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Abstract

We define an integral, the distributional integral of functions of one real variable, that is more general than the Lebesgue and the Denjoy–Perron–Henstock–Kurzweil integrals, and which allows the integration of functions with distributional values everywhere or nearly everywhere.

Our integral has the property that if f is locally distributionally integrable over the real line and $\psi \in \mathcal{D}(\mathbb{R})$ is a test function, then $f\psi$ is distributionally integrable, and the formula

$$\langle \mathsf{f}, \psi \rangle = (\mathfrak{dist}) \int_{-\infty}^{\infty} f(x)\psi(x)\,\mathrm{d}x,$$

defines a distribution $\mathsf{f} \in \mathcal{D}'(\mathbb{R})$ that has distributional point values almost everywhere and actually $\mathsf{f}(x) = f(x)$ almost everywhere.

The indefinite distributional integral $F(x) = (\mathfrak{dist}) \int_a^x f(t)\,\mathrm{d}t$ corresponds to a distribution with point values everywhere and whose distributional derivative has point values almost everywhere equal to $f(x)$.

The distributional integral is more general than the standard integrals, but it still has many of the useful properties of those standard ones, including integration by parts formulas, substitution formulas, even for infinite intervals (in the Cesàro sense), mean value theorems, and convergence theorems. The distributional integral satisfies a version of Hake's theorem. Unlike general distributions, locally distributionally integrable functions can be restricted to closed sets and can be multiplied by power functions with real positive exponents.

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1. Introduction

In this article we construct and study the properties of a general integration operator that can be applied to functions of one variable, $f : [a, b] \to \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$. We denote this integral as

$$(\mathfrak{dist}) \int_a^b f(x)\,\mathrm{d}x, \tag{1.1}$$

and call it the *distributional integral* of f. The space of distributionally integrable functions is a vector space and the operator (1.1) is a linear functional in this space.

The construction gives an integral with the following properties:

1. Any Denjoy–Perron–Henstock–Kurzweil integrable function is also distributionally integrable and the integrals coincide. In particular any Lebesgue integrable function is distributionally integrable and the integrals coincide. If the Denjoy–Perron–Henstock–Kurzweil integral can be assigned the value $+\infty$ (or $-\infty$) then the distributional integral can also be assigned the value $+\infty$ (or $-\infty$).

2. If a distribution $\mathsf{f} \in \mathcal{D}'(\mathbb{R})$ has distributional point values (as defined in Subsection 2.2) at all points of $[a, b]$ and if $f(x) = \mathsf{f}(x)$ is the function given by those point values, then f is distributionally integrable over $[a, b]$.

3. If $f : \mathbb{R} \to \overline{\mathbb{R}}$ is a function that is distributionally integrable over any compact interval, and if $\psi \in \mathcal{D}(\mathbb{R})$ is a test function, then the formula

$$\langle \mathsf{f}(x), \psi(x)\rangle = (\mathfrak{dist}) \int_{-\infty}^{\infty} f(x)\psi(x)\,\mathrm{d}x, \tag{1.2}$$

where the integral on the right is meant as the distributional integral on any compact interval that contains the support of ψ, defines a distribution $\mathsf{f} \in \mathcal{D}'(\mathbb{R})$. This distribution f has distributional point values almost everywhere and

$$\mathsf{f}(x) = f(x) \quad \text{a.e.} \tag{1.3}$$

If we start with a distribution $\mathsf{f}_0 \in \mathcal{D}'(\mathbb{R})$ that has values everywhere, then construct the function f given by those values, and then define a distribution $\mathsf{f} \in \mathcal{D}'(\mathbb{R})$ by formula (1.2), then we recover the initial distribution: $\mathsf{f} = \mathsf{f}_0$.

We call the integral a *general* integral because of property **1**, which says that it is more general than the standard integrals. We call it the *distributional* integral because of **2** and **3**, since these properties say that functions integrable in this sense are related to corresponding distributions in a very precise fashion.

In the same way that locally Lebesgue integrable functions f give rise to associated "regular" distributions f, $f \leftrightarrow \mathsf{f}$, locally distributionally integrable functions have associated "locally integrable distributions". Actually Denjoy–Perron–Henstock–Kurzweil integrable functions also have canonically associated distributions [32]. Observe, however, that for the purposes of this article is better to say that f and f are *associated* and employ different notations for the function and the distribution, instead of the standard practice of saying that f "is" f. The question of whether a distribution can be associated to a function or not was considered in the lecture [17]; understanding that distributions, in general, are regularizations of functions, and usually *not* uniquely determined [19] allows one to avoid common misunderstandings in the formulas used in mathematical physics [25].

Our construction of the integral is based upon a characterization of positive measures in terms of the properties of the ϕ-transform [51, 24, 11, 37], introduced in Section 3. Indeed, in Theorems 3.4, 3.5, and 3.7 we give conditions on the pointwise extreme values of a distribution that guarantee that it is a positive measure, and this allows us to consider the notions of major and minor distributional pairs and then, in Definition 4.4, define the distributional integral. In Section 4 we also show that the integral is a linear functional, that distributionally integrable functions are finite almost everywhere and measurable, and that the integrals of functions that are equal a.e. coincide.

In Section 5 we study the indefinite integral

$$F(x) = (\mathfrak{dist}) \int_a^x f(t)\,\mathrm{d}t, \tag{1.4}$$

of a distributionally integrable function f. We prove that F is a Łojasiewicz function (Definition 2.2), that is, it has point values everywhere. In general, F will not be continuous but it will be "continuous in an average sense". Other integration processes have discontinuous indefinite integrals [28, Sections 479–482], but they are not even linear operations. Any Łojasiewicz function has an associated unique distribution F, $F \leftrightarrow \mathsf{F}$, and thus we may consider its derivative, $\mathsf{f} = \mathsf{F}'$. We show that F' has distributional values almost everywhere and that actually $\mathsf{F}'(x) = f(x)$ a.e. This is a precise statement of the idea that f is the derivative of F almost everywhere. Later on, in Section 7, we are able to show that $\mathsf{f} = \mathsf{F}'$ is the same distribution given by (1.2).

In Section 6 we show that our integral is more general than the Lebesgue integral and than the Denjoy–Perron–Henstock–Kurzweil integral. In fact, more generally, our integral is capable of recovering a function from its higher order differential quotients, a problem originally considered by Denjoy in [9]. We also show that Łojasiewicz functions and distributionally regulated functions [48] are distributionally integrable, as are the distributional derivatives of Łojasiewicz distributions whose point values exist nearly everywhere. The relationship between locally distributionally integrable functions and distributions is studied in Section 7, not only in the space $\mathcal{D}'(\mathbb{R})$, but in other spaces such as $\mathcal{E}'(\mathbb{R})$, $\mathcal{S}'(\mathbb{R})$, or $\mathcal{K}'(\mathbb{R})$ as well.

According to Hake's theorem [27], there are no improper Denjoy–Perron–Henstock–Kurzweil integrals over finite intervals, since such integrals are actually ordinary Denjoy–Perron–Henstock–Kurzweil integrals. We prove a corresponding result, namely, if f is distributionally integrable over $[a, x]$ for any $x < b$, and if $(\mathfrak{dist}) \int_a^x f(t)\,\mathrm{d}t$ has a distributional limit L as $x \to b$, then f is integrable over $[a,b]$ and the integral is equal to L. We apply this result to show that if f is distributionally integrable over $[a,b]$ then so are the functions $(x-a)^{\alpha}(b-x)^{\beta}f(x)$ for any real numbers $\alpha > 0$ and $\beta > 0$.

We prove a bounded convergence theorem, a monotone convergence theorem, and a version of Fatou's lemma in Section 9. We examine changes of variables in Section 10, showing, in particular, that distributional integrals become Cesàro type integrals when the change sends a finite interval to an infinite one. The three mean value theorems of integral calculus are proved in Section 11.

In the final Section 12 we provide several examples that illustrate our ideas. We give examples of functions that are distributionally integrable but not Denjoy–Perron–Henstock–Kurzweil integrable, examples of distributionally integrable functions that are not Łojasiewicz functions, and examples of Łojasiewicz functions which are not indefinite integrals. We consider the boundary values of the Poisson integral of a distributionally integrable function. Moreover, we consider the Fourier series of periodic locally distributionally integrable functions and the Fourier transform of tempered locally distributionally integrable functions. We also explain why the Cauchy representation formula

$$F(z) = \frac{1}{2\pi i}(\mathfrak{dist})\int_{-\infty}^{\infty} \frac{f(\xi)}{\xi - z}\,\mathrm{d}\xi \tag{1.5}$$

holds for certain functions F analytic in $\operatorname{Im} z > 0$ whose boundary values on $\mathbb{R}$ come from locally integrable distributions (as $f(\xi) = \xi^{-1}e^{-i/\xi}$, for instance), and why such a formula does not hold, even in the principal value sense, for non-distributionally-integrable functions (as $f(\xi) = \xi^{-1}$, for instance).

There have been several studies that involve distributions and integration. Let us emphasize that our integral is a method to find the integral of *functions* as are, let us say, the Riemann or the Denjoy integrals. A completely different question is the integration of *distributions.* Indeed, observe, first of all, that the fact that any distribution $\mathsf{f} \in \mathcal{D}'(\mathbb{R})$ has a primitive $\mathsf{F} \in \mathcal{D}'(\mathbb{R})$, $\mathsf{F}' = \mathsf{f}$, is trivial. If F has values at $x = a$ and at $x = b$ then we say that f is integrable over $[a,b]$ and write ([7])

$$\int_a^b \mathsf{f}(x)\,\mathrm{d}x = \mathsf{F}(b) - \mathsf{F}(a). \tag{1.6}$$

Hence $\int_a^b \mathsf{f}(x)\,\mathrm{d}x$ is a *number.* This notion is due to the Polish school [2, 30] and has several applications, as in the theory of sampling theorems [58]. On the other hand, Silva and Sikorski, independently, used their definitions of the integral of distributions to write Fourier transforms and convolutions of distributions as integrals [41, 42]. Moreover, several authors [4, 44] have considered the class of continuously integrable distributions, that is, those distributions with a continuous primitive; observe, however, that continuously integrable distributions may not have values at any point, and thus are not really *functions,* in general.

We should point out that one can devise a simple procedure for the construction of primitives of *functions* by using the fact that *distributions* are known to have primitives. Indeed, start with a function f, associate to it a distribution f, construct the distributional primitive F, that is, $\mathsf{F}' = \mathsf{f}$, and then construct the function F associated to F. Then F

would be a primitive of f. Unfortunately, this procedure fails, in general, because there is no unique way to assign a distribution f to a given function f, as follows from Theorem 7.1. Interestingly, however, it does work sometimes, as we show, for instance, for Łojasiewicz functions, because in this case all the associations are unique [30].

2. Preliminaries

In this section we have collected several important ideas that will play a role in our construction of a general distributional integral.

2.1. Spaces. We use the term *smooth function* to mean a C^∞ function. The Schwartz spaces of test functions $\mathcal{D}$, $\mathcal{E}$, and $\mathcal{S}$ and the corresponding spaces of distributions are well known [2, 29, 40, 43, 55]. Recall that $\mathcal{E}$ consists of all smooth functions, while $\mathcal{D}$ and $\mathcal{S}$ stand, respectively, for the spaces of smooth compactly supported and rapidly decreasing test functions. In general [61], we call a topological vector space $\mathcal{A}$ a *space of test functions* if $\mathcal{D} \subseteq \mathcal{A} \subseteq \mathcal{E}$, where the inclusions are continuous and dense, and if $\frac{d}{dx}$ is a continuous operator on $\mathcal{A}$. A useful space, particularly in the study of distributional asymptotic expansions [21, 22, 36, 56] is $\mathcal{K}'(\mathbb{R})$, the dual of $\mathcal{K}(\mathbb{R})$. The test function space $\mathcal{K}(\mathbb{R})$ is given by $\mathcal{K}(\mathbb{R}) = \bigcup_{\alpha\in\mathbb{R}} \mathcal{K}_\alpha(\mathbb{R})$, the union having topological meaning, where each $\mathcal{K}_\alpha(\mathbb{R})$ consists of those smooth functions ϕ that satisfy

$$\phi^{(m)}(t) = O(|t|^{\alpha-m}) \quad \text{as } |t| \to \infty \qquad \forall m \in \mathbb{N}, \tag{2.1}$$

and is provided with the topology generated by the family of seminorms

$$\max\{\sup_{|t|\le 1} |\phi^{(m)}(t)|, \sup_{|t|\ge 1} |t|^{m-\alpha}|\phi^{(m)}(t)|\}. \tag{2.2}$$

The space $\mathcal{K}'(\mathbb{R})$ plays a fundamental role in the theory of summability of distributional evaluations [13].

We shall use the notation f, g, F, etc. to denote distributions, while f, g, F, etc. will denote functions. If f is a locally Lebesgue integrable function and f is the corresponding *regular* distribution, given by $\langle \mathsf{f}, \phi\rangle = \int_{-\infty}^{\infty} f(x)\phi(x)\,\mathrm{d}x$ for $\phi \in \mathcal{D}(\mathbb{R})$, then we shall use the notation $f \leftrightarrow \mathsf{f}$; naturally f is not really a function but an equivalence class of functions equal almost everywhere.

2.2. Point values. In [30, 31] Łojasiewicz defined the value of a distribution $\mathsf{f} \in \mathcal{D}'(\mathbb{R})$ at the point x_0 as the limit

$$\mathsf{f}(x_0) = \lim_{\varepsilon\to 0} \mathsf{f}(x_0 + \varepsilon x), \tag{2.3}$$

if the limit exists in $\mathcal{D}'(\mathbb{R})$, that is, if

$$\lim_{\varepsilon\to 0} \langle \mathsf{f}(x_0+\varepsilon x), \phi(x)\rangle = \mathsf{f}(x_0)\int_{-\infty}^{\infty} \phi(x)\,\mathrm{d}x \tag{2.4}$$

for each $\phi \in \mathcal{D}(\mathbb{R})$. It was shown by Łojasiewicz that the existence of the distributional point value $\mathsf{f}(x_0) = \gamma$ is equivalent to the existence of $n \in \mathbb{N}$ and a primitive of order

n of f, that is, $\mathsf{F}^{(n)} = \mathsf{f}$, which corresponds, near x_0, to a continuous function F that satisfies

$$\lim_{x\to x_0} \frac{n!F(x)}{(x-x_0)^n} = \gamma. \tag{2.5}$$

One can also define point values by using the operator

$$\partial_{x_0}(\mathsf{f}) = ((x-x_0)\mathsf{f}(x))', \tag{2.6}$$

since $\mathsf{f}_1(x_0) = \gamma$ if and only if $\mathsf{f}(x_0) = \gamma$, where $\mathsf{f} = \partial_{x_0}(\mathsf{f}_1)$. Therefore ([7]), f has a distributional value equal to γ at $x = x_0$ if and only if there exists $n \in \mathbb{N}$ and a function f_n, continuous at $x = x_0$, with $f_n(x_0) = \gamma$, such that $\mathsf{f} = \partial^n_{x_0}(\mathsf{f}_n)$ near x_0, where $f_n \leftrightarrow \mathsf{f}_n$.

Suppose that $\mathsf{f} \in \mathcal{S}'(\mathbb{R})$ has the Łojasiewicz point value $\mathsf{f}(x_0) = \gamma$. Initially, (2.4) is only supposed to hold for $\phi \in \mathcal{D}(\mathbb{R})$; however, it is shown in [15, 54] that (2.4) will remain true for all $\phi \in \mathcal{S}(\mathbb{R})$. Actually using the notion of the Cesàro behavior of a distribution at infinity [13] explained below, (2.4) will hold [15, 46, 51, 52] if $f(x) = O(|x|^\beta)$ (C) as $|x| \to \infty$, $\phi(x) = O(|x|^\alpha)$ strongly as $|x| \to \infty$ and $\alpha < -1$, $\alpha + \beta < -1$. An asymptotic estimate is *strong* if it remains valid after differentiation of any order, that is, if (2.1) is satisfied.

The notion of distributional point value introduced by Łojasiewicz has been shown to be of fundamental importance in analysis [7, 12, 33, 35, 48, 49, 57, 59, 60]. It seems to have originated in the idea of generalized differentials studied by Denjoy in [9]. There are other notions of distributional point values, as that of Campos Ferreira [7, 8], who also introduced the very useful concept of bounded distributions (see also [62]). A distribution f is said to be *distributionally bounded* at x_0 if $\mathsf{f}(x_0 + \varepsilon x) = O(1)$ as $\varepsilon \to 0$ in $\mathcal{D}'(\mathbb{R})$, i.e., for each test function ϕ, $\langle \mathsf{f}(x_0 + \varepsilon x), \phi(x)\rangle = O(1)$. Distributional boundedness admits a characterization [47] similar to that of Łojasiewicz point values, but this time one replaces (2.5) by $F(x) = O(|x - x_0|^n)$.

The distributional limit $\lim_{x\to x_0} \mathsf{f}(x)$ exists and equals L if

$$\lim_{\varepsilon\to 0} \langle \mathsf{f}(x_0 + \varepsilon x), \phi(x)\rangle = L \int_{-\infty}^{\infty} \phi(x)\,\mathrm{d}x \tag{2.7}$$

for all test functions ϕ with support contained in $\mathbb{R} \setminus \{0\}$. If the point value $\mathsf{f}(x_0)$ exists distributionally then the distributional limit $\lim_{x\to x_0} \mathsf{f}(x)$ exists and equals $\mathsf{f}(x_0)$. On the other hand, if $\lim_{x\to x_0} \mathsf{f}(x) = L$ distributionally then there exist constants $a_0, \ldots, a_n$ such that $\mathsf{f}(x) = \mathsf{f}_0(x) + \sum_{j=0}^{n} a_j \delta^{(j)}(x - x_0)$, where the distributional point value $\mathsf{f}_0(x_0)$ exists and equals L. Notice that the distributional limit $\lim_{x\to x_0} \mathsf{f}(x)$ can actually be defined for distributions $\mathsf{f} \in \mathcal{D}'(\mathbb{R} \setminus \{x_0\})$.

We may also consider lateral limits. We say that the *distributional lateral value* $\mathsf{f}(x_0^+)$ exists if $\mathsf{f}(x_0^+) = \lim_{\varepsilon\to 0^+} \mathsf{f}(x_0 + \varepsilon x)$ in $\mathcal{D}'(0, \infty)$, that is,

$$\lim_{\varepsilon\to 0^+} \langle \mathsf{f}(x_0 + \varepsilon x), \phi(x)\rangle = \mathsf{f}(x_0^+) \int_0^{\infty} \phi(x)\,\mathrm{d}x, \qquad \phi \in \mathcal{D}(0, \infty). \tag{2.8}$$

Similar definitions apply to $\mathsf{f}(x_0^-)$. Notice that the distributional limit $\lim_{x\to x_0} \mathsf{f}(x)$ exists if and only if the distributional lateral limits $\mathsf{f}(x_0^-)$ and $\mathsf{f}(x_0^+)$ exist and coincide.

2.3. The Cesàro behavior of distributions at infinity. The Cesàro behavior [13, 22] of a distribution at infinity is studied by using the order symbols $O(x^\alpha)$ and $o(x^\alpha)$ in the Cesàro sense. If $\mathsf{f} \in \mathcal{D}'(\mathbb{R})$ and $\alpha \in \mathbb{R}\backslash\{-1,-2,-3,\ldots\}$, we say that $\mathsf{f}(x) = O(x^\alpha)$ as $x \to \infty$ *in the Cesàro sense*, and write

$$\mathsf{f}(x) = O(x^\alpha) \quad (\mathrm{C}) \qquad \text{as } x \to \infty, \tag{2.9}$$

if there exists $N \in \mathbb{N}$ such that every primitive F of order N, i.e., $\mathsf{F}^{(N)} = \mathsf{f}$, corresponds for large arguments to a locally integrable function, $\mathsf{F} \leftrightarrow F$, that satisfies the ordinary order relation

$$F(x) = p(x) + O(x^{\alpha+N}) \qquad \text{as } x \to \infty \tag{2.10}$$

for a suitable polynomial p of degree at most $N-1$. Note that if $\alpha > -1$, then the polynomial p is irrelevant. If we want to specify the value N, we write (C, N) instead of just (C). A similar definition applies to the little o symbol. The definitions when $x \to -\infty$ are clear.

