|E| = \kappa$. Since $|E\cap X| < \nu$ for all $X \in \mathscr{D}\cup\mathscr{A}$, this shows that $\mathscr{D}\cup\mathscr{A}$ is not $[\kappa]^\kappa, \nu$-mad.

Finally, (ix) holds by Proposition 2.6. ∎

PROPOSITION 5.2. $\mathrm{MAD}_2(\kappa,\mu,\nu) = \mathrm{MAD}_1(\kappa,\kappa,\mu,\nu)$.

Proof. For $\subseteq$, let $f : \kappa\times\mu \to \kappa$ be one-one and onto. Let $\varrho \in \mathrm{MAD}_2(\kappa,\mu,\nu)$; say $\varrho = |\mathscr{A}|$, where $\forall\alpha<\kappa\ (\{\alpha\}\times\mu \notin \mathscr{A})$, and $\mathscr{A}\cup\{\{\alpha\}\times\mu : \alpha<\kappa\}$ is $[\kappa\times\mu]^\mu, \nu$-mad. Let $\mathscr{D} = \{f[\{\alpha\}\times\mu] : \alpha<\kappa\}$ and $\mathscr{B} = \{f[X] : X\in\mathscr{A}\}$. Thus $\mathscr{B}$ is as in the definition of $\mathrm{MAD}_1(\kappa,\kappa,\mu,\nu)$, and $|\mathscr{A}| = |\mathscr{B}|$, as desired.

For the other direction, suppose that $\varrho \in \mathrm{MAD}_1(\kappa,\kappa,\mu,\nu)$; say $|\mathscr{A}| = \varrho$, $\mathscr{D}$ is a partition of κ into κ sets of size μ, $\mathscr{A}\cap\mathscr{D} = 0$, and $\mathscr{A}\cup\mathscr{D}$ is $[\kappa]^\mu, \nu$-mad. Say $\mathscr{D} = \{X_\alpha : \alpha<\kappa\}$ without repetitions. Let $g : \kappa \to \kappa\times\mu$ be such that $g[X_\alpha] = \{\alpha\}\times\mu$ for all $\alpha<\kappa$. Then set $\mathscr{B} = \{g[Y] : Y\in\mathscr{A}\}$. ∎

PROPOSITION 5.3. $\mathrm{MAD}_1(\kappa,\kappa,\mu,\lambda) \subseteq \mathrm{MAD}(\kappa,\mu,\nu)$ *if* $1<\nu$, *and* ($\mu<\kappa$, $\nu<\kappa$, *or* κ *is regular*).

Proof. By Proposition 5.1(vii)–(ix). ∎

PROPOSITION 5.4. *If* κ *is regular*, *then* $\mathrm{MAD}(\kappa)\cap[\kappa,\infty) = \mathrm{MAD}_1(\kappa)$.

Proof. By Propositions 5.1(ix) and 5.3, it only remains to prove $\subseteq$. Suppose that $\mathscr{A}$ is $[\kappa]^\kappa, \kappa$-mad and $|\mathscr{A}| \ge \kappa$. By 2.6 we have $|\mathscr{A}| > \kappa$. Let $X \in {}^\kappa\mathscr{A}$ be one-one. We define $\langle\alpha_\zeta : \zeta<\kappa\rangle$. Let $\zeta<\kappa$ be given. For every $\xi<\zeta$ choose $\beta_\xi<\kappa$ such that $X_\xi\cap X_\zeta \subseteq \beta_\xi$. Let $\alpha_\zeta = (\sup_{\xi<\zeta}\beta_\xi)\cup(\zeta+1)$. Thus

($*$) For all $\xi,\zeta<\kappa$, if $\xi<\zeta$, then $X_\xi\cap X_\zeta \subseteq \alpha_\zeta$; moreover, $\zeta<\alpha_\zeta$.

Now define, for any $\zeta<\kappa$,

$$Y_\zeta = \begin{cases} (X_\zeta\setminus\alpha_\zeta)\cup\{\zeta\} & \text{if } \zeta\notin\bigcup_{\xi<\zeta}Y_\xi,\\ X_\zeta\setminus\alpha_\zeta & \text{otherwise.}\end{cases}$$

If $\xi<\zeta<\kappa$, there are two possibilities. If $\zeta \in \bigcup_{\lambda<\zeta}Y_\lambda$, then

$$Y_\xi\cap Y_\zeta \subseteq ((X_\xi\setminus\alpha_\xi)\cup\{\xi\})\cap(X_\zeta\setminus\alpha_\zeta) = 0.$$

If $\zeta\notin\bigcup_{\lambda<\zeta}Y_\lambda$, then

$$Y_\xi\cap Y_\zeta \subseteq ((X_\xi\setminus\alpha_\xi)\cup\{\xi\})\cap((X_\zeta\setminus\alpha_\zeta)\cup\{\zeta\}) = 0.$$

It follows that $\langle Y_\zeta : \zeta<\kappa\rangle$ is a partition. Let $\mathscr{A}' = \mathscr{A}\setminus\{X_\zeta : \zeta<\kappa\}$. Clearly $\mathscr{A}'\cup\{Y_\zeta : \zeta<\kappa\}$ is $[\kappa]^\kappa, \kappa$-mad. ∎

PROPOSITION 5.5. $\mathrm{MAD}(\kappa,\mu,\mu) = \mathrm{MAD}_1(\kappa,\kappa,\mu,\mu)$ *if* κ *is regular and* $\mu<\kappa$.

Proof. Again, we only need to prove $\subseteq$. Let $\mathscr{A}$ be $[\kappa]^\mu, \mu$-mad.

(1) We may assume that $\forall\alpha<\kappa\ \exists X\in\mathscr{A}\ (|X\cap\alpha|<\mu)$.

To prove this, we consider two cases.

Case 1: $\forall\alpha<\kappa\ \exists\beta>\alpha\ \exists X\in\mathscr{A}\ (|X\cap\beta|=\mu$ and $|X\setminus\beta|=\mu)$. Now we define $X_\zeta\in\mathscr{A}$ and $\beta_\zeta<\kappa$ by induction, for each $\zeta<\kappa$. Suppose defined for all $\xi<\zeta$. Choose

$$\beta_\zeta>\bigcup_{\xi<\zeta}\beta_\xi\cup\bigcup_{\xi<\zeta}\sup X_\xi$$

and $X_\zeta\in\mathscr{A}$ such that $|X_\zeta\cap\beta_\zeta|=\mu$ and $|X_\zeta\setminus\beta_\zeta|=\mu$. The second part of the definition of β_ζ ensures that the X_ζ's are distinct. In $\mathscr{A}$, replace each X_ζ by $X_\zeta\cap\beta_\zeta$ and $X_\zeta\setminus\beta_\zeta$. The resulting set $\mathscr{A}'$ is still $[\kappa]^\mu,\mu$-mad, and (1) now holds.

Case 2: $\exists\alpha<\kappa\ \forall\beta>\alpha\ \forall X\in\mathscr{A}\ (|X\cap\beta|<\mu$ or $|X\setminus\beta|<\mu)$. We show that (1) holds for $\mathscr{A}$ itself. Let $\gamma<\kappa$ be given. Choose $\beta>\gamma,\alpha$, and let $Y\in[\kappa\setminus\beta]^\mu$. Choose $X\in\mathscr{A}$ such that $|X\cap Y|\ge\mu$. Hence $|X\setminus\beta|\ge\mu$. It follows that $|X\cap\beta|<\mu$, and hence $|X\cap\gamma|<\mu$, as desired.

Thus (1) holds, and we make the indicated assumption.

Now we define $X_\xi\in\mathscr{A}$ for all $\xi<\kappa$. Suppose that X_ξ has been defined for all $\xi<\zeta$. Let $Y_\zeta=\bigcup_{\xi<\zeta}(X_\xi\cup\zeta)$. Note that $\sup(Y_\zeta)<\kappa$, since $\mu<\kappa$. Choose $X_\zeta\in\mathscr{A}$ such that $|X_\zeta\cap\sup(Y_\zeta)|<\mu$. This finishes the definition of the X_ζ's. Clearly they are all distinct. Next, define

$$X'_\zeta=(X_\zeta\cup\{\zeta\})\setminus Y_\zeta.$$

Then

(2) If $\xi<\zeta<\kappa$, then $X'_\xi\cap X'_\zeta=0$.

For, suppose that $\alpha\in X'_\xi\cap X'_\zeta$. Then $\alpha\in X_\xi\cup\{\xi\}\subseteq Y_\zeta$ because $\alpha\in X'_\xi$, and this contradicts $\alpha\in X'_\zeta$. So (2) holds.

Now

(3) $\zeta\in\bigcup_{\xi\le\zeta}X'_\xi$ for all $\zeta<\kappa$.

In fact, suppose that $\zeta\notin\bigcup_{\xi\le\zeta}X'_\xi$. Now $\zeta\in X_\zeta\cup\{\zeta\}$ but $\zeta\notin X'_\zeta$, so $\zeta\in Y_\zeta$. Hence there is a $\xi<\zeta$ such that $\zeta\in X_\xi$; take the least such ξ. Then $\zeta\notin Y_\xi$, so $\zeta\in X'_\xi$, contradiction.

Now let $\mathscr{A}'=(\mathscr{A}\setminus\{X_\zeta:\zeta<\kappa\})\cup\{X'_\zeta:\zeta<\kappa\}$. Clearly $\mathscr{A}'$ is still μ-ad. Suppose that $Y\in[\kappa]^\mu$. Choose $Z\in\mathscr{A}$ such that $|Y\cap Z|=\mu$. If $Z\notin\{X_\zeta:\zeta<\kappa\}$, this is fine. Suppose that $Z=X_\zeta$ with $\zeta<\kappa$. Now $X_\zeta\setminus Y_\zeta\subseteq X'_\zeta$, so $X_\zeta\setminus X'_\zeta\subseteq Y_\zeta$. Clearly $|X_\zeta\cap Y_\zeta|<\mu$, so $|X_\zeta\setminus X'_\zeta|<\mu$. Since $|Y\cap X_\zeta|=\mu$, it follows that $|Y\cap X'_\zeta|=\mu$.

So $\mathscr{A}'$ is $[\kappa]^\mu,\mu$-mad, and it includes a partition $\mathscr{C}$ of κ into κ sets of size μ, as desired (see also Proposition 5.1(vii)). ∎

Proposition 5.6. *If κ is regular, then* $\mathrm{MAD}(\kappa)\cap[\kappa,\infty)\subseteq\mathrm{MAD}(\kappa^+,\kappa,\kappa)$.

Proof. Let $\lambda\in\mathrm{MAD}(\kappa)\cap[\kappa,\infty)$. Note by Proposition 2.6 that $\kappa<\lambda$. Now we construct $\mathscr{A}_\alpha$ for $\kappa\le\alpha<\kappa^+$ so that the following conditions hold:

(1) $\mathscr{A}_\alpha$ is $[\alpha]^\kappa,\kappa$-mad.

(2) If $\kappa\le\beta<\alpha$, then $\mathscr{A}_\beta\subseteq\mathscr{A}_\alpha$.

(3) $|\mathscr{A}_\alpha|=\lambda$.

To start with, let $\mathscr{A}_\kappa$ be obtained by the definition of $\mathrm{MAD}(\kappa)$ so that (1) and (3) hold; (2) does not apply yet. Now suppose that $\kappa < \alpha < \kappa^+$ and $\mathscr{A}_\beta$ has been defined for all $\beta \in [\kappa, \alpha)$.

Case 1: α is a successor ordinal $\beta + 1$. Let $\mathscr{A}_\alpha = \mathscr{A}_\beta$. Clearly (1)–(3) continue to hold.

Case 2: α is a limit ordinal, and $\operatorname{cf}\alpha < \kappa$. Let $\mathscr{A}_\alpha = \bigcup_{\kappa \le \beta < \alpha} \mathscr{A}_\beta$. By (2) it is clear that $\mathscr{A}_\alpha$ is κ-ad. Suppose now that $\Gamma \in [\alpha]^\kappa$. Let $\langle \beta_\xi : \xi < \operatorname{cf}\alpha \rangle$ be a strictly increasing sequence of ordinals with supremum α, and with $\beta_0 \ge \kappa$. Then there is a $\xi < \operatorname{cf}\alpha$ such that $|\Gamma \cap \beta_\xi| = \kappa$. It follows that there is an $X \in \mathscr{A}_{\beta_\xi}$ such that $|\Gamma \cap X| = \kappa$. Thus $\mathscr{A}_\alpha$ is $[\alpha]^\kappa, \kappa$-mad. So (1) holds. Clearly (2) holds, as does (3).

Case 3: α is a limit ordinal, $\operatorname{cf}\alpha = \kappa$, and $\exists \beta < \alpha\ \forall \gamma \in (\beta, \alpha)\ [\operatorname{cf}\gamma < \kappa]$. Then α must have the form $\gamma + \kappa$ for some γ. Note that $\mathscr{A}_\delta = \mathscr{A}_\gamma$ for all $\delta \in \alpha \setminus \gamma$. Let $\mathscr{A}$ be $[\alpha \setminus \gamma]^\kappa, \kappa$-mad and of size λ. Then we set $\mathscr{A}_\alpha = \mathscr{A}_\gamma \cup \mathscr{A}$. Clearly (1)–(3) hold.

Case 4: α is a limit ordinal, $\operatorname{cf}\alpha = \kappa$, and $\forall \beta < \alpha\ \exists \gamma \in (\beta, \alpha)\ [\operatorname{cf}\gamma = \kappa]$. Then there is a continuous strictly increasing sequence $\langle \beta_\xi : \xi < \kappa \rangle$ of ordinals with supremum α, with $\kappa = \beta_0$, and with $|\beta_{\xi+1} \setminus \beta_\xi| = \kappa$ for every $\xi < \kappa$. By Proposition 5.4, $\lambda \in \mathrm{MAD}_1(\kappa)$. Hence let $\mathscr{A}$ and $\mathscr{D}$ be as in the definition of $\mathrm{MAD}_1(\kappa)$, with $|\mathscr{D}| = \kappa$ and $|\mathscr{A}| = \lambda$. Let $\langle D_\xi : \xi < \kappa \rangle$ be a one-one enumeration of $\mathscr{D}$. Let f be a one-one function mapping κ onto α such that $f[D_0] = \kappa$ and $f[D_{1+\xi}] = \beta_{\xi+1} \setminus \beta_\xi$ for every $\xi < \kappa$. Then we define

$$\mathscr{A}_\alpha = \bigcup_{\kappa \le \gamma < \alpha} \mathscr{A}_\gamma \cup \{f[a] : a \in \mathscr{A}\}.$$

(5) $\mathscr{A}_\alpha$ is κ-ad.

For, suppose that x and y are distinct elements of $\mathscr{A}_\alpha$. If both are in $\bigcup_{\kappa \le \gamma < \alpha} \mathscr{A}_\gamma$, then they are both in $\mathscr{A}_\gamma$ for some $\gamma \in [\kappa, \alpha)$, and so $|x \cap y| < \kappa$. Suppose that $x \in \mathscr{A}_\gamma$ with $\gamma \in [\kappa, \alpha)$, and $y = f[a]$ with $a \in \mathscr{A}$. Choose $\xi < \kappa$ so that $\gamma < \beta_\xi$. Then

$$x \cap y \subseteq \bigcup_{\eta < \xi} (\beta_{\eta+1} \setminus \beta_\eta) \cap y \subseteq f\Big[\Big(\bigcup_{\eta < \xi} D_\eta\Big) \cap a\Big],$$

which has size less than κ. If $x = f[a]$ and $y = f[b]$ with $a, b \in \mathscr{A}$, clearly $|x \cap y| < \kappa$. By symmetry, these are all possibilities.

(6) $\mathscr{A}_\alpha$ is $[\alpha]^\kappa, \kappa$-mad.

For, let $x \in [\alpha]^\kappa$.

Subcase 1: $|x \cap (\beta_{\xi+1} \setminus \beta_\xi)| = \kappa$ for some $\xi < \kappa$. Choose $y \in \mathscr{A}_{\beta_{\xi+1}}$ such that $|x \cap y| = \kappa$.

Subcase 2: $x \cap (\beta_{\xi+1} \setminus \beta_\xi)$ has size less than κ for all $\xi < \kappa$. So, $|f^{-1}[x] \cap D_\xi| < \kappa$ for all $\xi < \kappa$. Choose $a \in \mathscr{A}$ such that $|f^{-1}[x] \cap a| = \kappa$. So $|x \cap f[a]| = \kappa$.

The construction is completed. Clearly $\bigcup_{\alpha < \kappa^+} \mathscr{A}_\alpha$ is $[\kappa^+]^\kappa, \kappa$-mad, as desired. ∎

Corollary 5.7. *If κ is regular, then $\mathfrak{a}_\kappa = \mathfrak{a}_{\kappa^+\kappa\kappa}$.*

Proof. By Proposition 5.6 we have $\mathfrak{a}_\kappa \ge \mathfrak{a}_{\kappa^+\kappa\kappa}$, and by Proposition 4.10, $\mathfrak{a}_{\kappa^+\kappa\kappa} \ge \delta$ for some $\delta \in \mathrm{MAD}(\kappa) \cap [\kappa, \infty)$, so also $\mathfrak{a}_{\kappa^+\kappa\kappa} \ge \mathfrak{a}_\kappa$. ∎

CONSISTENCY RESULTS

There are several natural consistency questions and results concerning the existence of models of ZFC with special MAD properties. These concern:

(1) Models where $|\mathrm{MAD}(\kappa,\mu,\nu)|$ is small and all members of $\mathrm{MAD}(\kappa,\mu,\nu)$ are small.
(2) Models with $|\mathrm{MAD}(\kappa,\mu,\nu)|$ small, but the members of $\mathrm{MAD}(\kappa,\mu,\nu)$ large.
(3) Models where $\mathrm{MAD}(\kappa,\mu,\nu)$ has large members.
(4) Models where $|\mathrm{MAD}(\kappa,\mu,\nu)|$ is large.
(5) Models in which $\mathrm{MAD}(\kappa,\mu,\nu)$ is specified in advance.

As will be seen, there are several open questions in connection with the known results. We survey the various consistency results now. Those given with details in the literature will not be reproved here, but several of the consistency proofs for the countable case can be easily generalized, and we give the details.

- First, the notions can be completely described under GCH, and we give this description in Section 6. Models of GCH are of type (1).
- MA gives models of type (2). The following theorem is due to Wage [79].

