Introduction

Spaces of generalised smoothness have been considered by several mathematicians within different approaches. We refer to Gol'dman (using modulus of continuity, cf. [Gol76]), Kalyabin and Lizorkin (approximation theory, cf. [KL87]), Merucci, Cobos and Fernandez (interpolation theory, cf. [Mer84], [CF88]) among others. A survey has been given in [KL87]. More historical references can be found in [Leo98a].

Our approach is similar to that in [Leo98a], that is, we use the point of view of Fourier analysis and, moreover, consider the more general context of quasi-Banach spaces. The interest of Leopold in [Leo98a] was in using spaces of generalised smoothness of Besov type to handle embedding properties in delicate limiting situations. Our study was strongly motivated by the articles [ET98] and [ET99]. There, Edmunds and Triebel used spaces of generalised smoothness of Besov type when studying the behaviour of eigenvalues in problems which correspond to the vibration of a drum, the whole mass of which is concentrated on a fractal subset of the drum. In order to explain the relationship between fractals and function spaces we need some previous considerations. The fractals considered by Edmunds and Triebel in the above papers are (isotropic) perturbed d-sets, called (d, Ψ)-sets.

Let Γ be a non-empty closed subset of $\mathbb{R}^n$, $0 < d < n$ and Ψ a positive monotone function on the interval $(0, 1]$ with

$$c_1 \Psi(2^{-j}) \leq \Psi(2^{-2j}) \leq c_2 \Psi(2^{-j}), \qquad j \in \mathbb{N}_0, \tag{0.1}$$

for some positive constants c_1 and c_2. Then Γ is called a (d, Ψ)-set if there is a Radon measure μ with $\operatorname{supp} \mu = \Gamma$ and two positive constants c_1 and c_2 such that

$$c_1 r^d \Psi(r) \leq \mu(B(\gamma, r)) \leq c_2 r^d \Psi(r) \tag{0.2}$$

for any ball $B(\gamma, r)$ centred at $\gamma \in \Gamma$ of radius $r \in (0, 1)$. If, additionally, Ψ is decreasing with $\lim_{r \to 0} \Psi(r) = \infty$, and (0.2) holds for $d = n$, then Γ is called an (n, Ψ)-set.

Let Ω be a bounded C^∞ domain in $\mathbb{R}^n$ and let $-\Delta$ be the Dirichlet Laplacian in Ω. According to Theorem 2.28 and Corollary 2.30 of [ET99], the operator

$$B = (-\Delta)^{-1} \circ \operatorname{tr}^\Gamma \tag{0.3}$$

is a compact self-adjoint non-negative operator in $\mathring{W}_2^1(\Omega)$, where $\Gamma \subset \Omega$ is a (d, Ψ)-set with $n - 2 < d \leq n$ and tr^Γ is closely related to the trace tr_Γ of $\mathring{W}_2^1(\Omega)$ on Γ. Moreover, the positive eigenvalues μ_k of B, ordered so that $\mu_{k+1} \leq \mu_k$, $k \in \mathbb{N}$, and repeated according to their algebraic multiplicity, can be estimated as follows:

$$c_1 k^{-1} (k \Psi(k^{-1}))^{(n-2)/d} \leq \mu_k \leq c_2 k^{-1} (k \Psi(k^{-1}))^{(n-2)/d}, \qquad k \in \mathbb{N}, \tag{0.4}$$

for some positive constants c_1 and c_2.

If in the definition of a $(d,\varPsi)$-set, restricted to $0<d<n$, we take $\varPsi \sim 1$, then we get the concept of a d-set. The corresponding fractal drum problem was solved first by Triebel in his book [Tri97]. The method used there relies on the close connection between d-sets, in particular L_p-spaces on a d-set $\varGamma$, and some Besov spaces B^s_{pq}. The technique includes estimates for the entropy numbers of compact embeddings between function spaces on $\varGamma$, which once more relies on the machinery available for the usual Besov spaces, specially characterisations via atomic and subatomic decompositions.

For a generalisation to $(d,\varPsi)$-sets, we have to consider the spaces $B^{(s,\varPsi^a)}_{pq}$ where $0<p\le\infty$, $0<q\le\infty$ and the smoothness is now expressed by the couple $(s,\varPsi^a)$, $s\in\mathbb{R}$, $a\in\mathbb{R}$ and the above function $\varPsi$. For this reason as well as for some intrinsic interest it is worthwhile to extend to these generalised spaces of Besov type several results known for the usual Besov spaces. We do this in the first section including a parallel approach to the spaces of generalised smoothness of Triebel–Lizorkin type in $\mathbb{R}^n$. In the second section we begin by developing measure properties of $(d,\varPsi)$-sets. In particular, we show that, up to equivalence, there exists only one Radon measure related to a $(d,\varPsi)$-set, and that any $(d,\varPsi)$-set has Hausdorff dimension d and Lebesgue measure zero. We finish the second section by showing a deep relation between L_p-spaces on a $(d,\varPsi)$-set and some spaces $B^{(s,\varPsi^a)}_{pq}(\mathbb{R}^n)$. The third section is devoted to entropy numbers. We estimate entropy numbers of embeddings between some sequence spaces and then using also the results of the first section we get estimates for the entropy numbers of compact embeddings between spaces of Besov type on a $(d,\varPsi)$-set. Essentially we obtain an extension of Theorem 2.24 in [ET99] to $(d,\varPsi)$-sets in the light of [Tri97]. Having in mind Carl's inequality, these results can be used to estimate from above the eigenvalues of suitable bounded operators like (0.3). This is done in the fourth section.

1. Function spaces on $\mathbb{R}^n$

1.1. Introduction. Our aim in this section is to develop a detailed study of the spaces of generalised smoothness $B^{(s,\varPsi)}_{pq}(\mathbb{R}^n)$ and $F^{(s,\varPsi)}_{pq}(\mathbb{R}^n)$. They were introduced by D. Edmunds and H. Triebel in [ET98], in the context of spectral theory for isotropic fractal drums, and generalise the usual Besov and Triebel–Lizorkin spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$, respectively. Now a new parameter $\varPsi$ is coming in, but s remains the main smoothness parameter while $\varPsi$ stands for a finer tuning.

Spaces of generalised smoothness have been considered by several mathematicians within different approaches. We refer to Gol'dman (using modulus of continuity), Kalyabin (approximation theory), Merucci, Cobos and Fernandez (interpolation theory) among others. A survey has been given in [KL87]. More historical references can be found in [Leo98a].

1.2. Definitions and basic properties

1.2.1. *Basic notations.* As usual, $\mathbb{R}^n$ denotes the n-dimensional real euclidean space, $\mathbb{N}$ the collection of all natural numbers, $\mathbb{N}_0=\mathbb{N}\cup\{0\}$ and $\mathbb{C}$ stands for the complex numbers.

If $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$ is a multi-index its length is $|\alpha| = \sum_{j=1}^n \alpha_j$, the derivatives $D^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}$ have the usual meaning and if $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ then $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$.

Let $\mathcal{S}(\mathbb{R}^n)$ be the Schwartz space of all complex-valued rapidly decreasing infinitely differentiable functions on $\mathbb{R}^n$ equipped with the usual topology. By $\mathcal{S}'(\mathbb{R}^n)$ we denote its topological dual, the space of all tempered distributions on $\mathbb{R}^n$. If $\varphi \in \mathcal{S}(\mathbb{R}^n)$ then

$$\widehat{\varphi}(\xi) = (\mathcal{F}\varphi)(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\xi} \varphi(x)\, dx, \qquad \xi \in \mathbb{R}^n, \tag{1.5}$$

denotes the Fourier transform of φ. Then $\mathcal{F}^{-1}\varphi$ or $\check{\varphi}$ stands for the inverse Fourier transform, given by the right-hand side of (1.5) with i in place of $-i$. Of course, $x\xi$ denotes the scalar product on $\mathbb{R}^n$. Both $\mathcal{F}$ and $\mathcal{F}^{-1}$ are extended to $\mathcal{S}'(\mathbb{R}^n)$ in the standard way.

The collection of all complex-valued infinitely differentiable functions on $\mathbb{R}^n$ with compact support is denoted by $\mathcal{D}(\mathbb{R}^n)$, and $\mathcal{D}'(\mathbb{R}^n)$ stands for the set of all complex distributions on $\mathbb{R}^n$.

Let $0 < q \le \infty$. Then ℓ_q is the set of all sequences $b = (b_k)_{k \in \mathbb{N}_0}$ of complex numbers such that

$$\|b \,|\, \ell_q\| = \Big(\sum_{k=0}^\infty |b_k|^q \Big)^{1/q} < \infty$$

(modified to $\sup_{k\in\mathbb{N}_0} |b_k|$ if $q = \infty$). Of course, ℓ_q is a quasi-Banach space (a Banach space if $q \ge 1$). Let $0 < p, q \le \infty$, and let $f = (f_k(x))_{k\in\mathbb{N}_0}$ be a sequence of complex-valued Lebesgue measurable functions on $\mathbb{R}^n$. Then

$$\|f \,|\, L_p(\ell_q)\| = \Big(\int_{\mathbb{R}^n} \Big(\sum_{k=0}^\infty |f_k(x)|^q \Big)^{p/q} dx \Big)^{1/p},$$

$$\|f \,|\, \ell_q(L_p)\| = \Big(\sum_{k=0}^\infty \Big(\int_{\mathbb{R}^n} |f_k(x)|^p \, dx \Big)^{q/p} \Big)^{1/q}$$

(modified to $\operatorname{ess\,sup}_{x\in\mathbb{R}^n}$ if $p = \infty$ and to $\sup_{k\in\mathbb{N}_0}$ if $q = \infty$). Let $L_p(\ell_q) = L_p(\mathbb{R}^n, \ell_q)$ be the set of all sequences f such that $\|f \,|\, L_p(\ell_q)\| < \infty$, and let $\ell_q(L_p) = \ell_q(L_p(\mathbb{R}^n))$ be the set of all sequences f such that $\|f \,|\, \ell_q(L_p)\| < \infty$. In the scalar case the corresponding space is denoted by $L_p(\mathbb{R}^n)$, quasi-normed by

$$\|f \,|\, L_p(\mathbb{R}^n)\| = \Big(\int_{\mathbb{R}^n} |f(x)|^p \, dx \Big)^{1/p}$$

(modified to $\operatorname{ess\,sup}_{x\in\mathbb{R}^n} |f(x)|$ if $p = \infty$). $L_p(\ell_q)$, $\ell_q(L_p)$ and the scalar case $L_p(\mathbb{R}^n)$ are quasi-Banach spaces (Banach spaces if $p, q \ge 1$).

All unimportant constants are denoted by c, occasionally with additional subscripts within the same formulas. The equivalence $\sim$ in

$$a_k \sim b_k \quad \text{or} \quad \varphi(x) \sim \psi(x)$$

means that there are positive constants c_1 and c_2 such that

$$c_1 a_k \le b_k \le c_2 a_k \quad \text{or} \quad c_1\varphi(x) \le \psi(x) \le c_2 \varphi(x)$$

for all admitted values of the discrete variable k or the continuous variable x. Here a_k, b_k are positive numbers and $\varphi(x)$, $\psi(x)$ are positive functions. We adopt the following convention. A real function Ψ on the interval $(0,1]$ is said to be monotone if it is either decreasing or increasing, where decreasing (resp. increasing) means not increasing (resp. not decreasing). Finally, log is always taken to base 2.

1.2.2. *Definitions.* Let φ_0 be a C^∞ function on $\mathbb{R}^n$ with

$$\operatorname{supp}\varphi_0 \subset \{\xi \in \mathbb{R}^n : |\xi| \le 2\}, \qquad \varphi_0(\xi) = 1 \quad \text{if } |\xi| \le 1. \tag{1.6}$$

Let $j \in \mathbb{N}$ and

$$\varphi_j(\xi) = \varphi_0(2^{-j}\xi) - \varphi_0(2^{-j+1}\xi), \qquad \xi \in \mathbb{R}^n. \tag{1.7}$$

Then, since

$$\operatorname{supp}\varphi_j \subset \{\xi \in \mathbb{R}^n : 2^{j-1} \le |\xi| \le 2^{j+1}\}, \qquad j \in \mathbb{N}, \tag{1.8}$$

and

$$\sum_{j=0}^{\infty} \varphi_j(\xi) = 1 \qquad \text{for all } \xi \in \mathbb{R}^n, \tag{1.9}$$

$(\varphi_j)_{j\in\mathbb{N}_0}$ is a smooth dyadic resolution of unity. By the Paley–Wiener–Schwartz theorem $(\varphi_j\widehat{f}\,)^\vee$, $j \in \mathbb{N}_0$, is an entire analytic function on $\mathbb{R}^n$, for any $f \in \mathcal{S}'(\mathbb{R}^n)$. In particular $(\varphi_j\widehat{f}\,)^\vee$ makes sense pointwise. Moreover

$$f = \sum_{j=0}^{\infty} (\varphi_j\widehat{f}\,)^\vee \qquad (\text{convergence in } \mathcal{S}'(\mathbb{R}^n)). \tag{1.10}$$

Definition 1.1. A positive monotone function Ψ on the interval $(0,1]$ is called *admissible* if

$$\Psi(2^{-j}) \sim \Psi(2^{-2j}), \qquad j \in \mathbb{N}_0.$$

Example 1.2. Let $0 < c < 1$ and $b \in \mathbb{R}$. Then

$$\Psi(x) = |\log cx|^b, \qquad 0 < x \le 1,$$

is an admissible function.

Remark 1.3. Let Ψ be an admissible function. We have two cases:

(i) If Ψ is increasing, then there exists $\theta \in \mathbb{R}_0^+$ such that

$$\Psi(2^{-2j}) \le \Psi(2^{-j}) \le 2^\theta \Psi(2^{-2j}) \le 2^{\theta k}\Psi(2^{-2^k j}), \qquad j \in \mathbb{N}_0,\ k \in \mathbb{N}. \tag{1.11}$$

(ii) If Ψ is decreasing, then there exists $\theta' \in \mathbb{R}_0^+$ such that

$$\Psi(2^{-j}) \le \Psi(2^{-2^k j}) \le 2^{\theta' k}\Psi(2^{-j}), \qquad k,j \in \mathbb{N}_0. \tag{1.12}$$

In the next proposition we state some basic facts concerning admissible functions.

Proposition 1.4. *Let Ψ be an admissible function.*

(i) *Let $\chi \in \mathbb{R}$. Then Ψ^χ is also an admissible function.*

(ii) *There are non-negative numbers c_1, c_2, b and c, with $c \in (0,1)$ and $c_1, c_2 > 0$, such that*

$$c_1|\log cx|^{-b} \le \Psi(x) \le c_2|\log cx|^b, \qquad x \in (0,1].$$

(iii) *Let* $a \in \mathbb{R}^+$. *Then*

$$\lim_{x \to 0^+} x^a \Psi(x) = 0.$$

(iv) *If* $a \in \mathbb{R}^+$, *then there exists* $j_0 \in \mathbb{N}_0$ *such that for any* $j \in \mathbb{N}_0$ *with* $j \geq j_0$,

$$\Psi(a2^{-j}) \sim \Psi(2^{-j}) \quad \textit{and} \quad \Psi(2^{-aj}) \sim \Psi(2^{-j}).$$

(v) *There is a positive constant* c *such that*

$$\Psi(2x) \leq c\Psi(x), \qquad x \in (0, 1/2].$$

(vi) *There are non-negative numbers* c_1, c_2 *and* b, *with* $c_1, c_2 > 0$, *such that*

$$c_1(1 + j - k)^{-b} \leq \frac{\Psi(2^{-j})}{\Psi(2^{-k})} \leq c_2(1 + j - k)^b$$

for all $j, k \in \mathbb{N}_0$ *with* $j \geq k$.

Proof. Part (i) is obvious. For (ii) it is sufficient to prove it for Ψ decreasing (the other case then follows using (i) and $\chi = -1$). Recall that we have (1.12). Let

$$2^{-2^{k+1}} \leq x \leq 2^{-2^k} \quad \text{for some } k \in \mathbb{N}_0. \tag{1.13}$$

Then, with $b = \theta'$, according to (1.12) we have on the one hand

$$\Psi(x) \leq \Psi(2^{-2^{k+1}}) \leq 2^{bk}\Psi(2^{-2}) \leq \Psi(2^{-2})|\log x|^b$$

and on the other hand

$$\Psi(x) \geq \Psi(2^{-2^k}) \geq 2^{-bk}\Psi(2^{-2^{2k}}) \geq 2^{-bk}\Psi(1) \geq \Psi(1)|\log x|^{-b},$$

both for x satisfying (1.13). Now we check the remaining case. If $2^{-1} \leq x \leq 1$ then for $c = 2^{-1}$, cx fulfils (1.13) with $k = 0$. We get

$$\Psi(x) \leq \Psi(cx) \leq \Psi(2^{-2})|\log cx|^b$$

and

$$\Psi(x) \geq \Psi(1) \geq \frac{\Psi(1)}{\Psi(2^{-1})}2^{-b}\Psi(2^{-2}) \geq \frac{\Psi(1)}{\Psi(2^{-1})}2^{-b}\Psi(cx) \geq \Psi(1)2^{-b}|\log cx|^{-b}.$$

Note that for $c \in (0, 1)$ and x in (1.13) we have $|\log x| \leq |\log cx|$, hence the proof of (ii) is complete.

To show (iii), note that by the above, there are positive constants c_1, c_2 and b such that

$$c_1|\log x|^{-b} \leq \Psi(x) \leq c_2|\log x|^b, \qquad x \in (0, 1/2]. \tag{1.14}$$

For $a > 0$, we have

$$\lim_{x \to 0^+} x^a|\log x|^b = \lim_{x \to 0^+} x^a|\log x|^{-b} = 0.$$

This, together with (1.14), proves (iii).

Having in mind (i) for $\chi = -1$ it is enough to show (iv) for Ψ increasing. The proof of the first equivalence in (iv) is divided in two cases: $0 < a < 1$ and $a \geq 1$. For $0 < a < 1$, it is immediate that $\Psi(a\,2^{-j}) \leq \Psi(2^{-j})$, $j \in \mathbb{N}_0$. For $j \in \mathbb{N}$, we choose $k \in \mathbb{N}$ such that $k \geq \log\left(\frac{j - \log a}{j}\right)$. Then

$$\Psi(2^{-j}) \leq c^k\Psi(2^{-2^k j}) \leq c^k\Psi(a2^{-j}) \tag{1.15}$$

for some positive constant c, depending only on Ψ. As $\lim_{j\to\infty}\log\left(\frac{j-\log a}{j}\right)=0$, there exists $j_0\in\mathbb{N}$ such that $\log\left(\frac{j-\log a}{j}\right)<1$ for any $j\geq j_0$. For any such j we can take $k=1$ in (1.15). Hence, $\Psi(2^{-j})\leq c\Psi(a2^{-j})$, $j\geq j_0$, and this was the remaining inequality. If $a\geq 1$, we have $\Psi(2^{-j})\leq c\Psi(a2^{-j})$, $j\geq \log a$. For $j>[\log a]+1$, where $[x]$ denotes the largest integer not greater than x, and for $k\in\mathbb{N}$ such that $k\geq\log\left(\frac{j}{j-[\log a]-1}\right)$, we have

$$\Psi(a2^{-j})\leq\Psi(2^{-(j-[\log a]-1)})\leq c^k\Psi(2^{-2^k(j-[\log a]-1)})\leq c^k\Psi(2^{-j}),\tag{1.16}$$

where, once more, c is a positive constant which depends only on Ψ. Reasoning as above, for any j arbitrarily large we may choose $k=1$. Therefore, $\Psi(a2^{-j})\leq c\Psi(2^{-j})$, $j\geq j_0$.

The proof of the second equivalence in (iv) can be divided into three cases: $0<a<1$, $1\leq a\leq 2$ and $a>2$. If $0<a<1$, then obviously $\Psi(2^{-j})\leq\Psi(2^{-aj})$, $j\in\mathbb{N}_0$. Moreover, for any integer k with $k\geq\log(j/[aj])$, we have

$$\Psi(2^{-aj})\leq\Psi(2^{-[aj]})\leq c^k\Psi(2^{-2^k[aj]})\leq c^k\Psi(2^{-j}),\qquad j\in\mathbb{N}.\tag{1.17}$$

Note that there exists $j_0\in\mathbb{N}$ such that $\log(j/[aj])<\log(a^{-1})+1$ for any $j\geq j_0$. So, for any such j we may choose $k=[\log a^{-1}]+1$ in (1.17), which gives the remaining inequality for this first case. If $1\leq a\leq 2$ the assertion is a direct consequence of the monotonocity of Ψ and $\Psi(2^{-j})\sim\Psi(2^{-2j})$ from the definition of an admissible function. If $a>2$, then obviously $\Psi(2^{-aj})\leq\Psi(2^{-j})$, $j\in\mathbb{N}$. On the other hand

$$\Psi(2^{-j})\leq c^{[\log a]+1}\Psi(2^{-2^{[\log a]+1}j})\leq c^{[\log a]+1}\Psi(2^{-aj}),\qquad j\in\mathbb{N}_0.$$

This completes the proof of (iv).

For (v), if Ψ is decreasing, then obviously (v) is satisfied with $c=1$. If Ψ is increasing then, by Definition 1.1, there exists a positive constant c such that

$$\Psi(2^{-2j})\leq\Psi(2^{-j})\leq c\Psi(2^{-2j}),\qquad j\in\mathbb{N}_0.$$

Let $j\in\mathbb{N}_0$ be such that $2^{-(j+1)}\leq 2x\leq 2^{-j}$. Then $2^{-(j+2)}\leq x\leq 2^{-(j+1)}$, and hence

$$\Psi(2x)\leq\Psi(2^{-j})\leq c\Psi(2^{-2j})\leq c^2\Psi(2^{-4j})\leq c^2\Psi(2^{-(j+2)})\leq c^2\Psi(x).$$

To prove (vi) it is again enough to consider Ψ increasing. Since $j\geq k$ it is then obvious that

$$\frac{\Psi(2^{-j})}{\Psi(2^{-k})}\leq 1.$$

On the other hand, by (1.11), we have

$$\Psi(2^{-k})\leq c^\nu\Psi(2^{-2^\nu k}),\qquad \nu\in\mathbb{N}_0,\tag{1.18}$$

for some constant $c\geq 1$. If $k\neq 0$ and $\nu\in\mathbb{N}_0$ is chosen so that $2^\nu k\geq j$, then (1.18) implies

$$\Psi(2^{-k})\leq c^\nu\Psi(2^{-j}).\tag{1.19}$$

Otherwise, if $k=0$, instead of (1.18) we can write

$$\Psi(2^{-k})=\Psi(1)\leq\frac{\Psi(1)}{\Psi(2^{-1})}c^\nu\Psi(2^{-2^\nu\cdot 1}),\qquad \nu\in\mathbb{N}_0.$$

If we now choose $\nu \in \mathbb{N}_0$ such that $2^\nu \ge j$, we get, for $k = 0$,

$$\Psi(2^{-k}) = \Psi(1) \le \frac{\Psi(1)}{\Psi(2^{-1})} c^\nu \Psi(2^{-j}). \tag{1.20}$$

The value of $\nu = [\log(1 + j - k)] + 1$ can be used for both cases of k. This together with (1.18) and (1.20) yields

$$\Psi(2^{-k}) \le \frac{\Psi(1)}{\Psi(2^{-1})} c(1 + j - k)^{\log c} \Psi(2^{-j}),$$

which completes the proof. ■

Definition 1.5. (i) Let $0 < p, q \le \infty$, $s \in \mathbb{R}$ and Ψ an admissible function. Then $B^{(s,\Psi)}_{pq}(\mathbb{R}^n)$ is the collection of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f \mid B^{(s,\Psi)}_{pq}(\mathbb{R}^n)\|_\varphi = \Big(\sum_{j=0}^{\infty} 2^{sjq} \Psi(2^{-j})^q \|(\varphi_j \widehat{f})^\vee \mid L_p(\mathbb{R}^n)\|^q \Big)^{1/q} \tag{1.21}$$

(with the usual modification if $q = \infty$) is finite.

(ii) Let $0 < p < \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$ and Ψ an admissible function. Then $F^{(s,\Psi)}_{pq}(\mathbb{R}^n)$ is the collection of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f \mid F^{(s,\Psi)}_{pq}(\mathbb{R}^n)\|_\varphi = \Big\| \Big(\sum_{j=0}^{\infty} 2^{sjq} \Psi(2^{-j})^q |(\varphi_j \widehat{f})^\vee|^q \Big)^{1/q} \Big| L_p(\mathbb{R}^n) \Big\| \tag{1.22}$$

(with the usual modification if $q = \infty$) is finite.

Remark 1.6. If $\Psi \sim 1$ then the spaces $B^{(s,\Psi)}_{pq}(\mathbb{R}^n)$ and $F^{(s,\Psi)}_{pq}(\mathbb{R}^n)$ coincide with the usual Besov and Triebel–Lizorkin spaces, $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$, respectively. The theory of these last spaces has been developed in full extent in [Tri83] and [Tri92]. For more recent topics we refer to [ET96], [RuS96] and [Tri97]. If $\Psi(x) = (1 + |\log x|)^b$, $b \in \mathbb{R}$, we obtain the spaces $B^{s,b}_{pq}(\mathbb{R}^n)$ used by Leopold in [Leo98a].

Of course the quasi-norms in (1.21) and (1.22) depend on the function φ_0 chosen according to (1.6). But this is not the case for the spaces $B^{(s,\Psi)}_{pq}(\mathbb{R}^n)$ and $F^{(s,\Psi)}_{pq}(\mathbb{R}^n)$ (in the sense of equivalent quasi-norms). This can be proved in the usual way, using the multiplier theorem 1.6.3 of [Tri83] and the properties of the admissible function Ψ, and that is why we omit the subscript φ in our notation. Both $B^{(s,\Psi)}_{pq}(\mathbb{R}^n)$ and $F^{(s,\Psi)}_{pq}(\mathbb{R}^n)$ are quasi-Banach spaces (Banach spaces if $p \ge 1$ and $q \ge 1$).

1.2.3. *Equivalent quasi-norms.* Let $(\varphi_k)_{k \in \mathbb{N}_0} \subset \mathcal{S}(\mathbb{R}^n)$. We introduce the maximal functions

$$(\varphi_k^* f)_a(x) = \sup_{z \in \mathbb{R}^n} \frac{|(\varphi_k \widehat{f})^\vee (x - z)|}{1 + |2^k z|^a}, \qquad f \in \mathcal{S}'(\mathbb{R}^n),\ a > 0, \tag{1.23}$$

where $x \in \mathbb{R}^n$ and $k \in \mathbb{N}_0$. The result below is the counterpart of Theorem 2.3.2 of [Tri92] for the spaces $B^{(s,\Psi)}_{pq}(\mathbb{R}^n)$ and $F^{(s,\Psi)}_{pq}(\mathbb{R}^n)$; it is a simple consequence of Theorem 1.6.2 of [Tri83].

Theorem 1.7. *Let $(\varphi_k)_{k \in \mathbb{N}_0}$ be a system of functions as in* 1.2.2 *with the generating function φ_0.*

(i) *Let* $0<p,q\le\infty$, $s\in\mathbb{R}$, Ψ *an admissible function and* $a>n/p$. *Then*

$$B_{pq}^{(s,\Psi)}(\mathbb{R}^n)=\Big\{f\in\mathcal{S}'(\mathbb{R}^n):\Big(\sum_{j=0}^{\infty}2^{sjq}\Psi(2^{-j})^q\|(\varphi_j^*f)_a\,|\,L_p(\mathbb{R}^n)\|^q\Big)^{1/q}<\infty\Big\} \tag{1.24}$$

(*with the usual modification if* $q=\infty$) *in the sense of equivalent quasi-norms.*

(ii) *Let* $0<p<\infty$, $0<q\le\infty$, $s\in\mathbb{R}$, Ψ *an admissible function and* $a>n/\min(p,q)$. *Then*

$$F_{pq}^{(s,\Psi)}(\mathbb{R}^n)=\Big\{f\in\mathcal{S}'(\mathbb{R}^n):\Big\|\Big(\sum_{j=0}^{\infty}2^{sjq}\Psi(2^{-j})^q|(\varphi_j^*f)_a(\cdot)|^q\Big)^{1/q}\,\Big|\,L_p(\mathbb{R}^n)\Big\|<\infty\Big\} \tag{1.25}$$

(*with the usual modification if* $q=\infty$) *in the sense of equivalent quasi-norms.*

1.2.4. *Lifting property.* Let $\sigma\in\mathbb{R}$. Then

$$I_\sigma: f\mapsto(\langle\xi\rangle^\sigma\widehat{f}\,)^\vee, \tag{1.26}$$

with $\langle\xi\rangle=(1+|\xi|^2)^{1/2}$, is a one-to-one map of $\mathcal{S}(\mathbb{R}^n)$ onto itself and of $\mathcal{S}'(\mathbb{R}^n)$ onto itself. Obviously $I_\sigma I_\eta=I_{\sigma+\eta}$. For the B and F scales, I_σ acts as a lift:

Proposition 1.8. *Let* $s\in\mathbb{R}$, $\sigma\in\mathbb{R}$, $0<q\le\infty$ *and* Ψ *an admissible function.*

(i) *Let* $0<p\le\infty$. *Then* I_σ *maps* $B_{pq}^{(s,\Psi)}(\mathbb{R}^n)$ *isomorphically onto* $B_{pq}^{(s-\sigma,\Psi)}(\mathbb{R}^n)$ *and* $\|I_\sigma f\,|\,B_{pq}^{(s-\sigma,\Psi)}(\mathbb{R}^n)\|$ *is an equivalent quasi-norm on* $B_{pq}^{(s,\Psi)}(\mathbb{R}^n)$.

(ii) *Let* $0<p<\infty$. *Then* I_σ *maps* $F_{pq}^{(s,\Psi)}(\mathbb{R}^n)$ *isomorphically onto* $F_{pq}^{(s-\sigma,\Psi)}(\mathbb{R}^n)$ *and* $\|I_\sigma f\,|\,F_{pq}^{(s-\sigma,\Psi)}(\mathbb{R}^n)\|$ *is an equivalent quasi-norm on* $F_{pq}^{(s,\Psi)}(\mathbb{R}^n)$.

Proof. Step 1. We first prove (ii). Let $f\in F_{pq}^{(s,\Psi)}(\mathbb{R}^n)$. We have

$$\|I_\sigma f\,|\,F_{pq}^{(s-\sigma,\Psi)}(\mathbb{R}^n)\|=\|(2^{(s-\sigma)j}\Psi(2^{-j})(\varphi_j\langle\xi\rangle^\sigma\widehat{f}\,)^\vee)_{j\in\mathbb{N}_0}\,|\,L_p(\ell_q)\|. \tag{1.27}$$

Let $\phi\in\mathcal{S}(\mathbb{R}^n)$ with

$$\phi(x)=1\quad\text{if } 1/2\le|x|\le 2\quad\text{and}\quad \operatorname{supp}\phi\subset\{\xi\in\mathbb{R}^n:1/4\le|\xi|\le 4\}.$$

Then

$$(\varphi_j\langle\xi\rangle^\sigma\widehat{f}\,)^\vee=(\langle\xi\rangle^\sigma\phi(2^{-j}\xi)(\varphi_j\widehat{f}\,))^\vee,\qquad j\in\mathbb{N}.$$

Applying Theorem 1.6.3 of [Tri83] with $\eta\in\mathbb{N}$ such that $\eta>n/2+n/\min(p,q)$ and

$$M_j(\xi)=2^{-\sigma j}\,\langle\xi\rangle^\sigma\,\phi(2^{-j}\xi)$$

we get

$$\begin{aligned}&\|(2^{(s-\sigma)j}\Psi(2^{-j})(\varphi_j(1+|\xi|^2)^{\sigma/2}\widehat{f}\,)^\vee)_{j\in\mathbb{N}}\,|\,L_p(\ell_q)\|\\ &\qquad\le c\sup_{l\in\mathbb{N}}\|M_l(2^{l+2}\cdot)\,|\,H_2^\eta(\mathbb{R}^n)\|\cdot\|2^{sj}\Psi(2^{-j})(\varphi_j\widehat{f}\,)^\vee\,|\,L_p(\ell_q)\|.\end{aligned} \tag{1.28}$$

For a multi-index $\alpha\in\mathbb{N}_0^n$ with $|\alpha|\le\eta$ we have

$$\begin{aligned}&|D^\alpha[M_l(2^{l+2}\cdot)](x)|\\ &\quad\le 2^{2\sigma}\sum_{\beta\le\alpha}\binom{\alpha}{\beta}|D^\beta[(2^{-2(l+2)}+|x|^2)^{\sigma/2}]|\cdot|(D^{\alpha-\beta}\phi)(4x)|4^{|\alpha-\beta|}\\ &\quad\le 2^{2(\sigma+\eta)}\sup_{|\gamma|\le\eta}\sup_{y\in\mathbb{R}^n}|D^\alpha\phi(y)|\sum_{\beta\le\alpha}\binom{\alpha}{\beta}|D^\beta[(2^{-2(l+2)}+|x|^2)^{\sigma/2}]|.\end{aligned} \tag{1.29}$$

But

$$|D^\beta[(2^{-2(l+2)}+|x|^2)^{\sigma/2}]| \le c_{\sigma,\beta}(2^{-2(l+2)}+|x|^2)^{\sigma/2-|\beta|/2}, \tag{1.30}$$

and for $x \in \operatorname{supp} M_l(2^{l+2}\cdot)$ we have $1/16 \le |x| \le 1$. Recall that for $\eta \in \mathbb{N}$, $H_2^\eta(\mathbb{R}^n) = W_2^\eta(\mathbb{R}^n)$ is the usual Sobolev space normed by

$$\|f \,|\, W_2^\eta(\mathbb{R}^n)\| = \Big(\sum_{|\alpha|\le\eta} \|D^\alpha f \,|\, L_2(\mathbb{R}^n)\|^2\Big)^{1/2}.$$

Hence, using in (1.30) the fact that $|x| \le 1$ for the values of β with $|\beta| \le \sigma$ while $|x| \ge 1/16$ for $|\beta| > \sigma$, and by (1.29) we get

$$\sup_{l\in\mathbb{N}} \|M_l(2^{l+2}\cdot) \,|\, H_2^\eta(\mathbb{R}^n)\| < \infty.$$

Applying this in (1.28), together with the term corresponding to $j = 0$, which can be treated in a similar way, gives us

$$\|I_\sigma f \,|\, F_{pq}^{(s-\sigma,\Psi)}(\mathbb{R}^n)\| \le C\|f \,|\, F_{pq}^{(s,\Psi)}(\mathbb{R}^n)\|.$$

Observing that $I_\sigma I_{-\sigma} f = f$ completes the proof of (ii).

Step 2. The proof of (i) is similar and can be obtained by interchanging the roles of the L_p and ℓ_q quasi-norms in the proof above and using the scalar version of Theorem 1.6.3 of [Tri83]. ∎

1.2.5. *Embeddings.* We finish this subsection with some embedding assertions. This is the counterpart of Proposition 2.3.2/2 and Theorem 2.7.1 of [Tri83], p. 47 and p. 129. In the following "$\hookrightarrow$" always stands for topological embedding.

PROPOSITION 1.9. (i) *Let $0 < p \le \infty$, $0 < q_0 \le q_1 \le \infty$, $s \in \mathbb{R}$ and Ψ an admissible function. Then*

$$B_{pq_0}^{(s,\Psi)}(\mathbb{R}^n) \hookrightarrow B_{pq_1}^{(s,\Psi)}(\mathbb{R}^n),$$

and the corresponding assertion for the F-spaces holds with $0 < p < \infty$.

(ii) *Let $0 < p, q_0, q_1 \le \infty$, $s \in \mathbb{R}$, $\varepsilon > 0$, Ψ and $\widetilde{\Psi}$ admissible functions. Then*

$$B_{pq_0}^{(s+\varepsilon,\Psi)}(\mathbb{R}^n) \hookrightarrow B_{pq_1}^{(s,\widetilde{\Psi})}(\mathbb{R}^n),$$

and the corresponding assertion for the F-spaces holds with $0 < p < \infty$.

(iii) *Let $0 < q \le \infty$, $0 < p < \infty$, $s \in \mathbb{R}$ and Ψ an admissible function. Then*

$$B_{p\min(p,q)}^{(s,\Psi)}(\mathbb{R}^n) \hookrightarrow F_{pq}^{(s,\Psi)}(\mathbb{R}^n) \hookrightarrow B_{p\max(p,q)}^{(s,\Psi)}(\mathbb{R}^n).$$

(iv) *Let $0 < p_0 \le p_1 \le \infty$, $0 < q \le \infty$, Ψ an admissible function and $s_0, s_1 \in \mathbb{R}$ with*

$$s_0 - \frac{n}{p_0} = s_1 - \frac{n}{p_1}.$$

Then

$$B_{p_0q}^{(s_0,\Psi)}(\mathbb{R}^n) \hookrightarrow B_{p_1q}^{(s_1,\Psi)}(\mathbb{R}^n).$$

(v) *Let $0 < p_0 < p_1 < \infty$, $0 < q_0, q_1 \le \infty$, Ψ an admissible function and $s_0, s_1 \in \mathbb{R}$ with*

$$s_0 - \frac{n}{p_0} = s_1 - \frac{n}{p_1}.$$

Then

$$F^{(s_0,\Psi)}_{p_0q_0}(\mathbb{R}^n) \hookrightarrow F^{(s_1,\Psi)}_{p_1q_1}(\mathbb{R}^n).$$

(vi) *Let* $0 < p, q \le \infty$, $s \in \mathbb{R}$ *and* Ψ *an admissible function. Then*

$$B^{s+\varepsilon}_{pq}(\mathbb{R}^n) \hookrightarrow B^{(s,\Psi)}_{pq}(\mathbb{R}^n) \hookrightarrow B^{s}_{pq}(\mathbb{R}^n) \qquad \text{if } \Psi \text{ is decreasing},$$
$$B^{s}_{pq}(\mathbb{R}^n) \hookrightarrow B^{(s,\Psi)}_{pq}(\mathbb{R}^n) \hookrightarrow B^{s-\varepsilon}_{pq}(\mathbb{R}^n) \qquad \text{if } \Psi \text{ is increasing},$$

for any $\varepsilon > 0$; *and the corresponding assertion for the* F*-spaces holds with* $0 < p < \infty$.

(vii) *Let* $0 < p, q \le \infty$, $s \in \mathbb{R}$ *and* Ψ *an admissible function. Then*

$$\mathcal{S}(\mathbb{R}^n) \hookrightarrow B^{(s,\Psi)}_{pq}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n).$$

If in addition $\max(p,q) < \infty$, *then* $\mathcal{S}(\mathbb{R}^n)$ *is dense in* $B^{(s,\Psi)}_{pq}(\mathbb{R}^n)$. *The corresponding assertion is true for the* F*-spaces with* $0 < p < \infty$.

Proof. For (i), (iii), (v) and (vii) simply follow the proof for $\Psi \equiv 1$ in [Tri83], inserting the factor $\Psi(2^{-j})$. For (iv) proceed as before with $s_1 = 0$ and then use the lift according to Proposition 1.8.

For the proof of (ii) for B-spaces (similar for F-spaces): from (i), it is enough to take $q_0 = \infty$. Since Ψ and $\widetilde{\Psi}$ are admissible, by Proposition 1.4, there exist positive constants c_1, c_2, $\widetilde{c}_1$, $\widetilde{c}_2$, b and $\widetilde{b}$ such that

$$c_1 j^{-b} \le \Psi(2^{-j}) \le c_2 j^{b} \quad \text{and} \quad \widetilde{c}_1 j^{-\widetilde{b}} \le \widetilde{\Psi}(2^{-j}) \le \widetilde{c}_2 j^{\widetilde{b}}, \qquad j \in \mathbb{N}.$$

Let ε_1 be such that $0 < \varepsilon_1 < \varepsilon$. Then

$$\begin{aligned} 2^{sj}\widetilde{\Psi}(2^{-j}) &= 2^{(s+\varepsilon)j}\, 2^{(\varepsilon_1-\varepsilon)j} 2^{-\varepsilon_1 j}\, \widetilde{\Psi}(2^{-j}) \\ &\le c_\varepsilon 2^{(s+\varepsilon)j} j^{-(b+\widetilde{b})} 2^{-\varepsilon_1 j} \widetilde{c}_2 j^{\widetilde{b}} \le c_\varepsilon \frac{\widetilde{c}_2}{c_1} 2^{(s+\varepsilon)j} 2^{-\varepsilon_1 j} \Psi(2^{-j}). \end{aligned}$$

Hence

$$\begin{aligned} \|f \mid B^{(s,\widetilde{\Psi})}_{pq_1}(\mathbb{R}^n)\| &= \Big(\sum_{j=0}^{\infty} 2^{sjq_1} \widetilde{\Psi}(2^{-j})^{q_1} \|(\varphi_j \widehat{f}\,)^\vee \mid L_p(\mathbb{R}^n)\|^{q_1}\Big)^{1/q_1} \\ &\le c'_\varepsilon \Big(\sum_{j=0}^{\infty} 2^{-\varepsilon_1 q_1 j} 2^{(s+\varepsilon)jq_1} \Psi(2^{-j})^{q_1} \|(\varphi_j \widehat{f}\,)^\vee \mid L_p(\mathbb{R}^n)\|^{q_1}\Big)^{1/q_1} \\ &\le c'_\varepsilon \Big(\sum_{j=0}^{\infty} 2^{-\varepsilon_1 q_1 j}\Big)^{1/q_1} \sup_{j\in\mathbb{N}_0} 2^{(s+\varepsilon)j}\Psi(2^{-j}) \|(\varphi_j \widehat{f}\,)^\vee \mid L_p(\mathbb{R}^n)\| \\ &\le c''_\varepsilon \|f \mid B^{(s+\varepsilon,\Psi)}_{p\infty}(\mathbb{R}^n)\|. \end{aligned}$$

(vi) is a consequence of (ii) and the fact that $\Psi(1) \le \Psi(2^{-j})$, $j \in \mathbb{N}_0$, if Ψ is decreasing, while $\Psi(2^{-j}) \le \Psi(1)$, $j \in \mathbb{N}_0$, if Ψ is increasing. ∎

1.3. Characterisation by local means. Let $B = \{y \in \mathbb{R}^n : |y| \le 1\}$ be the unit ball in $\mathbb{R}^n$, and let k be a C^∞ function in $\mathbb{R}^n$ with $\operatorname{supp} k \subset B$. Then we introduce the local means

$$k(t,f)(x) = \int_{\mathbb{R}^n} k(y) f(x+ty)\, dy, \qquad x \in \mathbb{R}^n,\ t > 0, \tag{1.31}$$

which makes sense for any $f \in \mathcal{S}'(\mathbb{R}^n)$ (appropriately interpreted). Let k_0 and k^0 be two C^∞ functions in $\mathbb{R}^n$ with

$$\operatorname{supp} k_0 \subset B, \qquad \operatorname{supp} k^0 \subset B, \tag{1.32}$$

$$\widehat{k}_0(0) \neq 0, \qquad \widehat{k^0}(0) \neq 0. \tag{1.33}$$

For $N \in \mathbb{N}$, we define

$$k(y) = \Delta^N k^0(y) = \left(\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} \right)^N k^0(y), \qquad y \in \mathbb{R}^n. \tag{1.34}$$

Note that

$$\check{k}(x) = |x|^{2N} \check{k^0}(x), \qquad x \in \mathbb{R}^n. \tag{1.35}$$

We introduce some notations. For $0 < p, q \leq \infty$, let

$$\sigma_p = n\left(\frac{1}{p} - 1\right)_+ \quad \text{and} \quad \sigma_{pq} = n\left(\frac{1}{\min(p,q)} - 1\right)_+. \tag{1.36}$$

As usual for any $a \in \mathbb{R}$ we put $a_+ = \max(a, 0)$ and $[a]$ stands for the largest integer smaller than or equal to a.

THEOREM 1.10. *Let $0 < p < \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$ and Ψ an admissible function. Let $N \in \mathbb{N}_0$ with $2N > s$. Then there exists $h \in \mathbb{N}_0$ such that*

$$\|k_0(2^{-h}, f) \mid L_p(\mathbb{R}^n)\| + \left\| \left(\sum_{j=1}^{\infty} 2^{jsq} \Psi(2^{-j})^q |k(2^{-j}, f)(\cdot)|^q \right)^{1/q} \Big| L_p(\mathbb{R}^n) \right\| \tag{1.37}$$

(*with the usual modification if $q = \infty$*) *is an equivalent quasi-norm in $F_{pq}^{(s,\Psi)}(\mathbb{R}^n)$.*

Proof. The idea of the proof goes back to Theorem 2.4.1 of [Tri92]. Note that we always have

$$k(2^{-j}, f)(x) = (2\pi)^{n/2} (\check{k}(2^{-j}\cdot)\widehat{f}\,)^\vee(x), \qquad x \in \mathbb{R}^n,\ j \in \mathbb{N}, \tag{1.38}$$

and an analogous equality for k_0.