The elements of $\mathcal{S}'(\mathbb{R})$ can be characterized by their Cesàro behavior at $\pm\infty$; in fact, $\mathsf{f} \in \mathcal{S}'(\mathbb{R})$ if and only if there exists $\alpha \in \mathbb{R}$ such that $\mathsf{f}(x) = O(x^\alpha)$ (C) as $x \to \infty$, and $\mathsf{f}(x) = O(|x|^\alpha)$ (C) as $x \to -\infty$. On the other hand, this is true for all $\alpha \in \mathbb{R}$ if and only if $\mathsf{f} \in \mathcal{K}'(\mathbb{R})$.

Using these ideas, one can define the limit of a distribution at ∞ in the Cesàro sense. We say that $\mathsf{f} \in \mathcal{D}'(\mathbb{R})$ has a limit L at infinity *in the Cesàro sense*, and write

$$\lim_{x\to\infty} \mathsf{f}(x) = L \quad (\mathrm{C}), \tag{2.11}$$

if $\mathsf{f}(x) = L + o(1)$ (C) as $x \to \infty$.

The Cesàro behavior of a distribution f at infinity is related to the parametric behavior of $\mathsf{f}(\lambda x)$ as $\lambda \to \infty$. In fact, one can show [22, 45, 47] that if $\alpha > -1$, then $\mathsf{f}(x) = O(x^\alpha)$ (C) as $x \to \infty$ and $\mathsf{f}(x) = O(|x|^\alpha)$ (C) as $x \to -\infty$ if and only if

$$\mathsf{f}(\lambda x) = O(\lambda^\alpha) \qquad \text{as } \lambda \to \infty, \tag{2.12}$$

where the last relation holds weakly in $\mathcal{D}'(\mathbb{R})$, i.e., for all $\phi \in \mathcal{D}(\mathbb{R})$ fixed, $\langle \mathsf{f}(\lambda x), \phi(x)\rangle = O(\lambda^\alpha)$, $\lambda \to \infty$. A distribution f belongs to the space $\mathcal{K}'(\mathbb{R})$ if and only if it satisfies the moment asymptotic expansion [21, 22]

$$\mathsf{f}(\lambda x) \sim \sum_{n=0}^{\infty} \frac{(-1)^n \mu_n \delta^{(n)}(x)}{n!\lambda^{n+1}} \qquad \text{as } \lambda \to \infty, \tag{2.13}$$

where the $\mu_n = \langle \mathsf{f}(x), x^n\rangle$ are the moments of f.

2.4. Evaluations. Let $\mathsf{f} \in \mathcal{D}'(\mathbb{R})$ with support bounded on the left. If $\phi \in \mathcal{E}(\mathbb{R})$ then the evaluation $\langle \mathsf{f}(x), \phi(x)\rangle$ will not be defined, in general. We say that the evaluation exists *in the Cesàro sense* and equals L, written as

$$\langle \mathsf{f}(x), \phi(x)\rangle = L \quad (\mathrm{C}), \tag{2.14}$$

if $\mathsf{g}(x) = L + o(1)$ (C) as $x \to \infty$, where g is the primitive of $\mathsf{f}\phi$ with support bounded on the left. A similar definition applies if $\operatorname{supp} \mathsf{f}$ is bounded on the right. Observe that if f corresponds to a locally integrable function f with $\operatorname{supp} \mathsf{f} \subset [a, \infty)$ then (2.14) means that

$$\int_a^\infty f(x)\phi(x)\,\mathrm{d}x = L \quad \text{(C)}. \tag{2.15}$$

Naturally, this will hold for *any* integration method we use. If $\mathsf{f}(x) = \sum_{n=0}^\infty a_n\delta(x-n)$ then (2.14) tells us that

$$\sum_{n=0}^\infty a_n\phi(n) = L \quad \text{(C)}. \tag{2.16}$$

In the general case when the support of f extends to both $-\infty$ and $+\infty$, there are various different but related notions of evaluations in the Cesàro sense (or in any other summability sense, in fact). If f admits a representation of the form $\mathsf{f} = \mathsf{f}_1+\mathsf{f}_2$, with $\operatorname{supp}\mathsf{f}_1$ bounded on the left and $\operatorname{supp}\mathsf{f}_2$ bounded on the right, such that $\langle \mathsf{f}_j(x),\phi(x)\rangle = L_j$ (C) exist, then we say that the (C) evaluation $\langle \mathsf{f}(x),\phi(x)\rangle$ (C) exists and equals $L = L_1 + L_2$. This is clearly independent of the decomposition. The notation (2.14) is used in this situation as well.

It happens many times that $\langle \mathsf{f}(x),\phi(x)\rangle$ (C) does not exist, but the symmetric limit, $\lim_{x\to\infty}\{\mathsf{g}(x)-\mathsf{g}(-x)\} = L$, where g is any primitive of $\mathsf{f}\phi$, exists in the (C) sense. Then we say that the evaluation $\langle \mathsf{f}(x),\phi(x)\rangle$ exists *in the principal value Cesàro sense* [22, 53], and write

$$\text{p.v.}\langle \mathsf{f}(x),\phi(x)\rangle = L \quad \text{(C)}. \tag{2.17}$$

Observe that p.v. $\sum_{n=-\infty}^\infty a_n\phi(n) = L$ (C) if and only if $\sum_{-N}^N a_n\phi(n) \to L$ (C) as $N\to\infty$ while p.v. $\int_{-\infty}^\infty f(x)\phi(x)\,\mathrm{d}x = L$ (C) if and only if $\int_{-A}^A f(x)\phi(x)\,\mathrm{d}x \to L$ (C) as $A\to\infty$.

A very useful intermediate notion is the following [48, 49, 53]. If there exists k such that

$$\lim_{x\to\infty}\{\mathsf{g}(ax)-\mathsf{g}(-x)\} = L \quad (\mathrm{C},k) \qquad \forall a>0, \tag{2.18}$$

we say that the distributional evaluation exists *in the e.v. Cesàro sense* and write

$$\text{e.v.}\langle \mathsf{f}(x),\phi(x)\rangle = L\ (\mathrm{C},k), \tag{2.19}$$

or just e.v.$\langle \mathsf{f}(x),\phi(x)\rangle = L$ (C) if there is no need to emphasize the value of k.

2.5. Łojasiewicz distributions. There is a class of distributions that correspond to ordinary functions, the class of Łojasiewicz distributions. In general, Łojasiewicz distributions are not regular distributions, that is, they correspond to ordinary functions that are not locally Lebesgue integrable functions.

The simplest class of distributions that correspond to functions are those that come from continuous functions. If $f \leftrightarrow \mathsf{f}$ and f is continuous then it is an ordinary function: we can always say what $f(x_0)$ is for any x_0. The function f is not just defined almost everywhere but it is actually defined *everywhere*.

DEFINITION 2.1. A distribution f is a *Łojasiewicz distribution* if the distributional point value $\mathsf{f}(x_0)$ exists for every $x_0\in\mathbb{R}$.

Definition 2.2. A function f defined in $\mathbb{R}$ is called a *Łojasiewicz function* if there exists a Łojasiewicz distribution f such that

$$\mathsf{f}(x) = f(x) \qquad \forall x \in \mathbb{R}. \tag{2.20}$$

The correspondence $f \leftrightarrow \mathsf{f}$ is clearly and uniquely defined in the case of Łojasiewicz functions and distributions [30]. The Łojasiewicz functions can be considered as a distributional generalization of continuous functions. They are defined at all points, and furthermore the value at each given point is not arbitrary but the (distributional) limit of the function as one approaches the given point. The Łojasiewicz functions and distributions were introduced in [30].

If f is a Łojasiewicz distribution, and F is a primitive, $\mathsf{F}' = \mathsf{f}$, then F is also a Łojasiewicz distribution. If f is a Łojasiewicz distribution and ψ is a smooth function, then $\psi\mathsf{f}$ is a Łojasiewicz distribution and

$$(\psi\mathsf{f})(x) = \psi(x)\mathsf{f}(x). \tag{2.21}$$

If f is a Łojasiewicz function, $f \leftrightarrow \mathsf{f}$, then we can *define* its definite integral [2, 30] as

$$\int_a^b f(x)\,\mathrm{d}x = \mathsf{F}(b) - \mathsf{F}(a), \tag{2.22}$$

where $\mathsf{F}' = \mathsf{f}$. The evaluation of f on a test function ϕ, $\langle \mathsf{f}, \phi \rangle$, can actually be given as an integral, namely,

$$\langle \mathsf{f}, \phi \rangle = \int_{-\infty}^{\infty} f(x)\phi(x)\,\mathrm{d}x = \int_a^b f(x)\phi(x)\,\mathrm{d}x, \qquad \phi \in \mathcal{D}(\mathbb{R}), \tag{2.23}$$

where $\operatorname{supp}\phi \subset [a, b]$. We will give a rather constructive procedure below (Sections 4 and 6) to calculate (2.22).

If f_0 is a Łojasiewicz function, $f_0 \leftrightarrow \mathsf{f}_0$, defined for $x < a$, and f_1 is a Łojasiewicz function, $f_1 \leftrightarrow \mathsf{f}_1$, defined for $x > a$, and if the distributional lateral limits $\mathsf{f}_0(a - 0)$ and $\mathsf{f}_1(a + 0)$ exist and coincide, then there is a Łojasiewicz function f whose restriction to $(-\infty, a)$ is f_0 and whose restriction to (a, ∞) is f_1.

A typical example of a Łojasiewicz function is

$$s_{\alpha,\beta}(x) = \begin{cases} |x|^\alpha \sin |x|^{-\beta}, & x \neq 0, \\ 0, & x = 0, \end{cases} \tag{2.24}$$

for $\alpha \in \mathbb{C}$ and $\beta > 0$. If H is the Heaviside function, then the functions $H(\pm x)s_{\alpha,\beta}(x)$ and their linear combinations are also Łojasiewicz functions. It is not hard to see that this implies that derivatives of arbitrary order of $\mathsf{s}_{\alpha,\beta}$, where $s_{\alpha,\beta} \leftrightarrow \mathsf{s}_{\alpha,\beta}$, are also Łojasiewicz distributions. These are rapidly oscillating functions. However, not all fast oscillating functions are Łojasiewicz functions. Curiously, the regular distribution $\sin(\ln|x|)$ is not a Łojasiewicz distribution since the distributional value at $x = 0$ does not exist in the Łojasiewicz sense, even though it exists and equals 0 in the Campos Ferreira sense [7].

2.6. Distributionally regulated functions. Another case when a distribution corresponds to a function is the case of regulated distributions, introduced and studied in [48]. They are generalizations of the ordinary regulated functions [10], which are functions whose lateral limits exist at all points, although they may be different. They are related to the recently introduced "thick" points [20].

DEFINITION 2.3. A distribution f is called a *regulated distribution* if the distributional lateral limits

$$\mathsf{f}(x_0^+) \quad \text{and} \quad \mathsf{f}(x_0^-) \tag{2.25}$$

exist for all $x_0 \in \mathbb{R}$, and there are no delta functions at any point.

The statement that "there are no delta functions" at any point explicitly means that for each $\phi \in \mathcal{D}(\mathbb{R})$ and any $x_0 \in \mathbb{R}$,

$$\lim_{\varepsilon \to 0^+} \langle \mathsf{f}(x_0 + \varepsilon x), \phi(x) \rangle = \mathsf{f}(x_0^-) \int_{-\infty}^{0} \phi(x)\,\mathrm{d}x + \mathsf{f}(x_0^+) \int_{0}^{\infty} \phi(x)\,\mathrm{d}x. \tag{2.26}$$

The relation (2.26) is known as (*pointwise*) *distributional jump behavior* and has interesting applications in the theory of Fourier series [23, 50, 53].

If $\mathsf{f}(x_0^+) = \mathsf{f}(x_0^-)$ then $\mathsf{f}(x_0)$ exists, since these distributions do not have delta functions, and therefore we can define the function

$$f(x_0) = \mathsf{f}(x_0) \tag{2.27}$$

for these x_0. Then f is called a *distributionally regulated function.* The function f is defined in the set $\mathbb{R} \setminus \mathfrak{S}$, where $\mathfrak{S}$ is the set of points x_0 where $\mathsf{f}(x_0^+) \neq \mathsf{f}(x_0^-)$. The set $\mathfrak{S}$ has measure zero since in fact it is countable at the most [48]. One can actually define

$$f(x_0) = \frac{\mathsf{f}(x_0^+) + \mathsf{f}(x_0^-)}{2}, \tag{2.28}$$

and this is defined everywhere.

The basic properties of the distributionally regulated functions and the corresponding regulated distributions are the following. If f is a regulated distribution, and F is a primitive, $\mathsf{F}' = \mathsf{f}$, then F is a Łojasiewicz distribution. If f is a regulated distribution and ψ is a smooth function, then $\psi\mathsf{f}$ is a regulated distribution, too. If f is a regulated function, $f \leftrightarrow \mathsf{f}$, then we can *define* its definite integral as

$$\int_a^b f(x)\,\mathrm{d}x = \mathsf{F}(b) - \mathsf{F}(a), \tag{2.29}$$

where $\mathsf{F}' = \mathsf{f}$. Then

$$\langle \mathsf{f}, \phi \rangle = \int_{-\infty}^{\infty} f(x)\phi(x)\,\mathrm{d}x, \qquad \phi \in \mathcal{D}(\mathbb{R}). \tag{2.30}$$

As in the case of Łojasiewicz functions, the integral that we will define in Section 4 coincides with (2.29) for distributionally regulated functions (Theorem 6.6).

2.7. Romanovski's lemma. We shall apply the following useful result [38], Romanovski's lemma, in some of our proofs. See [26] for many interesting applications of this result, and [24] for generalizations to several variables.

THEOREM 2.4 (Romanovski's lemma). *Let $\mathfrak{F}$ be a family of open intervals in (a, b) with the following four properties:*

I. *If $(\alpha, \beta) \in \mathfrak{F}$ and $(\beta, \gamma) \in \mathfrak{F}$, then $(\alpha, \gamma) \in \mathfrak{F}$.*

II. *If* $(\alpha,\beta)\in\mathfrak{F}$ *and* $(\gamma,\delta)\subset(\alpha,\beta)$ *then* $(\gamma,\delta)\in\mathfrak{F}$.
III. *If* $(\alpha,\beta)\in\mathfrak{F}$ *for all* $[\alpha,\beta]\subset(c,d)$ *then* $(c,d)\in\mathfrak{F}$.
IV. *If all the intervals contiguous to a perfect closed set* $K\subset[a,b]$ *belong to* $\mathfrak{F}$ *then there exists an interval* $I\in\mathfrak{F}$ *with* $I\cap K\neq\emptyset$.

Then $(a,b)\in\mathfrak{F}$.

Observe that if we take $K=[a,b]$ in IV we obtain $\mathfrak{F}\neq\{\emptyset\}$, but it may be easier to show this separately.

2.8. Measures. We shall use the following nomenclature. A (*Radon*) *measure* would mean a *positive* functional on the space of compactly supported continuous functions, which would be denoted by integral notation such as $\mathrm{d}\mu$, or by distributional notation, $\mathsf{f}=\mathsf{f}_\mu$, so that

$$\langle \mathsf{f},\phi\rangle=\int_{\mathbb{R}}\phi(x)\,\mathrm{d}\mu(x), \tag{2.31}$$

and $\langle \mathsf{f},\phi\rangle\geq 0$ if $\phi\geq 0$. A *signed measure* is a real bounded functional on the space of compactly supported continuous functions, denoted as $\mathrm{d}\nu$, say, or as $\mathsf{g}=\mathsf{g}_\nu$. Observe that any signed measure can be written as $\nu=\nu_+-\nu_-$, where $\nu_\pm$ are measures concentrated on disjoint sets. We shall also use the Lebesgue decomposition, according to which any signed measure ν can be written as $\nu=\nu_{\mathrm{abs}}+\nu_{\mathrm{sig}}$, where ν_{abs} is absolutely continuous with respect to the Lebesgue measure, so that it corresponds to a regular distribution, while ν_{sig} is a signed measure concentrated on a set of Lebesgue measure zero. We shall also need to consider the measures $(\nu_{\mathrm{sig}})_\pm=(\nu_\pm)_{\mathrm{sig}}$, the positive and negative singular parts of ν.

3. The ϕ-transform

A very important tool in our definition of a general distributional integral is the ϕ-transform. The ϕ-transform [11, 37, 48, 51] in one variable is defined as follows. Let $\phi\in\mathcal{D}(\mathbb{R})$ be a fixed *normalized* test function, that is, one that satisfies

$$\int_{-\infty}^{\infty}\phi(x)\,\mathrm{d}x=1. \tag{3.1}$$

If $\mathsf{f}\in\mathcal{D}'(\mathbb{R})$ we introduce the function of two variables $F=F_\phi\{\mathsf{f}\}$ by the formula

$$F(x,t)=\langle \mathsf{f}(x+ty),\phi(y)\rangle, \tag{3.2}$$

where $(x,t)\in\mathbb{H}$, the half-plane $\mathbb{R}\times(0,\infty)$. Naturally the evaluation in (3.2) is with respect to the variable y. We call F the *ϕ-transform* of f. Whenever we consider ϕ-transforms we assume that ϕ satisfies (3.1).

The ϕ-transform converges to the distribution as $t\to0^+$ ([51, 52]): If $\phi\in\mathcal{D}(\mathbb{R})$ and $\mathsf{f}\in\mathcal{D}'(\mathbb{R})$ then

$$\lim_{t\to0^+}F(x,t)=\mathsf{f}(x) \tag{3.3}$$

distributionally in the space $\mathcal{D}'(\mathbb{R})$, that is, if $\rho \in \mathcal{D}(\mathbb{R})$ then

$$\lim_{t\to0^+} \langle F(x,t), \rho(x)\rangle = \langle \mathsf{f}(x), \rho(x)\rangle. \tag{3.4}$$

The definition of the ϕ-transform tells us that if the distributional point value [30] $\mathsf{f}(x_0)$ exists and equals γ then $F(x_0,t) \to \gamma$ as $t \to 0^+$, but actually $F(x,t) \to \gamma$ as $(x,t) \to (x_0,0)$ in an *angular* or *non-tangential* fashion, that is, if $|x-x_0| \le Mt$ for some $M > 0$ (just replace $\phi(y)$ by the compact set $\{\phi(y+\tau) : |\tau| \le M\}$).

The angular behavior of the ϕ-transform at a point $(x_0,0)$ gives us important information [11, 37, 51] about the nature of the distribution at $x = x_0$, even if the angular limit does not exist.

If $x_0 \in \mathbb{R}$ we shall denote by $C_{x_0,\theta}$ the cone in $\mathbb{H}$ starting at x_0 and of angle θ,

$$C_{x_0,\theta} = \{(x,t) \in \mathbb{H} : |x-x_0| \le (\tan\theta)t\}. \tag{3.5}$$

If $\mathsf{f} \in \mathcal{D}'(\mathbb{R})$ and $x_0 \in \mathbb{R}$ then we consider the upper and lower angular values of its ϕ-transform,

$$\mathsf{f}^+_{\phi,\theta}(x_0) = \limsup_{\substack{(x,t)\to(x_0,0)\\(x,t)\in C_{x_0,\theta}}} F(x,t), \tag{3.6}$$

$$\mathsf{f}^-_{\phi,\theta}(x_0) = \liminf_{\substack{(x,t)\to(x_0,0)\\(x,t)\in C_{x_0,\theta}}} F(x,t). \tag{3.7}$$

The quantities $\mathsf{f}^\pm_{\phi,\theta}(x_0)$ are well defined at all points x_0, but, of course, they could be infinite. For $\theta = 0$, we obtain the upper and lower *radial* limits of the ϕ-transform.