Theorem. (MA) *Assume that $\omega\le\mu<2^\omega$, and $\langle T_\alpha:\alpha<\mu\rangle$ is a system of countable almost disjoint subsets of some cardinal κ. Then there is an $M\subseteq\kappa$ such that $|M|=\kappa$ and $M\cap T_\alpha$ is finite for all $\alpha<\mu$.* ■

Corollary. (MA)
(i) *If $\lambda\in\mathrm{MAD}(\kappa,\omega,\omega)$, then $\lambda\ge 2^\omega$.*
(ii) $\mathrm{MAD}(\omega)=(0,\omega)\cup\{2^\omega\}$. ■

Problem 6. *Is there a model in which for every infinite cardinal κ, if $\kappa^\omega>\kappa,2^\omega$ then* $\mathrm{MAD}(\kappa,\omega,\omega)=\{\kappa^\omega\}$*?*

Problem 7. *What can one say along these lines for the general notion* $\mathrm{MAD}(\kappa,\mu,\nu)$*?*

- Theorem 6.1 of Baumgartner [76] is relevant to (3). That theorem implies that if M is a model of GCH, and $\lambda\le\mu\le\kappa\le\varrho$ are cardinals in M such that λ and κ are regular, then there is a generic extension preserving cofinalities in which $\mathfrak{a}_{\kappa\mu\lambda}\ge\varrho$.
- Modifying the argument of this theorem of Baumgartner, we can give a result of type (4) for $\mathrm{MAD}(\kappa,\kappa,\nu)$; this is done in Section 7.
- Blass [93] gave a result of type (5) for the case of all three cardinals equal to ω. We generalize this to $\mathrm{MAD}(\kappa)$ for κ regular, and to $\mathrm{MAD}(\kappa,\mu,\mu)$, in Sections 8 and 9.

6. MAD families under GCH

GCH treats question (1) in the above list. Theorem 3.4 of Baumgartner [76] implies the following:

Proposition 6.1. (GCH) $\mathrm{MAD}(\kappa,\mu,\nu)\cap(\kappa,\infty)\ne 0$ *iff* $\nu=\mu$ *and* $\mathrm{cf}\,\mu=\mathrm{cf}\,\kappa$.

Proof. $\Rightarrow$: Say $\lambda\in\mathrm{MAD}(\kappa,\mu,\nu)\cap(\kappa,\infty)$. Then $\kappa^+\in\mathrm{AD}(\kappa,\mu,\nu)$.

Suppose that $\nu < \mu$. Then there is a $\mu' \in [\nu, \mu]$ with $\operatorname{cf}\mu' \neq \operatorname{cf}\kappa$. In particular, $\mu' < \kappa$. By 1.2, $\kappa^+ \in \mathrm{AD}(\kappa, \mu', \nu)$. Thus $\operatorname{cf}\kappa \neq \operatorname{cf}\kappa^+$ and $\operatorname{cf}\kappa \neq \operatorname{cf}\mu'$, so by 2.2 there is a $\varrho \in [\mu', \kappa)$ such that $\kappa^+ \in \mathrm{AD}(\varrho, \mu', \nu)$. Now $\kappa^+ \leq \varrho^\nu$ by 2.4, $\varrho^\nu \leq \varrho^+$ by GCH, and $\varrho^+ \leq \kappa$, contradiction. Thus $\nu = \mu$.

Suppose that $\operatorname{cf}\mu \neq \operatorname{cf}\kappa$. By 2.2, there is a $\varrho \in [\mu, \kappa)$ such that $\kappa^+ \in \mathrm{AD}(\varrho, \mu, \nu)$. Hence $\kappa^+ \leq \varrho^\nu$ by 2.4, and $\varrho^\nu \leq \varrho^+ \leq \kappa$ by GCH, contradiction.

$\Leftarrow$: If κ is regular, then $\mu \leq \kappa$ and $\operatorname{cf}\mu = \operatorname{cf}\kappa$ imply that $\mu = \kappa$. We then have $\mathrm{MAD}(\kappa, \kappa, \kappa) \cap (\kappa, \infty) \neq 0$ by 2.6. Suppose that κ is singular. Then the desired conclusion holds by 4.16 and 4.17. ■

Together with other results above, this gives a complete description of MAD under GCH, where the sets are as small as possible:

Proposition 6.2. (GCH)

(i) *If κ is regular and $\mu < \kappa$, then* $\mathrm{MAD}(\kappa, \mu, \nu) = \{\kappa\}$.
(ii) *If κ is singular and $\nu < \mu < \kappa$, then* $\mathrm{MAD}(\kappa, \mu, \nu) = \{\kappa\}$.
(iii) *If κ is singular, $\mu < \kappa$, and* $\operatorname{cf}\mu \neq \operatorname{cf}\kappa$, *then* $\mathrm{MAD}(\kappa, \mu, \mu) = \{\kappa\}$.
(iv) *If κ is singular, $\mu < \kappa$, and* $\operatorname{cf}\mu = \operatorname{cf}\kappa$, *then* $\mathrm{MAD}(\kappa, \mu, \mu) = \{\kappa^+\}$.
(v) *If $\nu < \kappa$, then* $\mathrm{MAD}(\kappa, \kappa, \nu) = [1, \kappa]$.
(vi) *If κ is regular, then* $\mathrm{MAD}(\kappa) = [1, \kappa) \cup \{\kappa^+\}$.
(vii) *If κ is singular, then* $\mathrm{MAD}(\kappa) = [1, \operatorname{cf}\kappa) \cup (\operatorname{cf}\kappa, \kappa^+]$.

Proof. (i): By 1.1(ii) and 6.1.

(ii): By 1.1(ii) and 6.1.

(iii): By 1.1(ii) and 6.1.

(iv): By 1.1(ii), 2.4, 4.16, and 6.1.

(v): By 1.1(iii) and 6.1.

(vi): By 1.1(iv), 2.4, and 2.6.

(vii): By 1.1(iv) we have $[1, \operatorname{cf}\kappa) \subseteq \mathrm{MAD}(\kappa)$. By 4.18, $\operatorname{cf}\kappa \notin \mathrm{MAD}(\kappa)$. If ν is regular and $\operatorname{cf}\kappa < \nu < \kappa$, by 4.13 choose $\delta \in \mathrm{MAD}(\kappa)$ such that $\nu \leq \delta \leq \nu^{\operatorname{cf}\kappa}$. But $\nu^{\operatorname{cf}\kappa} = \nu$ by GCH, so $\nu \in \mathrm{MAD}(\kappa)$. By 4.6, all singular cardinals in $(\operatorname{cf}\kappa, \kappa)$ are in $\mathrm{MAD}(\kappa)$. By 4.15, also $\kappa \in \mathrm{MAD}(\kappa)$. Finally, $\kappa^+ \in \mathrm{MAD}(\kappa)$ by 6.1. ■

7. Many members of $\mathrm{MAD}(\kappa, \kappa, \nu)$

For the remainder of the notes we shall refer to Kunen [80] for basic notions and results concerning forcing.

Theorem 7.1. *Suppose that M is a model of GCH, and in M we have infinite cardinals κ, ν, and λ with $\nu \leq \kappa < \lambda$, where ν and κ are regular. Then there is a generic extension $M[G]$ of M preserving cofinalities and cardinals such that in $M[G]$, every regular cardinal in $(\kappa, \lambda]$ is a member of* $\mathrm{MAD}(\kappa, \kappa, \nu)$.

The proof of this theorem will occupy all of this section. As mentioned above, it is a generalization of Theorem 6.1 of Baumgartner [76], and the proof also follows the lines of his proof. He worked with only one cardinal λ and produced a generic extension preserving cofinalities such that in the extension $\lambda \in \mathrm{AD}(\kappa,\kappa,\nu)$. This was done by starting with an extension in which $\lambda \in \mathrm{AD}(\kappa,\kappa,\kappa)$ and thinning out the almost disjoint family. So, we do something similar, except in starting with a whole set of cardinals.

To start with, we work in ZFC. Assume that κ,ν,λ are infinite cardinals with ν and κ regular, $\nu \le \kappa < \lambda$. We define

$$\mathbf{O} = \{\alpha : \alpha \text{ is a limit ordinal and } \alpha \le \lambda \text{ and } \kappa < \operatorname{cf}\alpha\}.$$

This is the set which takes the place of Baumgartner's λ. Let $F := \langle F(\alpha,\beta) : \alpha \in \mathbf{O}, \beta < \alpha\rangle$ be a sequence of subsets of κ, possibly with repetitions. Now for μ regular and $\nu \le \mu \le \kappa$, let $Q'(\kappa,\mu,F)$ be the set of all functions f such that the following two conditions hold:

(1) $\mathrm{dmn}(f) \in [\mathbf{O}]^{<\mu}$;
(2) for all $\alpha \in \mathrm{dmn}(f)$, f_α is a function, $\mathrm{dmn}(f_\alpha) \in [\alpha]^{<\mu}$, and for all $\beta \in \mathrm{dmn}(f_\alpha)$, $f_\alpha(\beta) \in [F(\alpha,\beta)]^{<\mu}$.

For $f,g \in Q'(\kappa,\mu,F)$ we write $f \le g$ iff the following conditions hold:

(3) $\mathrm{dmn}(g) \subseteq \mathrm{dmn}(f)$;
(4) for every $\alpha \in \mathrm{dmn}(g)$,
   (a) $\mathrm{dmn}(g_\alpha) \subseteq \mathrm{dmn}(f_\alpha)$;
   (b) $g_\alpha(\beta) \subseteq f_\alpha(\beta)$ for all $\beta \in \mathrm{dmn}(g_\alpha)$;
   (c) $g_\alpha(\beta) \cap g_\alpha(\gamma) = f_\alpha(\beta) \cap f_\alpha(\gamma)$ for all distinct $\beta,\gamma \in \mathrm{dmn}(g_\alpha)$.

Let K be the set of all regular cardinals μ such that $\nu \le \mu \le \kappa$. Suppose that $\mu \in K$. Then $Q(\kappa,\mu,F)$ is the set of all functions $f \in \prod_{\mu \le \varrho \in K} Q'(\kappa,\varrho,F)$ such that the following condition holds (where for clarity the value of f at ϱ is written as f^ϱ):

(5) If $\varrho,\varrho' \in K$ and $\mu \le \varrho \le \varrho'$, then $\mathrm{dmn}(f^\varrho) \subseteq \mathrm{dmn}(f^{\varrho'})$, and for all $\alpha \in \mathrm{dmn}(f^\varrho)$, $\mathrm{dmn}(f^\varrho_\alpha) \subseteq \mathrm{dmn}(f^{\varrho'}_\alpha)$, and for any $\beta \in \mathrm{dmn}(f^\varrho_\alpha)$, $f^\varrho_\alpha(\beta) \subseteq f^{\varrho'}_\alpha(\beta)$.

For $f,g \in Q(\kappa,\mu,F)$ we write $f \le g$ iff $f^\varrho \le g^\varrho$ for all $\varrho \in K$ such that $\mu \le \varrho$. For any $\mu \in K$ we let

$$Q_\mu(\kappa,F) = \{f\restriction[\nu,\mu]\cap K) : f \in Q(\kappa,\nu,F)\}.$$

For $f,g \in Q_\mu(\kappa,F)$ we define $f \le g$ iff $f^\varrho \le g^\varrho$ for all $\varrho \in [\nu,\mu] \cap K$.

Finally, $Q(\kappa,\mu)$ stands for $Q(\kappa,\mu,F)$ with $F(\alpha,\beta) = \kappa$ for all $\beta < \alpha \in \mathbf{O}$; similarly for $Q'(\kappa,\mu)$ and $Q_\mu(\kappa)$.

Lemma 7.2. *If $\mu \in K$, $\mu^{<\mu} = \mu$, and $|F(\alpha,\beta) \cap F(\alpha,\gamma)| \le \mu$ whenever $\alpha \in \mathbf{O}$ and $\beta < \lambda < \alpha$, then $Q_\mu(\kappa,F)$ has the μ^+-chain condition. In particular, if $\kappa^{<\kappa} = \kappa$, then $Q(\kappa,\nu)$ has the κ^+-chain condition.*

Proof. Suppose to the contrary that I is a set of pairwise incompatible elements of $Q_\mu(\kappa,F)$, with $|I| = \mu^+$. Now for any $f \in I$ we have $f = g\restriction([\nu,\mu] \cap K)$ for some $g \in Q(\kappa,\nu,F)$, and so $f^\mu = g^\mu \in Q'(\kappa,\mu,F)$, and hence the function f^μ is a member of $Q'(\kappa,\mu,F)$, and so its domain has size $< \mu$. By the Δ-system theorem, we may assume that $\langle \mathrm{dmn}(f^\mu) : f \in I\rangle$ is a Δ-system, say with kernel D. For all $f \in I$ we have

$|\bigcup_{\alpha\in\mathrm{dmn}(f^\mu)}\mathrm{dmn}(f^\mu_\alpha)|<\mu$, so we may assume that

$$\Big\langle\bigcup_{\alpha\in\mathrm{dmn}(f^\mu)}\mathrm{dmn}(f^\mu_\alpha):f\in I\Big\rangle$$

is a Δ-system, say with kernel D'. And for each $f\in I$ the set $\bigcup_{\alpha\in\mathrm{dmn}(f^\mu)}\bigcup_{\beta\in\mathrm{dmn}(f^\mu_\alpha)}f^\mu_\alpha(\beta)$ has size less than μ, so we may assume that

$$\Big\langle\bigcup_{\alpha\in\mathrm{dmn}(f^\mu)}\bigcup_{\beta\in\mathrm{dmn}(f^\mu_\alpha)}f^\mu_\alpha(\beta):f\in I\Big\rangle$$

is a Δ-system, say with kernel R. Let

$$X=\bigcup_{\alpha\in D,\,\beta,\gamma\in D',\,\beta\neq\gamma}F(\alpha,\beta)\cap F(\alpha,\gamma).$$

Since $|D|<\mu$, $|D'|<\mu$, and each $|F(\alpha,\beta)\cap F(\alpha,\gamma)|\le\mu$, we have $|X|\le\mu$. Let $Y=X\cup R$. So $|Y|\le\mu$. Now we claim:

$(*)$ There exist $I'\in[I]^{\mu^+}$ and a function $f\in J:=\prod_{\varrho\in[\nu,\mu]\cap K}{}^{D}({}^{D'}([Y]^{<\mu}))$ such that

$\forall g\in I'\ \forall\varrho\in\mathrm{dmn}(g)\ \forall\alpha\in D\cap\mathrm{dmn}(g^\varrho)\ \forall\beta\in D'\cap\mathrm{dmn}(g^\varrho_\alpha)\ [g^\varrho_\alpha(\beta)\cap Y=f^\varrho_\alpha(\beta)]$.

To prove this, first suppose that $\mu=\sigma^+$ for some σ. For every $g\in I$ define a function ${}_gf$ as follows: $\mathrm{dmn}({}_gf)=K\cap[\nu,\mu]$. For each $\varrho\in[\nu,\mu]\cap K$ let $\mathrm{dmn}({}_gf^\varrho)=D$, and for any $\alpha\in D$ let $\mathrm{dmn}({}_gf^\varrho_\alpha)=D'$. Then for $\varrho\in[\nu,\mu]\cap K$, $\alpha\in D$, and $\beta\in D'$ let

$${}_gf^\varrho_\alpha(\beta)=\begin{cases}g^\varrho_\alpha(\beta)\cap Y & \text{if }\alpha\in\mathrm{dmn}(g^\varrho)\text{ and }\beta\in\mathrm{dmn}(g^\varrho_\alpha),\\ \emptyset & \text{otherwise.}\end{cases}$$

Note that $g\in Q_\mu(\kappa,F)$, and hence there is a $k\in Q(\kappa,\nu,F)$ such that $g=k\restriction([\nu,\mu]\cap K)$. Hence $g^\varrho=k^\varrho\in Q'(\kappa,\varrho,F)$, and $g^\varrho_\alpha(\beta)\in[F(\alpha,\beta)]^{<\mu}$. So ${}_gf^\varrho_\alpha(\beta)\in[Y]^{<\mu}$.

Thus ${}_gf\in J$. Note that J has size at most μ since $|[\nu,\mu]|\le\sigma<\mu$. Hence $(*)$ follows in the case that μ is a successor cardinal.

Now assume that μ is a limit cardinal, and hence is weakly inaccessible. Temporarily fix $g\in I$. We claim:

$(**)$ There is a $\tau(g)\in[\nu,\mu)$ such that $g^{\tau(g)}=g^\xi$ for all $\xi\in[\tau(g),\mu)$.

We can see this step by step as follows. If $\nu\le\sigma\le\tau<\mu$ and $\sigma,\tau\in K$, then $\mathrm{dmn}(g^\sigma)\subseteq\mathrm{dmn}(g^\tau)\subseteq\mathrm{dmn}(g^\mu)$ and $|\mathrm{dmn}(g^\mu)|<\mu$. Hence there is a $\sigma(0,g)<\mu$ with $\sigma(0,g)\in K$ such that $\mathrm{dmn}(g^{\sigma(0,g)})=\mathrm{dmn}(g^\tau)$ for all $\tau\in[\sigma(0,g),\mu)\cap K$. Now for all $\alpha\in\mathrm{dmn}(g^{\sigma(0,g)})$ and any $\tau\in[\sigma(0,g),\mu)\cap K$ we have $\mathrm{dmn}(g_\alpha^{\sigma(0,g)})\subseteq\mathrm{dmn}(g^\tau_\alpha)\subseteq\mathrm{dmn}(g^\mu_\alpha)$ and $|\mathrm{dmn}(g^\mu_\alpha)|<\mu$, so there is a $\sigma(1,\alpha,g)\in[\sigma(0,g),\mu)\cap K$ such that $\mathrm{dmn}(g_\alpha^{\sigma(1,\alpha,g)})=\mathrm{dmn}(g^\tau_\alpha)$ for all $\tau\in[\sigma(1,\alpha,g),\mu)$. Let $\sigma(2,g)=\sup_{\alpha\in\mathrm{dmn}(g^{\sigma(0,g)})}\sigma(1,\alpha,g)$. So $\sigma(2,g)<\mu$, and for all $\alpha\in\mathrm{dmn}(g^{\sigma(2,g)})$ and all $\tau\in[\sigma(2,g),\mu)$ we have $\mathrm{dmn}(g_\alpha^{\sigma(2,g)})=\mathrm{dmn}(g^\tau_\alpha)$. For all $\alpha\in\mathrm{dmn}(g^{\sigma(2,g)})$, $\tau\in[\sigma(2,g),\mu)$, and $\beta\in\mathrm{dmn}(g_\alpha^{\sigma(2,g)})$ we have $g_\alpha^{\sigma(2,g)}(\beta)\subseteq g^\tau_\alpha(\beta)\subseteq g^\mu_\alpha(\beta)$, and $|g^\mu_\alpha(\beta)|<\mu$, so there is a $\sigma(3,\alpha,\beta,g)\in[\sigma(2,g),\mu)\cap K$ such that $g^\xi_\alpha(\beta)=g_\alpha^{\sigma(3,\alpha,\beta,g)}(\beta)$ for all $\xi\in[\sigma(3,\alpha,\beta,g),\mu)$, $\alpha\in\mathrm{dmn}(g^{\sigma(3,\alpha,\beta,g)})$, and $\beta\in\mathrm{dmn}(g_\alpha^{\sigma(3,\alpha,\beta,g)})$. Let $\tau(g)=\sup_{\alpha\in\mathrm{dmn}(g^{\sigma(2,g)})}\sup_{\beta\in\mathrm{dmn}(g_\alpha^{\sigma(2,g)})}\sigma(3,\alpha,\beta,g)$. Then $\tau(g)<\mu$, and for all $\xi\in[\tau(g),\mu)$, $\alpha\in\mathrm{dmn}(g^{\tau(g)})$, and $\beta\in\mathrm{dmn}(g_\alpha^{\tau(g)})$ we have $g_\alpha^{\tau(g)}(\beta)=g^\xi_\alpha(\beta)$. Hence $g^{\tau(g)}=g^\xi$ for all $\xi\in[\tau(g),\mu)$, as desired in $(**)$.