Step 1. Let $f \in F_{pq}^{(s,\Psi)}(\mathbb{R}^n)$. In the first two steps we prove that the quasi-norm in (1.37) can be estimated from above by $c\|f \mid F_{pq}^{(s,\Psi)}(\mathbb{R}^n)\|$. Let $(\varphi_k)_{k\in\mathbb{N}_0}$ be the dyadic resolution of unity introduced in 1.2.2 and let $\varphi_l = 0$ if $-l \in \mathbb{N}$. We write

$$\begin{aligned} 2^{sj}\Psi(2^{-j})k(2^{-j}, f)(x) &= (2\pi)^{n/2} 2^{sj} \Psi(2^{-j}) \sum_{l=-\infty}^{M} (\check{k}(2^{-j}\cdot)\varphi_{j+l}\widehat{f}\,)^\vee(x) \\ &\quad + (2\pi)^{n/2} 2^{sj} \Psi(2^{-j}) \sum_{l=M+1}^{\infty} (\check{k}(2^{-j}\cdot)\varphi_{j+l}\widehat{f}\,)^\vee(x), \qquad j \in \mathbb{N}, \end{aligned} \tag{1.39}$$

where $M \in \mathbb{N}$ will be chosen later on. We take for granted that the convergence in (1.39) is not only in $\mathcal{S}'(\mathbb{R}^n)$ but also pointwise a.e. (to be proved later on in Step 3). We estimate the first sum in (1.39), where there is no problem of convergence, because the sum is finite ($\varphi_{j+l} = 0$ if $l < -j$). Let

$$\widetilde{\varphi}_j(x) = |2^{-j}x|^{2N} \varphi_j(x), \qquad j \in \mathbb{N}_0. \tag{1.40}$$

By Proposition 1.4(vi), there exist constants $c > 0$ and $b \geq 0$ such that

$$\Psi(2^{-j}) \leq c(1+|l|)^b\, \Psi(2^{-(j+l)}) \tag{1.41}$$

for any $j \in \mathbb{N}$ and $l \in \mathbb{Z}$. Recalling also (1.35), for $j \in \mathbb{N}$ we have

$$\begin{aligned} &\Big|\sum_{l=-\infty}^{M} 2^{sj}\Psi(2^{-j})(\check{k}(2^{-j}\cdot)\varphi_{j+l}\widehat{f}\,)^\vee(x)\Big| \\ &\qquad \leq c\sum_{l=-\infty}^{M} 2^{(2N-s)l}(1+|l|)^b|(\check{k^0}(2^{-j}z)2^{(j+l)s}\Psi(2^{-(j+l)})\widetilde{\varphi}_{j+l}(z)\widehat{f}\,)^\vee(x)|. \end{aligned} \tag{1.42}$$

But

$$\begin{aligned} &|(\check{k^0}(2^{-j}z)2^{(j+l)s}\Psi(2^{-(j+l)})\widetilde{\varphi}_{j+l}(z)\widehat{f}\,)^\vee(x)| \\ &\qquad \leq (2\pi)^{-n/2}\int\limits_{\mathbb{R}^n} |(\check{k^0}(2^{-j}\cdot))^\vee(y)|\cdot|(2^{(j+l)s}\Psi(2^{-(j+l)})\widetilde{\varphi}_{j+l}\widehat{f}\,)^\vee(x-y)|\,dy \\ &\qquad = (2\pi)^{-n/2}\int\limits_{\mathbb{R}^n} |k^0(-\xi)|\cdot|(2^{(j+l)s}\Psi(2^{-(j+l)})\widetilde{\varphi}_{j+l}\widehat{f}\,)^\vee(x-2^{-j}\xi)|\,d\xi. \end{aligned} \tag{1.43}$$

Let $a > n/\min(p,q)$. Obviously

$$|(\widetilde{\varphi}_{j+l}\widehat{f}\,)^\vee(x-2^{-j}\xi)| \leq (\widetilde{\varphi}^*_{j+l}f)_a(x)(1+|2^l\xi|^a). \tag{1.44}$$

Using (1.44) in (1.43) leads us to

$$\begin{aligned} &|(\check{k^0}(2^{-j}z)2^{(j+l)s}\Psi(2^{-(j+l)})\widetilde{\varphi}_{j+l}(z)\widehat{f}\,)^\vee(x)| \\ &\qquad \leq (2\pi)^{-n/2}2^{(j+l)s}\Psi(2^{-(j+l)})(\widetilde{\varphi}^*_{j+l}f)_a(x)\int\limits_{\mathbb{R}^n}|k^0(-\xi)|(1+|2^M\xi|^a)\,d\xi \\ &\qquad \leq c2^{(j+l)s}\Psi(2^{-(j+l)})(\widetilde{\varphi}^*_{j+l}f)_a(x) \quad \text{for } l \leq M, \end{aligned} \tag{1.45}$$

since $k^0 \in \mathcal{D}(\mathbb{R}^n)$. Putting (1.45) in (1.42) gives

$$\begin{aligned} &\Big|\sum_{l=-\infty}^{M} 2^{sj}\,\Psi(2^{-j})(\check{k}(2^{-j}\cdot)\varphi_{j+l}\widehat{f}\,)^\vee(x)\Big| \\ &\qquad \leq c\sum_{l=-\infty}^{M} 2^{(2N-s)l}(1+|l|)^b2^{(j+l)s}\Psi(2^{-(j+l)})(\widetilde{\varphi}^*_{j+l}f)_a(x), \qquad j \in \mathbb{N}. \end{aligned} \tag{1.46}$$

We first apply in (1.46) the ℓ_q-quasi-norm with respect to j and then the L_p-quasi-norm with respect to x. Because $2N > s$ we obtain

$$\begin{aligned} &\Big\|\Big(\sum_{j=1}^{\infty}\Big|\sum_{l=-\infty}^{M} 2^{sj}\Psi(2^{-j})(\check{k}(2^{-j}\cdot)\varphi_{j+l}\widehat{f}\,)^\vee(x)\Big|^q\Big)^{1/q}\Big|\,L_p(\mathbb{R}^n)\Big\| \\ &\qquad \leq c\Big\|\Big(\sum_{m=0}^{\infty} 2^{smq}\Psi(2^{-m})^q(\widetilde{\varphi}^*_m f)^q_a(x)\Big)^{1/q}\Big|\,L_p(\mathbb{R}^n)\Big\|. \end{aligned} \tag{1.47}$$

We use Theorem 2.2.4(i) of [Tri92] to estimate the right-hand side of (1.47). Notice that

$$2^{sm}\Psi(2^{-m})(\widetilde{\varphi}^*_m f)_a(x) \leq 2^{2a}\sup_{z\in\mathbb{R}^n}\frac{|(2^{sm}\Psi(2^{-m})\widetilde{\varphi}_m\widehat{f}\,)^\vee(x-z)|}{1+|2^{m+2}z|^a}. \tag{1.48}$$

Since $a > n/\min(p,q)$, the number $r = n/a$ satisfies $0 < r < \min(p,q)$. In order to apply that theorem we must be sure that $(2^{sm}\Psi(2^{-m})(\widetilde{\varphi}_m \widehat{f}\,)^\vee)_{m=0}^\infty$ belongs to $L_p(\ell_q)$. But this is a consequence of Theorem 2.2.4(ii) of [Tri92]. In fact, we have

$$2^{sm}\Psi(2^{-m})(\widetilde{\varphi}_m \widehat{f}\,)^\vee = (|2^{-m}z|^{2N} H(2^{-m}z) 2^{sm}\Psi(2^{-m})\varphi_m(z)\widehat{f}\,)^\vee, \qquad m \in \mathbb{N}_0, \tag{1.49}$$

where H is a function in $\mathcal{D}(\mathbb{R}^n)$ such that

$$H(x) = 1 \quad \text{if } |x| \le 2. \tag{1.50}$$

Take

$$M_m(z) = |2^{-m}z|^{2N} H(2^{-m}z), \qquad z \in \mathbb{R}^n,\ m \in \mathbb{N}_0, \tag{1.51}$$

and choose $\kappa > n/2 + n/\min(p,q)$. Then

$$\sup_{m\in\mathbb{N}_0} \|M_m(2^{m+2}\cdot) \,|\, H_2^\kappa(\mathbb{R}^n)\| = \big\| |4z|^{2N} H(4z) \,\big|\, H_2^\kappa(\mathbb{R}^n)\big\| < \infty, \tag{1.52}$$

since $|4z|^{2N} H(4z) \in \mathcal{D}(\mathbb{R}^n)$. Because $f \in F_{pq}^{(s,\Psi)}(\mathbb{R}^n)$, we have

$$(2^{sm}\Psi(2^{-m})(\varphi_m \widehat{f}\,)^\vee)_{m=0}^\infty \in L_p(\ell_q). \tag{1.53}$$

From (1.49), (1.51)–(1.53) and Theorem 2.2.4(ii) of [Tri92], there exists a positive constant c such that

$$\Big\| \Big(\sum_{m=0}^\infty 2^{smq}\Psi(2^{-m})^q |(\widetilde{\varphi}_m \widehat{f}\,)^\vee(\cdot)|^q \Big)^{1/q} \,\Big|\, L_p(\mathbb{R}^n) \Big\| \le c \|f \,|\, F_{pq}^{(s,\Psi)}(\mathbb{R}^n)\|. \tag{1.54}$$

By (1.48) and (1.54), applying Theorem 2.2.4(i) of [Tri92], we get

$$\Big\| \Big(\sum_{m=0}^\infty 2^{smq}\Psi(2^{-m})^q (\widetilde{\varphi}_m^* f)_a^q(\cdot) \Big)^{1/q} \,\Big|\, L_p(\mathbb{R}^n) \Big\| \le c \|f \,|\, F_{pq}^{(s,\Psi)}(\mathbb{R}^n)\|. \tag{1.55}$$

Finally, by (1.55) and (1.47), we obtain

$$\Big\| \Big(\sum_{j=1}^\infty \Big| \sum_{l=-\infty}^M 2^{sj}\Psi(2^{-j})(\check{k}(2^{-j}\cdot)\varphi_{j+l}\widehat{f}\,)^\vee(\cdot) \Big|^q \Big)^{1/q} \,\Big|\, L_p(\mathbb{R}^n) \Big\| \le c \|f \,|\, F_{pq}^{(s,\Psi)}(\mathbb{R}^n)\|. \tag{1.56}$$

Step 2. We estimate the second sum in (1.39) and we have to make sure now that (1.39) converges a.e. and in some $L_r(\mathbb{R}^n)$ with $1 \le r \le \infty$. However the latter comes as a by-product. Let $s_0 \in \mathbb{R}$ be such that

$$s_0 + 2\sigma_{pq} < s, \tag{1.57}$$

and introduce

$$\varphi'_j(x) = |2^{-j}x|^{s_0}\varphi_j(x), \qquad x \in \mathbb{R}^n,\ j \in \mathbb{N}. \tag{1.58}$$

By (1.35) and (1.41) we have

$$\begin{aligned} &\Big| \sum_{l=M+1}^\infty 2^{sj}\Psi(2^{-j})(\check{k}(2^{-j}\cdot)\varphi_{j+l}\widehat{f}\,)^\vee(x) \Big| \\ &\quad \le c \sum_{l=M+1}^\infty 2^{(s_0-s)l}(1+l)^b |(\check{k}^0(2^{-j}z)|2^{-j}z|^{2N-s_0} 2^{(j+l)s}\Psi(2^{-(j+l)})\varphi'_{j+l}\widehat{f}\,)^\vee(x)|. \end{aligned} \tag{1.59}$$

Let χ be a function in $\mathcal{D}(\mathbb{R}^n)$ such that

$$\chi(x) = 1 \quad \text{if } 1/2 \le |x| \le 2 \quad \text{and} \quad \operatorname{supp}\chi \subset \{\xi \in \mathbb{R}^n : 1/4 \le |\xi| \le 4\}. \tag{1.60}$$

Each term in (1.59) can be estimated from above as follows:

$$\begin{aligned} &|(\check{k^0}(2^{-j}z)|2^{-j}z|^{2N-s_0}2^{(j+l)s}\Psi(2^{-(j+l)})\varphi'_{j+l}(z)\widehat{f}\,)^\vee(x)| \\ &\le (2\pi)^{-n/2}\int_{\mathbb{R}^n}|(\check{k^0}(2^{-j}\cdot)|2^{-j}\cdot|^{2N-s_0}\chi(2^{-j-l}\cdot))^\vee(y)| \\ &\quad \times |(2^{(j+l)s}\Psi(2^{-(j+l)})\varphi'_{j+l}\widehat{f}\,)^\vee(x-y)|\,dy \\ &\le 2(2\pi)^{-n/2}2^{(j+l)s}\Psi(2^{-(j+l)})(\varphi'^*_{j+l}f)_a(x)\int_{\mathbb{R}^n}\left|\left(\frac{\check{k}(2^l\cdot)}{|2^l\cdot|^{s_0}}\chi(\cdot)\right)^\vee(\xi)\right|(1+|\xi|)^a\,d\xi, \end{aligned} \tag{1.61}$$

with a as in Step 1. By Theorem 4.1 of [Far00] the integral in (1.61) can be estimated from above as follows:

$$\begin{aligned} &\int_{\mathbb{R}^n}\left|\left(\frac{\check{k}(2^l z)}{|2^l z|^{s_0}}\chi(z)\right)^\vee(\xi)\right|(1+|\xi|)^a\,d\xi \\ &\qquad \le c2^{-ls_0}\|\check{k}(2^l z)\chi(z)\,|\,H_2^\lambda(\mathbb{R}^n)\|\left\|(1+|\xi|)^a\left(\frac{h(z)}{|z|^{s_0}}\right)^\vee(\xi)\,\Big|\,L_1(\mathbb{R}^n)\right\|, \end{aligned} \tag{1.62}$$

where $h \in \mathcal{D}(\mathbb{R}^n)$ is such that

$$h(x) = 1 \quad \text{if } 1/4 \le |x| \le 4 \quad \text{and} \quad \operatorname{supp} h \subset \{\xi \in \mathbb{R}^n : 1/8 \le |\xi| \le 8\}, \tag{1.63}$$

and $\lambda > a + n/2$. The second factor in (1.62) is obviously constant since $h(z)/|z|^{s_0} \in \mathcal{D}(\mathbb{R}^n)$. For the other factor in (1.62) it can be proved that

$$\sup_{l\in\mathbb{N}} 2^{-ls_0}\|\check{k}(2^l z)\chi(z)\,|\,H_2^\lambda(\mathbb{R}^n)\| < \infty \tag{1.64}$$

(see Remark 1.11). From what has been said and from (1.61), (1.62), there exists a positive constant c such that

$$\begin{aligned} &|(\check{k^0}(2^{-j}z)|2^{-j}z|^{2N-s_0}2^{(j+l)s}\Psi(2^{-(j+l)})\varphi'_{j+l}(z)\widehat{f})^\vee(x)| \\ &\qquad \le c2^{(j+l)s}\Psi(2^{-(j+l)})(\varphi'^*_{j+l}f)_a(x). \end{aligned} \tag{1.65}$$

Applying (1.65) in (1.59) we obtain

$$\begin{aligned} &\Big|\sum_{l=M+1}^{\infty}2^{sj}\Psi(2^{-j})(\check{k}(2^{-j}\cdot)\varphi_{j+l}\widehat{f}\,)^\vee(x)\Big| \\ &\qquad \le c\sum_{l=M+1}^{\infty}2^{(s_0-s)l}(1+l)^b2^{(j+l)s}\Psi(2^{-(j+l)})(\varphi'^*_{j+l}f)_a(x), \qquad j \in \mathbb{N}. \end{aligned} \tag{1.66}$$

We take in (1.66) first the ℓ_q-quasi-norm with respect to j and afterwards the L_p-quasi-norm with respect to x. Since $s > s_0$ we get

$$(1.67)\quad \Big\|\Big(\sum_{j=1}^{\infty}\Big|\sum_{l=M+1}^{\infty} 2^{sj}\Psi(2^{-j})(\check{k}(2^{-j}\cdot)\varphi_{j+l}\widehat{f})^{\vee}(x)\Big|^{q}\Big)^{1/q}\Big| L_p(\mathbb{R}^n)\Big\| \le c2^{(s_0-s)M/2}\Big\|\Big(\sum_{m=1}^{\infty} 2^{smq}\Psi(2^{-m})^q(\varphi_m'^{*}f)_a^q(x)\Big)^{1/q}\Big| L_p(\mathbb{R}^n)\Big\|.$$

Acting as in Step 1, from (1.67) we obtain

$$(1.68)\quad \Big\|\Big(\sum_{j=1}^{\infty}\Big|\sum_{l=M+1}^{\infty} 2^{sj}\Psi(2^{-j})(\check{k}(2^{-j}\cdot)\varphi_{j+l}\widehat{f})^{\vee}(x)\Big|^{q}\Big)^{1/q}\Big| L_p(\mathbb{R}^n)\Big\| \le c2^{(s_0-s)M/2}\|f\,|\,F_{pq}^{(s,\Psi)}(\mathbb{R}^n)\|.$$

Now by (1.39), (1.56) and (1.68), using the quasi-triangular inequality in the space $L_p(\ell_q)$, we get

$$(1.69)\quad \Big\|\Big(\sum_{j=1}^{\infty} 2^{sjq}\Psi(2^{-j})^q\,|k(2^{-j},f)(x)|^q\Big)^{1/q}\Big| L_p(\mathbb{R}^n)\Big\| \le c\|f\,|\,F_{pq}^{(s,\Psi)}(\mathbb{R}^n)\|.$$

In an analogous way one can prove that

$$(1.70)\quad \|k_0(2^{-h},f)\,|\,L_p(\mathbb{R}^n)\| \le c\|f\,|\,F_{pq}^{(s,\Psi)}(\mathbb{R}^n)\|.$$

With (1.69) and (1.70) we have proved one of the desired inequalities between the quasi-norm (1.37) and $\|\cdot\,|\,F_{pq}^{(s,\Psi)}(\mathbb{R}^n)\|$.

Step 3. We have to care about the convergence on the right-hand side of (1.39) pointwise a.e. and in some $L_r(\mathbb{R}^n)$, $1\le r\le\infty$. We can rewrite (1.66) as follows:

$$(1.71)\quad \Big|\sum_{l=M+1}^{L} 2^{sj}\Psi(2^{-j})(\check{k}(2^{-j}\cdot)\varphi_{j+l}\widehat{f})^{\vee}(x)\Big| \le c\sum_{l=M+1}^{L} 2^{(s_0-s)l}(1+l)^b 2^{(j+l)s}\Psi(2^{-(j+l)})(\varphi_{j+l}'^{*}f)_a(x),\qquad j\in\mathbb{N},$$

with $L>M$. Using $s_0-s<0$ and $\ell_q\hookrightarrow\ell_1$ if $0<q\le 1$, or the Hölder inequality if $1<q\le\infty$, we conclude that if M is large enough then the right-hand side of (1.71) can be estimated from above by

$$(1.72)\quad \varepsilon\Big(\sum_{l=M+1}^{\infty} 2^{lsq}\Psi(2^{-l})^q(\varphi_l'^{*}f)_a^q(x)\Big)^{1/q},$$

for given $\varepsilon>0$. Because $f\in F_{pq}^{(s,\Psi)}(\mathbb{R}^n)$, Theorem 1.7 and considerations as in Step 2 give us

$$\Big\|\Big(\sum_{m=1}^{\infty} 2^{smq}\Psi(2^{-m})^q\,(\varphi_m'^{*}f)_a(x)\Big)^{1/q}\Big| L_p(\mathbb{R}^n)\Big\| < \infty.$$

Therefore, the expression in (1.72) is finite a.e. and this proves the desired pointwise convergence. Next we prove the $\mathcal{S}'$ convergence and assume $0<p<1$. Let $\sigma=s-\sigma_p$.

Putting this in (1.71) we obtain

$$(1.73)\quad \Big|\sum_{l=M+1}^{L} 2^{\sigma j}\Psi(2^{-j})(\check{k}(2^{-j}\cdot)\varphi_{j+l}\widehat{f}\,)^{\vee}(x)\Big| \le c\sum_{l=M+1}^{L} 2^{(s_0-\sigma)l}(1+l)^b 2^{(j+l)\sigma}\Psi(2^{-(j+l)})(\varphi'^{*}_{j+l}f)_a(x).$$

From (1.57) and $\sigma_{pq} \ge \sigma_p$ (recall (1.36)), we have $\sigma = s - \sigma_p > s_0$. Using this last inequality instead of $s > s_0$, and proceeding as to obtain (1.72) we conclude that there exists M sufficiently large such that the right-hand side of (1.73) can be estimated from above by

$$(1.74)\qquad \varepsilon\Big(\sum_{l=M+1}^{\infty} 2^{l\sigma q}\Psi(2^{-l})^q(\varphi'^{*}_l f)^q_a(x)\Big)^{1/q},$$

for given $\varepsilon > 0$. From the embedding $F^{(s,\Psi)}_{p,q}(\mathbb{R}^n) \hookrightarrow F^{(\sigma,\Psi)}_{1,q}(\mathbb{R}^n)$, a consequence of Proposition 1.9(v), we have

$$\Big\|\Big(\sum_{m=1}^{\infty} 2^{m\sigma q}\Psi(2^{-m})^q(\varphi'^{*}_m f)^q_a(\cdot)\Big)^{1/q}\Big|\, L_1(\mathbb{R}^n)\Big\| < \infty.$$

And then, from (1.73) and (1.74),

$$\Big\|\sum_{l=M+1}^{L} 2^{\sigma j}\Psi(2^{-j})(\check{k}(2^{-j}\cdot)\varphi_{j+l}\widehat{f}\,)^{\vee}(\cdot)\,|\,L_1(\mathbb{R}^n)\Big\| \le \varepsilon\Big\|\Big(\sum_{l=M+1}^{\infty} 2^{l\sigma q}\Psi(2^{-l})^q(\varphi'^{*}_l f)^q_a(\cdot)\Big)^{1/q}\Big|\, L_1(\mathbb{R}^n)\Big\|,$$

for given $\varepsilon > 0$. It follows that (1.73) and hence (1.39) converges in $L_1(\mathbb{R}^n)$.

If $1 \le p < \infty$, then by (1.71) and (1.72),

$$\Big\|\sum_{l=M+1}^{L} 2^{sj}\Psi(2^{-j})(\check{k}(2^{-j}\cdot)\varphi_{j+l}\widehat{f}\,)^{\vee}(\cdot)\Big|\, L_p(\mathbb{R}^n)\Big\| \le \varepsilon\Big\|\Big(\sum_{l=M+1}^{\infty} 2^{lsq}\Psi(2^{-l})^q(\varphi'^{*}_l f)^q_a(\cdot)\Big)^{1/q}\Big|\, L_p(\mathbb{R}^n)\Big\|,$$

where $\varepsilon > 0$ is given. So (1.71) and hence (1.39) converges in $L_p(\mathbb{R}^n)$, therefore in $\mathcal{S}'(\mathbb{R}^n)$.

Step 4. Let $f \in F^{(s,\Psi)}_{pq}(\mathbb{R}^n)$. We now want to prove that $\|f\,|\,F^{(s,\Psi)}_{pq}(\mathbb{R}^n)\|$ can be estimated from above by the quasi-norm in (1.37). By hypothesis $\widehat{k_0}(0) \ne 0$ and $\widehat{k^0}(0) \ne 0$. Then also $\check{k}_0(0) \ne 0$ and $\check{k}^0(0) \ne 0$. Since $k_0, k^0 \in \mathcal{S}(\mathbb{R}^n)$, $\check{k}_0, \check{k}^0 \in \mathcal{S}(\mathbb{R}^n)$ are C^∞ functions. So, there exists a neighbourhood of the origin where both $\check{k}_0$ and $\check{k}^0$ are non-zero. Recall (1.35). Therefore, there exists $\varepsilon > 0$ such that

$$(1.75)\quad \check{k}^0(x) \ne 0 \text{ for } |x| \le 2\varepsilon,\ \ \check{k}_0(x) \ne 0 \text{ for } |x| \le 2\varepsilon \text{ and } \check{k}(x) \ne 0 \text{ for } \varepsilon/2 \le |x| \le 2\varepsilon.$$

If useful one can choose ε to be of the form $\varepsilon = 2^{-h}$ for some fixed $h \in \mathbb{N}_0$. Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ be a function with

$$\operatorname{supp}\phi \subset \{\xi \in \mathbb{R}^n : |\xi| \le 2^{M+1}\} \quad\text{and}\quad \phi(x) = 1 \quad \text{if } |x| \le 2^M,$$

where the natural number M will be chosen later on. By (1.6), (1.8) and (1.75), we have

$$|(\varphi_j\widehat{f}\,)^\vee(x)| = |(\varphi_j\phi(2^{-j}\cdot)\widehat{f}\,)^\vee(x)| \le (2\pi)^{-n/2}\int_{\mathbb{R}^n}\left|\left(\frac{\varphi_j}{\check{k}(\varepsilon 2^{-j}\cdot)}\right)^\vee(y)(\check{k}(\varepsilon 2^{-j}\cdot)\phi(2^{-j}\cdot)\widehat{f}\,)^\vee(x-y)\right|dy, \quad j\in\mathbb{N}. \tag{1.76}$$

A corresponding estimate holds for $j=0$, in this case with k_0 instead of k. We assume this latter modification for $j=0$ throughout this step. For fixed $x\in\mathbb{R}^n$ the Fourier transform of the y-function in the integral in (1.76) has support contained in $\{\xi\in\mathbb{R}^n : |\xi|\le 2^{M+j+2}\}$. Let $0<r<\min(1,p,q)$. Using an inequality of Plancherel–Pólya–Nikol'skiĭ type as in [Tri83, 1.3.2/(5)] we obtain

$$|(\varphi_j\widehat{f}\,)^\vee(x)|^r \le c2^{(M+j)n(1-r)} \times\int_{\mathbb{R}^n}\left|\left(\frac{\varphi_j}{\check{k}(\varepsilon 2^{-j}\cdot)}\right)^\vee(y)(\check{k}(\varepsilon 2^{-j}\cdot)\phi(2^{-j}\cdot)\widehat{f}\,)^\vee(x-y)\right|^r dy, \quad j\in\mathbb{N}_0. \tag{1.77}$$

If $j\in\mathbb{N}$, then $\varphi_j(x)=\overline{\varphi}(2^{-j}x)$ with $\overline{\varphi}(x)=\varphi_0(x)-\varphi_0(2x)$ (see (1.7)), hence

$$\left|\left(\frac{\varphi_j}{\check{k}(\varepsilon 2^{-j}\cdot)}\right)^\vee(y)\right|^r = 2^{jnr}\left|\left(\frac{\overline{\varphi}}{\check{k}(\varepsilon\cdot)}\right)^\vee(2^jy)\right|^r \le c_\eta 2^{jnr}(1+|2^jy|)^{-\eta}, \tag{1.78}$$

where $\eta\in\mathbb{N}$ is at our disposal, since $(\overline{\varphi}/\check{k}(\varepsilon\cdot))^\vee\in\mathcal{S}(\mathbb{R}^n)$. Putting (1.78) in (1.77) leads to

$$|(\varphi_j\widehat{f}\,)^\vee(x)|^r \le c'_\eta 2^{(M+j)n(1-r)+jnr} \times\sum_{l=0}^\infty 2^{-\eta l}\int_{\{\xi\in\mathbb{R}^n:|\xi|\le 2^{-j+l}\}}|(\check{k}(\varepsilon 2^{-j}\cdot)\phi(2^{-j}\cdot)\widehat{f}\,)^\vee(x-y)|^r\,dy. \tag{1.79}$$

Now we estimate from above each integral in (1.79):

$$\int_{\{\xi\in\mathbb{R}^n:|\xi|\le 2^{-j+l}\}}|(\check{k}(\varepsilon 2^{-j}\cdot)\phi(2^{-j}\cdot)\widehat{f}\,)^\vee(x-y)|^r\,dy \le 2^{(-j+l)n}\mathcal{M}[|(\check{k}(\varepsilon 2^{-j}\cdot)\phi(2^{-j}\cdot)\widehat{f}\,)^\vee|^r](x) \tag{1.80}$$

where $\mathcal{M}$ stands for the Hardy–Littlewood maximal function. We apply this estimate in (1.79), and choosing $\eta\in\mathbb{N}$ such that $\eta>n$ we arrive at

$$|(\varphi_j\widehat{f}\,)^\vee(x)|^r \le c2^{Mn(1-r)}\,\mathcal{M}[|(\check{k}(\varepsilon 2^{-j}\cdot)\phi(2^{-j}\cdot)\widehat{f}\,)^\vee|^r](x). \tag{1.81}$$

Since $0<r<\min(1,p,q)$, we have $1<p/r<\infty$ and $1<q/r\le\infty$. We multiply (1.81) with $2^{sjr}\Psi(2^{-j})^r$, apply the $\ell_{q/r}$-norm with respect to j and afterwards the $L_{p/r}$-norm with respect to x; then by Theorem 2.2.2 of [Tri92] we obtain

$$\begin{aligned}\left\|\Big(\sum_{j=0}^\infty 2^{sjq}\Psi(2^{-j})^q|(\varphi_j\widehat{f}\,)^\vee(x)|^q\Big)^{1/q}\,\Big|\,L_p(\mathbb{R}^n)\right\|^r &\le c2^{Mn(1-r)}\|2^{sjr}\Psi(2^{-j})^r|(\check{k}(\varepsilon 2^{-j}\cdot)\phi(2^{-j}\cdot)\widehat{f}\,)^\vee|^r\,|\,L_{p/r}(\ell_{q/r})\|\\ &= c2^{Mn(1-r)}\left\|\Big(\sum_{j=0}^\infty 2^{sjq}\Psi(2^{-j})^q|(\check{k}(\varepsilon 2^{-j}\cdot)\phi(2^{-j}\cdot)\widehat{f}\,)^\vee(\cdot)|^q\Big)^{1/q}\,\Big|\,L_p(\mathbb{R}^n)\right\|^r,\end{aligned} \tag{1.82}$$

where c is a positive constant independent of M. Because

$$\check{k}(\varepsilon 2^{-j}\cdot)\phi(2^{-j}\cdot) = \check{k}(\varepsilon 2^{-j}\cdot) - \check{k}(\varepsilon 2^{-j}\cdot)(1-\phi(2^{-j}\cdot)),$$

and using the quasi-triangular inequality in $L_p(\ell_q)$, the right-hand side of (1.82) can be estimated from above by

$$\begin{aligned}(1.83)\qquad & c2^{Mn(1-r)}\Big\|\Big(\sum_{j=0}^{\infty} 2^{sjq}\Psi(2^{-j})^q |(\check{k}(\varepsilon 2^{-j}\cdot)\widehat{f})^\vee(\cdot)|^q\Big)^{1/q} \Big| L_p(\mathbb{R}^n)\Big\|^r \\ & + c2^{Mn(1-r)}\Big\|\Big(\sum_{j=0}^{\infty} 2^{sjq}\Psi(2^{-j})^q |(\check{k}(\varepsilon 2^{-j}\cdot)(1-\phi(2^{-j}\cdot))\widehat{f})^\vee(\cdot)|^q\Big)^{1/q} \Big| L_p(\mathbb{R}^n)\Big\|^r.\end{aligned}$$

The first term in (1.83) is precisely what we want. The additional term in (1.83) can be treated as in Step 2 and estimated from above by

$$(1.84)\qquad c2^{Mn(1-r)}2^{(s_0-s)Mr/2}\|f\,|\,F^{(s,\Psi)}_{pq}(\mathbb{R}^n)\|^r.$$

By (1.57), we may choose r such that

$$n\Big(\frac{1}{r}-1\Big) + \Big(\frac{s_0-s}{2}\Big) < 0.$$

Recall that the natural number M is at our disposal. We take M large enough so that (1.84) can be estimated from above by

$$\frac{1}{2}\|f\,|\,F^{(s,\Psi)}_{pq}(\mathbb{R}^n)\|^r.$$

Applying this, (1.84) and (1.83) in (1.82) gives

$$\begin{aligned}(1.85)\qquad \|f\,|\,F^{(s,\Psi)}_{pq}(\mathbb{R}^n)\| \le c\Big[& \|(\check{k}_0(\varepsilon\cdot)\widehat{f})^\vee\,|\,L_p(\mathbb{R}^n)\| \\ & + \Big\|\Big(\sum_{j=1}^{\infty} 2^{sjq}\Psi(2^{-j})^q |(\check{k}(\varepsilon 2^{-j}\cdot)\widehat{f})^\vee(\cdot)|^q\Big)^{1/q} \Big| L_p(\mathbb{R}^n)\Big\|\Big].\end{aligned}$$

As mentioned at the beginning of Step 4, we can take $\varepsilon = 2^{-h}$, for some fixed $h \in \mathbb{N}_0$. Therefore

$$\begin{aligned}\|f\,|\,F^{(s,\Psi)}_{pq}(\mathbb{R}^n)\| \le c\Big[& \|k_0(2^{-h}, f)\,|\,L_p(\mathbb{R}^n)\| \\ & + \Big\|\Big(\sum_{j=1}^{\infty} 2^{sjq}\Psi(2^{-j})^q |(\check{k}(2^{-j}\cdot)\widehat{f})^\vee(\cdot)|^q\Big)^{1/q} \Big| L_p(\mathbb{R}^n)\Big\|\Big],\end{aligned}$$

which completes the proof. ■

Remark 1.11. We prove (1.64). Recall that the function $\chi \in \mathcal{D}(\mathbb{R}^n)$ satisfies (1.60). Let

$$\Omega = \{\xi \in \mathbb{R}^n : 1/4 \le |\xi| \le 4\} \quad \text{and} \quad \chi_l(x) = \check{k}(2^l x)\chi(x), \qquad l \in \mathbb{N}.$$

If $m \in \mathbb{N}$ is so large that $m > 1 + [\lambda]$, then there exists a constant $c > 0$ such that

$$(1.86)\qquad \|\chi_l\,|\,H^\lambda_2(\mathbb{R}^n)\| \le c \sum_{|\alpha|\le m} \|D^\alpha \chi_l\,|\,L_\infty(\mathbb{R}^n)\|.$$

For $|\alpha| \le m$ and $x \in \Omega$,

$$(1.87) \qquad |(D^\alpha \chi_l)(x)| \le \sum_{\beta \le \alpha} \binom{\alpha}{\beta} |(D^\beta \check{k})(2^l x)| 2^{l|\beta|} |(D^{\alpha-\beta}\chi)(x)| \\ \le c 2^{l|\alpha|} \Big(\sum_{\beta \le \alpha} \binom{\alpha}{\beta} |(D^\beta \check{k})(2^l x)| \Big),$$

since $\chi \in \mathcal{D}(\mathbb{R}^n)$. Let $m_1 \in \mathbb{N}$ be so large that $m - m_1 \le s_0$. As $\check{k} \in \mathcal{S}(\mathbb{R}^n)$, there exists a constant $c > 0$ such that

$$(1.88) \qquad |(D^\beta \check{k})(x)| \le c(1+|x|)^{-m_1}, \qquad \forall x \in \mathbb{R}^n,\ \forall \beta \in \mathbb{N}_0^n : |\beta| \le m.$$

Putting (1.88) in (1.87), we arrive at

$$|(D^\alpha \chi_l)(x)| \le c 2^{l|\alpha|} \Big(\max_{|\alpha| \le m} \sum_{\beta \le \alpha} \binom{\alpha}{\beta} \Big) (1 + 2^l |x|)^{-m_1} \le c' 2^{2m_1} 2^{l(m-m_1)} \le c' 2^{2m_1} 2^{ls_0}.$$

So,

$$\sum_{|\alpha| \le m} \|D^\alpha \chi_l \,|\, L_\infty(\mathbb{R}^n)\| \le c 2^{ls_0}.$$

This in (1.86) gives (1.64).

Theorem 1.12. *Let $0 < p, q \le \infty$, $s \in \mathbb{R}$ and Ψ an admissible function. Let $N \in \mathbb{N}$ with $2N > s$. Then there exists $h \in \mathbb{N}_0$ such that*

$$(1.89) \qquad \|k_0(2^{-h}, f) \,|\, L_p(\mathbb{R}^n)\| + \Big(\sum_{j=1}^\infty 2^{jsq} \Psi(2^{-j})^q \|k(2^{-j}, f)(\cdot) \,|\, L_p(\mathbb{R}^n)\|^q \Big)^{1/q}$$

(with the usual modification if $q = \infty$) is an equivalent quasi-norm in $B_{pq}^{(s,\Psi)}(\mathbb{R}^n)$.

Proof. This is the counterpart of Theorem 1.10 for $B_{pq}^{(s,\Psi)}(\mathbb{R}^n)$; we modify its proof.

Step 1. We again have the splitting (1.39) and the estimate (1.42). But from (1.42) we still have

$$(1.90) \qquad \Big| \sum_{l=-\infty}^{M} 2^{sj} \Psi(2^{-j}) (\check{k}(2^{-j}\cdot) \varphi_{j+l} \widehat{f}\,)^\vee(x) \Big| \\ \le c \sum_{l=-\infty}^{M} 2^{(2N-s)l} (1+|l|)^b 2^{jn} |[k^0(-2^j .) * (2^{(j+l)s} \Psi(2^{-(j+l)}) \widetilde{\varphi}_{j+l} \widehat{f}\,)^\vee](x)|.$$

Let first $1 \le p \le \infty$. We apply the L_p-norm to (1.90), use the triangle inequality and Young's inequality and obtain

$$(1.91) \qquad \Big\| \sum_{l=-\infty}^{M} 2^{sj} \Psi(2^{-j}) (\check{k}(2^{-j}\cdot) \varphi_{j+l} \widehat{f}\,)^\vee(\cdot) \,\Big|\, L_p(\mathbb{R}^n) \Big\| \\ \le c \sum_{l=-\infty}^{M} 2^{(2N-s)l} (1+|l|)^b \|(2^{(j+l)s} \Psi(2^{-(j+l)}) \widetilde{\varphi}_{j+l} \widehat{f}\,)^\vee \,|\, L_p(\mathbb{R}^n)\|,$$

since $k^0 \in \mathcal{S}(\mathbb{R}^n) \subset L_1(\mathbb{R}^n)$. Applying the ℓ_q-quasi-norm in (1.91), because $2N > s$, we get

$$\Big(\sum_{j=1}^{\infty}\Big\|\sum_{l=-\infty}^{M}2^{sj}\varPsi(2^{-j})(\check{k}(2^{-j}\cdot)\varphi_{j+l}\widehat{f})^{\vee}(\cdot)\,\Big|\,L_p(\mathbb{R}^n)\Big\|^q\Big)^{1/q} \\ \le c\Big(\sum_{m=0}^{\infty}2^{smq}\varPsi(2^{-m})^q\|(\widetilde{\varphi}_m\widehat{f})^{\vee}\,|\,L_p(\mathbb{R}^n)\|^q\Big)^{1/q}. \tag{1.92}$$

Now let $0<p<1$. For each term in (1.42) we have

$$|(\check{k^0}(2^{-j}\cdot)2^{(j+l)s}\varPsi(2^{-(j+l)})\widetilde{\varphi}_{j+l}\widehat{f})^{\vee}(x)| \\ \le (2\pi)^{-n/2}\int_{\mathbb{R}^n}|(\check{k^0}(2^{-j}\cdot)\alpha(c_12^{-j}\cdot))^{\vee}(y)(2^{(j+l)s}\varPsi(2^{-(j+l)})\widetilde{\varphi}_{j+l}\widehat{f})^{\vee}(x-y)|\,dy \tag{1.93}$$

where $c_1=2^{(M+1)}$ and $\alpha\in\mathcal{D}(\mathbb{R}^n)$ is such that

$$\alpha(x)=1 \text{ if } |x|\le 1 \quad\text{and}\quad \operatorname{supp}\alpha\subset\{x\in\mathbb{R}^n : |x|\le 2\}.$$

The Fourier transform of the y-function inside the integral in (1.93) has compact support contained in $\{\xi\in\mathbb{R}^n : |\xi|\le 6\cdot 2^{j+M}\}$. Since now $0<p<1$, we apply an inequality of Plancherel–Pólya–Nikol'skiĭ type (cf. [Tri83, 1.3.2/5]), and obtain

$$\int_{\mathbb{R}^n}|(\check{k^0}(2^{-j}\cdot)\alpha(c_12^{-j}\cdot))^{\vee}(y)(2^{(j+l)s}\varPsi(2^{-(j+l)})\widetilde{\varphi}_{j+l}\widehat{f})^{\vee}(x-y)|\,dy \\ \le c_2 2^{(j+M)n(1/p-1)} \\ \times\Big[\int_{\mathbb{R}^n}|(\check{k^0}(2^{-j}\cdot)\alpha(c_12^{-j}\cdot))^{\vee}(y)(2^{(j+l)s}\varPsi(2^{-(j+l)})\widetilde{\varphi}_{j+l}\widehat{f})^{\vee}(x-y)|^p\,dy\Big]^{1/p} \tag{1.94}$$

where the positive constant c_2 is independent of j. Putting (1.94) together with (1.93) in (1.42) and then applying the L_p-quasi-norm, we get

$$\Big\|\sum_{l=-\infty}^{M}2^{sj}\varPsi(2^{-j})(\check{k}(2^{-j}\cdot)\varphi_{j+l}\widehat{f}))^{\vee}(\cdot)\,\Big|\,L_p(\mathbb{R}^n)\Big\| \\ \le c2^{Mn(1/p-1)}\Big[\sum_{l=-\infty}^{M}2^{(2N-s)pl}(1+|l|)^{bp}\|2^{(j+l)s}\varPsi(2^{-(j+l)})(\widetilde{\varphi}_{j+l}\widehat{f})^{\vee}(\cdot)\,|\,L_p(\mathbb{R}^n)\|^p\Big]^{1/p}. \tag{1.95}$$

We have used $(\check{k^0}\alpha(c_1\cdot))^{\vee}\in\mathcal{S}(\mathbb{R}^n)$. Recall that $1/p>1$. Let p_1 be its conjugate exponent. Because $2N>s$ and using Hölder's inequality we estimate the right-hand side of (1.95) by

$$c2^{Mn(1/p-1)}\Big(\sum_{l=-\infty}^{M}2^{(2N-s)pp_1l/2}(1+|l|)^{bpp_1}\Big)^{1/(p_1p)} \\ \times\sum_{l=-\infty}^{M}2^{(2N-s)l/2}\|2^{(j+l)s}\varPsi(2^{-(j+l)})(\widetilde{\varphi}_{j+l}\widehat{f})^{\vee}(\cdot)\,|\,L_p(\mathbb{R}^n)\| \\ \le c'2^{Mn(1/p-1)}\sum_{l=-\infty}^{M}2^{(2N-s)l/2}\|2^{(j+l)s}\varPsi(2^{-(j+l)})(\widetilde{\varphi}_{j+l}\widehat{f})^{\vee}(\cdot)\,|\,L_p(\mathbb{R}^n)\|. \tag{1.96}$$

Putting (1.96) in (1.95) and applying the ℓ_q-quasi-norm with respect to j we arrive at

$$(1.97)\quad \Big(\sum_{j=1}^{\infty}\Big\|\sum_{l=-\infty}^{M}2^{sj}\Psi(2^{-j})(\check{k}(2^{-j}\cdot)\varphi_{j+l}\widehat{f})^{\vee}(\cdot)\,\Big|\,L_p(\mathbb{R}^n)\Big\|^q\Big)^{1/q} \\ \leq c2^{Mn(1/p-1)}\Big(\sum_{m=0}^{\infty}2^{smq}\Psi(2^{-m})^q\|(\widetilde{\varphi}_m\widehat{f})^{\vee}\,|\,L_p(\mathbb{R}^n)\|^q\Big)^{1/q}.$$

This was already obtained also in case $1\leq p\leq\infty$, in (1.92). Recall that

$$(\widetilde{\varphi}_m\widehat{f})^{\vee}=(|2^{-m}\cdot|^{2N}H(2^{-m}\cdot)[(\varphi_m\widehat{f})^{\vee}]^{\wedge})^{\vee},\qquad m\in\mathbb{N}_0,$$

with $H\in\mathcal{D}(\mathbb{R}^n)$ as in (1.50). As a consequence of Theorem 1.5.2 of [Tri83, (13)], for $\nu>n(1/\min(p,1)-1/2)$, we have

$$\|(\widetilde{\varphi}_m\widehat{f})^{\vee}\,|\,L_p(\mathbb{R}^n)\|\leq c\||2\cdot|^{2N}H(2\cdot)\,|\,H_2^{\nu}(\mathbb{R}^n)\|\cdot\|(\varphi_m\widehat{f})^{\vee}\,|\,L_p(\mathbb{R}^n)\|,$$

where c is a positive constant independent of $m\in\mathbb{N}_0$. Applying this in (1.97) we obtain

$$(1.98)\quad \Big(\sum_{j=1}^{\infty}\Big\|\sum_{l=-\infty}^{M}2^{sj}\Psi(2^{-j})(\check{k}(2^{-j}\cdot)\varphi_{j+l}\widehat{f})^{\vee}(\cdot)\,\Big|\,L_p(\mathbb{R}^n)\Big\|^q\Big)^{1/q} \\ \leq c2^{Mn(1/p-1)}\|f\,|\,B_{pq}^{(s,\Psi)}(\mathbb{R}^n)\|.$$

Step 2. We estimate the second sum in (1.39); we have to make sure that (1.39) converges a.e. and in some L_r-space with $1\leq r\leq\infty$. However the latter comes as a by-product. Following Step 2 of the proof for $F_{pq}^{(s,\Psi)}(\mathbb{R}^n)$, we also have (1.59), with s_0 such that

$$(1.99)\quad s_0+4\sigma_p<s.$$

Let $1\leq p\leq\infty$, and χ as in (1.60). Then we apply the L_p-norm to (1.59), and use Young's inequality to obtain

$$(1.100)\quad \Big\|\sum_{l=M+1}^{\infty}2^{sj}\Psi(2^{-j})(\check{k}(2^{-j}\cdot)\varphi_{j+l}\widehat{f})^{\vee}(\cdot)\,|\,L_p(\mathbb{R}^n)\Big\| \\ \leq c\sum_{l=M+1}^{\infty}2^{(s_0-s)l}(1+l)^b\Big\|\Big(\frac{\check{k}(2^l\cdot)}{|2^l\cdot|^{s_0}}\chi(\cdot)\Big)^{\vee}\,\Big|\,L_1(\mathbb{R}^n)\Big\|\|(2^{(j+l)s}\Psi(2^{-(j+l)})\varphi'_{j+l}\widehat{f})^{\vee}\,|\,L_p(\mathbb{R}^n)\| \\ \leq c'\sum_{l=M+1}^{\infty}2^{(s_0-s)l}(1+l)^b\|(2^{(j+l)s}\Psi(2^{-(j+l)})\varphi'_{j+l}\widehat{f})^{\vee}\,|\,L_p(\mathbb{R}^n)\|.$$