The following simple result would be useful.

Lemma 3.1. *Let $\mathsf{f} \in \mathcal{D}'(\mathbb{R})$ and $x_0 \in \mathbb{R}$. If*

$$\mathsf{f}^+_{\phi,0}(x_0) = \mathsf{f}^-_{\phi,0}(x_0) = \gamma \tag{3.8}$$

for all normalized positive test functions $\phi \in \mathcal{D}(\mathbb{R})$, then the distributional point value $\mathsf{f}(x_0)$ exists and equals γ.

Proof. Indeed, (3.8) implies that $\lim_{\varepsilon\to0}\langle \mathsf{f}(x_0+\varepsilon x), \phi(x)\rangle$ exists and equals γ for any positive normalized test function. If we multiply by a constant, we see that the limit exists and equals $\gamma\int_{-\infty}^{\infty}\phi(x)\,\mathrm{d}x$ for any positive test function. The result now follows because any test function is the difference of two positive test functions. Indeed, given an arbitrary test function $\phi \in \mathcal{D}(\mathbb{R})$, let $M = \max_{x\in\mathbb{R}}|\phi(x)|$. Find a positive $\varphi \in \mathcal{D}(\mathbb{R})$ so that $\varphi(x) = 1$ for $x \in \operatorname{supp}\phi$. Then $\phi_1 = M\varphi$ and $\phi_2 = \phi + \phi_1$ are positive test functions with $\phi = \phi_2 - \phi_1$. ∎

We shall need several characterizations of positive measures in terms of the extreme values $\mathsf{f}^\pm_{\phi,\theta}(x)$ of a distribution f. The following result was proved in [51].

Theorem 3.2. *Let $\mathsf{f} \in \mathcal{D}'(\mathbb{R})$. Let U be an open set. Then f is a measure in U if and only if its ϕ-transform $F = F_\phi\{\mathsf{f}\}$ with respect to a given normalized, positive test function $\phi \in \mathcal{D}(\mathbb{R})$ satisfies*

$$\mathsf{f}^-_{\phi,\theta}(x) \ge 0 \quad \forall x \in U, \tag{3.9}$$

for all angles θ. Moreover, if the support of ϕ is contained in $[-R,R]$ and if (3.9) *holds for a single value of $\theta > \arctan R$, then f is a measure in U.*

We should also point out that if there exists a constant $M > 0$ such that $\mathsf{f}^{-}_{\phi,\theta}(x) \geq -M$ for all $x \in U$, where $\theta > \arctan R$, then f is a signed measure in U whose singular part is *positive* [51]. It is easy to see that these results are not true if we use *radial* limits instead of angular ones. An example is provided by taking $\mathsf{f}(x) = -\delta'(x)$ and $\phi \in \mathcal{D}(\mathbb{R})$ with $\phi'(0) > 0$. Actually this example shows that if (3.9) holds for a value of $\theta < \arctan R$, then f might not be a measure.

Using Romanovski's lemma we were able to prove the ensuing stronger result in [24].

THEOREM 3.3. *Let $\mathsf{f} \in \mathcal{D}'(\mathbb{R})$. Let U be an open set. Suppose its ϕ-transform $F = F_\phi\{\mathsf{f}\}$ with respect to a given normalized, positive test function $\phi \in \mathcal{D}(\mathbb{R})$ with* $\operatorname{supp}\phi \subset [-R,R]$ *satisfies*

$$\mathsf{f}^{+}_{\phi,0}(x) \geq 0 \quad \textit{almost everywhere in } U, \tag{3.10}$$

while for each $x \in U$ there is a constant $M_x > 0$ such that

$$\mathsf{f}^{-}_{\phi,\theta}(x) \geq -M_x, \tag{3.11}$$

where $\theta > \arctan R$. Then f is a measure in U.

Furthermore, one needs the inequality (3.11) to be true at *all* points of U, as the example $\mathsf{f}(x) = -\delta(x-a)$, where $a \in U$, shows. However, in our construction of the general distributional integral we shall need to consider the case when $\mathsf{f}^{-}_{\phi,\theta}(x) = -\infty$ for $x \in E$ where E is a small set in the sense that $|E| \leq \aleph_0$. We have a corresponding result in this case if we ask that any primitive of f be a Łojasiewicz distribution.

THEOREM 3.4. *Let $\mathsf{f} \in \mathcal{D}'(\mathbb{R})$. Suppose that $\mathsf{f} = \mathsf{F}'$, where F is a Łojasiewicz distribution. Let U be an open set. Suppose the ϕ-transform $F = F_\phi\{\mathsf{f}\}$ with respect to a given normalized, positive test function $\phi \in \mathcal{D}(\mathbb{R})$ with* $\operatorname{supp}\phi \subset [-R,R]$ *satisfies*

$$\mathsf{f}^{+}_{\phi,0}(x) \geq 0 \quad \textit{almost everywhere in } U, \tag{3.12}$$

while there exist a countable set E such that for each $x \in U\setminus E$ there is a constant $M_x > 0$ such that

$$\mathsf{f}^{-}_{\phi,\theta}(x) \geq -M_x, \qquad x \in U \setminus E, \tag{3.13}$$

where $\theta > \arctan R$. Then f is a measure in U.

Proof. Suppose that U is an open interval. Let $\mathfrak{U}$ be the family of open subintervals V of U such that the restriction $\mathsf{f}|_V$ is a measure. We shall use Theorem 2.4 to prove that $U \in \mathfrak{U}$. Let us first show that $\mathfrak{U} \neq \{\emptyset\}$. Suppose that $E \subseteq \{x_n : 1 \leq n < \infty\}$. Let $t_0 \geq 1$ be fixed and put

$$g_n(x) = \min\{F(y,t) : |y-x| \leq (\tan\theta)t,\ n^{-1} \leq t \leq t_0\}. \tag{3.14}$$

The functions g_n are continuous and, because of (3.13), for each $x \in U \setminus E$ there exists a constant $M'_x > 0$ such that $g_n(x) \geq -M'_x$ for all n. Hence if

$$W_k = \{x \in U : g_n(x) \geq -k\ \forall n \in \mathbb{N}\} \cup \{x_1,\ldots,x_k\}, \tag{3.15}$$

then $U = \bigcup_{k=1}^{\infty} W_k$. If we now employ the Baire theorem, we obtain the existence of $k \in \mathbb{N}$ such that W_k has non-empty interior, and thus the interior of the set

$$\{x \in U : g_n(x) \geq -k \ \forall n \in \mathbb{N}\} \tag{3.16}$$

is also non-empty. Hence there is a non-empty open interval $V \subset U$ and a constant $M > 0$ such that $F(x,t) \geq -M$ for all $(y,t) \in C_{x,\theta}$ with $x \in V$ and $0 < t \leq t_0$, and hence $\mathsf{f}^-_{\phi,\theta}(x) \geq -M$ for $x \in V$. Theorem 3.3 then implies that $\mathsf{f}|_V$ is a measure. Therefore $V \in \mathfrak{U}$, and so $\mathfrak{U} \neq \{\emptyset\}$.

Condition I of Theorem 2.4 follows from the fact that if $\mathsf{f}|_{(\alpha,\beta)}$ and $\mathsf{f}|_{(\beta,\gamma)}$ are measures, then $\mathsf{F}|_{(\alpha,\beta)}$ and $\mathsf{F}|_{(\beta,\gamma)}$ are distributions corresponding to increasing continuous functions, and since F is a Łojasiewicz distribution it follows that F, $F \leftrightarrow \mathsf{F}$, must also be continuous at $x = \beta$, so that F is a continuous increasing function in (α,γ) and consequently $\mathsf{f}|_{(\alpha,\gamma)}$ is a measure.

It is clear that II and III are satisfied.

In order to prove IV, let $K \subset \overline{U}$ be a perfect closed set such that all the intervals contiguous to K belong to $\mathfrak{U}$. Then by the Baire theorem again, there exists an open interval $V \subset U$ and a constant $M > 0$ such that $\mathsf{f}^-_{\phi,\theta}(x) \geq -M$ for all $x \in K \cap V \neq \emptyset$. But f is a measure in $V \setminus K$, and thus $\mathsf{f}^-_{\phi,\theta}(x) \geq 0$ for $x \in V \setminus K$. Theorem 3.3 allows us to conclude that $\mathsf{f}|_V$ is a measure, and thus $V \in \mathfrak{U}$; this proves IV. ∎

Observe that if the hypotheses of Theorem 3.4 are satisfied then f is a measure in U, and thus $\mathsf{f}^-_{\phi,\theta}(x) \geq 0$ at all points of U and for all angles, not just radially almost everywhere, and similarly the set E where $\mathsf{f}^-_{\phi,\theta}(x) = -\infty$ is actually empty.

We shall also employ characterizations merely in terms of *radial* limits of the ϕ-transform. The following is such a result for the lower radial limits of a harmonic function.

THEOREM 3.5. *Let $H(x,y)$ be a harmonic function defined in the upper half-plane $\mathbb{H}$. Suppose that $\lim_{(x,y)\to\infty} H(x,y) = 0$. Also suppose that the distributional limit of $H(x,y)$ as $y \to 0^+$ exists and equals $\mathsf{f} \in \mathcal{E}'(\mathbb{R})$; suppose that $\mathsf{f} = \mathsf{F}'$, where F is a Łojasiewicz distribution. If*

$$\limsup_{y\to 0^+} H(x,y) \geq 0 \qquad \text{almost everywhere in } \mathbb{R}, \tag{3.17}$$

and there exists a countable set E and constants $M_x < \infty$ for $x \in \mathbb{R} \setminus E$ such that

$$\liminf_{y\to 0^+} H(x,y) \geq -M_x, \qquad x \in \mathbb{R} \setminus E, \tag{3.18}$$

then f is a measure and $H(x,y) \geq 0$ for all $(x,y) \in \mathbb{H}$.

Proof. We shall employ Romanovski's lemma (Theorem 2.4) to prove that f is a measure in $\mathbb{R}$. Let (a,b) be an open interval with $\operatorname{supp} \mathsf{f} \subset (a,b)$. Let $\mathfrak{U}$ be the family of open subintervals of (a,b) where the restriction of f is a measure; clearly $\mathfrak{U}$ contains non-empty intervals. Observe that if $(c,d) \in \mathfrak{U}$, then F is an increasing continuous function in $[c,d]$, where $F \leftrightarrow \mathsf{F}$; condition I follows from this observation. Conditions II and III are easy. For condition IV, suppose that K is a perfect compact subset of (a,b) such that $(a,b) \setminus K = \bigcup_{n=1}^{\infty} (a_n,b_n)$, with $(a_n,b_n) \in \mathfrak{U}$. Let $m = \min_{x\in\mathbb{R}} H(x,1)$. By the Baire theorem, there exists a constant M, with $M > 0$ and $M > -m$, and an open interval I such that $I \cap K \neq \emptyset$ and $H(x,y) \geq -M$ for $x \in \overline{I} \cap K$ and for $0 < y \leq 1$. If $(a_n,b_n) \subset I$, then the harmonic function H is bounded below by $-M$ in the boundary of the rectangle $(a_n, b_n) \times (0,1) \subset \mathbb{H}$, except perhaps at the corners a_n and b_n, but since f is the derivative of a Łojasiewicz distribution we obtain the bound $H(x,y) = o(((x-x_0)^2 + y^2)^{-1/2})$ as $(x,y) \to x_0$, for any $x_0 \in \mathbb{R}$, and this allows us to conclude that H is bounded below by $-M$ in the rectangle $[a_n, b_n] \times (0,1]$. Actually if H were not bounded below in the rectangle then at one of the corners, $x_0 = a_n$ or $x_0 = b_n$, H would grow at least as fast as $((x-x_0)^2 + y^2)^{-1}$, as follows from the results of [15, Section 4] when applied to the harmonic function $\tilde{H}(\xi) = H(\sqrt{\xi - x_0})$. Therefore $H(x,y) \geq -M$ for all $x \in \overline{I}$ and all $0 < y \leq 1$, and we conclude that $I \in \mathfrak{U}$. ∎

It is convenient to define some classes of test functions.

Definition 3.6. The class $\mathcal{T}_0$ consists of all positive normalized functions $\phi \in \mathcal{E}(\mathbb{R})$ that satisfy the following condition:

$$\exists \alpha < -1 \text{ such that} \quad \phi(x) = O(|x|^\alpha) \quad \text{strongly as } |x| \to \infty. \tag{3.19}$$

The class $\mathcal{T}_1$ is the subclass of $\mathcal{T}_0$ consisting of those functions that also satisfy

$$x\phi'(x) \leq 0 \qquad \text{for all } x \in \mathbb{R}. \tag{3.20}$$

Observe that the ϕ-transform is well defined when $f \in \mathcal{E}'(\mathbb{R})$ and $\phi \in \mathcal{T}_0$. Since the Poisson kernel $\varphi(x) = \pi^{-1}(1+x^2)^{-1}$ belongs to $\mathcal{T}_1$ and the ϕ-transform $H = F_\varphi\{\mathsf{f}\}$ with respect to this function φ is the harmonic function $H(x,y)$ defined for $(x,y) \in \mathbb{H}$, that vanishes at infinity, and that satisfies $H(x, 0^+) = \mathsf{f}(x)$ distributionally, we arrive at the following result, corollary of Theorem 3.5.

Theorem 3.7. *Let $\mathsf{f} \in \mathcal{E}'(\mathbb{R})$. Suppose that $\mathsf{f} = \mathsf{F}'$, where F is a Łojasiewicz distribution. Suppose that the ϕ-transform $F = F_\phi\{\mathsf{f}\}$ with respect to any $\phi \in \mathcal{T}_1$ satisfies*

$$\mathsf{f}^{+}_{\phi,0}(x) \geq 0 \qquad \textit{almost everywhere in } \mathbb{R}, \tag{3.21}$$

$$\mathsf{f}^{-}_{\phi,0}(x) \geq -M_x > -\infty, \qquad x \in \mathbb{R} \setminus E, \tag{3.22}$$

where E is a countable set. Then f is a measure in $\mathbb{R}$.

4. The definite integral

Let f be a function defined in $[a,b]$ with values in $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$. We now proceed to define its integral. We start with the concepts of major and minor pairs.

Definition 4.1. A pair (u, U) is called a *major distributional pair* for the function f if:

(1) $\mathsf{u} \in \mathcal{E}'[a,b]$, $\mathsf{U} \in \mathcal{D}'(\mathbb{R})$, and

$$\mathsf{U}' = \mathsf{u}. \tag{4.1}$$

(2) U is a Łojasiewicz distribution with $\mathsf{U}(a) = 0$.

(3) There exists a set E with $|E| \leq \aleph_0$ and a set Z of null Lebesgue measure, $m(Z) = 0$, such that for all $x \in [a,b] \setminus Z$ and all $\phi \in \mathcal{T}_0$ we have

$$(\mathsf{u})^{-}_{\phi,0}(x) \geq f(x), \tag{4.2}$$

while for $x \in [a,b] \setminus E$ and all $\phi \in \mathcal{T}_1$,

$$(\mathsf{u})^-_{\phi,0}(x) > -\infty. \tag{4.3}$$

The definition of a minor distributional pair is similar.

DEFINITION 4.2. A pair (v,V) is called a *minor distributional pair* for the function f if:

(1) $\mathsf{v} \in \mathcal{E}'[a,b]$, $\mathsf{V} \in \mathcal{D}'(\mathbb{R})$, and

$$\mathsf{V}' = \mathsf{v}. \tag{4.4}$$

(2) V is a Łojasiewicz distribution with $\mathsf{V}(a) = 0$.

(3) There exists a set E with $|E| \le \aleph_0$ and a set Z of null Lebesgue measure, $m(Z)=0$, such that for all $x \in [a,b] \setminus Z$ and all $\phi \in \mathcal{T}_0$ we have

$$(\mathsf{v})^+_{\phi,0}(x) \le f(x), \tag{4.5}$$

while for $x \in [a,b] \setminus E$ and all $\phi \in \mathcal{T}_1$,

$$(\mathsf{v})^+_{\phi,0}(x) < \infty. \tag{4.6}$$

Naturally, we may always assume in Definitions 4.1 and 4.2 that the countable set satisfies $E \subset Z$.

Employing the results of Theorem 3.7, we immediately obtain the following useful result.

LEMMA 4.3. *If* (u,U) *is a major distributional pair and* (v,V) *is a minor distributional pair for* f, *then* $\mathsf{u}-\mathsf{v}$ *is a positive measure and* $U-V$ *is a continuous increasing function, where* $U \leftrightarrow \mathsf{U}$ *and* $V \leftrightarrow \mathsf{V}$.

If (u,U) is a major distributional pair and (v,V) is a minor distributional pair for f, then U and V are constant in the interval $[b,\infty)$, and $\mathsf{V}(b) \le \mathsf{U}(b)$.

DEFINITION 4.4. A function $f : [a,b] \to \overline{\mathbb{R}}$ is called *distributionally integrable* if it has both major and minor distributional pairs and if

$$\sup_{(\mathsf{v},\mathsf{V}) \text{ minor pair}} \mathsf{V}(b) = \inf_{(\mathsf{u},\mathsf{U}) \text{ major pair}} \mathsf{U}(b). \tag{4.7}$$

When this is the case, this common value is the *integral* of f over $[a,b]$ and is denoted as

$$(\mathfrak{dist}) \int_a^b f(x)\,\mathrm{d}x, \tag{4.8}$$

or just as $\int_a^b f(x)\,\mathrm{d}x$ if there is no risk of confusion.

We shall show in Section 6 that any Lebesgue integrable function and, more generally, any Denjoy–Perron–Henstock–Kurzweil integrable function is distributionally integrable, and the integrals are the same. Therefore the symbol $\int_a^b f(x)\,\mathrm{d}x$ will have only one possible meaning if the function f is Denjoy–Perron–Henstock–Kurzweil integrable or Lebesgue integrable. In some cases we shall, however, use the notation $(\mathfrak{dist}) \int_a^b f(x)\,\mathrm{d}x$ to emphasize that we are dealing with the integral defined in this article. Occasionally, we shall also use the notation $(\mathfrak{DPHK}) \int_a^b f(x)\,\mathrm{d}x$ for a Denjoy–Perron–Henstock–Kurzweil integral and $(\mathfrak{Leb}) \int_a^b f(x)\,\mathrm{d}x$ for a Lebesgue integral.

Observe that the function f is distributionally integrable over $[a,b]$ if and only if for each $\varepsilon>0$ there are minor and major pairs, (v,V) and (u,U), such that

$$\mathsf{U}(b)-\mathsf{V}(b)<\varepsilon. \tag{4.9}$$

We shall first show that the distributional integral has the standard properties of an integral.