Now we unfix g. Let $I''' \in [I]^{\mu^+}$ be a subset of I on which $\tau(g)$ has a constant value τ. Now for any $g \in I'''$ define ${}_gf \in J' := \prod_{\varrho\in[\nu,\tau]\cap K} {}^D({}^{D'}([Y]^{<\mu}))$ as follows. For any $\varrho \in [\nu,\tau]\cap K$, $\alpha \in D$, $\beta \in D'$, let

$$ {}_gf^\varrho_\alpha(\beta) = \begin{cases} g^\varrho_\alpha(\beta)\cap Y & \text{if } \alpha \in \mathrm{dmn}(g^\varrho) \text{ and } \beta \in \mathrm{dmn}(g^\varrho_\alpha),\\ 0 & \text{otherwise.}\end{cases} $$

Since $|J'| \le \mu$, let $I'' \in [I''']^{\mu^+}$ be such that ${}_gf$ is constant, say with value f', on I''.

For any $g \in I''$ define ${}_gh \in {}^D({}^{D'}([Y]^{<\mu}))$ by setting, for any $\alpha \in D$ and $\beta \in D'$,

$$ {}_gh_\alpha(\beta) = \begin{cases} g^\mu_\alpha(\beta)\cap Y & \text{if } \alpha \in \mathrm{dmn}(g^\mu) \text{ and } \beta \in \mathrm{dmn}(g^\mu_\alpha),\\ 0 & \text{otherwise.}\end{cases} $$

Now $|{}^D({}^{D'}([Y]^{<\mu}))| \le \mu$, so there is an $I' \in [I'']^{<\mu}$ such that ${}_gh$ is constant, say equal to h, on I'.

Define $f \in J$ by: $f^\varrho = (f')^\varrho$ for $\varrho \in [\nu,\tau]$, $f^\varrho = (f')^\tau$ for $\varrho \in (\tau,\mu)$, and $f^\mu = h$. Then $(*)$ holds. Namely, if $g \in I'$, $\varrho \in \mathrm{dmn}(g)$, $\alpha \in D\cap\mathrm{dmn}(g^\varrho)$, and $\beta \in D'\cap\mathrm{dmn}(g^\varrho_\alpha)$, then

$$ g^\varrho_\alpha(\beta)\cap Y = \begin{cases} {}_gf^\varrho_\alpha(\beta) = (f')^\varrho_\alpha(\beta) = f^\varrho_\alpha(\beta) & \text{if } \nu \le \varrho \le \tau,\\ g^\tau_\alpha(\beta)\cap Y = {}_gf^\tau_\alpha(\beta) = (f')^\tau_\alpha(\beta) = f^\varrho_\alpha(\beta) & \text{if } \tau < \varrho < \mu,\\ {}_gh_\alpha(\beta) = h_\alpha(\beta) = f^\mu_\alpha(\beta) & \text{if } \varrho = \mu.\end{cases} $$

Thus, indeed, $(*)$ holds also when μ is inaccessible.

Choose I' and f as in $(*)$. Take any two distinct $g,h \in I'$. We claim that they are compatible (contradiction!). To see this, for $\varrho \in [\nu,\mu]\cap K$ let $\mathrm{dmn}(k^\varrho) = \mathrm{dmn}(g^\varrho)\cup\mathrm{dmn}(h^\varrho)$. For each $\alpha \in \mathrm{dmn}(k^\varrho)$ let

$$ k^\varrho_\alpha = \begin{cases} g^\varrho_\alpha & \text{if } \alpha \in \mathrm{dmn}(g^\varrho)\setminus\mathrm{dmn}(h^\varrho),\\ h^\varrho_\alpha & \text{if } \alpha \in \mathrm{dmn}(h^\varrho)\setminus\mathrm{dmn}(g^\varrho),\\ s & \text{if } \alpha \in \mathrm{dmn}(g^\varrho)\cap\mathrm{dmn}(h^\varrho),\end{cases} $$

where $\mathrm{dmn}(s) = \mathrm{dmn}(g^\varrho_\alpha)\cup\mathrm{dmn}(h^\varrho_\alpha)$, and for every $\beta \in \mathrm{dmn}(s)$,

$$ s(\beta) = \begin{cases} g^\varrho_\alpha(\beta) & \text{if } \beta \in \mathrm{dmn}(g^\varrho_\alpha)\setminus\mathrm{dmn}(h^\varrho_\alpha),\\ h^\varrho_\alpha(\beta) & \text{if } \beta \in \mathrm{dmn}(h^\varrho_\alpha)\setminus\mathrm{dmn}(g^\varrho_\alpha),\\ g^\varrho_\alpha(\beta)\cup h^\varrho_\alpha(\beta) & \text{if } \beta \in \mathrm{dmn}(g^\varrho_\alpha)\cap\mathrm{dmn}(h^\varrho_\alpha).\end{cases} $$

For any $\varrho \in (\mu,\kappa]\cap K$ let $k^\varrho = k^\mu$. Clearly $k \in Q(\kappa,\nu,F)$. Let $l = k\restriction([\nu,\mu]\cap K)$. By symmetry it suffices to show that $l \le g$. Take any $\varrho \in [\nu,\mu]\cap K$. Only condition (4)(c) is problematic. Take any $\alpha \in \mathrm{dmn}(g^\varrho)$ and distinct $\beta,\gamma \in \mathrm{dmn}(g^\varrho_\alpha)$.

Case 1: $\alpha \notin \mathrm{dmn}(h^\varrho)$. Clear.

Case 2: $\alpha \in \mathrm{dmn}(h^\varrho)$. Thus $\alpha \in D$.

Subcase 2.1: $\beta,\gamma \notin h^\varrho_\alpha$. Clear.

Subcase 2.2: $\beta \notin \mathrm{dmn}(h^\varrho_\alpha)$ but $\gamma \in \mathrm{dmn}(h^\varrho_\alpha)$. So $\gamma \in D'$, and $g^\varrho_\alpha(\beta)\cap h^\varrho_\alpha(\gamma) \subseteq R \subseteq Y$. Hence

$$ \begin{aligned} l^\varrho_\alpha(\beta)\cap l^\varrho_\alpha(\gamma) &= g^\varrho_\alpha(\beta)\cap[g^\varrho_\alpha(\gamma)\cup h^\varrho_\alpha(\gamma)] = [g^\varrho_\alpha(\beta)\cap g^\varrho_\alpha(\gamma)]\cup[g^\varrho_\alpha(\beta)\cap h^\varrho_\alpha(\gamma)]\\ &= [g^\varrho_\alpha(\beta)\cap g^\varrho_\alpha(\gamma)]\cup[g^\varrho_\alpha(\beta)\cap h^\varrho_\alpha(\gamma)\cap Y]\\ &= [g^\varrho_\alpha(\beta)\cap g^\varrho_\alpha(\gamma)]\cup[g^\varrho_\alpha(\beta)\cap f^\varrho_\alpha(\gamma)] \qquad \text{(by $(*)$ for h)}\\ &= [g^\varrho_\alpha(\beta)\cap g^\varrho_\alpha(\gamma)]\cup[g^\varrho_\alpha(\beta)\cap g^\varrho_\alpha(\gamma)\cap Y] \qquad \text{(by $(*)$ for g)}\\ &= g^\varrho_\alpha(\beta)\cap g^\varrho_\alpha(\gamma). \end{aligned} $$

Subcase 2.3: $\beta, \gamma \in \operatorname{dmn}(h^\varrho_\alpha)$. So $\beta, \gamma \in D'$, $g^\varrho_\alpha(\beta) \cap h^\varrho_\alpha(\gamma) \subseteq R \subseteq Y$, $g^\varrho_\alpha(\gamma) \cap h^\varrho_\alpha(\beta) \subseteq Y$, and $h^\varrho_\alpha(\beta) \cap h^\varrho_\alpha(\gamma) \subseteq X \subseteq Y$. Hence

$$\begin{aligned} l^\varrho_\alpha(\beta) \cap l^\varrho_\alpha(\gamma) &= [g^\varrho_\alpha(\beta) \cup h^\varrho_\alpha(\beta)] \cap [g^\varrho_\alpha(\gamma) \cup h^\varrho_\alpha(\gamma)] \\ &= [g^\varrho_\alpha(\beta) \cap g^\varrho_\alpha(\gamma)] \cup [g^\varrho_\alpha(\beta) \cap h^\varrho_\alpha(\gamma)] \cup [h^\varrho_\alpha(\beta) \cap g^\varrho_\alpha(\gamma)] \cup [h^\varrho_\alpha(\beta) \cap h^\varrho_\alpha(\gamma)] \\ &= [g^\varrho_\alpha(\beta) \cap g^\varrho_\alpha(\gamma)] \cup [g^\varrho_\alpha(\beta) \cap h^\varrho_\alpha(\gamma) \cap Y] \\ &\quad \cup [h^\varrho_\alpha(\beta) \cap g^\varrho_\alpha(\gamma) \cap Y] \cup [h^\varrho_\alpha(\beta) \cap h^\varrho_\alpha(\gamma) \cap Y] \\ &= [g^\varrho_\alpha(\beta) \cap g^\varrho_\alpha(\gamma)] \cup [g^\varrho_\alpha(\beta) \cap f^\varrho_\alpha(\gamma)] \cup [f^\varrho_\alpha(\beta) \cap g^\varrho_\alpha(\gamma)] \cup [f^\varrho_\alpha(\beta) \cap f^\varrho_\alpha(\gamma)] \\ &= [g^\varrho_\alpha(\beta) \cap g^\varrho_\alpha(\gamma)] \cup [g^\varrho_\alpha(\beta) \cap g^\varrho_\alpha(\gamma) \cap Y] = g^\varrho_\alpha(\beta) \cap g^\varrho_\alpha(\gamma). \ \blacksquare \end{aligned}$$

Lemma 7.3. *If $\mu \in K$, then $Q(\kappa, \mu)$ is μ-closed.*

Proof. Suppose that $\nu < \mu$ and $\langle {}_\xi f : \xi < \nu \rangle$ is a sequence of members of $Q(\kappa, \mu)$ such that ${}_\eta f \leq {}_\xi f$ whenever $\xi < \eta < \nu$. For any $\varrho \in [\mu, \kappa] \cap K$ let $\operatorname{dmn}(g^\varrho) = \bigcup_{\xi<\nu} \operatorname{dmn}({}_\xi f^\varrho)$. For each $\alpha \in \operatorname{dmn}(g^\varrho)$ let $\operatorname{dmn}(g^\varrho_\alpha) = \bigcup_{\xi<\nu,\, \alpha \in \operatorname{dmn}({}_\xi f^\varrho)} \operatorname{dmn}({}_\xi f^\varrho_\alpha)$. For each $\beta \in \operatorname{dmn}(g^\varrho_\alpha)$ let

$$g^\varrho_\alpha(\beta) = \bigcup \{ {}_\xi f^\varrho_\alpha(\beta) : \xi < \nu,\ \alpha \in \operatorname{dmn}({}_\xi f^\varrho),\ \beta \in \operatorname{dmn}({}_\xi f^\varrho_\alpha) \}.$$

Clearly $g \in Q(\kappa, \mu)$. We check (4)(c). Suppose that $\xi < \nu$, $\varrho \in [\mu, \kappa] \cap K$, $\alpha \in \operatorname{dmn}({}_\xi f^\varrho)$, and β and γ are distinct members of $\operatorname{dmn}({}_\xi f^\varrho_\alpha)$. Suppose that $\delta \in g^\varrho_\alpha(\beta) \cap g^\varrho_\alpha(\gamma)$. Then there is an $\eta \in (\xi, \nu)$ such that $\alpha \in \operatorname{dmn}({}_\eta f^\varrho)$, $\beta, \gamma \in \operatorname{dmn}({}_\eta f^\varrho_\alpha)$, and $\delta \in {}_\eta f^\varrho_\alpha(\beta) \cap {}_\eta f^\varrho_\alpha(\gamma)$. So $\delta \in {}_\xi f^\varrho_\alpha(\beta) \cap {}_\xi f^\varrho_\alpha(\gamma)$ since ${}_\eta f \leq {}_\xi f$. $\blacksquare$

At a certain point in the proof of the next lemma we will need the following general fact about forcing. Recall from Kunen [80, VII.2.12] the definition of the *standard name* for a generic filter.

Fact 7.4. *Let Γ be the standard name for a generic filter. Then $s \Vdash \exists f \in \Gamma\ [\chi(f)]$ iff the set*

$$\{ r : \exists f\ [r \leq f \text{ and } r \Vdash \chi(f)] \}$$

is dense below s.

Proof. $\Rightarrow$: Assume the lhs, and suppose that $t \leq s$. Let G be generic with $t \in G$. By the lhs, choose $f \in G$ such that $M[G] \models \chi(f)$. Choose $k \in G$ such that $k \Vdash \chi(f)$. Take $r \leq f, k, t$. Clearly r is as desired.

$\Leftarrow$: Assume the rhs. Let $s \in G$, with G generic. By the rhs, choose r in the indicated set with $r \leq s$ and $r \in G$. Then choose f as indicated. Then $f \in G$ and $M[G] \models \chi(f)$, as desired. $\blacksquare$

Lemma 7.5. *Let G be $Q(\kappa, \nu)^M$-generic over M, and suppose that μ is a regular cardinal of M such that $\nu \leq \mu < \kappa$. Let*

$$H = \{ f \restriction ([\mu^+, \kappa] \cap K) : f \in G \}, \qquad J = \{ f \restriction ([\nu, \mu] \cap K) : f \in G \}.$$

Then

(6) $M[G] = M[H][J]$;
(7) *H is $Q(\kappa, \mu^+)^M$-generic over M;*

(8) *J is $Q_\mu(\kappa,F)^{M[H]}$-generic over $M[H]$, where*

$$F(\alpha,\beta)=\bigcup\{f^{\mu^+}_\alpha(\beta): f\in H,\ \alpha\in\mathrm{dmn}(f^{\mu^+}),\ \beta\in\mathrm{dmn}(f^{\mu^+}_\alpha)\}$$

for all $\beta<\alpha\in\mathbf{O}$.

Proof. First we check (7). Clearly if $f,g\in H$ then there is an $h\in H$ such that $h\le f,g$. Now suppose that $f\in H$ and $f\le g\in Q(\kappa,\mu^+)$; we want to show that $g\in H$. Say $f=f'\restriction([\mu^+,\kappa]\cap K)$ with $f'\in G$. It suffices to define an $h\in Q(\kappa,\nu)$ such that $f'\le h$ and $h\restriction([\mu^+,\kappa]\cap K)=g$. So we let $g\subseteq h$, and for $\varrho\in[\nu,\mu]\cap K$ define $\mathrm{dmn}(h^\varrho)=\mathrm{dmn}(g^{\mu^+})\cap\mathrm{dmn}(f'^\varrho)$, and for any $\alpha\in\mathrm{dmn}(h^\varrho)$ let $\mathrm{dmn}(h^\varrho_\alpha)=\mathrm{dmn}(g^{\mu^+}_\alpha)\cap\mathrm{dmn}(f'^\varrho_\alpha)$, and for every $\beta\in\mathrm{dmn}(h^\varrho_\alpha)$ let $h^\varrho_\alpha(\beta)=g^{\mu^+}_\alpha(\beta)\cap f'^\varrho_\alpha(\beta)$.

Take any $\varrho\in K$; we check that $h^\varrho\in Q'(\kappa,\nu)$. For $\varrho\in[\mu^+,\kappa]$ we have $h^\varrho=g^\varrho$, so this is given. Suppose that $\varrho\in[\nu,\mu]$. Then (1) and (2) are clear for h^ϱ. So always $h^\varrho\in Q'(\kappa,\nu)$.

Next, we check (5). Suppose that $\varrho,\sigma\in K$ and $\varrho\le\sigma$. If $\mu^+\le\varrho$, then (5) is OK since $h^\varrho=g^\varrho$ and $h^\sigma=g^\sigma$. If $\sigma\le\mu$, then (5) holds since $f'\in Q(\kappa,\nu)$. So, suppose that $\varrho\le\mu<\mu^+\le\sigma$. Then

$$\mathrm{dmn}(h^\varrho)=\mathrm{dmn}(g^{\mu^+})\cap\mathrm{dmn}(f'^\varrho)\subseteq\mathrm{dmn}(g^{\mu^+})\subseteq\mathrm{dmn}(g^\sigma)=\mathrm{dmn}(h^\sigma).$$

For $\alpha\in\mathrm{dmn}(h^\varrho)$,

$$\mathrm{dmn}(h^\varrho_\alpha)=\mathrm{dmn}(g^{\mu^+}_\alpha)\cap\mathrm{dmn}(f'^\varrho_\alpha)\subseteq\mathrm{dmn}(g^{\mu^+}_\alpha)\subseteq\mathrm{dmn}(g^\sigma_\alpha)=\mathrm{dmn}(h^\sigma_\alpha).$$

For $\beta\in\mathrm{dmn}(h^\varrho_\alpha)$,

$$h^\varrho_\alpha(\beta)=g^{\mu^+}_\alpha(\beta)\cap f'^\varrho_\alpha(\beta)\subseteq g^{\mu^+}_\alpha(\beta)\subseteq g^\sigma_\alpha(\beta)=h^\sigma_\alpha(\beta).$$

So $h\in Q(\kappa,\nu)$.

Now we check that $f'\le h$. Let $\varrho\in K$. If $\varrho\in[\mu^+,\kappa]$, then $f'^\varrho=f^\varrho\le g^\varrho=h^\varrho$, as desired. Suppose that $\varrho\in[\nu,\mu]\cap K$. Then $\mathrm{dmn}(h^\varrho)\subseteq\mathrm{dmn}(f'^\varrho)$. Suppose that $\alpha\in\mathrm{dmn}(h^\varrho)$. Then $\mathrm{dmn}(h^\varrho_\alpha)\subseteq\mathrm{dmn}(f'^\varrho_\alpha)$. If $\beta\in\mathrm{dmn}(h^\varrho_\alpha)$, then $h^\varrho_\alpha(\beta)\subseteq f'^\varrho_\alpha(\beta)$. Finally, to check (4)(c), suppose that β and γ are distinct members of $\mathrm{dmn}(h^\varrho_\alpha)$. Then

$$\begin{aligned}h^\varrho_\alpha(\beta)\cap h^\varrho_\alpha(\gamma)&=g^{\mu^+}_\alpha(\beta)\cap f'^\varrho_\alpha(\beta)\cap g^{\mu^+}_\alpha(\gamma)\cap f'^\varrho_\alpha(\gamma)\\&=g^{\mu^+}_\alpha(\beta)\cap g^{\mu^+}_\alpha(\gamma)\cap f'^\varrho_\alpha(\beta)\cap f'^\varrho_\alpha(\gamma)\\&=f^{\mu^+}_\alpha(\beta)\cap f^{\mu^+}_\alpha(\gamma)\cap f'^\varrho_\alpha(\beta)\cap f'^\varrho_\alpha(\gamma)\quad\text{(using (4)(c), since } f\le g)\\&=f'^\varrho_\alpha(\beta)\cap f'^\varrho_\alpha(\gamma).\end{aligned}$$

So, indeed, $f'\le h$. It follows that $g\in H$.