The last inequality is due to

$$\sup_{l\in\mathbb{N}}\Big\|\Big(\frac{\check{k}(2^l\cdot)}{|2^l\cdot|^{s_0}}\chi(\cdot)\Big)^{\vee}\,\Big|\,L_1(\mathbb{R}^n)\Big\|<\infty,$$

which can be proved introducing inside the inverse Fourier transform the function h of (1.63), applying Theorem 2.2.3 of [Tri92] and using (1.64). Then, applying the ℓ_q-quasi-norm to (1.100), because $s_0-s<0$, we get

$$(1.101)\quad \Big(\sum_{j=1}^{\infty}\Big\|\sum_{l=M+1}^{\infty}2^{sj}\Psi(2^{-j})(\check{k}(2^{-j})\,\varphi_{j+l}\widehat{f})^{\vee}(\cdot)\,\Big|\,L_p(\mathbb{R}^n)\Big\|^q\Big)^{1/q} \\ \leq c2^{(s_0-s)M/2}\Big(\sum_{m=1}^{\infty}2^{smq}\Psi(2^{-m})^q\|(\varphi'_m\widehat{f})^{\vee}\,|\,L_p(\mathbb{R}^n)\|^q\Big)^{1/q}.$$

Let $0 < p < 1$. Each term in (1.59) can be estimated from above by

$$\left|\left(\frac{\check{k}(2^{-j}\cdot)}{|2^{-j}\cdot|^{s_0}}\chi(2^{-j-l}\cdot)2^{s(j+l)}\Psi(2^{-(j+l)})\varphi'_{j+l}\widehat{f}\right)^\vee(x)\right| \tag{1.102}$$
$$\leq (2\pi)^{-n/2}\int_{\mathbb{R}^n}\left|\left(\frac{\check{k}(2^{-j}\cdot)}{|2^{-j}\cdot|^{s_0}}\chi(2^{-j-l}\cdot)\right)^\vee(y)(2^{s(j+l)}\Psi(2^{-(j+l)})\varphi'_{j+l}\widehat{f})^\vee(x-y)\right|dy.$$

But the Fourier transform of the y-function inside the integral in (1.102) has compact support contained in $\{\xi \in \mathbb{R}^n : |\xi| \leq 6\cdot 2^{j+l}\}$, and since now $0 < p < 1$, we use the Theorem of [Tri83, 1.3.2/(5)]:

$$\left|\left(\frac{\check{k}(2^{-j}\cdot)}{|2^{-j}\cdot|^{s_0}}\chi(2^{-j-l}\cdot)2^{s(j+l)}\Psi(2^{-(j+l)})\varphi'_{j+l}\widehat{f}\right)^\vee(x)\right| \leq c2^{(j+l)n(1/p-1)} \tag{1.103}$$
$$\times\left[\int_{\mathbb{R}^n}\left|\left(\frac{\check{k}(2^{-j}\cdot)}{|2^{-j}\cdot|^{s_0}}\chi(2^{-j-l}\cdot)\right)^\vee(y)(2^{s(j+l)}\Psi(2^{-(j+l)})\varphi'_{j+l}\widehat{f})^\vee(x-y)\right|^p dy\right]^{1/p},$$

where c is independent of l and j. Putting the estimates (1.102) and (1.103) in (1.59), and then applying the L_p-quasi-norm we get

$$\left\|\sum_{l=M+1}^{\infty} 2^{sj}\Psi(2^{-j})(\check{k}(2^{-j}\cdot)\varphi_{j+l}\widehat{f})^\vee(\cdot)\,\Big|\, L_p(\mathbb{R}^n)\right\| \tag{1.104}$$
$$\leq c_1\Bigg[\sum_{l=M+1}^{\infty}2^{(s_0-s)pl}(1+l)^{bp}\int_{\mathbb{R}^n}\left|\left(\frac{\check{k}(2^{l}\cdot)}{|2^{l}\cdot|^{s_0}}\chi(\cdot)\right)^\vee(\xi)\right|^p d\xi$$
$$\times\,\|2^{(j+l)s}\Psi(2^{-(j+l)})(\varphi'_{j+l}\widehat{f})^\vee \,|\, L_p(\mathbb{R}^n)\|^p\Bigg]^{1/p}$$
$$\leq c_2\Big(\sum_{l=M+1}^{\infty}2^{(s_0-s)pl}(1+l)^{bp}\|2^{(j+l)s}\Psi(2^{-(j+l)})(\varphi'_{j+l}\widehat{f})^\vee \,|\, L_p(\mathbb{R}^n)\|^p\Big)^{1/p}.$$

We have used

$$\sup_{l\in\mathbb{N}}\left\|\left(\frac{\check{k}(2^{l}\cdot)}{|2^{l}\cdot|^{s_0}}\chi(\cdot)\right)^\vee \,\Big|\, L_p(\mathbb{R}^n)\right\|^p < \infty,$$

which can be proved introducing inside the inverse Fourier transform the function h of (1.63), applying Theorem 2.2.3 of [Tri92] and using (1.64). Recall that $1/p > 1$. Let p_1 be its conjugate exponent. Using Hölder's inequality we estimate the right-hand side of (1.104) by

$$c\sum_{l=M+1}^{\infty}2^{(s_0-s)l/2}\|2^{(j+l)s}\Psi(2^{-(j+l)})(\varphi'_{j+l}\widehat{f})^\vee \,|\, L_p(\mathbb{R}^n)\|. \tag{1.105}$$

Now, by (1.104) and (1.105), applying the ℓ_q-quasi-norm and because $s_0 - s < 0$ we obtain (1.101) for any value of $0 < p \leq \infty$. Recall that

$$(\varphi'_m\widehat{f})^\vee = (|2^{-m}\cdot|^{s_0}\chi(2^{-m}\cdot)[(\varphi_m\widehat{f})^\vee]^\wedge)^\vee, \qquad m\in\mathbb{N},$$

where $\chi \in \mathcal{D}(\mathbb{R}^n)$ is as in (1.60). As a consequence of Theorem 1.5.2 of [Tri83, (13)], with $\nu > n(1/\min(p,1) - 1/2)$, we have

$$\|(\varphi'_m\widehat{f})^\vee \,|\, L_p(\mathbb{R}^n)\| \leq c\||2\cdot|^{s_0}\chi(2\cdot)\,|\, H_2^\nu(\mathbb{R}^n)\|\cdot\|(\varphi_m\widehat{f})^\vee \,|\, L_p(\mathbb{R}^n)\|,$$

where c is independent of $m \in \mathbb{N}$. Applying this in (1.101) we obtain

$$\Big(\sum_{j=1}^{\infty} 2^{sjq}\Psi(2^{-j})^q \Big\| \sum_{l=M+1}^{\infty} (\check{k}(2^{-j}\cdot)\varphi_{j+l}\widehat{f}\,)^\vee(\cdot) \,\Big|\, L_p(\mathbb{R}^n)\Big\|^q\Big)^{1/q} \le c 2^{(s_0-s)M/4} \|f \,|\, B^{(s,\Psi)}_{pq}(\mathbb{R}^n)\|. \tag{1.106}$$

From (1.39), (1.98) and (1.106), and using the quasi-triangular inequality in the space $\ell_q(L_p)$, we get

$$\Big(\sum_{j=1}^{\infty} 2^{sjq}\Psi(2^{-j})^q \|k(2^{-j}, f) \,|\, L_p(\mathbb{R}^n)\|^q\Big)^{1/q} \le c\|f \,|\, B^{(s,\Psi)}_{pq}(\mathbb{R}^n)\|. \tag{1.107}$$

One can also prove that

$$\|k_0(2^{-h}, f) \,|\, L_p(\mathbb{R}^n)\| \le c'\|f \,|\, B^{(s,\Psi)}_{pq}(\mathbb{R}^n)\|. \tag{1.108}$$

With (1.107) and (1.108) we have proved one of the desired inequalities between the quasi-norm (1.89) and $\|\cdot \,|\, B^{(s,\Psi)}_{pq}(\mathbb{R}^n)\|$.

Step 3. We prove the convergence on the right-hand side of (1.39) in some space $L_r(\mathbb{R}^n)$, $1 < r \le \infty$. Let $1 \le p \le \infty$. We can rewrite (1.100) as

$$\Big\| \sum_{l=M+1}^{L} 2^{sj}\Psi(2^{-j})(\check{k}(2^{-j}\cdot)\varphi_{j+l}\widehat{f}\,)^\vee(\cdot) \,\Big|\, L_p(\mathbb{R}^n)\Big\| \le c \sum_{l=M+1}^{L} 2^{(s_0-s)l}(1+l)^b \|(2^{(j+l)s}\Psi(2^{-(j+l)})\varphi'_{j+l}\widehat{f}\,)^\vee \,|\, L_p(\mathbb{R}^n)\|, \tag{1.109}$$

with $L > M$. Using $s_0 - s < 0$ and $\ell_q \hookrightarrow \ell_1$ if $0 < q \le 1$, or the Hölder inequality if $1 < q \le \infty$, we conclude that if M is large enough then the right-hand side of (1.109) can be estimated from above by

$$\varepsilon\Big(\sum_{l=M+1}^{\infty} 2^{slq}\Psi(2^{-l})^q \|(\varphi'_l\widehat{f}\,)^\vee \,|\, L_p(\mathbb{R}^n)\|^q\Big)^{1/q}, \tag{1.110}$$

for given $\varepsilon > 0$. Since $f \in B^{(s,\Psi)}_{pq}(\mathbb{R}^n)$, as in Step 2,

$$\Big(\sum_{m=1}^{\infty} 2^{smq}\Psi(2^{-m})^q \|(\varphi'_m\widehat{f}\,)^\vee \,|\, L_p(\mathbb{R}^n)\|^q\Big)^{1/q} < \infty.$$

Therefore, by (1.109) and (1.110), we conclude that the right-hand side of (1.39) converges in $L_p(\mathbb{R}^n)$, hence pointwise a.e. and also in $\mathcal{S}'(\mathbb{R}^n)$. If $0 < p < 1$, we can rewrite (1.59) as

$$\Big| \sum_{l=M+1}^{L} 2^{sj}\Psi(2^{-j})(\check{k}(2^{-j}\cdot)\varphi_{j+l}\widehat{f}\,)^\vee(x)\Big| \le c \sum_{l=M+1}^{L} 2^{(s_0-s)l}(1+l)^b |(\check{k^0}(2^{-j}z)|2^{-j}z|^{2N-s_0} 2^{(j+l)s}\Psi(2^{-(j+l)})\varphi'_{j+l}\widehat{f}\,)^\vee(x)|. \tag{1.111}$$

with $L > M$. Applying the L_1-norm, using Fubini's theorem and a suitable change of variables we get

$$\begin{aligned}(1.112)\quad &\Big\| \sum_{l=M+1}^{L} 2^{sj}\,\Psi(2^{-j})(\check{k}(2^{-j}\cdot)\varphi_{j+l}\widehat{f})^\vee(\cdot)\,\Big|\, L_1(\mathbb{R}^n)\Big\| \\ &\qquad \le c_1 \sum_{l=M+1}^{L} 2^{(s_0-s)l}(1+l)^b \Big\| \Big(\frac{\check{k}^0(2^l\cdot)}{|2^l\cdot|^{s_0}}\chi(\cdot)\Big)^\vee \,\Big|\, L_1(\mathbb{R}^n)\Big\| \\ &\qquad\quad \times \|2^{(j+l)s}\Psi(2^{-(j+l)})(\varphi'_{j+l}\widehat{f})^\vee \,|\, L_1(\mathbb{R}^n)\| \\ &\qquad \le c_2 \sum_{l=M+1}^{L} 2^{(s_0-s)l}(1+l)^b \|2^{(j+l)s}\Psi(2^{-(j+l)})(\varphi'_{j+l}\widehat{f})^\vee \,|\, L_1(\mathbb{R}^n)\|,\end{aligned}$$

since

$$\sup_{l\in\mathbb{N}_0} \Big\| \Big(\frac{\check{k}^0(2^l\cdot)}{|2^l\cdot|^{s_0}}\chi(\cdot)\Big)^\vee \,\Big|\, L_1(\mathbb{R}^n)\Big\| < \infty.$$

Let $\sigma = s - \sigma_p$. Then by Proposition 1.9(iv), we have the embedding $B^{(s,\Psi)}_{p,q}(\mathbb{R}^n) \hookrightarrow B^{(\sigma,\Psi)}_{1,q}(\mathbb{R}^n)$. Putting $s = \sigma + \sigma_p$ in (1.112), we have

$$\begin{aligned}(1.113)\quad &\Big\| \sum_{l=M+1}^{L} 2^{\sigma j}\Psi(2^{-j})(\check{k}(2^{-j}\cdot)\varphi_{j+l}\widehat{f})^\vee(\cdot)\,\Big|\, L_1(\mathbb{R}^n)\Big\| \\ &\qquad \le c \sum_{l=M+1}^{L} 2^{(s_0-\sigma)l}(1+l)^b \|2^{(j+l)\sigma}\Psi(2^{-(j+l)})(\varphi'_{j+l}\widehat{f})^\vee \,|\, L_1(\mathbb{R}^n)\|.\end{aligned}$$

Since from (1.99), $\sigma > s_0$, there exists M large enough such that the right-hand side of (1.113) can be estimated from above by

$$\varepsilon\Big(\sum_{l=M+1}^{\infty} 2^{\sigma l q}\Psi(2^{-l})^q \|(\varphi'_l\widehat{f})^\vee \mid L_1(\mathbb{R}^n)\|^q\Big)^{1/q}$$

for any given $\varepsilon > 0$. From the embedding mentioned below, and since $f \in B^{(s,\Psi)}_{pq}(\mathbb{R}^n)$ (and using arguments as in Step 2) we have

$$\Big(\sum_{m=1}^{\infty} 2^{\sigma m q}\Psi(2^{-m})^q \|(\varphi'_m\widehat{f})^\vee \,|\, L_1(\mathbb{R}^n)\|^q\Big)^{1/q} < \infty.$$

Hence the right-hand side of (1.39) converges in $L_1(\mathbb{R}^n)$.

Step 4. Let $f \in B^{(s,\Psi)}_{pq}(\mathbb{R}^n)$. We now want to prove that $\|\cdot \,|\, B^{(s,\Psi)}_{pq}(\mathbb{R}^n)\|$ can be estimated from above by the quasi-norm in (1.89). We follow Step 4 of the proof for $F^{(s,\Psi)}_{pq}(\mathbb{R}^n)$. We can rewrite everything from (1.75) up to (1.81), but now it will be sufficient that $0 < r < \min(1, p)$. Since $p/r \ge 1$, we apply in (1.81) the $L_{p/r}$-norm and use the scalar Hardy–Littlewood maximal inequality as in [Tri83, 1.2.3/(4)]. Then we have

$$(1.114)\qquad \|(\varphi_j\widehat{f})^\vee \,|\, L_p(\mathbb{R}^n)\|^r \le c2^{Mn(1-r)}\|(\check{k}(\varepsilon 2^{-j}\cdot)\phi(2^{-j})\widehat{f})^\vee \,|\, L_p(\mathbb{R}^n)\|^r,$$

always with k_0 instead of k if $j = 0$. Multiplying (1.114) by $2^{sjr}\Psi(2^{-j})^r$ and applying the $\ell_{q/r}$-quasi-norm we get

$$(1.115)\qquad \Big(\sum_{j=0}^{\infty} 2^{sjq}\Psi(2^{-j})^q \|(\varphi_j \widehat{f}\,)^\vee \mid L_p(\mathbb{R}^n)\|^q\Big)^{1/q} \\ \le c2^{Mn(1/r-1)}\Big(\sum_{j=0}^{\infty} 2^{sjq}\Psi(2^{-j})^q \|(\check{k}(\varepsilon 2^{-j}\cdot)\phi(2^{-j})\widehat{f}\,)^\vee \mid L_p(\mathbb{R}^n)\|^q\Big)^{1/q}.$$

Because

$$\check{k}(\varepsilon 2^{-j}\cdot)\phi(2^{-j}\cdot) = \check{k}(\varepsilon 2^{-j}\cdot) - \check{k}(\varepsilon 2^{-j}\cdot)(1-\phi(2^{-j}\cdot)), \qquad j \in \mathbb{N},$$

and by the quasi-triangular inequality in $\ell_q(L_p)$, (1.115) can be estimated from above by

$$(1.116)\qquad c2^{Mn(1/r-1)}\Big(\sum_{j=0}^{\infty} 2^{sjq}\Psi(2^{-j})^q \|(\check{k}(\varepsilon 2^{-j}\cdot)\widehat{f}\,)^\vee \mid L_p(\mathbb{R}^n)\|^q\Big)^{1/q} \\ + c2^{Mn(1/r-1)}\Big(\sum_{j=0}^{\infty} 2^{sjq}\Psi(2^{-j})^q \|(\check{k}(\varepsilon 2^{-j}\cdot)(1-\phi(2^{-j}\cdot))\widehat{f}\,)^\vee \mid L_p(\mathbb{R}^n)\|^q\Big)^{1/q}.$$

The first term in (1.116) is precisely what we want. The additional term in (1.116) can be treated as in Step 2 and estimated from above by

$$(1.117)\qquad c2^{Mn(1/r-1)}2^{(s_0-s)M/4}\|f \mid B^{(s,\Psi)}_{pq}(\mathbb{R}^n)\|.$$

By (1.99), we may choose r such that

$$n\Big(\frac{1}{r}-1\Big) + \frac{s_0-s}{4} < 0.$$

Recall that the natural number M is at our disposal. We can take M so large that (1.117) can be estimated from above by

$$\tfrac{1}{2}\|f \mid B^{(s,\Psi)}_{pq}(\mathbb{R}^n)\|.$$

Applying this fact, (1.117) and (1.116) in (1.115) gives

$$(1.118)\qquad \|f \mid B^{(s,\Psi)}_{pq}(\mathbb{R}^n)\| \le c\|(\check{k}_0(\varepsilon\cdot)\widehat{f}\,)^\vee \mid L_p(\mathbb{R}^n)\| \\ + c\Big(\sum_{j=1}^{\infty} 2^{sjq}\Psi(2^{-j})^q \|(\check{k}(\varepsilon 2^{-j}\cdot)\widehat{f}\,)^\vee \mid L_p(\mathbb{R}^n)\|^q\Big)^{1/q}.$$

As observed in the proof for $F^{(s,\Psi)}_{pq}(\mathbb{R}^n)$, one can take $\varepsilon = 2^{-h}$, for some $h \in \mathbb{N}_0$ fixed. As there, from (1.118) we come to

$$\|f \mid B^{(s,\Psi)}_{pq}(\mathbb{R}^n)\| \le c\|k_0(2^{-h}, f) \mid L_p(\mathbb{R}^n)\| \\ + c\Big(\sum_{j=1}^{\infty} 2^{sjq}\Psi(2^{-j})^q \|(\check{k}(2^{-j}\cdot)\widehat{f}\,)^\vee | L_p(\mathbb{R}^n)\|^q\Big)^{1/q},$$

which completes the proof. ∎

REMARK 1.13. (i) If we replace k_0 by the new function $2^{hn}k_0(2^h\cdot)$, then in (1.37) and (1.89) there will appear simply $\|k_0(1,f) \mid L_p(\mathbb{R}^n)\|$ instead of $\|k_0(2^{-h},f) \mid L_p(\mathbb{R}^n)\|$.

(ii) If $s < 0$, then $N = 0$ is admitted in Theorems 1.10 and 1.12. That means that only one kernel $k_0 = k = k^0$ is sufficient.

1.4. Atomic and subatomic decompositions. Recall that $\mathbb{Z}^n$ stands for the lattice of all points in $\mathbb{R}^n$ with integer components. Furthermore, $Q_{\nu m}$ denotes a cube in $\mathbb{R}^n$ with sides parallel to the axes, centred at $2^{-\nu}m = (2^{-\nu}m_1 \dots, 2^{-\nu}m_n)$, and with side length $2^{-\nu}$, where $m = (m_1, \dots, m_n) \in \mathbb{Z}^n$ and $\nu \in \mathbb{N}_0$. If Q is a cube in $\mathbb{R}^n$ and $r > 0$ then rQ is the cube in $\mathbb{R}^n$ concentric with Q and with side length r times that of Q.

DEFINITION 1.14. (i) Let $K \in \mathbb{N}_0$ and $c > 1$. A K times differentiable complex-valued function $a(x)$ in $\mathbb{R}^n$ (continuous if $K = 0$) is called a 1_K-*atom* if

$$\operatorname{supp} a \subset c\,Q_{0m} \quad \text{for some } m \in \mathbb{Z}^n, \tag{1.119}$$

$$|D^\alpha a(x)| \le 1 \quad \text{for } |\alpha| \le K. \tag{1.120}$$

(ii) Let $s \in \mathbb{R}$, $0 < p \le \infty$, Ψ an admissible function, $K \in \mathbb{N}_0$, $L + 1 \in \mathbb{N}_0$ and $c > 1$. A K times differentiable complex-valued function $a(x)$ in $\mathbb{R}^n$ (continuous if $K = 0$) is called an $(s, p, \Psi)_{K,L}$-*atom* if for some $\nu \in \mathbb{N}_0$,

$$\operatorname{supp} a \subset cQ_{\nu m} \quad \text{for some } m \in \mathbb{Z}^n, \tag{1.121}$$

$$|D^\alpha a(x)| \le 2^{-\nu(s-n/p)+|\alpha|\nu}\Psi(2^{-\nu})^{-1} \quad \text{for } |\alpha| \le K, \tag{1.122}$$

and

$$\int_{\mathbb{R}^n} x^\beta a(x)\,dx = 0 \quad \text{if } |\beta| \le L. \tag{1.123}$$

Note that Q_{0m} is a cube with side length 1. If the atom $a(x)$ is located at $Q_{\nu m}$, i.e.,

$$\operatorname{supp} a \subset c\,Q_{\nu m} \quad \text{with } \nu \in \mathbb{N}_0,\ m \in \mathbb{Z}^n,\ c > 1,$$

then we write it $a_{\nu m}(x)$. The value of $c > 1$ in (1.119) and (1.121) is unimportant. It simply makes it clear that at level ν some controlled overlapping of the supports of $a_{\nu m}(x)$ must be allowed. The moment conditions (1.123) can be reformulated as

$$(D^\beta \widehat{a})(0) = 0 \quad \text{if } |\beta| \le L,$$

which shows that a sufficiently strong decay of $\widehat{a}(\xi)$ at the origin is required. If $L = -1$ then (1.123) simply means that there are no moment conditions. The reason for the normalising factor in (1.120) and (1.122) is that there exists a constant $c > 0$ such that for all these atoms we have $\|a \,|\, B^{(s,\Psi)}_{pq}(\mathbb{R}^n)\| \le c$, $\|a \,|\, F^{(s,\Psi)}_{pq}(\mathbb{R}^n)\| \le c$. Hence, atoms are normalising building blocks satisfying some moment conditions.

We now introduce the sequence spaces b_{pq} and f_{pq}. If $\nu \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$ and $Q_{\nu m}$ is a cube as above let $\chi_{\nu m}$ be the characteristic function of $Q_{\nu m}$. If $0 < p \le \infty$ let $\chi^{(p)}_{\nu m} = 2^{\nu n/p}\chi_{\nu m}$ (with the obvious modification if $p = \infty$) be the L_p-normalised characteristic function of $Q_{\nu m}$, that is,

$$\|\chi^{(p)}_{\nu m} \,|\, L_p(\mathbb{R}^n)\| = 1.$$

DEFINITION 1.15. Let $0 < p, q \le \infty$ and $\lambda = \{\lambda_{\nu m} \in \mathbb{C} : \nu \in \mathbb{N}_0,\ m \in \mathbb{Z}^n\}$. Then

$$b_{pq} = \Big\{\lambda : \|\lambda \,|\, b_{pq}\| = \Big(\sum_{\nu=0}^{\infty}\Big(\sum_{m\in\mathbb{Z}^n} |\lambda_{\nu m}|^p\Big)^{q/p}\Big)^{1/q} < \infty\Big\}, \tag{1.124}$$

$$f_{pq} = \Big\{\lambda : \|\lambda \,|\, f_{pq}\| = \Big\|\Big(\sum_{\nu=0}^{\infty}\sum_{m\in\mathbb{Z}^n} |\lambda_{\nu m}\chi^{(p)}_{\nu m}(\cdot)|^q\Big)^{1/q} \,\Big|\, L_p(\mathbb{R}^n)\Big\| < \infty\Big\} \tag{1.125}$$

(with the usual modification if $p = \infty$ or/and $q = \infty$).

REMARK 1.16. Observe that

$$\|\lambda \mid b_{pq}\| = \Big(\sum_{\nu=0}^{\infty} \Big\| \sum_{m\in\mathbb{Z}^n} \lambda_{\nu m}\, \chi_{\nu m}^{(p)}(\cdot) \,\Big|\, L_p(\mathbb{R}^n) \Big\|^q \Big)^{1/q}; \tag{1.126}$$

since the $\chi_{\nu m}^{(p)}$'s have disjoint supports a.e., we see that the b and f quasi-norms are obtained from each other by interchanging the L_p and ℓ_q quasi-norms (as in the B and F case).

PROPOSITION 1.17 [Tri97, 13.6, p. 75]. *Let $0 < p, q \le \infty$. Then b_{pq} and f_{pq} are quasi-Banach spaces. Furthermore*

$$b_{p,\min(p,q)} \hookrightarrow f_{pq} \hookrightarrow b_{p,\max(p,q)},$$

and, in particular, $b_{pp} = f_{pp}$.

Recall the notations introduced in (1.36).

THEOREM 1.18. (i) *Let $0 < p < \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$ and Ψ an admissible function. Fix $K \in \mathbb{N}_0$ and $L + 1 \in \mathbb{N}_0$ with*

$$K \ge (1 + [s])_+ \quad \textit{and} \quad L \ge \max(-1, [\sigma_{pq} - s]). \tag{1.127}$$

Then $f \in \mathcal{S}'(\mathbb{R}^n)$ belongs to $F_{pq}^{(s,\Psi)}(\mathbb{R}^n)$ if, and only if, it can be represented as

$$f = \sum_{\nu=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \lambda_{\nu m} a_{\nu m}(x) \quad (\textit{convergence in } \mathcal{S}'(\mathbb{R}^n)), \tag{1.128}$$

where $a_{\nu m}(x)$ are 1_K-atoms ($\nu = 0$) or $(s, p, \Psi)_{K,L}$-atoms ($\nu \in \mathbb{N}$) and $\lambda \in f_{pq}$. Furthermore

$$\inf \|\lambda \mid f_{pq}\|, \tag{1.129}$$

where the infimum is taken over all admissible representations (1.128), *is an equivalent quasi-norm in $F_{pq}^{(s,\Psi)}(\mathbb{R}^n)$.*

(ii) *Let $0 < p, q \le \infty$, $s \in \mathbb{R}$ and Ψ an admissible function. Fix $K \in \mathbb{N}_0$ and $L+1 \in \mathbb{N}_0$ with*

$$K \ge (1 + [s])_+ \quad \textit{and} \quad L \ge \max(-1, [\sigma_p - s]). \tag{1.130}$$

Then $f \in \mathcal{S}'(\mathbb{R}^n)$ belongs to $B_{pq}^{(s,\Psi)}(\mathbb{R}^n)$ if, and only if, it can be represented as in (1.128) *where $a_{\nu m}(x)$ are 1_K-atoms ($\nu = 0$) or $(s, p, \Psi)_{K,L}$-atoms ($\nu \in \mathbb{N}$) and $\lambda \in b_{pq}$. Furthermore*

$$\inf \|\lambda \mid b_{pq}\|, \tag{1.131}$$

where the infimum is taken over all admissible representations (1.128), *is an equivalent quasi-norm in $B_{pq}^{(s,\Psi)}(\mathbb{R}^n)$.*

We refer to the above theorem as the *atomic decomposition theorem*. For more references to this subject we refer to [FrJ85], [FJW91], [Tri97] and [Far00], the first three dealing with the usual Besov and Triebel–Lizorkin spaces and the latter with the anisotropic case. A proof of Theorem 1.18 will be provided later on. Now we mention that the convergence in $\mathcal{S}'(\mathbb{R}^n)$ of the right-hand side of (1.128) is ensured by the required properties of the atoms involved and $\lambda \in b_{pq}$ or $\lambda \in f_{pq}$. In particular, convergence in $\mathcal{S}'(\mathbb{R}^n)$ in

(1.128) is not an additional assumption but a result. Before giving a precise statement of this, we need the following lemma:

LEMMA 1.19. *Fix $c \geq 1$ and $\nu \in \mathbb{N}_0$. Then any $x \in \mathbb{R}^n$ belongs to at most N cubes $c\,Q_{\nu m}$, $m \in \mathbb{Z}^n$, where N is independent of ν and m (it only depends on c and on the dimension n).*

Proof. For $x \in \mathbb{R}^n$ there surely exists $m \in \mathbb{Z}^n$ such that $x \in Q_{\nu m}$. Assume $x \in c\,Q_{\nu m'}$ for some $m' \in \mathbb{Z}^n$. We have

$$|x_i - 2^{-\nu} m_i| \leq 2^{-\nu-1} \quad \text{and} \quad |x_i - 2^{-\nu} m_i'| \leq c 2^{-\nu-1}, \qquad i = 1, \dots, n.$$

This gives

$$|m_i - m_i'| \leq \frac{c+1}{2}, \qquad i = 1, \dots, n,$$

which means that m' belongs to the cube centred at m and with side length $c+1$. The number of such $m' \in \mathbb{Z}^n$ is $N = ([c]+1)^n$. ∎

PROPOSITION 1.20. (i) *Let $0 < p < \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$ and Ψ an admissible function. Fix $K \in \mathbb{N}_0$ and $L+1 \in \mathbb{N}_0$ with* (1.127). *If $a_{\nu m}(x)$ are 1_K-atoms ($\nu = 0$) or $(s,p,\Psi)_{K,L}$-atoms ($\nu \in \mathbb{N}$) and $\lambda \in f_{pq}$ then*

$$\sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} a_{\nu m}(x) \tag{1.132}$$

converges in $\mathcal{S}'(\mathbb{R}^n)$.

(ii) *Let $0 < p, q \leq \infty$, $s \in \mathbb{R}$ and Ψ an admissible function. Fix $K \in \mathbb{N}_0$ and $L+1 \in \mathbb{N}_0$ with* (1.130). *If $a_{\nu m}(x)$ are 1_K-atoms ($\nu = 0$) or $(s,p,\Psi)_{K,L}$-atoms ($\nu \in \mathbb{N}$) and $\lambda \in b_{pq}$ then* (1.132) *converges in $\mathcal{S}'(\mathbb{R}^n)$.*

Proof. By the above lemma, for fixed $\nu \in \mathbb{N}_0$, we have only a controlled overlapping of the supports of the atoms $a_{\nu m}$. Therefore, the convergence in $\mathcal{S}'(\mathbb{R}^n)$ of (1.132) means

$$\lim_{\mu \to \infty} \sum_{\nu=0}^{\mu} \Big(\sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} a_{\nu m} \Big)$$

where, as can be seen through the proof, the inner sum causes no problem.

Step 1. We first prove (ii). We may assume $L \neq -1$, otherwise we have to modify a little the following considerations, in particular using $s > \sigma_p$ instead of $L \geq [\sigma_p - s]$. Assume first $1 \leq p \leq \infty$ and let $\varphi \in \mathcal{S}(\mathbb{R}^n)$. By Definition 1.14, in particular (1.123), and Taylor expansion of φ up to order L with respect to the off-points $2^{-\nu} m$ we obtain for fixed $\nu \in \mathbb{N}_0$,

$$\begin{aligned} \int_{\mathbb{R}^n} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} a_{\nu m}(y) \varphi(y)\, dy = \int_{\mathbb{R}^n} \sum_{m \in \mathbb{Z}^n} & \lambda_{\nu m} 2^{-\nu(L+1)} a_{\nu m}(y) \\ & \times \Big(\varphi(y) - \sum_{|\beta| \leq L} \frac{|D^\beta \varphi(2^{-\nu} m)|}{\beta!} (y - 2^{-\nu} m)^\beta \Big) 2^{\nu(L+1)}\, dy. \end{aligned} \tag{1.133}$$

For an appropriate ξ lying on the line segment joining y and $2^{-\nu}m$ we have the following estimate for the last factor in (1.133):

$$\Big|\varphi(y) - \sum_{|\beta|\le L} \frac{|D^\beta\varphi(2^{-\nu}m)|}{\beta!}(y-2^{-\nu}m)^\beta\Big| 2^{\nu(L+1)} \le \sum_{|\gamma|=L+1} \frac{|D^\gamma\varphi(\xi)|}{\gamma!}|y-2^{-\nu}m|^{L+1}2^{\nu(L+1)} \le c_1' \sum_{|\gamma|=L+1}\frac{|D^\gamma\varphi(\xi)|}{\gamma!}. \tag{1.134}$$

In the last inequality, we have used $|y-2^{-\nu}m| \le \sqrt{n}\, c2^{-\nu-1}$, due to $y \in \operatorname{supp} a_{\nu m} \subset cQ_{\nu m}$. We also remark that $\xi \in cQ_{\nu m}$, and so $|y-\xi| \le \sqrt{n}\, c2^{-\nu}$. Then some calculations show that for any $M > 0$,

$$\langle y\rangle^M \le (3+2c^2n)^{M/2}\langle \xi\rangle^M. \tag{1.135}$$

Using (1.135) in (1.134) we get

$$\Big|\varphi(y) - \sum_{|\beta|\le L} \frac{|D^\beta\varphi(2^{-\nu}m)|}{\beta!}(y-2^{-\nu}m)^\beta\Big| 2^{\nu(L+1)} \le c_2'\langle y\rangle^{-M} \sup_{x\in\mathbb{R}^n}\langle x\rangle^M \sum_{|\gamma|\le L+1}|D^\gamma\varphi(x)| \tag{1.136}$$

where $c_2' > 0$ depends only on M, L, c and n. Because $a_{\nu m}$ is an $(s,p,\Psi)_{K,L}$-atom, $\nu\in\mathbb{N}$, we have

$$2^{-\nu(L+1)}|a_{\nu m}(y)| \le 2^{\nu n/p}2^{-\nu(L+1+s)}\Psi(2^{-\nu})^{-1}\widetilde{\chi}_{\nu m}(y), \tag{1.137}$$

where $\widetilde{\chi}_{\nu m}$ is the characteristic function of the cube $cQ_{\nu m}$. By the properties of admissible functions (cf. Proposition 1.4(i),(iii)), for any $\varepsilon > 0$ there exists $c_\varepsilon > 0$ such that

$$\Psi(2^{-\nu})^{-1} \le c_\varepsilon 2^{\varepsilon\nu}, \qquad \nu\in\mathbb{N}_0. \tag{1.138}$$

Since L satisfies (1.130), we have $L+1 > \sigma_p - s \ge -s$. We choose ε with $0<\varepsilon<L+1+s$. With this choice, putting (1.138) in (1.137) we get

$$2^{-\nu(L+1)}|a_{\nu m}(y)| \le c_\varepsilon 2^{-\nu\theta}2^{\nu n/p}\widetilde{\chi}_{\nu m}(y) \tag{1.139}$$

with $\theta = L+1+s-\varepsilon > 0$. Applying (1.139) and (1.136) in (1.134), with p' the conjugate exponent of p, M chosen such that $Mp' > n/2$ and using Hölder's inequality we obtain

$$\begin{aligned} &\Big|\int_{\mathbb{R}^n}\sum_{m\in\mathbb{Z}^n}\lambda_{\nu m}a_{\nu m}(y)\varphi(y)\,dy\Big| \\ &\le c_1 2^{-\nu\theta}\sup_{x\in\mathbb{R}^n}\langle x\rangle^M \sum_{|\gamma|\le L+1}|D^\gamma\varphi(x)| \int_{\mathbb{R}^n}\sum_{m\in\mathbb{Z}^n}|\lambda_{\nu m}|2^{\nu n/p}\widetilde{\chi}_{\nu m}(y)\langle y\rangle^{-M}\,dy \\ &\le c_2 2^{-\nu\theta}\sup_{x\in\mathbb{R}^n}\langle x\rangle^M \sum_{|\gamma|\le L+1}|D^\gamma\varphi(x)|\Big[\int_{\mathbb{R}^n}\Big(\sum_{m\in\mathbb{Z}^n}|\lambda_{\nu m}|2^{\nu n/p}\widetilde{\chi}_{\nu m}(y)\Big)^p dy\Big]^{1/p} \\ &\le c_2 2^{-\nu\theta}\sup_{x\in\mathbb{R}^n}\langle x\rangle^M \sum_{|\gamma|\le L+1}|D^\gamma\varphi(x)|\Big(\int_{\mathbb{R}^n}2^{(p-1)N}\sum_{m\in\mathbb{Z}^n}|\lambda_{\nu m}|^p 2^{\nu n}\widetilde{\chi}_{\nu m}(y)\,dy\Big)^{1/p} \\ &\le c_3 2^{-\nu\theta}\Big(\sum_{m\in\mathbb{Z}^n}|\lambda_{\nu m}|^p\Big)^{1/p}\sup_{x\in\mathbb{R}^n}\langle x\rangle^M\sum_{|\gamma|\le L+1}|D^\gamma\varphi(x)|. \end{aligned} \tag{1.140}$$

We have used the above lemma, which tells us that each $y \in \mathbb{R}^n$ belongs to at most N (only depending on c and n) cubes $c\,Q_{\nu m}$, $m \in \mathbb{Z}^n$. Since $\theta > 0$ and $\lambda \in b_{p,q} \subset b_{p,\infty}$, by (1.140) the convergence of (1.132) in $\mathcal{S}'(\mathbb{R}^n)$ is now clear.

Now let $0 < p < 1$. Since L satisfies (1.130), we have $L+1 > \sigma_p = n/p - n - s$. The value of ε in (1.138) is chosen so that $0 < \varepsilon < L+1+s-n/p+n$. Then the substitute of (1.139) in this case is

$$2^{-\nu(L+1)}|a_{\nu m}(y)| \le c_\varepsilon 2^{-\nu\eta} 2^{\nu n} \widetilde{\chi}_{\nu m}(y), \tag{1.141}$$

where $\eta = L+1+s-n/p+n-\varepsilon > 0$. Applying (1.141) and (1.136) in (1.134), we get

$$\begin{aligned} &\Big| \int_{\mathbb{R}^n} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} a_{\nu m}(y) \varphi(y)\, dy \Big| \\ &\quad \le c_1 2^{-\nu\eta} \sup_{x \in \mathbb{R}^n} \langle x \rangle^M \sum_{|\gamma| \le L+1} |D^\gamma \varphi(x)| \int_{\mathbb{R}^n} \sum_{m \in \mathbb{Z}^n} |\lambda_{\nu m}| 2^{\nu n} \widetilde{\chi}_{\nu m}(y) \, \langle y \rangle^{-M} \, dy \\ &\quad \le c_2 2^{-\nu\eta} \Big(\sum_{m \in \mathbb{Z}^n} |\lambda_{\nu m}| \Big) \sup_{x \in \mathbb{R}^n} \langle x \rangle^M \sum_{|\gamma| \le L+1} |D^\gamma \varphi(x)|. \end{aligned} \tag{1.142}$$

Since $\eta > 0$ and $\lambda \in b_{p,q} \subset b_{1,\infty}$ $(0 < p < 1)$, by (1.142) the convergence of (1.132) in $\mathcal{S}'(\mathbb{R}^n)$ is clear.

Step 2. (i) follows from (ii), $\sigma_{pq} \ge \sigma_p$ and $f_{p,q} \subset b_{p,\max(p,q)}$. ■

In Theorem 1.18 no information is given about the possibility to obtain atomic decompositions in which the atoms are constructed with the help of dilations and translations from one smooth function Φ having compact support. In order to present the subatomic decomposition we need to introduce some special building blocks called quarks.

DEFINITION 1.21. Let $\Phi \in \mathcal{S}(\mathbb{R}^n)$ be such that, for some $d > 1$,

$$\operatorname{supp} \Phi \subset dQ_{00} \quad \text{and} \quad \sum_{m \in \mathbb{Z}^n} \Phi(x-m) = 1 \quad \text{for } x \in \mathbb{R}^n. \tag{1.143}$$

Let $s \in \mathbb{R}$, $0 < p \le \infty$, Ψ an admissible function, $(L+1)/2 \in \mathbb{N}_0$, $\beta \in \mathbb{N}_0^n$ and $\Phi^\beta(x) = x^\beta \Phi(x)$. Then

$$(\beta qu)^L_{\nu m}(x) = 2^{-\nu(s-n/p)} \Psi(2^{-\nu})^{-1} ((-\Delta)^{(L+1)/2} \Phi^\beta)(2^\nu x - m) \tag{1.144}$$

is called an $(s,p,\Psi)_L$-β-*quark* related to $Q_{\nu m}$. When $L = -1$, let $(\beta qu)^L_{\nu m} = (\beta qu)_{\nu m}$ denote an (s,p,Ψ)-β-*quark*.

The quarks are specialised atoms, as we prove in the next lemma.

LEMMA 1.22. *Let $s \in \mathbb{R}$, $0 < p \le \infty$, Ψ an admissible function, $(L+1)/2 \in \mathbb{N}_0$ and $\beta \in \mathbb{N}_0^n$. Up to normalising constants the $(s,p,\Psi)_L$-β-quarks are $(s,p,\Psi)_{K,L}$-atoms for any $K \in \mathbb{N}_0$. Moreover, the normalising constants by which an $(s,p,\Psi)_L$-β-quark must be divided to become an $(s,p,\Psi)_{K,L}$-atom can be estimated from above by $c2^{\kappa|\beta|}$, where c and κ are positive constants independent of β (but may depend on K and L).*

Proof. From (1.144) and (1.143), we get

$$\operatorname{supp} (\beta qu)^L_{\nu m} \subset \{x \in \mathbb{R}^n : 2^\nu x - m \in \operatorname{supp} \Phi\} \subset dQ_{\nu m}. \tag{1.145}$$

Now fix $K \in \mathbb{N}_0$ and let $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \le K$. We have

(1.146) $$|D^\alpha(\beta qu)^L_{\nu m}(x)| = 2^{-\nu(s-n/p)+\nu|\alpha|}\Psi(2^{-\nu})^{-1}|(D^\alpha(-\Delta)^{(L+1)/2}\Phi^\beta)(2^\nu x - m)|.$$

For any $\lambda \in \mathbb{N}_0^n$ with $|\lambda| \le K + L + 1$, Leibniz's rule gives

$$(D^\lambda \Phi^\beta)(x) = \sum_{\gamma \le \lambda} \binom{\lambda}{\gamma} D^\gamma(x^\beta)(D^{\lambda-\gamma}\Phi)(x)$$

where

$$D^\gamma(x^\beta) = \frac{\beta!}{(\beta-\gamma)!} x^{\beta-\gamma} \quad \text{for } \gamma \le \beta,$$

while $D^\gamma(x^\beta) = 0$ if $\gamma_i > \beta_i$ for some $i \in \{1, \ldots, n\}$. Moreover, for $\gamma \le \beta$,

$$\frac{\beta!}{(\beta-\gamma)!} = \prod_{j=1}^n \frac{\beta_j!}{(\beta_j-\gamma_j)!} = \prod_{j=1}^n \beta_j(\beta_j-1)\ldots(\beta_j-\gamma_j+1) \le |\beta|^{|\gamma|} \le c_\varepsilon 2^{\varepsilon|\beta|}$$

for all $\varepsilon > 0$. Since

$$\max_{|\delta| \le K+L+1} \max_{x \in dQ_{00}} |D^\delta \Phi(x)| < \infty,$$

using (1.145) we get

(1.147) $$\begin{aligned} |(D^\lambda \Phi^\beta)(x)| &\le c_1 2^{\varepsilon|\beta|} \sum_{\gamma\le\lambda, \gamma\le\beta} \binom{\lambda}{\gamma} \prod_{j=1}^n |x_j|^{\beta_j-\gamma_j} \chi_{dQ_{00}}(x) \\ &\le c_1 2^{\varepsilon|\beta|} \sum_{\gamma\le\lambda, \gamma\le\beta} \binom{\lambda}{\gamma} d^{|\beta|-|\gamma|} \le c_2 2^{(\varepsilon+\log d)|\beta|}, \end{aligned}$$

where the positive constant c_2 depends only on ε, K, L and Φ. We put (1.147) in (1.146) to arrive at

(1.148) $$|D^\alpha(\beta qu)^L_{\nu m}(x)| \le c_3 2^{(\varepsilon+\log d)|\beta|} 2^{-\nu(s-n/p)+\nu|\alpha|}\Psi(2^{-\nu})^{-1}$$

with c_3 independent of β (depending only on n, ε, K, L and Φ). By (1.144) and integration by parts, it is obvious that

(1.149) $$\int_{\mathbb{R}^n} x^\gamma (\beta qu)^L_{\nu m}(x)\, dx = 0 \quad \text{if } |\gamma| \le L.$$

By (1.145), (1.148) and (1.149) the proof is complete by taking $\kappa > \log d$ and c the corresponding constant c_3 in (1.148). ∎

Below we will use the sequence spaces b_{pq} and f_{pq} with respect to the sequences

(1.150) $$\lambda^\beta = \{\lambda^\beta_{\nu m} \in \mathbb{C} : \nu \in \mathbb{N}_0,\ m \in \mathbb{Z}^n\}$$

where now $\beta \in \mathbb{N}_0^n$.