Proposition 4.5. *If f is distributionally integrable over $[a,b]$ then it is distributionally integrable over any subinterval $[c,d]\subset[a,b]$.*

Proof. Let $\varepsilon>0$, and choose minor and major pairs for f over $[a,b]$, (v,V) and (u,U), such that $\mathsf{U}(b)-\mathsf{V}(b)<\varepsilon$. Let $U\leftrightarrow\mathsf{U}$ and $V\leftrightarrow\mathsf{V}$. Let now $\widetilde{\mathsf{U}}$ and $\widetilde{\mathsf{V}}$ be the Łojasiewicz distributions corresponding to the Łojasiewicz functions $\widetilde{U}$ and $\widetilde{V}$ given by

$$\widetilde{U}(x)=\begin{cases}0, & x<c,\\ U(x)-U(c), & c\le x\le d,\\ U(d)-U(c), & x>d,\end{cases} \tag{4.10}$$

$$\widetilde{V}(x)=\begin{cases}0, & x<c,\\ V(x)-V(c), & c\le x\le d,\\ V(d)-V(c), & x>d.\end{cases} \tag{4.11}$$

Then $(\widetilde{\mathsf{V}}',\widetilde{\mathsf{V}})$ and $(\widetilde{\mathsf{U}}',\widetilde{\mathsf{U}})$ are minor and major pairs for f over $[c,d]$, and $\widetilde{\mathsf{U}}(d)-\widetilde{\mathsf{V}}(d)<\varepsilon$. ∎

We now consider the integrals of functions that are equal almost everywhere. As is the case with other integrals, the integral can actually be defined as a functional on the space of equivalence classes of functions equal a.e., and each class has elements that are finite everywhere.

Proposition 4.6. *If f is distributionally integrable over $[a,b]$ then it is finite almost everywhere.*

Proof. Let A be the set of points where $|f(x)|=\infty$. Let (v,V) and (u,U) be minor and major pairs for f over $[a,b]$, and let E be the denumerable set outside of which $(\mathsf{u})^-_{\phi,0}(x)>-\infty$ and $(\mathsf{v})^+_{\phi,0}(x)<\infty$ for all $\phi\in\mathcal{T}_1$. Consider the increasing continuous function $\rho(x)=\mathsf{U}(x)-\mathsf{V}(x)$. Using (4.3) and (4.6) we deduce that if $x\in A\setminus E$ then $\rho'(x)=\infty$; but the set of points where the derivative of an increasing continuous function is infinite has measure 0. ∎

The ensuing result allows us to consider distributional integration of functions that are defined almost everywhere.

Proposition 4.7. *If f is distributionally integrable over $[a,b]$ and $g(x)=f(x)$ a.e. then g is also distributionally integrable over $[a,b]$ and*

$$\int_a^b g(x)\,\mathrm{d}x=\int_a^b f(x)\,\mathrm{d}x. \tag{4.12}$$

Proof. Indeed, any major or minor pair for f is also a major or a minor pair for g, and conversely. ∎

The integral has the expected linear properties.

PROPOSITION 4.8. *If f_1 and f_2 are distributionally integrable over $[a,b]$ then so is f_1+f_2 and*

$$\int_a^b (f_1(x)+f_2(x))\,\mathrm{d}x = \int_a^b f_1(x)\,\mathrm{d}x + \int_a^b f_2(x)\,\mathrm{d}x. \tag{4.13}$$

Proof. From Propositions 4.6 and 4.7 it follows that we may assume that both f_1 and f_2 are finite everywhere, so that their sum is also defined everywhere. Then we just observe that the sum of major pairs for f_1 and f_2 is a major pair for $f_1 + f_2$, and similarly for the sum of minor pairs. ∎

PROPOSITION 4.9. *If f is distributionally integrable over $[a,b]$ then so is kf for any constant k and*

$$\int_a^b kf(x)\,\mathrm{d}x = k\int_a^b f(x)\,\mathrm{d}x. \tag{4.14}$$

Proof. This is a consequence of the following observations. If $k > 0$ then multiplying a major pair for f with k gives a major pair for kf, and similarly for minor pairs. If $k < 0$ then multiplication with k transforms major pairs for f into minor pairs for kf and minor pairs for f into major pairs for kf. ∎

It follows from the previous results that the set of distributionally integrable functions over $[a,b]$ is a linear space and that the integral is a linear functional.

We also have the following easy result.

PROPOSITION 4.10. *Suppose $a < c < b$. A function f defined in $[a,b]$ is distributionally integrable there if and only if it is distributionally integrable over $[a,c]$ and $[c,b]$, and when this is the case,*

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x. \tag{4.15}$$

If $A \subset [a,b]$ then we say that f is *distributionally integrable over A* if $\chi_A f$, where χ_A is the characteristic function of A, is distributionally integrable, and we use the notation

$$(\mathfrak{Dist})\int_A f(x)\,\mathrm{d}x. \tag{4.16}$$

As with any non-absolute integral, f will not be integrable over all measurable subsets of $[a,b]$, but if A has measure 0 then the distributional integral exists and equals 0. Also, according to Proposition 4.5, if f is distributionally integrable over $[a,b]$ then it is integrable over any of its subintervals.

5. The indefinite integral

We shall now study the indefinite integral function

$$F(x) = (\mathfrak{Dist})\int_a^x f(t)\,\mathrm{d}t \tag{5.1}$$

of a function f that is distributionally integrable over $[a,b]$. We are interested in the case when $a \le x \le b$, but sometimes it would be convenient to extend the domain of F by putting $F(x) = 0$ for $x < a$ and $F(x) = F(b)$ for $x > b$.

The indefinite integral of a Lebesgue integrable function is absolutely continuous, while that of a Denjoy–Perron–Henstock–Kurzweil integrable function is continuous. We shall show that (5.1) defines a Łojasiewicz function, with associated Łojasiewicz distribution F, $F \leftrightarrow \mathsf{F}$. We shall also show that the derivative $\mathsf{f} = \mathsf{F}'$ is a distribution that has Łojasiewicz distributional point values almost everywhere and actually $\mathsf{f}(x) = f(x)$ a.e.

We start with some useful results.

Lemma 5.1. *Let* (v,V) *and* (u,U) *be minor and major pairs for a distributionally integrable function* f *over* $[a,b]$. *Let* $U \leftrightarrow \mathsf{U}$ *and* $V \leftrightarrow \mathsf{V}$. *Then* $U - F$ *and* $F - V$ *are both continuous increasing functions that vanish at* $x = a$.

Proof. Observe that if $a \le c < d \le b$ then (4.10) gives a major pair $(\widetilde{\mathsf{U}}', \widetilde{\mathsf{U}})$ for f over $[c,d]$ with $\widetilde{U}(t) = U(t) - U(c)$ for $c \le x \le d$. Thus

$$F(d) - F(c) = \int_c^d f(x)\,\mathrm{d}x \le \widetilde{U}(d) = U(d) - U(c),$$

and so

$$U(c) - F(c) \le U(d) - F(d). \tag{5.2}$$

Similarly one shows that $F - V$ is increasing.

Observe now that $U - V = (U - F) + (F - V)$ is a continuous increasing function (Lemma 4.3) written as the sum of two increasing functions: we conclude that both $U - F$ and $F - V$ are continuous. ∎

Using the lemma we see that $F = (F - V) + V$ is the sum of a continuous function and a Łojasiewicz function and thus it is a Łojasiewicz function.

Theorem 5.2. *Let* f *be a distributionally integrable function over* $[a,b]$, *with indefinite integral* F. *Then* F *is a Łojasiewicz function.*

Observe that one may consider f as an equivalence class of functions defined almost everywhere, and thus the value $f(x)$ for a particular x may or may not have a useful meaning. However, F is a Łojasiewicz function, and this implies that the value $F(x)$ has a clear interpretation for *all* numbers x.

Since F is a Łojasiewicz function, it has an associated Łojasiewicz distribution F. The distributional derivative $\mathsf{f} = \mathsf{F}'$ is a well defined distribution with $\operatorname{supp} \mathsf{f} \subset [a,b]$. The relationship between f and f is as follows.

Theorem 5.3. *Let* f *be a distributionally integrable function over* $[a,b]$, *with indefinite integral* F, *let* $F \leftrightarrow \mathsf{F}$, *and let* $\mathsf{f} = \mathsf{F}'$. *Then* f *has point values almost everywhere and*

$$\mathsf{f}(x) = f(x) \qquad a.e. \tag{5.3}$$

Proof. Let $\varepsilon, \eta > 0$. Let (u,U) be a major pair for f over $[a,b]$ with

$$U(b) - F(b) < \varepsilon\eta, \tag{5.4}$$

where $U \leftrightarrow \mathsf{U}$. Let $\rho = U - F$, an increasing continuous function.

Consider the set $A = \{x \in [a,b] : \rho'(x) \geq \varepsilon\}$. Since

$$\varepsilon m(A) \leq \int_a^b \rho'(x)\,\mathrm{d}x \leq \rho(b) < \varepsilon\eta,$$

it follows that $m(A) < \eta$, where $m(A)$ is the Lebesgue measure.

Notice now that if $x \in [a,b] \setminus (A \cup Z)$, where Z is the null set outside of which $(\mathsf{u})^-_{\phi,0}(x) \geq f(x) > -\infty$ for all $\phi \in \mathcal{T}_0$, then

$$(\mathsf{f})^-_{\phi,0}(x) = (\mathsf{u})^-_{\phi,0}(x) - \rho'(x) > f(x) - \varepsilon.$$

Hence

$$m(\{x \in [a,b] : (\mathsf{f})^-_{\phi,0}(x) \leq f(x) - \varepsilon \ \forall \phi \in \mathcal{T}_0\}) < \eta. \tag{5.5}$$

But η is arbitrary, and thus the set where $(\mathsf{f})^-_{\phi,0}(x) \leq f(x) - \varepsilon$ has measure 0, and since ε is also arbitrary we see that $(\mathsf{f})^-_{\phi,0}(x) \geq f(x)$ a.e.

Using a similar analysis involving minor pairs one likewise deduces that $(\mathsf{f})^+_{\phi,0}(x) \leq f(x)$ a.e. If we now use Lemma 3.1 then (5.3) follows. ■

The following consequence of the preceding theorem is worth mentioning.

Corollary 5.4. *If f is distributionally integrable over $[a,b]$ then it is measurable.*

Proof. Let $\phi \in \mathcal{D}(\mathbb{R})$ be a normalized test function. Then the sequence of continuous functions

$$f_n(x) = \langle \mathsf{f}(x + y/n), \phi(y) \rangle \tag{5.6}$$

converges to f almost everywhere, namely where (5.3) holds, and the measurability of f is thus obtained. ■

If we now use Theorem 5.3, combined with Lemma 5.1, we obtain more information on the nature of major and minor pairs.

Proposition 5.5. *Let (v, V) and (u, U) be minor and major pairs for a distributionally integrable function f over $[a,b]$. Then the distributional point values $\mathsf{v}(t)$ and $\mathsf{u}(t)$ exist almost everywhere in $[a,b]$. If $\widetilde{v}$ is a function given by the point values of v, that is, $\widetilde{v}(t) = \mathsf{v}(t)$ when the value exists, extended in any way to a function over $[a,b]$, then $\widetilde{v}$ is distributionally integrable over $[a,b]$. Similarly the function $\widetilde{u}(t) = \mathsf{u}(t)$, when the value exists, is distributionally integrable over $[a,b]$. Furthermore,*

$$\mathsf{V}(d) - \mathsf{V}(c) \leq \int_c^d \widetilde{v}(x)\,\mathrm{d}x \leq \int_c^d f(x)\,\mathrm{d}x, \tag{5.7}$$

$$\int_c^d f(x)\,\mathrm{d}x \leq \int_c^d \widetilde{u}(x)\,\mathrm{d}x \leq \mathsf{U}(d) - \mathsf{U}(c). \tag{5.8}$$

Proof. Let $U \leftrightarrow \mathsf{U}$, $V \leftrightarrow \mathsf{V}$, and $F \leftrightarrow \mathsf{F}$. Since $F - V$ is an increasing continuous function, it follows that $(\mathsf{F} - \mathsf{V})' = \mathsf{f} - \mathsf{v}$ is a positive measure, and thus it has distributional values almost everywhere, and since f has a.e. distributional values (equal to f), it follows that likewise v has distributional values a.e. The function $\tilde{v}$ is distributionally integrable because $\tilde{v}(t) = f(t) - h(t)$ a.e., where h is the Lebesgue integrable function which corresponds to the absolutely continuous part of $\mathsf{f} - \mathsf{v}$ (see Theorem 6.1 below). The inequality

(5.7) is obtained from the fact that

$$0 \leq \int_c^d (f(x) - \widetilde{v}(x))\,\mathrm{d}x \leq (\mathsf{F}(d) - \mathsf{V}(d)) - (\mathsf{F}(c) - \mathsf{V}(c)). \tag{5.9}$$

The results for the major pair are obtained in a similar fashion. ■

This proposition suggests an alternative approach to the distributional integral. Call a pair (u, U) a major pair v.2 (*version* 2) if it satisfies all the conditions of Definition 4.1 plus the extra requirement that $\mathsf{u}(x)$ exists almost everywhere in $[a, b]$. Define, analogously, minor pairs v.2 and an integral in terms of major and minor pairs v.2. Then this integral would be identical to the distributional integral we have been considering, because any major or minor pair in the original sense is actually a pair in the v.2 sense. However, the use of the v.2 definition allows one to obtain some proofs, as those of Theorems 5.2 and 5.3, in a rather simple way.

Proposition 5.5 also has the following consequence on the major and minor distributional pairs.

Corollary 5.6. *Let* (v, V) *and* (u, U) *be minor and major pairs for a distributionally integrable function* f *over* $[a, b]$. *Then there exists a set of null Lebesgue measure* Z *such that for all* $x \in [a, b] \setminus Z$, *all* $\phi \in \mathcal{T}_0$, *and all angles* θ *we have*

$$(\mathsf{u})^-_{\phi,\theta}(x) \geq f(x), \tag{5.10}$$

$$(\mathsf{v})^+_{\phi,\theta}(x) \leq f(x). \tag{5.11}$$

Proof. Let Z be the complement in $[a, b]$ of the set on which the distributional point values of u, v, and f exist. Then Z has null Lebesgue measure and (5.10) and (5.11) are both valid on $[a, b] \setminus Z$. ■

Corollary 5.6 implicitly suggests a third variant yet for the definition of the distributional integral. Let us say that (u, U) is a major pair v.3 (*version* 3) if it satisfies the conditions of Definition 4.1 and additionally we replace the radial condition (4.3) by the stronger requirement (5.10), assumed to hold for all $x \in [a, b] \setminus Z$, $m(Z) = 0$, all $\phi \in \mathcal{T}_0$, and all angles θ. Likewise, one defines minor pairs v.3. If we define an integral in terms of major and minor pairs v.3, then we obtain nothing new, because in view of Corollary 5.6 this integral coincides with the distributional integral defined in Section 4.

6. Comparison with other integrals

We shall now consider the relationship of the distributional integral to the Lebesgue integral, to the Denjoy–Perron–Henstock–Kurzweil integral, and to the Łojasiewicz method (see (2.22)). We also give a constructive solution to Denjoy's problem on the reconstruction of functions from their higher order differential quotients [9].

Let us start with Lebesgue integration.

Theorem 6.1. *Any Lebesgue integrable function over* $[a, b]$ *is also distributionally integrable over* $[a, b]$ *and the integrals coincide.*

Proof. Let $\varepsilon > 0$. If f is a Lebesgue integrable function over $[a, b]$, we can apply the Vitali–Carathéodory theorem [39, III, (7.6)] to find a lower semicontinuous function u with $u(x) \geq f(x)$ for all x, and with

$$(\mathfrak{Leb}) \int_a^b (u(x) - f(x))\,\mathrm{d}x < \frac{\varepsilon}{2}. \tag{6.1}$$

If $U(x) = \int_a^x u(x)\,\mathrm{d}x$, then the pair $(\mathsf{U}', \mathsf{U})$, where $U \leftrightarrow \mathsf{U}$, is a distributional major pair for f with

$$U(b) < (\mathfrak{Leb}) \int_a^b f(x)\,\mathrm{d}x + \frac{\varepsilon}{2}. \tag{6.2}$$

Similarly, employing minor functions and upper semicontinuous functions, we can find a minor distributional major pair $(\mathsf{V}', \mathsf{V})$ for f with

$$V(b) > (\mathfrak{Leb}) \int_a^b f(x)\,\mathrm{d}x - \frac{\varepsilon}{2}. \tag{6.3}$$

The distributional integrability of f and the fact that

$$(\mathfrak{dist}) \int_a^b f(x)\,\mathrm{d}x = (\mathfrak{Leb}) \int_a^b f(x)\,\mathrm{d}x \tag{6.4}$$

then follow. ■

The Perron method of integration uses major and minor *functions* [26, 34, 39]. We shall show that these functions give major and minor distributional pairs in a natural way.

THEOREM 6.2. *Any Denjoy–Perron–Henstock–Kurzweil integrable function over* $[a, b]$ *is also distributionally integrable over* $[a, b]$ *and the integrals coincide.*

Proof. Let U be a continuous major function for a Denjoy–Perron–Henstock–Kurzweil integrable function f over $[a, b]$. Then the pair $(\mathsf{U}', \mathsf{U})$, where $U \leftrightarrow \mathsf{U}$, is a distributional major pair for f. Indeed, the derivative $U'(x)$ exists a.e. in $[a, b]$, and at those points the distributional value $\mathsf{U}'(x)$ exists, and thus $(\mathsf{U}')^-_{\phi,0}(x) = \mathsf{U}'(x) = U'(x) \geq f(x)$ for all $\phi \in \mathcal{T}_0$.

Furthermore, for any $x \in [a, b]$,

$$\liminf_{y \to x} \frac{U(y) - U(x)}{y - x} > -\infty. \tag{6.5}$$

But if $(y - x)^{-1}(U(y) - U(x)) \geq M$ for $|x - y| < c$, then we can write $\mathsf{U} = \mathsf{U}_1 + \mathsf{U}_2$, where $\mathsf{U}_1(y) = \chi_{(x-c,x+c)}(y)\mathsf{U}_1(y)$. Let $\phi \in \mathcal{T}_1$. Since $\mathsf{U}_2(y) = 0$ in a neighborhood of $y = x$, it follows that $\langle \mathsf{U}_2'(x + \varepsilon y), \phi(y)\rangle \to 0$. Also,

$$\begin{aligned}\liminf_{\varepsilon \to 0^+} \langle \mathsf{U}_1'(x + \varepsilon y), \phi(y)\rangle &= -\liminf_{\varepsilon \to 0^+} \frac{1}{\varepsilon}\langle \mathsf{U}_1(x + \varepsilon y) - \mathsf{U}_1(x), \phi'(y)\rangle \\ &\geq -M \int_{-\infty}^{\infty} y\phi'(y)\,\mathrm{d}y = M.\end{aligned}$$

Hence $(\mathsf{U}')^-_{\phi,0}(x) \geq M > -\infty$.

Similarly, if V is a continuous minor function for f, then $(\mathsf{V}', \mathsf{V})$, where $V \leftrightarrow \mathsf{V}$, is a distributional minor pair for f. The fact that f is distributionally integrable over $[a, b]$

and that the distributional and the Denjoy–Perron–Henstock–Kurzweil integrals coincide is now clear. ∎

On the other hand, one does not need to go beyond the Lebesgue integral when considering positive functions.

THEOREM 6.3. *Let f be distributionally integrable over $[a,b]$. If $f(x)\ge 0$ for all $x\in[a,b]$ then f is Lebesgue integrable over $[a,b]$.*

Suppose that f_1 and f_2 are distributionally integrable over $[a,b]$ and $f_1(x)\ge f_2(x)$ for all $x\in[a,b]$. Then f_1 is Lebesgue integrable over $[a,b]$ if and only if f_2 is Lebesgue integrable over $[a,b]$. Similarly, f_1 is Denjoy–Perron–Henstock–Kurzweil integrable over $[a,b]$ if and only if f_2 is Denjoy–Perron–Henstock–Kurzweil integrable over $[a,b]$.

Proof. Let (u,U) be a major pair for f. Then because $f(x)\ge 0$ for all x it follows that $(\mathsf{0},\mathsf{0})$ is a minor pair for f. Therefore, u is a positive measure, the point values $\widetilde{u}(x)=\mathsf{u}(x)$ exist almost everywhere, and satisfy $\widetilde{u}(x)\ge f(x)$ almost everywhere. Since (5.8) yields $\int_a^b \widetilde{u}(x)\,\mathrm{d}x<\infty$ we deduce by comparison that f is Lebesgue integrable over $[a,b]$.