Now suppose that D is a dense subset of $Q(\kappa,\mu^+)$ in M. Let

$$D'=\{f\in Q(\kappa,\nu): f\restriction([\mu^+,\kappa]\cap K)\in D\}\quad\text{(in } M)$$

We claim that D' is dense in $Q(\kappa,\nu)$. For, let $g\in Q(\kappa,\nu)$. Choose $f\in D$ so that $f\le g\restriction([\mu^+,\kappa]\cap K)$. Define f' with domain K by: $f\subseteq f'$ and for $\varrho\in[\nu,\mu]\cap K$, $f'^\varrho=g^\varrho$. To show that $f'\in Q(\kappa,\nu)$, the only questionable point is (5) for $\nu\le\varrho\le\mu<\mu^+\le\sigma\le\kappa$. Then

$$\mathrm{dmn}(f'^\varrho)=\mathrm{dmn}(g^\varrho)\subseteq\mathrm{dmn}(g^\sigma)\subseteq\mathrm{dmn}(f^\sigma)=\mathrm{dmn}(f'^\sigma),$$

and for any $\alpha \in \mathrm{dmn}(f'^{\varrho})$,

$$\mathrm{dmn}(f'^{\varrho}_{\alpha}) = \mathrm{dmn}(g^{\varrho}_{\alpha}) \subseteq \mathrm{dmn}(g^{\sigma}_{\alpha}) \subseteq \mathrm{dmn}(f^{\sigma}_{\alpha}) = \mathrm{dmn}(f'^{\sigma}_{\alpha}),$$

and for any $\beta \in \mathrm{dmn}(f'^{\varrho}_{\alpha})$,

$$f'^{\varrho}_{\alpha}(\beta) = g^{\varrho}_{\alpha}(\beta) \subseteq g^{\sigma}_{\alpha}(\beta) \subseteq f^{\sigma}_{\alpha}(\beta) = f'^{\sigma}_{\alpha}(\beta).$$

So $f' \in Q(\kappa,\nu)$, hence $f' \in D'$. Clearly $f' \leq g$. This shows that D' is dense in $Q(\kappa,\nu)$.

Hence choose $f \in D' \cap G$. So $f\restriction([\mu^+,\kappa] \cap K) \in D \cap H$. So (7) is proved.

Now we turn to (8). Let θ be a name in M for $Q_\mu(\kappa,F)$. Note

(a) $\theta^{M[H]} \subseteq M$.

In fact, work in $M[H]$. Let $f \in \theta^{M[H]}$. Let

$$L = \{(\varrho,\alpha,\beta) : \varrho \in \mathrm{dmn}(f),\ \alpha \in \mathrm{dmn}(f^{\varrho}),\ \beta \in \mathrm{dmn}(f^{\varrho}_{\alpha})\}.$$

Thus $L \subseteq K \times \lambda \times \lambda$, and $|L| \leq \mu$. Let g be a mapping of μ onto L.

We no longer work in $M[H]$. By (7), 7.3 and Kunen [80, VII.6.14], $g \in M$, and hence $L \in M$. Now we claim that if $(\varrho,\alpha,\beta) \in L$, then $f^{\varrho}_{\alpha}(\beta) \in M$. If $f^{\varrho}_{\alpha}(\beta) = 0$, this is obvious. If $f^{\varrho}_{\alpha}(\beta) \neq 0$, let h be a mapping of μ onto $f^{\varrho}_{\alpha}(\beta)$. By (7), 7.3 and Kunen [80, VII.6.14], $h \in M$ and hence $f^{\varrho}_{\alpha}(\beta) \in M$. This proves our claim. Now define $f'(\varrho,\alpha,\beta) = f^{\varrho}_{\alpha}(\beta)$ for any $(\varrho,\alpha,\beta) \in L$. Then f' maps L into $\mathscr{P}^M(\kappa)$, and so by (7), 7.3 and Kunen [80, VII.6.14], $f' \in M$. Hence $f \in M$, proving (a).

Now let Γ be the standard name for a generic filter. Then the following formula $\varphi(x,\alpha,\beta)$ expresses that $x \subseteq F(\alpha,\beta)$:

$$\forall y \in x\ \exists f \in \Gamma\ [\alpha \in \mathrm{dmn}(f^{\mu^+}) \wedge \beta \in \mathrm{dmn}(f^{\mu^+}_{\alpha}) \wedge y \in f^{\mu^+}_{\alpha}(\beta)].$$

Thus $\varphi^{M[H]}(x,\alpha,\beta)$ iff $x \subseteq F(\alpha,\beta)$.

Now let $\psi(z)$ be the following formula:

$$\begin{aligned}\mathrm{dmn}(z) = [\nu,\mu] \cap K \wedge \exists w\ [w \in Q(\kappa,\nu) \wedge \forall \varrho \in [\nu,\mu] \cap K\ (z^{\varrho} = w^{\varrho})\\ \wedge \forall \alpha \in \mathrm{dmn}(w^{\mu})\ \forall \beta \in \mathrm{dmn}(w^{\mu}_{\alpha})\ \varphi(w^{\mu}_{\alpha}(\beta),\alpha,\beta)].\end{aligned}$$

Thus $\psi^{M[H]}(z)$ iff $z \in Q_\mu(\kappa,F)$. Now we claim:

(b) In M, if $f \in Q(\kappa,\mu^+)$ and $g \in M$, then $f \Vdash \psi(g)$ iff $g \in Q_\mu(\kappa)$ and the set $\{h \in Q(\kappa,\mu^+) : g \cup h \in Q(\kappa,\nu)\}$ is dense below f.

For $\Rightarrow$, suppose that $f \in Q(\kappa,\mu^+)$, $g \in M$, and $f \Vdash \psi(g)$. Take any $h \leq f$. Choose w and $k \leq h$ so that $w \in Q(\kappa,\nu)$, $g = w\restriction([\nu,\mu] \cap K)$, and for all $\alpha \in \mathrm{dmn}(w^{\mu})$, and all $\beta \in \mathrm{dmn}(w^{\mu}_{\alpha})$, $k \Vdash \varphi(w^{\varrho}_{\alpha}(\beta),\alpha,\beta)$. Clearly then $g \in Q_\mu(\kappa)$. Note that $\mathrm{dmn}(g) = [\nu,\mu] \cap K$, while for each $h \in Q(\kappa,\mu^+)$ we have $\mathrm{dmn}(h) = [\mu^+,\kappa] \cap K$. Now for all $\alpha \in \mathrm{dmn}(g^{\mu})$ and all $\beta \in \mathrm{dmn}(g^{\mu}_{\alpha})$ we have

$$k \Vdash \forall \gamma \in g^{\mu}_{\alpha}(\beta)\ \exists l \in \Gamma\ [\alpha \in \mathrm{dmn}(l^{\mu^+}) \wedge \beta \in \mathrm{dmn}(l^{\mu^+}_{\alpha}) \wedge \gamma \in l^{\mu^+}_{\alpha}(\beta)].$$

Fix $\alpha \in \mathrm{dmn}(g^{\mu})$, $\beta \in \mathrm{dmn}(g^{\mu}_{\alpha})$, and $\gamma \in g^{\mu}_{\alpha}(\beta)$. Then

$$k \Vdash \exists l \in \Gamma\ [\alpha \in \mathrm{dmn}(l^{\mu^+}) \wedge \beta \in \mathrm{dmn}(l^{\mu^+}_{\alpha}) \wedge \gamma \in l^{\mu^+}_{\alpha}(\beta)].$$

By Fact 7.4,

$$\{r : \exists l\ [r \leq l \wedge \alpha \in \mathrm{dmn}(l^{\mu^+}) \wedge \beta \in \mathrm{dmn}(l^{\mu^+}_{\alpha}) \wedge \gamma \in l^{\mu^+}_{\alpha}(\beta)]\}$$

is dense below k. Note that if $r \le l$, $\alpha \in \mathrm{dmn}(l^{\mu^+})$, $\beta \in \mathrm{dmn}(l^{\mu^+}_\alpha)$, and $\gamma \in l^{\mu^+}_\alpha(\beta)$, then $\alpha \in \mathrm{dmn}(r^{\mu^+})$, $\beta \in \mathrm{dmn}(r^{\mu^+}_\alpha)$, and $\gamma \in r^{\mu^+}_\alpha(\beta)$. So

$$\{r : \alpha \in \mathrm{dmn}(r^{\mu^+}),\ \beta \in \mathrm{dmn}(r^{\mu^+}_\alpha),\ \gamma \in r^{\mu^+}_\alpha(\beta)\}$$

is dense below k. By μ^+-closedness, it follows that for each $\alpha \in \mathrm{dmn}(g^\mu)$ and $\beta \in \mathrm{dmn}(g^\mu_\alpha)$ the set

$$\{r : \alpha \in \mathrm{dmn}(r^{\mu^+}),\ \beta \in \mathrm{dmn}(r^{\mu^+}_\alpha),\ g^\mu_\alpha(\beta) \subseteq r^{\mu^+}_\alpha(\beta)\}$$

is dense below k. By μ^+-closedness two more times,

$$\{r : \mathrm{dmn}(g^\mu) \subseteq \mathrm{dmn}(r^{\mu^+}) \text{ and } \forall \alpha \in \mathrm{dmn}(g^\mu)\ [\mathrm{dmn}(g^\mu_\alpha) \subseteq \mathrm{dmn}(r^{\mu^+}_\alpha) \text{ and } \forall \beta \in \mathrm{dmn}(g^\mu_\alpha)\ [g^\mu_\alpha(\beta) \subseteq r^{\mu^+}_\alpha(\beta)]]\}$$

is dense below k. For l in this set, $g \cup l \in Q(\kappa,\nu)$, as desired in (b).

For the other direction, assume the condition, and suppose that $f \in L$, with L $Q(\kappa,\mu^+)$-generic. Choose $h \le f$ in the indicated set, $h \in L$. So for all $\alpha \in \mathrm{dmn}(g^\mu)$ and all $\beta \in \mathrm{dmn}(g^\mu_\alpha)$ we have $\alpha \in \mathrm{dmn}(h^{\mu^+})$, $\beta \in \mathrm{dmn}(h^{\mu^+}_\alpha)$, and $g^\mu_\alpha(\beta) \subseteq h^{\mu^+}_\alpha(\beta)$. Thus $\varphi(g^\mu_\alpha(\beta),\alpha,\beta)$ holds. Hence $\psi(g)$ holds, as desired.

Now we start actually proving (8). Clearly if $f, g \in J$, then there is an $h \in J$ such that $h \le f, g$. Now suppose that $f \in J$ and $f \le g \in Q_\mu(\kappa,F)^{M[H]}$. Choose $h \in H$ such that $h \Vdash \psi(g)$. So by (b), there is a $k \in H$ with $k \le h$ such that $g \cup k \in Q(\kappa,\nu)$. Say $k = k' \restriction ([\mu^+,\kappa] \cap K)$ with $k' \in G$, and say $f = f' \restriction ([\nu,\mu] \cap K)$ with $f' \in G$. Choose $l \in G$ with $l \le k', f'$. We claim that $l \le g \cup k$ (hence $g \cup k \in G$ and so $g \in J$). In fact, if $\varrho \in [\nu,\mu] \cap K$, then $l^\varrho \le f'^\varrho = f^\varrho \le g^\varrho$, and for $\varrho \in [\mu^+,\kappa] \cap K$, $l^\varrho \le k'^\varrho = k^\varrho$.

Next, suppose that D is $Q_\mu(\kappa,F)^{M[H]}$-dense in $M[H]$. Let τ be a term such that $\tau^H = D$. Now we introduce some notation for an arbitrary $f \in Q(\kappa,\nu)$:

$$f^{(\mu)} = f \restriction ([\mu^+,\kappa] \cap K), \qquad f_{(\mu)} = f \restriction ([\nu,\mu] \cap K).$$

Now in $M[H]$ we have: $D \subseteq Q_\mu(\kappa,F) \wedge \forall h \in Q_\mu(\kappa,F)\ \exists k \in D\ (k \le h)$. Hence choose $f \in H$ such that

(c) $f \Vdash \forall k \in \tau\ \psi(k) \wedge \forall h\ [\psi(h) \to \exists k \in \tau\ (k \le h)]$.

Say $f = g^{(\mu)}$ with $g \in G$. We now claim (in M):

(d) $\{h \in Q(\kappa,\nu) : h^{(\mu)} \Vdash h_{(\mu)} \in \tau\}$ is dense below g.

To prove this, take any $r \le g$. Now $r^{(\mu)} \le f$, so

(e) $r^{(\mu)} \Vdash \forall k \in \tau\ \psi(k) \wedge \forall h\ [\psi(h) \to \exists k \in \tau\ (k \le h)]$.

Now

(f) $r_{(\mu)} \in Q_\mu(\kappa)$ and $\{h \in Q(\kappa,\mu^+) : r_{(\mu)} \cup h \in Q(\kappa,\nu)\}$ is dense below $r^{(\mu)}$.

For, the first statement is clear. Now suppose that $s \le r^{(\mu)}$. Then $r_{(\mu)} \cup s \in Q(\kappa,\nu)$. For,

$$\mathrm{dmn}((r_{(\mu)})^\mu) = \mathrm{dmn}(r^\mu) \subseteq \mathrm{dmn}(r^{\mu^+}) \subseteq \mathrm{dmn}(s^{\mu^+})$$

and for any $\alpha \in \mathrm{dmn}((r_{(\mu)})^\mu)$,

$$\mathrm{dmn}((r_{(\mu)})^\mu_\alpha) = \mathrm{dmn}(r^\mu_\alpha) \subseteq \mathrm{dmn}(r^{\mu^+}_\alpha) \subseteq \mathrm{dmn}(s^{\mu^+}_\alpha),$$

and for any $\beta \in \mathrm{dmn}((r^{(\mu)})^\mu_\alpha)$,

$$(r^{(\mu)})^\mu_\alpha(\beta) = r^\mu_\alpha(\beta) \subseteq r^{\mu^+}_\alpha(\beta) \subseteq s^{\mu^+}_\alpha(\beta).$$

So, (f) holds.

By (f) and (b) we have $r^{(\mu)} \Vdash \psi(r_{(\mu)})$. Hence by (e), $r^{(\mu)} \Vdash \exists k \in \tau\ [k \le r_{(\mu)} \wedge \psi(k)]$. Let L be $Q(\kappa,\mu^+)$-generic over M such that $r^{(\mu)} \in L$. Choose $k \in \tau_L$ such that $k \le r_{(\mu)}$ and $\psi^L(k)$. Note that $k \in M$ by the version of (a) for $M[L]$. Choose $s \in L$ such that $s \Vdash k \in \tau \wedge \psi(k)$ and $s \le r^{(\mu)}$. By (b), choose $l \le s$ such that $t := k \cup l \in Q(\kappa,\nu)$. Thus $t^{(\mu)} = l \Vdash \psi(k)$, and $k = t_{(\mu)}$, so $t^{(\mu)} \Vdash \psi(t_{(\mu)})$. Moreover, $t \le r$ since if $\varrho \in [\nu,\mu] \cap K$ then $t^\varrho = k^\varrho \le (r_{(\mu)})^\varrho = r^\varrho$, while if $\varrho \in [\mu^+,\kappa] \cap K$ then $t^\varrho = l^\varrho \le s^\varrho \le (r^{(\mu)})^\varrho = r^\varrho$. So (d) holds.

By (d), there is a $u \in G$ such that $u \le g$ and $u^{(\mu)} \Vdash u_{(\mu)} \in \tau$. Hence $u^{(\mu)} \in H$, so $u_{(\mu)} \in J \cap D$, as desired.

Thus (8) holds.

Finally, we turn to (6), where we apply Kunen [80, VII.2.9]. We have $M \subseteq M[G]$ and $H \in M[G]$, so $M[H] \subseteq M[G]$. And $J \in M[G]$, so $M[H][J] \subseteq M[G]$.

For the other inclusion it suffices to show that $G \in M[H][J]$. For any function f,

$$\begin{aligned} f \in G \quad \text{iff} \quad & f\restriction([\mu^+,\kappa]\cap K) \in H \text{ and } f\restriction([\nu,\mu]\cap K) \in J \text{ and } \mathrm{dmn}(f^\mu) \subseteq \mathrm{dmn}(f^{\mu^+}) \\ & \text{and } \forall \alpha \in \mathrm{dmn}(f^\mu)\,[\mathrm{dmn}(f^\mu_\alpha) \subseteq \mathrm{dmn}(f^{\mu^+}_\alpha) \\ & \qquad\qquad \text{and } \forall \beta \in \mathrm{dmn}(f^\mu_\alpha)\ [f^\mu_\alpha(\beta) \subseteq f^{\mu^+}_\alpha(\beta)]]. \end{aligned}$$

This finishes the proof of (6) and of Lemma 7.5. ■

Lemma 7.6. *Cofinalities are preserved in $M[G]$, where G is $Q(\kappa,\nu)$-generic over M.*

Proof. By Kunen [80, VII.5.9] it suffices to get a contradiction upon assuming that τ is regular in M but singular in $M[G]$. Let $\mu = \mathrm{cf}\,\tau$ in $M[G]$. Thus μ is a cardinal in M. If $\mu < \nu$, this contradicts Lemma 7.3 and Kunen [80, VII.6.14]. If $\kappa \le \mu$, then $Q(\kappa,\nu)$ has the κ^+-cc by Lemma 7.2, hence the μ^+-cc, so this contradicts Kunen [80, VII.6.9] since $\mathrm{cf}(\tau)^M = \tau \ge \mu^+$. So, assume that $\nu \le \mu < \kappa$.

Let H, J, and F be as in Lemma 7.5. By Lemma 7.3, $Q(\kappa,\mu^+)$ is μ^+-closed. Hence by Kunen [80, VII.6.14], $\mathrm{cf}\,\tau \ge \mu^+$ in $M[H]$. Also, $\mu^{<\mu} = \mu$ in $M[H]$ by Kunen [80, VII.6.14]. Next,

(1) $|F(\alpha,\beta) \cap F(\alpha,\gamma)| \le \mu$ whenever $\beta < \gamma < \alpha \in \mathbf{O}$.

To prove this, we may assume that $F(\alpha,\beta) \cap F(\alpha,\gamma) \ne \emptyset$. Then we claim

(2) There is a $k \in H$ such that $\alpha \in \mathrm{dmn}(k^{\mu^+})$ and $\beta,\gamma \in \mathrm{dmn}(k^{\mu^+}_\alpha)$.

In fact, take any $\varepsilon \in F(\alpha,\beta) \cap F(\alpha,\gamma)$. Then there are $s,t \in H$ such that $\alpha \in \mathrm{dmn}(s^{\mu^+})$, $\beta \in \mathrm{dmn}(s^{\mu^+}_\alpha)$, and $\varepsilon \in s^{\mu^+}_\alpha(\beta)$; and $\alpha \in \mathrm{dmn}(t^{\mu^+})$, $\gamma \in \mathrm{dmn}(t^{\mu^+}_\alpha)$, and $\varepsilon \in t^{\mu^+}_\alpha(\gamma)$. Take $k \in H$ such that $k \le s,t$. Clearly (2) holds for this k.