THEOREM 1.23. (i) *Let $0 < p < \infty$, $0 < q \le \infty$, Ψ an admissible function and $s \in \mathbb{R}$ be such that*

(1.151) $$s > \sigma_{pq}.$$

There exists $\kappa > 0$ with the following property: let $\mu > \kappa$; then $f \in \mathcal{S}'(\mathbb{R}^n)$ belongs to $F^{(s,\Psi)}_{pq}(\mathbb{R}^n)$ if, and only if, it can be represented as

$$f = \sum_{\beta \in \mathbb{N}_0^n} \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda^\beta_{\nu m} (\beta qu)_{\nu m}(x), \tag{1.152}$$

convergence being in $\mathcal{S}'(\mathbb{R}^n)$, where $(\beta qu)_{\nu m}$ are (s,p,Ψ)-β-quarks and

$$\sup_{\beta \in \mathbb{N}_0^n} 2^{\mu|\beta|} \|\lambda^\beta \,|\, f_{pq}\| < \infty. \tag{1.153}$$

Furthermore, the infimum of (*the left-hand side of*) (1.153) *over all representations* (1.152) *is an equivalent quasi-norm in $F^{(s,\Psi)}_{pq}(\mathbb{R}^n)$.*

(ii) *Let $0 < p, q \le \infty$, Ψ an admissible function and $s \in \mathbb{R}$ be such that*

$$s > \sigma_p. \tag{1.154}$$

There exists $\kappa > 0$ with the following property: let $\mu > \kappa$; then $f \in \mathcal{S}'(\mathbb{R}^n)$ belongs to $B^{(s,\Psi)}_{pq}(\mathbb{R}^n)$ if, and only if, it can be represented as in (1.152), *convergence being in $\mathcal{S}'(\mathbb{R}^n)$, where $(\beta qu)_{\nu m}$ are (s,p,Ψ)-β-quarks and*

$$\sup_{\beta \in \mathbb{N}_0^n} 2^{\mu|\beta|} \|\lambda^\beta \,|\, b_{pq}\| < \infty. \tag{1.155}$$

Furthermore, the infimum of (1.155) *over all representations* (1.152) *is an equivalent quasi-norm in $B^{(s,\Psi)}_{pq}(\mathbb{R}^n)$.*

REMARK 1.24. As for the atomic case, convergence of the subatomic sum (1.152) under the assumptions (1.153) or (1.155) is always true, i.e. it is not a further requirement of the theorem. Moreover, as we see below, in certain circumstances the convergence is really nice.

I. We begin by studying the convergence of (1.152) for the situation described in (ii) of the above theorem, i.e. $0 < p, q \le \infty$, Ψ an admissible function, $s > \sigma_p$, $(\beta qu)_{\nu m}$ are (s,p,Ψ)-β-quarks, $\mu > \kappa$ and

$$\sup_{\beta \in \mathbb{N}_0^n} 2^{\mu|\beta|} \|\lambda^\beta \,|\, b_{pq}\| < \infty.$$

I.1. Let first $p = \infty$. Then $\sigma_p = 0$ and so $s > 0$. Having in mind Lemma 1.22 we have

$$|f(x)| \le c_1 \sum_{\beta \in \mathbb{N}_0^n} \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} |\lambda^\beta_{\nu m}| 2^{\kappa|\beta|}\, 2^{-\nu s} \Psi(2^{-\nu})^{-1}\, \widetilde{\chi}_{\nu m}(x),$$

where $\widetilde{\chi}_{\nu m}$ is the characteristic function of the cube $d\,Q_{\nu m}$. Then, with $\mu > \kappa$ and using Lemma 1.19, we get

$$\begin{aligned} |f(x)| &\le c_1 \Big(\sum_{\beta \in \mathbb{N}_0^n} 2^{(\kappa-\mu)|\beta|} \Big) \sup_{\beta \in \mathbb{N}_0^n} 2^{\mu|\beta|} \Big(\sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} |\lambda^\beta_{\nu m}| 2^{-\nu s} \Psi(2^{-\nu})^{-1} \widetilde{\chi}_{\nu m}(x) \Big) \\ &\le c_2 \sup_{\beta \in \mathbb{N}_0^n} 2^{\mu|\beta|} \Big(\sum_{\nu=0}^{\infty} 2^{-\nu s} \Psi(2^{-\nu})^{-1} \sup_{m \in \mathbb{Z}^n} |\lambda^\beta_{\nu m}| \Big). \end{aligned}$$

If $0 < q < 1$, we use $\ell_q \hookrightarrow \ell_1$ and $2^{-\nu s}\Psi(2^{-\nu})^{-1} \le c$ for all $\nu \in \mathbb{N}_0$ (consequence of Proposition 1.4(i), (iii) and $s > 0$). If $1 \le q \le \infty$, with q' its conjugate exponent, we use the Hölder inequality and the convergence of the series $\sum_{\nu=0}^{\infty} 2^{-\nu s q'} \Psi(2^{-\nu})^{-q'}$ (guaranteed by Proposition 1.4(ii) and $s > 0$, with the usual modification if $q' = \infty$). In both cases of q we arrive at

$$|f(x)| \le C \sup_{\beta \in \mathbb{N}_0^n} 2^{\mu|\beta|} \|\lambda^\beta \,|\, b_{\infty q}\|.$$

Therefore, for $p = \infty$, (1.152) converges pointwise uniformly and absolutely and $f(x)$ is a bounded uniformly continuous function in $\mathbb{R}^n$.

I.2. Let $1 \le p < \infty$. Then also $\sigma_p = 0$ and so $s > 0$. In a similar way to the case above, for all $\varepsilon > 0$ we have

$$\begin{aligned} |f(x)| &\le c_1 \sum_{\beta \in \mathbb{N}_0^n} \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} |\lambda_{\nu m}^\beta| 2^{\kappa|\beta|} 2^{-\nu(s-n/p)} \Psi(2^{-\nu})^{-1} \widetilde{\chi}_{\nu m}(x), \\ &\le c_2 \sup_{\beta \in \mathbb{N}_0^n} 2^{(\kappa+\varepsilon)|\beta|} \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} |\lambda_{\nu m}^\beta| 2^{-\nu(s-n/p)} \Psi(2^{-\nu})^{-1} \widetilde{\chi}_{\nu m}(x) \\ &\le c_3 \sup_{\beta \in \mathbb{N}_0^n} \sup_{\nu \in \mathbb{N}_0} \sup_{m \in \mathbb{Z}^n} 2^{(\kappa+\varepsilon)|\beta|} 2^{-\nu(s-n/p-\varepsilon)} |\lambda_{\nu m}^\beta| \Psi(2^{-\nu})^{-1} \widetilde{\chi}_{\nu m}(x). \end{aligned}$$

We choose ε so small that $0 < \varepsilon < \min(\mu - \kappa, s)$. We get

$$\begin{aligned} |f(x)|^p &\le c_1' \sup_{\beta \in \mathbb{N}_0^n} \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} 2^{(\kappa+\varepsilon)|\beta| p} 2^{-\nu(s-\varepsilon)p + \nu n} \Psi(2^{-\nu})^{-p} |\lambda_{\nu m}^\beta|^p \widetilde{\chi}_{\nu m}(x). \\ &\le c_2' \sup_{\beta \in \mathbb{N}_0^n} 2^{\mu|\beta| p} \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} 2^{-\nu(s-\varepsilon)p + \nu n} \Psi(2^{-\nu})^{-p} |\lambda_{\nu m}^\beta|^p \widetilde{\chi}_{\nu m}(x). \end{aligned}$$

Integration gives

$$\|f \,|\, L_p(\mathbb{R}^n)\|^p \le c_3' \sup_{\beta \in \mathbb{N}_0^n} 2^{\mu|\beta| p} \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} 2^{-\nu(s-\varepsilon)p} \Psi(2^{-\nu})^{-p} |\lambda_{\nu m}^\beta|^p.$$

If $0 < q \le p$, we use $\ell_{q/p} \hookrightarrow \ell_1$ and $2^{-\nu(s-\varepsilon)p} \Psi(2^{-\nu})^{-p} \le c$ for all $\nu \in \mathbb{N}_0$ (consequence of Proposition 1.4(i),(iii) and $s > \varepsilon$). If $0 < p < q$, $q/p > 1$ with t its conjugate exponent, we use Hölder's inequality and the convergence of the series $\sum_{\nu=0}^{\infty} 2^{-\nu(s-\varepsilon)pt} \Psi(2^{-\nu})^{-pt}$ (guaranteed by Proposition 1.4(ii) and $s > \varepsilon$). In both cases of q we arrive at

$$\|f \,|\, L_p(\mathbb{R}^n)\| \le C \sup_{\beta \in \mathbb{N}_0^n} 2^{\mu|\beta|} \|\lambda^\beta \,|\, b_{pq}\|.$$

Therefore, for $1 \le p < \infty$, (1.152) converges in $L_p(\mathbb{R}^n)$.

I.3. Let $0 < p < 1$. Then $s > n/p - n > 0$. For all $\mu > \kappa$ we have

$$\begin{aligned} |f(x)| &\le c_1 \sum_{\beta \in \mathbb{N}_0^n} \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} |\lambda_{\nu m}^\beta| 2^{\kappa|\beta|} 2^{-\nu(s-n/p)} \Psi(2^{-\nu})^{-1} \widetilde{\chi}_{\nu m}(x), \\ &\le c_2 \sup_{\beta \in \mathbb{N}_0^n} 2^{\mu|\beta|} \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} |\lambda_{\nu m}^\beta| 2^{-\nu(s-n/p)} \Psi(2^{-\nu})^{-1} \widetilde{\chi}_{\nu m}(x). \end{aligned}$$

Integration gives

$$\|f \mid L_1(\mathbb{R}^n)\| \le c_3 \sup_{\beta\in\mathbb{N}_0^n} 2^{\mu|\beta|} \sum_{\nu=0}^{\infty} \sum_{m\in\mathbb{Z}^n} |\lambda_{\nu m}^{\beta}| 2^{-\nu(s-n/p+n)} \Psi(2^{-\nu})^{-1}.$$

Using arguments similar to the ones in I.2 we conclude that

$$\|f \mid L_1(\mathbb{R}^n)\| \le C \sup_{\beta\in\mathbb{N}_0^n} 2^{\mu|\beta|} \|\lambda^{\beta} \mid b_{pq}\|.$$

Therefore, for $0 < p < 1$, (1.152) converges in $L_1(\mathbb{R}^n)$.

II. The convergence of (1.152) for the situation described in (i) of the theorem above, i.e. $0 < p < \infty$, $0 < q \le \infty$, Ψ an admissible function, $s > \sigma_{pq}$, $(\beta qu)_{\nu m}$ are (s, p, Ψ)-β-quarks, $\mu > \kappa$ and

$$\sup_{\beta\in\mathbb{N}_0^n} 2^{\mu|\beta|} \|\lambda^{\beta} \mid f_{pq}\| < \infty,$$

is covered by I if we have in mind that $\sigma_{pq} \ge \sigma_p$ and $f_{pq} \subset b_{p,\max(p,q)}$.

REMARK 1.25. To show that $f \in F_{pq}^{(s,\Psi)}(\mathbb{R}^n)$ (respectively, $f \in B_{pq}^{(s,\Psi)}(\mathbb{R}^n)$) can be decomposed as (1.152) with (1.153) (respectively, (1.152) with (1.155)), we do not need the assumptions (1.151) (respectively, (1.154)). These restrictions are needed to prove the converse assertion.

Next we present the proof of (i) in Theorems 1.18 and 1.23. The proof of (ii) is somewhat similar but technically simpler. Nevertheless, occasionally we say a word about the modification corresponding to (ii).

Proof. Step 1 (if-part of atomic decomposition). The proof relies on the equivalent quasi-norm in $F_{pq}^{(s,\Psi)}(\mathbb{R}^n)$ given by (1.37), and the underlying local means according to (1.32)–(1.34), where N with $2N > s$ may be chosen arbitrarily large. We follow [Tri97, 13.8] with appropriate modifications. Let $a_{\nu m}(x)$ with $\nu \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$ be an 1_K-atom ($\nu = 0$) or an $(s, p, \Psi)_{K,L}$-atom ($\nu \in \mathbb{N}$), where K and L are fixed integers satisfying (1.127) such that we have (1.128) with $\lambda \in f_{pq}$. For $j \in \mathbb{N}$ we have

$$2^{js}\Psi(2^{-j}) k_N(2^{-j}, a_{\nu m})(x) = 2^{js}\Psi(2^{-j}) \int_{\mathbb{R}^n} a_{\nu m}(x + 2^{-j}y) k_N(y)\, dy \tag{1.156}$$
$$= 2^{js}\Psi(2^{-j}) \int_{\mathbb{R}^n} a_{\nu m}(x + 2^{-j}y)(\Delta^N k^0)(y)\, dy.$$

We have to distinguish between $j \ge \nu$ and $j < \nu$. The exceptional values $\nu = 0$ and/or $j = 0$ corresponding to 1_K-atoms and the first summand in (1.37), respectively, can be incorporated in the following considerations after necessary modifications. Assume in the following that $\nu \in \mathbb{N}$ and $j \in \mathbb{N}$.

Let $j \ge \nu$. We put

$$a^{\nu m}(y) = 2^{\nu(s-n/p)} \Psi(2^{-\nu})\, a_{\nu m}(2^{-\nu}(y + m)). \tag{1.157}$$

Then $a^{\nu m}(x)$ is a 1_K-atom with respect to the unit cube at the origin. Let $K = 2M$ with $M \in \mathbb{N}_0$ for simplicity. The modifications necessary for K odd will be clear. We insert (1.157) in (1.156), choose $N > M$, and obtain by partial integration

$$(1.158)\quad \begin{aligned} &2^{js}\Psi(2^{-j})k_N(2^{-j},a_{\nu m})(x)\\ &= 2^{js}\Psi(2^{-j})2^{-\nu(s-n/p)}\Psi(2^{-\nu})^{-1}\int_{\mathbb{R}^n} a^{\nu m}(2^\nu(x+2^{-j}y)-m)(\Delta^N k^0)(y)\,dy\\ &= 2^{-(K-s)(j-\nu)}2^{n\nu/p}\Psi(2^{-j})\Psi(2^{-\nu})^{-1}\\ &\quad\times\int_{\mathbb{R}^n}(\Delta^{N-M}k^0)(y)(\Delta^M a^{\nu m})(2^\nu x-m+2^{\nu-j}y)\,dy. \end{aligned}$$

Since both k^0 and $\Delta^M a^{\nu m}$ have supports in a ball centred at the origin of some radius $c_1>0$, Proposition 1.4(vi) shows that

$$(1.159)\quad 2^{js}\Psi(2^{-j})|k_N(2^{-j},a_{\nu m})(x)|\le c_2 2^{-(K-s)(j-\nu)}(1+j-\nu)^b\widetilde{\chi}^{(p)}_{\nu m}(x),\qquad j\ge\nu,$$

where $\widetilde{\chi}^{(p)}_{\nu m}(x)$ is the p-normalised characteristic function of the cube $4c_1Q_{\nu m}$, $c_2>0$ and $b\ge 0$ are constants independent of j and ν.

Let now $j<\nu$ and put again $k_N(y)=(\Delta^N k^0)(y)$. Then the integration in

$$(1.160)\quad 2^{js}\Psi(2^{-j})\,k_N(2^{-j},a_{\nu m})(x)=2^{js}\Psi(2^{-j})2^{jn}\int_{\mathbb{R}^n}k_N(2^j y)a_{\nu m}(x+y)\,dy$$

can be restricted to $\{y\in\mathbb{R}^n : |y|\le c_1 2^{-j}\}$. Furthermore with L given by (1.127) we expand $k_N(2^j y)$ up to order L with respect to the off-point $2^{-\nu}m-x$ and obtain

$$(1.161)\quad k_N(2^j y)=\sum_{|\beta|\le L}c_\beta(x)(y-2^{-\nu}m+x)^\beta+2^{j(L+1)}\mathcal{O}(|y-2^{-\nu}m+x|^{L+1}).$$

We insert (1.161) in (1.160). By (1.123) the terms with $|\beta|\le L$ vanish. Since

$$(1.162)\quad |a_{\nu m}(x+y)|\le 2^{-\nu(s-n/p)}\Psi(2^{-\nu})^{-1}\widetilde{\chi}_{\nu m}(x+y),$$

where $\widetilde{\chi}_{\nu m}(x)$ is the characteristic function of $cQ_{\nu m}$, we obtain

$$(1.163)\quad \begin{aligned} &2^{js}\Psi(2^{-j})|k_N(2^{-j},a_{\nu m})(x)|\\ &\le 2^{(s+n)j}2^{-\nu(s-n/p)}\Psi(2^{-j})\Psi(2^{-\nu})^{-1}\int_{\{y:|y|\le c_1 2^{-j}\}}2^{j(L+1)}\widetilde{\chi}_{\nu m}(x+y)\mathcal{O}(|y-2^{-\nu}m+x|^{L+1})\,dy\\ &\le c_3 2^{(s+n)j}2^{-\nu(s-n/p)}\Psi(2^{-j})\Psi(2^{-\nu})^{-1}2^{(j-\nu)(L+1)}\int_{\{y:|y|\le c_1 2^{-j}\}}\widetilde{\chi}_{\nu m}(x+y)\,dy\\ &\le c_3' 2^{(s+n)j}2^{-\nu(s-n/p)}2^{(j-\nu)(L+1)}(1+\nu-j)^b\int_{\{y:|y|\le c_1 2^{-j}\}}\widetilde{\chi}_{\nu m}(x+y)\,dy. \end{aligned}$$

The last inequality in (1.163) is justified by Proposition 1.4(vi). Recall that $j<\nu$. The integral in (1.163) is at most $c^n 2^{-\nu n}$ and it is zero if x is outside a cube $c_4 2^{\nu-j}Q_{\nu m}$ (centred at $2^{-\nu}m$ and with side length $c_4 2^{-j}$). Hence,

$$(1.164)\quad \int_{|y|\le c_1 2^{-j}}\widetilde{\chi}_{\nu m}(x+y)\,dy\le c^n 2^{-\nu n}\chi(c_4 2^{\nu-j}Q_{\nu m})(x),$$

where $\chi(c_4 2^{\nu-j}\,Q_{\nu m})(x)$ is the characteristic function of the indicated cube. For $x\in c_4 2^{\nu-j}Q_{\nu m}$ we have

$$(1.165)\quad (\mathcal{M}\chi_{\nu m})(x)\ge |c_4 2^{\nu-j}\,Q_{\nu m}|^{-1}\int_{c_4 2^{\nu-j}Q_{\nu m}}\chi_{\nu m}(y)\,dy\ge c_4^{-n}2^{-(\nu-j)n}.$$

Let $0 < w < \min(1, p, q)$. From (1.165) and (1.164) we get

$$(1.166)\qquad \int\limits_{\{y:|y|\le c_1 2^{-j}\}} \widetilde{\chi}_{\nu m}(x+y)\,dy \le c_1^n 2^{-\nu n} c_4^{n/w} 2^{(\nu-j)n/w} (\mathcal{M}\chi_{\nu m})^{1/w}(x)$$
$$\le c_5 2^{-\nu n} 2^{(\nu-j)n/w} (\mathcal{M}\chi_{\nu m})^{1/w}(x), \qquad x \in \mathbb{R}^n.$$

Replacing $\chi_{\nu m}$ in (1.166) by $\chi_{\nu m}^{(p)}$ and inserting the estimate (1.166) in (1.163) we obtain

$$(1.167)\qquad 2^{sj}\Psi(2^{-j})|k_N(2^{-j}, a_{\nu m})(x)|$$
$$\le c2^{-(\nu-j)(L+1+s+n-n/w)}(1+\nu-j)^b (\mathcal{M}\chi_{\nu m}^{(p)w})^{1/w}(x), \qquad x \in \mathbb{R}^n.$$

Since $L \ge [\sigma_{pq} - s]$ the number w can be chosen in such a way that $\eta = L+1+s+n-n/w > 0$. Hence

$$(1.168)\qquad 2^{sj}\Psi(2^{-j})|k_N(2^{-j}, a_{\nu m})(x)| \le c2^{-(\nu-j)\eta}(1+\nu-j)^b (\mathcal{M}\chi_{\nu m}^{(p)w})^{1/w}(x), \qquad j < \nu,$$

with $\eta > 0$. Combining (1.159) and (1.168) we obtain, for $q \le 1$,

$$(1.169)\qquad \Big|2^{sj}\Psi(2^{-j})k_N\Big(2^{-j}, \sum_{\nu,m}\lambda_{\nu m}a_{\nu m}\Big)(x)\Big|^q$$
$$\le c\sum_{\nu\le j}\sum_m |\lambda_{\nu m}|^q 2^{-\varrho(j-\nu)q}(1+j-\nu)^{bq}\widetilde{\chi}_{\nu m}^{(p)q}(x)$$
$$+ c'\sum_{\nu>j}\sum_m |\lambda_{\nu m}|^q 2^{-\eta(\nu-j)q}(1+\nu-j)^{bq}(\mathcal{M}\chi_{\nu m}^{(p)w})^{1/w}(x)$$

for some $\varrho, \eta > 0$. We sum over j, take the $1/q$-power and afterwards the $L_p(\mathbb{R}^n)$-quasi-norm and arrive at

$$(1.170)\qquad \Big\|\Big(\sum_{j=1}^{\infty} 2^{jsq}\Psi(2^{-j})^q\Big|k_N\Big(2^{-j}, \sum_{\nu,m}\lambda_{\nu m}\,a_{\nu m}\Big)(x)\Big|^q\Big)^{1/q}\Big| L_p(\mathbb{R}^n)\Big\|$$
$$\le c\Big\|\Big(\sum_{\nu,m}|\lambda_{\nu m}|^q\widetilde{\chi}_{\nu m}^{(p)q}(\cdot)\Big)^{1/q}\Big| L_p(\mathbb{R}^n)\Big\|$$
$$+ c\Big\|\Big(\sum_{\nu,m}|\lambda_{\nu m}|^q(\mathcal{M}\chi_{\nu m}^{(p)w})^{q/w}(\cdot)\Big)^{1/q}\Big| L_p(\mathbb{R}^n)\Big\|.$$

We have also used the convergence of the series

$$\sum_{k=0}^{\infty} 2^{-\eta kq}(1+k)^{bq} \quad \text{and} \quad \sum_{k=0}^{\infty} 2^{-\varrho kq}(1+k)^{bq},$$

since $\eta, \varrho > 0$. The modification of (1.169) if $1 < q \le \infty$ is clear, by the Hölder inequality. Hence (1.170) holds for any $0 < q \le \infty$. The first summand on the right-hand side is just what we want, since $\widetilde{\chi}_{\nu m}^{(p)}$ can be replaced by $\chi_{\nu m}^{(p)}$. With $g_{\nu m}(x) = \lambda_{\nu m}\,\chi_{\nu m}^{(p)}(x)$ the second summand on the right-hand side can be written as

$$(1.171)\qquad c\Big\|\Big(\sum_{\nu m}(\mathcal{M}g_{\nu m}^w)(\cdot)^{q/w}\Big)^{w/q}\Big| L_{p/w}(\mathbb{R}^n)\Big\|^{1/w}.$$

Since $1 < q/w \le \infty$ and $1 < p/w < \infty$ we can apply the vector-valued maximal inequality of Fefferman and Stein (see [Tri92, 2.2.2, p. 89]). Then (1.171) can be estimated from above by

$$c\Big\|\Big(\sum_{\nu,m}(g^w_{\nu m}(\cdot))^{q/w}\Big)^{w/q}\,\Big|\,L_{p/w}(\mathbb{R}^n)\Big\|^{1/w} = c\|\lambda\,|\,f_{pq}\|. \tag{1.172}$$

This gives the required estimate. As already mentioned, the terms with $\nu = 0$ and/or $j = 0$ are also covered by this technique. Thus we obtain

$$\|f\,|\,F^{(s,\Psi)}_{pq}(\mathbb{R}^n)\| \le c\|\lambda\,|\,f_{pq}\|.$$

Let us just mention that in the corresponding proof of Step 1 for the $B^{(s,\Psi)}_{pq}(\mathbb{R}^n)$ spaces it is sufficient that w satisfies $0 < w < \min(1,p)$, since one only needs the scalar Hardy–Littlewood maximal theorem which holds also for $p = \infty$ (see [Tri83, Remark 1.2.3, p. 15]). This is the reason for the modification in (1.127) to (1.130).

Step 2 (only-if part of subatomic decomposition). Let $f \in F^{(s,\Psi)}_{pq}(\mathbb{R}^n)$. By (1.10) we have

$$\widehat{f} = \sum_{\nu=0}^{\infty}\varphi_\nu\widehat{f} \quad \text{(convergence in } \mathcal{S}'(\mathbb{R}^n)). \tag{1.173}$$

Let Q_ν be the cube in $\mathbb{R}^n$ centred at the origin and with side length $2\pi 2^\nu$. In particular we have $\operatorname{supp}\varphi_\nu \subset Q_\nu$. We interpret $\varphi_\nu\widehat{f}$ as a periodic distribution and expand it in Q_ν by

$$(\varphi_\nu\widehat{f})(\xi) = \sum_{k\in\mathbb{Z}^n} b_{\nu k}\exp(-i2^{-\nu}k\xi), \qquad \xi \in Q_\nu, \tag{1.174}$$

with

$$b_{\nu k} = (2\pi)^{-n}2^{-\nu n}\int_{Q_\nu}(\varphi_\nu\widehat{f})(\xi)\exp(-i2^{-\nu}k\xi)\,d\xi = (2\pi)^{-n/2}2^{-\nu n}(\varphi_\nu\widehat{f})^\vee(2^{-\nu}k). \tag{1.175}$$

Let $\varrho \in \mathcal{S}(\mathbb{R}^n)$ with $\varrho(x) = 1$ if $|x| \le 2$ and $\operatorname{supp}\varrho \subset \pi\,Q_{00}$ and let $\varrho_\nu(\xi) = \varrho(2^{-\nu}\xi)$, $\nu \in \mathbb{N}_0$. Then $\varrho_\nu(\xi) = 1$ if $\xi \in \operatorname{supp}\varphi_\nu$ and $\operatorname{supp}\varrho_\nu \subset Q_\nu$. We multiply (1.174) by ϱ_ν and extend it from Q_ν to $\mathbb{R}^n$. Hence

$$(\varphi_\nu\widehat{f})^\vee(x) = \sum_{k\in\mathbb{Z}^n} b_{\nu k}[\exp(-i2^{-\nu}k\cdot)\varrho_\nu(\cdot)]^\vee(x) = \sum_{k\in\mathbb{Z}^n} d_{\nu k}2^{-\nu(s-n/p)}\check{\varrho}(2^\nu x - k) \tag{1.176}$$

with

$$d_{\nu k} = (2\pi)^{-n/2}2^{\nu(s-n/p)}(\varphi_\nu\widehat{f})^\vee(2^{-\nu}k). \tag{1.177}$$

The entire function $\check{\varrho} \in \mathcal{S}(\mathbb{R}^n)$ can be extended from $\mathbb{R}^n$ to $\mathbb{C}^n$. Furthermore, by the Paley–Wiener–Schwartz theorem (see e.g. [Tri83, 1.2.1, p. 13]), for any $\lambda > 0$ and appropriate $c_\lambda > 0$,

$$|\check{\varrho}(x+iy)| \le c_\lambda e^{c|y|}(1+|x|)^{-\lambda}, \qquad x, y \in \mathbb{R}^n. \tag{1.178}$$

Iterative application of Cauchy's representation theorem in the complex plane yields

$$\check{\varrho}(z_1,\dots,z_n) = (2\pi i)^{-n}\int_{|\xi_1-z_1|=1}\cdots\int_{|\xi_n-z_n|=1}\frac{\check{\varrho}(\xi_1,\dots,\xi_n)}{(\xi_1-z_1)\dots(\xi_n-z_n)}\,d\xi_1\dots\xi_n, \tag{1.179}$$

where $z_k \in \mathbb{C}$. By (1.178) we obtain

$$|D^\alpha\check{\varrho}(x)| \le c'_\lambda\alpha!(1+|x|)^{-\lambda}, \qquad x \in \mathbb{R}^n, \tag{1.180}$$

where c'_λ is independent of $x \in \mathbb{R}^n$ and $\alpha \in \mathbb{N}_0^n$. Let Φ be as in Definition 1.21 and let as there $\Phi^\beta(x) = x^\beta \Phi(x)$ where $\beta \in \mathbb{N}_0^n$. With $\mu \in \mathbb{N}_0$ fixed, from (1.176),

$$(\varphi_\nu \widehat{f}\,)^\vee(x) = \sum_{k\in\mathbb{Z}^n} d_{\nu k} 2^{-\nu(s-n/p)} \sum_{m\in\mathbb{Z}^n} \check{\varrho}(2^\nu x - k)\Phi(2^{(\nu+\mu)}x - m). \tag{1.181}$$

We expand $\check{\varrho}(2^\nu \cdot -k)$ at the point $2^{-(\nu+\mu)}m$, where $m \in \mathbb{Z}^n$ and $\mu \in \mathbb{N}_0$ are fixed. Then we obtain

$$\check{\varrho}(2^\nu x - k) = \sum_{\beta\in\mathbb{N}_0^n} \frac{2^{\nu|\beta|}}{\beta!}(D^\beta \check{\varrho})(2^{-\mu}m - k)(x - 2^{-(\nu+\mu)}m)^\beta. \tag{1.182}$$

Putting (1.182) in (1.181) gives

$$(\varphi_\nu \widehat{f}\,)^\vee(x) = 2^{-\nu(s-n/p)} \sum_{k\in\mathbb{Z}^n} d_{\nu k} \sum_{m\in\mathbb{Z}^n} \sum_{\beta\in\mathbb{N}_0^n} \frac{2^{-\mu|\beta|}}{\beta!}(D^\beta\check{\varrho})(2^{-\mu}m - k)\Phi^\beta(2^{(\nu+\mu)}x - m).$$

We insert this last equality in (1.173) to get

$$\begin{aligned} f = \sum_{\beta\in\mathbb{N}_0^n} \sum_{\nu=0}^\infty \sum_{m\in\mathbb{Z}^n} & 2^{\mu(s-n/p)}\Psi(2^{-(\nu+\mu)})^{-1}2^{-(\nu+\mu)(s-n/p)}\Phi^\beta(2^{(\nu+\mu)}x - m) \\ & \times \sum_{k\in\mathbb{Z}^n} d_{\nu k}\frac{(D^\beta\check{\varrho})(2^{-\mu}m-k)}{\beta!}2^{-\mu|\beta|}\Psi(2^{-(\nu+\mu)}) \\ = \sum_{\beta\in\mathbb{N}_0^n} \sum_{\nu=0}^\infty \sum_{m\in\mathbb{Z}^n} & \lambda^\beta_{\nu+\mu,m}2^{\mu(s-n/p)}(\beta qu)_{\nu+\mu,m}(x), \end{aligned} \tag{1.183}$$

where $(\beta qu)_{\nu+\mu,m}(x)$ are (s,p,Ψ)-β-quarks and

$$\lambda^\beta_{\nu+\mu,m} = 2^{-\mu|\beta|}\sum_{k\in\mathbb{Z}^n} d_{\nu k}\frac{(D^\beta\check{\varrho})(2^{-\mu}m-k)}{\beta!}\Psi(2^{-(\nu+\mu)}) = 2^{-\mu|\beta|}\vartheta^\beta_{\nu+\mu,m}. \tag{1.184}$$

We may replace in (1.183) $\nu+\mu$ by ν and obtain (1.152). As in (1.150) we denote by λ^β or ϑ^β the collection of all respective coefficients in (1.184). We wish to prove that there is $c_\mu > 0$ (independent of β) such that for all $\beta \in \mathbb{N}_0^n$,

$$2^{\mu|\beta|}\|\lambda^\beta \mid f_{pq}\| \le c_\mu \|f \mid F^{(s,\Psi)}_{pq}(\mathbb{R}^n)\|. \tag{1.185}$$

We prove (1.185) in two steps. On the one hand we prove the existence of a constant $c' > 0$, independent of β, such that

$$\|\{d_{\nu k}\Psi(2^{-\nu})\}_{\nu,k} \mid f_{pq}\| \le c'\|f \mid F^{(s,\Psi)}_{pq}(\mathbb{R}^n)\|, \tag{1.186}$$

and on the other hand that there exists a constant $c > 0$, independent of β, such that

$$\|\vartheta^\beta \mid f_{pq}\| \le c\|\{d_{\nu k}\Psi(2^{-\nu})\}_{\nu,k} \mid f_{pq}\|. \tag{1.187}$$

Let us begin the proof of (1.186). For fixed $\nu \in \mathbb{N}_0$ we have

$$\begin{aligned} \sum_{k\in\mathbb{Z}^n} |d_{\nu k}\Psi(2^{-\nu})\chi^{(p)}_{\nu k}(x)|^q &\le (2\pi)^{-nq/2}2^{\nu sq}\Psi(2^{-\nu})^q \sum_{k\in\mathbb{Z}^n}\chi_{\nu k}(x)\big(\sup_{y\in Q_{\nu k}}|(\varphi_\nu\widehat{f}\,)^\vee(y)|\big)^q \\ &\le (1+\sqrt{n})^{aq}(2\pi)^{-nq/2}2^{\nu sq}\Psi(2^{-\nu})^q(\varphi^*_\nu f)^q_a(x), \end{aligned}$$

since for $x, y \in Q_{\nu k}$, $|x-y| \le \sqrt{n}\, 2^{-\nu}$, $\sum_{k\in\mathbb{Z}^n} \chi_{\nu k}(x) = 1$, and $(\varphi_\nu^* f)_a$ is the Peetre maximal function with $a > n/\min(p,q)$. It follows that

$$\sum_{\nu=0}^{\infty} \sum_{k\in\mathbb{Z}^n} |d_{\nu k} \Psi(2^{-\nu}) \chi_{\nu k}^{(p)}(x)|^q \le c_1'^q \sum_{\nu=0}^{\infty} 2^{\nu s q} \Psi(2^{-\nu})^q (\varphi_\nu^* f)_a^q(x).$$

Now, using Theorem 1.7, we get

$$\begin{aligned} \|\{d_{\nu k}\Psi(2^{-\nu})\}_{\nu,k} \mid f_{pq}\| &\le c_1' \Big\| \Big(\sum_{\nu=0}^{\infty} 2^{\nu s q} \Psi(2^{-\nu})^q (\varphi_\nu^* f)_a^q(x) \Big)^{1/q} \Big| L_p(\mathbb{R}^n) \Big\| \\ &\le c' \|f \mid F_{pq}^{(s,\Psi)}(\mathbb{R}^n)\| \end{aligned}$$

with $c' > 0$ independent of β, which completes the proof of (1.186).

Let us now prepare to show (1.187). Fix $\nu \in \mathbb{N}_0$ and $k \in \mathbb{Z}^n$. Recall that by Proposition 1.4(vi),

$$\Psi(2^{-(\nu+\mu)}) \le c(1+\mu)^b \Psi(2^{-\nu}).$$

By (1.180) there exists a positive constant c_λ'', independent of β (but may depend on μ, λ and Ψ), with

$$\begin{aligned} |\vartheta_{\nu+\mu,m}^{\beta}| &\le \sum_{k\in\mathbb{Z}^n} \frac{|(D^\beta \check{\varrho})(2^{-\mu}m - k)|}{\beta!} |d_{\nu k}| \Psi(2^{-(\nu+\mu)}) \\ &\le c_\lambda'' \sum_{k\in\mathbb{Z}^n} (1 + |2^\nu(2^{-(\nu+\mu)}m) - k)|)^{-\lambda} |d_{\nu k}|\, \Psi(2^{-\nu}). \end{aligned} \tag{1.188}$$

We set $x_m = 2^{-(\nu+\mu)}m$ and let $k_m \in \mathbb{Z}^n$ be such that $x_m \in Q_{\nu,k_m}$; then clearly $|2^\nu x_m - k_m| \le \sqrt{n}/2$. We decompose $\mathbb{Z}^n$ into the sets

$$E_j = \{k \in \mathbb{Z}^n : 2^j - 1 \le |k - k_m| \le 2^{j+1} - 1\}, \qquad j \in \mathbb{N}_0.$$

If $j \in \mathbb{N}_0$ is fixed, for $k \in E_j$ we have on the one hand

$$2^j \le 1 + |k - k_m| \le 1 + |k - 2^\nu x_m| + |2^\nu x_m - k_m| \le \max(\sqrt{n}, 2)(1 + |2^\nu x_m - k|)$$

and so

$$(1 + |2^\nu x_m - k|)^{-\lambda} \le c_1 2^{-j\lambda}, \tag{1.189}$$

where $c_1 > 0$ is independent of ν, k and m. On the other hand, if $x \in Q_{\nu+\mu,m}$ and $y \in Q_{\nu k}$, then

$$\begin{aligned} |y - x| &\le |y - 2^{-\nu}k| + 2^{-\nu}|k - k_m| + |2^{-\nu}k_m - x_m| + |x_m - x| \\ &\le \sqrt{n}(1 + 2^{-\mu-1})2^{-\nu}(1 + |k - k_m|) \le c_2 2^{-\nu+j} \end{aligned} \tag{1.190}$$

where $c_2 > 0$ is independent of ν, k, m but may depend on μ and n. Choose now $0 < w < \min(1, p, q)$. For a fixed $\nu \in \mathbb{N}_0$ the cubes $Q_{\nu k}$ have volume $2^{-\nu n}$ and are disjoint so that using the embedding $\ell_w \hookrightarrow \ell_1$ and (1.190) we obtain

$$\begin{aligned} \sum_{k\in E_j} |d_{\nu k}| &\le \Big(\sum_{k\in E_j} |d_{\nu k}|^w \Big)^{1/w} = \Big(2^{\nu n} \int_{|y-x|\le c_2 2^{j-\nu}} \Big(\sum_{k\in E_j} |d_{\nu k}| \chi_{\nu k}(y) \Big)^w dy \Big)^{1/w} \\ &\le c_3 \Big(2^{jn} \mathcal{M}\Big(\sum_{k\in\mathbb{Z}^n} |d_{\nu k}|\, \chi_{\nu k} \Big)^w (x) \Big)^{1/w} \end{aligned} \tag{1.191}$$

for $x \in Q_{\nu+\mu,m}$, where $\mathcal{M}$ stands for the Hardy–Littlewood maximal function and c_3 is a constant independent of ν, m, k. Using (1.191) and (1.189) in (1.188) and assuming that $\lambda > n/w$ is sufficiently large we have

$$
\begin{aligned}
(1.192)\quad & |\vartheta^{\beta}_{\nu+\mu,m}\chi^{(p)}_{\nu+\mu,m}(x)| \\
&\le c''_\lambda \sum_{k\in\mathbb{Z}^n} (1+|2^\nu x_m - k)|)^{-\lambda}|d_{\nu k}|\Psi(2^{-\nu})2^{(\nu+\mu)n/p}\chi_{\nu+\mu,m}(x) \\
&\le c_4 \sum_{j=0}^{\infty} 2^{-j\lambda} \sum_{k\in E_j} |d_{\nu k}|\Psi(2^{-\nu})2^{(\nu+\mu)n/p}\chi_{\nu+\mu,m}(x) \\
&\le c_5 \sum_{j=0}^{\infty} 2^{-j(\lambda-\frac{n}{w})}\Big(\mathcal{M}\Big(\sum_{k\in\mathbb{Z}^n}|d_{\nu k}|\chi^{(p)}_{\nu k}\Big)^{w}(x)\Big)^{1/w}\Psi(2^{-\nu})\chi_{\nu+\mu,m}(x) \\
&\le c_6\Psi(2^{-\nu})\Big(\mathcal{M}\Big(\sum_{k\in\mathbb{Z}^n}|d_{\nu k}|\chi^{(p)}_{\nu k}\Big)^{w}(x)\Big)^{1/w}\chi_{\nu+\mu,m}(x)
\end{aligned}
$$

where the constants above do not depend on ν, m but may depend on μ. In (1.192) we take the q-power, sum over $m \in \mathbb{Z}^n$ and then over $\nu \in \mathbb{N}_0$ to get

$$\sum_{\nu=0}^{\infty}\sum_{m\in\mathbb{Z}^n}|\vartheta^{\beta}_{\nu+\mu,m}\chi^{(p)}_{\nu+\mu,m}(x)|^q \le c_6^q \sum_{\nu=0}^{\infty}(h^w_\nu(x))^{q/w}$$

where $h_\nu = \Psi(2^{-\nu})\sum_{k\in\mathbb{Z}^n}|d_{\nu k}|\,\chi^{(p)}_{\nu k}$ (with the usual modification if $q=\infty$). Taking the $1/q$-power and the L_p-quasi-norm, and applying the Fefferman–Stein inequality, as in [Tri92, 2.2.2, p. 89], since $1 < p/w < \infty$ and $1 < q/w \le \infty$, we arrive at

$$
\begin{aligned}
\|\vartheta^\beta \mid f_{pq}\| &\le c_6\,\|\mathcal{M}(h^w_\nu(\cdot))^{1/w} \mid L_p(\ell_q)\| = c_6\|\mathcal{M}(h^w_\nu(\cdot)) \mid L_{p/w}(\ell_{q/w})\|^{1/w} \\
&\le c\|h^w_\nu(\cdot) \mid L_{p/w}(\ell_{q/w})\|^{1/w} = c\|h_\nu(\cdot) \mid L_p(\ell_q)\| = c\|\Psi(2^{-\nu})d_{\nu k} \mid f_{pq}\|,
\end{aligned}
$$

which finishes the proof of (1.187). And so the only-if part of the subatomic decomposition for $F^{(s,\Psi)}_{pq}(\mathbb{R}^n)$ is complete.

In what concerns the corresponding proof for $B^{(s,\Psi)}_{pq}(\mathbb{R}^n)$, one has to obtain analogous inequalities to (1.186) and (1.187) (of course with b_{pq} instead of f_{pq}). The counterpart of (1.186) can be proved using the arguments in [Tri97, 14.15, p. 102]. For the counterpart of (1.187), one can use (1.126) and in the proof it is sufficient to choose $0 < w < \min(1,p)$ and use the scalar Hardy–Littlewood maximal inequality which holds also for $p=\infty$ (see [Ste70, 1.3, p. 5]).

Step 3 (only-if part of atomic decomposition). Let $f \in F^{(s,\Psi)}_{pq}(\mathbb{R}^n)$. First consider $s > \sigma_{pq}$ and fix $K \in \mathbb{N}_0$ and $L=-1$ satisfying (1.127). By Step 2,

$$f = \sum_{\beta\in\mathbb{N}_0^n}\sum_{\nu=0}^{\infty}\sum_{m\in\mathbb{Z}^n}\lambda^\beta_{\nu m}(\beta qu)_{\nu m}(x), \tag{1.193}$$

where $(\beta qu)_{\nu m}$ are (s,p,Ψ)-β-quarks and

$$\sup_{\beta\in\mathbb{N}_0^n} 2^{\mu|\beta|}\|\lambda^\beta \mid f_{pq}\| \le c_\mu\|f \mid F^{(s,\Psi)}_{pq}(\mathbb{R}^n)\| \tag{1.194}$$

for any $\mu > 0$. Let

$$a_{\nu m}(x) = \sum_{\beta \in \mathbb{N}_0^n} \frac{\lambda^\beta_{\nu m}}{\Lambda_{\nu m}} (\beta q u)_{\nu m}(x), \qquad \nu \in \mathbb{N}_0,\ m \in \mathbb{Z}^n, \tag{1.195}$$

with

$$\Lambda_{\nu m} = c \sum_{\beta \in \mathbb{N}_0^n} |\lambda^\beta_{\nu m}| 2^{\kappa|\beta|} \tag{1.196}$$

(with the additional factor $\Psi(1)$ on the right-hand side if $\nu = 0$) with κ and c being positive constants as in Lemma 1.22 (the constants are independent of β but may depend on K, L and n). Then

$$f = \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \Lambda_{\nu m} a_{\nu m}(x). \tag{1.197}$$

By straightforward calculations, using (1.145) and (1.148), $a_{\nu m}$ are $(s, p, \Psi)_{K,-1}$-atoms if $\nu \in \mathbb{N}$, and 1_K-atoms if $\nu = 0$. Finally, we will show that there exists a constant $c > 0$ such that

$$\|\Lambda \mid f_{pq}\| \le c \|f \mid F^{(s,\Psi)}_{pq}(\mathbb{R}^n)\|,$$

where

$$\Lambda = \{\Lambda_{\nu m} : \nu \in \mathbb{N}_0,\ m \in \mathbb{Z}^n\}.$$

This will be done by showing that

$$\|\Lambda \mid f_{pq}\| \le c' \sup_{\beta \in \mathbb{N}_0^n} 2^{\mu|\beta|} \|\lambda^\beta \mid f_{pq}\|$$

for some $\mu > 0$ sufficiently large.

