Contents

Abstract

We prove sufficiency of conditions on pairs of measures μ and ν, defined respectively on the interior and the boundary of a bounded Lipschitz domain Ω in d-dimensional Euclidean space, which ensure that, if u is the solution of the Dirichlet problem.

$$\Delta u = 0 \quad \text{in } \Omega,$$
$$u|_{\partial\Omega} = f,$$

with f belonging to a reasonable test class, then

$$\left(\int_\Omega |\nabla u|^q \, d\mu\right)^{1/q} \leq \left(\int_{\partial\Omega} |f|^p \, d\nu\right)^{1/p},$$

where $1 < p \leq q < \infty$ and $q \geq 2$. Our sufficiency conditions resemble those found by Wheeden and Wilson for the Dirichlet problem on $\mathbb{R}^{d+1}_+$. As in that case we attack the problem by means of Littlewood–Paley theory. However, the lack of translation invariance forces us to use a general result of Wilson, which must then be translated into the setting of homogeneous spaces. We also consider what can be proved when a strictly elliptic divergence form operator replaces the Laplacian.

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Introduction

We are interested in the following general question: To what extent is the interior smoothness of the solution of a PDE controlled by the size of its boundary values? To be more specific, suppose (for now) that $\Omega \subset \mathbb{R}^{d+1}$ is a nice domain, μ is a positive measure supported in Ω, and v is a non-negative measurable function defined on $\partial\Omega$. If $f : \partial\Omega \to \mathbb{R}$ is reasonable (say, a continuous function with compact support), we let $u : \Omega \to \mathbb{R}$ be the solution of the classical Dirichlet problem with boundary values equal to f. (We are implicitly assuming that Ω is nice enough to have this make sense!) Let p and q be real numbers lying strictly between 1 and infinity.

When is it the case that

$$\left(\int_{\Omega} |\nabla u|^q \, d\mu\right)^{1/q} \le \left(\int_{\partial\Omega} |f|^p \, v \, ds\right)^{1/p}$$

holds for all such f? Here ds denotes surface measure on $\partial\Omega$, but we could easily replace it by some other measure—as indeed we will (see below).

For "classical" domains—half-spaces and disks—this problem has been studied extensively. In the case where $\Omega = \mathbb{R}^{d+1}_+$ and $v \equiv 1$, complete characterizations of the right μ's have been given for all $0 < p,\ q < \infty$ (with f's L^p norm being replaced by a Hardy space H^p norm when $0 < p \le 1$). These results can be found in [Lu1], [Lu2], [Sh1], [Sh2], [Ve], and [Vi].

In [WhWi], Wheeden and Wilson continued this line of research in their study of weighted norm inequalities of the form

$$\left(\int_{\mathbb{R}^{d+1}_+} |\nabla u|^q \, d\mu\right)^{1/q} \le \left(\int_{\mathbb{R}^d} |f|^p v \, dx\right)^{1/p}. \tag{0.1}$$

Here u is the Poisson extension of f, which is assumed to belong to some natural test class, v is a weight (i.e., a non-negative function in $L^1_{\mathrm{loc}}(\mathbb{R}^d)$), and μ is a positive Borel measure on $\mathbb{R}^{d+1}_+$. They proved sufficient conditions, depending on p and q, on μ and v which ensured that (0.1) would hold for all "reasonable" f, for $1 < p \le q < \infty$ and $q \ge 2$. Their sufficient conditions, which we will give presently, were quantitative statements of the fact that (0.1) should hold if μ does not put too much mass near places where v is small, taking into account the interactions between p, q, and the rates of decay of the convolutions kernels that "generate" the components of ∇u.

In this paper, we take the first steps in generalizing these results to the setting of bounded Lipschitz domains. As part of this investigation we also demonstrate three different paths to almost-orthogonality. This is accomplished in Sections 2.2, 3.1, and 4. Our fundamental Littlewood–Paley type inequalities, Theorems 1.1 and 2.1, depend on certain functions having an "almost-orthogonality" property. Fortunately, almost-orthogonality is not rare. We give three proofs of this property, for appropriate sets of functions, under three significantly different sets of hypotheses. In particular the first, most abstract proof, Theorem 2.2 below, treats functions defined on a general homogeneous space [CoWeis]. The proof of this property for a general doubling measure is one of the major new results of this paper. The proof is accomplished by means of the intrinsic square function. This function, the ISF, was introduced by J. M. Wilson in [Wi2] & [Wi3]. The ISF plays a role analogous to the Hardy–Littlewood maximal function. It dominates many classical "square functions", in the same way that the Hardy–Littlewood maximal function dominates many convolution operators, but it is not essentially larger (or harder to estimate) than any one of them. The other two proofs of almost-orthogonality are tailored to functions defined on the boundary of a Lipschitz domain.