Now take any $\delta \in F(\alpha,\beta) \cap F(\alpha,\gamma)$. Then there exist $f,g \in H$ such that $\alpha \in \mathrm{dmn}(f^{\mu^+})$, $\alpha \in \mathrm{dmn}(g^{\mu^+})$, $\beta \in \mathrm{dmn}(f^{\mu^+}_\alpha)$, $\gamma \in \mathrm{dmn} g^{\mu^+}_\alpha$, and $\delta \in f^{\mu^+}_\alpha(\beta) \cap g^{\mu^+}_\alpha(\gamma)$. Take $h \in H$ such that $h \le f,g,k$. Then

$$\delta \in f^{\mu^+}_\alpha(\beta) \cap g^{\mu^+}_\alpha(\gamma) \subseteq h^{\mu^+}_\alpha(\beta) \cap h^{\mu^+}_\alpha(\gamma) = k^{\mu^+}_\alpha(\beta) \cap k^{\mu^+}_\alpha(\gamma).$$

This proves (1).

By Lemma 7.2, $Q_\mu(\kappa, F)$ has the μ^+-cc. So $\mathrm{cf}\,\tau \geq \mu^+$ in $M[H][J]$ by Kunen [80, VII.6.9], contradiction. ∎

Proof of Theorem 7.1. Preservation of cardinalities and cofinalities follows from Lemma 7.6. Now for $\alpha \in \mathbf{O}$ and $\beta < \alpha$, let

$$G^\alpha_\beta = \bigcup_{f\in G,\,\alpha\in\mathrm{dmn}(f^\nu),\,\beta\in\mathrm{dmn}(f^\nu_\alpha)} f^\nu_\alpha(\beta).$$

We claim that for each $\alpha \in \mathbf{O}$, the system $\langle G^\alpha_\beta : \beta < \alpha\rangle$ is $[\kappa]^\kappa, \nu$-mad. (This will finish the proof.) To prove this, first fix α and β, with $\beta < \alpha \in \mathbf{O}$. Then the following set is dense:

$$D := \{f \in Q(\kappa, \nu) : \alpha \in \mathrm{dmn}(f^\nu) \text{ and } \beta \in \mathrm{dmn}(f^\nu_\alpha)\}.$$

To see this, let $g \in Q(\kappa, \nu)$ be arbitrary. For each $\mu \in K$ let $\mathrm{dmn}(f^\mu) = \mathrm{dmn}(g^\mu) \cup \{\alpha\}$. Then let

$$\mathrm{dmn}(f^\mu_\alpha) = \begin{cases} \mathrm{dmn}(g^\mu_\alpha) \cup \{\beta\} & \text{if } \alpha \in \mathrm{dmn}(g^\mu),\\ \{\beta\} & \text{otherwise.}\end{cases}$$

Finally, define

$$f^\mu_\alpha(\beta) = \begin{cases} g^\mu_\alpha(\beta) & \text{if } \alpha \in \mathrm{dmn}(g^\mu) \text{ and } \beta \in \mathrm{dmn}(g^\mu_\alpha),\\ 0 & \text{otherwise.}\end{cases}$$

It is clear that $f \in D$ and $f \leq g$. So D is dense.

Now for each $\gamma < \kappa$ the following set is dense:

$$E := \{f \in Q(\kappa, \nu) : \alpha \in \mathrm{dmn}(f^\nu) \text{ and } \beta \in \mathrm{dmn}(f^\nu_\alpha) \text{ and } f^\nu_\alpha(\beta) \cap (\gamma, \kappa) \neq 0\}.$$

For, given $g \in Q(\kappa, \nu)$, by the density of D we may assume that $\alpha \in \mathrm{dmn}(g^\nu)$ and $\beta \in \mathrm{dmn}(g^\nu_\alpha)$. Choose $\delta \in (\gamma, \kappa) \setminus \bigcup_{\varepsilon\in\mathrm{dmn}(g^\kappa)} \bigcup \mathrm{rng}(g^\kappa_\varepsilon)$. Now define f by setting $\mathrm{dmn}(f^\mu) = \mathrm{dmn}(g^\mu)$ for all $\mu \in K$, and for any $\xi \in \mathrm{dmn}(f^\mu)$, $\mathrm{dmn}(f^\mu_\xi) = \mathrm{dmn}(g^\mu_\xi)$, and for any $\eta \in \mathrm{dmn}(f^\nu_\xi)$,

$$f^\nu_\xi(\eta) = \begin{cases} g^\nu_\xi(\eta) & \text{if } \xi \neq \alpha \text{ or } \eta \neq \beta,\\ g^\nu_\alpha(\beta) \cup \{\delta\} & \text{if } \xi = \alpha \text{ and } \eta = \beta.\end{cases}$$

Clearly $f \in Q(\kappa, \nu)$. To show that $f \leq g$, only (4)(c) is a problem. Suppose that $\eta \neq \beta$ and $\varphi \in f^\mu_\alpha(\beta) \cap f^\mu_\alpha(\eta)$. Thus $\varphi \in g^\mu_\alpha(\eta)$, so $\varphi \neq \delta$, and hence also $\varphi \in g^\mu_\alpha(\beta)$, as desired.

By the density of E we have $|G^\alpha_\beta| = \kappa$.

Next, suppose that $\beta < \gamma < \alpha \in \mathbf{O}$. We claim that $|G^\alpha_\beta \cap G^\alpha_\gamma| < \nu$. For, choose $f \in G$ with $\alpha \in \mathrm{dmn}(f^\nu)$ and $\beta, \gamma \in \mathrm{dmn}(f^\nu_\alpha)$. Then $|G^\alpha_\beta \cap G^\alpha_\gamma| \leq |f^\nu_\alpha(\beta) \cap f^\nu_\alpha(\gamma)| < \nu$. In fact, suppose that $\delta \in G^\alpha_\beta \cap G^\alpha_\gamma$. Then there exist $g, h \in G$ such that $\alpha \in \mathrm{dmn}(g^\nu) \cap \mathrm{dmn}(h^\nu)$, $\beta \in \mathrm{dmn}(g^\nu_\alpha)$, $\gamma \in \mathrm{dmn}(g^\nu_\beta)$, $\delta \in g^\nu_\alpha(\beta)$, and $\delta \in h^\nu_\alpha(\gamma)$. Choose $k \in G$ so that $k \leq f, g, h$. Then

$$\delta \in g^\nu_\alpha(\beta) \cap h^\nu_\alpha(\gamma) \subseteq k^\nu_\alpha(\beta) \cap k^\nu_\alpha(\gamma) = f^\nu_\alpha(\beta) \cap f^\nu_\alpha(\gamma).$$

It only remains to show that for each $\alpha \in \mathbf{O}$, $\langle G^\alpha_\beta : \beta < \alpha\rangle$ is maximal. First, some notation. We write Θ^α_β for the term

$$\bigcup_{f\in\Gamma,\,\alpha\in\mathrm{dmn}(f^\nu),\,\beta\in\mathrm{dmn}(f^\nu_\alpha)} f^\nu_\alpha(\beta),$$

where Γ is the standard name for a generic filter. Note that $(\Theta^\alpha_\beta)_G = G^\alpha_\beta$.

(9) If $\alpha \in \mathrm{dmn}(f^\nu)$, $\beta \in \mathrm{dmn}(f^\nu_\alpha)$, and $\gamma \in f^\nu_\alpha(\beta)$, then $f \Vdash \gamma \in \Theta^\alpha_\beta$.

For, suppose that $f \in H$ with H generic. Then $f \in \Gamma_H$. So (9) holds.

Now suppose that $X \in [\kappa]^\kappa$ and $|X \cap G^\alpha_\beta| < \nu$ for all $\beta < \alpha$; we want to get a contradiction. Let τ be a nice name for a subset of κ such that $\tau_G = X$. Say $\tau = \bigcup_{\gamma\in\kappa}\{\gamma\}\times B_\gamma$, where each B_γ is a collection of pairwise incompatible elements of $Q(\kappa,\nu)$. So $|B_\gamma| \le \kappa$ for all $\gamma \in \kappa$, by 7.2. Choose

$$\beta \in \alpha \setminus \bigcup_{\gamma\in\kappa} \bigcup_{f\in B_\gamma,\, \alpha\in\mathrm{dmn}(f^\kappa)} \mathrm{dmn}(f^\kappa_\alpha).$$

Now $X \cap G^\alpha_\beta \in [\kappa]^{<\nu}$, so by Kunen [80, VII.6.14] we have $X \cap G^\alpha_\beta \in M$. Let $\Omega = X \cap G^\alpha_\beta$. Choose $f \in G$ such that

$$f \Vdash |\tau| = \kappa \wedge \forall\gamma \in \tau\ (\gamma \in \Theta^\alpha_\beta \leftrightarrow \gamma \in \Omega).$$

Let $g \le f$ with $g \in G$, $\alpha \in \mathrm{dmn}(g^\nu)$, and $\beta \in \mathrm{dmn}(g^\nu_\alpha)$. Define

$$\Xi = \Omega \cup \bigcup_{\gamma\in\mathrm{dmn}(g^\kappa_\alpha)} g^\kappa_\alpha(\gamma) \cup \bigcup_{\gamma\in\mathrm{dmn}(g^\kappa_\alpha)} (X \cap G^\alpha_\gamma).$$

Note that $|\Xi| < \kappa$. Hence we can choose $\delta \in X \setminus \Xi$. Since $\delta \in X$, there is an $r \in B_\delta \cap G$. Hence also there is an $h \in G$ such that $h \le g, r$.

(10) If $\alpha \in \mathrm{dmn}(r^\kappa)$, then $\beta \notin \mathrm{dmn}(r^\kappa_\alpha)$.

This holds by the choice of β.

(11) If $\alpha \in \mathrm{dmn}(r^\kappa)$ and $\tau \in \mathrm{dmn}(r^\kappa_\alpha) \cap \mathrm{dmn}(g^\kappa_\alpha)$, then $\delta \notin r^\kappa_\alpha(\tau)$.

For, otherwise we get $r \Vdash \delta \in \Theta^\alpha_\tau$ by (9); also $r \Vdash \delta \in \tau$. Since $r \in G$, this implies that $\delta \in X \cap G^\alpha_\tau$, contradicting the choice of δ.

Now we define a function l as follows: $\mathrm{dmn}(l) = K$, $\mathrm{dmn}(l^\mu) = \mathrm{dmn}(g^\mu) \cup \mathrm{dmn}(r^\mu)$ for all $\mu \in K$, and for any $\varepsilon \in \mathrm{dmn}(l^\mu)$,

$$l^\mu_\varepsilon = \begin{cases} g^\mu_\varepsilon & \text{if } \varepsilon \in \mathrm{dmn}(g^\mu) \setminus \mathrm{dmn}(r^\mu) \text{ and } \varepsilon \ne \alpha, \\ t^\mu & \text{if } \varepsilon \in \mathrm{dmn}(g^\mu) \setminus \mathrm{dmn}(r^\mu) \text{ and } \varepsilon = \alpha, \\ r^\mu_\varepsilon & \text{if } \varepsilon \in \mathrm{dmn}(r^\mu) \setminus \mathrm{dmn}(g^\mu), \\ s^\mu & \text{if } \varepsilon \in \mathrm{dmn}(g^\mu) \cap \mathrm{dmn}(r^\mu), \end{cases}$$

where $\mathrm{dmn}(t^\mu) = \mathrm{dmn}(g^\mu_\alpha)$ and for all $\theta \in \mathrm{dmn}(t^\mu)$,

$$t^\mu(\theta) = \begin{cases} g^\mu_\alpha(\theta) & \text{if } \theta \ne \beta, \\ g^\mu_\alpha(\beta) \cup \{\delta\} & \text{if } \theta = \beta, \end{cases}$$

and $\mathrm{dmn}(s^\mu) = \mathrm{dmn}(g^\mu_\varepsilon) \cup \mathrm{dmn}(r^\mu_\varepsilon)$, and for any $\theta \in \mathrm{dmn}(s^\mu)$,

$$s^\mu(\theta) = \begin{cases} g^\mu_\varepsilon(\theta) & \text{if } \theta \in \mathrm{dmn}(g^\mu_\varepsilon) \setminus \mathrm{dmn}(r^\mu_\varepsilon) \text{ and } (\theta \ne \beta \text{ or } \varepsilon \ne \alpha), \\ r^\mu_\varepsilon(\theta) & \text{if } \theta \in \mathrm{dmn}(r^\mu_\varepsilon) \setminus \mathrm{dmn}(g^\mu_\varepsilon), \\ g^\mu_\varepsilon(\theta) \cup r^\mu_\varepsilon(\theta) & \text{if } \theta \in \mathrm{dmn}(g^\mu_\varepsilon) \cap \mathrm{dmn}(r^\mu_\varepsilon), \\ g^\mu_\varepsilon(\beta) \cup \{\delta\} & \text{if } \varepsilon = \alpha \text{ and } \theta = \beta. \end{cases}$$

It is a straightforward but lengthy matter to check that $l \in Q(\kappa,\nu)$. We claim that $l \le g$ and $l \le r$. Again the hard part of checking this is condition (4)(c), and we do one of the harder cases here. Suppose that $\mu \in K$, $\alpha \in \mathrm{dmn}(g^\mu) \cap \mathrm{dmn}(r^\mu)$, and $\theta, \beta \in \mathrm{dmn}(g^\mu_\alpha)$,

$\theta \in \mathrm{dmn}(r^\mu_\alpha)$, and $\theta \neq \beta$; we want to show that $l^\mu_\alpha(\theta) \cap l^\mu_\alpha(\beta) \subseteq g^\mu_\alpha(\theta) \cap g^\mu_\alpha(\beta)$. We have

$$l^\mu_\alpha(\theta) \cap l^\mu_\alpha(\beta) = (g^\mu_\alpha(\theta) \cup r^\mu_\alpha(\theta)) \cap (g^\mu_\alpha(\beta) \cup \{\delta\}).$$

Since $\delta \notin g^\mu_\alpha(\theta) \cup r^\mu_\alpha(\theta)$ by construction and (11), we get

$$\begin{aligned} l^\mu_\alpha(\theta) \cap l^\mu_\alpha(\beta) &= (g^\mu_\alpha(\theta) \cup r^\mu_\alpha(\theta)) \cap g^\mu_\alpha(\beta) = (g^\mu_\alpha(\theta) \cap g^\mu_\alpha(\beta)) \cup (r^\mu_\alpha(\theta) \cap g^\mu_\alpha(\beta)) \\ &\subseteq (g^\mu_\alpha(\theta) \cap g^\mu_\alpha(\beta)) \cup (h^\mu_\alpha(\theta) \cap h^\mu_\alpha(\beta)) \subseteq g^\mu_\alpha(\theta) \cap g^\mu_\alpha(\beta), \end{aligned}$$

as desired.

Now we can finish the proof. Since $l \leq r$ and $r \Vdash \delta \in \tau$, we have $l \Vdash \delta \in \tau$. But also $l \leq g$, so $l \Vdash \delta \in \Theta^\alpha_\beta \to \delta \in \Omega$. By construction, $\delta \in l^\mu_\alpha(\beta)$, so by (9) we get $l \Vdash \delta \in \Theta^\alpha_\beta$. These facts contradict $\delta \notin \Omega$.

The proof of Theorem 7.1 is finished. ∎

8. Specifying $\mathrm{MAD}(\kappa)$

Let κ be a regular cardinal, and let C be a set of cardinals satisfying the following conditions:

(1) each member of C is greater than κ;
(2) C is closed;
(3) C contains the immediate successor of each of its members of cofinality between ω and κ inclusive;
(4) C contains all cardinals in $[\kappa^+, |C|]$;
(5) $\kappa^+ \in C$.

The aim of this section is to prove the following theorem, which generalizes a theorem in Blass [93].

Theorem. *Assume the above about κ and C, in a countable transitive model M of GCH. Then there is a partial ordering P such that if G is P-generic over M, then cofinalities and cardinalities are preserved in $M[G]$, and in $M[G]$, $\mathrm{MAD}(\kappa) = C$.*

Note that the set C is not quite arbitrary. We do not know to what extent this theorem can be generalized to other sets.

The proof of the theorem follows Blass [93] as well. We begin by defining the partial order P.

Let P be the set of all functions p such that

(6) $\mathrm{dmn}(p) \in [C]^{<\kappa}$;
(7) for all $\lambda \in \mathrm{dmn}(p)$, p_λ is a function, $\mathrm{dmn}(p_\lambda) \in [\lambda]^{<\kappa}$, and for all $\beta \in \mathrm{dmn}(p_\lambda)$, $p_\lambda(\beta) \in [\kappa]^{<\kappa}$.

For $p, q \in P$ we write $p \leq q$ iff

(8) $\mathrm{dmn}(q) \subseteq \mathrm{dmn}(p)$;
(9) for all $\lambda \in \mathrm{dmn}(q)$,
  (a) $\mathrm{dmn}(q_\lambda) \subseteq \mathrm{dmn}(p_\lambda)$;
  (b) for all $\beta \in \mathrm{dmn}(q_\lambda)$, $q_\lambda(\beta) \subseteq p_\lambda(\beta)$;

(c) for all distinct $\beta,\gamma \in \mathrm{dmn}(q_\lambda)$,

$$q_\lambda(\beta)\cap q_\lambda(\gamma) = p_\lambda(\beta)\cap p_\lambda(\gamma).$$

Lemma 8.1. *If $\kappa^{<\kappa}=\kappa$, then P has the κ^+-chain condition.*

Proof. Suppose that $I \subseteq P$ is pairwise incompatible and $|I| = \kappa^+$. Without loss of generality,

$$\langle \mathrm{dmn}(p) : p \in I\rangle$$

is a Δ-system, say with kernel D, and

$$\Big\langle \bigcup_{\lambda\in\mathrm{dmn}(p)} \mathrm{dmn}(p_\lambda) : p \in I\Big\rangle$$

is a Δ-system, say with kernel E. Then

(10) there is an $f \in {}^D({}^E([\kappa]^{<\kappa}))$ such that

$$I' := \{p \in I : \forall\lambda\in D\ \forall\beta\in E\ [p_\lambda(\beta) = f_\lambda(\beta)]\}$$

has size κ^+.

We now claim that any two distinct members p,q of I' are compatible (contradiction!).