If $0 < q < 1$, then with q_1 the conjugate exponent of $1/q$ and $\varrho > \kappa$, we have

$$\begin{aligned} \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} |\Lambda_{\nu m} \chi^{(p)}_{\nu m}(x)|^q &\le c^q \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \sum_{\beta \in \mathbb{N}_0^n} |\lambda^\beta_{\nu m} \chi^{(p)}_{\nu m}(x)|^q 2^{\kappa|\beta| q} \\ &\le c^q \Big(\sum_{\beta \in \mathbb{N}_0^n} 2^{(\kappa-\varrho)|\beta| q q_1} \Big)^{1/q_1} \Big\{ \sum_{\beta \in \mathbb{N}_0^n} \Big(2^{\varrho|\beta| q} \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} |\lambda^\beta_{\nu m} \chi^{(p)}_{\nu m}(x)|^q \Big)^{1/q} \Big\}^q \\ &\le c_1^q \Big\{ \sum_{\beta \in \mathbb{N}_0^n} 2^{\varrho|\beta| q} \Big(\sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} |\lambda^\beta_{\nu m} \chi^{(p)}_{\nu m}(x)|^q \Big)^{1/q} \Big\}^q. \end{aligned} \tag{1.198}$$

Taking the $1/q$-power of (1.198) gives

$$\Big(\sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} |\Lambda_{\nu m} \chi^{(p)}_{\nu m}(x)|^q \Big)^{1/q} \le c_1 \sum_{\beta \in \mathbb{N}_0^n} 2^{\varrho|\beta| q} \Big(\sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} |\lambda^\beta_{\nu m} \chi^{(p)}_{\nu m}(x)|^q \Big)^{1/q}. \tag{1.199}$$

Letting $\overline{p} = \min(1, p)$, from (1.199) we get

$$\begin{aligned} &\Big\| \Big(\sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} |\Lambda_{\nu m} \chi^{(p)}_{\nu m}(x)|^q \Big)^{1/q} \Big| L_p(\mathbb{R}^n) \Big\|^{\overline{p}} \\ &\qquad \le c_1 \sum_{\beta \in \mathbb{N}_0^n} 2^{\varrho|\beta| q} \Big\| \Big(\sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} |\lambda^\beta_{\nu m} \chi^{(p)}_{\nu m}(x)|^q \Big)^{1/q} \Big| L_p(\mathbb{R}^n) \Big\|^{\overline{p}}. \end{aligned}$$

Hence, for any $\mu > \varrho$, we have

$$\|\Lambda \,|\, f_{pq}\| \le c_2 \sup_{\beta\in\mathbb{N}_0^n} 2^{\mu|\beta|}\|\lambda^\beta \,|\, f_{pq}\|. \tag{1.200}$$

If $1 \le q \le \infty$, let q' be its conjugate exponent and $\varepsilon > 0$. Then

$$\begin{aligned}\sum_{\nu=0}^{\infty}\sum_{m\in\mathbb{Z}^n} |\Lambda_{\nu m}\chi_{\nu m}^{(p)}(x)|^q \\ &= c^q \sum_{\nu=0}^{\infty}\sum_{m\in\mathbb{Z}^n}\Big(\sum_{\beta\in\mathbb{N}_0^n} 2^{-\varepsilon|\beta|}2^{(\kappa+\varepsilon)|\beta|}|\lambda_{\nu m}^\beta\chi_{\nu m}^{(p)}(x)|\Big)^q \\ &\le c^q \sum_{\nu=0}^{\infty}\sum_{m\in\mathbb{Z}^n}\Big(\sum_{\beta\in\mathbb{N}_0^n} 2^{-\varepsilon|\beta|q'}\Big)^{q/q'}\sum_{\beta\in\mathbb{N}_0^n} 2^{(\kappa+\varepsilon)|\beta|q}|\lambda_{\nu m}^\beta\chi_{\nu m}^{(p)}(x)|^q \\ &\le c_2^q \sum_{\beta\in\mathbb{N}_0^n} 2^{(\kappa+\varepsilon)|\beta|q}\sum_{\nu=0}^{\infty}\sum_{m\in\mathbb{Z}^n} |\lambda_{\nu m}^\beta\chi_{\nu m}^{(p)}(x)|^q.\end{aligned} \tag{1.201}$$

Taking the $1/q$-power of (1.201), we obtain

$$\Big(\sum_{\nu=0}^{\infty}\sum_{m\in\mathbb{Z}^n} |\Lambda_{\nu m}\chi_{\nu m}^{(p)}(x)|^q\Big)^{1/q} \le c_2 \sum_{\beta\in\mathbb{N}_0^n} 2^{(\kappa+\varepsilon)|\beta|}\Big(\sum_{\nu=0}^{\infty}\sum_{m\in\mathbb{Z}^n} |\lambda_{\nu m}^\beta\chi_{\nu m}^{(p)}(x)|^q\Big)^{1/q}, \tag{1.202}$$

which is analogous to (1.199); the rest follows as in case $0 < q < 1$. So we get an inequality as in (1.200) for $\mu > \kappa + \varepsilon$ in this case.

Now let $s \in \mathbb{R}$ be arbitrary and fix $K, L \in \mathbb{N}_0$ satisfying (1.127). Choose $M \in \mathbb{N}$ such that $2M > L$. As remarked in [Tri97, 13.8, p. 80] we can change the lift described in (1.26) to $I_{2M} f = ((1+|\xi|^{2M})\widehat{f}\,)^\vee$ with inverse

$$\mathrm{id} + (-\Delta)^M : F_{pq}^{(s+2M,\Psi)}(\mathbb{R}^n) \to F_{pq}^{(s,\Psi)}(\mathbb{R}^n).$$

So $f \in F_{pq}^{(s,\Psi)}(\mathbb{R}^n)$ can be represented as

$$f = g + (-\Delta)^M g \quad \text{with} \quad \|f \,|\, F_{pq}^{(s,\Psi)}(\mathbb{R}^n)\| \sim \|g \,|\, F_{pq}^{(s+2M,\Psi)}(\mathbb{R}^n)\|. \tag{1.203}$$

We apply this argument to g with $s + 2M$ in place of s. Iteration yields

$$f = f_1 + (-\Delta)^M f_2 \quad \text{with} \quad f_1 \in F_{pq}^{(s+2jM,\Psi)}(\mathbb{R}^n),\ f_2 \in F_{pq}^{(s+2M,\Psi)}(\mathbb{R}^n), \tag{1.204}$$

and

$$\|f_1 \,|\, F_{pq}^{(s+2jM,\Psi)}(\mathbb{R}^n)\| \sim \|f \,|\, F_{pq}^{(s,\Psi)}(\mathbb{R}^n)\| \sim \|f_2 \,|\, F_{pq}^{(s+2M,\Psi)}(\mathbb{R}^n)\|, \tag{1.205}$$

where j can be chosen arbitrarily large. We choose $\sigma > K$ and iterate as indicated above until the level j such that $s + 2jM - n/p > \sigma$. Hence, by Proposition 1.9, we have the embeddings

$$F_{p,q}^{(s+2jM,\Psi)}(\mathbb{R}^n) \hookrightarrow B_{\infty,\infty}^\sigma(\mathbb{R}^n) = \mathcal{C}^\sigma(\mathbb{R}^n),$$

and the inequalities

$$\|f_1 \,|\, \mathcal{C}^\sigma(\mathbb{R}^n)\| \le c\|f_1 \,|\, F_{pq}^{(s+2jM,\Psi)}(\mathbb{R}^n)\| \le c'\|f \,|\, F_{pq}^{(s,\Psi)}(\mathbb{R}^n)\|,$$

where $\mathcal{C}^\sigma(\mathbb{R}^n)$ is the Hölder–Zygmund space (see e.g. [Tri97, 10.5(iv)]). We decompose

$$f_1(x) = \sum_{m\in\mathbb{Z}^n} \lambda_{0m} a_{0m}(x) \tag{1.206}$$

with

$$\lambda_{0m} = c_1' \sum_{|\alpha|\le K} \sup_{|y-m|\le\sqrt{n}d/2} |D^\alpha f_1(y)|, \qquad m\in\mathbb{Z}^n, \tag{1.207}$$

and

$$a_{0m}(x) = \lambda_{0m}^{-1} \varPhi(x-m) f_1(x), \tag{1.208}$$

provided that $\lambda_{0m}\neq 0$ (otherwise we set $a_{0m}=0$), where

$$c_1' = \sup_{x\in\mathbb{R}^n}\sup_{|\delta|\le K} |D^\delta \varPhi(x)| \sup_{|\alpha|\le K}\sup_{\beta\le\alpha}\binom{\alpha}{\beta},$$

$\varPhi$ and $d>1$ as in (1.143). It follows by straightforward calculations that $a_{0m}(x)$ are 1_K-atoms. Note that $s+2M>\sigma_{pq}$, due to the choice of M and to the fact that L satisfies (1.127). Hence, as proved above in the first part of Step 3, $f_2\in F_{pq}^{(s+2M,\varPsi)}(\mathbb{R}^n)$ has an atomic decomposition

$$f_2 = \sum_{\nu=0}^\infty \sum_{m\in\mathbb{Z}^n} \varLambda_{\nu m}\, a_{\nu m}(x)$$

with $a_{\nu m}(x)$ being $(s+2M,p,\varPsi)_{K+2M,-1}$-atoms and

$$\|\varLambda \mid f_{pq}\| \le c\,\|f_2 \mid F_{pq}^{s+2M}(\mathbb{R}^n)\|.$$

So,

$$(-\varDelta)^M f_2 = \sum_{\nu=0}^\infty \sum_{m\in\mathbb{Z}^n} \varLambda_{\nu m}\,(-\varDelta)^M a_{\nu m}(x). \tag{1.209}$$

It can be easily seen that $(-\varDelta)^M a_{\nu m}(x)$ are $(s,p,\varPsi)_{K,L}$-atoms, where we use $2M>L$. Furthermore, we have

$$\|\varLambda \mid f_{pq}\| \le c\|f_2 \mid F_{pq}^{s+2M}(\mathbb{R}^n)\| \le c'\|f \mid F_{pq}^{s}(\mathbb{R}^n)\|. \tag{1.210}$$

To complete this step we still have to prove the inequality

$$\Big\|\Big(\sum_{m\in\mathbb{Z}^n} |\lambda_{0m}\chi_{0m}^{(p)}(\cdot)|^q\Big)^{1/q} \Big| L_p(\mathbb{R}^n)\Big\| \le c\|f \mid F_{pq}^{(s,\varPsi)}(\mathbb{R}^n)\|.$$

We have

$$\begin{aligned}\sum_{m\in\mathbb{Z}^n} |\lambda_{0m}\chi_{0m}^{(p)}(x)|^q &= c_1'^q \sum_{m\in\mathbb{Z}^n}\Big(\sum_{|\alpha|\le K}\sup_{|y-m|\le\sqrt{n}d/2}|D^\alpha f_1(y)|\Big)^q \chi_{0m}(x)\\ &\le c_2' \sum_{m\in\mathbb{Z}^n}\sum_{|\alpha|\le K}(\sup_{|x-y|\le\sqrt{n}d}|D^\alpha f_1(y)|)^q\chi_{0m}(x)\\ &= c_2'\sum_{|\alpha|\le K}(\sup_{|x-y|\le\sqrt{n}d}|D^\alpha f_1(y)|)^q.\end{aligned}$$

Taking the $1/q$-power of the last inequality and then the L_p-quasi-norm we get

$$(1.211)\quad \Big\| \Big(\sum_{m\in\mathbb{Z}^n} |\lambda_{0m} \chi_{0m}^{(p)}(x)|^q \Big)^{1/q} \Big| L_p(\mathbb{R}^n) \Big\| \\ \leq c_3' \sum_{|\alpha|\leq K} \| \sup_{|x-y|\leq\sqrt{n}d} |D^\alpha f_1(y)| \,|\, L_p(\mathbb{R}^n)\| \\ \leq c_4' \|f_1 \,|\, F_{pq}^{(s+2jM,\Psi)}(\mathbb{R}^n)\| \leq c_5' \|f \,|\, F_{pq}^{(s,\Psi)}(\mathbb{R}^n)\|.$$

We have also made use of formula (13.62) of [Tri97, p. 81]. From (1.204), (1.206) and (1.209)–(1.211) we get what we wanted.

Step 4 (if-part of subatomic decomposition). Assume that $f \in \mathcal{S}'(\mathbb{R}^n)$ satisfies (1.152) and (1.153). We will show that $f \in F_{pq}^{(s,\Psi)}(\mathbb{R}^n)$ and

$$(1.212)\quad \|f \,|\, F_{pq}^{(s,\Psi)}(\mathbb{R}^n)\| \leq c' \sup_{\beta\in\mathbb{N}_0^n} 2^{\mu|\beta|} \|\lambda^\beta \,|\, f_{pq}\|$$

for some positive constant c'. We decompose the representation (1.152) as

$$f = \sum_{\beta\in\mathbb{N}_0^n} f^\beta$$

with

$$(1.213)\quad f^\beta = \sum_{\nu=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \lambda_{\nu m}^\beta (\beta qu)_{\nu m}(x).$$

Let $K \in \mathbb{N}$ with $K > s$ and $L = -1$. By Lemma 1.22 the (s,p,Ψ)-β-quarks are $(s,p,\Psi)_{K,L}$-atoms multiplied by $c2^{\kappa|\beta|}$, where $c, \kappa > 0$ are independent of β. It follows by Step 1 that (1.213) converges in $\mathcal{S}'(\mathbb{R}^n)$, $f^\beta \in F_{pq}^{(s,\Psi)}(\mathbb{R}^n)$ and

$$(1.214)\quad \|f^\beta \,|\, F_{pq}^{(s,\Psi)}(\mathbb{R}^n)\| \leq c_1 2^{\kappa|\beta|} \|\lambda^\beta \,|\, f_{pq}\|,$$

where $c_1, \kappa > 0$ are independent of β. So, for $\mu > \kappa$,

$$(1.215)\quad \|f^\beta \,|\, F_{pq}^{(s,\Psi)}(\mathbb{R}^n)\| \leq c_1 2^{(\kappa-\mu)|\beta|} \sup_{\beta\in\mathbb{N}_0^n} 2^{\mu|\beta|} \|\lambda^\beta \,|\, f_{pq}\|.$$

Applying the t-triangle inequality, where $t = \min(1,p,q)$, and using (1.215) we get

$$\Big\| \sum_{\beta\in\mathbb{N}_0^n} f^\beta \Big| F_{pq}^{(s,\Psi)}(\mathbb{R}^n) \Big\| \leq \Big(\sum_{\beta\in\mathbb{N}_0^n} \|f^\beta \,|\, F_{pq}^{(s,\Psi)}(\mathbb{R}^n)\|^t \Big)^{1/t} \\ \leq c_1 \Big(\sum_{\beta\in\mathbb{N}_0^n} 2^{(\kappa-\mu)t|\beta|} \Big)^{1/t} \sup_{\beta\in\mathbb{N}_0^n} 2^{\mu|\beta|} \|\lambda^\beta \,|\, f_{pq}\| \\ \leq c_2 \sup_{\beta\in\mathbb{N}_0^n} 2^{\mu|\beta|} \|\lambda^\beta \,|\, f_{pq}\|,$$

and this is just (1.212). We remark that in this step the restriction (1.151) was essential for the use of atomic decomposition with no moment conditions on the atoms. ■

REMARK 1.26. The coefficients $\lambda_{\nu m}^\beta$ depend linearly on f. This follows from (1.184) and (1.177).

We now state the subatomic decomposition for an arbitrary smoothness parameter s.

COROLLARY 1.27. (i) *Let* $0 < p < \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$ *and* Ψ *an admissible function. Fix* $\sigma \in \mathbb{R}$ *and* L *with* $(L+1)/2 \in \mathbb{N}_0$ *such that*

$$\sigma > \max(\sigma_{pq}, s) \quad \text{and} \quad L \geq \max(-1, [\sigma_{pq} - s]). \tag{1.216}$$

Let $(\beta qu)_{\nu m}$ *be* (σ, p, Ψ)*-*β*-quarks and let* $(\beta qu)^L_{\nu m}$ *be* $(s, p, \Psi)_L$*-*β*-quarks. There exists* $\kappa > 0$ *with the following property: let* $\mu > \kappa$*; then* $f \in \mathcal{S}'(\mathbb{R}^n)$ *belongs to* $F^{(s,\Psi)}_{pq}(\mathbb{R}^n)$ *if, and only if, it can be represented as*

$$f = \sum_{\beta \in \mathbb{N}_0^n} \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda^\beta_{\nu m} (\beta qu)_{\nu m}(x) + \varrho^\beta_{\nu m} (\beta qu)^L_{\nu m}(x), \tag{1.217}$$

convergence being in $\mathcal{S}'(\mathbb{R}^n)$*, with*

$$\sup_{\beta \in \mathbb{N}_0^n} 2^{\mu|\beta|} (\|\lambda^\beta \,|\, f_{pq}\| + \|\varrho^\beta \,|\, f_{pq}\|) < \infty. \tag{1.218}$$

Furthermore, the infimum of (1.218) *over all representations* (1.217) *is an equivalent quasi-norm in* $F^{(s,\Psi)}_{pq}(\mathbb{R}^n)$.

(ii) *Let* $0 < p, q \leq \infty$, $s \in \mathbb{R}$ *and* Ψ *an admissible function. Fix* $\sigma \in \mathbb{R}$ *and* L *with* $(L+1)/2 \in \mathbb{N}_0$ *such that*

$$\sigma > \max(\sigma_p, s) \quad \text{and} \quad L \geq \max(-1, [\sigma_p - s]). \tag{1.219}$$

Let $(\beta qu)_{\nu m}$ *be* (σ, p, Ψ)*-*β*-quarks and let* $(\beta qu)^L_{\nu m}$ *be* $(s, p, \Psi)_L$*-*β*-quarks. There exists* $\kappa > 0$ *with the following property: let* $\mu > \kappa$*; then* $f \in \mathcal{S}'(\mathbb{R}^n)$ *belongs to* $B^{(s,\Psi)}_{pq}(\mathbb{R}^n)$ *if, and only if, it can be represented as in* (1.217) *and*

$$\sup_{\beta \in \mathbb{N}_0^n} 2^{\mu|\beta|} (\|\lambda^\beta \,|\, b_{pq}\| + \|\varrho^\beta \,|\, b_{pq}\|) < \infty. \tag{1.220}$$

Furthermore, the infimum of (1.220) *over all representations* (1.217) *is an equivalent quasi-norm in* $B^{(s,\Psi)}_{pq}(\mathbb{R}^n)$.

Proof. We prove (i). Obvious modifications also prove (ii).

Step 1. Let f be represented by (1.217) with (1.218). Let

$$f_1 = \sum_{\beta \in \mathbb{N}_0^n} \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda^\beta_{\nu m} (\beta qu)_{\nu m}(x),$$

and for fixed $\beta \in \mathbb{N}_0^n$,

$$f_2^\beta = \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \varrho^\beta_{\nu m} (\beta qu)^L_{\nu m}(x).$$

Since from (1.218), $\sup_{\beta \in \mathbb{N}_0^n} 2^{\mu|\beta|} \|\lambda^\beta \,|\, f_{pq}\| < \infty$ and $(\beta qu)_{\nu m}$ are (σ, p, Ψ)-β-quarks with $\sigma > \sigma_{pq}$, by the if-part of Theorem 1.23(i) we find that $f_1 \in F^{(\sigma,\Psi)}_{pq}(\mathbb{R}^n)$. Hence $f_1 \in F^{(s,\Psi)}_{pq}(\mathbb{R}^n)$, because $\sigma > s$. On the other hand, by (1.218), for fixed $\beta \in \mathbb{N}_0^n$, $\varrho^\beta \in f_{pq}$. Moreover, $(\beta qu)^L_{\nu m}$ are $(s, p, \Psi)_{K,L}$-atoms (for all $K \in \mathbb{N}_0$), multiplied by a constant not greater than $c2^{\kappa|\beta|}$, where $c, \kappa > 0$ do not depend on β (see Lemma 1.22). Since L satisfies (1.216), by the if-part of Theorem 1.18(i) we get $f_2^\beta \in F^{(s,\Psi)}_{pq}(\mathbb{R}^n)$ and $\|f_2^\beta \,|\, F^{(s,\Psi)}_{pq}(\mathbb{R}^n)\| \leq$

$c2^{\kappa|\beta|}\|\varrho^\beta \mid f_{pq}\|$, with $c > 0$ independent of β. Taking $\mu > \kappa$ and using the t-triangle inequality with $t = \min(1,p,q)$ we get

$$\Big\| \sum_{\beta\in\mathbb{N}_0^n} f_2^\beta \,\Big|\, F_{pq}^{(s,\Psi)}(\mathbb{R}^n)\Big\| \le c_1\Big(\sum_{\beta\in\mathbb{N}_0^n} 2^{\kappa|\beta|t}\|\varrho^\beta \mid f_{pq}\|^t\Big)^{1/t} \le c_2 \sup_{\beta\in\mathbb{N}_0^n} 2^{\mu|\beta|}\|\varrho^\beta \mid f_{pq}\|.$$

Hence

$$f_2 = \sum_{\beta\in\mathbb{N}_0^n}\sum_{\nu=0}^{\infty}\sum_{m\in\mathbb{Z}^n} \varrho_{\nu m}^\beta (\beta qu)_{\nu m}^L(x)$$

belongs to $F_{pq}^{(s,\Psi)}(\mathbb{R}^n)$. Therefore, $f = f_1 + f_2 \in F_{pq}^{(s,\Psi)}(\mathbb{R}^n)$, and

$$\|f \mid F_{pq}^{(s,\Psi)}(\mathbb{R}^n)\| \le c \sup_{\beta\in\mathbb{N}_0^n} 2^{\mu|\beta|}(\|\lambda^\beta \mid f_{pq}\| + \|\varrho^\beta \mid f_{pq}\|)$$

for any $\mu > 0$ sufficiently large, and this completes the proof of the if-part of (i).

Step 2. We now prove the only-if part of (i). We use the lift described in Step 3 of the proof of Theorem 1.18, i.e.

$$\mathrm{id} + (-\Delta)^M : F_{pq}^{(s+2M,\Psi)}(\mathbb{R}^n) \to F_{pq}^{(s,\Psi)}(\mathbb{R}^n),$$

taking now $M = (L+1)/2$. Iteration yields that for $f \in F_{pq}^{(s,\Psi)}(\mathbb{R}^n)$, we have

$$f = f_1 + (-\Delta)^{(L+1)/2} f_2 \tag{1.221}$$

with $f_1 \in F_{pq}^{(s+j(L+1),\Psi)}(\mathbb{R}^n)$ and $f_2 \in F_{pq}^{(s+L+1,\Psi)}(\mathbb{R}^n)$ ($j \in \mathbb{N}$). We stop when j is such that $s + j(L+1) > \sigma$. So, we obtain the decomposition (1.221) with $f_1 \in F_{pq}^{(\sigma,\Psi)}(\mathbb{R}^n)$. Due to (1.216), $\sigma > \sigma_{pq}$ and hence, in view of Theorem 1.23(i), f_1 can be written as

$$f_1 = \sum_{\beta\in\mathbb{N}_0^n}\sum_{\nu=0}^{\infty}\sum_{m\in\mathbb{Z}^n} \lambda_{\nu m}^\beta\, (\beta qu)_{\nu m}(x)$$

with $(\beta qu)_{\nu m}$ being (σ, p, Ψ)-β-quarks and

$$\sup_{\beta\in\mathbb{N}_0^n} 2^{\mu_1|\beta|}\|\lambda^\beta \mid f_{pq}\| < \infty$$

for any $\mu_1 > 0$ sufficiently large. Since by (1.216), $s + L + 1 > \sigma_{pq}$, and also in view of Theorem 1.23(i), we can write

$$f_2 = \sum_{\beta\in\mathbb{N}_0^n}\sum_{\nu=0}^{\infty}\sum_{m\in\mathbb{Z}^n} \varrho_{\nu m}^\beta\, (\beta qu)_{\nu m}(x)$$

now with $(\beta qu)_{\nu m}$ being $(s+L+1, p, \Psi)$-β-quarks and

$$\sup_{\beta\in\mathbb{N}_0^n} 2^{\mu_2|\beta|}\|\varrho^\beta \mid f_{pq}\| < \infty$$

for any $\mu_2 > 0$ sufficiently large. Then

$$(-\Delta)^{(L+1)/2} f_2 = \sum_{\beta\in\mathbb{N}_0^n}\sum_{\nu=0}^{\infty}\sum_{m\in\mathbb{Z}^n} \varrho_{\nu m}^\beta (-\Delta)^{(L+1)/2}(\beta qu)_{\nu m}(x).$$

We only need to prove that $(-\Delta)^{(L+1)/2}(\beta qu)_{\nu m}$ are $(s,p,\Psi)_L$-β-quarks. In fact, we have

$$(\beta qu)_{\nu m}(x) = 2^{-(s+L+1-n/p)\nu}\Psi(2^{-\nu})^{-1}\Phi^\beta(2^\nu x - m)$$

and hence

$$(-\Delta)^{(L+1)/2}(\beta qu)_{\nu m} = 2^{-(s-n/p)\nu}\Psi(2^{-\nu})^{-1}((-\Delta)^{(L+1)/2}\Phi^\beta)(2^\nu x - m).$$

Furthermore,

$$\sup_{\beta\in\mathbb{N}_0^n} 2^{\mu|\beta|}(\|\lambda^\beta \mid f_{pq}\| + \|\varrho^\beta \mid f_{pq}\|) < \infty$$

for any $\mu > 0$ sufficiently large, and the proof is complete. ∎

In the following corollary we consider distributions with compact support.

Corollary 1.28. (i) *Let $0 < p < \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$ and Ψ an admissible function. Let σ, L and κ as in Corollary* 1.27(i). *Let $\mu > \kappa$. Then $f \in \mathcal{S}'(\mathbb{R}^n)$ with compact support belongs to $F_{pq}^{(s,\Psi)}(\mathbb{R}^n)$ if, and only if, it can be represented as*

$$f = \sum_{\beta\in\mathbb{N}_0^n}\sum_{\nu=0}^{\infty}\sum_{m\in\mathbb{Z}^n} \lambda_{\nu m}^\beta\,\varphi(x)(\beta qu)_{\nu m}(x) + \varrho_{\nu m}^\beta(-\Delta)^{(L+1)/2}[\varphi(\beta qu)_{\nu m}^*](x), \tag{1.222}$$

where $\varphi \in \mathcal{S}(\mathbb{R}^n)$ is such that

$$\varphi(x) = 1 \quad \text{if } x \in (\operatorname{supp} f)_\varepsilon \quad \text{and} \quad \operatorname{supp}\varphi \subset (\operatorname{supp} f)_{2\varepsilon} \tag{1.223}$$

for some $\varepsilon > 0$, $(\beta qu)_{\nu m}$ are (σ, p, Ψ)-β-quarks, $(\beta qu)_{\nu m}^*$ are $(s + L + 1, p, \Psi)_L$-β-quarks and

$$\sup_{\beta\in\mathbb{N}_0^n} 2^{\mu|\beta|}(\|\lambda^\beta \mid f_{pq}\| + \|\varrho^\beta \mid f_{pq}\|) < \infty. \tag{1.224}$$

Again, the infimum of (1.224) *over all representations* (1.222) *is an equivalent quasi-norm in $F_{pq}^{(s,\Psi)}(\mathbb{R}^n)$. If, in addition, $s > \sigma_{pq}$, then* (1.222) *and* (1.224) *can be replaced by*

$$f = \sum_{\beta\in\mathbb{N}_0^n}\sum_{\nu=0}^{\infty}\sum_{m\in\mathbb{Z}^n} \lambda_{\nu m}^\beta\,\varphi(x)(\beta qu)_{\nu m}(x), \tag{1.225}$$

where now $(\beta qu)_{\nu m}$ are (s, p, Ψ)-β-quarks, and

$$\sup_{\beta\in\mathbb{N}_0^n} 2^{\mu|\beta|}\|\lambda^\beta \mid f_{pq}\| < \infty, \tag{1.226}$$

respectively.

(ii) *Let $0 < p, q \le \infty$, $s \in \mathbb{R}$ and Ψ an admissible function. Let σ, L and κ be as in Corollary* 1.27(ii). *Let $\mu > \kappa$. Then $f \in \mathcal{S}'(\mathbb{R}^n)$ with compact support belongs to $B_{pq}^{(s,\Psi)}(\mathbb{R}^n)$ if, and only if, it can be represented as in* (1.222) *with*

$$\sup_{\beta\in\mathbb{N}_0^n} 2^{\mu|\beta|}(\|\lambda^\beta \mid b_{pq}\| + \|\varrho^\beta \mid b_{pq}\|) < \infty. \tag{1.227}$$

Again, the infimum of (1.227) *over all representations* (1.222) *is an equivalent quasi-norm in $B_{pq}^{(s,\Psi)}(\mathbb{R}^n)$. If, in addition, $s > \sigma_p$, then* (1.222) *and* (1.227) *can be replaced by* (1.225) *and*

$$\sup_{\beta\in\mathbb{N}_0^n} 2^{\mu|\beta|}\,\|\lambda^\beta \mid b_{pq}\| < \infty, \tag{1.228}$$

respectively.

Proof. We prove (i). Obvious modifications also prove (ii).

Step 1. We start by proving the if-part. Let f be given by (1.222) with (1.224). For fixed $\beta \in \mathbb{N}_0^n$, let

$$f_1^\beta = \sum_{\nu=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \lambda_{\nu m}^\beta \varphi(x)(\beta qu)_{\nu m}(x), \tag{1.229}$$

$$f_2^\beta = \sum_{\nu=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \varrho_{\nu m}^\beta (-\Delta)^{(L+1)/2}[\varphi(\beta qu)_{\nu m}^*](x). \tag{1.230}$$

Some calculations, similar to the ones in the proof of Lemma 1.22, yield that up to normalising constants, $\varphi(\beta qu)_{\nu m}$ and $(-\Delta)^{(L+1)/2}[\varphi(\beta qu)_{\nu m}^*]$ are $(\sigma, p, \Psi)_{K,-1}$-atoms and $(s, p, \Psi)_{K,L}$-atoms, respectively, for any $K \in \mathbb{N}_0$. Moreover, in both cases the constant by which they have to be divided to become, respectively, a $(\sigma, p, \Psi)_{K,-1}$ or an $(s, p, \Psi)_{K,L}$-atom can be estimated from above by $c2^{\kappa|\beta|}$, where $c, \kappa > 0$ are independent of β. By Theorem 1.18(i), as $\sigma > \sigma_{pq}$, we conclude that

$$f_1^\beta \in F_{pq}^{(\sigma,\Psi)}(\mathbb{R}^n) \quad \text{and} \quad \|f_1^\beta \,|\, F_{pq}^{(\sigma,\Psi)}(\mathbb{R}^n)\| \le c_1 2^{\kappa|\beta|}\, \|\lambda^\beta \,|\, f_{pq}\|.$$

But $\sigma > s$, hence

$$f_1^\beta \in F_{pq}^{(s,\Psi)}(\mathbb{R}^n) \quad \text{and} \quad \|f_1^\beta \,|\, F_{pq}^{(s,\Psi)}(\mathbb{R}^n)\| \le c_2 2^{\kappa|\beta|}\, \|\lambda^\beta \,|\, f_{pq}\|.$$

For any $\mu > \kappa$ and $t = \min(1, p, q)$, the t-triangle inequality yields

$$\begin{aligned} \Big\| \sum_{\beta\in\mathbb{N}_0^n} f_1^\beta \,\Big|\, F_{pq}^{(s,\Psi)}(\mathbb{R}^n)\Big\| &\le c_1 \Big(\sum_{\beta\in\mathbb{N}_0^n} 2^{\kappa|\beta|t} \|\lambda^\beta \,|\, f_{pq}\|^t \Big)^{1/t} \\ &\le c_2 \sup_{\beta\in\mathbb{N}_0^n} 2^{\mu|\beta|} \|\lambda^\beta \,|\, f_{pq}\|. \end{aligned} \tag{1.231}$$

Also by Theorem 1.18(i),

$$f_2^\beta \in F_{pq}^{(s,\Psi)}(\mathbb{R}^n) \quad \text{and} \quad \|f_2^\beta \,|\, F_{pq}^{(s,\Psi)}(\mathbb{R}^n)\| \le c_3 2^{\kappa|\beta|} \|\lambda^\beta \,|\, f_{pq}\|,$$

and, in the same way, for all $\mu > \kappa$,

$$\Big\| \sum_{\beta\in\mathbb{N}_0^n} f_1^\beta \,\Big|\, F_{pq}^{(s,\Psi)}(\mathbb{R}^n)\Big\| \le c_4 \sup_{\beta\in\mathbb{N}_0^n} 2^{\mu|\beta|} \|\lambda^\beta \,|\, f_{pq}\|. \tag{1.232}$$

By (1.222) and (1.229)–(1.232) we obtain

$$f \in F_{pq}^{(s,\Psi)}(\mathbb{R}^n) \quad \text{and} \quad \|f \,|\, F_{pq}^{(s,\Psi)}(\mathbb{R}^n)\| \le c \sup_{\beta\in\mathbb{N}_0^n} 2^{\mu|\beta|} (\|\lambda^\beta \,|\, f_{pq}\| + \|\varrho^\beta \,|\, f_{pq}\|)$$

for $\mu > \kappa$.

Step 2. We now prove the only-if assertion of (i). Let $f \in F_{pq}^{(s,\Psi)}(\mathbb{R}^n)$. We assume $L \ne -1$, otherwise we can skip this first part of Step 2. Put $I = L + 1 + ([\sigma - s] - L)_+$ and let $\phi_k \in \mathcal{S}(\mathbb{R}^n)$, $k = 1, \dots, I$, be such that

$$\phi_k(x) = 1 \text{ if } x \in (\operatorname{supp} f)_{k\varepsilon/(2I)} \quad \text{and} \quad \operatorname{supp} \phi_k \subset (\operatorname{supp} f)_{(k+1)\varepsilon/(2I)}. \tag{1.233}$$

In particular $\phi_{k+1}(x) = 1$ for x in a neighbourhood of $\operatorname{supp} \phi_k$. We consider once more the lift $\mathrm{id} + (-\Delta)^{(L+1)/2}$. On the one hand we have

$$f = g_1 + (-\Delta)^{(L+1)/2} g_1 \quad \text{with} \quad \|f \,|\, F_{pq}^{(s,\Psi)}(\mathbb{R}^n)\| \sim \|g_1 \,|\, F_{pq}^{(s+L+1,\Psi)}(\mathbb{R}^n)\|, \tag{1.234}$$

and on the other hand $f = \phi_1 f$. Hence

$$f = \phi_1\, g_1 + \phi_1(-\Delta)^{(L+1)/2} g_1 \tag{1.235}$$
$$= \phi_1 g_1 + (-\Delta)^{(L+1)/2}(\phi_1\, g_1) + \sum_{\substack{|\alpha|+|\beta|=L+1 \\ |\beta|\le L}} c_1^{\alpha,\beta}(D^\alpha\phi_1)(D^\beta g_1).$$

We denote the last summand in (1.235) by f_1. We remark that $f_1 \in F_{pq}^{(s+1,\Psi)}(\mathbb{R}^n)$ and $\operatorname{supp} f_1 \subset \operatorname{supp}\phi_1$. We can apply the same argument to f_1 in place of f, with $s+1$ in place of s and using ϕ_2 instead of ϕ_1. Iteration yields

$$f = F_1 + (-\Delta)^{(L+1)/2} F_2 \quad \text{with} \tag{1.236}$$

$$F_i \in F_{pq}^{(s+L+1,\Psi)}(\mathbb{R}^n) \quad \text{and} \quad \operatorname{supp} F_i \subset \operatorname{supp}\phi_{L+1},\ i=1,2. \tag{1.237}$$

If $s+L+1<\sigma$, we have to apply the above kind of iteration to F_1. Now $[\sigma - s] - L$ iterations will be enough and we get

$$f = H_1 + (-\Delta)^{(L+1)/2} H_2 \quad \text{with} \tag{1.238}$$

$$H_1 \in F_{pq}^{(\sigma,\Psi)}(\mathbb{R}^n), \quad H_2 \in F_{pq}^{(s+L+1,\Psi)}(\mathbb{R}^n), \quad \operatorname{supp} H_i \subset (\operatorname{supp} f)_\varepsilon,\ i=1,2. \tag{1.239}$$

By Theorem 1.23(i) for $H_1 \in F_{pq}^{(\sigma,\Psi)}(\mathbb{R}^n)$ we have

$$H_1 = \sum_{\beta\in\mathbb{N}_0^n}\sum_{\nu=0}^{\infty}\sum_{m\in\mathbb{Z}^n} \lambda_{\nu m}^\beta (\beta qu)_{\nu m}(x), \tag{1.240}$$

with $(\beta qu)_{\nu m}$ being (σ,p,ψ)-β-quarks and

$$\sup_{\beta\in\mathbb{N}_0^n} 2^{\mu_1|\beta|}\|\lambda^\beta \mid f_{pq}\| < \infty \tag{1.241}$$

for any $\mu_1>0$ large. In the same way, by Theorem 1.23(i) and Remark 1.25, for $H_2 \in F_{pq}^{(s+L+1,\Psi)}(\mathbb{R}^n)$ we have

$$H_2 = \sum_{\beta\in\mathbb{N}_0^n}\sum_{\nu=0}^{\infty}\sum_{m\in\mathbb{Z}^n} \varrho_{\nu m}^\beta (\beta qu)^*_{\nu m}(x), \tag{1.242}$$

with $(\beta qu)^*_{\nu m}$ being $(s+L+1,p,\psi)$-β-quarks and

$$\sup_{\beta\in\mathbb{N}_0^n} 2^{\mu_2|\beta|}\|\varrho^\beta \mid f_{pq}\| < \infty \tag{1.243}$$

for any $\mu_2>0$ large. Thanks to (1.239), $H_i = \varphi\, H_i,\ i=1,2$. With this remark (1.222) is a consequence of (1.238), (1.240) and (1.242). Moreover, (1.224) comes from (1.241) and (1.243).

Step 3. The special case in (i) for $s>\sigma_{pq}$ is an easy consequence of Theorem 1.23(i). ■

2. Function spaces on fractals

2.1. (d,Ψ)-sets

2.1.1. *Introduction.* The notion of a (d,Ψ)-set was introduced by D. Edmunds and H. Triebel in [ET98, ET99] and it generalises the concept of a d-set.

A closed non-empty subset Γ of $\mathbb{R}^n$ is called a *d-set*, for $0 < d \leq n$, if there exist a Borel measure μ in $\mathbb{R}^n$ with $\operatorname{supp}\mu = \Gamma$ and two positive constants c_1 and c_2 such that

$$c_1 r^d \leq \mu(B(\gamma, r)) \leq c_2 r^d \tag{2.1}$$

for any closed ball $B(\gamma, r)$ in $\mathbb{R}^n$, centred at $\gamma \in \Gamma$ and of radius $r \in (0, 1)$. The notion of a d-set occurs both in the theory of function spaces and in fractal geometry. We refer to [JW84], [Mat95] and [Tri97], among others. Some self-similar fractals are outstanding examples of d-sets. For instance, the ordinary (middle third) Cantor set in $\mathbb{R}^1$ is a d-set for $d = \log 2/\log 3$ (this example extends to generalised Cantor sets in $\mathbb{R}^n$), and the von Koch curve in $\mathbb{R}^2$ is a d-set for $d = \log 4/\log 3$.

It is well known that the measure μ in (2.1) is even a Radon measure and that any two such measures μ_1 and μ_2 related to a d-set Γ are equivalent (see e.g. Proposition 1 in [JW84] on p. 30), in the sense that there are two positive constants c_1 and c_2 such that

$$c_1 \mu_1(A) \leq \mu_2(A) \leq c_2 \mu_1(A) \tag{2.2}$$

for any Borel set $A \subset \mathbb{R}^n$. One can get a canonical measure related to a d-set Γ by means of the restriction to Γ of the usual d-dimensional Hausdorff measure. This paves the way to proving that a d-set Γ with $0 < d < n$ has Hausdorff dimension d, $\dim_{\mathrm{H}}(\Gamma) = d$, and Lebesgue measure zero, $|\Gamma| = 0$. We refer to proofs in [JW84, Chapter II, §1.2, pp. 30–33] and [Tri97, Theorem 3.4, p. 5]. It is mentioned in [ET99, Remark 2.6, p. 86] that if Γ is a (d, Ψ)-set with $0 < d \leq n$, then also

$$\dim_{\mathrm{H}}(\Gamma) = d \quad \text{and} \quad |\Gamma| = 0. \tag{2.3}$$

As mentioned above, with the exception of the case $d = n$, this is the counterpart for (d, Ψ)-sets of known results for d-sets. Our aim in this subsection is to give a proof of (2.3). In particular, we prove that any two measures related to a (d, Ψ)-set are equivalent and we find a canonical measure that, in this case, can be obtained by means of a generalised Hausdorff measure.

2.1.2. *Definition and properties of a (d, Ψ)-set*

DEFINITION 2.1. Let Γ be a non-empty closed subset of $\mathbb{R}^n$.

(i) Let $0 < d < n$ and let Ψ be an admissible function. Then Γ is called a *(d, Ψ)-set* if there exist a Radon measure μ on $\mathbb{R}^n$, with $\operatorname{supp}\mu = \Gamma$, and two positive constants c_1 and c_2 such that

$$c_1 r^d \Psi(r) \leq \mu(B(\gamma, r)) \leq c_2 r^d \Psi(r) \tag{2.4}$$

for any ball $B(\gamma, r)$ in $\mathbb{R}^n$ centred at $\gamma \in \Gamma$ and of radius $r \in (0, 1)$.

(ii) Let Ψ be a decreasing admissible function with $\lim_{r \to 0^+} \Psi(r) = \infty$. Then Γ is called an *(n, Ψ)-set* if there is a Radon measure μ with the above properties and $d = n$ in (2.4).

EXAMPLE 2.2. Obviously any d-set with $0 < d < n$ is a (d, Ψ)-set for $\Psi = 1$. For any couple (d, Ψ) with $0 < d \leq n$ and Ψ an admissible function (as in Definition 2.1(ii) if $d = n$), there exists a (d, Ψ)-set. We refer to Proposition 2.8 of [ET99]. In the case of d-sets, in which case $\Psi(r) = 1$, for any d with $0 < d < n$ there is even a self-similar d-set as an attractor of a suitable family of contractions, or iterated function schemes;

see [Tri97, §4], [Mat95, 4.13] and [Fal90, 9.1], among others. The examples of (d,Ψ)-sets given in [ET99] (pseudo self-similar sets) are created in a similar way, but the dilation factors of the contractions involved may vary from step to step.

REMARK 2.3. In this remark we state some easy consequences of Definition 2.1.

(i) If Γ is a (d,Ψ)-set with $0 < d \le n$, then the right-hand inequality of (2.4) is even true for any $\gamma \in \mathbb{R}^n$, but now with $r \in (0,1/2)$ and another constant c_2. This follows from the observation: given $y \in \mathbb{R}^n$ and $r \in (0,1/2)$, either $B(y,r) \cap \Gamma = \emptyset$ which gives $\mu(B(y,r)) = 0$, or there exists $\gamma \in B(y,r) \cap \Gamma$ which gives $B(y,r) \subset B(\gamma,2r)$, and hence $\mu(B(y,r)) \le c_2 (2r)^d \Psi(2r) \le c_3\, r^d \Psi(r)$. We have used (2.4) and Proposition 1.4(v).

(ii) An immediate consequence of Proposition 1.4(v) is

$$\mu(B(\gamma,2r)) \le c\mu(B(\gamma,r)), \qquad \gamma \in \Gamma,\ r \in (0,1/2),$$

for some positive constant c.

(iii) The relation (2.4) also implies that, for some positive constant c, we have

$$\mu(B(x,r)) \le cr^n, \qquad x \in \mathbb{R}^n,\ r \ge 1.$$

This follows because $B(x,r)$ can be covered by $c_1 r^n$ balls of radius $1/2$.

(iv) In Definition 2.1 it is sufficient to assume that μ is a Borel measure. Then we can easily prove that μ turns out to be a Radon measure.

PROPOSITION 2.4. *Let Γ be a (d,Ψ)-set in $\mathbb{R}^n$ with $0 < d \le n$. Let μ_1 and μ_2 be two Radon measures related to Γ according to* (2.4)*. Then μ_1 and μ_2 are equivalent in the sense described in* (2.2).

Proof. Take an open set O with $\mu_1(O) > 0$ and let t be such that $0 < t < \mu_1(O)$. Since μ_1 is a Radon measure, there exists a compact set K with $K \subset O$ and $\mu_1(K) > t$. We can cover $K \cap \Gamma$ by finitely many open balls $B(\gamma_i, r_i) \subset O$, $i \in I$, with centres $\gamma_i \in K \cap \Gamma$ and arbitrarily small radius $r_i \in (0,1/4)$. By a standard argument (see Lemma 7.3 of [Rud87, p. 137]), we can choose a subcollection $\{B(\gamma_i,r_i)\}_{i\in I'}$ of $\{B(\gamma_i,r_i)\}_{i\in I}$, $I' \subset I$, such that the balls $B(\gamma_i,r_i)$ with $i \in I'$ are disjoint and

$$\bigcup_{i\in I} B(\gamma_i,r_i) \subset \bigcup_{i\in I'} B(\gamma_i,4r_i).$$

We get

$$\begin{aligned} t < \mu_1(K) &\le \mu_1\Big(\bigcup_{i\in I} B(\gamma_i,r_i)\Big) \le \mu_1\Big(\bigcup_{i\in I'} B(\gamma_i,4r_i)\Big) \le \sum_{i\in I'} \mu_1(B(\gamma_i,4\,r_i)) \\ &\le c_1 \sum_{i\in I'} (4\,r_i)^d \Psi(4r_i) \le c_2 \sum_{i\in I'} r_i^d \Psi(r_i) \le c_3 \sum_{i\in I'} \mu_2(B(\gamma_i,r_i)) \\ &= c_3\mu_2\Big(\bigcup_{i\in I'} B(\gamma_i,r_i)\Big) \le c_3\mu_2\Big(\bigcup_{i\in I} B(\gamma_i,r_i)\Big) \le c_3\mu_2(O). \end{aligned}$$

We have used the properties of μ_i, $i = 1,2$, and Proposition 1.4(v). Letting t tend to $\mu_1(O)$ we conclude that $\mu_1(O) \le c_3\,\mu_2(O)$. For an arbitrary Borel set E,

$$\mu_i(E) = \inf\{\mu_i(O) : O \text{ open},\ E \subset O\}, \qquad i = 1,2.$$

But, for any open set O with $E \subset O$, we have $\mu_1(E) \leq \mu_1(O) \leq c_3\mu_2(O)$. Taking the infimum over all such O we get $\mu_1(E) \leq c_3\mu_2(E)$. Since we get an inequality in the other direction in the same way, the proof is finished. ∎

We can get a canonical measure related to a (d,Ψ)-set by means of a generalised Hausdorff measure. Next we recall some facts concerning measure theory. We follow [Mat95, §4.1,4.2] and [Tri97, §2].

Let $\mathcal{F}$ be a family of subsets of $\mathbb{R}^n$ and ζ a non-negative function on $\mathcal{F}$ with the properties:

(I) For every $\delta > 0$ there are $E_j \in \mathcal{F}$ such that $\mathbb{R}^n = \bigcup_{j=1}^{\infty} E_j$ and $\operatorname{diam}(E_j) \leq \delta$.

(II) For every $\delta > 0$ there exists an $E \in \mathcal{F}$ such that $\zeta(E) \leq \delta$ and $\operatorname{diam}(E) \leq \delta$.