The second part follows by writing $f_1=f_2+(f_1-f_2)$ and observing that f_1-f_2 is Lebesgue integrable over $[a,b]$. ∎

From this theorem it follows that if f is distributionally integrable over $[a,b]$ and $f(x)\ge 0$ for all $x\in[a,b]$ then actually the distribution $\mathsf{f}=\mathsf{F}'$ is a positive measure (indeed, a regular distribution).

We also have the following results, natural for non-absolute integrals.

THEOREM 6.4. *If f is distributionally integrable over any measurable subset of $[a,b]$ then f is Lebesgue integrable over $[a,b]$.*

If f is measurable and $|f|$ is distributionally integrable over $[a,b]$ then f is Lebesgue integrable over $[a,b]$.

Proof. Indeed, in the first case both $f_+=f\chi_{A^+}=(f+|f|)/2$ and $f_-=f\chi_{A^-}=(|f|-f)/2$, where $A^\pm$ is the set where $\pm f(t)>0$, are distributionally integrable, and since they are positive they must be Lebesgue integrable. Then $f=f_+-f_-$ would be Lebesgue integrable.

If now $|f|$ is distributionally integrable over $[a,b]$ then it is Lebesgue integrable over $[a,b]$. Since $0\le f_\pm\le |f|$, it follows that both $f_\pm$ are Lebesgue integrable, and so is $f=f_+-f_-$. ∎

Our next task is to consider Łojasiewicz functions.

THEOREM 6.5. *Let G be a Łojasiewicz function, with associated distribution G. Let $\mathsf{g}=\mathsf{G}'$, and suppose the distributional point values*

$$g(x)=\mathsf{g}(x) \tag{6.6}$$

exist for all $x\in[a,b]\setminus E$, where $|E|\le\aleph_0$. Then g is distributionally integrable over $[a,b]$ and

$$(\mathfrak{dist})\int_c^d g(x)\,\mathrm{d}x=G(d)-G(c) \tag{6.7}$$

for $[c,d]\subset[a,b]$.

Proof. Let H be the Łojasiewicz distribution equal to 0 in $(-\infty, a)$, equal to $\mathsf{G} - \mathsf{G}(a)$ in (a, b), and equal to the constant $\mathsf{G}(b)-\mathsf{G}(a)$ in (b, ∞). Then the pair $(\mathsf{H}', \mathsf{H})$ is both a major and a minor pair for g. ■

This theorem applies to Łojasiewicz functions, and more generally, to distributionally regulated functions.

THEOREM 6.6. *Any Łojasiewicz function is distributionally integrable. Any distributionally regulated function is distributionally integrable.*

Observe that the integral of a Łojasiewicz function, obtained from Definition 4.4, is equal to (2.22), the definition given in [30]. Similarly the integral of a distributionally regulated function reduces to the definition (2.29).

We can generalize Theorem 6.5 by considering a Łojasiewicz distribution whose derivative has values almost everywhere if we assume distributional boundedness at the other points.

THEOREM 6.7. *Let G be a Łojasiewicz function, with associated distribution* G. *Let* $\mathsf{g} = \mathsf{G}'$, *and suppose the distributional point values*

$$g(x) = \mathsf{g}(x) \tag{6.8}$$

exist almost everywhere in $[a, b]$, while g *is distributionally bounded at all $x \in [a, b] \setminus E$, where $|E| \le \aleph_0$. Then g is distributionally integrable over $[a, b]$ and*

$$(\mathfrak{Dist}) \int_c^d g(x)\,\mathrm{d}x = G(d) - G(c) \tag{6.9}$$

for $[c, d] \subseteq [a, b]$.

Proof. Let g be an extension of the function $\mathsf{g}(t)$, defined in the set of full measure where the values exist, to $[a, b]$. If, as before, H is the Łojasiewicz distribution equal to 0 in $(-\infty, a)$, equal to $\mathsf{G} - \mathsf{G}(a)$ in (a, b), and equal to the constant $\mathsf{G}(b)-\mathsf{G}(a)$ in (b, ∞), then the pair $(\mathsf{H}', \mathsf{H})$ is both a major and a minor pair for g because of the distributional boundedness of g on $[a, b] \setminus E$. ■

We now apply the ideas of this section to reconstruct functions from their higher order *Peano generalized derivatives.* Let f be continuous on $[a, b]$. We say that f has a *Peano nth derivative at $x \in (a, b)$* if there are n numbers $f_1(x), \ldots, f_n(x)$ such that

$$f(x + h) = f(x) + f_1(x)h + \cdots + f_n(x)\frac{h^n}{n!} + o(h^n) \quad \text{as } h \to 0. \tag{6.10}$$

We call each $f_j(x)$ its *Peano jth derivative at x.* The same notion makes sense at $x = a$ or $x = b$ if we only ask (6.10) to hold as $h \to 0^+$ or $h \to 0^-$, respectively. Notice that the ordinary first order derivative of f must exist at x, and actually $f'(x) = f_1(x)$. We set $f_0(x) = f(x)$.

Suppose that (6.10) holds everywhere in $[a, b]$. Naturally, the everywhere existence of the Peano nth derivative does not even imply that f is C^1. On the other hand, if the distribution f corresponds to f, where f has been extended to $\mathbb{R}$ as $f(x) = \sum_{j=0}^{n}(f_j(a)/j!)(x - a)^j$ for $x \le a$ and $f(x) = \sum_{j=0}^{n}(f_j(b)/j!)(x - b)^j$ for $b \le x$, then the

$\mathsf{f}^{(j)}$ are Łojasiewicz distributions for all $0 \le j \le n$ and, indeed, $\mathsf{f}^{(j)}(x) = f_j(x)$ for all $x \in [a,b]$. Thus, the functions f_j are distributionally integrable over $[a,b]$ and

$$f_{j-1}(x) = f_{j-1}(a) + (\mathfrak{dist}) \int_a^x f_j(x)\,\mathrm{d}x, \qquad j = n, \ldots, 1. \tag{6.11}$$

The relations (6.11) allow us to reconstruct f from f_n as an n-times iterated integral. Furthermore, we obtain the ensuing stronger result if we employ Theorem 6.7.

THEOREM 6.8. *Let f be continuous on $[a,b]$. Suppose that f has the Peano $(n-1)$th derivative at every point of $[a,b]$. Furthermore, assume that there is a denumerable set E such that for all $x \in [a,b] \setminus E$,*

$$f(x+h) = f(x) + f_1(x)h + \cdots + f_{n-1}(x)\frac{h^{n-1}}{(n-1)!} + O(h^n) \qquad \text{as } h \to 0. \tag{6.12}$$

If the Peano nth derivative $f_n(x)$ of f exists almost everywhere in $[a,b]$, then f_n is distributionally integrable over $[a,b]$ and

$$f(x) = \sum_{j=0}^{n-1} \frac{f_j(a)}{j!}(x-a)^j + (\mathfrak{dist}) \int_a^x \int_a^{t_{n-1}} \cdots \int_a^{t_2} f_n(t_1)\,\mathrm{d}t_1 \cdots \mathrm{d}t_{n-1}\,\mathrm{d}t_n. \tag{6.13}$$

In particular, if $f_n(x) = 0$ a.e., then f is a polynomial of degree at most $n-1$.

REMARK 6.9. The last integration step in (6.13) may be made with the Denjoy–Perron–Henstock–Kurzweil integral; however, in general, the previous integrals do not have to exist in the sense of Denjoy–Perron–Henstock–Kurzweil.

7. Distributions and integration

If f is a distributionally integrable function over $[a,b]$, with indefinite integral F, which in turn has an associated distribution F, $F \leftrightarrow \mathsf{F}$, then we proved in Theorem 5.3 that the distribution $\mathsf{f} = \mathsf{F}'$ has distributional point values almost everywhere and actually $\mathsf{f}(x) = f(x)$ almost everywhere in $[a,b]$. Our aim is to show that the association $f \leftrightarrow \mathsf{f}$ is a *natural* one, in the same way that Lebesgue integrable functions are associated to regular distributions, by showing that

$$\langle \mathsf{f}, \psi \rangle = (\mathfrak{dist}) \int_a^b f(x)\psi(x)\,\mathrm{d}x \tag{7.1}$$

for all test functions $\psi \in \mathcal{D}(\mathbb{R})$.

Observe, first of all, that if a distribution f has point values *everywhere* then there is a well defined association between f, the function given by those values, and f. That (7.1) is satisfied in this case was proved by Łojasiewicz [30]. However, when we extend this idea to values that exist almost everywhere, we need to proceed with care. For instance, the Dirac delta function $\delta(x)$ has distributional values almost everywhere equal to 0, but is not the zero distribution, or the distributional derivative of the Cantor function is a measure concentrated on the Cantor set, and thus it has values a.e. equal to 0 without being the null distribution. *Unless* (7.1) *is satisfied one cannot associate a distribution to the function given by its point values, even if those values exist almost everywhere.*

Actually the almost everywhere values of a distribution tell us very little about the nature of the distribution.

THEOREM 7.1. *Let f be any finite measurable function defined in $[a, b]$. Then there are infinitely many distributions* g *that have distributional values almost everywhere and satisfy*

$$\mathsf{g}(x) = f(x) \qquad a.e. \tag{7.2}$$

Proof. Existence follows at once from Luzin's theorem [39, Section VII.2], which says that if f is any finite measurable function defined in $[a, b]$ then there exists a continuous function F such that $F'(x) = f(x)$ almost everywhere. We then consider the distribution $\mathsf{g}_0 = \mathsf{F}'$, where $F \leftrightarrow \mathsf{F}$. If g_1 is any distribution whose support has measure 0 then $\mathsf{g} = \mathsf{g}_0 + \mathsf{g}_1$ satisfies (7.2). ■

We now proceed to the proof of the formula (7.1) when f is distributionally integrable over $[a, b]$. First we prove that $f\psi$ is distributionally integrable whenever ψ is C^∞ on $[a, b]$. Since any smooth function defined in $[a, b]$ can be extended to the whole real line, and since the integral of $f\psi$ over $[a, b]$ does not depend on how we do the extension, it is convenient to assume that ψ is actually smooth in $\mathbb{R}$.

THEOREM 7.2. *Let f be distributionally integrable over $[a, b]$ and let ψ be any smooth function defined in $\mathbb{R}$. Then $f\psi$ is distributionally integrable over $[a, b]$ and*

$$(\mathfrak{dist}) \int_a^b f(x)\psi(x)\,\mathrm{d}x = F(b)\psi(b) - (\mathfrak{dist}) \int_a^b F(x)\psi'(x)\,\mathrm{d}x, \tag{7.3}$$

where F is the indefinite integral $F(x) = (\mathfrak{dist}) \int_a^x f(t)\,\mathrm{d}t$.

Proof. Observe, first, that F is a Łojasiewicz function, and hence so is $F\psi'$. Thus $F\psi'$ is distributionally integrable over $[a, b]$ and consequently the right side of the equation (7.3) is well defined.

We start with the case when $\psi(x) > 0$ for all $x \in [a, b]$. Let (u, U) be major pair for f in $[a, b]$. Let H_+ be the Łojasiewicz distribution that satisfies $\mathsf{H}'_+ = \psi'\mathsf{U}$ in (a, b), equal to 0 in $(-\infty, a)$, and constant in (b, ∞). Then the pair $(\psi\mathsf{u}, \psi\mathsf{U} - \mathsf{H}_+)$ is a distributional major pair for ψf. Indeed, conditions (1) and (2) of Definition 4.1 are clear, while for (3) we observe for a fixed x

$$\mathsf{U}(x + ty) = \mathsf{U}(x) + o(1) \quad \text{as } t \to 0^+ \quad \text{in } \mathcal{D}'(\mathbb{R}), \tag{7.4}$$

and thus

$$\mathsf{u}(x + ty) = o(1/t) \quad \text{as } t \to 0^+ \quad \text{in } \mathcal{D}'(\mathbb{R}). \tag{7.5}$$

Hence

$$\begin{aligned} \psi(x + ty)\mathsf{u}(x + ty) &= (\psi(x) + O(t))\mathsf{u}(x + ty) \\ &= \psi(x)\mathsf{u}(x + ty) + o(1) \quad \text{in } \mathcal{D}'(\mathbb{R}). \end{aligned} \tag{7.6}$$

Since $\psi(x) > 0$ we deduce that if $\mathsf{u}^-_{\phi,0}(x) > -\infty$, $\phi \in \mathcal{T}_1$, then

$$(\psi\mathsf{u})^-_{\phi,0}(x) > -\infty, \tag{7.7}$$

while if $\mathsf{u}^-_{\phi,0}(x) \geq f(x)$, $\phi \in \mathcal{T}_0$, then

$$(\psi\mathsf{u})^-_{\phi,0}(x) \geq \psi(x)f(x). \tag{7.8}$$

Similarly, if (v,V) is a minor pair for f in $[a,b]$, then $(\psi\mathsf{v},\psi\mathsf{V}-\mathsf{H}_-)$ is a minor pair for ψf, where H_- is the Łojasiewicz distribution that satisfies $\mathsf{H}'_- = \psi'\mathsf{V}$ in (a,b), equal to 0 in $(-\infty,a)$, and constant in (b,∞).

Let $\varepsilon > 0$, and choose the major and minor pairs (u,U) and (v,V) in such a way that

$$\mathsf{U}(b)-\mathsf{V}(b) < \varepsilon\left(\psi(b)+\int_a^b |\psi'(x)|\,\mathrm{d}x\right)^{-1}. \tag{7.9}$$

Then the major and minor pairs $(\psi\mathsf{u},\psi\mathsf{U}-\mathsf{H}_+)$ and $(\psi\mathsf{v},\psi\mathsf{V}-\mathsf{H}_-)$ for ψf satisfy

$$(\psi(b)\mathsf{U}(b)-\mathsf{H}_+(b)) - (\psi(b)\mathsf{V}(b)-\mathsf{H}_-(b)) < \varepsilon, \tag{7.10}$$

where we have used Lemma 4.3. This proves the distributional integrability of ψf.

If we take the infimum of $\psi(b)\mathsf{U}(b)-\mathsf{H}_+(b)$, or the supremum of $\psi(b)\mathsf{V}(b)-\mathsf{H}_-(b)$, we arrive at $\psi(b)F(b)-\int_a^b \psi'(x)F(x)\,\mathrm{d}x$, and this yields the integration by parts formula (7.3).

For a general function $\psi \in C^\infty(\mathbb{R})$ we can find a constant $k>0$ such that $k+\psi(x)>0$ for all $x\in[a,b]$. The distributional integrability of $\psi f = (k+\psi)f - kf$ follows, while formula (7.3) is obtained because it holds for both $(k+\psi)f$ and kf. ∎

We can now prove that the association $f \leftrightarrow \mathsf{f}$ is a natural one.

Theorem 7.3. *Let f be distributionally integrable function over $[a,b]$, its indefinite integral be F, with associated distribution F, $F \leftrightarrow \mathsf{F}$, and let $\mathsf{f}=\mathsf{F}' \in \mathcal{E}'(\mathbb{R})$, so that $\mathsf{f}(x)=f(x)$ almost everywhere in $[a,b]$. Then for any $\psi\in\mathcal{E}(\mathbb{R})$,*

$$\langle \mathsf{f},\psi\rangle = (\mathfrak{dist})\int_a^b f(x)\psi(x)\,\mathrm{d}x. \tag{7.11}$$

Proof. Let χ be the characteristic function of $[a,b]$. Then $\chi\mathsf{F}$ is distributionally regulated, with a jump of magnitude $-F(b)$ at $x=b$. Thus

$$(\chi(x)\mathsf{F}(x))' = \mathsf{f}(x) - F(b)\delta(x-b), \tag{7.12}$$

and this yields

$$\begin{aligned}\langle \mathsf{f},\psi\rangle &= \langle (\chi\mathsf{F})' + F(b)\delta(x-b),\psi\rangle = F(b)\psi(b) - \langle \chi\mathsf{F},\psi'\rangle \\ &= F(b)\psi(b) - (\mathfrak{dist})\int_a^b F(x)\psi'(x)\,\mathrm{d}x = (\mathfrak{dist})\int_a^b f(x)\psi(x)\,\mathrm{d}x,\end{aligned}$$

as required. ∎

Remark 7.4. Theorem 7.2 actually shows that the integration by parts formula

$$(\mathfrak{dist})\int_a^b f(x)\psi(x)\,\mathrm{d}x = G(x)\psi(x)\,\Big|_{x=a}^{x=b} - (\mathfrak{dist})\int_a^b G(x)\psi'(x)\,\mathrm{d}x \tag{7.13}$$

holds for any Łojasiewicz function G, $G\leftrightarrow\mathsf{G}$, with $\mathsf{f}=\mathsf{G}'$, and ψ smooth.

We say that a function f defined in $\mathbb{R}$ is *locally distributionally integrable* if f is integrable over any compact *interval* of $\mathbb{R}$. For such a function we define the improper integral

$$(\mathfrak{dist})\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \lim_{\substack{a\to-\infty\\ b\to\infty}} (\mathfrak{dist})\int_{a}^{b} f(x)\,\mathrm{d}x, \tag{7.14}$$

if the limit exists.

The previous theorem treats the case of $\mathcal{E}'(\mathbb{R})$; for $\mathcal{D}'(\mathbb{R})$ we have a corresponding result.

THEOREM 7.5. *Let f be locally distributionally integrable over $\mathbb{R}$. Then the formula*

$$\psi \rightsquigarrow (\mathfrak{dist})\int_{-\infty}^{\infty} f(x)\psi(x)\,\mathrm{d}x, \tag{7.15}$$

for $\psi \in \mathcal{D}(\mathbb{R})$, defines a distribution $\mathsf{f} \in \mathcal{D}'(\mathbb{R})$. Actually for any fixed $a \in \mathbb{R}$, we have $\mathsf{f} = \mathsf{F}'$, where $F \leftrightarrow \mathsf{F}$, and $F(x) = (\mathfrak{dist})\int_a^x f(t)\,\mathrm{d}t$.

Furthermore, $\mathsf{f}(x) = f(x)$ almost everywhere in $\mathbb{R}$.

Proof. This follows at once from Theorem 7.3. Observe that for any ψ the integral in (7.15) is not really an improper integral, but actually an integral over a compact interval. ∎

We can now extend the notion of association between a function and a distribution. From our results, we can associate to any locally distributionally integrable function over $\mathbb{R}$ a unique distribution. We shall call those distributions *locally integrable distributions.* The association

$$f \leftrightarrow \mathsf{f} \tag{7.16}$$

between locally distributionally integrable functions and locally integrable distributions is characterized by the equation

$$\langle \mathsf{f}, \psi\rangle = (\mathfrak{dist})\int_{-\infty}^{\infty} f(x)\psi(x)\,\mathrm{d}x, \qquad \psi \in \mathcal{D}(\mathbb{R}). \tag{7.17}$$

This association generalizes the association between locally Lebesgue integrable functions and regular distributions as well as the association between Łojasiewicz functions and Łojasiewicz distributions.

We also have similar integral representation results for other spaces of distributions. In fact, since all evaluations in $\mathcal{S}'(\mathbb{R})$ and $\mathcal{K}'(\mathbb{R})$ are Cesàro evaluations [52] we immediately obtain the following.

THEOREM 7.6. *Let f be a locally integrable distribution, $f \leftrightarrow \mathsf{f}$. If $\mathsf{f} \in \mathcal{S}'(\mathbb{R})$ then there exists $k \in \mathbb{N}$ such that for all $\phi \in \mathcal{S}(\mathbb{R})$,*

$$\langle \mathsf{f}, \phi\rangle = (\mathfrak{dist})\int_{-\infty}^{\infty} f(x)\phi(x)\,\mathrm{d}x \quad (\mathrm{C}, k). \tag{7.18}$$

If $\mathsf{f} \in \mathcal{K}'(\mathbb{R})$ and $\phi \in \mathcal{K}(\mathbb{R})$, then (7.18) holds for some $k \in \mathbb{N}$ that depends on ϕ.