Define r as follows: $\mathrm{dmn}(r) = \mathrm{dmn}(p)\cup\mathrm{dmn}(q)$. For any $\lambda \in \mathrm{dmn}(r)$,

$$r_\lambda = \begin{cases} p_\lambda & \text{if } \lambda\in\mathrm{dmn}(p)\setminus\mathrm{dmn}(q),\\ q_\lambda & \text{if } \lambda\in\mathrm{dmn}(q)\setminus\mathrm{dmn}(p),\\ s & \text{if } \lambda\in\mathrm{dmn}(p)\cap\mathrm{dmn}(q),\end{cases}$$

where $\mathrm{dmn}(s) = \mathrm{dmn}(p_\lambda)\cup\mathrm{dmn}(q_\lambda)$, and for any $\beta\in\mathrm{dmn}(s)$,

$$s(\beta) = \begin{cases} p_\lambda(\beta) & \text{if } \beta\in\mathrm{dmn}(p_\lambda)\setminus\mathrm{dmn}(q_\lambda),\\ q_\lambda(\beta) & \text{if } \beta\in\mathrm{dmn}(q_\lambda)\setminus\mathrm{dmn}(p_\lambda),\\ p_\lambda(\beta)\cup q_\lambda(\beta) & \text{if } \beta\in\mathrm{dmn}(p_\lambda)\cap\mathrm{dmn}(q_\lambda).\end{cases}$$

Clearly $r\in P$. By symmetry we show only that $r\le p$. Only (9)(c) is a problem. Suppose that $\lambda\in\mathrm{dmn}(p)$ and β and γ are distinct members of $\mathrm{dmn}(p_\lambda)$. If $\lambda\notin\mathrm{dmn}(q)$ the conclusion is clear. Assume that $\lambda\in\mathrm{dmn}(q)$. If $\beta,\gamma\notin\mathrm{dmn}(q_\lambda)$ the conclusion is clear. Suppose that $\beta\in\mathrm{dmn}(q_\lambda)$ and $\gamma\notin\mathrm{dmn}(q_\lambda)$. Then $\lambda\in D$ and $\beta\in E$, so $q_\lambda(\beta)=p_\lambda(\beta)$ and the conclusion is clear. The other cases are similar. ∎

Lemma 8.2. *P is κ-closed.* ∎

Theorem 8.3. *Assume the above about κ and C, in a countable transitive model M of GCH. Let G be P-generic over M. Then cofinalities and cardinalities are preserved in $M[G]$, and in $M[G]$, $\mathrm{MAD}(\kappa)=C$.*

Proof. By the lemmas, cofinalities and cardinalities are preserved. Now for $\lambda\in C$ and $\beta<\lambda$, let

$$A^\lambda_\beta = \{\gamma<\kappa : \exists p\in G\ [\lambda\in\mathrm{dmn}(p),\ \beta\in\mathrm{dmn}(p_\lambda),\ \gamma\in p_\lambda(\beta)]\}.$$

(11) If $\lambda\in C$ and $\beta<\lambda$, then $D := \{p\in P : \lambda\in\mathrm{dmn}(p), \beta\in\mathrm{dmn}(p_\lambda)\}$ is dense.

In fact, let $p\in P$ be given. If $\lambda\notin\mathrm{dmn}(p)$, let $q = p\cup\{(\lambda,\{(\beta,0)\})\}$. Clearly $q\in D$ and $q\le p$. If $\lambda\in\mathrm{dmn}(p)$ but $\beta\notin\mathrm{dmn}(p_\lambda)$ one proceeds similarly.

(12) If $\lambda \in C$, $\beta < \lambda$, and $\gamma < \kappa$, then

$$\{p \in P : \lambda \in \mathrm{dmn}(p),\ \beta \in \mathrm{dmn}(p_\lambda),\ p_\lambda(\beta) \cap (\gamma, \kappa) \neq \emptyset\}$$

is dense.

For, suppose that $q \in P$. By (11) we may assume that $\lambda \in \mathrm{dmn}(q)$ and $\beta \in \mathrm{dmn}(q_\lambda)$. Choose

$$\delta \in (\gamma, \kappa) \setminus \bigcup_{\xi \in \mathrm{dmn}(q_\lambda)} q_\lambda(\xi).$$

Let p be like q except that $p_\lambda(\beta) = q_\lambda(\xi) \cup \{\delta\}$. Clearly $p \in P$. To show that $p \leq q$, the only sticky point is to prove that if ε is a member of $\mathrm{dmn}(q_\lambda)$ different from β, then

$$q_\lambda(\beta) \cap q_\lambda(\varepsilon) = p_\lambda(\beta) \cap p_\lambda(\varepsilon).$$

Since $\delta \notin q_\lambda(\varepsilon)$, this is clear.

So (12) holds. Hence $|A^\lambda_\beta| = \kappa$ for all $\lambda \in C$, $\beta < \lambda$.

(13) If $\lambda \in C$ and β and γ are distinct members of λ, then $|A^\lambda_\beta \cap A^\lambda_\gamma| < \kappa$.

For, by (11) choose $p \in G$ such that $\lambda \in \mathrm{dmn}(p)$ and $\beta, \gamma \in \mathrm{dmn}(p_\lambda)$. We claim that

$$A^\lambda_\beta \cap A^\lambda_\gamma = p_\lambda(\beta) \cap p_\lambda(\gamma).$$

In fact, $\supseteq$ is clear. Now suppose that δ is in the lhs. Then there is a $q \in G$ such that $q \leq p$ and $\delta \in q_\lambda(\beta) \cap q_\lambda(\gamma)$. Hence $\delta \in p_\lambda(\beta) \cap p_\lambda(\gamma)$ by (9)(c). So (13) holds.

Now fix $\lambda \in C$; we show that $\{A^\lambda_\beta : \beta < \lambda\}$ is maximal. Suppose that $X \in [\kappa]^\kappa$ and $|X \cap A^\lambda_\beta| < \kappa$ for all $\beta < \lambda$. Let τ be a nice name for a subset of κ such that $\tau^G = X$. Say $\tau = \bigcup_{\gamma\in\kappa}\{\gamma\} \times B_\gamma$, each B_γ pairwise incompatible. By (11) we may assume for all $\gamma \in \kappa$ that $\lambda \in \mathrm{dmn}(r)$ for all $r \in B_\gamma$. Now $|B| \leq \kappa$ for all $\gamma \in \kappa$, by Lemma 8.1. Choose

$$\beta \in \lambda \setminus \bigcup_{\gamma\in\kappa} \bigcup_{p \in B_\gamma} \mathrm{dmn}(p_\lambda).$$

This is possible by (1). Let Γ be the standard name for a generic filter. For each $\gamma < \lambda$ let

$$\Theta_\gamma = \bigcup\{p_\lambda(\gamma) : p \in \Gamma,\ \lambda \in \mathrm{dmn}(p),\ \gamma \in \mathrm{dmn}(p_\lambda)\}.$$

Choose $p \in G$ such that

$$p \Vdash |\tau| = \kappa \wedge \forall\gamma < \lambda\ [|\tau \cap \Theta_\gamma| < \kappa].$$

So

$$p \Vdash \exists\theta < \kappa\ \forall\gamma \in \tau\ (\gamma \in \Theta_\beta \to \gamma < \theta).$$

Hence choose $\theta < \kappa$ and $q \in G$, $q \leq p$, such that

$$q \Vdash \forall\gamma \in \tau\ (\gamma \in \Theta_\beta \to \gamma < \theta).$$

By (11) we may assume that $\lambda \in \mathrm{dmn}(q)$ and $\beta \in \mathrm{dmn}(q_\lambda)$. Let

$$\Xi = \theta \cup \bigcup_{\gamma \in \mathrm{dmn}(q_\lambda)} q_\lambda(\gamma).$$

So $|\Xi| < \kappa$. Now $|\bigcup_{\gamma\in\mathrm{dmn}(q_\lambda)} A^\lambda_\gamma \cap X| < \kappa$, hence there is a $\delta \in X$ such that $\delta \notin \Xi$ and $\delta \notin \bigcup_{\gamma\in\mathrm{dmn}(q_\lambda)} A^\lambda_\gamma$. So there is a $k \in G$ and an $r \in B_\delta$ such that $k \le r, q$ and for all $\gamma \in \mathrm{dmn}(q_\lambda)$, $k \Vdash \delta \notin \Theta_\gamma$. Then

(14) if $\gamma \in \mathrm{dmn}(q_\lambda)$ and $\gamma \in \mathrm{dmn}(r_\lambda)$, then $\delta \notin r_\lambda(\gamma)$.

For, otherwise $r \Vdash \delta \in \Theta_\gamma$, contradicting $k \le r$.

Now define l as follows: $\mathrm{dmn}(l) = \mathrm{dmn}(q) \cup \mathrm{dmn}(r)$. For any $\mu \in \mathrm{dmn}(l)$,

$$l_\mu = \begin{cases} q_\mu & \text{if } \mu \in \mathrm{dmn}(q) \setminus \mathrm{dmn}(r), \\ r_\mu & \text{if } \mu \in \mathrm{dmn}(r) \setminus \mathrm{dmn}(q), \\ s & \text{if } \mu \in \mathrm{dmn}(q) \cap \mathrm{dmn}(r), \end{cases}$$

where $\mathrm{dmn}(s) = \mathrm{dmn}(q_\mu) \cup \mathrm{dmn}(r_\mu)$, and for any $\gamma \in \mathrm{dmn}(s)$,

$$s(\gamma) = \begin{cases} q_\mu(\gamma) & \text{if } \gamma \in \mathrm{dmn}(q_\mu) \setminus \mathrm{dmn}(r_\mu) \text{ and } (\mu,\gamma) \neq (\lambda,\beta), \\ r_\mu(\gamma) & \text{if } \gamma \in \mathrm{dmn}(r_\mu) \setminus \mathrm{dmn}(q_\mu), \\ q_\mu(\gamma) \cup r_\mu(\gamma) & \text{if } \gamma \in \mathrm{dmn}(q_\mu) \cap \mathrm{dmn}(r_\mu), \\ q_\lambda(\beta) \cup \{\delta\} & \text{if } (\mu,\gamma) = (\lambda,\beta). \end{cases}$$

Note that $\beta \notin \mathrm{dmn}(r_\lambda)$ by the choice of β. Clearly $l \in P$.

(15) $l \le q$.

To prove this, only (9)(c) is problematic. Suppose that $\mu \in \mathrm{dmn}(q)$, and φ, ψ are distinct members of $\mathrm{dmn}(q_\mu)$. To show

$$q_\mu(\varphi) \cap q_\mu(\psi) = l_\mu(\varphi) \cap l_\mu(\psi)$$

let $\varepsilon \in l_\mu(\varphi) \cap l_\mu(\psi)$. If $(\mu,\varphi), (\mu,\psi) \neq (\lambda,\beta)$, then $\varepsilon \in q_\mu(\varphi) \cap q_\mu(\psi)$, as desired. Suppose that $(\mu,\varphi) = (\lambda,\beta)$. Thus $\psi \neq \beta$. If $\psi \in \mathrm{dmn}(r_\lambda)$, by (14), $\delta \notin r_\lambda(\psi)$. Now $\delta \notin q_\lambda(\psi)$ since $\delta \notin \Xi$, so $\varepsilon \neq \delta$, and the desired conclusion follows as before.

(16) $l \le r$.

Since $\beta \notin \mathrm{dmn}(r_\lambda)$, this holds since $k \le q, r$.

Now $q \Vdash \delta \in \tau \wedge \theta \le \delta \to \delta \notin \Theta_\beta$. So l forces the same thing. Now $\theta \le \delta$, $r \Vdash \delta \in \tau$, and $l \le r$, so $l \Vdash \delta \in \tau \wedge \theta \le \delta$. Hence $l \Vdash \delta \notin \Theta_\beta$. This is a contradiction, since $\delta \in l_\lambda(\beta)$.

We have thus shown that $\{A^\lambda_\beta : \beta < \lambda\}$ is maximal.

Now suppose that $\lambda > \kappa$ and $\lambda \notin C$. Suppose that, in $M[G]$, $\langle X_\alpha : \alpha < \lambda\rangle$ is a mad family of members of $[\kappa]^\kappa$; we want to get a contradiction. Choose $p^0 \in G$ so that

$$p^0 \Vdash \dot{X} \text{ is a MAD } \lambda\text{-sequence of members of } [\kappa]^\kappa.$$

Let $\mu = \sup(C \cap \lambda)$. This is well defined since $\kappa^+ \in C$, and so $\kappa^+ < \lambda$. Since C is closed, it follows that $\mu \in C$, and hence $\mu < \lambda$. By (3), $\mathrm{cf}\,\mu > \kappa$. Hence $\mu^\kappa = \mu$.

Next, for each $\alpha < \lambda$ and $\beta < \kappa$ let A^α_β be such that

(17) for all $q \in A^\alpha_\beta$, $q \Vdash \beta \in \dot{X}_\alpha$;
(18) A^α_β is pairwise incompatible;
(19) A^α_β is maximal such that (17) and (18) hold.

For each $\alpha < \lambda$ let $\tau_\alpha = \bigcup\{\{\beta\} \times A^\alpha_\beta : \beta < \kappa\}$. Then

(20) for each $\alpha < \lambda$, τ_α is a nice name for a subset of κ and $\tau^G_\alpha = X_\alpha$.

For, let $\alpha < \lambda$. Clearly τ_α is a nice name for a subset of κ. Now suppose that $\beta \in \tau_\alpha^G$. Then there is a $q \in A^\alpha_\beta \cap G$. So $\beta \in X_\alpha$. Conversely, suppose that $\beta \in X_\alpha$. Choose $q \in G$ such that $q \Vdash \beta \in \dot{X}_\alpha$. Then

(21) $\{s : \exists r \in A^\alpha_\beta\ (s \le r)\}$ is dense below q.

For, if $t \le q$, then $t \Vdash \beta \in \dot{X}_\alpha$, so there is an $r \in A^\alpha_\beta$ such that r and t are compatible. Say $s \le r, t$. Thus s is as desired. So (21) holds.

Choose $s \in G$, $s \le q$, $s \le r \in A^\alpha_\beta$. So $r \in G$, hence $\beta \in \tau_\alpha^G$. Thus (20) holds.

We define some sets needed below:

$$L = \{(\nu, \gamma) : \nu \in \mathrm{dmn}(p^0) \text{ and } \gamma \in \mathrm{dmn}(p^0_\nu)\};$$

$$I_\nu = \{(\nu, \alpha) : \alpha < \nu\} \quad \text{for each } \nu \in C;$$

$$I = \bigcup_{\nu \in C} I_\nu;$$

$$J_\alpha = \{(\nu, \gamma) : \exists \beta < \kappa\ \exists p \in A^\alpha_\beta\ [\nu \in \mathrm{dmn}(p) \text{ and } \gamma \in \mathrm{dmn}(p_\nu)]\} \quad \text{for each } \alpha < \lambda;$$

$$S = L \cup \bigcup_{\alpha < \lambda} J_\alpha \cup \bigcup \{I_\nu : \nu \le \mu,\ \nu \in C\}.$$

Clearly $|J_\alpha| \le \kappa$ for all $\alpha < \lambda$, $|I_\nu| = \nu$ for all $\nu \in C$, $|L| < \kappa$, $S \subseteq I$, and $|S| \le \lambda$.

We say that a subset J of I is a *support* of an element $p \in P$ if $\nu \in \mathrm{dmn}(p)$ and $\alpha \in \mathrm{dmn}(p_\nu)$ imply that $(\nu, \alpha) \in J$. Thus L supports p^0, and J_α supports each member of $\bigcup_{\beta<\kappa} A^\alpha_\beta$.

Now we will define a sequence $\langle N_\xi : \xi < \kappa^+\rangle$ of subsets of λ, each of size at most μ. We define $N_0 = \emptyset$, and for ξ limit, $N_\xi = \bigcup_{\tau<\xi} N_\tau$. Now assume that N_ξ has been defined.

Temporarily fix $K \subseteq S$ such that $|K| \le \mu$ and $I_\nu \subseteq K$ for all $\nu \in C$ with $\nu \le \mu$.

A *special set* is a subset $J \subseteq I$ such that:

(22) $|J| \le \kappa$;
(23) for all $\nu \in C$, if $J \cap I_\nu \setminus K \ne 0$, then $|J \cap I_\nu \setminus K| = \kappa$.

If $\sigma = \bigcup_{\beta<\kappa} \{\beta\} \times B_\beta$ is a nice name for a subset of κ, a *support* for σ is a special set J which supports each member of $\bigcup_{\beta<\kappa} B_\beta$.

Then we have:

(24) Every nice name for a subset of κ has a support.

In fact, let $\sigma = \bigcup_{\beta<\kappa} \{\beta\} \times B_\beta$ be a nice name for a subset of κ. We define

$$R = \{(\nu, \alpha) : \exists \beta < \kappa\ \exists p \in B_\beta\ [\nu \in \mathrm{dmn}(p) \text{ and } \alpha \in \mathrm{dmn}(p_\nu)]\}.$$

Clearly $|R| \le \kappa$. Now if $\nu \in C$ and $R \cap I_\nu \setminus K \ne \emptyset$, then $|I_\nu| = \nu > \kappa$, so there is a $Q_\nu \subseteq I_\nu \setminus K$ such that $|Q_\nu| = \kappa$. We then define

$$J = R \cup \bigcup \{Q_\nu : R \cap I_\nu \setminus K \ne \emptyset\}.$$

Clearly J is a support for σ.

Let $\mathscr{G}_\xi$ be the group of all permutations of I that map each I_ν to itself and fix all members of K. For each $g \in \mathscr{G}_\xi$ and $\nu \in C$ define $g^\nu : \nu \to \nu$ by $g(\nu, \alpha) = (\nu, g^\nu(\alpha))$. Now each $g \in \mathscr{G}_\xi$ induces an automorphism $\widehat{g}$ of P. Namely, if $p \in P$, we define $\mathrm{dmn}(\widehat{g}(p)) = \mathrm{dmn}(p)$, for any $\nu \in \mathrm{dmn}(p)$, $\mathrm{dmn}(\widehat{g}(p)_\nu) = g^\nu[\{\alpha : \alpha \in \mathrm{dmn}(p_\nu)\}]$, and for any $\alpha \in$

$\mathrm{dmn}(p_\nu)$, $\widehat{g}(p)_\nu(g^\nu(\alpha)) = p_\nu(\alpha)$. It is straightforward to check that $\widehat{g}$ is an automorphism of P. For example, suppose that $p, q \in P$ and $p \leq q$; we check condition (9)(c) for showing that $\widehat{g}(p) \leq \widehat{g}(q)$. So, suppose that $\nu \in \mathrm{dmn}(\widehat{g}(q))$ (thus $\nu \in \mathrm{dmn}(q)$) and β and γ are distinct members of $\mathrm{dmn}(\widehat{g}(q)_\nu)$. Say $\beta = g^\nu(\beta')$ and $\gamma = g^\nu(\gamma')$, with β' and γ' distinct members of $\mathrm{dmn}(q_\nu)$. Then

$$\begin{aligned}\widehat{g}(p)_\nu(\beta) \cap \widehat{g}(p)_\nu(\gamma) &= \widehat{g}(p)_\nu(g^\nu(\beta')) \cap \widehat{g}(p)_\nu(g^\nu(\gamma')) = p_\nu(\beta') \cap p_\nu(\gamma') = q_\nu(\beta') \cap q_\nu(\gamma') \\ &= \widehat{g}(q)_\nu(g^\nu(\beta')) \cap \widehat{g}(q)_\nu(g^\nu(\gamma')) = \widehat{g}(q)_\nu(\beta) \cap \widehat{g}(q)_\nu(\gamma).\end{aligned}$$

Also $\widehat{}$ clearly takes inverses to inverses. So $\widehat{g}$ is indeed an automorphism of P.

(25) If J is a special subset of I and $g \in \mathscr{G}_\xi$, then $g[J]$ is a special subset of I.

For, $|g[J]| = |J| \leq \kappa$. Now suppose that $g[J] \cap I_\nu \setminus K \neq \emptyset$; choose $(\nu, \alpha) \in g[J] \cap I_\nu \setminus K$. Write $(\nu, \alpha) = g(\nu, \beta)$ with $(\nu, \beta) \in J$. Then $(\nu, \beta) \in J \cap I_\nu \setminus K$, since g is the identity on K. Thus $J \cap I_\nu \setminus K \neq \emptyset$. Hence $|J \cap I_\nu \setminus K| = \kappa$, and so also $|g[J] \cap I_\nu \setminus K| = |g[J \cap I_\nu \omega K]| = \kappa$. So (25) holds.