For $0 < \delta < \infty$ and $A \subset \mathbb{R}^n$ we define

$$\psi_\delta(A) = \inf\Big\{\sum_{j=1}^{\infty}\zeta(E_j) : A \subset \bigcup_{j=1}^{\infty} E_j, \operatorname{diam}(E_j) \leq \delta,\ E_j \in \mathcal{F}\Big\}. \tag{2.5}$$

Of course, $\psi_\delta(A)$ is monotone,

$$\psi_\delta(A) \leq \psi_\varepsilon(A) \quad \text{when } 0 < \varepsilon < \delta < \infty, \tag{2.6}$$

and hence $\psi = \psi(\mathcal{F},\zeta)$, given by

$$\psi(A) = \lim_{\delta\to 0}\psi_\delta(A) = \sup_{\delta>0}\psi_\delta(A), \quad A \subset \mathbb{R}^n, \tag{2.7}$$

makes sense. The measure ψ is the result of Carathéodory's construction from ζ on $\mathcal{F}$. This kind of construction is also described extensively in [Fed69, 2.10]. Theorem 4.2 of [Mat95, p. 55] and Theorem 2.3 of [Tri97, p. 3] state the following characterisation of the measure ψ.

THEOREM 2.5. (i) *ψ is a Borel measure on $\mathbb{R}^n$.*

(ii) *If the members of $\mathcal{F}$ are Borel sets, then ψ is a Borel regular measure on $\mathbb{R}^n$.*

(iii) *If the members of $\mathcal{F}$ are Borel sets, and A is a ψ-measurable set with $\psi(A) < \infty$, then $\psi_{|A}$ is a Radon measure on $\mathbb{R}^n$.*

One way of constructing such a measure is by means of a non-negative function $h : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ with $h(0) = \lim_{t\to 0^+} h(t) = 0$, and $\mathcal{F}$ the family of all closed sets in $\mathbb{R}^n$ (see e.g. [Fal90, 2.5, p. 33] and [Gar72, p. 58]). Note that, by (2.5)–(2.7), what matters is the behaviour of h in a neighbourhood of 0. Then the function ζ defined by

$$\zeta(E) = h(\operatorname{diam}(E)), \quad E \subset \mathbb{R}^n,$$

satisfies (I) and (II) above. We denote the corresponding measure ψ by Λ_h. For $h(t) = t^s$, $0 \leq s < \infty$, we get the usual s-dimensional Hausdorff measure, usually denoted by $\mathcal{H}^s$.

It is known (see, for instance, [Tri97, 3.4, p. 5]) that a canonical measure related to a d-set Γ, $0 < d \leq n$, is $\mathcal{H}^d_{|\Gamma}$, the restriction to Γ of the d-dimensional Hausdorff measure. We prove that for a (d,Ψ)-set we get, in an analogous way, a related measure by means of Λ_h, where

$$h(t) = t^d\Psi(t), \quad 0 < t \leq 1, \tag{2.8}$$

and $h(0) = \lim_{t\to 0^+} h(t) = 0$ (recall Proposition 1.4(iii)).

In the special case of $h(t) = t^s$, $0 \le s < \infty$, but with $\mathcal{F}$, the family of all closed sets in $\mathbb{R}^n$, replaced by $\mathcal{B}$, the family of all closed balls in $\mathbb{R}^n$, we get the so-called s-dimensional spherical measure $\mathcal{S}^s$. The following relation is well known (cf. [Tri97, 2.5, p. 4]):

$$\mathcal{H}^s(E) \le \mathcal{S}^s(E) \le 2^s \mathcal{H}^s(E), \qquad E \subset \mathbb{R}^n.$$

Such type of relation is also true between the corresponding measures Λ_h and $\mathcal{S}_h$, constructed by means of the same function h in (2.8), but with $\mathcal{F}$ or $\mathcal{B}$, respectively.

Lemma 2.6. *Let h be given by* (2.8). *There exists a positive constant c, only depending on d and Ψ, such that*

$$\Lambda_h(E) \le \mathcal{S}_h(E) \le c\Lambda_h(E), \qquad E \subset \mathbb{R}^n. \tag{2.9}$$

Proof. The first inequality in (2.9) is obvious thanks to $\mathcal{B} \subset \mathcal{F}$. If we have a δ-covering, $0 < \delta < 1/2$, of E by closed sets $\{E_j\}_{j=1}^\infty$, $E \subset \bigcup_{j=1}^\infty E_j$, then $E \subset \bigcup_{j=1}^\infty B_j$, where B_j are closed balls of diameter $2 \operatorname{diam} E_j$ (see e.g. [Fed69, §2.10.41, p. 200]). Hence,

$$\sum_{j=1}^\infty h(\operatorname{diam} B_j) = \sum_{j=1}^\infty (2 \operatorname{diam} E_j)^d \Psi(2 \operatorname{diam} E_j) \le 2^d C \sum_{j=1}^\infty (\operatorname{diam} E_j)^d \Psi(\operatorname{diam} E_j).$$

We have used Proposition 1.4(v). The last inequality implies $\mathcal{S}_h(E) \le 2^d C \Lambda_h(E)$, and so the proof is complete. ∎

The following result relates the measures constructed from two different functions h and g. We refer to [Gar72, Lemma 1.2].

Lemma 2.7. *For any bounded set E, we have*

$$\Lambda_h(E) \le \left(\limsup_{t \to 0^+} \frac{h(t)}{g(t)} \right) \Lambda_g(E).$$

We are now ready to state and prove the following proposition:

Proposition 2.8. *Let Γ be a (d, Ψ)-set on $\mathbb{R}^n$ with $0 < d \le n$. Then the restriction to Γ of the measure Λ_h, with h given by* (2.8), *satisfies* (2.4), *that is, $\Lambda_{h|\Gamma}$ is a measure related to the (d, Ψ)-set Γ.*

Proof. Let μ denote a measure related to the (d, Ψ)-set Γ according to Definition 2.1. Let $\gamma \in \Gamma$, $0 < r < 1$ and define $\Gamma(\gamma, r) = B(\gamma, r) \cap \Gamma$. Let $\{B_j\}_{j=1}^\infty$ be a countable family of closed balls with radius $r_j < 1/2$ which covers $\Gamma(\gamma, r)$. We have

$$c_1 r^d \Psi(r) \le \mu(\Gamma(\gamma, r)) \le \mu\Big(\bigcup_{j=1}^\infty B_j \Big) \le \sum_{j=1}^\infty \mu(B_j) \le c_2 \sum_{j=1}^\infty r_j^d \Psi(r_j).$$

However, by Lemma 2.6, for any $\varepsilon > 0$ the last sum is, for a suitable choice of $\{B_j\}$, less than $c_3[\varepsilon + \Lambda_h(\Gamma(\gamma, r))]$, where the constant c_3 depends only on d and Ψ. This gives

$$\Lambda_h(\Gamma(\gamma, r)) \ge \frac{c_1}{c_3} r^d \Psi(r), \qquad \gamma \in \Gamma, \ 0 < r < 1,$$

which proves one of the desired inequalities. Now take $0 < t < \Lambda_h(\Gamma(\gamma, r))$ and $0 < \varepsilon < \min(1-r, 1/16)$. We can cover $\Gamma(\gamma, r)$ by finitely many open balls $S_j \subset B(\gamma, r+\varepsilon)$, $j \in I$, with centres in $\Gamma(\gamma, r)$ and radius $r_j \le \varepsilon$. We can choose a disjoint subcollection $\{B_j\}_{j \in I'}$ of $\{S_j\}_{j \in I}$, $I' \subset I$, such that $\bigcup_{j \in I} S_j \subset \bigcup_{j \in I'} B'_j$, where B'_j is the ball concentric with

B_j whose radius is four times the radius r_j of B_j (see Lemma 7.3 in [Rud87, p. 137]). Since $\Gamma(\gamma, r) \subset \bigcup_{j\in I} S_j \subset \bigcup_{j\in I'} B'_j$ we get

$$t < \sum_{j\in I'} (8r_j)^d \Psi(8r_j) \tag{2.10}$$

if $\varepsilon > 0$ is small enough. On the other hand, by the properties of μ, we have

$$c_1 \sum_{j\in I'} r_j^d \Psi(r_j) \le \sum_{j\in I'} \mu(B_j) = \mu\Big(\bigcup_{j\in I'} B_j\Big) \le \mu(B(\gamma, r+\varepsilon)) \le c_2(r+\varepsilon)^d \Psi(r+\varepsilon). \tag{2.11}$$

By Proposition 1.4(v), we have $\Psi(8r_j) \le c^3 \Psi(r_j)$ provided that $r_j \le \varepsilon \le 1/16$. This together with (2.10) and (2.11) gives $t \le c_3\,(r+\varepsilon)^d \Psi(r+\varepsilon)$ if $\varepsilon > 0$ is small enough. Letting ε tend to zero we obtain

$$t \le c_3 r^d \lim_{\varepsilon\to 0^+} \Psi(r+\varepsilon) \le c_4 r^d \Psi(r). \tag{2.12}$$

Letting t tend to $\Lambda_h(\Gamma(\gamma, r))$ we get $\Lambda_h(\Gamma(\gamma, r)) \le c_4 r^d \Psi(r)$, which completes the proof, if we show the last inequality in (2.12). Note that the monotonicity of Ψ yields the existence of $\lim_{\varepsilon\to 0^+} \Psi(r+\varepsilon)$. If the admissible function Ψ is decreasing then $\lim_{\varepsilon\to 0^+} \Psi(r+\varepsilon) \le \Psi(r)$. Otherwise, if Ψ is increasing, then $\lim_{\varepsilon\to 0^+} \Psi(r+\varepsilon) \le c_5 \Psi(r)$, for some positive constant c_5, independent of r. In fact:

- If $0 < r < 1/2$, there is $j \in \mathbb{N}$ such that $2^{-2j} \le r \le 2^{-j}$; then
$$\lim_{\varepsilon\to 0^+} \Psi(r+\varepsilon) \le \Psi(2^{-j}) \le c\Psi(2^{-2j}) \le c\,\Psi(r).$$
- If $1/2 < r < 1$, then
$$\lim_{\varepsilon\to 0^+} \Psi(r+\varepsilon) \le \Psi(1) \le \frac{\Psi(1)}{\Psi(2^{-1})}\,\Psi(r). \quad ■$$

From Propositions 2.8 and 2.4 it makes sense, up to equivalence, to speak about the measure associated with a (d, Ψ)-set Γ, having always in mind $\Lambda_{h|\Gamma}$.

Corollary 2.9. *Let Γ be a (d, Ψ)-set in $\mathbb{R}^n$ with $0 < d \le n$. Then*

$$\dim_{\mathrm{H}}(\Gamma \cap B(\gamma, r)) = d$$

for any $\gamma \in \Gamma$ and $r > 0$.

Proof. Let first $0 < r < 1$. By Proposition 2.8 we know that $0 < \Lambda_h(\Gamma \cap B(\gamma, r)) < \infty$. For $s > d$, using Lemma 2.7 (Ψ^{-1} is also an admissible function, by Proposition 1.4(i)), we have

$$\mathcal{H}^s(\Gamma \cap B(\gamma, r)) \le \left(\limsup_{t\to 0^+} \frac{t^s}{t^d \Psi(t)}\right) \Lambda_h(\Gamma \cap B(\gamma, r)),$$

and by Proposition 1.4(iii), we get $\mathcal{H}^s(\Gamma \cap B(\gamma, r)) = 0$. In an analogous way, for $s < d$ we have

$$0 < \Lambda_h(\Gamma \cap B(\gamma, r)) \le \left(\limsup_{t\to 0^+} \frac{t^d \Psi(t)}{t^s}\right) \mathcal{H}^s(\Gamma \cap B(\gamma, r)),$$

and hence $\mathcal{H}^s(\Gamma \cap B(\gamma, r)) = \infty$. Therefore, by the definition of Hausdorff dimension,

$$\dim_{\mathrm{H}}(\Gamma \cap B(\gamma, r)) = \inf\{s \ge 0 : \mathcal{H}^s(\Gamma \cap B(\gamma, r)) = 0\} = d.$$

Now consider the case $r \geq 1$. We can cover $B(\gamma, r)$ by cr^n balls of radius $1/4$, say $\{B(x_i, 1/4)\}_{i=1}^{cr^n}$. It can happen that $\Gamma \cap B(x_i, 1/4) = \emptyset$, or there exists $\gamma_i \in B(x_i, 1/4) \cap \Gamma$, which implies

$$\Gamma \cap B(x_i, 1/4) \subset \Gamma \cap B(\gamma_i, 1/2),$$

and then

$$\Gamma \cap B(\gamma, 1/2) \subset \Gamma \cap B(\gamma, r) \subset \bigcup_{i=1}^{cr^n} \Gamma \cap B(\gamma_i, 1/2).$$

By the properties of the Hausdorff dimension (cf. [Mat95, p. 59]), and the first part of the proof, we obtain

$$d \leq \dim_{\mathrm{H}}(\Gamma \cap B(\gamma, r)) \leq \sup_{i=1,\ldots,cr^n} \dim_{\mathrm{H}}(\Gamma \cap B(\gamma_i, 1/2)) = d.$$

Therefore $\dim_{\mathrm{H}}(\Gamma \cap B(\gamma, r)) = d$ for any $\gamma \in \Gamma$ and $r > 0$, and the proof is complete. ∎

Proposition 2.8 and Corollary 2.9 with the additional assumption on the boundedness of Γ enable us to prove (2.3).

Corollary 2.10. *If Γ is a compact (d, Ψ)-set in $\mathbb{R}^n$ with $0 < d \leq n$, then*

$$\dim_{\mathrm{H}}(\Gamma) = d \quad \textit{and} \quad |\Gamma| = 0.$$

Proof. Obviously we can write

$$\Gamma = \bigcup_{z \in \mathbb{Z}^n} B(z, \sqrt{n}\,) \cap \Gamma.$$

Only for finitely many $z \in \mathbb{Z}^n$ do we have $B(z, \sqrt{n}) \cap \Gamma \neq \emptyset$. If that is the case, there exists $\gamma \in B(z, \sqrt{n}) \cap \Gamma$, which implies $\Gamma \cap B(z, \sqrt{n}) \subset \Gamma \cap B(\gamma, 2\sqrt{n})$. Hence, we can even write

$$\Gamma = \bigcup_{j=1}^{N} B(\gamma_j, 2\sqrt{n}) \cap \Gamma$$

with $\gamma_j \in \Gamma$ and some $N \in \mathbb{N}$. By Corollary 2.9, it follows that

$$\dim_{\mathrm{H}}(\Gamma) = \sup_{j=1,\ldots,N} \dim_{\mathrm{H}}(B(\gamma_j, 2\sqrt{n})) = d.$$

For the second part of the proof, we need to recall the equality $\mathcal{H}^n = c\mathcal{L}^n$, where c is some positive constant and $\mathcal{L}^n$ denotes the Lebesgue measure in $\mathbb{R}^n$ (see [Fed69, 2.10.35, p. 197]). If $d < n$, since $\dim_{\mathrm{H}}(\Gamma) = d$, we have $\mathcal{H}^n(\Gamma) = 0$, and so $\mathcal{L}^n(\Gamma) = |\Gamma| = 0$. If $d = n$ we will also prove that $\mathcal{H}^n(\Gamma) = 0$, but in this case this is not an immediate consequence of $\dim_{\mathrm{H}}(\Gamma) = n$. It is important that Γ is compact. In fact, since Γ is bounded, we have $\Gamma \subset B(O, R)$ for some $R > 1$. Then with h as in (2.8), by Proposition 2.8 and Remark 2.3(iii), we have

$$\Lambda_h(\Gamma) \leq \Lambda_h(B(O, R)) \leq cR^n < \infty. \tag{2.13}$$

By Lemma 2.7, we get

$$\mathcal{H}^n(\Gamma) \leq \left(\limsup_{t \to 0^+} \frac{t^n}{t^n \Psi(t)} \right) \Lambda_h(\Gamma). \tag{2.14}$$

But, by Definition 2.1(ii), $\lim_{t\to 0^+}\Psi(t) = +\infty$. This, together with (2.13) and (2.14), gives $\mathcal{H}^n(\Gamma) = 0 = |\Gamma|$. Hence, also in the case $d = n$ we get $|\Gamma| = 0$. ∎

2.2. Spaces on (d,Ψ)-sets

2.2.1. $L_p(\Gamma)$ *as spaces of distributions.* In this subsubsection we always assume that Γ is a compact (d,Ψ)-set in $\mathbb{R}^n$. As pointed out in the previous subsection, the Radon measure μ related to the (d,Ψ)-set Γ is unique up to equivalence and we can always think of μ as being the measure $\Lambda_{h|\Gamma}$ described in 2.1.2. If $0 < p \le \infty$, then $L_p(\Gamma)$ is the usual complex quasi-Banach space (Banach space if $p \ge 1$) with respect to the related Radon measure μ, quasi-normed by

$$\|f \mid L_p(\Gamma)\| = \Big(\int_\Gamma |f(\gamma)|^p\,\mu(d\gamma)\Big)^{1/p} \tag{2.15}$$

(with the usual modification if $p = \infty$).

Any $f^\Gamma \in L_p(\Gamma)$ with $1 \le p \le \infty$ can be interpreted as a (uniquely determined) tempered distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ given by

$$f(\varphi) = \int_\Gamma f^\Gamma(\gamma)\varphi_{|\Gamma}(\gamma)\,\mu(d\gamma), \qquad \varphi \in \mathcal{S}(\mathbb{R}^n), \tag{2.16}$$

where $\varphi_{|\Gamma}$ is the pointwise trace of φ on Γ. The interpretation (2.16) paves the way for the identification of some spaces $L_p(\Gamma)$ with suitable subspaces of some spaces $B^{(s,\Psi)}_{pq}(\mathbb{R}^n)$ which will be introduced now.

Definition 2.11. Let Γ be a non-empty closed subset of $\mathbb{R}^n$ with $|\Gamma| = 0$. Suppose that $0 < p, q \le \infty$, $s \in \mathbb{R}$ and Ψ is an admissible function. Then

$$B^{(s,\Psi),\Gamma}_{pq}(\mathbb{R}^n) = \{f \in B^{(s,\Psi)}_{pq}(\mathbb{R}^n) : f(\varphi) = 0 \text{ if } \varphi \in \mathcal{S}(\mathbb{R}^n) \text{ and } \varphi_{|\Gamma} = 0\}. \tag{2.17}$$

This definition generalises Definition 17.2 of [Tri97] and coincides essentially with Definition 2.14 of [ET99]. If $f \in B^{(s,\Psi),\Gamma}_{pq}(\mathbb{R}^n)$, then

$$\operatorname{supp} f \subset \Gamma. \tag{2.18}$$

However, the assertion (2.18) is necessary for $f \in B^{(s,\Psi),\Gamma}_{pq}(\mathbb{R}^n)$ but not sufficient (for an example see [Tri97, p. 126]). We also refer to [Bri00], where moreover certain type of sets Γ are described for which the above condition turns out to be both sufficient and necessary.

Since $|\Gamma| = 0$ the spaces $B^{(s,\Psi),\Gamma}_{pq}(\mathbb{R}^n)$ are trivial if $B^{(s,\Psi)}_{pq}(\mathbb{R}^n)$ is a subset of $L_1^{\mathrm{loc}}(\mathbb{R}^n)$. In other words, in any case, with the exception of the zero distribution, $B^{(s,\Psi),\Gamma}_{pq}(\mathbb{R}^n)$ consists of singular distributions. Recall that for $\varepsilon > 0$, by Proposition 1.9, we have

$$B^{(s+\varepsilon,\Psi)}_{pq}(\mathbb{R}^n) \hookrightarrow B^s_{pq}(\mathbb{R}^n)$$

and on the other hand (see e.g. [RuS96, 2.2.4]),

$$B^s_{pq}(\mathbb{R}^n) \hookrightarrow L_1^{\mathrm{loc}}(\mathbb{R}^n)$$

provided that $s > \sigma_p$ (recall the notation in (1.36)). Hence, $B^{(s,\Psi),\Gamma}_{pq}(\mathbb{R}^n)$ is trivial if $0 < p, q \le \infty$, Ψ is an admissible function and $s > \sigma_p$.

Analogously to (2.17) one can introduce the corresponding spaces $F^{(s,\Psi),\Gamma}_{pq}(\mathbb{R}^n)$.

Having in mind the identification specified in (2.16), we have the following:

PROPOSITION 2.12. *Let Γ be a compact (d,Ψ)-set in $\mathbb{R}^n$ with $0<d\le n$. Let $1\le p\le\infty$ and denote by p' its conjugate exponent. Then*

$$L_p(\Gamma)\subset B^{(-(n-d)/p',\Psi^{-1/p'}),\Gamma}_{p,\infty}(\mathbb{R}^n). \tag{2.19}$$

Proof. This proof is adapted from the proof of Theorem 18.2 in [Tri97]. Let $f^\Gamma\in L_p(\Gamma)$ with $1\le p\le\infty$ and let $f\in\mathcal{S}'(\mathbb{R}^n)$ be given by (2.16). We show that $f\in B^{(-(n-d)/p',\Psi^{-1/p'}),\Gamma}_{p,\infty}(\mathbb{R}^n)$ and

$$\|f\,|\,B^{(-(n-d)/p',\Psi^{-1/p'}),\Gamma}_{p,\infty}(\mathbb{R}^n)\|\le c\,\|f^\Gamma\,|\,L_p(\Gamma)\|$$

for some $c>0$ which is independent of f^Γ. Let k be a suitable kernel according to Theorem 1.12. Using Hölder's inequality we get

$$\begin{aligned}|k(2^{-j},f)(x)|&=2^{jn}\left|\int_\Gamma f^\Gamma(\gamma)k\left(\frac{\gamma-x}{2^{-j}}\right)\mu(d\gamma)\right|\\&\le 2^{jn}\left(\int_\Gamma|f^\Gamma(\gamma)|^p\left|k\left(\frac{\gamma-x}{2^{-j}}\right)\right|\mu(d\gamma)\right)^{1/p}\left(\int_\Gamma\left|k\left(\frac{\gamma-x}{2^{-j}}\right)\right|\mu(d\gamma)\right)^{1/p'}.\end{aligned}\tag{2.20}$$

Since $\operatorname{supp}k\subset\{\xi\in\mathbb{R}^n:|\xi|\le1\}$, the second integral on the right-hand side of (2.20) can be restricted over $\Gamma\cap B(x,2^{-j})$. Since Γ is a (d,Ψ)-set, it follows that

$$\mu(\Gamma\cap B(x,2^{-j}))\le c2^{-jd}\Psi(2^{-j}),\qquad j\in\mathbb{N}.$$

Moreover $\sup_{x\in\mathbb{R}^n}|k(x)|<\infty$. Then

$$\begin{aligned}&|k(2^{-j},f)(x)|\\&\quad\le c2^{j(n-d/p')}\Psi(2^{-j})^{1/p'}\left(\int_\Gamma|f^\Gamma(\gamma)|^p\left|k\left(\frac{\gamma-x}{2^{-j}}\right)\right|\mu(d\gamma)\right)^{1/p},\quad j\in\mathbb{N}.\end{aligned}\tag{2.21}$$

From (2.21) using Fubini's theorem and a suitable change of variables, we get

$$\begin{aligned}\int_{\mathbb{R}^n}|k(2^{-j},f)(x)|^p\,dx&\le c2^{jp(n-d/p')}\Psi(2^{-j})^{p/p'}\int_{\mathbb{R}^n}\int_\Gamma|f^\Gamma(\gamma)|^p\left|k\left(\frac{\gamma-x}{2^{-j}}\right)\right|\mu(d\gamma)\,dx\\&\le c2^{jp(n-d/p')}\Psi(2^{-j})^{p/p'}\int_\Gamma|f^\Gamma(\gamma)|^p\,\mu(d\gamma)\int_{\mathbb{R}^n}2^{-jn}|k(y)|\,dy\\&=c'2^{jp(n-d)/p'}\Psi(2^{-j})^{p/p'}\int_\Gamma|f^\Gamma(\gamma)|^p\,\mu(d\gamma).\end{aligned}$$

Taking the $1/p$-power, we obtain

$$\|k(2^{-j},f)(\cdot)\,|\,L_p(\mathbb{R}^n)\|\le c2^{j(n-d)/p'}\Psi(2^{-j})^{1/p'}\|f^\Gamma\,|\,L_p(\Gamma)\|,\qquad j\in\mathbb{N}.$$

Hence

$$\sup_{j\in\mathbb{N}_0}2^{j-(n-d)/p'}\Psi(2^{-j})^{-1/p'}\|k(2^{-j},f)(\cdot)\,|\,L_p(\mathbb{R}^n)\|\le c\|f^\Gamma\,|\,L_p(\Gamma)\|,$$

because the term corresponding to $j=0$ can be treated in a similar way. Moreover it is obvious that (2.16) implies $f(\varphi)=0$ for any φ in $\mathcal{S}(\mathbb{R}^n)$ with $\varphi_{|\Gamma}=0$. Therefore $f\in B^{(-(n-d)/p',\Psi^{-1/p'}),\Gamma}_{p,\infty}(\mathbb{R}^n)$, and the proof is complete. ■

REMARK 2.13. The proposition above is included in Theorem 2.16 of [ET99], which generalises Theorem 18.2 of [Tri97] from d-sets to (d,Ψ)-sets. Concerning Theorem 2.16 in [ET99] it is moreover stated that the inclusion (2.19) can be replaced by equality if $p>1$ and either (i) $d<n$ or (ii) $d=n$ and $\sum_{j=0}^{\infty}\Psi(2^{-j})^{-1/p'}<\infty$. For a detailed proof of this fact see also [Bri00]. Concerning the special case of the last proposition

$$L_1(\Gamma)\subset B^{(0,\Psi^0),\Gamma}_{1,\infty}(\mathbb{R}^n)=B^{0,\Gamma}_{1,\infty}(\mathbb{R}^n),$$

we refer for further comments to [Tri97, 18.3].

2.2.2. *Traces.* First we recall what is meant by traces. Let Γ be a compact set in $\mathbb{R}^n$ and let μ be a Radon measure on $\mathbb{R}^n$ with $\operatorname{supp}\mu=\Gamma$. Of course, $L_p(\Gamma)$ are the related L_p-spaces. Let $\operatorname{tr}_\Gamma\varphi=\varphi_{|\Gamma}$ be the pointwise trace of $\varphi\in\mathcal{S}(\mathbb{R}^n)$ on Γ. Suppose that for some space $B^{(s,\Psi)}_{pq}(\mathbb{R}^n)$ with $\max(p,q)<\infty$, there exists a constant $c>0$ such that for all $\varphi\in\mathcal{S}(\mathbb{R}^n)$,

$$\|\operatorname{tr}_\Gamma\varphi\,|\,L_p(\Gamma)\|\le c\|\varphi\,|\,B^{(s,\Psi)}_{pq}(\mathbb{R}^n)\|. \tag{2.22}$$

Due to $\max(p,q)<\infty$, $\mathcal{S}(\mathbb{R}^n)$ is dense in $B^{(s,\Psi)}_{pq}(\mathbb{R}^n)$, hence the definition of tr_Γ on the whole space $B^{(s,\Psi)}_{pq}(\mathbb{R}^n)$ is a matter of completion. The statement

$$L_p(\Gamma)=\operatorname{tr}_\Gamma B^{(s,\Psi)}_{pq}(\mathbb{R}^n)$$

should be understood in the sense that any $f^\Gamma\in L_p(\Gamma)$ is the trace on Γ of some $g\in B^{(s,\Psi)}_{pq}(\mathbb{R}^n)$ and $\|f^\Gamma\,|\,L_p(\Gamma)\|$ is equivalent to

$$\inf\{\|g\,|\,B^{(s,\Psi)}_{pq}(\mathbb{R}^n)\|:\operatorname{tr}_\Gamma g=f^\Gamma\}.$$

PROPOSITION 2.14. *Let Γ be a compact (d,Ψ)-set in $\mathbb{R}^n$ with $0<d\le n$. Then*

$$\operatorname{tr}_\Gamma B^{((n-d)/p,\Psi^{1/p})}_{pq}(\mathbb{R}^n)\hookrightarrow L_p(\Gamma) \tag{2.23}$$

for $0<p<\infty$ and $0<q\le\min(1,p)$.

Proof. We modify Step 1 of the proof of Theorem 18.6 in [Tri97, p. 139]. Let $\varphi\in\mathcal{S}(\mathbb{R}^n)$. Obviously $\varphi\in B^{((n-d)/p,\Psi^{1/p})}_{pq}(\mathbb{R}^n)$, and by Theorem 1.18(ii) we can write

$$\varphi=\sum_{\nu=0}^{\infty}\sum_{m\in\mathbb{Z}^n}\lambda_{\nu m}a_{\nu m}\quad\text{ in }\mathcal{S}'(\mathbb{R}^n),$$

where the sequence $\lambda=\{\lambda_{\nu m}:\nu\in\mathbb{N}_0,\ m\in\mathbb{Z}^n\}\in b_{pq}$, and $a_{\nu m}$ are $((n-d)/p,p,\Psi^{1/p})_{K,L}$-atoms for some $K\in\mathbb{N}_0$, $L+1\in\mathbb{N}_0$. In particular,

$$\|\lambda\,|\,b_{pq}\|\le c\|\varphi\,|\,B^{((n-d)/p,\Psi^{1/p})}_{pq}(\mathbb{R}^n)\|, \tag{2.24}$$

where c is a positive constant independent of φ. Moreover, for $\nu\in\mathbb{N}$ and $m\in\mathbb{Z}^n$,

$$|a_{\nu m}(x)|\le 2^{\nu d/p}\Psi(2^{-\nu})^{-1/p}\widetilde{\chi}_{\nu m}(x), \tag{2.25}$$

where $\widetilde{\chi}_{\nu m}$ is the characteristic function of the cube $c\,Q_{\nu m}$ which contains the support of $a_{\nu m}$. Let $0<p\le 1$. It follows that

$$(2.26)\qquad \|\mathrm{tr}_\Gamma\,\varphi\,|\,L_p(\Gamma)\|^p = \int_\Gamma |\varphi(\gamma)|^p\,\mu(d\gamma) \le \sum_{\nu=0}^\infty \int_\Gamma \Big|\sum_{m\in\mathbb{Z}^n} \lambda_{\nu m}\,a_{\nu m}(\gamma)\Big|^p\,\mu(d\gamma)$$
$$\le \sum_{\nu=0}^\infty 2^{\nu d}\Psi(2^{-\nu})^{-1}\int_\Gamma \Big|\sum_{m\in\mathbb{Z}^n} \lambda_{\nu m}\widetilde{\chi}_{\nu m}(\gamma)\Big|^p\,\mu(d\gamma).$$

With $\chi_{\nu m}$ the characteristic function of $Q_{\nu m}$ and c_1 a positive constant independent of ν, m (recall Lemma 1.19), we have

$$(2.27)\qquad \|\mathrm{tr}_\Gamma\,\varphi\,|\,L_p(\Gamma)\|^p \le c_1\sum_{\nu=0}^\infty 2^{\nu d}\Psi(2^{-\nu})^{-1}\int_\Gamma \Big|\sum_{m\in\mathbb{Z}^n} |\lambda_{\nu m}|\chi_{\nu m}(\gamma)\Big|^p\,\mu(d\gamma)$$
$$\le c_1\sum_{\nu=0}^\infty 2^{\nu d}\Psi(2^{-\nu})^{-1}\int_\Gamma \sum_{m\in\mathbb{Z}^n} |\lambda_{\nu m}|^p\chi_{\nu m}(\gamma)\,\mu(d\gamma)$$
$$\le c_1\sum_{\nu=0}^\infty \sum_{m\in\mathbb{Z}^n} 2^{\nu d}\Psi(2^{-\nu})^{-1}|\lambda_{\nu m}|^p\mu(\Gamma\cap Q_{\nu m})$$
$$\le c_2\sum_{\nu=0}^\infty \sum_{m\in\mathbb{Z}^n} |\lambda_{\nu m}|^p \le c_3\Big(\sum_{\nu=0}^\infty\Big(\sum_{m\in\mathbb{Z}^n} |\lambda_{\nu m}|^p\Big)^{q/p}\Big)^{p/q}$$
$$= c_3\|\lambda\,|\,b_{pq}\|^p \le c_4\|\varphi\,|\,B_{pq}^{((n-d)/p,\Psi^{1/p})}(\mathbb{R}^n)\|^p.$$

We have made use of (2.24) and $\ell_{q/p}\hookrightarrow\ell_1$, due to $0<q\le p$ for $0<p\le 1$. The result follows from (2.27) by completion. If $1<p<\infty$, then the first inequality in (2.26) must be replaced by the usual triangle inequality and afterwards we need to use $\ell_q\hookrightarrow\ell_1$ (instead of $\ell_{q/p}\hookrightarrow\ell_1$), which comes from $0<q\le 1$. ■

Remark 2.15. (i) According to Theorem 2.19 of [ET99] the assertion (2.23) is sharp for $1<p<\infty$ and $q=1$, if either (i) $d<n$ or (ii) $d=n$ and $\sum_{j=0}^\infty \Psi(2^{-j})^{-1/p}<\infty$. This means that under these circumstances we even have equality in (2.23). In case $d<n$, (2.23) is sharp also for $0<p<\infty$ and $0<q<\min(1,p)$ (cf. [Bri00]). In any case the inclusion in Proposition 2.14 will be enough for our purpose.

(ii) We can complement (2.23) for $p=\infty$, because

$$(2.28)\qquad B_{\infty,1}^{(0,\Psi^0)}(\mathbb{R}^n) = B_{\infty,1}^0(\mathbb{R}^n)$$

consists of continuous functions and the trace is taken pointwise. Moreover, by Proposition 1.9(i),

$$B_{\infty,q}^0(\mathbb{R}^n)\hookrightarrow B_{\infty,1}^0(\mathbb{R}^n)$$

for any $0<q\le 1$. Concerning the first statement, see Section 20.1 of [Tri97] and the references given there.

(iii) By the embedding assertion in Proposition 1.9(ii), we have

$$(2.29)\qquad B_{p,q}^{(s+\varepsilon,\widetilde{\Psi})}(\mathbb{R}^n)\hookrightarrow B_{p,\min(1,p)}^{(s,\Psi^{1/p})}(\mathbb{R}^n)$$

for any $s\in\mathbb{R}$, $0<p,q\le\infty$, Ψ, $\widetilde{\Psi}$ admissible functions and $\varepsilon>0$. From (2.29) with $s=(n-d)/p$, (2.28) and (2.23) it makes sense to speak about traces on Γ for all spaces $B_{pq}^{(\sigma,\Psi)}(\mathbb{R}^n)$ with $0<p,q\le\infty$ and $\sigma>(n-d)/p$, as subspaces of $L_p(\Gamma)$.

So we are now able to generalise Definition 2.21 of [ET99] of the Besov spaces on a compact $(d,\varPsi)$-set for an arbitrary $p \in (0,\infty]$.

Definition 2.16. Let $\varGamma$ be a compact $(d,\varPsi)$-set in $\mathbb{R}^n$. Let $0 < p,q \le \infty$, $s > 0$ and $a \in \mathbb{R}$. Then

$$\mathbb{B}_{pq}^{(s,\varPsi^a)}(\varGamma) = \mathrm{tr}_\varGamma\, B_{pq}^{(s+(n-d)/p,\varPsi^{1/p+a})}(\mathbb{R}^n) \tag{2.30}$$

equipped with the quasi-norm

$$\|f \,|\, \mathbb{B}_{pq}^{(s,\varPsi^a)}(\varGamma)\| = \inf \|g \,|\, B_{pq}^{(s+(n-d)/p,\varPsi^{1/p+a})}(\mathbb{R}^n)\| \tag{2.31}$$

where the infimum is taken over all $g \in B_{pq}^{(s+(n-d)/p,\varPsi^{1/p+a})}(\mathbb{R}^n)$ with $\mathrm{tr}_\varGamma\, g = f$.

Lemma 2.17. *Let $\varGamma$ be a compact $(d,\varPsi)$-set in $\mathbb{R}^n$ and $r \ge 1$. For fixed $\nu \in \mathbb{N}_0$ let M_ν be the number of cubes $Q_{\nu m}$ such that $r\,Q_{\nu m} \cap \varGamma \neq \emptyset$. Then:*

(i) $M_\nu \sim 2^{\nu d}\varPsi(2^{-\nu})^{-1}$, $\nu \in \mathbb{N}_0$,
(ii) $\varPsi(2^{-\nu}) \sim \varPsi((2M_\nu)^{-1})$, $\nu \ge \nu_0$.

Proof. Step 1. Let μ denote a Radon measure related to the $(d,\varPsi)$-set $\varGamma$. For fixed $\nu \in \mathbb{N}_0$ let

$$\mathbb{Z}^{n,\varGamma,\nu} = \{m \in \mathbb{Z}^n : rQ_{\nu m} \cap \varGamma \neq \emptyset\}.$$

For each $m \in \mathbb{Z}^{n,\varGamma,\nu}$ we choose $\gamma_{\nu m} \in r\,Q_{\nu m} \cap \varGamma$. We have

$$rQ_{\nu m} \subset B(\gamma_{\nu m}, r\sqrt{n}\,2^{-\nu}), \qquad m \in \mathbb{Z}^{n,\varGamma,\nu},$$

and so $\{B(\gamma_{\nu m}, r\sqrt{n}\,2^{-\nu}) : m \in \mathbb{Z}^{n,\varGamma,\nu}\}$ covers $\varGamma$. By the properties of the admissible function $\varPsi$, namely Proposition 1.4(iv), there exists $\nu_0 \in \mathbb{N}$ such that for any natural number $\nu \ge \nu_0$,

$$\begin{aligned}\mu(\varGamma) &\le \mu\Big(\bigcup_{m\in\mathbb{Z}^{n,\varGamma,\nu}} B(\gamma_{\nu m}, r\sqrt{n}\,2^{-\nu})\Big) \le \sum_{m\in\mathbb{Z}^{n,\varGamma,\nu}} \mu(B(\gamma_{\nu,m}, r\,\sqrt{n}\,2^{-\nu}))\\ &\le c_1 \sum_{m\in\mathbb{Z}^{n,\varGamma,\nu}} (r\sqrt{n}\,2^{-\nu})^d \varPsi(r\sqrt{n}\,2^{-\nu}) \le c_2 M_\nu 2^{-\nu d}\varPsi(2^{-\nu}).\end{aligned}$$

Maybe with another constant, we obtain $M_\nu \ge c2^{\nu d}\varPsi(2^{-\nu})^{-1}$, $\nu \in \mathbb{N}_0$.

Step 2. For fixed $\nu \in \mathbb{N}$, let N_ν denote the largest possible number of disjoint balls centred at $\varGamma$ of radius $r2^{-\nu-2}$. Let $B_1,\ldots,B_{N_\nu}$ be a collection of such balls. Let B'_j denote the ball concentric with B_j with radius $r2^{-\nu-1}$, $j = 1,\ldots,N_\nu$. Note that $\{B'_j\}_{j=1}^{N_\nu}$ covers $\varGamma$: each $\gamma \in \varGamma$ must be within $r2^{-\nu-2}$ of one of the B_j, $j \in \{1,\ldots,N_\nu\}$, otherwise the ball $B(\gamma, r2^{-\nu-2})$ can be added to form a larger collection of disjoint balls. Moreover, each B'_j has diameter $r2^{-\nu}$ and therefore it intersects at most $(4[r]+1)^n$ cubes of side length $r2^{-\nu}$. Hence, $M_\nu \le (4[r]+1)^n\, N_\nu$. Using also again the properties of the admissible function $\varPsi$ we find $\nu_0 \in \mathbb{N}$ such that for any natural $\nu \ge \nu_0$,

$$\begin{aligned}\mu(\varGamma) &\ge \sum_{j=1}^{N_\nu} \mu(B_j) \ge c_1 \sum_{j=1}^{N_\nu} (r2^{-\nu-2})^d \varPsi(r2^{-\nu-2}) \ge c_2 \sum_{j=1}^{N_\nu} 2^{-\nu d}\varPsi(2^{-\nu})\\ &= c_2 N_\nu 2^{-\nu d}\varPsi(2^{-\nu}) \ge c_3 M_\nu 2^{-\nu d}\varPsi(2^{-\nu}).\end{aligned}$$

Maybe with another constant, we obtain $M_\nu \le c' 2^{\nu d} \Psi(2^{-\nu})^{-1}$, $\nu \in \mathbb{N}_0$. So the proof of (i) is finished.

Step 3. Since Ψ is an admissible function, by Proposition 1.4(i),(ii), there are positive constants c_1, c_2 and b such that

$$c_1 \nu^{-b} \le \Psi(2^{-\nu})^{-1} \le c_2 \nu^b, \qquad \nu \in \mathbb{N}. \tag{2.32}$$

We remark that given $\varepsilon > 0$, there exists a positive constant c_ε such that $\nu^b \le c_\varepsilon 2^{\varepsilon\nu}$ for all $\nu \in \mathbb{N}_0$. Hence, taking $0 < \varepsilon < d$ and using (2.32) as well as the assertion (i) proved in Steps 1 and 2, we get

$$c_1' 2^{-a_1\nu} \le (2\, M_\nu)^{-1} \le c_2' 2^{-a_2\nu}, \qquad \nu \in \mathbb{N}_0, \tag{2.33}$$

for some positive constants c_1', c_2', a_1, a_2. Then Proposition 1.4(iv) and (2.33) yield the desired inequalities. ∎

PROPOSITION 2.18. *Let Γ be a compact (d, Ψ)-set in $\mathbb{R}^n$. Let $0 < p_2 < p_1 \le \infty$, $0 < q \le \infty$, $s > 0$, $a \in \mathbb{R}$. Then we have the embedding*

$$\mathbb{B}^{(s,\Psi^a)}_{p_1 q}(\Gamma) \hookrightarrow \mathbb{B}^{(s,\Psi^a)}_{p_2 q}(\Gamma).$$

Proof. Let $f_\Gamma \in \mathbb{B}^{(s,\Psi^a)}_{p_1 q}(\Gamma)$. Then there exists $f \in B^{(s+(n-d)/p_1, \Psi^{1/p_1+a})}_{p_1 q}(\mathbb{R}^n)$ such that $\mathrm{tr}_\Gamma f = f_\Gamma$. By Corollary 1.27 and Definition 1.21 we can write

$$\begin{aligned} f = \sum_{\beta\in\mathbb{N}_0^n} \sum_{\nu=0}^{\infty} \sum_{m\in\mathbb{Z}^n} & \lambda^\beta_{\nu m} 2^{-\nu(\sigma-d/p_1)} \Psi(2^{-\nu})^{-1/p_1-a} \Phi^\beta(2^\nu x - m) \\ & + \varrho^\beta_{\nu m} 2^{-\nu(s-d/p_1)} \Psi(2^{-\nu})^{-1/p_1-a} ((-\Delta)^{(L+1)/2}\Phi^\beta)(2^\nu x - m) \end{aligned} \tag{2.34}$$

for $\sigma > \max(\sigma_{p_2}, s)$, $(L+1)/2 \in \mathbb{N}_0$ with $L \ge \max(-1, [\sigma_{p_2} - s])$ and

$$\sup_{\beta\in\mathbb{N}_0^n} 2^{\mu|\beta|} (\|\lambda^\beta \,|\, b_{p_1 q}\| + \|\varrho^\beta \,|\, b_{p_1 q}\|) < \infty \tag{2.35}$$

for any $\mu > 0$ large. The part relevant for the trace has (m, ν) such that $cQ_{\nu m} \cap \Gamma \ne \emptyset$. Let $\mathbb{Z}^{n,\Gamma,\nu} = \{m \in \mathbb{Z}^n : cQ_{\nu m} \cap \Gamma \ne \emptyset\}$. Having in mind Lemma 2.17(i), for fixed $\nu \in \mathbb{N}_0$, with M_ν the number of elements of $\mathbb{Z}^{n,\Gamma,\nu}$, we have $M_\nu \sim 2^{\nu d}\Psi(2^{-\nu})^{-1}$. With $1/p_1 + 1/r = 1/p_2$, we have

$$\begin{aligned} \Big(\sum_{m\in\mathbb{Z}^{n,\Gamma,\nu}} |\lambda^\beta_{\nu m}|^{p_2} \Big)^{1/p_2} &\le \Big(\sum_{m\in\mathbb{Z}^{n,\Gamma,\nu}} 1 \Big)^{1/r} \Big(\sum_{m\in\mathbb{Z}^{n,\Gamma,\nu}} |\lambda^\beta_{\nu m}|^{p_1} \Big)^{1/p_1} \\ &\le c(2^{\nu d}\Psi(2^{-\nu})^{-1})^{1/r} \Big(\sum_{m\in\mathbb{Z}^{n,\Gamma,\nu}} |\lambda^\beta_{\nu m}|^{p_1} \Big)^{1/p_1}. \end{aligned} \tag{2.36}$$

We can rewrite (2.34) as follows:

$$\begin{aligned} f = \sum_{\beta\in\mathbb{N}_0^n} \sum_{\nu=0}^{\infty} \sum_{m\in\mathbb{Z}^n} & 2^{-\nu d/r}\Psi(2^{-\nu})^{1/r} \lambda^\beta_{\nu m} 2^{-\nu(\sigma-d/p_2)} \Psi(2^{-\nu})^{-1/p_2-a} \Phi^\beta(2^\nu x - m) \\ & + 2^{-\nu d/r}\Psi(2^{-\nu})^{1/r} \varrho^\beta_{\nu m} 2^{-\nu(s-d/p_2)} \Psi(2^{-\nu})^{-1/p_2-a} ((-\Delta)^{(L+1)/2}\Phi^\beta)(2^\nu x - m). \end{aligned}$$

From (2.36) we get

$$\|2^{-\nu d/r}\Psi(2^{-\nu})^{1/r} \lambda^\beta_{\nu m} \,|\, b_{p_2 q}\| \le c \|\lambda^\beta \,|\, b_{p_1 q}\|$$

and, in a similar way

$$\|2^{-\nu d/r}\Psi(2^{-\nu})^{1/r}\varrho^{\beta}_{\nu m}\,|\,b_{p_2q}\| \le c\|\varrho^{\beta}\,|\,b_{p_1q}\|.$$

This together with (2.35) shows that $f_\Gamma \in \mathbb{B}^{(s,\Psi^a)}_{p_2q}(\Gamma)$. Moreover,

$$\|f_\Gamma\,|\,\mathbb{B}^{(s,\Psi^a)}_{p_2q}(\Gamma)\| \le \|f_\Gamma\,|\,\mathbb{B}^{(s,\Psi^a)}_{p_1q}(\Gamma)\|. \ \blacksquare$$

3. Entropy numbers

The aim of this section is to generalise Theorem 2.24 of [ET99]. On the one hand we include the case with $0 < p < 1$ and on the other hand, besides L_p, we also consider a $\mathbb{B}$-space as target space. The idea is the one developed by Triebel in [Tri97]: we use the knowledge about the entropy numbers of embeddings between general weighted sequence spaces, together with the techniques of subatomic decompositions developed in the first section, to estimate the entropy numbers of embeddings between Besov spaces of generalised smoothness on fractals. We need to recall basic results concerning entropy numbers, which is done in the next subsection.