Using the results of [13, 18] (see [22, Chap. 6]), we also obtain the ensuing useful characterization.

Theorem 7.7. *Let* f *be a locally integrable distribution,* $f \leftrightarrow \mathsf{f}$. *Then* $\mathsf{f} \in \mathcal{K}'(\mathbb{R})$ *if and only if the integrals*

$$(\mathfrak{dist}) \int_{-\infty}^{\infty} f(x) x^n \, \mathrm{d}x \quad (\mathrm{C}) \tag{7.19}$$

exist in the Cesàro sense for all $n \in \mathbb{N}$.

At this point we present a useful local bound for locally integrable distributions.

Proposition 7.8. *Let* f *be a locally integrable distribution. Then for any* $x \in \mathbb{R}$,

$$\mathsf{f}(x + \varepsilon y) = o(1/\varepsilon), \qquad \varepsilon \to 0^+, \tag{7.20}$$

in the space $\mathcal{D}'(\mathbb{R})$, *that is, if* $\psi \in \mathcal{D}(\mathbb{R})$, *then*

$$\langle \mathsf{f}(x + \varepsilon y), \psi(y) \rangle = o(1/\varepsilon), \qquad \varepsilon \to 0^+. \tag{7.21}$$

Proof. Indeed, if F is a primitive for f, $\mathsf{F}' = \mathsf{f}$, then F is a Łojasiewicz distribution, and thus the point value $\mathsf{F}(x)$ exists. Hence in $\mathcal{D}'(\mathbb{R})$,

$$\mathsf{F}(x + \varepsilon y) = \mathsf{F}(x) + o(1) \quad \text{ as } \varepsilon \to 0^+. \tag{7.22}$$

Differentiation of (7.22) yields (7.20). ■

Suppose that f is a locally integrable distribution with compact support contained in $[a, b]$. Then while (7.21) is valid at the endpoints $x = a$ and $x = b$ if $\psi \in \mathcal{D}(\mathbb{R})$, it is enough to consider the distributional limits $\mathsf{f}(a + \varepsilon y)$ as $\varepsilon \to 0^+$, only for $y > 0$, and $\mathsf{f}(b + \varepsilon y)$ as $\varepsilon \to 0^+$, only for $y < 0$. This means that if $x = a$, it suffices to require (7.21) to hold if $\psi \in \mathcal{D}(\mathbb{R})$ satisfies $\operatorname{supp}\psi \subset (0, \infty)$, or, when needed, if $\operatorname{supp}\psi \subset [0, \infty)$. Similarly at $x = b$ one just needs to consider test functions with $\operatorname{supp}\psi \subset (-\infty, 0)$ or $\operatorname{supp}\psi \subset (-\infty, 0]$.

8. Improper integrals

It is well known that if f is Lebesgue integrable over $[a, c]$ for any $c < b$, then the improper integral

$$\int_a^b f(x) \, \mathrm{d}x = \lim_{c \to b^-} \int_a^c f(x) \, \mathrm{d}x \tag{8.1}$$

may exist even when f is not Lebesgue integrable over $[a, b]$.

On the other hand, according to Hake's theorem [27] (see [3], [26], or [39]), if f is Denjoy–Perron–Henstock–Kurzweil integrable over $[a, c]$ for any $c < b$, and the improper integral (8.1) exists, then f must be Denjoy–Perron–Henstock–Kurzweil integrable over $[a, b]$. In other words, there is no such thing as improper Denjoy–Perron–Henstock–Kurzweil integrals over a finite interval.

For the distributional integral we have the following result.

Theorem 8.1. *Let* f *be distributionally integrable over* $[a, c]$ *for any* $c < b$. *Let* $F(x) = (\mathfrak{dist}) \int_a^x f(t) \, \mathrm{d}t$, $x < b$, *be its indefinite integral, and let* F *be the corresponding Łojasiewicz distribution defined for* $x < b$, $F \leftrightarrow \mathsf{F}$. *Suppose that the distributional limit*

$$\lim_{c \to b^-} \mathsf{F}(c) = L \tag{8.2}$$

exists. Then f is distributionally integrable over $[a,b]$ and

$$(\mathfrak{dist})\int_a^b f(x)\,\mathrm{d}x = L. \tag{8.3}$$

Proof. Let $\varepsilon>0$. Let $\{c_n\}_{n=1}^{\infty}$ be a strictly increasing sequence with $c_1=a$ and $c_n\nearrow b$. For each n let $(\mathsf{u}_n,\mathsf{U}_n)$ and $(\mathsf{v}_n,\mathsf{V}_n)$ be major and minor pairs for f over $[c_n,c_{n+1}]$ that satisfy $\mathsf{U}_n(c_{n+1})-\mathsf{V}_n(c_{n+1})<\varepsilon/2^n$. The two series

$$\widetilde{\mathsf{U}}=\sum_{n=1}^{\infty}\mathsf{U}_n,\qquad \widetilde{\mathsf{V}}=\sum_{n=1}^{\infty}\mathsf{V}_n \tag{8.4}$$

converge distributionally in the interval $(-\infty,b)$, since in any interval of the form $(-\infty,c)$ for $c<b$ the series become finite sums. The distributions $\widetilde{\mathsf{U}}$ and $\widetilde{\mathsf{V}}$ are Łojasiewicz distributions for $x<b$, and for each $c<b$ they yield major and minor pairs $(\mathsf{U}'_{(c)},\mathsf{U}_{(c)})$ and $(\mathsf{V}'_{(c)},\mathsf{V}_{(c)})$ for f over $[a,c]$ by taking $\mathsf{U}_{(c)}$ and $\mathsf{V}_{(c)}$ to be Łojasiewicz distributions over $\mathbb{R}$ that equal $\widetilde{\mathsf{U}}$ and $\widetilde{\mathsf{V}}$, respectively, over $(-\infty,c)$ and constant over (c,∞). Also, $\widetilde{\mathsf{U}}(c)-\widetilde{\mathsf{V}}(c)<\varepsilon$ for $c<b$.

Observe now that $\widetilde{\mathsf{U}}-\mathsf{F}$ and $\mathsf{F}-\widetilde{\mathsf{V}}$ are both Łojasiewicz distributions over $(-\infty,b)$, corresponding to continuous increasing functions. Since F has a distributional limit from the left at $x=b$, the same is true of both $\widetilde{\mathsf{U}}$ and $\widetilde{\mathsf{V}}$, and thus one can extend them as Łojasiewicz distributions over $\mathbb{R}$ by asking the extensions, say U and V, to be constant over (b,∞). Then (U',U) and (V',V) are major and minor pairs for f over $[a,b]$ with $\mathsf{U}(b)-\mathsf{V}(b)\le\varepsilon$, and the distributional integrability of f over $[a,b]$ follows. Furthermore, we also obtain the bounds

$$L-\varepsilon\le\mathsf{V}(b)\le\mathsf{U}(b)\le L+\varepsilon,$$

which immediately yield (8.3). ∎

One can rephrase the previous theorem by simply saying that the distributional integral $(\mathfrak{dist})\int_a^b f(x)\,\mathrm{d}x$ exists, and is finite, if and only if the distributional limit of $(\mathfrak{dist})\int_a^c f(x)\,\mathrm{d}x$ as $c\to b^-$ exists. We may also reformulate Theorem 8.1 if we use local Cesàro limits. Let g be distributionally integrable over $[a,c]$ for any $c<b$. Define its sequence of n-primitives $\{g_a^{(-n)}\}_{n=0}^{\infty}$ on $[a,b)$ recursively as

$$g_a^{(0)}(x)=g(x),\quad g_a^{(-n-1)}(x)=(\mathfrak{dist})\int_a^x g_a^{(-n)}(t)\,\mathrm{d}t,\qquad x\in[a,b). \tag{8.5}$$

We say that g has a *Cesàro limit* as $c\to b_-$, and write

$$\lim_{c\to b^-} g(c)=L\quad(\mathrm{C}), \tag{8.6}$$

if there exist $d\in[a,b)$, $n\in\mathbb{N}$, and a polynomial p of degree at most $n-1$ such that $g_a^{(-n)}$ is continuous on (d,b) and

$$\lim_{c\to b^-}\frac{g_a^{(-n)}(c)-p(c)}{(c-b)^n}=\frac{L}{n!}. \tag{8.7}$$

Let $g\leftrightarrow \mathsf{g}\in\mathcal{E}'[a,b]$. Because of Łojasiewicz's characterization of distributional limits [30, 47, 54], we know that (8.6) is equivalent to the distributional lateral limit $\lim_{c\to b^-}\mathsf{g}(c)$

$= L$. This yields immediately the following version of Theorem 8.1 in which we replace (8.2) by a Cesàro limit.

THEOREM 8.2. *Let f be distributionally integrable over $[a, c]$ for any $c < b$. Then f is distributionally integrable over $[a, b]$ if and only if the following Cesàro limit exists and is finite:*

$$\lim_{c \to b^-} (\mathfrak{dist}) \int_a^c f(x)\,\mathrm{d}x = L \quad (\mathrm{C}). \tag{8.8}$$

In this case (8.3) *holds.*

Theorem 8.2 therefore tells us that improper Cesàro distributional integrals are always definite integrals, and conversely, any definite integral may be computed by the Cesàro limit (8.8). Thus, we have the following analogy with the Denjoy–Perron–Henstock–Kurzweil integral: there are *no improper* Cesàro distributional integrals over finite intervals.

If the integrability of f in $[a, c]$ for all $c < b$ is known, then we may determine the integrability of f over $[a, b]$ from the behavior of f, where $f \leftrightarrow \mathsf{f}$, near $x = b$. One result in this direction is the following.

THEOREM 8.3. *Let f be distributionally integrable over $[a, c]$ for any $c < b$. Let $f \leftrightarrow \mathsf{f}$, where f is a distribution in $\mathcal{D}'(-\infty, b)$. Suppose that*

$$\mathsf{f}(b + \varepsilon x) = O(\varepsilon^{\alpha}), \qquad \varepsilon \to 0^+, \tag{8.9}$$

for some $\alpha > -1$ in the space $\mathcal{D}'((-\infty, 0))$, that is, for $\psi \in \mathcal{D}(\mathbb{R})$,

$$\langle \mathsf{f}(b + \varepsilon y), \psi(y) \rangle = O(\varepsilon^{\alpha}), \quad \varepsilon \to 0^+, \qquad \text{whenever } \operatorname{supp} \psi \subset (-\infty, 0). \tag{8.10}$$

Then f is distributionally integrable over $[a, b]$.

Proof. Let $F(x) = (\mathfrak{dist}) \int_a^x f(t)\,\mathrm{d}t$, $x < b$, be the indefinite integral of f, and let F be the corresponding Łojasiewicz distribution defined for $x < b$, $F \leftrightarrow \mathsf{F}$. We need to show that L, the distributional limit of $\mathsf{F}(c)$ as $c \to b^-$, exists, that is,

$$\lim_{\varepsilon \to 0^+} \langle \mathsf{F}(b + \varepsilon y), \psi(y) \rangle = L \int_{-\infty}^{\infty} \psi(x)\,\mathrm{d}x \tag{8.11}$$

whenever $\operatorname{supp} \psi \subset (-\infty, 0)$.

Observe first that if $\int_{-\infty}^{\infty} \psi(x)\,\mathrm{d}x = 0$ then $\psi = \varphi'$, where $\varphi \in \mathcal{D}(\mathbb{R})$, $\operatorname{supp} \varphi \subset (-\infty, 0)$. Thus

$$\langle \mathsf{F}(b + \varepsilon y), \psi(y) \rangle = \varepsilon \langle \mathsf{f}(b + \varepsilon y), \varphi(y) \rangle = O(\varepsilon^{\alpha+1})$$

as $\varepsilon \to 0^+$, so that (8.11) holds with any L if the integral of ψ vanishes.

Let ψ_0 be a fixed test function in $\mathcal{D}(\mathbb{R})$, with $\operatorname{supp} \psi_0 \subset (-\infty, 0)$, that satisfies

$$\int_{-\infty}^{\infty} \psi_0(x)\,\mathrm{d}x = 1. \tag{8.12}$$

If $\psi \in \mathcal{D}(\mathbb{R})$, $\operatorname{supp} \psi \subset (-\infty, 0)$, we can write $\psi = c\psi_0 + \psi_1$, where $c = \int_{-\infty}^{\infty} \psi(x)\,\mathrm{d}x$, and where $\int_{-\infty}^{\infty} \psi_1(x)\,\mathrm{d}x = 0$. Therefore,

$$\langle \mathsf{F}(b + \varepsilon y), \psi(y) \rangle = \rho(\varepsilon) \int_{-\infty}^{\infty} \psi(x)\,\mathrm{d}x + O(\varepsilon^{\alpha+1}) \tag{8.13}$$

as $\varepsilon \to 0^+$, where

$$\rho(\varepsilon) = \langle \mathsf{F}(b+\varepsilon y), \psi_0(y)\rangle. \tag{8.14}$$

If $a > 0$ then

$$\rho(a\varepsilon) = \rho(\varepsilon) + O(\varepsilon^{\alpha+1}), \tag{8.15}$$

since

$$\begin{aligned}\rho(a\varepsilon) &= \langle \mathsf{F}(b+a\varepsilon y), \psi_0(y)\rangle = \frac{1}{a}\langle \mathsf{F}(b+\varepsilon y), \psi_0(y/a)\rangle \\ &= \frac{\rho(\varepsilon)}{a}\int_{-\infty}^{\infty} \psi_0(x/a)\,\mathrm{d}x + O(\varepsilon^{\alpha+1}) = \rho(\varepsilon) + O(\varepsilon^{\alpha+1}).\end{aligned}$$

The asymptotic identity (8.15) is actually valid uniformly in a if $a \in [A,B]$ and $0 < A < B < \infty$ because weak convergence yields strong convergence in spaces of distributions and thus (8.10) holds uniformly on ψ if ψ belongs to a compact subset of $\mathcal{D}(\mathbb{R})$. Hence L, the limit of $\rho(\varepsilon)$ as $\varepsilon \to 0^+$, exists (and actually $\rho(\varepsilon) = L + O(\varepsilon^{\alpha+1})$). The required formula (8.11) then follows from (8.13). ■

It is interesting to observe that the condition $\mathsf{f}(b+\varepsilon x) = o(1/\varepsilon)$ as $\varepsilon \to 0^+$ is not enough to give integrability of f over $[a,b]$. Take $f(x) = ((b-x)\ln(b-x))^{-1}$, $x < b$, for instance. The preceding proof does not work because (8.15) becomes $\rho(a\varepsilon) = \rho(\varepsilon) + o(1)$, so that ρ is asymptotically homogeneous of degree 0 [22, 46, 54], and some asymptotically homogeneous functions may tend to infinity as $\varepsilon \to 0^+$.

Using Theorem 8.3 it is possible to give a clear meaning to some irregular operations involving integrable distributions. If $\mathsf{f} \in \mathcal{D}'(\mathbb{R})$ and χ is the characteristic function of an interval $[c,d]$ then in general there is no *canonical* way of defining a distribution $\chi\mathsf{f}$ in the space $\mathcal{D}'(\mathbb{R})$ [14]; however, if f is a locally integrable distribution then Proposition 4.5 says that $\chi\mathsf{f}$ is defined in a natural way. The following result gives another such natural definition, namely, that of $(b-x)^\beta\mathsf{f}(x)$ if $\mathsf{f} \in \mathcal{E}'[a,b]$ is integrable and $\beta > 0$.

Proposition 8.4. *Let f be distributionally integrable over $[a,b]$. If $\beta > 0$ then the function*

$$f_\beta(x) = (b-x)^\beta f(x)$$

is distributionally integrable over $[a,b]$. Similarly, $(x-a)^\beta f(x)$ is also distributionally integrable over $[a,b]$.

Proof. This follows at once from Theorem 8.3, since using Proposition 7.8 we see that (8.9) holds for f_β with $\alpha = \beta - 1$. ■

9. Convergence theorems

We shall now show that the usual convergence theorems, namely, the bounded convergence theorem, the monotone convergence theorem, and Fatou's lemma are valid for the distributional integral.

It is convenient to first introduce the notation for integrals that have an infinite value. If f is measurable in $[a,b]$, $f(x)\geq 0$ almost everywhere, and f is not integrable, we put

$$(\mathfrak{dist})\int_a^b f(x)\,\mathrm{d}x=\infty. \tag{9.1}$$

More generally, we use (9.1) if $f=f_1+f_2$, where f_1 is distributionally integrable and f_2 is positive a.e. but not integrable. The notation $(\mathfrak{dist})\int_a^b g(x)\,\mathrm{d}x=-\infty$ is interpreted in a corresponding fashion.

Given a measurable function f defined in $[a,b]$, there are three possibilities, namely, f may be distributionally integrable, in which case $(\mathfrak{dist})\int_a^b f(x)\,\mathrm{d}x$ is a real number, or $(\mathfrak{dist})\int_a^b f(x)\,\mathrm{d}x=\pm\infty$, or the distributional integral is undefined. This is also the case for other integrals, such as the Lebesgue integral or the Denjoy–Perron–Henstock–Kurzweil integral. If f is distributionally integrable but not Denjoy–Perron–Henstock–Kurzweil integrable then the symbol $(\mathfrak{DPHK})\int_a^b f(x)\,\mathrm{d}x$ is undefined, but even more, if f is decomposed as $f=f_1+f_2$ and the symbols $(\mathfrak{DPHK})\int_a^b f_j(x)\,\mathrm{d}x$ are defined for $j=1$ or 2, then one of them is $+\infty$ and the other is $-\infty$. Similarly, if the Lebesgue integral of f is undefined, but f is Denjoy–Perron–Henstock–Kurzweil integrable then whenever $f=f_1+f_2$ and the symbols $(\mathfrak{Leb})\int_a^b f_j(x)\,\mathrm{d}x$ are defined for $j=1$ or 2, one of them must be $+\infty$ and the other $-\infty$. In a sense, going to a more general integral means that a method to solve some indefinite forms $+\infty-\infty$ has been included in the definition of the more general and refined integral.

We now consider the following comparison results.

Proposition 9.1. *Let f and g be measurable on $[a,b]$ and suppose that $f(x)\geq g(x)$ almost everywhere. If g is distributionally integrable, then f is also distributionally integrable or $\int_a^b f(x)\,\mathrm{d}x=\infty$. Similarly, if f is distributionally integrable, then either so is g, or $\int_a^b g(x)\,\mathrm{d}x=-\infty$.*

Observe that the proposition implies that if g is distributionally integrable and $f\geq g$, then $(\mathfrak{dist})\int_a^b f(x)\,\mathrm{d}x$ will always be defined, as a number in $\mathbb{R}\cup\{\infty\}$. If f is just a measurable function, without such an inequality, however, then $(\mathfrak{dist})\int_a^b f(x)\,\mathrm{d}x$ would in general be meaningless.

Proposition 9.2. *Let f, g and h be measurable on $[a,b]$ and suppose that*

$$f(x)\geq g(x)\geq h(x)\qquad a.e. \tag{9.2}$$

Suppose f and h are distributionally integrable. Then so is g.

If, in addition, one of the functions is Lebesgue integrable, then so are the other two. Similarly if one of the functions is Denjoy–Perron–Henstock–Kurzweil integrable then the other two are as well.

Proof. If f and h are distributionally integrable, then so is $f-h$, which being positive, must be Lebesgue integrable. By comparison, $g-h$ is also Lebesgue integrable. It follows that $g=h+(g-h)$ is distributionally integrable. The second part is obtained directly from Theorem 6.3. ∎

We now give the bounded convergence theorem.