Given $J \subseteq I$, let

$$\overline{J} = \{\nu \in C : J \cap I_\nu \setminus K \neq 0\}.$$

If J, J' are special sets, $J \cap K = J' \cap K$, and $\overline{J} = \overline{J}'$, then there is a $g \in \mathscr{G}_\xi$ such that $g[J] = J'$. Now $|J \cap K| \leq \kappa$ and $|K| \leq \mu$, so there are only $\mu^\kappa = \mu$ possibilities for $J \cap K$. Also, $\overline{J} \in [C]^{\leq\kappa}$ and $|C| \leq \mu$ (if $|C| > \mu$, then $\lambda \in C$ by (4), contradiction). So there are only $\mu^\kappa = \mu$ possibilities for $\overline{J}$. So there are at most μ $\mathscr{G}_\xi$-orbits of special sets.

(26) For each special set J' there is a special set J in the same $\mathscr{G}_\xi$-orbit such that $J \cap S = J \cap K$.

In fact, we define $g \in \mathscr{G}_\xi$ as follows. Let $\nu \in C$. If $J' \cap I_\nu \cap S \subseteq K$, let $g \restriction I_\nu$ be the identity. Now suppose that $J' \cap I_\nu \cap S \nsubseteq K$. Then $\lambda < \nu$, as otherwise $\nu \leq \lambda$, hence $\nu \leq \mu$, and hence $I_\nu \subseteq K$, contradiction. Now $|K| \leq \mu < \nu$ and $|J' \cap I_\nu \setminus K| \leq |J'| \leq \kappa < \nu$, so we can take $g \restriction \nu$ to be a permutation of I_ν that is the identity on $I_\nu \cap K$ and maps $J' \cap I_\nu \setminus K$ out of S. This finishes the definition of g; clearly $g \in \mathscr{G}_\xi$.

Now set $J = g[J']$. If $(\nu, \alpha) \in J \cap S$, then $(\nu, \alpha) \in J \cap I_\nu \cap S$. Choose $(\nu, \beta) \in J'$ such that $g(\nu, \beta) = (\nu, \alpha)$. Thus $(\nu, \beta) \in J' \cap I_\nu$, and $g(\nu, \beta) \in S$, hence by construction, $(\nu, \beta) \in K$, hence $\alpha = \beta$ and $(\nu, \alpha) \in K$. So (26) holds.

Let $\mathbb{S}$ be the set of all orbits of special sets. By the remark before (26) we have $|\mathbb{S}| \leq \mu$. For each $Q \in \mathbb{S}$ let $L_Q \in Q$ be such that $L_Q \cap S = L_Q \cap K$; it exists by (26). The set of all L_Q is called the set of *standard* sets; so there are at most μ standard sets.

If $\sigma = \bigcup_{\beta<\kappa} \{\beta\} \times B_\beta$ is a nice name for a subset of κ, and $g \in \mathscr{G}_\xi$, we define

$$g^*(\sigma) = \bigcup_{\beta<\kappa} \{\beta\} \times \widehat{g}[B_\beta].$$

Clearly $g^*(\sigma)$ is a nice name for a subset of κ.

(27) If $J \subseteq I$ supports a nice name σ for a subset of κ, and $g \in \mathscr{G}_\xi$, then $g[J]$ supports the nice name $g^*(\sigma)$.

For, suppose that $\beta < \kappa$, $p \in \widehat{g}[B_\beta]$, $\nu \in \mathrm{dmn}(p)$, and $\alpha \in \mathrm{dmn}(p_\nu)$. Write $p = \widehat{g}(q)$ with $q \in B_\beta$. Thus $\nu \in \mathrm{dmn}(q)$. Choose γ so that $\alpha = g^\nu(\gamma)$ and $\gamma \in \mathrm{dmn}(q_\nu)$. So $(\nu,\gamma) \in J$ since J supports σ, and $(\nu,\alpha) = (\nu, g^\nu(\gamma)) = g(\nu,\gamma) \in g[J]$, as desired in (27).

(28) If σ is a nice name for a member of $[\kappa]^\kappa$, then there is an $A \in [\lambda]^{\le\kappa}$ such that

$$p^0 \Vdash \exists\alpha \in A\ [|\sigma \cap \dot{X}_\alpha| = \kappa].$$

For, let B be a maximal pairwise incompatible subset of

$$\{q \le p^0 : \exists\alpha < \lambda\ [q \Vdash |\sigma \cap \dot{X}_\alpha| = \kappa]\}.$$

Now $\{r : \exists q \in B\ (r \le q)\}$ is dense below p^0. For, suppose that $s \le p^0$. Then we have $s \Vdash \exists\alpha < \lambda\ [|\sigma \cap \dot{X}_\alpha| = \kappa]$, so there exist a $q \le s$ and an $\alpha < \lambda$ such that $q \Vdash |\sigma \cap \dot{X}_\alpha| = \kappa$. Say q, r compatible, $r \in B$. Then say $t \le q, r$. Then t is as desired.

For all $q \in B$ choose $\alpha_q < \lambda$ such that $q \Vdash |\sigma \cap \dot{X}_{\alpha_q}| = \kappa$. Let $A = \{\alpha_q : q \in B\}$. So $A \in [\lambda]^{\le\kappa}$. Now let $p^0 \in H$, H generic. Choose $q \in H \cap B$; this is possible by the indicated denseness. Then $q \Vdash |\sigma \cap \dot{X}_{\alpha_q}| = \kappa$, so $|\sigma^H \cap \dot{X}^H_{\alpha_q}| = \kappa$. Thus A is as desired in (28).

Let J be a standard set. Suppose that $Q \in [J]^{<\kappa}$. Let

$$\begin{aligned} s_Q = \{p \in P : \mathrm{dmn}(p) = \{\nu \in C : (\nu,\alpha) \in Q \text{ for some } \alpha\}\\ \text{and for all } \nu \in \mathrm{dmn}(p), \mathrm{dmn}(p_\nu) = \{\alpha < \nu : (\nu,\alpha) \in Q\}\}. \end{aligned}$$

There is a one-one function F from s_Q into $\prod_{(\nu,\alpha)\in Q}[\kappa]^{<\kappa}$, namely one can let $F(p)_{\nu\alpha} = p_\nu(\alpha)$. It follows that $|s_Q| \le \kappa$. Now if $p \in P$ and J supports p, let $Q = \{(\nu,\alpha) : \nu \in \mathrm{dmn}(p) \text{ and } \alpha \in \mathrm{dmn}(p_\nu)\}$; then $Q \in [J]^{<\kappa}$ and $p \in s_Q$. It follows that there are at most μ elements $p \in P$ with support contained in J, hence at most μ pairwise incompatible sets all members of which have support contained in J, hence at most μ nice names for members of $[\kappa]^\kappa$ with support contained in J. For each nice name σ for a member of $[\kappa]^\kappa$ with standard support, choose A_σ as in (28). Let B_K be the union of all such. So $|B_K| \le \mu$ since there are at most μ standard sets.

We now define $N_{\xi+1}$: let $N_{\xi+1} = B_{K_\xi} \cup N_\xi$, where

$$K_\xi = L \cup \bigcup_{\alpha\in N_\xi} J_\alpha \cup \bigcup_{\mu\ge\nu\in C} I_\nu.$$

(Note that $|K_\xi| \le \mu$.)

Let $Q = \bigcup_{\xi<\kappa^+} N_\xi$. Thus also $|Q| \le \mu < \lambda$. Choose $\beta \in \lambda \setminus Q$. Recall that J_β is a support for τ_β, and $|J_\beta| \le \kappa$. Define $K_\infty = \bigcup_{\tau<\kappa^+} K_\tau$. Fix $\tau < \kappa^+$ such that $J_\beta \cap K_\infty \subseteq K_\tau$. Now consider the step from N_τ to $N_{\tau+1}$. Choose $g \in \mathscr{G}_\tau$ such that $g[J_\beta]$ is standard. Thus g is the identity on K_τ. By $J_\beta \cap K_\infty \subseteq K_\tau$,

(29) $J_\beta \cap (K_{\tau+1} \setminus K_\tau) = 0$;
(30) $g[J_\beta] \cap (K_{\tau+1} \setminus K_\tau) = 0$.

This is true since $g[J_\beta] \cap S = g[J_\beta] \cap K_\tau$ and $K_{\tau+1} \subseteq S$.

Choose $h \in \mathscr{G}$ such that $h\restriction J = g\restriction J$ and h is the identity on $K_{\tau+1} \setminus K_\tau$. So h fixes $K_{\tau+1}$ pointwise. Now

$$p^0 \Vdash \exists\alpha \in A_{h^*(\tau_\beta)}\ [|h^*(\tau_\beta) \cap \dot{\tau}_\alpha| = \kappa],$$

so

$$p^0 \Vdash \exists\alpha \in A_{h^*(\tau_\beta)}\ [|\tau_\beta \cap \dot{\tau}_\alpha| = \kappa].$$

This is true since $A_{h^*(\tau_\beta)} \subseteq B_{K_\tau} \subseteq N_{\tau+1}$, and hence $J_\alpha \subseteq K_{\tau+1}$, so h fixes J_α pointwise. Recall that J_α is a support of τ_α. Also, $K_{\tau+1}$ supports p^0. Thus $p^0 \Vdash \exists\alpha \in Q\ [|\tau_\beta \cap \dot{\tau}_\alpha| = \kappa]$. But $\alpha \neq \beta$, contradiction. ∎

9. Specifying $\mathrm{MAD}(\kappa, \mu, \mu)$

THEOREM 9.1. *Suppose that M is a countable transitive model of GCH, and in M, $\mu < \kappa$ are infinite regular cardinals. Suppose that C is a set of cardinals satisfying the following conditions*:

(i) $\kappa^+ \in C$;
(ii) *C is closed*;
(iii) *C contains the immediate successor of each of its members of cofinality between ω and κ inclusive*;
(iv) $[\kappa^+, |C|] \subseteq C$.

Then there is a generic extension $M[G]$ preserving cofinalities such that in $M[G]$, we have $\mathrm{MAD}(\kappa, \mu, \mu) \cap [\kappa^+, \infty) = C$.

Proof. The proof is similar to that in the previous section, so some details will be omitted.

We work within M for a while. Let $\langle B_\alpha : \alpha < \kappa\rangle$ enumerate $[\kappa]^\mu$. For all $\alpha < \kappa$, let t_α be a one-one function mapping B_α onto μ. Let P consist of all functions f such that

(1) $\mathrm{dmn}(f) \in [C]^{<\mu}$;
(2) for all $\nu \in \mathrm{dmn}(f)$, f_ν is a function, $\mathrm{dmn}(f_\nu) \in [\kappa]^{<\mu}$, and
  (a) for all $\alpha \in \mathrm{dmn}(f_\nu)$, $f_{\nu\alpha}$ is a function, $\mathrm{dmn}(f_{\nu\alpha}) \in [\nu]^{<\mu}$, and
    (α) for all $\beta \in \mathrm{dmn}(f_{\nu\alpha})$, $f_{\nu\alpha}(\beta) \in [B_\alpha]^{<\mu}$.

For $f, g \in P$, we define $f \leq g$ iff

(3) $\mathrm{dmn}(g) \subseteq \mathrm{dmn}(f)$, for all $\nu \in \mathrm{dmn}(g)$, $\mathrm{dmn}(g_\nu) \subseteq \mathrm{dmn}(f_\nu)$, for all $\alpha \in \mathrm{dmn}(g_\nu)$, $\mathrm{dmn}(g_{\nu\alpha}) \subseteq \mathrm{dmn}(f_{\nu\alpha})$, for all $\beta \in \mathrm{dmn}(g_{\nu\alpha})$, $g_{\nu\alpha}(\beta) \subseteq f_{\nu\alpha}(\beta)$, and
  (a) for all $\alpha, \varphi \in \mathrm{dmn}(g_\nu)$ and all $\beta \in \mathrm{dmn}(g_{\nu\alpha})$ and $\psi \in \mathrm{dmn}(g_{\nu\varphi})$, if $(\alpha, \beta) \neq (\varphi, \psi)$, then

$$g_{\nu\alpha}(\beta) \cap g_{\nu\varphi}(\psi) = f_{\nu\alpha}(\beta) \cap f_{\nu\varphi}(\psi). \tag{4}$$

Clearly

(5) P is μ-closed;
(6) P has the μ^+-chain condition.

For, suppose that I is a collection of pairwise incompatible conditions with $|I| = \mu^+$. We may assume:

$$\langle \mathrm{dmn}(f) : f \in I\rangle$$

is a Δ-system, say with kernel D_0, and

$$\Big\langle \bigcup_{\nu\in\mathrm{dmn}(f)} \mathrm{dmn}(f_\nu) : f \in I\Big\rangle$$

is a Δ-system, say with kernel D_1, and

$$\Big\langle \bigcup_{\nu\in\mathrm{dmn} f} \Big(\bigcup_{\alpha\in\mathrm{dmn}(f_\nu)} \mathrm{dmn}(f_{\nu\alpha})\Big) : f \in I\Big\rangle$$

is a Δ-system, say with kernel D_2. Then we may assume that if g and h are in I, $\nu \in D_0$, $\alpha \in D_1$, $\beta \in D_2$, then $t_\alpha[g_{\nu\alpha}(\beta)] = t_\alpha[h_{\nu\alpha}(\beta)]$, and hence $g_{\nu\alpha}(\beta) = h_{\nu\alpha}(\beta)$. Then any two distinct members g and h of I are compatible (contradiction). We omit the proof.

Now let G be P-generic over M. By (5) and (6), $M[G]$ preserves cofinalities and cardinals. Now if $\nu \in C$, $\alpha \in \kappa$, and $\beta < \nu$, define

$$A^\nu_{\alpha\beta} = \bigcup\{f_{\nu\alpha}(\beta) : f \in G,\ \nu \in \mathrm{dmn}(f),\ \alpha \in \mathrm{dmn}(f_\nu),\ \beta \in \mathrm{dmn}(f_{\nu\alpha})\}. \tag{7}$$

We claim that for each $\nu \in C$,

$$\langle A^\nu_{\alpha\beta} : \alpha \in \kappa,\ \beta \in \nu\rangle \tag{8}$$

is the desired $[\kappa]^\mu$, μ-mad family.

(9) For any $\nu \in C$, $\{f \in P : \nu \in \mathrm{dmn}(f)\}$ is dense.

In fact, if $g \in P$, assume that $\nu \notin \mathrm{dmn}(g)$. Then let f be like g except that $\nu \in \mathrm{dmn}(f)$ and $f_\nu = \emptyset$. Clearly $f \in P$ and $f \le g$, proving (9).

(10) For any $\nu \in C$ and $\alpha \in \kappa$, the set

$$\{f \in P : \nu \in \mathrm{dmn}(f) \text{ and } \alpha \in \mathrm{dmn}(f_\nu)\}$$

is dense.

For, let g be given. By (9) we may assume that $\nu \in \mathrm{dmn}(g)$. And we may assume that $\alpha \notin \mathrm{dmn}(g_\nu)$. Let f be like g except that $\alpha \in \mathrm{dmn}(f_\nu)$ and $f_{\nu\alpha} = \emptyset$. Clearly $f \in P$ and $f \le g$.

Similarly,

(11) for any $\nu \in C$, $\alpha \in \kappa$, and $\beta \in \nu$, the set

$$\{f \in P : \nu \in \mathrm{dmn}(f) \text{ and } \alpha \in \mathrm{dmn}(f_\nu) \text{ and } \beta \in \mathrm{dmn}(f_{\nu\alpha})\}$$

is dense;

(12) for any $\nu \in C$, $\alpha \in \kappa$, $\beta \in \nu$, and $\gamma < \mu$ the set

$$\{f \in P : \nu \in \mathrm{dmn}(f),\ \alpha \in \mathrm{dmn}(f_\nu),\ \beta \in \mathrm{dmn}(f_{\nu\alpha}),\ \text{and } t_\alpha[f_{\nu\alpha}(\beta)] \cap (\gamma,\mu) \neq 0\}$$

is dense.

For, assume the hypotheses of (12), and suppose that $g \in P$. We may assume that $\nu \in \mathrm{dmn}(g)$, $\alpha \in \mathrm{dmn}(g_\nu)$, and $\beta \in \mathrm{dmn}(g_{\nu\alpha})$. Choose

$$\delta \in (\gamma,\mu) \setminus t_\alpha\Big[\bigcup_{\varphi\in\mathrm{dmn}(g_\nu),\,\psi\in\mathrm{dmn}(g_{\nu\varphi})} (B_\alpha \cap g_{\nu\varphi}(\psi))\Big].$$

Now let f be like g except that $f_{\nu\alpha}(\beta) = g_{\nu\alpha}(\beta) \cup \{t_\alpha^{-1}(\delta)\}$. Clearly $f \in P$. To show that $f \leq g$, only (3)(a) is a problem. So suppose that $\sigma \in \mathrm{dmn} g$, $\varphi, \psi \in \mathrm{dmn} g_\sigma$, $\xi \in \mathrm{dmn}(g_{\sigma\varphi})$, $\eta \in \mathrm{dmn}(g_{\sigma\psi})$, and $(\varphi, \xi) \neq (\psi, \eta)$; we want to show

$$g_{\sigma\varphi}(\xi) \cap g_{\sigma\psi}(\eta) = f_{\sigma\varphi}(\xi) \cap f_{\sigma\psi}(\eta).$$

This is clear unless one of the triples (σ, φ, ξ), (σ, ψ, η) is equal to (ν, α, β). Say $(\sigma, \varphi, \xi) = (\nu, \alpha, \beta)$. Now $t_\alpha^{-1}(\delta) \notin g_{\nu\psi}(\eta)$, so the equality follows. Hence (12) holds. By (12), each set $A^\nu_{\alpha\beta}$ has size μ.

(13) If $\nu \in C$, $\alpha, \varphi \in \kappa$, $\xi, \eta \in \nu$, and $(\alpha, \xi) \neq (\varphi, \eta)$, then $|A^\nu_{\alpha\xi} \cap A^\nu_{\varphi\eta}| < \mu$.

To see this, assume the hypotheses. Clearly there is an $f \in G$ such that $\nu \in \mathrm{dmn}(f)$, $\alpha, \varphi \in \mathrm{dmn}(f_\nu)$, $\xi \in \mathrm{dmn}(f_{\nu\alpha})$, and $\eta \in \mathrm{dmn}(f_{\nu\varphi})$. We claim that

$$A^\nu_{\alpha\xi} \cap A^\nu_{\varphi\eta} = f_{\nu\alpha}(\xi) \cap f_{\nu\varphi}(\eta),$$

which will prove (13). To see this, $\supseteq$ is clear. Now take any $\beta \in A^\nu_{\alpha\xi} \cap A^\nu_{\varphi\eta}$. Then there is a $g \in G$ with $g \leq f$ and $\beta \in g_{\nu\alpha}(\xi) \cap g_{\nu\varphi}(\eta)$. Clearly then $\beta \in f_{\nu\alpha}(\xi) \cap f_{\nu\varphi}(\eta)$, as desired. So (13) holds.