3.1. Definition and elementary properties. In this subsection we recall basic facts concerning entropy numbers. We follow closely [ET96]. Other related references are [CS90] and [EE87]. If A, B are quasi-Banach spaces then $L(A,B)$ denotes the family of all bounded linear maps from A into B and $U_A = \{a \in A : \|a\,|\,A\| \le 1\}$.

DEFINITION 3.1. Let A, B be quasi-Banach spaces and let $T \in L(A,B)$. Then for all $k \in \mathbb{N}$, the kth *entropy number* $e_k(T)$ of T is defined by

$$e_k(T) = \inf\Big\{\varepsilon > 0 : T(U_A) \subset \bigcup_{j=1}^{2^{k-1}} (b_j + \varepsilon U_B) \text{ for some } b_j \in B,\ j \in \{1,\ldots,2^{k-1}\}\Big\}.$$

The following proposition gives some elementary properties of entropy numbers. We refer to Lemma 1 of [ET96, 1.3.1, pp. 7–8], where a simple proof may be found.

PROPOSITION 3.2. *Let A, B, C be quasi-Banach spaces, let $S,T \in L(A,B)$ and suppose that $R \in L(B,C)$.*

(i) $\|T\| \ge e_1(T) \ge e_2(T) \ge \ldots \ge 0$; $e_1(T) = \|T\|$ *if B is a Banach space.*

(ii) *For all $k,l \in \mathbb{N}$,*

$$e_{k+l-1}(R \circ S) \le e_k(R)\, e_l(S).$$

(iii) *If B is a p-Banach space, where $0 < p \le 1$, then for all $k,l \in \mathbb{N}$,*

$$e^p_{k+l-1}(S+T) \le e^p_k(S) + e^p_l(T).$$

REMARK 3.3. Since the $e_k(T)$ decrease as k increases, and are non-negative, $\lim_{k\to\infty} e_k(T)$ exists and plainly equals

$$\inf\{\varepsilon > 0 : T(U_A) \text{ can be covered by finitely many } B\text{-balls of radius } \varepsilon\}.$$

Hence, $T \in L(A,B)$ is compact if, and only if, $\lim_{k\to\infty} e_k(T) = 0$.

An important application of entropy numbers is to spectral theory. If $T \in L(B)$ is a compact operator on the quasi-Banach space B, then the spectrum of T, apart from the point 0, consists solely at most of a countable infinite number of eigenvalues of finite algebraic multiplicity. We refer to [ET96, pp. 3–7]. Let $(\mu_k(T))_{k\in\mathbb{N}}$ be the sequence of all non-zero eigenvalues of T, repeated according to algebraic multiplicity and ordered so that

$$|\mu_1(T)| \geq |\mu_2(T)| \geq \ldots \to 0.$$

If T has only m $(< \infty)$ distinct eigenvalues and M is the sum of their algebraic multiplicities we put $\mu_n(T) = 0$ for all $n > M$. A connection between $\mu_k(T)$ and $e_k(T)$ is provided by the following:

THEOREM 3.4 [CT80]. *Let B be a quasi-Banach space, $T \in L(B)$ a compact operator and $(\mu_k(T))_{k\in\mathbb{N}}$ as above. Then*

$$\Big(\prod_{m=1}^{k} |\mu_k(T)|\Big)^{1/k} \leq \inf_{n\in\mathbb{N}} 2^{n/(2k)} e_n(T), \qquad k \in \mathbb{N}.$$

An immediate consequence is Carl's inequality:

COROLLARY 3.5. *For all $k \in \mathbb{N}$, $|\mu_k(T)| \leq \sqrt{2}\, e_k(T)$.*

3.2. Entropy numbers of embeddings between weighted sequence spaces. In this subsection we follow closely [Leo00a], [Leo98b] and [Tri97, §8,9]. We begin by introducing sequence spaces with which we will be concerned.

DEFINITION 3.6. Let $0 < p, q \leq \infty$, $\{\beta_j\}_{j=0}^{\infty}$ a general weight sequence and $\{M_j\}_{j=0}^{\infty}$ a sequence of natural numbers. Then $\ell_q(\beta_j \ell_p^{M_j})$ is the collection of all complex sequences $x = (x_{j,l} : j \in \mathbb{N}_0,\ l = 1, \ldots, M_j)$ such that the quasi-norm

$$\|x \mid \ell_q(\beta_j\, \ell_p^{M_j})\| = \Big(\sum_{j=0}^{\infty} \beta_j^q \Big(\sum_{l=1}^{M_j} |x_{j,l}|^p\Big)^{q/p}\Big)^{1/q}$$

is finite (with obvious modifications if $p = \infty$ and/or $q = \infty$). In case $\beta_j = 1$, $j \in \mathbb{N}_0$, we write $\ell_q(\ell_p^{M_j})$.

Following some suggestions from Professor Leopold, Theorem 1 of [Leo00a] and its proof can be modified in order to obtain the next proposition. The cited result in [Leo00a] was not sufficient for our case and for completeness we present the next proposition and its proof which turns out to be sufficient for our purposes. However, we mention that a generalisation of both Theorem 1 of [Leo00a] and the proposition below can now be found in the recent paper of Leopold [Leo00b].

PROPOSITION 3.7. *Let $0 < p_1 \leq p_2 \leq \infty$, $0 < q_1, q_2 \leq \infty$, $\{M_j\}_{j=0}^{\infty}$ a sequence of natural numbers satisfying*

$$M_j \sim 2^{jd} \Psi^{-1}(2^{-j}), \qquad j \in \mathbb{N}_0, \tag{3.1}$$

and $\beta_j = 2^{j\delta}\Psi^b(2^{-j})$, $j \in \mathbb{N}_0$, a weight sequence where $d, \delta \in \mathbb{R}^+$, $b \in \mathbb{R}$ and Ψ is an admissible function. Then

$$e_{2M_L}[\mathrm{id} : \ell_{q_1}(\beta_j\, \ell_{p_1}^{M_j}) \to \ell_{q_2}(\ell_{p_2}^{M_j})] \sim \beta_L^{-1}\, M_L^{-(1/p_1 - 1/p_2)}, \qquad L \in \mathbb{N}_0. \tag{3.2}$$

Proof. *Step 1.* According to Theorem 1 in [Leo98b] and since $p_1 \le p_2$, the embedding

$$\mathrm{id} : \ell_{q_1}(\beta_j\, \ell_{p_1}^{M_j}) \to \ell_{q_2}(\ell_{p_2}^{M_j}) \tag{3.3}$$

exists and is bounded if, and only if, $(\beta_j^{-1})_{j\in\mathbb{N}_0} \in \ell_{q^*}$ where $1/q^* = (1/q_2 - 1/q_1)_+$. By Proposition 1.4(i),(ii), there are positive constants c_1, c_2 and b' such that

$$c_1 j^{-b'} \le \Psi^b(2^{-j}) \le c_2 j^{b'}, \qquad j \in \mathbb{N}.$$

This together with $\delta > 0$ gives us $(\beta_j^{-1})_{j\in\mathbb{N}_0} \in \ell_{q^*}$ for any $q^* \in (0,\infty]$. Therefore, the embedding (3.3) exists and is bounded. Moreover, a direct application of Lemma 1 of [Leo98b] provides the desired estimate from below for its entropy numbers.

Step 2. We decompose the embedding in (3.3) as

$$\mathrm{id} = \sum_{j=0}^{\infty} \mathrm{id}_j, \tag{3.4}$$

where

$$\mathrm{id}_j\, x = (\delta_{jk}\, x_{k,l})_{k\in\mathbb{N}_0,\, l=1,\dots M_k} = (0,\dots,0,x_{j,1},\dots,x_{j,M_j},0,\dots,0). \tag{3.5}$$

We have

$$\Big\| \Big(\mathrm{id} - \sum_{j=0}^{N} \mathrm{id}_j \Big) x \;\Big|\; \ell_{q_2}(\ell_{p_2}^{M_j}) \Big\| \le R_N \| x \,|\, \ell_{q_1}(\beta_j\, \ell_{p_1}^{M_j}) \| \tag{3.6}$$

with

$$R_N = \Big(\sum_{j=N+1}^{\infty} \beta_j^{-q^*} \Big)^{1/q^*} \tag{3.7}$$

(with the usual modification if $q^* = \infty$), q^* being such that $1/q^* = (1/q_2 - 1/q_1)_+$. Let $\varrho = \min(1,p_2,q_2)$; then $\ell_{q_2}(\ell_{p_2}^{M_j})$ is a ϱ-Banach space. Using (3.4), (3.6) and Proposition 3.2(i),(ii), we get

$$e_k^{\varrho}(\mathrm{id}) \le R_N^{\varrho} + \sum_{j=0}^{L} e_{k_j}^{\varrho}(\mathrm{id}_j) + \sum_{j=L+1}^{N} e_{k_j}^{\varrho}(\mathrm{id}_j) \tag{3.8}$$

where

$$k = \sum_{j=0}^{N} k_j - (N+1). \tag{3.9}$$

The splitting of k into the k_j is not fixed at the moment and L is a natural number between 0 and N which will also be chosen later.

Step 3. For each $j \in \mathbb{N}_0$, we consider the commutative diagram

$$\begin{tikzcd} \ell_{q_1}(\beta_j \ell_{p_1}^{M_j}) \arrow[r, "\mathrm{id}_j"] \arrow[d, "T_j"'] & \ell_{q_2}(\ell_{p_2}^{M_j}) \\ \ell_{p_1}^{M_j} \arrow[r, "\mathrm{id}^{(j)}"] & \ell_{p_2}^{M_j} \arrow[u, "E_j"'] \end{tikzcd}$$

where

$$T_j x = (x_{j,l})_{l=1}^{M_j} \quad \text{and} \quad E_j((y_l)_{l=1}^{M_j}) = (0,\dots,0,\widehat{y}_{j,1},\dots,\widehat{y}_{j,M_j},0,\dots,0) \quad \text{with} \quad \widehat{y}_{j,l} = y_l.$$

We have

$$\|T_j : \ell_{q_1}(\beta_j \ell_{p_1}^{M_j}) \to \ell_{p_1}^{M_j}\| = \beta_j^{-1}, \qquad \|E_j : \ell_{p_2}^{M_j} \to \ell_{q_2}(\ell_{p_2}^{M_j})\| = 1,$$

and $\mathrm{id}_j = E_j \circ \mathrm{id}^{(j)} \circ T_j$. By Proposition 3.2(i),(ii) we get

$$e_k(\mathrm{id}_j) \le \beta_j^{-1} e_{k_j}[\mathrm{id}^{(j)} : \ell_{p_2}^{M_j} \to \ell_{p_2}^{M_j}]. \tag{3.10}$$

Step 4. For $j = 0, \dots, L$, let k_j be natural numbers such that

$$k_j - 1 < 2M_j 2^{(L-j)d/2} \le k_j.$$

Then $k_j \ge 2M_j$, $j = 0, \dots, L$. Moreover, using (3.1), $j \le L$, Proposition 1.4(vi) and $d > 0$, we obtain

$$\begin{aligned}\sum_{j=0}^{L} k_j &\le \sum_{j=0}^{L} 2M_j 2^{(L-j)d/2} + (L+1) \le c_1(2M_L) \sum_{j=0}^{L} 2^{-(L-j)d/2}(1+L-j)^c + (L+1)\\ &\le c_2(2M_L) + (L+1).\end{aligned}$$

By (3.10) and Proposition 7.3 in [Tri97], we get

$$\begin{aligned}e_{k_j}(\mathrm{id}_j) &\le c\beta_j^{-1} 2^{-k_j/(2M_j)} (2M_j)^{-(1/p_1 - 1/p_2)}\\ &\le c\beta_L^{-1}(2M_L)^{-(1/p_1-1/p_2)} \frac{\beta_L}{\beta_j} \left(\frac{M_j}{M_L}\right)^{-(1/p_1-1/p_2)} 2^{-2^{(L-j)d/2}}.\end{aligned}$$

Summation gives

$$\sum_{j=0}^{L} e_{k_j}^{\varrho}(\mathrm{id}_j) \le c^{\varrho}\, \beta_L^{-\varrho}(2M_L)^{-(1/p_1-1/p_2)\varrho} R_{L,\varrho},$$

with

$$\begin{aligned}R_{L,\varrho} &= \sum_{j=0}^{L} \left(\frac{\beta_L}{\beta_j}\right)^{\varrho} \left(\frac{M_j}{M_L}\right)^{-(1/p_1-1/p_2)\varrho} 2^{-\varrho 2^{(L-j)d/2}}\\ &\le c_1 \sum_{j=0}^{L} 2^{(L-j)\varrho(\delta + d(1/p_1-1/p_2))} (1+L-j)^{\varrho(c_2 + c_3(1/p_1-1/p_2))} 2^{-\varrho 2^{(L-j)d/2}} < \infty\end{aligned}$$

as $d > 0$. Therefore

$$\sum_{j=0}^{L} e_{k_j}^{\varrho}(\mathrm{id}_j) \le c\beta_L^{-\varrho}(2M_L)^{-(1/p_1-1/p_2)\varrho}$$

for every natural L with c being a positive constant independent of L.

Step 5. The aim is to estimate the remaining sum in (3.8) by an expression which depends on L and other parameters, but is independent of N in such a way that

$$\sum_{j=L+1}^{N} e_{k_j}^{\varrho}(\mathrm{id}_j) \le c\beta_L^{-\varrho}(2M_L)^{-(1/p_1-1/p_2)\varrho}$$

and

$$\sum_{j=L+1}^{N} k_j \le c(2M_L) + N - L,$$

with c a positive constant also independent of N. First of all, we remark the existence of positive constants c_i, $i = 1, 2, 3$, and $c_4 \geq 0$ such that

$$c_1 2^{kd} \Psi(2^{-k})^{-1} \leq M_k \leq c_2 2^{kd} \Psi(2^{-k})^{-1}, \qquad k \in \mathbb{N}_0,$$

and

$$c_3'(1+j-k)^{-c_4} \leq \frac{\Psi(2^{-j})}{\Psi(2^{-k})} \leq c_3(1+j-k)^{c_4}, \qquad j,k \in \mathbb{N}_0,\ j \geq k.$$

Let $j = L+1, \ldots, N$ and k_j be natural numbers such that

$$k_j - 1 < CM_L(1+j-L)^{-\kappa} \leq k_j$$

with $C = C(M, \Psi) = c_1/(c_2 c_3)$ and $\kappa \geq \max(c_4, 2)$. Then

$$k_j < 1 + C\, M_L(1+j-L)^{-\kappa} \leq 1 + M_j 2^{-(j-L)d}(1+j-L)^{c_4-\kappa} \leq 1 + M_j \leq 2M_j.$$

Moreover

$$\begin{aligned}\sum_{j=L+1}^{N} k_j &\leq CM_L \sum_{j=L+1}^{N} (1+j-L)^{-\kappa} + (N-L) \\ &\leq CM_L \sum_{k=1}^{\infty} (1+k)^{-2} + (N-L) \leq c2M_L + (N-L)\end{aligned}$$

where c is a positive constant independent of L and N. Because $k_j \leq 2M_j$ for $j = L+1, \ldots, N$, and by Proposition 7.3 in [Tri97], we get

$$\begin{aligned}e_{k_j}(\mathrm{id}_j) &\leq c\beta_j^{-1}\left(k_j^{-1} \log\left(1 + \frac{2M_j}{k_j}\right)\right)^{1/p_1 - 1/p_2} \\ &\leq c'(2M_L)^{-(1/p_1-1/p_2)} \beta_j^{-1} (1+j-L)^{\kappa(1/p_1-1/p_2)} \\ &\quad \times ((c_4+\kappa)\log(1+j-L) + (j-L)d)^{-(1/p_1-1/p_2)}.\end{aligned}$$

Summation gives

$$\sum_{j=L+1}^{N} e_{k_j}^{\varrho}(id_j) \leq c'^{\varrho}(2M_L)^{-\varrho(1/p_1-1/p_2)} \beta_L^{-\varrho} R_{N,L,\varrho},$$

with

$$R_{N,L,\varrho} = \sum_{j=L+1}^{N} \left(\frac{\beta_L}{\beta_j}\right)^{\varrho} (1+j-L)^{\kappa(1/p_1-1/p_2)}((c_4+\kappa)\log(1+j-L) + (j-L)d)^{(1/p_1-1/p_2)}.$$

We have

$$\begin{aligned}R_{N,L,\varrho} \leq\ & c_6 \sum_{j=L+1}^{N} 2^{\delta\varrho(L-j)}(1+j-L)^{\varrho c_5 + \kappa\varrho(1/p_1-1/p_2)}((c_4+\kappa)\log(1+j-L))^{1/p_1-1/p_2} \\ &+ c_6 \sum_{j=L+1}^{N} 2^{\delta\varrho(L-j)}(1+j-L)^{\varrho c_5 + \kappa\varrho(1/p_1-1/p_2)}((j-L)d)^{(1/p_1-1/p_2)\varrho}\end{aligned}$$

$$\leq c_6(c_4+\kappa)^{(1/p_1-1/p_2)\varrho}\sum_{k=0}^{\infty}2^{-\delta\varrho k}(1+k)^{\varrho c_5+\kappa\varrho(1/p_1-1/p_2)}\log(1+k)^{1/p_1-1/p_2}$$
$$+\ c_6\sum_{k=0}^{\infty}2^{-\delta\varrho k}(1+k)^{\varrho c_5+\kappa\varrho(1/p_1-1/p_2)}(kd)^{(1/p_1-1/p_2)\varrho}<\infty$$

since $\delta\varrho>0$.

Step 6. By the previous two steps we get

$$k=\sum_{j=0}^{N}k_j-(N+1)\leq c2M_L,$$

which put in (3.8) gives

$$e^{\varrho}_{c2M_L}(\mathrm{id})\leq e^{\varrho}_k(\mathrm{id})\leq R^{\varrho}_N+C\beta_L^{-\varrho}(2M_L)^{-(1/p_1-1/p_2)\varrho}.$$

We now choose N in such a way that

$$R_N\sim\beta_L^{-1}(2M_L)^{-(1/p_1-1/p_2)}.$$

We can always do so because

$$0<\beta_L^{-1}(2M_L)^{-(1/p_1-1/p_2)}\leq R_L$$

and $(R_N)_{N\in\mathbb{N}}$ is a decreasing sequence with $\lim_{N\to\infty}R_N=0$. Therefore

$$e_{c2M_L}(\mathrm{id})\leq c'\beta_L^{-1}(2M_L)^{-(1/p_1-1/p_2)},\qquad L\in\mathbb{N}. \tag{3.11}$$

Step 7. By (3.1) and Proposition 1.4(vi), for $l,j\in\mathbb{N}_0$ with $j\geq l$, we have

$$\frac{M_{j-l}}{M_j}\leq c_1 2^{-ld}\,\frac{\Psi(2^{-j})}{\Psi(2^{-(j-l)})}\leq c_2 2^{-ld}(1+l)^{c_3}, \tag{3.12}$$

with $c_1,c_2>0$ and $c_3\geq 0$ constants independent of j and l. The right-hand side of (3.12) tends to zero as l goes to infinity. With c the positive constant in (3.11), we can assure the existence of $l_0\in\mathbb{N}$ such that the right-hand side of (3.12) is less than or equal to c^{-1} for any $l\geq l_0$. Hence

$$cM_{j-l_0}\leq M_j,\qquad j\geq l_0. \tag{3.13}$$

Reasoning as above, we remark that

$$\frac{M_{j-l_0}}{M_j}\geq c_1 2^{-l_0 d}(1+l_0)^{-c_2}\geq c_3\quad\text{ and }\quad\frac{\beta_{j-l_0}}{\beta_j}\geq c_1' 2^{-l_0\delta}(1+l_0)^{-c_2'}\geq c_3'$$

where c_3 and c_3' are positive constants independent of $j\geq l_0$. Using these last inequalities, (3.11) and (3.13), we obtain

$$e_{2M_j}(\mathrm{id})\leq e_{c2M_{j-l_0}}(\mathrm{id})\leq c'\beta_{j-l_0}^{-1}M_{j-l_0}^{-(1/p_1-1/p_2)}\leq c''\beta_j^{-1}M_j^{-(1/p_1-1/p_2)},\qquad j\geq l_0.$$

Maybe with another positive constant c'', we get the inequality

$$e_{2M_L}(\mathrm{id})\leq c''\beta_L^{-1}M_L^{-(1/p_1-1/p_2)}$$

for any $L\in\mathbb{N}_0$, and the proof is now complete. ■

Proposition 3.7 is not completely sufficient for our later purposes. We need some kind of ℓ_u-version of it.

Definition 3.8. Let $0 < p, q, u \le \infty$, $\mu \ge 0$, $\{\beta_j\}_{j=0}^{\infty}$ a general weight sequence and $\{M_j\}_{j=0}^{\infty}$ a sequence of natural numbers. Then $\ell_u[2^{\mu m}\ell_q(\beta_j\,\ell_p^{M_j})^2]$ is the collection of all $\ell_q(\beta_j\,\ell_p^{M_j})^2$-valued sequences $x = (x_1^m, x_2^m)$, $m \in \mathbb{N}_0$, such that the quasi-norm

$$\|x\,|\,\ell_u[2^{\mu m}\ell_q(\beta_j\,\ell_p^{M_j})^2]\| = \Big(\sum_{m=0}^{\infty} 2^{\mu m u}(\|x_1^m\,|\,\ell_q(\beta_j\,\ell_p^{M_j})\| + \|x_2^m\,|\,\ell_q(\beta_j\,\ell_p^{M_j})\|)^u\Big)^{1/u}$$

is finite (with obvious modifications if $u = \infty$).

Proposition 3.9. *Let $0 < p_1 \le p_2 \le \infty$, $0 < q_1, q_2, u_1, u_2 \le \infty$, $\mu > 0$, $\{M_j\}_{j=0}^{\infty}$ a sequence of natural numbers satisfying* (3.1) *and $\beta_j = 2^{j\delta}\Psi^b(2^{-j})$, $j \in \mathbb{N}_0$, a weight sequence where $d, \delta \in \mathbb{R}^+$, $b \in \mathbb{R}$ and Ψ is an admissible function. Then the identity map*

$$\mathrm{id} : \ell_{u_1}[2^{\mu m}\ell_{q_1}(\beta_j\,\ell_{p_1}^{M_j})^2] \to \ell_{u_2}[\ell_{q_2}(\ell_{p_2}^{M_j})^2] \tag{3.14}$$

is compact and for the related entropy numbers we have

$$e_{2M_L}(\mathrm{id}) \sim \beta_L^{-1} M_L^{-(1/p_1 - 1/p_2)}, \qquad L \in \mathbb{N}_0. \tag{3.15}$$

Proof. Step 1. To prove that (3.14) is compact we use the decomposition

$$\mathrm{id} = \sum_{m=0}^{\infty} \mathrm{id}_m, \qquad \mathrm{id}_m = \mathrm{id}_{m,1} + \mathrm{id}_{m,2} \tag{3.16}$$

where

$$\mathrm{id}_{m,i}\,x = (y_1^k, y_2^k)_{k\in\mathbb{N}_0}, \quad \text{with} \quad y_j^k = \delta_{ji}\delta_{km}x_j^k \quad \text{and} \quad x = (x_1^k, x_2^k)_{k\in\mathbb{N}_0}.$$

We have

$$\begin{aligned} \|\mathrm{id}_m\,x\,|\,\ell_{u_2}[\ell_{q_2}(\ell_{p_2}^{M_j})^2]\| &= \|(x_1^m, x_2^m)\,|\,\ell_{q_2}(\ell_{p_2}^{M_j})^2\| \\ &= \|x_1^m\,|\,\ell_{q_2}(\ell_{p_2}^{M_j})\| + \|x_2^m\,|\,\ell_{q_2}(\ell_{p_2}^{M_j})\| \\ &\le c(\|x_1^m\,|\,\ell_{q_1}(\beta_j\,\ell_{p_1}^{M_j})\| + \|x_2^m\,|\,\ell_{q_1}(\beta_j\,\ell_{p_1}^{M_j})\|) \\ &\le c2^{-\mu m}\|x\,|\,\ell_{u_1}[2^{\mu m}\ell_{q_1}(\beta_j\,\ell_{p_1}^{M_j})^2]\|. \end{aligned} \tag{3.17}$$

Now by (3.16), (3.17) and $\mu > 0$, it follows that id is compact.

Step 2. In this step we prove the estimate from above for the entropy numbers of the identity map (3.14). In the commutative diagram

$$\begin{array}{ccc} \ell_{q_1}(\beta_j\,\ell_{p_1}^{M_j}) & \xrightarrow{\widetilde{\mathrm{id}}} & \ell_{q_2}(\ell_{p_2}^{M_j}) \\ {\scriptstyle E_{m,i}}\big\downarrow & & \big\uparrow{\scriptstyle T_{m,i}} \\ \ell_{u_1}[2^{\mu m}\ell_{q_1}(\ell_{p_1}^{M_j})^2] & \xrightarrow{\mathrm{id}} & \ell_{u_2}[\ell_{q_2}(\ell_{p_2}^{M_j})^2] \end{array}$$

the operator $E_{m,i}$ is defined by

$$E_{m,i}z = (y_1^k, y_2^k)_{k\in\mathbb{N}_0} \quad \text{with} \quad y_l^k = \delta_{li}\delta_{km}z,$$

the operator $T_{m,i}$ by

$$T_{m,i}\,x = x_i^m \quad \text{for } x = (x_1^k, x_2^k)_{k\in\mathbb{N}_0},$$

for $i = 1, 2$, and $\widetilde{\mathrm{id}}$ denotes the identity map between the indicated spaces. We have

$$\widetilde{\mathrm{id}} = T_{m,i} \circ \mathrm{id} \circ E_{m,i}, \qquad m \in \mathbb{N}_0. \tag{3.18}$$

Plainly

$$\|E_{m,i}\| = 2^{\mu m} \quad \text{and} \quad \|T_{m,i}\| = 1,$$

and consequently by the multiplication property of entropy numbers, Proposition 3.2(ii) and (i), and Proposition 3.7, we get

$$\begin{aligned} c\beta_L^{-1} M_L^{-(1/p_1-1/p_2)} &\le e_{2M_L}(\widetilde{\mathrm{id}}) \le \|R_{m,1}\| e_{2M_L}(\mathrm{id}) \, \|E_{m,1}\| \\ &\le 2^{\mu m} e_{2M_L}(\mathrm{id}), \qquad m, L \in \mathbb{N}_0. \end{aligned}$$

In particular,

$$e_{2M_L}(\mathrm{id}) \ge c\beta_L^{-1} M_L^{-(1/p_1-1/p_2)}, \qquad L \in \mathbb{N}_0. \tag{3.19}$$

Step 3. Let, for brevity, $a = \delta/d + 1/p_1 - 1/p_2$, which is greater than zero since $\delta > 0$ and $p_1 \le p_2$. Let $L \in \mathbb{N}_0$ and

$$N = \left[\frac{\log(\beta_L M_L^{1/p_1-1/p_2})}{\mu} \right] + 1.$$

Recall (3.16) and (3.17). It follows that

$$\left\| \mathrm{id} - \sum_{m=0}^{N} \mathrm{id}_m \right\| \le c 2^{-\mu N} \le c' \beta_L^{-1} M_L^{-(1/p_1-1/p_2)}. \tag{3.20}$$

Let $\varrho = \min(1, p_2, q_2, u_2)$. Then $\ell_{u_2}[\ell_{q_2}(\ell_{p_2}^{M_j})^2]$ is a ϱ-Banach space. By (3.16), Proposition 3.2(iii),(i) and (3.20) we obtain

$$\begin{aligned} e_k^\varrho(\mathrm{id}) &\le \left\| \mathrm{id} - \sum_{m=0}^{N} \mathrm{id}_m \right\|^\varrho + \sum_{m=0}^{N} e_{k_m}^\varrho(\mathrm{id}_m) \\ &\le c'^\varrho \beta_L^{-\varrho} M_L^{-(1/p_1-1/p_2)\varrho} + \sum_{m=0}^{N} e_{k_m}^\varrho(\mathrm{id}_m), \end{aligned} \tag{3.21}$$

where $k = \sum_{m=0}^{N} k_m$. For $m \in \mathbb{N}_0$ and $i = 1, 2$ we have the commutative diagram

$$\begin{tikzcd} \ell_{u_1}[2^{\mu m}\ell_{q_1}(\beta_j \ell_{p_1}^{M_j})^2] \arrow[r, "\mathrm{id}_{m,i}"] \arrow[d, "T_{m,i}"'] & \ell_{u_2}[\ell_{q_2}(\ell_{p_2}^{M_j})^2] \\ \ell_{q_1}(\beta_j \ell_{p_1}^{M_j}) \arrow[r, "\widetilde{\mathrm{id}}"] & \ell_{q_2}(\ell_{p_2}^{M_j}) \arrow[u, "E_{m,i}"'] \end{tikzcd}$$

where the operators $\mathrm{id}_{m,i}$, $E_{m,i}$, $T_{m,i}$ were defined in the previous steps. Hence $\mathrm{id}_{m,i} = E_{m,i} \circ \widetilde{\mathrm{id}} \circ T_{m,i}$. Then $e_k(\mathrm{id}_{m,i}) \le 2^{-\mu m} e_k(\widetilde{\mathrm{id}})$ and therefore

$$e_{2k}(\mathrm{id}_m) \le e_k(\mathrm{id}_{m,1}) + e_k(\mathrm{id}_{m,2}) \le 2^{-\mu m+1} e_k(\widetilde{\mathrm{id}}). \tag{3.22}$$

Now we choose

$$k_m = 4M_{J_m}, \qquad m = 0, \dots, N, \tag{3.23}$$

where

$$J_m = \inf\{J \in \mathbb{N} : 2M_L 2^{-m\varepsilon} \le 2M_J\} \tag{3.24}$$

and $\varepsilon > 0$ is such that $a\varepsilon < \mu$. In particular, we have

$$2M_{J_m-1} < 2M_L 2^{-m\varepsilon} \le 2M_{J_m} \quad \text{and} \quad J_m \le L, \quad m = 0, \dots, N.$$

We remark that (3.1) and the properties of Ψ, namely Proposition 1.4(vi), give us

$$\frac{M_k}{M_{k-1}} \leq c_1 2^d \frac{\Psi(2^{-(k+1)})}{\Psi(2^{-k})} \leq c_2, \qquad k \in \mathbb{N},$$

where c_2 is a positive constant independent of k. Then we have

$$\sum_{m=0}^{N} k_m = 2 \sum_{m=0}^{N} \frac{M_{J_m}}{M_{J_m-1}} 2M_{J_m-1} \leq c \sum_{m=0}^{N} (2M_L) 2^{-m\varepsilon} \leq c' 2M_L \tag{3.25}$$

and

$$\sum_{m=0}^{N} k_m \geq 4M_L \sum_{m=0}^{N} 2^{-\varepsilon m} \geq 2(2M_L). \tag{3.26}$$

By (3.22), (3.23) and Proposition 3.7, we get

$$e_{k_m}(\mathrm{id}_m) \leq 2^{-\mu m+1} e_{2M_{J_m}}(\widetilde{\mathrm{id}}) \leq c 2^{-\mu m} \beta_{J_m}^{-1} M_{J_m}^{-(1/p_1-1/p_2)}, \qquad m = 0, \ldots, N. \tag{3.27}$$

Hence

$$\sum_{m=0}^{N} e_{k_m}^{\varrho}(\mathrm{id}_m) \leq c^{\varrho} \beta_L^{-\varrho} M_L^{-(1/p_1-1/p_2)\varrho} R_{N,\varrho}, \tag{3.28}$$

where

$$R_{N,\varrho} = \sum_{m=0}^{N} 2^{-\mu m \varrho} \left(\frac{\beta_L}{\beta_{J_m}} \right)^{\varrho} \left(\frac{M_L}{M_{J_m}} \right)^{(1/p_1-1/p_2)\varrho}. \tag{3.29}$$

By definition of the sequence $(\beta_j)_{j\in\mathbb{N}_0}$ and Proposition 1.4(vi), we have

$$\frac{\beta_L}{\beta_{J_m}} \leq c \left(\frac{M_L}{M_{J_m}} \right)^{\delta/d} \left(\frac{\Psi(2^{-L})}{\Psi(2^{-J_m})} \right)^{b+\delta/d} \leq c' \left(\frac{M_L}{M_{J_m}} \right)^{\delta/d} (1 + L - J_m)^{\eta}, \tag{3.30}$$

where the constants $c, c' > 0$ and $\eta \geq 0$ are independent of L and m. We are now concerned with the estimation from above of $L - J_m$. On the one hand, we have

$$\frac{M_L}{M_{J_m}} \leq 2^{\varepsilon m}, \qquad m = 0, \ldots, N, \tag{3.31}$$

and on the other hand, (3.1) and Proposition 1.4(vi) give us

$$\frac{M_L}{M_{J_m}} \geq c 2^{(L-J_m)d} \frac{\Psi(2^{-J_m})}{\Psi(2^{-L})} \geq c' \frac{2^{(L-J_m)d}}{(1 + L - J_m)^{\sigma}}, \qquad m = 0, \ldots, N, \tag{3.32}$$

where $c, c' > 0$ and $\sigma \geq 0$ are independent of L and m. There exists a suitable constant $c^* > 0$, only depending on σ and d, such that $(1 + y)^{\sigma} \leq c^* 2^{dy/2}$ for all $y \geq 0$. Putting this in (3.32) gives

$$\frac{M_L}{M_{J_m}} \geq c 2^{(L-J_m)d/2}, \qquad m = 0, \ldots, N, \tag{3.33}$$

for some positive constant $c < 1$, which is again independent of L and m. From (3.31) and (3.33) we can conclude that

$$L - J_m \leq c_1 m + c_2, \qquad m = 0, \ldots, N, \tag{3.34}$$

with constants $c_1, c_2 > 0$ independent of L and m. Back to (3.29), using (3.30), (3.31) and (3.34) and since $\varepsilon a - \mu < 0$, we get

$$R_{N,\varrho} \le c \sum_{m=0}^{N} 2^{(\varepsilon a-\mu)m\varrho}(1 + c_1 m + c_2)^{\eta} < \infty.$$

Having this in mind, by (3.21), (3.28) and (3.25), we can write

$$e_{c2M_L}(\mathrm{id}) \le c' \beta_L^{-1} M_L^{-(1/p_1-1/p_2)}, \quad L \in \mathbb{N}_0,$$

for some positive constants c and c'. Acting as in Step 7 of the proof of Proposition 3.7 we complete the proof. ∎

Corollary 3.10. *Let p_1, p_2, q_1, q_2, u_1, u_2, μ, $\{M_j\}_{j=0}^{\infty}$, $\{\beta_j\}_{j=0}^{\infty}$, d, δ, b and Ψ be as in Proposition* 3.9. *Moreover, assume that the sequence $\{M_j\}_{j=0}^{\infty}$ is increasing. Then for the entropy numbers of the identity map* (3.14) *we have*

$$e_k(\mathrm{id}) \sim (k\Psi(k^{-1}))^{-(\delta/d+1/p_1-1/p_2)}\Psi(k^{-1})^{-b+1/p_1-1/p_2}, \quad k \in \mathbb{N}. \tag{3.35}$$

Proof. Let $k \in \mathbb{N}$ with $k \ge \max(\nu_0, 2M_0)$, where ν_0 is a natural number as in Lemma 2.17(ii). Since $\{M_j\}_{j=0}^{\infty}$ is increasing, there exists $L \in \mathbb{N}_0$ such that $2M_L \le k \le 2M_{L+1}$. Thanks to (3.1) and Proposition 1.4(vi) we have

$$c \le \frac{M_{k+1}}{M_k} \le c', \quad k \in \mathbb{N}_0, \tag{3.36}$$

for some positive constants c, c' independent of k. Moreover, $\Psi(2^{-\nu}) \sim \Psi(2^{-(\nu+1)})$, $\nu \in \mathbb{N}_0$, and by Lemma 2.17(ii),

$$\Psi(2^{-\nu}) \sim \Psi((2M_\nu)^{-1}), \quad \nu \ge \nu_0. \tag{3.37}$$

Using the monotonicity of entropy numbers, Proposition 3.9 and (3.36), we have on the one hand

$$\begin{aligned} e_k(\mathrm{id}) &\le e_{2M_L}(\mathrm{id}) \le c_1\beta_L^{-1}M_L^{-(1/p_1-1/p_2)} \le c_2 M_L^{-(\delta/d+1/p_1-1/p_2)}\Psi(2^{-L})^{-b-\delta/d} \\ &\le c_3 M_{L+1}^{-(\delta/d+1/p_1-1/p_2)}\Psi(k^{-1})^{-b-\delta/d} \\ &\le c_4(k\Psi(k^{-1}))^{-(\delta/d+1/p_1-1/p_2)}\Psi(k^{-1})^{-b+1/p_1-1/p_2} \end{aligned}$$

and on the other hand

$$\begin{aligned} e_k(\mathrm{id}) &\ge e_{2M_{L+1}}(\mathrm{id}) \ge c_1'\beta_{L+1}^{-1}M_{L+1}^{-(1/p_1-1/p_2)} \ge c_2' M_{L+1}^{-(\delta/d+1/p_1-1/p_2)}\Psi(2^{-(L+1)})^{-b-\delta/d} \\ &\ge c_3' M_L^{-(\delta/d+1/p_1-1/p_2)}\Psi(k^{-1})^{-b-\delta/d} \\ &\ge c_4'(k\Psi(k^{-1}))^{-(\delta/d+1/p_1-1/p_2)}\Psi(k^{-1})^{-b+1/p_1-1/p_2}. \end{aligned}$$

We have proved (3.35) for all $k \in \mathbb{N}$ except finitely many, but the final statement for all $k \in \mathbb{N}$ comes easily, possibly with other positive constants c_4 and c_4'. ∎

Remark 3.11. For the embeddings in Proposition 3.9 and Corollary 3.10 we considered only weights on one of the spaces, but this is sufficient. In particular, we can replace in (3.14) the weight $2^{\mu m}$, $\mu > 0$, on the source space by the weights $2^{\mu_1 m}$ and $2^{\mu_2 m}$, with $\mu_1 > \mu_2$, on the source and on the target space, respectively. This can be easily seen from the proof, just following the role of μ.

3.3. Entropy numbers of embeddings between spaces on fractals. We are now prepared to the subject announced at the beginning of the section, that is, to generalise Theorem 2.24 of [ET99]. This provides a generalisation of Theorem 20.6 of [Tri97] from d-sets to (d,Ψ)-sets.

PROPOSITION 3.12. *Let Γ be a compact (d,Ψ)-set in $\mathbb{R}^n$ with $0<d\le n$. Let $\mathbb{B}^{(s,\Psi^a)}_{pq}(\Gamma)$ be the spaces introduced in Definition 2.16, notationally complemented by $\mathbb{B}^{(0,1)}_{pq}(\Gamma)=L_p(\Gamma)$ for any $0<p,q\le\infty$. Let $0<p_1,p_2,q_1,q_2\le\infty$, $a_1,a_2\in\mathbb{R}$ and $s_1,s_2\in\mathbb{R}^+_0$ be such that*

$$\delta_+ = s_1-s_2-d\left(\frac{1}{p_1}-\frac{1}{p_2}\right)_+ > 0.$$

Then the embedding

$$\mathrm{id}:\mathbb{B}^{(s_1,\Psi^{a_1})}_{p_1q_1}(\Gamma)\to\mathbb{B}^{(s_2,\Psi^{a_2})}_{p_2q_2}(\Gamma)$$

is compact and there exists a positive constant c such that the related entropy numbers satisfy

$$e_k[\mathrm{id}:\mathbb{B}^{(s_1,\Psi^{a_1})}_{p_1q_1}(\Gamma)\to\mathbb{B}^{(s_2,\Psi^{a_2})}_{p_2q_2}(\Gamma)]\le c(k\Psi(k^{-1}))^{-(s_1-s_2)/d}\Psi(k^{-1})^{a_2-a_1},\qquad k\in\mathbb{N}.$$

Proof. Step 1. Let $p_1\le p_2$. With

$$\sigma_1=s_1+\frac{n-d}{p_1},\qquad \sigma_2=s_2+\frac{n-d}{p_2},\qquad \delta=\delta_+, \tag{3.38}$$

we have

$$\sigma_1-\frac{n}{p_1}=s_1-\frac{d}{p_1}=\delta+s_2-\frac{d}{p_2}=\delta+\sigma_2-\frac{n}{p_2}. \tag{3.39}$$

Let $f\in\mathbb{B}^{(s_1,\Psi^a)}_{p_1q_1}(\Gamma)$. By Definition 2.16, in particular (2.30) and (2.31), there exists a (non-linear) bounded extension operator $\mathrm{ext}\, f=g$ such that

$$\mathrm{tr}_\Gamma\, g=f\quad\text{and}\quad \|g\,|\,B^{(\sigma_1,\Psi^{1/p_1+a_1})}_{p_1q_1}(\mathbb{R}^n)\|\le 2\|f\,|\,\mathbb{B}^{(s_1,\Psi^{a_1})}_{p_1q_1}(\Gamma)\|. \tag{3.40}$$

We expand g according to the subatomic representation theorem (Corollary 1.27 or Theorem 1.23) in terms of $(N_1,p_1,\Psi^{1/p_1+a_1})$-$\beta$-quarks and $(\sigma_1,p_1,\Psi^{1/p_1+a_1})_L$-$\beta$-quarks, with $N_1\in\mathbb{R}$ and $L+1\in\mathbb{N}_0$ fixed satisfying

$$N_1>\max\left(\sigma_{p_2}+\delta+n\left(\frac{1}{p_1}-\frac{1}{p_2}\right),\sigma_1\right),\qquad L\ge\max(-1,[\sigma_{p_1}-\sigma_1],[\sigma_{p_2}-\sigma_2]). \tag{3.41}$$

We have

$$\begin{aligned} g=\sum_{\beta\in\mathbb{N}_0^n}\sum_{\nu=0}^\infty\sum_{m\in\mathbb{Z}^n}&\lambda^\beta_{\nu m}2^{-\nu(N_1-n/p_1)}\Psi(2^{-\nu})^{-(1/p_1+a_1)}\Phi^\beta(2^\nu x-m)\\ &+\varrho^\beta_{\nu m}2^{-\nu(\sigma_1-n/p_1)}\Psi(2^{-\nu})^{-(1/p_1+a_1)}((-\Delta)^{(L+1)/2}\Phi^\beta)(2^\nu x-m)\end{aligned} \tag{3.42}$$

and

$$\sup_{\beta\in\mathbb{N}_0^n}2^{\mu_1|\beta|}(\|\lambda^\beta\,|\,b_{p_1q_1}\|+\|\varrho^\beta\,|\,b_{p_1q_1}\|)\le C\|g\,|\,B^{(\sigma_1,\Psi^{1/p_1+a_1})}_{p_1q_1}(\mathbb{R}^n)\| \tag{3.43}$$

for all $\mu_1>0$ large, $\lambda^\beta=\{\lambda^\beta_{\nu m}:\nu\in\mathbb{N}_0,\ m\in\mathbb{Z}^n\}$ and $\varrho^\beta=\{\varrho^\beta_{\nu m}:\nu\in\mathbb{N}_0,\ m\in\mathbb{Z}^n\}$.