THEOREM 9.3. *Let f and h be distributionally integrable on $[a,b]$ and suppose that $\{g_n\}_{n=1}^{\infty}$ is a sequence of distributionally integrable functions that satisfies*

$$f(x) \geq g_n(x) \geq h(x) \qquad a.e. \tag{9.3}$$

If $g_n \to g$ almost everywhere then g is distributionally integrable and

$$\lim_{n\to\infty} (\mathfrak{dist}) \int_a^b g_n(x)\,\mathrm{d}x = (\mathfrak{dist}) \int_a^b g(x)\,\mathrm{d}x. \tag{9.4}$$

Proof. Observe that g also satisfies $f(x) \geq g(x) \geq h(x)$ almost everywhere, and thus the comparison result in Proposition 9.2 shows that g is distributionally integrable. Notice now that $|g(x)-g_n(x)| \leq f(x)-h(x)$ almost everywhere, $f-h$ is Lebesgue integrable, and $|g-g_n| \to 0$ almost everywhere. We conclude from the Lebesgue bounded convergence theorem that $\int_a^b |g(x)-g_n(x)|\,\mathrm{d}x \to 0$, that is, $\{g-g_n\}_{n=1}^{\infty}$ converges to 0 in $L^1[a,b]$, and in particular, that (9.4) holds. ∎

We also have a monotone convergence theorem.

THEOREM 9.4. *Let h be distributionally integrable on $[a,b]$ and let $\{g_n\}_{n=1}^{\infty}$ be a monotone sequence of measurable functions that satisfies*

$$g_{n+1}(x) \geq g_n(x) \geq h(x) \qquad a.e. \tag{9.5}$$

Let $g(x) = \lim_{n\to\infty} g_n(x)$. Then g is distributionally integrable if and only if

$$\lim_{n\to\infty} (\mathfrak{dist}) \int_a^b g_n(x)\,\mathrm{d}x < \infty, \tag{9.6}$$

and if that is the case then $(\mathfrak{dist}) \int_a^b g_n(x)\,\mathrm{d}x \to (\mathfrak{dist}) \int_a^b g(x)\,\mathrm{d}x$ as $n \to \infty$.

Proof. Suppose first that g is distributionally integrable. Then $g(x) \geq g_n(x) \geq h(x)$ almost everywhere, and from the bounded convergence theorem, Theorem 9.3, we conclude that the increasing numerical sequence $\{\int_a^b g_n(x)\,\mathrm{d}x\}_{n=1}^{\infty}$ converges to $\int_a^b g(x)\,\mathrm{d}x$, and (9.6) follows.

Conversely, if (9.6) holds, then $\{g_n - h\}_{n=1}^{\infty}$ is a Cauchy sequence in the space $L^1[a,b]$, because

$$\begin{aligned}
&\lim_{\substack{n,m\to\infty\\ n\geq m}} \int_a^b |(g_n(x)-h(x)) - (g_m(x)-h(x))|\,\mathrm{d}x \\
&\quad = \lim_{\substack{n,m\to\infty\\ n\geq m}} \int_a^b (g_n(x)-g_m(x))\,\mathrm{d}x = \lim_{\substack{n,m\to\infty\\ n\geq m}} \left(\int_a^b g_n(x)\,\mathrm{d}x - \int_a^b g_m(x)\,\mathrm{d}x\right) = 0.
\end{aligned}$$

Since $\{g_n - h\}_{n=1}^{\infty}$ converges a.e. to $g-h$, it must also converge to $g-h$ in $L^1[a,b]$; we also obtain the convergence of $\int_a^b (g_n(x)-h(x))\,\mathrm{d}x$ to $\int_a^b (g(x)-h(x))\,\mathrm{d}x$. Thus $\int_a^b (g(x)-h(x))\,\mathrm{d}x < \infty$, and so $g = (g-h)+h$ is distributionally integrable. The convergence of $\int_a^b g_n(x)\,\mathrm{d}x$ to $\int_a^b g(x)\,\mathrm{d}x$ is now clear. ∎

Observe that the monotone convergence theorem also says that if g is not distributionally integrable, so that $(\mathfrak{dist}) \int_a^b g(x)\,\mathrm{d}x = \infty$, then $(\mathfrak{dist}) \int_a^b g_n(x)\,\mathrm{d}x \nearrow \infty$.

Fatou's lemma takes the following form.

Theorem 9.5. *Let h be distributionally integrable on $[a,b]$ and let $\{g_n\}_{n=1}^{\infty}$ be a sequence of measurable functions that satisfies*

$$g_n(x) \geq h(x) \qquad a.e. \tag{9.7}$$

Suppose that

$$\liminf_{n\to\infty} (\mathfrak{dist}) \int_a^b g_n(x)\,\mathrm{d}x < \infty. \tag{9.8}$$

Then the function defined by $g_*(x) = \liminf_{n\to\infty} g_n(x)$ is distributionally integrable and

$$(\mathfrak{dist}) \int_a^b g_*(x)\,\mathrm{d}x \leq \liminf_{n\to\infty} (\mathfrak{dist}) \int_a^b g_n(x)\,\mathrm{d}x. \tag{9.9}$$

Proof. Let $h_n(x) = \inf\{g_j(x) : n \leq j < \infty\}$. Then $h \leq h_n \leq g_j$ for $n \leq j$, and since (9.8) implies that for each n there are indices j with $n \leq j$ such that g_j is distributionally integrable, it follows that h_n is distributionally integrable for all n. Notice also that the sequence $\{h_n\}$ is increasing. Since $\int_a^b h_n(x)\,\mathrm{d}x \leq \int_a^b g_n(x)\,\mathrm{d}x$ we obtain

$$\lim_{n\to\infty} \int_a^b h_n(x)\,\mathrm{d}x \leq \liminf_{n\to\infty} \int_a^b g_n(x)\,\mathrm{d}x < \infty.$$

If we now use the fact that $g_*(x) = \lim_{n\to\infty} h_n(x)$ and Theorem 9.4, we find that g_* is distributionally integrable and

$$\int_a^b g_*(x)\,\mathrm{d}x = \lim_{n\to\infty} \int_a^b h_n(x)\,\mathrm{d}x \leq \liminf_{n\to\infty} \int_a^b g_n(x)\,\mathrm{d}x,$$

as required. ∎

It is interesting to observe that if a sequence $\{\mathsf{f}_n\}_{n=1}^{\infty}$ of integrable distributions in the space $\mathcal{E}'[a,b]$ converges distributionally to f, and f is integrable, then trivially

$$\lim_{n\to\infty} (\mathfrak{dist}) \int_a^b f_n(x)\,\mathrm{d}x = (\mathfrak{dist}) \int_a^b f(x)\,\mathrm{d}x, \tag{9.10}$$

where $f \leftrightarrow \mathsf{f}$, $f_n \leftrightarrow \mathsf{f}_n$, since $\langle \mathsf{f}_n, \psi\rangle \to \langle \mathsf{f}, \psi\rangle$ for all test functions ψ. However, in general, if $\{\mathsf{f}_n\}_{n=1}^{\infty}$ converges distributionally to f, then f does not have to be integrable. Actually, $\{f_n\}_{n=1}^{\infty}$ could even converge a.e. to a function f, but f and f cannot be associated if f is not integrable; (9.10) may or may not hold in such a case. For example, if $I_n = [1/n, 2/n]$, and $f_n = n\chi_{I_n}$, then $\{\mathsf{f}_n\}_{n=1}^{\infty}$ converges distributionally to $\delta(x)$, but $\{f_n\}_{n=1}^{\infty}$ converges everywhere to $f = 0$. Naturally (9.10) does not hold if $a < 0 < b$.

10. Change of variables

We now consider changes of variables in the integral. Let us start with a function

$$\rho : [c,d] \to [a,b] \tag{10.1}$$

that is of class C^∞, even at the endpoints, and satisfies $|\rho'(t)| > 0$ for all $t \in [c,d]$. Then ρ induces an isomorphism

$$T_\rho : \mathcal{E}'[a,b] \to \mathcal{E}'[c,d], \tag{10.2}$$

given by $T_\rho\{\mathsf{f}\}(t) = \mathsf{f}(\rho(t))$ for $\mathsf{f} \in \mathcal{E}'[a,b]$. Observe that $T_\rho^{-1} = T_{\rho^{-1}}$. In this case we say that ρ is a *change of variables of type I.*

For changes of type I it is easy to see [7, 30] that the distributional point value $\mathsf{f}(x)$ exists at $x = x_0$ if and only if the distributional point value $\mathsf{f}(\rho(t))$ exists at $t = t_0$, where $x_0 = \rho(t_0)$, and when both values exist they coincide. Also, if f is a function defined in $[a,b]$, $\rho' > 0$, and (u, U) is a major distributional pair for f in $[a,b]$, then $(\mathsf{u}(\rho(t))\rho'(t), \mathsf{U}(\rho(t)))$ is a major pair for $f(\rho(t))\rho'(t)$ in $[c,d]$, and similarly for minor pairs. Thus, we immediately obtain the following result.

Proposition 10.1. *Let $\rho : [c,d] \to [a,b]$ be a change of variables of type I. A function f is distributionally integrable over $[a,b]$ if and only if $f(\rho(t))\rho'(t)$ is distributionally integrable over $[c,d]$ and if $a = \rho(c)$, $b = \rho(d)$,*

$$(\mathfrak{dist})\int_a^b f(x)\,\mathrm{d}x = (\mathfrak{dist})\int_c^d f(\rho(t))\rho'(t)\,\mathrm{d}t. \tag{10.3}$$

The change of variables formula (10.3) remains valid under more general conditions on the function ρ. It holds for $\rho(t) = t^\alpha$ in $[0,d]$, if $\alpha > 0$; this change of variables is not of type I.

Lemma 10.2. *Let $\mathsf{f} \in \mathcal{D}'(0,b)$ and let $\alpha > 0$. Then the distributional limit of f from the right at $x = 0$ exists and equals γ if and only if the distributional limit of $\mathsf{f}(t^\alpha)$ from the right at $t = 0$ exists and equals γ.*

A function f is distributionally integrable over $[0,d]$ if and only if $f(t^\alpha)t^{\alpha-1}$ is distributionally integrable over $[0, d^{1/\alpha}]$ and

$$(\mathfrak{dist})\int_0^d f(x)\,\mathrm{d}x = (\mathfrak{dist})\int_0^{d^{1/\alpha}} \alpha f(t^\alpha)t^{\alpha-1}\,\mathrm{d}t. \tag{10.4}$$

Proof. The distributional limit of f from the right at $x = 0$ exists and equals γ if and only if

$$\lim_{\varepsilon\to 0^+}\langle \mathsf{f}(\varepsilon x), \psi(x)\rangle = \gamma\int_0^\infty \psi(x)\,\mathrm{d}x \tag{10.5}$$

for all $\psi \in \mathcal{D}(0,\infty)$. But if (10.5) holds then

$$\begin{aligned}\lim_{\varepsilon\to 0^+}\langle \mathsf{f}((\varepsilon t)^\alpha), \psi(t)\rangle &= \frac{1}{\alpha}\lim_{\varepsilon\to 0^+}\langle \mathsf{f}(\varepsilon^\alpha x), \psi(x^{1/\alpha})x^{1/\alpha-1}\rangle = \frac{\gamma}{\alpha}\int_0^\infty \psi(x^{1/\alpha})x^{1/\alpha-1}\,\mathrm{d}x \\ &= \gamma\int_0^\infty \psi(t)\,\mathrm{d}t,\end{aligned}$$

and it follows that the distributional limit of $\mathsf{f}(t^\alpha)$ from the right at $t = 0$ exists and equals γ.

Suppose now that f is distributionally integrable over $[0,d]$. Using Proposition 10.1 we deduce that $f(t^\alpha)t^{\alpha-1}$ is distributionally integrable over $[c, d^{1/\alpha}]$ for any $c > 0$, and if $F(c) = (\mathfrak{dist})\int_c^d f(x)\,\mathrm{d}x$ is the indefinite integral of f, then the indefinite integral of $\alpha f(t^\alpha)t^{\alpha-1}$ is $F(c^\alpha) = (\mathfrak{dist})\int_c^{d^{1/\alpha}} \alpha f(t^\alpha)t^{\alpha-1}\,\mathrm{d}t$. Now $\mathsf{F}(c)$, where $\mathsf{F} \leftrightarrow F$, has a distributional limit from the right at $c = 0$, equal to $(\mathfrak{dist})\int_0^{d^{1/\alpha}} f(x)\,\mathrm{d}x$, and it follows that the distributional limit of $\mathsf{F}(c^\alpha)$, which corresponds to the function $(\mathfrak{dist})\int_c^{d^{1/\alpha}} \alpha f(t^\alpha)t^{\alpha-1}\,\mathrm{d}t$, as $c \to 0^+$ also exists and equals $(\mathfrak{dist}) \int_0^d f(x)\,\mathrm{d}x$. The integrability of $\alpha f(t^\alpha)t^{\alpha-1}$ over $[0, d^{1/\alpha}]$ and formula (10.4) then follow from Theorem 8.1. ■

Introduce the *changes of variables of type II* as those continuous functions ρ from $[c, d]$ to $[a, b]$ that are of type I in $[c_1, d_1]$ whenever $c < c_1 < d_1 < d$, and are such that at the endpoints there exist $\alpha > 0$ and $\beta > 0$ for which $|\rho(x) - \rho(c)|^\alpha$ is of type I in $[c, d_1]$ and $|\rho(x) - \rho(d)|^\beta$ is of type I in $[c_1, d]$. Then using Lemma 10.2 we find that Proposition 10.1 also holds for changes of variables of type II.

Actually Proposition 10.1 remains valid for *changes of variables of type III*, which are those continuous functions ρ from $[c, d]$ to $\mathbb{R}$ for which there are numbers $\{c_j\}_{j=0}^n$ with $c = c_0 < c_1 < \cdots < c_{n-1} < c_n = d$ such that ρ is of type II in each of the subintervals $[c_j, c_{j+1}]$ for $0 \le j < n$.

PROPOSITION 10.3. *Let $\rho : [c, d] \to \mathbb{R}$ be a change of variables of type III. A function f is distributionally integrable over $\rho([c, d])$ if and only if $f(\rho(t))\rho'(t)$ is distributionally integrable over $[c, d]$ and*

$$(\mathfrak{dist}) \int_{\rho(c)}^{\rho(d)} f(x)\,\mathrm{d}x = (\mathfrak{dist}) \int_c^d f(\rho(t))\rho'(t)\,\mathrm{d}t. \tag{10.6}$$

Let us now consider changes of variables with an infinite range or domain.

LEMMA 10.4. *Let $\mathsf{f} \in \mathcal{D}'(0, 1)$. Then the distributional limit of f from the right at $x = 0$ exists and equals γ if and only if the Cesàro limit of $\mathsf{f}(1/t)$ as $t \to \infty$ exists and equals γ.*

A function f is distributionally integrable over $[0, 1]$ if and only if $f(t^{-1})t^{-2}$ is distributionally Cesàro integrable over $[1, \infty)$ and

$$(\mathfrak{dist}) \int_0^1 f(x)\,\mathrm{d}x = (\mathfrak{dist}) \int_1^\infty f(t^{-1})t^{-2}\,\mathrm{d}t \quad (\mathrm{C}). \tag{10.7}$$

Proof. The first part follows from the results of [22, Chap. 6]. The second part is obtained because by Proposition 10.1, f is distributionally integrable over $[c, 1]$ if and only if $f(t^{-1})t^{-2}$ is distributionally integrable over $[1, 1/c]$, and if $F(c) = (\mathfrak{dist}) \int_c^1 f(x)\,\mathrm{d}x$ then $F(1/c) = (\mathfrak{dist}) \int_1^{1/c} f(t^{-1})t^{-2}\,\mathrm{d}t$. But the distribution corresponding to $F(c)$ has a distributional lateral limit from the right at $c = 0$ if and only if the distribution corresponding to $F(1/c)$ has a Cesàro limit at infinity, and the limits coincide; in the first case the integral $(\mathfrak{dist}) \int_0^1 f(x)\,\mathrm{d}x$ exists, while in the second the Cesàro integral $(\mathfrak{dist}) \int_1^\infty f(t^{-1})t^{-2}\,\mathrm{d}t$ (C) exists. ■

Let us say that a function $\rho : [c, d) \to [a, \infty)$ with $\lim_{x\to d^-} \rho(x) = \infty$ is a *change of variables of type IV* if whenever $c < x < d$ then ρ is of type II in $[c, x]$ and $1/\rho$ is of type II in $[x, d]$. Our previous results immediately yield the ensuing change of variables formula.

PROPOSITION 10.5. *Let $\rho : [c, d) \to [a, \infty)$ be a change of variables of type IV. A function f is Cesàro distributionally integrable over $[a, \infty)$ if and only if $f(\rho(t))\rho'(t)$ is distributionally integrable over $[c, d]$ and*

$$(\mathfrak{dist}) \int_{\rho(c)}^\infty f(x)\,\mathrm{d}x = (\mathfrak{dist}) \int_c^d f(\rho(t))\rho'(t)\,\mathrm{d}t \quad (\mathrm{C}). \tag{10.8}$$

11. Mean value theorems

In this section we shall show how the usual three mean value theorems of integral calculus have versions for the distributional integral.

PROPOSITION 11.1. *Let f be a Łojasiewicz function on $[a,b]$ and let ψ be smooth and positive in $[a,b]$. Then there exists $\xi \in (a,b)$ such that*

$$(\mathfrak{dist})\int_a^b f(x)\psi(x)\,\mathrm{d}x = f(\xi)\int_a^b \psi(x)\,\mathrm{d}x. \tag{11.1}$$

Proof. Observe that since ψ is C^∞ it follows that $f\psi$ is also a Łojasiewicz function, and thus integrable. Since $\psi \geq 0$, we get

$$m\int_a^b \psi(x)\,\mathrm{d}x \leq (\mathfrak{dist})\int_a^b f(x)\psi(x)\,\mathrm{d}x \leq M\int_a^b \psi(x)\,\mathrm{d}x, \tag{11.2}$$

where $m = \inf\{f(x) : x \in [a,b]\}$, $M = \sup\{f(x) : x \in [a,b]\}$. Naturally m and M do not have to be real numbers in this case, $-\infty \leq m \leq M \leq \infty$. Notice now [30] that any Łojasiewicz function has the Darboux or intermediate value property, that is, $[f(c), f(d)] \subset f([c,d])$ for any subinterval $[c,d]$ of $[a,b]$. Hence there exists $\xi \in (a,b)$ such that $f(\xi) = (\int_a^b f(x)\psi(x)\,\mathrm{d}x)/(\int_a^b \psi(x)\,\mathrm{d}x)$. ∎

We also have the following second mean value theorem.

PROPOSITION 11.2. *Let f be distributionally integrable over $[a,b]$ and let ψ be smooth and monotonic. Then there exists $\xi \in (a,b)$ such that*

$$\int_a^b f(x)\psi(x)\,\mathrm{d}x = \psi(a)\int_a^\xi f(x)\,\mathrm{d}x + \psi(b)\int_\xi^b f(x)\,\mathrm{d}x. \tag{11.3}$$

Proof. Let $F(x) = (\mathfrak{dist})\int_a^x f(t)\,\mathrm{d}t$ be the indefinite integral of f. Then applying Proposition 11.1 to $\int_a^b F(x)\psi'(x)\,\mathrm{d}x$ we obtain the existence of $\xi \in (a,b)$ such that

$$\begin{aligned}\int_a^b f(x)\psi(x)\,\mathrm{d}x &= F(b)\psi(b) - \int_a^b F(x)\psi'(x)\,\mathrm{d}x = F(b)\psi(b) - F(\xi)\int_a^b \psi'(x)\,\mathrm{d}x\\ &= \psi(b)(F(b) - F(\xi)) + \psi(a)F(\xi),\end{aligned}$$

as required. ∎

The Bonnet form of the mean value theorem is as follows.