Next, we show maximality. Suppose that $\nu \in C$, $X \in [\kappa]^\mu$, and $|X \cap A^\nu_{\alpha\beta}| < \mu$ whenever $\alpha \in \kappa$ and $\beta \in \nu$; we want to get a contradiction. Now by the μ^+-chain condition, X is contained in some subset in M of κ of size μ. One can see this by applying Kunen [80, VII.6.8], as follows. Let $f : \mu \to \kappa$ be one-one. By Kunen [80, VII.6.8] there is an $F : \mu \to \mathscr{P}(\kappa)$ such that $F \in M$, $f(\alpha) \in F(\alpha)$ for all $\alpha < \mu$, and $|F(\alpha)| \leq \mu$ for all $\alpha < \mu$. Then $\bigcup_{\alpha<\mu} F(\alpha)$ is the desired set in M. Let B_α be such a subset. Let τ be a nice name for a subset of B_α such that $\tau_G = X$. Say $\tau = \bigcup_{\gamma \in B_\alpha} \{\gamma\} \times C_\gamma$, each C_γ pairwise incompatible. So each C_γ has size at most μ. Choose

$$\beta \in \nu \setminus \bigcup_{\gamma \in B_\alpha} \Big(\bigcup_{f \in C_\gamma,\, \nu \in \mathrm{dmn}(f),\, \alpha \in \mathrm{dmn}(f_\nu)} \mathrm{dmn}(f_{\nu\alpha}) \Big).$$

For any $\varphi \in \kappa$ and $\gamma \in \nu$ let $\Theta^\nu_{\varphi\gamma}$ be the term

$$\bigcup \{f_{\nu\varphi}(\gamma) : f \in \Gamma,\ \nu \in \mathrm{dmn}(f),\ \varphi \in \mathrm{dmn}(f_\nu),\ \gamma \in \mathrm{dmn}(f_{\nu\varphi})\}.$$

Here Γ is the standard name for a generic filter; see Kunen [80, VII.2.12]. Thus $(\Theta^\nu_{\varphi\gamma})_G = A^\nu_{\varphi\gamma}$.

Choose $f \in G$ such that

$$f \Vdash |\tau| = \mu \wedge \forall \varphi \in \kappa\ \forall \gamma \in \nu\ [|\tau \cap \Theta^\nu_{\varphi\gamma}| < \mu].$$

Hence

$$f \Vdash \exists \theta < \mu\ \forall \gamma \in \tau\ [\gamma \in \Theta^\nu_{\alpha\beta} \to t_\alpha(\gamma) < \theta].$$

So, choose $\theta < \mu$ and $g \leq f$ such that $g \in G$ and

$$g \Vdash \forall \gamma \in \tau\ [\gamma \in \Theta^\nu_{\alpha\beta} \to t_\alpha(\gamma) < \theta].$$

Without loss of generality $\nu \in \mathrm{dmn}(g)$, $\alpha \in \mathrm{dmn}(g_\nu)$, and $\beta \in \mathrm{dmn}(g_{\nu\alpha})$. Let

$$\Xi = \theta \cup \bigcup_{\varphi \in \mathrm{dmn}(g_\nu),\, \gamma \in \mathrm{dmn}(g_{\nu\varphi})} t_\alpha[g_{\nu\varphi}(\gamma) \cap B_\alpha].$$

So $|\Xi| < \mu$. Now

$$\Big|\bigcup_{\varphi\in\mathrm{dmn}(g_\nu),\,\psi\in\mathrm{dmn}(g_{\nu\varphi})} A^\nu_{\varphi\psi}\cap X\Big| < \mu,$$

so there is a $\delta \in X$ such that $t_\alpha(\delta) \notin \Xi$ and

$$\delta \notin \bigcup_{\varphi\in\mathrm{dmn}(g_\nu),\,\psi\in\mathrm{dmn}(g_{\nu\varphi})} A^\nu_{\varphi\psi}\cap X.$$

Hence there exist $h \le g$ with $h \in G$ and

$$h \Vdash \delta \in \tau \wedge \forall\varphi \in \mathrm{dmn}(g_\nu)\ \forall\psi \in \mathrm{dmn}(g_{\nu\varphi})\ (\delta \notin \Theta^\nu_{\varphi\psi}).$$

So there exist $k \in G$ and $r \in C_\delta$ such that $k \le h, r$ and for all $\varphi \in \mathrm{dmn}(g_\nu)$ and $\psi \in \mathrm{dmn}(g_{\nu\varphi})$, $k \Vdash \delta \notin \Theta^\nu_{\varphi\psi}$. Now

(14) if $\varphi \in \mathrm{dmn}(g_\nu)$, $\psi \in \mathrm{dmn}(g_{\nu\varphi})$, $\nu \in \mathrm{dmn}(r)$, $\varphi \in \mathrm{dmn}(r_\nu)$, and $\psi \in \mathrm{dmn}(r_{\nu\varphi})$, then $\delta \notin r_{\nu\varphi}(\psi)$.

For, otherwise $r \Vdash \delta \in \Theta^\nu_{\varphi\psi}$, contradicting $k \le r$.

Now we define l. It is "$g \cup r$" except that $l_{\nu\alpha}(\beta) = g_{\nu\alpha}(\beta) \cup \{\delta\}$. Note that it is not the case that ($\nu \in \mathrm{dmn}(r)$ and $\alpha \in \mathrm{dmn}(r)$ and $\beta \in \mathrm{dmn}(r_{\nu\alpha})$), by the choice of β.

$l \le g$: Assume that $\varphi \in \mathrm{dmn}(g_\nu)$, $\psi \in \mathrm{dmn}(g_{\nu\varphi})$, and $(\alpha,\beta) \ne (\varphi,\psi)$. Now $\delta \notin g_{\nu\varphi}(\psi)$ since $t_\alpha(\delta) \notin \Xi$. And $\delta \notin r_{\nu\varphi}(\psi)$ by (14). Hence it is clear that $l_{\nu\alpha}(\beta) \cap l_{\nu\varphi}(\psi) = g_{\nu\alpha}(\beta) \cap g_{\nu\varphi}(\psi)$.

$l \le r$: This is clear by the choice of β.

Now $g \Vdash \delta \in \tau \wedge \theta \le t_\alpha(\delta) \to \delta \notin \Theta^\nu_{\alpha\beta}$. So l forces the same thing. Now $\theta \le t_\alpha(\delta)$ by the choice of δ, and $l \le r$, so $l \Vdash \delta \in \tau \wedge \theta \le t_\alpha(\delta)$. So $l \Vdash \delta \notin \Theta^\nu_{\alpha\beta}$. This contradicts $l \le r$.

Thus we have shown maximality.

Now suppose that $\lambda > \kappa$ and $\lambda \notin C$. Suppose that, in $M[G]$, $\langle X_\alpha : \alpha < \lambda\rangle$ is a mad family of members of $[\kappa]^\mu$; we want to get a contradiction. Choose $p^0 \in G$ so that

$$p^0 \Vdash \dot X \text{ is a MAD } \lambda\text{-sequence of members of } [\kappa]^\mu.$$

Let $L = \{(\nu,\gamma) : \nu \in \mathrm{dmn}(p^0) \text{ and } \gamma \in \mathrm{dmn}(p^0_\nu)\}$. Let $\varrho = \sup(C\cap\lambda)$. This is well defined since $\kappa^+ \in C$, and so $\kappa^+ < \lambda$. Since C is closed, $\varrho < \lambda$, and by (iii), $\mathrm{cf}\,\varrho > \kappa$. It follows that $\varrho^\kappa = \kappa$ in M.

Now we will define a sequence $\langle N_\xi : \xi < \kappa^+\rangle$ of subsets of λ, each of size at most ϱ. We define $N_0 = 0$, and for ξ limit, $N_\xi = \bigcup_{\tau<\xi} N_\tau$. Now assume that N_ξ has been defined.

(15) There is a function $\tau \in M$ with domain λ such that for each $\alpha < \lambda$, τ_α is a nice name for a subset of κ and $\tau^G_\alpha = X_\alpha$.

To see this, for each $\alpha < \lambda$ and $\beta < \kappa$ let A^α_β be such that

(16) for all $q \in A^\alpha_\beta$, $q \Vdash \beta \in \dot X_\alpha$;
(17) A^α_β is pairwise incompatible;
(18) A^α_β is maximal such that (16) and (17) hold.

Let $\tau_\alpha = \bigcup\{\{\beta\} \times A^\alpha_\beta : \beta < \kappa\}$. Suppose that $\beta \in \tau^G_\alpha$. Then there is a $q \in A^\alpha_\beta \cap G$. So $\beta \in X_\alpha$. Conversely, suppose that $\beta \in X_\alpha$. Choose $q \in G$ such that $q \Vdash \beta \in \dot{X}_\alpha$.Then

$$\{s : \exists r \in A^\alpha_\beta \ (s \le r)\}$$

is dense below q. For, if $t \le q$, then $t \Vdash \beta \in \dot{X}_\alpha$, so there is an $r \in A^\alpha_\beta$ such that r and t are compatible. Say $s \le r, t$. Thus s is as desired.

Choose $s \in G$, $s \le q$, $s \le r \in A^\alpha_\beta$. So $r \in G$, hence $\beta \in \tau^G_\alpha$. Thus (15) holds.

We take the sets A^α_β as in the proof of (15). Thus $|A^\alpha_\beta| \le \mu$.

Now for each $\nu \in C$ let $I_\nu = \{(\nu, \alpha) : \alpha < \nu\}$, and let $I = \bigcup_{\nu \in C} I_\nu$. For each $\alpha < \lambda$ let

$$J_\alpha = \{(\nu, \gamma) : \exists \beta < \kappa \ \exists p \in A^\alpha_\beta \ [\nu \in \operatorname{dmn}(p) \text{ and } \gamma \in \operatorname{dmn}(p_\nu)]\}.$$

Then we set

$$S = L \cup \bigcup_{\alpha<\lambda} J_\alpha \cup \bigcup\{I_\nu : \nu \le \varrho,\ \nu \in C\}.$$

Thus $|S| \le \lambda$.

Temporarily fix $K \subseteq S$ such that $|K| \le \varrho$ and $I_\nu \subseteq K$ for all $\nu \in C$ with $\nu \le \varrho$.

A *special set* is a subset $J \subseteq I$ such that:

(19) $|J| \le \kappa$;
(20) for all $\nu \in C$, if $J \cap I_\nu \setminus K \ne 0$, then $|J \cap I_\nu \setminus K| = \kappa$.

If $\sigma = \bigcup_{\beta<\kappa} \{\beta\} \times B_\beta$ is a nice name for a subset of κ, a *support* for σ is a special set J such that

(21) if $\beta < \kappa$, $p \in B_\beta$, $\nu \in \operatorname{dmn}(p)$, and $\alpha \in \operatorname{dmn}(p_\nu)$, then $(\nu, \alpha) \in J$.

Clearly every nice name for a subset of κ has a support. Also, we say that a subset J of I is a *support* of an element $p \in P$ if $\nu \in \operatorname{dmn}(p)$ and $\alpha \in \operatorname{dmn}(p_\nu)$ imply that $(\nu, \alpha) \in J$.

Let $\mathscr{G}_\xi$ be the group of all permutations of I that map each I_ν to itself and fix all members of K. For each $\nu \in C$ define $g^\nu : \nu \to \nu$ by: $g(\nu, \alpha) = (\nu, g^\nu(\alpha))$. Clearly each $g \in \mathscr{G}_\xi$ induces an automorphism of P. Namely, if $p \in P$, we define $\operatorname{dmn}(g(p)) = \operatorname{dmn}(p)$, for any $\nu \in \operatorname{dmn}(g(p))$, $\operatorname{dmn}(g(p)_\nu) = \operatorname{dmn}(p_\nu)$, for any $\alpha \in \operatorname{dmn}(g(p)_\nu)$, $\operatorname{dmn}(g(p)_{\nu\alpha}) = g^\nu[\operatorname{dmn}(p_{\nu\alpha})]$, and for any $\beta \in \operatorname{dmn}(p_{\nu\alpha})$, $(g(p)_{\nu\alpha})(g^\nu(\beta)) = p_{\nu\alpha}(\beta)$.

Given $J \subseteq I$, let

$$\overline{J} = \{\nu \in C : J \cap I_\nu \setminus K \ne 0\}.$$

If J, J' are special sets, $J \cap K = J' \cap K$, and $\overline{J} = \overline{J}'$, then there is a $g \in \mathscr{G}_\xi$ such that $g[J] = J'$. Now $|J \cap K| \le \kappa$ and $|K| \le \varrho$, so there are only $\varrho^\kappa = \varrho$ possibilities for $J \cap K$ (since $\operatorname{cf} \varrho > \kappa$). Also, $\overline{J} \in [C]^{\le\kappa}$ and $|C| \le \varrho$ (if $|C| > \varrho$, then $\lambda \in C$ by (4), contradiction). So there are only $\varrho^\kappa = \varrho$ possibilities for $\overline{J}$. So there are at most ϱ $\mathscr{G}_\xi$-orbits of special sets.

(22) For each special set J' there is a special set J in the same $\mathscr{G}_\xi$-orbit such that $J \cap S = J \cap K$.

For, if $\nu \in C$ and $J' \cap I_\nu \cap S \setminus K \ne 0$, then $\lambda < \nu$ (otherwise $I_\nu \subseteq K$), so $|I_\nu| > \lambda$, and hence there is a permutation of I_ν fixing $I_\nu \cap K$ and mapping $J' \cap I_\nu \setminus K$ out of S. Let g combine all such, and set $J = g[J']$. If $(\nu, \alpha) \in J \cap S$, then $(\nu, \alpha) \in J \cap I_\nu \cap S$.

Choose $(\nu,\beta)\in J'$ such that $g(\nu,\beta)=(\nu,\alpha)$. Thus $(\nu,\beta)\in J'\cap I_\nu$, and $g(\nu,\beta)\in S$, so by construction, $(\nu,\beta)\in K$, hence $\alpha=\beta$ and $(\nu,\alpha)\in K$. So (22) holds.

For each orbit Q of special sets, choose $L_Q\in Q$ satisfying (24). Each set L_Q is called a *standard* set. There are at most ϱ standard sets.

If $g\in\mathscr{G}_\xi$ and J is a support for a nice name σ for a subset of κ, then $g[J]$ is the support for the nice name $g(\sigma)$.

(23) If σ is a nice name for a member of $[\kappa]^\mu$, then there is an $A\in[\lambda]^{\leq\kappa}$ such that

$$p^0\Vdash\exists\alpha\in A\ [|\sigma\cap\dot{X}_\alpha|\geq\mu].$$

For, let B be a maximal pairwise incompatible subset of

$$\{q\leq p^0:\exists\alpha<\lambda\ [q\Vdash|\sigma\cap\dot{X}_\alpha|\geq\mu]\}.$$

Now $\{r:\exists q\in B\ (r\leq q)\}$ is dense below p^0. For, suppose that $s\leq p^0$. Then we have $s\Vdash\exists\alpha<\lambda\,[|\sigma\cap\dot{X}_\alpha|\geq\mu]$, so there exist a $q\leq s$ and an $\alpha<\lambda$ such that $q\Vdash|\sigma\cap\dot{X}_\alpha|\geq\mu$. Say q,r compatible, $r\in B$. Then say $t\leq q,r$. Then t is as desired.

For all $q\in B$ choose $\alpha_q<\lambda$ such that $q\Vdash|\sigma\cap\dot{X}_\alpha|\geq\mu$. Let $A=\{\alpha_q:q\in B\}$. So $A\in[\lambda]^{\leq\kappa}$. Now let $p^0\in H$, H generic. Choose $q\in H\cap B$. Then $q\Vdash|\sigma\cap\dot{X}_{\alpha_q}|\geq\mu$, so $|\sigma^H\cap\dot{X}^H_{\alpha_q}|\geq\mu$. Thus A is as desired in (23).

Let J be a standard set. Suppose that $Q\in[J]^{<\mu}$. Let

$$\begin{aligned}s_Q=\{p\in P:\operatorname{dmn}(p)=\{\nu\in C:(\nu,\alpha)\in Q\text{ for some }\alpha\}\\ \text{and for all }\nu\in\operatorname{dmn}(p),\ \operatorname{dmn}(p_\nu)=\{\alpha<\nu:(\nu,\alpha)\in Q\}\}.\end{aligned}$$

There is a one-one function F from s_Q into $\prod_{(\nu,\alpha)\in Q}[\kappa]^{<\mu}$, namely one can let $F(p)_{\nu\alpha}=p_\nu(\alpha)$. It follows that $|s_Q|\leq\kappa$. Hence there are at most ϱ elements $p\in P$ with support contained in J, hence at most ϱ pairwise incompatible sets all members of which have support contained in J, hence at most ϱ nice names for members of $[\kappa]^\mu$ with support contained in J. For each nice name σ for a member of $[\kappa]^\mu$ with standard support, choose A_σ as in (23). Let B be the union of all such. So $|B|\leq\varrho$.

Now we unfix K. So $B=B_K$ depends on K. We now define $M_{\sigma+1}$: let $M_{\sigma+1}=B_{K_\sigma}\cup M_\sigma$, where

$$K_\sigma=L\cup\bigcup_{\alpha\in M_\sigma}J_\alpha\cup\bigcup_{\varrho\geq\nu\in C}I_\nu.$$

(Note that $|K_\sigma|\leq\varrho$.)

Let $M=\bigcup_{\sigma<\kappa^+}M_\sigma$. Thus also $|M|\leq\varrho$. Now suppose that σ is a nice name for a member of $[\kappa]^\mu$. Let J be a support for σ. Define $K_\infty=\bigcup_{\tau<\kappa^+}K_\tau$. Fix $\tau<\kappa^+$ such that $J\cap K_\infty\subseteq K_\tau$. Now consider the step from M_τ to $M_{\tau+1}$. Choose $g\in\mathscr{G}_\tau$ such that $g[J]$ is standard. By $J\cap K_\infty\subseteq K_\tau$,

(24) $J\cap(K_{\tau+1}\setminus K_\tau)=0$;
(25) $g[J]\cap(K_{\tau+1}\setminus K_\tau)=0$.

This is true since $g[J]\cap S=g[J]\cap K_\tau$ and $K_{\tau+1}\subseteq S$.

Choose $h\in\mathscr{G}_\xi$ such that $h\restriction J=g\restriction J$ and h is the identity on $K_{\tau+1}\setminus K_\tau$. So h fixes $K_{\tau+1}$ pointwise. Now

$$p^0\Vdash\exists\alpha\in A_{h(\sigma)}\ [|h(\sigma)\cap\dot{X}_\alpha|\geq\mu],$$

so

$$p^0 \Vdash \exists\alpha \in A_{h(\sigma)}\ [|\sigma \cap \dot{X}_\alpha| \geq \mu].$$

This is true since $A_{h(\sigma)} \subseteq B_{K_\tau} \subseteq M_{\tau+1}$, and hence $J_\alpha \subseteq K_{\tau+1}$ for each $\alpha \in A_{h(\sigma)}$, so h fixes J_α pointwise. Thus $p^0 \Vdash \exists\alpha \in M\ [|\sigma \cap \dot{X}_\alpha| \geq \mu]$. Since $|M| < \lambda$, this is a contradiction, as σ is arbitrary.

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