Assume $c > 1$ is fixed and let

$$\begin{aligned} \lambda^{\beta,\Gamma} &= \{\lambda^{\beta}_{\nu m} : \nu \in \mathbb{N}_0,\ m \in \mathbb{Z}^n,\ cQ_{\nu m} \cap \Gamma \neq \emptyset\}, \\ \varrho^{\beta,\Gamma} &= \{\varrho^{\beta}_{\nu m} : \nu \in \mathbb{N}_0,\ m \in \mathbb{Z}^n, cQ_{\nu m} \cap \Gamma \neq \emptyset\}. \end{aligned} \tag{3.44}$$

For fixed $\nu \in \mathbb{N}_0$ let M_ν be the number of cubes $Q_{\nu m}$ such that $cQ_{\nu m} \cap \Gamma \neq \emptyset$. According to Lemma 2.17(i),

$$M_\nu \sim 2^{\nu d} \Psi(2^{-\nu})^{-1}, \qquad \nu \in \mathbb{N}_0. \tag{3.45}$$

We introduce the linear operator S,

$$S : B^{(\sigma_1, \Psi^{1/p_1+a_1})}_{p_1 q_1}(\mathbb{R}^n) \to \ell_\infty[2^{\mu_1|\beta|}\, \ell_{q_1}(2^{\nu\delta}\Psi(2^{-\nu})^b \ell^{M_\nu}_{p_1})^2] \tag{3.46}$$

defined by

$$Sg = (\eta, \tau), \qquad \eta = \{\eta^{\beta,\Gamma} : \beta \in \mathbb{N}^n_0\}, \qquad \tau = \{\tau^{\beta,\Gamma} : \beta \in \mathbb{N}^n_0\} \tag{3.47}$$

with

$$\begin{aligned} \eta^{\beta,\Gamma} &= \{2^{-\nu\delta}\Psi(2^{-\nu})^{-b}\lambda^{\beta}_{\nu m} : \nu \in \mathbb{N}_0,\ m \in \mathbb{Z}^n,\ cQ_{\nu m} \cap \Gamma \neq \emptyset\}, \\ \tau^{\beta,\Gamma} &= \{2^{-\nu\delta}\Psi(2^{-\nu})^{-b}\tau^{\beta}_{\nu m} : \nu \in \mathbb{N}_0,\ m \in \mathbb{Z}^n,\ cQ_{\nu m} \cap \Gamma \neq \emptyset\} \end{aligned} \tag{3.48}$$

and $b = a_1 + 1/p_1 - a_2 - 1/p_2$. Recall that the expansion (3.42) is not unique, but this does not matter. By (3.43) it follows that S is a bounded operator. Then we take the embedding

$$\mathrm{id} : \ell_\infty[2^{\mu_1|\beta|}\, \ell_{q_1}(2^{\nu\delta}\Psi(2^{-\nu})^b\, \ell^{M_\nu}_{p_1})^2] \to \ell_\infty[2^{\mu_2|\beta|}\, \ell_{q_2}(\ell^{M_\nu}_{p_2})^2] \tag{3.49}$$

with $\mu_1 > \mu_2$; and afterwards the linear operator

$$T : \ell_\infty[2^{\mu_2|\beta|}\ell_{q_2}(\ell^{M_\nu}_{p_2})^2] \to B^{(\sigma_2, \Psi^{1/p_2+a_2})}_{p_2 q_2}(\mathbb{R}^n) \tag{3.50}$$

defined by

$$\begin{aligned} T(\chi, \xi) = \sum_{\beta \in \mathbb{N}^n_0} \sum_{\nu=0}^{\infty} \sum_{m} & \chi^{\beta}_{\nu m} 2^{-\nu(N_2 - n/p_2)} \Psi(2^{-\nu})^{-(1/p_2+a_2)} \Phi^\beta(2^\nu x - m) \\ & + \xi^{\beta}_{\nu m} 2^{-\nu(\sigma_2 - n/p_2)} \Psi(2^{-\nu})^{-(1/p_2+a_2)} ((-\Delta)^{(L+1)/2}\Phi^\beta)(2^\nu x - m) \end{aligned} \tag{3.51}$$

where $N_2 = N_1 - \delta + n(1/p_2 - 1/p_1)$, $\chi = \{\chi^{\beta,\Gamma} : \beta \in \mathbb{N}^n_0\}$, $\xi = \{\xi^{\beta,\Gamma} : \beta \in \mathbb{N}^n_0\}$ and the sum over m in (3.51) is taken according to (3.44). Note that (3.41) implies

$$N_2 > \max(\sigma_{p_2}, \sigma_2) \quad \text{and} \quad L \geq \max(-1, [\sigma_{p_2} - \sigma_2]).$$

It follows from Corollary 1.27(ii) that T is a bounded linear map. Finally we consider the trace

$$\mathrm{tr}_\Gamma : B^{(\sigma_2, \Psi^{1/p_2+a_2})}_{p_2 q_2}(\mathbb{R}^n) \to \mathbb{B}^{(s_2, \Psi^{a_2})}_{p_2 q_2}(\Gamma), \tag{3.52}$$

which is also a continuous map. We claim

$$\mathrm{id}(\mathbb{B}^{(s_1, \Psi^{a_1})}_{p_1 q_1}(\Gamma) \to \mathbb{B}^{(s_2, \Psi^{a_2})}_{p_2 q_2}(\Gamma)) = \mathrm{tr}_\Gamma \circ T \circ \mathrm{id} \circ S \circ \mathrm{ext}\,. \tag{3.53}$$

We follow the constructions. Let $f \in \mathbb{B}^{(s_1, \Psi^{a_1})}_{p_1 q_1}(\Gamma)$. Then we have (3.40) and (3.42). Checking the coefficients of $\Phi^\beta(2^\nu x - m)$ and $((-\Delta)^{(L+1)/2}\Phi^\beta)(2^\nu x - m)$ in (3.51), we have

$$\chi^{\beta}_{\nu m} 2^{-\nu(N_2-n/p_2)}\Psi(2^{-\nu})^{-(1/p_2+a_2)} = \lambda^{\beta}_{\nu m} 2^{-\nu\delta}\Psi(2^{-\nu})^{-b} 2^{-\nu(N_2-n/p_2)}\Psi(2^{-\nu})^{-(1/p_2+a_2)}$$
$$= \lambda^{\beta}_{\nu m} 2^{-\nu(N_1-n/p_1)}\Psi(2^{-\nu})^{-(1/p_1+a_1)}$$

and similarly

$$\xi^{\beta}_{\nu m} 2^{-\nu(\sigma_2-n/p_2)}\Psi(2^{-\nu})^{-(1/p_2+a_2)} = \varrho^{\beta}_{\nu m} 2^{-\nu(\sigma_1-n/p_1)}\Psi(2^{-\nu})^{-(1/p_1+a_1)},$$

where we have used (3.39). Hence taking finally tr_Γ we obtain f by (3.40), where we started from. This proves (3.53). The unit ball in $\mathbb{B}^{(s_1,\Psi^{a_1})}_{p_1q_1}(\Gamma)$ is mapped by $S\circ \mathrm{ext}$ into a bounded set in

$$\ell_\infty[2^{\mu_1|\beta|}\ell_{q_1}(2^{\nu\delta}\Psi(2^{-\nu})^b \ell^{M_\nu}_{p_1})^2].$$

By (3.49) this set is mapped into a pre-compact set in $\ell_\infty[2^{\mu_2|\beta|}\ell_{q_2}(\ell^{M_\nu}_{p_2})^2]$ which can be covered by 2^{k-1} balls of radius $ce_k(\mathrm{id})$ with

$$e_k(\mathrm{id}) \le c(k\Psi(k^{-1}))^{-(\delta/d+1/p_1-1/p_2)}\Psi(k^{-1})^{-b+1/p_1-1/p_2}. \tag{3.54}$$

This follows from Corollary 3.10 and Remark 3.11 upon using $p_1 \le p_2$. The two bounded linear maps T and tr_Γ do not change this covering assertion, up to constants. Hence, we arrive at a covering of the unit ball in $\mathbb{B}^{(s_1,\Psi^{a_1})}_{p_1q_1}(\Gamma)$ by 2^{k-1} balls of radius $c\,e_k(\mathrm{id})$ in $\mathbb{B}^{(s_2,\Psi^{a_2})}_{p_2q_2}(\Gamma)$. We insert $\delta=\delta_+$ and b to obtain

$$e_k[\mathrm{id}:\mathbb{B}^{(s_1,\Psi^{a_1})}_{p_1q_1}(\Gamma)\to \mathbb{B}^{(s_2,\Psi^{a_2})}_{p_2q_2}(\Gamma)] \le c'(k\Psi(k^{-1}))^{-(s_1-s_2)/d}\Psi(k^{-1})^{a_2-a_1}, \quad k\in\mathbb{N}. \tag{3.55}$$

Step 2. Let $p_2 < p_1$. By Proposition 2.18 we have $\mathbb{B}^{(s_2,\Psi^{a_2})}_{p_1q_2}(\Gamma) \hookrightarrow \mathbb{B}^{(s_2,\Psi^{a_2})}_{p_2q_2}(\Gamma)$. The rest follows using the multiplication property of the entropy numbers and Step 1. In fact,

$$e_k[\mathrm{id}:\mathbb{B}^{(s_1,\Psi^{a_1})}_{p_1q_1}(\Gamma)\to \mathbb{B}^{(s_2,\Psi^{a_2})}_{p_2q_2}(\Gamma)] \le ce_k[\mathrm{id}:\mathbb{B}^{(s_1,\Psi^{a_1})}_{p_1q_1}(\Gamma)\to \mathbb{B}^{(s_2,\Psi^{a_2})}_{p_1q_2}(\Gamma)]$$
$$\le c'(k\Psi(k^{-1}))^{-(s_1-s_2)/d}\Psi(k^{-1})^{a_2-a_1}, \quad k\in\mathbb{N}. \blacksquare$$

THEOREM 3.13. *Let Γ be a compact (d,Ψ)-set in $\mathbb{R}^n$ with $0<d\le n$. Let $0<p_1,p_2,q_1,q_2 \le\infty$, $a_1,a_2\in\mathbb{R}$ and $s_1,s_2\in\mathbb{R}^+_0$ be such that*

$$\delta_+ = s_1-s_2-d\left(\frac{1}{p_1}-\frac{1}{p_2}\right)_+ > 0.$$

Then the embedding

$$\mathrm{id}:\mathbb{B}^{(s_1,\Psi^{a_1})}_{p_1q_1}(\Gamma)\to \mathbb{B}^{(s_2,\Psi^{a_2})}_{p_2q_2}(\Gamma)$$

is compact and the related entropy numbers satisfy

$$e_k[\mathrm{id}:\mathbb{B}^{(s_1,\Psi^{a_1})}_{p_1q_1}(\Gamma)\to \mathbb{B}^{(s_2,\Psi^{a_2})}_{p_2q_2}(\Gamma)] \sim (k\Psi(k^{-1}))^{-(s_1-s_2)/d}\Psi(k^{-1})^{a_2-a_1}, \quad k\in\mathbb{N}. \tag{3.56}$$

Proof. Step 1. By Proposition 3.12 it remains to prove that there exists a positive constant c such that for all $k\in\mathbb{N}$,

$$e_k[\mathrm{id}:\mathbb{B}^{(s_1,\Psi^{a_1})}_{p_1q_1}(\Gamma)\to \mathbb{B}^{(s_2,\Psi^{a_2})}_{p_2q_2}(\Gamma)] \ge c(k\Psi(k^{-1}))^{-(s_1-s_2)/d}\Psi(k^{-1})^{a_2-a_1}. \tag{3.57}$$

Assume that there is no such $c>0$. Then we find a sequence $(k_j)_{j\in\mathbb{N}}$ of natural numbers tending to infinity such that

$$e_{k_j}[\mathrm{id}:\mathbb{B}^{(s_1,\Psi^{a})}_{p_1q_1}(\Gamma)\to \mathbb{B}^{(s_2,\Psi^{b})}_{p_2q_2}(\Gamma)]k_j^{(s_1-s_2)/d}\Psi(k_j^{-1})^{(s_1-s_2)/d+a_1-a_2}\to 0 \quad \text{as } j\to\infty. \tag{3.58}$$

In this step we show that we may assume $s_2 = 0$, $a_2 = 0$ and $1 < p_1 \leq \infty$ in (3.58). If $s_2 > 0$, using the multiplication property of entropy numbers, described in Proposition 3.2, and by Proposition 3.12, we get

$$\begin{aligned} &e_{2k_j}[\mathrm{id} : \mathbb{B}^{(s_1,\Psi^a)}_{p_1q_1}(\Gamma) \to L_{p_2}(\Gamma)] \\ &\qquad \leq e_{k_j}[\mathrm{id} : \mathbb{B}^{(s_1,\Psi^{a_1})}_{p_1q_1}(\Gamma) \to \mathbb{B}^{(s_2,\Psi^{a_2})}_{p_2q_2}(\Gamma)] e_{k_j}[\mathrm{id} : \mathbb{B}^{(s_2,\Psi^{a_2})}_{p_2q_2}(\Gamma) \to L_{p_2}(\Gamma)] \\ &\qquad \leq c k_j^{-s_2/d} \Psi(k_j^{-1})^{-s_2/d-a_2} e_{k_j}[\mathrm{id} : \mathbb{B}^{(s_1,\Psi^{a_1})}_{p_1q_1}(\Gamma) \to \mathbb{B}^{(s_2,\Psi^{a_2})}_{p_2q_2}(\Gamma)] \end{aligned}$$

and so

$$\begin{aligned} (3.59)\quad & k_j^{s_1/d}\Psi(k_j^{-1})^{s_1/d+a_1} e_{2k_j}[\mathrm{id} : \mathbb{B}^{(s_1,\Psi^{a_1})}_{p_1q_1}(\Gamma) \to L_{p_2}(\Gamma)] \\ &\leq c k_j^{(s_1-s_2)/d} \Psi(k_j^{-1})^{(s_1-s_2)/d+a_1-a_2} e_{k_j}[\mathrm{id} : \mathbb{B}^{(s_1,\Psi^{a_1})}_{p_1q_1}(\Gamma) \to \mathbb{B}^{(s_2,\Psi^{a_2})}_{p_2q_2}(\Gamma)]. \end{aligned}$$

This justifies that we may assume in (3.58) that $\mathbb{B}^{(s_2,\Psi^{a_2})}_{p_2q_2}(\Gamma) = L_{p_2}(\Gamma)$, which corresponds to $s_2 = 0$ and $a_2 = 0$.

If $0 < p_1 \leq 1$, let p_3 be such that $1 < p_3 \leq \infty$. Since then $p_1 < p_3$, by Proposition 2.18, we have the embedding

$$\mathbb{B}^{(s_1,\Psi^{a_1})}_{p_3q_1}(\Gamma) \hookrightarrow \mathbb{B}^{(s_1,\Psi^{a_1})}_{p_1q_1}(\Gamma).$$

By the multiplication property of entropy numbers, we have

$$e_{k_j}[\mathrm{id} : \mathbb{B}^{(s_1,\Psi^{a_1})}_{p_3q_1}(\Gamma) \to L_{p_2}(\Gamma)] \leq c' e_{k_j}[\mathrm{id} : \mathbb{B}^{(s_1,\Psi^{a_1})}_{p_1q_1}(\Gamma) \to L_{p_2}(\Gamma)].$$

Hence, (3.58), already with $s_2 = a_2 = 0$, would imply

$$k_j^{s_1/d}\Psi(k_j^{-1})^{s_1/d+a_1} e_{k_j}[\mathrm{id} : \mathbb{B}^{(s_1,\Psi^{a_1})}_{p_3q_1}(\Gamma) \to L_{p_2}(\Gamma)] \to 0 \quad \text{as } j \to \infty.$$

This shows that we may also assume that $1 < p_1 \leq \infty$ in (3.58).

Step 2. In this step we prove that there exists a constant $c > 0$ such that

$$(3.60)\qquad e_k[\mathrm{id} : \mathbb{B}^{(s,\Psi^a)}_{p_1q}(\Gamma) \to L_{p_2}(\Gamma)] \geq c k^{-s/d}\Psi(k^{-1})^{-s/d-a}, \qquad k \in \mathbb{N},$$

for

$$0 < q \leq \infty, \quad 0 < p_1 \leq \infty, \quad 1 \leq p_2 \leq \infty, \quad a \in \mathbb{R}, \quad s > d\left(\frac{1}{p_1} - \frac{1}{p_2}\right)_+.$$

Since Γ is a compact (d,Ψ)-set, for fixed $j \in \mathbb{N}$ we find $M_j \sim 2^{jd}\Psi(2^{-j})^{-1}$ disjoint balls $B_{j,r}$ of radius 2^{-j}, centred at $x^{j,r} \in \Gamma$, $r = 1, \ldots, M_j$. Let φ and $\widetilde{\varphi}$ be two non-negative C^∞ functions in $\mathbb{R}^n$, non-vanishing at the origin with supports in the unit ball. Note that

$$\begin{aligned} (3.61)\quad & \int_\Gamma \varphi(2^j(\gamma - x^{j,r}))\widetilde{\varphi}(2^j(\gamma - x^{j,r}))\,\mu(d\gamma) \\ &\qquad \leq (\max_{|y|\leq 1} \varphi(y)\widetilde{\varphi}(y))\mu(\Gamma \cap B_{j,r}) \leq c 2^{-jd}\Psi(2^{-j}), \qquad j \in \mathbb{N}. \end{aligned}$$

On the other hand, there exists a neighbourhood of the origin where $\varphi\widetilde{\varphi}$ is positive, say

$$\varphi(x)\widetilde{\varphi}(x) \geq L > 0 \quad \text{if } |x| \leq \delta,$$

for some $0 < \delta < 1$. Then we have

$$\int_{\Gamma} \varphi(2^j(\gamma - x^{j,r}))\widetilde{\varphi}(2^j(\gamma - x^{j,r}))\,\mu(d\gamma) \tag{3.62}$$
$$\geq \int_{\Gamma\cap\delta B_{j,r}} \varphi(2^j(\gamma - x^{j,r}))\widetilde{\varphi}(2^j(\gamma - x^{j,r}))\,\mu(d\gamma)$$
$$\geq L\mu(\Gamma\cap\delta B_{j,r}) \geq c2^{-jd}\Psi(2^{-j}), \quad j\in\mathbb{N}.$$

Let $c_{j,r}$, $j\in\mathbb{N}$, $r = 1,\dots,M_j$, be such that

$$c_{j,r}2^{jd}\Psi(2^{-j})^{-1}\int_{\Gamma}\varphi(2^j(\gamma - x^{j,r}))\widetilde{\varphi}(2^j(\gamma - x^{j,r}))\,\mu(d\gamma) = 1. \tag{3.63}$$

From the observations (3.61) and (3.62) above, there are two positive constants $0 < c_1 \leq c_2 < \infty$ such that

$$c_1 \leq c_{j,r} \leq c_2 \quad \text{for all } j\in\mathbb{N},\ r = 1,\dots,M_j. \tag{3.64}$$

In the commutative diagram

$$\begin{array}{ccc} \ell_{p_1}^{M_j} & \xrightarrow{A} & \mathbb{B}_{p_1q}^{(s,\Psi^a)}(\Gamma) \\ {\scriptstyle 2^{-j(d/p_2+s-d/p_1)}\Psi(2^{-j})^{1/p_2-1/p_1-a}\,\mathrm{id}^j}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_\Gamma} \\ \ell_{p_2}^{M_j} & \xleftarrow[B]{} & L_{p_2}(\Gamma) \end{array}$$

let the operators A and B be given by

$$A(a_r : r = 1,\dots,M_j) = \sum_{r=1}^{M_j} a_r 2^{-(s-d/p_1)j}\Psi(2^{-j})^{-(1/p_1+a)}\varphi(2^j(x - x^{j,r}))|\Gamma \tag{3.65}$$

and

$$Bf = \Big(2^{-jd(1/p_2-1)}\Psi(2^{-j})^{(1/p_2-1)}c_{j,r}\int_{\Gamma} f(\gamma)\widetilde{\varphi}(2^j(\gamma - x^{j,r}))\,\mu(d\gamma) : r = 1,\dots,M_j\Big). \tag{3.66}$$

Furthermore, id_Γ is the embedding indicated and $\mathrm{id}^j : \ell_{p_1}^{M_j} \to \ell_{p_2}^{M_j}$ the identity operator. We may interpret (3.65) as an atomic decomposition in $B_{p_1,q}^{(s+(n-d)/p_1,\Psi^{a+1/p_1})}(\mathbb{R}^n)$. Notice that there are no moment conditions required for the atoms, because

$$s + \frac{n-d}{p_1} > \sigma_{p_1} = \left(\frac{1}{p_1} - 1\right)_+$$

as $s > d(1/p_1 - 1/p_2)_+$ and $1 < p_2 \leq \infty$. Hence we obtain

$$\|A(a_r)_{r=1}^{M_j} \,|\, \mathbb{B}_{p_1q}^{(s,\Psi^a)}(\Gamma)\| \tag{3.67}$$
$$\leq \Big\| \sum_{r=1}^{M_j} a_r 2^{-(s-d/p_1)j}\Psi(2^{-j})^{-(1/p_1+a)}\varphi(2^j(x - x^{j,r})) \,\Big|\, B_{p_1,q}^{(s+(n-d)/p_1,\Psi^{a+1/p_1})}(\mathbb{R}^n)\Big\|$$
$$\leq c\|(a_r)_{r=1}^{M_j} \,|\, \ell_{p_1}^{M_j}\|,$$

where c is a positive constant independent of j. Denote by $b_{j,r}$ the coefficients in brackets in (3.66) and let p_2' be the conjugate exponent of p_2, i.e. $1/p_2 + 1/p_2' = 1$. Applying Hölder's inequality, using the fact that for fixed j the balls $B_{j,r}$ are disjoint and (3.64), we get

$$|b_{j,r}|^{p_2} \le c_{j,r}^{p_2} 2^{jd(p_2-1)} \Psi(2^{-j})^{p_2-1} \Big(\int_{\Gamma\cap B_{j,r}} |f(\gamma)| \widetilde{\varphi}(2^j(\gamma - x^{j,r}))\, \mu(d\gamma) \Big)^{p_2}$$
$$\le c 2^{jd(p_2-1)} \Psi(2^{-j})^{p_2-1} \mu(\Gamma\cap B_{j,r})^{p_2/p_2'} \int_{\Gamma\cap B_{j,r}} |f(\gamma)|^{p_2}\, \mu(d\gamma)$$
$$\le c' \int_{\Gamma\cap B_{j,r}} |f(\gamma)|^{p_2}\, \mu(d\gamma),$$

and then

$$\|Bf \,|\, \ell_{p_2}^{M_j}\| = \Big(\sum_{r=1}^{M_j} |b_{j,r}|^{p_2} \Big)^{1/p_2} \le c \Big(\sum_{r=1}^{M_j} \int_{\Gamma\cap B_{j,r}} |f(\gamma)|^{p_2}\, \mu(d\gamma) \Big)^{1/p_2} \le c' \|f \,|\, L_{p_2}(\Gamma)\|, \tag{3.68}$$

where again the constant in (3.68) is independent of j. In other words, both A and B are bounded linear operators whose norms can be estimated independently of j. By (3.63) we have

$$B \circ \mathrm{id}_\Gamma \circ A = 2^{-j(d/p_2+s-d/p_1)} \Psi(2^{-j})^{1/p_2-1/p_1-a}\, \mathrm{id}^j\,. \tag{3.69}$$

By (3.69) and the remark on the norms of A and B we get

$$2^{-j(d/p_2+s-d/p_1)} \Psi(2^{-j})^{1/p_2-1/p_1-a} e_k(\mathrm{id}^j) \le c e_k(\mathrm{id}_\Gamma), \qquad k \in \mathbb{N}, \tag{3.70}$$

where the constant c is independent of j. By Proposition 7.2 of [Tri97, p. 36] with $k = 2M_j$, and using the fact that $M_j \sim 2^{jd}\Psi(2^{-j})^{-1}$, we deduce from (3.70) that

$$e_{2M_j}(\mathrm{id}_\Gamma) \ge c_1 2^{-jd(1/p_2+s/d-1/p_1)} \Psi(2^{-j})^{1/p_2-1/p_1-a} (2M_j)^{1/p_2-1/p_1}$$
$$\ge c_2 (2M_j)^{-s/d} \Psi(2^{-j})^{-s/d-a} \ge c_3 (2M_j)^{-s/d} \Psi((2M_j)^{-1})^{-s/d-a}.$$

We have proved (3.60) for $k = 2M_j$. Reasoning as in the proof of Corollary 3.10 it turns out that (3.60) holds for any $k \in \mathbb{N}$.

Step 3. It remains to prove (3.60) for $0 < p_2 < 1$. Let $0 < p_2 < 1$ and $1 \le p_1 \le \infty$. Suppose that (3.60) does not hold. Then as in Step 1, we find a sequence $k_j \to \infty$ such that

$$e_{k_j}[\mathrm{id} : \mathbb{B}_{p_1 q}^{(s,\Psi^a)}(\Gamma) \to L_{p_2}(\Gamma)] k_j^{s/d} \Psi(k_j^{-1})^{s/d+a} \to 0 \quad \text{as } j \to \infty. \tag{3.71}$$

For all $f \in L_{p_1}(\Gamma)$,

$$\|f \,|\, L_p(\Gamma)\| \le \|f \,|\, L_{p_1}(\Gamma)\|^{1-\theta} \|f \,|\, L_{p_2}(\Gamma)\|^{\theta}, \tag{3.72}$$

where

$$0 < \theta < 1 \quad \text{and} \quad \frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}. \tag{3.73}$$

Then, by the interpolation property of entropy numbers (see e.g. [ET96, 1.3.2]) and

Proposition 3.12, we have

$$(3.74)\quad \begin{aligned} &e_{2k_j}[\mathrm{id} : \mathbb{B}^{(s,\Psi^a)}_{p_1q}(\Gamma) \to L_p(\Gamma)] \\ &\qquad \le c e_{k_j}[\mathrm{id} : \mathbb{B}^{(s,\Psi^a)}_{p_1q}(\Gamma) \to L_{p_1}(\Gamma)]^{1-\theta} e_{k_j}[\mathrm{id} : \mathbb{B}^{(s,\Psi^a)}_{p_1q}(\Gamma) \to L_{p_2}(\Gamma)]^{\theta} \\ &\qquad \le c'(k_j^{-s/d}\Psi(k_j^{-1})^{-s/d-a})^{1-\theta} e_{k_j}[\mathrm{id} : \mathbb{B}^{(s,\Psi^a)}_{p_1q}(\Gamma) \to L_{p_2}(\Gamma)]^{\theta}. \end{aligned}$$

Obviously we can rewrite (3.74) as

$$\begin{aligned} &k_j^{s/d}\Psi(k_j^{-1})^{s/d+a} e_{2k_j}[\mathrm{id} : \mathbb{B}^{(s,\Psi^a)}_{p_1q}(\Gamma) \to L_p(\Gamma)] \\ &\qquad \le c'(k_j^{s/d}\Psi(k_j^{-1})^{s/d+a} e_{k_j}[\mathrm{id} : \mathbb{B}^{(s,\Psi^a)}_{p_1q}(\Gamma) \to L_{p_2}(\Gamma)])^{\theta}. \end{aligned}$$

Then by (3.71) we would get

$$e_{2k_j}[\mathrm{id} : \mathbb{B}^{(s,\Psi^a)}_{p_1q}(\Gamma) \to L_p(\Gamma)] k_j^{s/d}\Psi(k_j^{-1})^{s/d+a} \to 0 \quad \text{as } j \to \infty.$$

But (3.73) enables us to choose $p > 1$ (take $0 < \theta < (1 - 1/p_1)/(1/p_2 - 1/p_1)$ which is less than one), and this contradicts what was proved in Step 2. ■

4. Applications

4.1. Fractal drums. Our aim is this section is to show an application of the assertions in the previous sections to the fractal drum problem. We follow [ET99].

Throughout this section, Ω denotes a bounded C^∞ domain in $\mathbb{R}^n$. As usual, $\mathcal{D}(\Omega)$ is the collection of all compactly supported complex-valued C^∞ functions in Ω. By $\mathcal{D}'(\Omega)$ we denote the dual space of all distributions on Ω. We assume that Γ is a compact (d,Ψ)-set in $\mathbb{R}^n$, according to Definition 1.1, with $\Gamma \subset \Omega$, and μ the related Radon measure.

Definition 4.1. Let Ω be a bounded C^∞ domain in $\mathbb{R}^n$. Let $0 < p, q \le \infty$, $s \in \mathbb{R}$ and Ψ an admissible function according to Definition 1.1. Then $B^{(s,\Psi)}_{pq}(\Omega)$ is the restriction of $B^{(s,\Psi)}_{pq}(\mathbb{R}^n)$ to Ω, which means

$$B^{(s,\Psi)}_{pq}(\Omega) = \{f \in \mathcal{D}'(\Omega) : \text{there exists a } g \in B^{(s,\Psi)}_{pq}(\mathbb{R}^n) \text{ with } g|\Omega = f\},$$
$$\|f \mid B^{(s,\Psi)}_{pq}(\Omega)\| = \inf \|g \mid B^{(s,\Psi)}_{pq}(\mathbb{R}^n)\|,$$

where the infimum is taken over all $g \in B^{(s,\Psi)}_{pq}(\mathbb{R}^n)$ whose restriction to Ω, denoted by $g|\Omega$, coincides in $\mathcal{D}'(\Omega)$ with f.

By Definition 4.1 the embedding assertions for $B^{(s,\Psi)}_{pq}$-spaces on $\mathbb{R}^n$ summarised in Proposition 1.9 can be carried over to the spaces $B^{(s,\Psi)}_{pq}(\Omega)$. By the boundedness of Ω, using the monotonicity of the L_p-spaces on bounded domains and the characterisation by local means presented in the first section we even have

$$B^{(s,\Psi)}_{p_1q}(\Omega) \hookrightarrow B^{(s,\Psi)}_{p_0q}(\Omega) \quad \text{if } 0 < p_0 \le p_1 \le \infty.$$

Let

$$(4.1)\qquad (\mathrm{tr}^\Gamma \varphi)(\psi) = \int_\Gamma \varphi(\gamma)\psi(\gamma)\,\mu(d\gamma), \qquad \varphi, \psi \in \mathcal{D}(\Omega).$$

This defines a mapping from $\mathcal{D}(\Omega)$ into $\mathcal{D}'(\Omega)$. Formalising the interpretation (2.16) as

$$\mathrm{id}_\Gamma : f^\Gamma \mapsto f$$

we have

$$\mathrm{tr}^\Gamma = \mathrm{id}_\Gamma \circ \mathrm{tr}_\Gamma\,. \tag{4.2}$$

Combining Proposition 2.14 and (2.19) we can extend tr^Γ to

$$\mathrm{tr}^\Gamma : B^{((n-d)/p,\Psi^{1/p})}_{p,q}(\Omega) \to B^{(-(n-d)/p',\Psi^{-1/p'})}_{p,\infty}(\Omega), \tag{4.3}$$

with $1 \le p \le \infty$ and $0 < q \le 1$. Independently of p, the loss of smoothness is always $(n-d,\Psi^{-1})$. The operator tr^Γ can be generalised to

$$\mathrm{tr}^\Gamma_b = \mathrm{id}_\Gamma \circ b \circ \mathrm{tr}_\Gamma \qquad \text{where } b \in L_r(\Gamma) \tag{4.4}$$

with

$$1 \le p, r \le \infty, \qquad 0 < q \le 1, \qquad \frac{1}{t} = \frac{1}{p} + \frac{1}{r} \le 1.$$

By Proposition 2.14, (2.19) and Hölder's inequality we have

$$\mathrm{tr}^\Gamma_b : B^{((n-d)/p,\Psi^{1/p})}_{p,q}(\Omega) \to B^{(-(n-d)/t',\Psi^{-1/t'})}_{t,\infty}(\Omega). \tag{4.5}$$

Obviously, $-\Delta = -\sum_{j=1}^n \partial^2/\partial x_j^2$ stands for the Laplacian. If

$$1 \le p, q \le \infty, \qquad s > 1/p, \tag{4.6}$$

then the Dirichlet Laplacian $-\Delta$ generates an isomorphic map

$$-\Delta : B^{(s,\Psi)}_{pq,0}(\Omega) \to B^{(s-2,\Psi)}_{pq}(\Omega), \tag{4.7}$$

where $B^{(s,\Psi)}_{pq,0}(\Omega) = \{g \in B^{(s,\Psi)}_{pq}(\Omega) : \mathrm{tr}_{\partial\Omega}\, g = 0\}$. Let $(-\Delta)^{-1}$ be the inverse of the Dirichlet Laplacian $-\Delta$; it will be clear from the context between which spaces $(-\Delta)^{-1}$ acts. Let

$$B = (-\Delta)^{-1} \circ \mathrm{tr}^\Gamma, \tag{4.8}$$

where any space continuously embedded in the source space in (4.3) can be admitted and where we assume that $(-\Delta)^{-1}$ can be applied to the target space in (4.3). In addition after application of tr^Γ and $(-\Delta)^{-1}$ we wish to return to the space we started from. This is ensured if $d > n-2$, because then

$$2 - \frac{n-d}{p'} > \frac{n-d}{p} \quad \text{and} \quad 2 - \frac{n-d}{p'} > \frac{1}{p}.$$

In particular, if $d > n-2$, then B is a continuous operator in $B^{(2-(n-d)/p',\Psi^{-1/p'})}_{p,\infty}(\Omega)$ for $1 \le p \le \infty$.

It can be easily proved that the operator B in (4.8) is compact in $B^{(s,\widetilde{\Psi})}_{pq}(\Omega)$ for $0 < q \le \infty$, $\widetilde{\Psi}$ an admissible function, $1 \le p \le \infty$ and $(n-d)/p < s < 2-(n-d)/p'$, with p' the conjugate exponent of p. Moreover, B is a spectral invariant, i.e. its eigenvalues and root spaces do not depend on the underlying space in which B is considered.

THEOREM 4.2 [ET99, Theorem 2.28 & Corollary 2.30]. *Let Ω be a bounded C^∞ domain in $\mathbb{R}^n$ and Γ a compact (d,Ψ)-set such that $\Gamma \subset \Omega$ and $n-2 < d \le n$ (with $0 < d \le 1$*

when $n = 1$). *Then* $B = (-\Delta)^{-1} \circ \mathrm{tr}^\Gamma$ *is a non-negative, compact, self-adjoint operator in* $\mathring{W}_2^1(\Omega)$ *with null-space*

$$N(B) = \{f \in \mathring{W}_2^1(\Omega) : \mathrm{tr}_\Gamma f = 0\}.$$

The positive eigenvalues μ_k *of* B, *repeated according to multiplicity and ordered by magnitude, satisfy*

$$\mu_k \sim k^{-1}(k\Psi(k^{-1}))^{(n-2)/d}, \qquad k \in \mathbb{N}.$$

Furthermore, B *is generated by the quadratic form*

$$\int_\Gamma (\mathrm{tr}_\Gamma)(\gamma)\,\overline{(\mathrm{tr}_\Gamma\, g)(\gamma)}\;\mu(d\gamma) = (Bf, g)_{\mathring{W}_2^1(\Omega)} \qquad \textit{where } f, g \in \mathring{W}_2^1(\Omega).$$

Proof. Step 1. By [Tri97, 27.11] and the references given there we know that

$$(-\Delta)^{1/2} : \mathring{W}_2^1(\Omega) \to L_2(\Omega)$$

is an isomorphic map. We then consider in $\mathring{W}_2^1(\Omega)$ the norm

$$\|f \,|\, \mathring{W}_2^1(\Omega)\| := \|(-\Delta)^{1/2} f \,|\, L_2(\Omega)\| \sim \|f \,|\, W_2^1(\Omega)\|. \tag{4.9}$$

$\mathring{W}_2^1(\Omega)$ turns out to be a Hilbert space with respect to the corresponding scalar product. As $d > n-2$, we have

$$W_2^1(\Omega) = B^1_{2,2}(\Omega) \hookrightarrow B^{((n-d)/2, \Psi^{1/2})}_{2,1}(\Omega).$$

Then by (2.19), we get

$$\|\mathrm{tr}_\Gamma f \,|\, L_2(\Gamma)\| \le c\|f \,|\, B^{((n-d)/2,\Psi^{1/2})}_{2,1}(\Omega)\| \le c'\|f \,|\, W_2^1(\Omega)\| \le c''\|f \,|\, \mathring{W}_2^1(\Omega)\| \tag{4.10}$$

for any $f \in \mathring{W}_2^1(\Omega)$. Let

$$a(f,g) = \int_\Gamma (\mathrm{tr}_\Gamma f)(\gamma)\,\overline{(\mathrm{tr}_\Gamma\, g)(\gamma)}\,\mu(d\gamma), \qquad f, g \in \mathring{W}_2^1(\Omega). \tag{4.11}$$

This defines a non-negative bounded quadratic form in $\mathring{W}_2^1(\Omega)$. Hence, there exists a uniquely determined non-negative self-adjoint bounded operator B in $\mathring{W}_2^1(\Omega)$ such that

$$a(f,g) = (Bf, g)_{\mathring{W}_2^1(\Omega)}, \qquad f, g \in \mathring{W}_2^1(\Omega).$$

Furthermore,

$$\|\sqrt{B} f \,|\, \mathring{W}_2^1(\Omega)\|^2 = (Bf, f)_{\mathring{W}_2^1(\Omega)} = a(f,f) = \|\mathrm{tr}_\Gamma f \,|\, L_2(\Gamma)\|^2, \qquad f \in \mathring{W}_2^1(\Omega). \tag{4.12}$$

This shows that

$$N(B) = \{f \in \mathring{W}_2^1(\Omega) : Bf = 0\} = \{f \in \mathring{W}_2^1(\Omega) : \mathrm{tr}_\Gamma f = 0\}.$$

Step 2. We prove that B is the operator $(-\Delta)^{-1} \circ \mathrm{tr}^\Gamma$. Let $g \in \mathcal{D}(\Omega)$ and $f \in \mathring{W}_2^1(\Omega)$. We have

$$\begin{aligned}\langle \mathrm{tr}^\Gamma f, g\rangle &= \int_\Gamma (\mathrm{tr}_\Gamma f)(\gamma) g(\gamma)\,\mu(d\gamma) = a(f, \overline{g}) = (Bf, \overline{g})_{\mathring{W}_2^1(\Omega)} \\ &= ((-\Delta)^{1/2} Bf, (-\Delta)^{1/2}\overline{g})_{L_2(\Omega)} = ((-\Delta)Bf, \overline{g})_{L_2(\Omega)} = \langle (-\Delta)Bf, g\rangle,\end{aligned}$$

where we denote by $\langle \cdot, \cdot \rangle$ the dual pairing $\mathcal{D}'(\Omega) \leftrightarrow \mathcal{D}(\Omega)$. Hence, $(-\Delta)Bf = \mathrm{tr}^\Gamma f$, $f \in \mathring{W}_2^1(\Omega)$.

Step 3. We estimate from above the eigenvalues μ_k of B. As mentioned before, the eigenvalues μ_k including their algebraic multiplicity are independent of the admissible space in which B can be considered. We choose $B_{2,\infty}^{(2-(n-d)/2,\Psi^{-1/2})}(\Omega)$ as basic space and decompose B as

$$B = (-\Delta)^{-1} \circ \mathrm{id}_\Gamma \circ \mathrm{id} \circ \mathrm{tr}_\Gamma$$

with

$$\begin{aligned} \mathrm{tr}_\Gamma &: B_{2,\infty}^{(2-(n-d)/2,\Psi^{-1/2})}(\Omega) \to \mathbb{B}_{2,\infty}^{(2-n+d,\Psi^{-1})}(\Gamma), \\ \mathrm{id} &: \mathbb{B}_{2,\infty}^{(2-n+d,\Psi^{-1})}(\Gamma) \to L_2(\Gamma), \\ \mathrm{id}_\Gamma &: L_2(\Gamma) \to B_{2,\infty}^{(-(n-d)/2,\Psi^{-1/2})}(\Omega), \\ (-\Delta)^{-1} &: B_{2,\infty}^{(-(n-d)/2,\Psi^{-1/2})}(\Omega) \to B_{2,\infty}^{(2-(n-d)/2,\Psi^{-1/2})}(\Omega). \end{aligned}$$

By Definition 2.16, tr_Γ is a bounded operator. By Theorem 3.13 the embedding id is compact and its entropy numbers satisfy

$$e_k(\mathrm{id}) \sim (k\Psi(k^{-1}))^{-(2-n+d)/d}\Psi(k^{-1}), \qquad k \in \mathbb{N}.$$

Moreover, both id_Γ and $(-\Delta)^{-1}$ are also bounded operators, by (2.19) and (4.7), respectively. Therefore, using the properties of entropy numbers and Carl's inequality (cf. Proposition 3.2 and Corollary 3.5), we get

$$\mu_k = |\mu_k| \le \sqrt{2}\, e_k(B) \le c e_k(\mathrm{id}) \le c'(k\Psi(k^{-1}))^{-(2-n+d)/d}\Psi(k^{-1}), \qquad k \in \mathbb{N}.$$

Step 4. It remains to prove the estimate from below for the eigenvalues μ_k of B. For this, the Hilbert space setting and the use of approximation numbers are essential. Since Γ is a (d,Ψ)-set, for each $j \in \mathbb{N}_0$ there exist at most N_j disjoint balls $B_{j,l}$, $l = 1,\dots,N_j$, centred at $x_{j,l} \in \Gamma$ and of radius 2^{-j} with $N_j \sim 2^{jd}\Psi(2^{-j})^{-1}$. We may assume that these balls are subsets of Ω (possibly upon replacing $j \in \mathbb{N}_0$ by $j \ge j_0$ for some $j_0 \in \mathbb{N}$). Let φ be a non-negative C^∞ function with

$$\operatorname{supp}\varphi \subset \{\xi \in \mathbb{R}^n : |\xi| \le 1/4\} \quad \text{and} \quad \varphi(x) > 0 \quad \text{for } |x| \le \delta$$

for some $0 < \delta < 1/4$. Let

$$\varphi_{j,l}(x) = \varphi(2^j(x - x_{j,l})), \qquad j \in \mathbb{N}_0,\ l = 1,\dots,N_j.$$

Then $\operatorname{supp}\varphi_{j,l} \subset B_{j,l}$. Hence, for fixed $j \in \mathbb{N}_0$, the functions $\varphi_{j,l}$, $l = 1,\dots,N_j$, have disjoint supports. By the localisation property in [ET96, 2.3.2, pp. 35–36], we have

$$\Big\| \sum_{l=1}^{N_j} c_{j,l}\varphi_{j,l} \,\Big|\, W_2^1(\Omega)\Big\| \sim \Big(\sum_{l=1}^{N_j} |c_{j,l}|^2 \Big)^{1/2} 2^{j(1-n/2)}, \qquad j \in \mathbb{N}. \tag{4.13}$$

Due to (4.10) we get

$$\Big\|\sqrt{B}\Big(\sum_{l=1}^{N_j} c_{j,l}\varphi_{j,l}\Big)\,\Big|\,W_2^1(\Omega)\Big\| = \Big\|\sum_{l=1}^{N_j} c_{j,l}\varphi_{j,l}\,\Big|\,L_2(\Gamma)\Big\|$$
$$= \Big(\sum_{l=1}^{N_j} |c_{j,l}|^2 \varphi(2^j(\gamma - x_{j,l}))^2\,\mu(d\gamma)\Big)^{1/2} \geq \inf_{|x|\geq\delta}\varphi(x)\Big(\sum_{l=1}^{N_j}|c_{j,l}|^2\mu(\Gamma\cap\delta B_{j,l})\Big)^{1/2}$$
$$\geq c2^{-jd/2}\Psi(2^{-j})^{1/2}\Big(\sum_{l=1}^{N_j}|c_{j,l}|^2\Big)^{1/2} \geq c' 2^{-j(2-n+d)/2}\Psi(2^{-j})^{1/2}\Big\|\sum_{l=1}^{N_j} c_{j,l}\varphi_{j,l}\,\Big|\,W_2^1(\Omega)\Big\|.$$

We assume that the dimension of the span of the functions

$$g_j = \sum_{l=1}^{N_j} c_{j,l}\varphi_{j,l}$$

is N_j. If $T \in L(\mathring{W}_2^1(\Omega))$ has rank less than N_j, there exists g_j such that $\|g_j \,|\, \mathring{W}_2^1(\Omega)\| = 1$ and $T(g_j) = 0$. Then

$$\|\sqrt{B} - T\| \geq \|(\sqrt{B} - T)g_j \,|\, \mathring{W}_2^1(\Omega)\| \geq c' 2^{-j(2-n+d)/2}\Psi(2^{-j})^{1/2}.$$

Hence, for the approximation numbers of $\sqrt{B}$ we get

$$a_{N_j}(\sqrt{B}) = \inf\{\|\sqrt{B} - P\| : P \in L(\mathring{W}_2^1(\Omega)), \operatorname{rank} P < N_j\}$$
$$\geq c2^{-j(2-n+d)/2}\Psi(2^{-j})^{1/2} \geq c' N_j^{-(2-n+d)/(2d)}\Psi(2^{-j})^{(n-2)/(2d)}.$$

Let $k \in \mathbb{N}$ with $k \geq N_0$. There exists $L \in \mathbb{N}_0$ such that $N_L \leq k \leq N_{L+1}$. Then since $N_j \sim 2^{jd}\Psi(2^{-j})^{-1}$ we obtain

$$a_k(\sqrt{B}) \geq ck^{-(2-n+d)/(2d)}\Psi(k^{-1})^{(n-2)/(2d)}.$$

Because $\sqrt{B}$ is a compact self-adjoint non-negative operator in the Hilbert space $\mathring{W}_2^1(\Omega)$, its eigenvalues coincide with its approximation numbers (cf. [Tri97, 24.5, p. 192]). Maybe with another positive constant c we arrive at

$$\mu_k \geq ck^{-(2-n+d)/d}\Psi(k^{-1})^{(n-2)/d}, \qquad k \in \mathbb{N}. \ \blacksquare$$

Using the same kind of arguments as in the proof of Theorem 4.2 and replacing (4.3) by (4.5) one can show in a similar way the following theorem.

THEOREM 4.3 [[ET99, Theorem 2.33] (Sintered drum)]. *Let Ω be a bounded C^∞ domain in $\mathbb{R}^n$ and Γ a compact (d,Ψ)-set such that $\Gamma \subset \Omega$ and $n-2 < d \leq n$ (with $0 < d \leq 1$ when $n = 1$). Let $b(\gamma)$ be a non-negative function on Γ such that*

$$b \in L_r(\Gamma) \quad \textit{for some } r > 1 \textit{ with } 0 \leq \frac{1}{r} < 1 - \frac{n-2}{d},$$

and for some $c > 0$,

$$b(\gamma) \geq c \qquad \textit{if } \gamma \in \Gamma_0$$

where Γ_0 is a (d,Ψ)-set with $\Gamma_0 \subset \Gamma$. Then $B = (-\Delta)^{-1}\circ \mathrm{tr}_b^\Gamma$ is a non-negative, compact, self-adjoint operator in $\mathring{W}_2^1(\Omega)$ with eigenvalues μ_k satisfying

$$\mu_k \sim k^{-1}(k\Psi(k^{-1}))^{(n-2)/d}, \qquad k \in \mathbb{N}.$$

Furthermore, B is generated by the quadratic form

$$\int_\Gamma b(\gamma)(\mathrm{tr}_\Gamma f)(\gamma)\,\overline{\mathrm{tr}_\Gamma g(\gamma)}\,\mu(d\gamma) = (Bf, g)_{\mathring{W}_2^1(\Omega)} \qquad \textit{where } f, g \in \mathring{W}_2^1(\Omega).$$

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