PROPOSITION 11.3. *Let f be distributionally integrable over $[a,b]$ and let ψ be smooth, positive, and increasing. Then there exists $\xi \in (a,b)$ such that*

$$(\mathfrak{dist})\int_a^b f(x)\psi(x)\,\mathrm{d}x = \psi(b)\,(\mathfrak{dist})\int_\xi^b f(x)\,\mathrm{d}x. \tag{11.4}$$

Proof. Let $a' < a$, and extend f and ψ to $[a', b]$ as follows. Put $f(x) = 0$ for $a' \leq x < a$, and let ψ be an extension that is smooth, positive, increasing, and with $\psi(a') = 0$. Then employing Proposition 11.2 for the integral $(\mathfrak{dist})\int_{a'}^b f(x)\psi(x)\,\mathrm{d}x$ we obtain (11.4). ∎

We also have the other form of the Bonnet mean value theorem, namely, if f is distributionally integrable over $[a,b]$ and ψ is smooth, positive, and *decreasing,* then there exists $\omega \in (a, b)$ such that

$$(\mathfrak{dist}) \int_a^b f(x)\psi(x)\,\mathrm{d}x = \psi(a)\,(\mathfrak{dist}) \int_a^\omega f(x)\,\mathrm{d}x. \tag{11.5}$$

We now give an example that shows how several standard arguments can still be used with the distributional integral.

Example 11.4. Let $a > 0$, and let f be a locally distributionally integrable function defined in $[a, \infty)$. Suppose that the indefinite integral $F(x) = (\mathfrak{dist}) \int_a^x f(t)\,\mathrm{d}t$ of f is a bounded function in $[a, \infty)$. Let now ψ be a C^∞ function defined in $[a, \infty)$ that decreases to 0, $\lim_{x\to\infty} \psi(x) = 0$. *Then the improper distributional integral*

$$(\mathfrak{dist}) \int_a^\infty f(x)\psi(x)\,\mathrm{d}x = \lim_{b\to\infty} \int_a^b f(x)\psi(x)\,\mathrm{d}x \tag{11.6}$$

converges.

Naturally, this is a well known result for locally Lebesgue integrable functions, but our aim is to show that the usual ideas also work for the distributional integral. Indeed, let

$$G(b) = (\mathfrak{dist}) \int_a^b f(x)\psi(x)\,\mathrm{d}x. \tag{11.7}$$

Then if $a \le b < b'$ there exists $\xi \in (b, b')$ such that

$$G(b') - G(b) = \psi(b)(F(\xi) - F(b)) + \psi(b')(F(b') - F(\xi)),$$

and since F is bounded and ψ tends to 0, we conclude that $G(b') - G(b) \to 0$ as $b, b' \to \infty$, that is, G satisfies the Cauchy criterion at ∞. Thus G has a limit at infinity, as we wanted to show.

12. Examples

We shall now give several examples of functions that are or are not distributionally integrable. We shall also consider several illustrations of our ideas.

Example 12.1. Observe, first of all, that for positive functions distributional integration is equivalent to Lebesgue integration, and thus nothing new arises in this case. Let $\alpha \in \mathbb{R}$. The positive function $|x|^\alpha$ is distributionally integrable over $\mathbb{R}$ if and only if $\alpha > -1$; the same is true for the function $x^\alpha H(x)$. There is a well defined and well known distribution, x_+^α, whenever $\alpha \neq -1, -2, -3, \ldots$; the distribution x_+^α is a regularization of the function $x^\alpha H(x)$, but the association $x^\alpha H(x) \leftrightarrow x_+^\alpha$ between an *integrable* function and the corresponding distribution holds only for $\alpha > -1$.

Functions that are distributionally integrable but not Lebesgue integrable need to be oscillatory.

Example 12.2. Let us now consider the distribution $\mathsf{s}_\alpha(x)$ that corresponds to the function $|x|^\alpha \sin(1/x)$ for $\alpha \in \mathbb{C}$. If $\Re e\,\alpha > -1$ then the function $|x|^\alpha \sin(1/x)$ is locally

Lebesgue integrable and thus it yields a regular distribution given by

$$\langle \mathsf{s}_\alpha(x), \psi(x)\rangle = \int_{-\infty}^{\infty} |x|^\alpha \sin(1/x)\,\psi(x)\,\mathrm{d}x, \qquad \psi \in \mathcal{D}(\mathbb{R}). \tag{12.1}$$

It is easy to show that s_α admits an analytic continuation from the right side half-plane $\Re e\,\alpha > -1$ to the whole complex plane. If $-1 \geq \Re e\,\alpha > -2$ then the function $|x|^\alpha \sin(1/x)$ is not Lebesgue integrable near $x = 0$ but it is Denjoy–Perron–Henstock–Kurzweil integrable. The function $|x|^\alpha \sin(1/x)$ is locally distributionally integrable for all $\alpha \in \mathbb{C}$, since actually it is a Łojasiewicz function because (see [30]) the distributional value $\mathsf{s}_\alpha(0)$ exists and equals 0 for all α. The association $|x|^\alpha \sin(1/x) \leftrightarrow \mathsf{s}_\alpha(x)$ holds for all $\alpha \in \mathbb{C}$.

Similarly, $|x|^\alpha \cos(1/x)$ is locally distributionally integrable for all $\alpha \in \mathbb{C}$ and it thus defines a distribution $\mathsf{c}_\alpha(x)$ given by

$$\langle \mathsf{c}_\alpha(x), \psi(x)\rangle = (\mathfrak{dist}) \int_{-\infty}^{\infty} |x|^\alpha \cos(1/x)\,\psi(x)\,\mathrm{d}x, \qquad \psi \in \mathcal{D}(\mathbb{R}). \tag{12.2}$$

The generalized function c_α is an entire function of α.

Example 12.3. Making a change of variables, we find that the functions $|x|^\alpha \sin |x|^{-\beta}$ and $|x|^\alpha \cos |x|^{-\beta}$ are locally distributionally integrable for all $\alpha \in \mathbb{C}$ and for all $\beta > 0$. We can multiply locally distributionally integrable functions by the characteristic functions of intervals and still obtain locally distributionally integrable functions. Thus $x^\alpha H(x) \sin |x|^{-\beta}$ and $x^\alpha H(x) \cos |x|^{-\beta}$, as well as $|x|^\alpha \operatorname{sgn} x \sin |x|^{-\beta}$ and $|x|^\alpha \operatorname{sgn} x \cos |x|^{-\beta}$, are also locally distributionally integrable functions for all $\alpha \in \mathbb{C}$ and for all $\beta > 0$.

Example 12.4. Let g be a locally distributionally integrable function, $g \leftrightarrow \mathsf{g}$, where $\mathsf{g} \in \mathcal{K}'(\mathbb{R})$. Then a change of variables shows that for each $a \in \mathbb{R}$ the function $f(x) = g((x-a)^{-1})$ is likewise locally distributionally integrable over $\mathbb{R}$, even at $x = a$.

Functions like $H(x-a)(x-a)^\alpha J_\nu((x-a)^{-\beta})$ will be locally distributionally integrable over $\mathbb{R}$ for all $\alpha \in \mathbb{C}$, $\beta > 0$, and $\nu \in \mathbb{R}$.

In particular, if h is a locally distributionally integrable function, periodic, and with zero mean, then $\mathsf{h} \in \mathcal{K}'(\mathbb{R})$, where $h \leftrightarrow \mathsf{h}$. Thus functions like $|x-a|^\alpha h(|x-a|^{-\beta})$ are locally distributionally integrable over $\mathbb{R}$ for all $\alpha \in \mathbb{C}$ and $\beta > 0$. If $h(x)$ is $\sin x$ or $\cos x$ we recover the previous examples; another example is $|x|^\alpha(\{|x|^{-\beta}\} - 1/2)$, where $\{x\}$ is the fractional part of x.

Example 12.5. Let $\{c_n\}_{n=1}^\infty$ be a numerical sequence, and define the function

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0 \text{ or } x \geq 1, \\ c_n & \text{if } \frac{1}{n+1} \leq x < \frac{1}{n}. \end{cases} \tag{12.3}$$

Let $a_n = c_n(\frac{1}{n} - \frac{1}{n+1})$. Then f is Lebesgue integrable at $x = 0$ if and only if the series $\sum_{n=1}^\infty a_n$ is absolutely convergent, while f is Denjoy–Perron–Henstock–Kurzweil integrable if and only if the series is convergent [3]. We now see that f is distributionally integrable at $x = 0$ if and only if $\sum_{n=1}^\infty a_n$ is Cesàro summable, and in that case

$$(\mathfrak{dist}) \int_0^1 f(x)\,\mathrm{d}x = \sum_{n=1}^\infty a_n \quad (\mathrm{C}). \tag{12.4}$$

In particular, if $c_n = (-1)^n n(n+1)$, we obtain $(\mathfrak{dist}) \int_0^1 f(x)\,\mathrm{d}x = -1/2$.

Example 12.6. In general the sum of a Lebesgue integrable function and a Łojasiewicz function is neither Lebesgue integrable nor a Łojasiewicz function, but it is certainly distributionally integrable. For instance, if F is a closed set with empty interior and positive Lebesgue measure in $[a,b]$ then $\chi_F(x) + J_\nu((x^2-(a+b)x+ab)^{-1})$ is a distributionally integrable function that is neither Lebesgue integrable nor a distributionally regulated function in $[a,b]$. It is worth pointing out that decompositions of distributions as sums of terms involving a Łojasiewicz distribution have shown to be of great importance in the study of distributional composition operations [1].

Example 12.7. If $a<b$, denote by $f_{a,b}$ the function

$$f_{a,b}(x) = [\chi_{[a,b]}(x)\sin((x^2-(a+b)x+ab)^{-1})]'. \tag{12.5}$$

Let $\{(a_n,b_n)\}_{n=1}^{\infty}$ be a sequence of mutually disjoint open intervals and let $\sum_{n=1}^{\infty} M_n$ be an absolutely convergent series. Let

$$f = \sum_{n=1}^{\infty} M_n f_{a_n,b_n}.$$

Then f is a locally distributionally integrable function and

$$(\mathfrak{dist})\int_c^d f(x)\,\mathrm{d}x = F(d) - F(c), \tag{12.6}$$

where $F(x) = M_n\sin((x^2-(a_n+b_n)x+a_nb_n)^{-1})$ if $x \in (a_n,b_n)$, while $F(x)=0$ if $x \notin \bigcup_{n=1}^{\infty}(a_n,b_n)$.

Example 12.8. It is well known that there are continuous functions whose derivatives do not exist at any point. Actually, one can show in many cases that the distributional derivative does not have values at any point. For instance [49] if $\{a_n\}_{n\in\mathbb{Z}}$ is a lacunary sequence such that $a_n \neq o(1)$ but $a_n = O(1)$, $|n|\to\infty$, then

$$G(x) = \sum_{n=-\infty}^{\infty} \frac{a_n}{n} e^{inx} \tag{12.7}$$

is continuous, but if $G \leftrightarrow \mathsf{G}$, then $\mathsf{g} = \mathsf{G}'$ does not have distributional point values at any point. It follows that G is not the indefinite integral of a distributionally integrable function in any interval.

Example 12.9. The Heaviside function $H(x)$ does not have a value at the origin, of course, and thus it is not the indefinite integral of a distributionally integrable function. Naturally, if $H \leftrightarrow \mathsf{H}$, then $\mathsf{H}'(x) = \delta(x)$ is *not* a function. Similarly, the continuous increasing Cantor function defined in $[0,1]$ is not the indefinite integral of a distributionally integrable function in any interval that meets the Cantor set, since its distributional derivative is not a function but rather a measure concentrated on the Cantor set.

Example 12.10. Let f be a locally distributionally integrable function such that if $f \leftrightarrow \mathsf{f}$, then $\mathsf{f} \in \mathcal{D}'(\mathbb{R})$ satisfies

$$\mathsf{f}(x) = O(|x|^{\beta}) \quad (\mathrm{C}), \qquad |x|\to\infty, \tag{12.8}$$

for some $\beta < 1$. Let

$$F(x,y) = \frac{y}{\pi}\,(\mathfrak{dist})\int_{-\infty}^{\infty} \frac{f(\xi)\,\mathrm{d}\xi}{(x-\xi)^2+y^2} \quad (\mathrm{C}), \tag{12.9}$$

for $x \in \mathbb{R}$ and $y > 0$. The function F is the ϕ-transform of f with respect to the function

$$\phi(x) = \frac{1}{\pi(x^2+1)}. \tag{12.10}$$

Naturally, $F(x,y)$ is the Poisson integral of f, which is the harmonic function with $F(x,0^+) = \mathsf{f}(x)$ that satisfies $F(x,y) = O(|x|^\beta)$ (C), $|x| \to \infty$, for each fixed $y > 0$.

The boundary behavior of F is as follows: $F(x,y) \to f(w)$ as $(x,y) \to (w,0)$ in any sector $y \geq m|x-w|$ for $m > 0$, almost everywhere with respect to $w \in \mathbb{R}$; this holds for all $w \in \mathbb{R}$ whenever f is a Łojasiewicz function.

Example 12.11. Let us now consider the Fourier transform of tempered locally integrable distributions. The characterization of the Fourier series of those periodic distributions that have a distributional point value was given in [12]: If $\mathsf{f}(\theta) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta}$ in the space $\mathcal{D}'(\mathbb{R})$ then

$$\mathsf{f}(\theta_0) = \gamma \quad \text{distributionally} \tag{12.11}$$

if and only if there exists k such that

$$\lim_{x\to\infty} \sum_{-x \leq n \leq ax} a_n e^{in\theta_0} = \gamma \quad (\mathrm{C}, k) \quad \forall a > 0. \tag{12.12}$$

Therefore, if f is a periodic locally distributionally integrable function, of period 2π, then the coefficients

$$a_n = \frac{1}{2\pi}\,(\mathfrak{dist})\int_0^{2\pi} f(\theta) e^{-in\theta}\,\mathrm{d}\theta \tag{12.13}$$

are well defined for all $n \in \mathbb{Z}$, and

$$\lim_{x\to\infty} \sum_{-x \leq n \leq ax} a_n e^{in\theta} = f(\theta) \quad (\mathrm{C}, k) \quad \forall a > 0, \tag{12.14}$$

almost everywhere with respect to θ. If f is a Łojasiewicz function then (12.14) holds for all $\theta \in \mathbb{R}$.

The characterization of the values of general Fourier transforms [48, 49] is as follows: Let $\mathsf{f} \in \mathcal{S}'(\mathbb{R})$, and let $x_0 \in \mathbb{R}$; then

$$\mathsf{f}(x_0) = \gamma \quad \text{distributionally} \tag{12.15}$$

if and only if

$$\text{e.v.}\ \langle \widehat{\mathsf{f}}(u), e^{-iux_0} \rangle = 2\pi\gamma \quad (\mathrm{C}). \tag{12.16}$$

We have chosen the constants in the Fourier transform in such a way that

$$\widehat{f}(u) = \int_{-\infty}^{\infty} f(x) e^{ixu}\,\mathrm{d}x, \tag{12.17}$$

if the integral makes sense. In case $\widehat{f}$ is locally distributionally integrable this means that

$$\text{e.v.}\ (\mathfrak{dist})\int_{-\infty}^{\infty} \widehat{f}(u) e^{-iux_0}\,\mathrm{d}u = 2\pi\gamma \quad (\mathrm{C}). \tag{12.18}$$

Suppose now that f is a locally integrable tempered distribution, $f \leftrightarrow \mathsf{f}$. Then

$$\text{e.v.}\,\langle \widehat{\mathsf{f}}(u), e^{-iux_0}\rangle = 2\pi f(x_0) \quad (\mathrm{C}), \tag{12.19}$$

almost everywhere with respect to x_0, and actually everywhere if f is a Łojasiewicz function. When $\widehat{\mathsf{f}}$ is also locally integrable, $\widehat{f} \leftrightarrow \widehat{\mathsf{f}}$, then (12.19) becomes

$$\text{e.v.}\,(\mathfrak{dist})\int_{-\infty}^{\infty} \widehat{f}(u)e^{-iux_0}\,\mathrm{d}u = 2\pi f(x_0) \quad (\mathrm{C}).$$

Example 12.12. Let f be a distribution defined in the complement of the origin, $\mathsf{f} \in \mathcal{D}'(\mathbb{R}\setminus\{0\})$. There may or may not be a distribution $\mathsf{g}_0 \in \mathcal{D}'(\mathbb{R})$ whose restriction to $\mathbb{R}\setminus\{0\}$ is f, which we call a *regularization* of f, but if one regularization exists then there are infinitely many regularizations, since $\mathsf{g}(x) = \mathsf{g}_0(x) + \sum_{j=0}^{m} a_j\delta^{(j)}(x)$ is also a regularization for any constants $a_0,\dots,a_m$. It is known [14] that there is no continuous way to choose the regularization g. Suppose now that f corresponds to a function f, locally distributionally integrable in $\mathbb{R}\setminus\{0\}$; if f is distributionally integrable at $x=0$ then it has an associated distribution $\mathsf{g}_0 \in \mathcal{D}'(\mathbb{R})$, and then g_0 is the *canonical* regularization of the distribution f.

Example 12.13. Consider now an analytic function $F(z)$ defined in the upper half-plane $\Im m\, z > 0$, which we assume to vanish at ∞, $F(z) \to 0$ as $z \to \infty$ angularly in the half-plane. Suppose that the distributional limit $\mathsf{g}(x) = F(x+i0)$ exists in $\mathcal{D}'(\mathbb{R})$ ([5, 6]), and let f be its restriction to $\mathbb{R}\setminus\{0\}$, which we assume to correspond to a locally distributionally integrable function f in $\mathbb{R}\setminus\{0\}$. If f is not distributionally integrable at $x=0$ then it does not define a canonical distribution in the whole real line, but f will have at least a regularization g_0. However, if f is also distributionally integrable at $x=0$ then a corresponding distribution $\mathsf{g}_0 \in \mathcal{D}'(\mathbb{R})$ is defined by the association $f \leftrightarrow \mathsf{g}_0$. Both g and g_0 are regularizations of f; in general they do not coincide, but if f is distributionally integrable at $x=0$ then the results of [16] immediately yield $\mathsf{g} = \mathsf{g}_0$, so that, if the integral converges at infinity, we have the Cauchy representation

$$F(z) = \frac{1}{2\pi i}\,(\mathfrak{dist})\int_{-\infty}^{\infty} \frac{f(\xi)}{\xi - z}\,\mathrm{d}\xi, \qquad \Im m\, z > 0. \tag{12.20}$$

Take $F(z) = 1/z$. In this case $f(x) = 1/x$ is not distributionally integrable at $x=0$. The standard regularization of $\mathsf{f} \in \mathcal{D}'(\mathbb{R}\setminus\{0\})$, $1/x \leftrightarrow \mathsf{f}$, is $\mathsf{g}_0 = (\ln|x|)' = \text{p.v.}(1/x)$. However, g is another regularization, namely, $\mathsf{g}(x) = (x+i0)^{-1} = \mathsf{g}_0(x) - \pi i\delta(x)$. Formula (12.20) does not even make sense in this case.

On the other hand, if $F(z) = e^{-i/z}/z$, then $f(x) = e^{-i/x}/x$ is actually distributionally integrable at $x=0$, and thus has an associated distribution $\mathsf{g}_0 \in \mathcal{D}'(\mathbb{R})$. In this case $\mathsf{g} = \mathsf{g}_0$, and (12.20) becomes

$$\frac{e^{-i/z}}{z} = \frac{1}{2\pi i}\,(\mathfrak{dist})\int_{-\infty}^{\infty} \frac{e^{-i/\xi}}{\xi(\xi - z)}\,\mathrm{d}\xi \tag{12.21}$$

in the half-plane $\Im m\, z > 0$.

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