Our overall approach will parallel that of [WhWi], and our results will have a similar form. It is appropriate that we review the main results from [WhWi], along with a little of their development.

We need to recall some standard definitions. If $Q \subset \mathbb{R}^d$ is a cube with sides parallel to the coordinate axes, and with side length $\ell(Q)$, we set

$$T(Q) \equiv \{(x,y) \in \mathbb{R}^{d+1}_+ : x \in Q,\, \ell(Q/2) \le y < \ell(Q)\},$$

which people commonly visualize as the "top half" of the so-called Carleson box $\hat{Q} \equiv Q \times (0, \ell(Q))$. We let x_Q denote the center of Q. If v is a weight and $1 < p < \infty$, we set $\sigma \equiv v^{1-p'}$, where p' is p's dual exponent. It is important to note that σ gets big where v gets small.

By looking at dyadic analogues of (0.1), one can easily come up with a plausible first approximation to [WhWi]'s sufficient condition; namely, that

$$\frac{\mu(T(Q))^{1/q}\sigma(Q)^{1/p'}}{\ell(Q)^{d+1}}$$

should be bounded by a constant independent of Q. The trouble with this condition, of course, is that it does not deal with the "tails" of the kernels; also, for technical reasons, the weight σ is not quite what one wants.

Let $\eta > p'/2$ and let w be any weight such that

$$\int_Q \sigma(x) \log^\eta(e + \sigma(x)/\sigma_Q)\, dx \le \int_Q w(x)\, dx$$

for all Q. (Such weights *usually* exist: let $w = cM^k\sigma$, where M^k denotes a k-fold application of the Hardy–Littlewood maximal operator, with $k \sim \eta$, and c is an appropriate positive constant.) In [WhWi] it is shown that there exist a positive constant $c = c(p,q,d,\eta)$ and an exponent $M = M(p,q,d)$ such that, if

$$\mu(T(Q))^{1/q}\bigg(\int_{\mathbb{R}^d} \frac{w(x)}{(1 + |x - x_Q|/\ell(Q))^M}\, dx\bigg)^{1/p'} \le c\ell(Q)^{d+1} \tag{0.2}$$

for all cubes Q, then (0.1) holds for all "reasonable" f (say, $f \in L^\infty$ with bounded support).

The analogue of (0.1) we consider in the present paper is

$$\left(\int_{\Omega}|\nabla u|^q\,d\mu\right)^{1/q}\le\left(\int_{\partial\Omega}|f|^p v\,d\omega\right)^{1/p},\tag{0.3}$$

where Ω is a bounded Lipschitz domain in $\mathbb{R}^d$, u is the solution of the Dirichlet problem with boundary data f, and ω is harmonic measure for some fixed point $X_0\in\Omega$. We assume that μ is a positive Borel measure defined on Ω and $v\in L^1(\partial\Omega,d\omega)$ is non-negative.

For technical reasons, we will also be assuming that μ is supported near the boundary of Ω, in the "band" $\Omega_\delta\equiv\{x\in\Omega: d(x,\partial\Omega)\le\delta\}$. Thus, we will actually be seeking sufficient conditions on μ and v such that, for *some* $\delta>0$,

$$\left(\int_{\Omega_\delta}|\nabla u|^q\,d\mu\right)^{1/q}\le\left(\int_{\partial\Omega}|f|^p\,v\,d\omega\right)^{1/p}\tag{0.4}$$

for all f in our test class. In this paper we define a Lipschitz domain as a bounded domain in $\mathbb{R}^{d+1}$ whose boundary can be described as a finite union of regions that are the rotation and/or translation of a Lipschitz graph in $\mathbb{R}^d$. The interior of the domain may not be covered by the top halves of the Carleson boxes that appear in the measure condition in Theorem 3.1; therefore we state the weighted inequality for the region Ω_δ, although it is valid over the entire domain Ω by a slight extension of the measure condition as explained in Section 3.

We will prove sufficient conditions on v and μ (depending on p, q, Ω, and X_0) which ensure that, for some $\delta>0$, (0.4) holds for all $f\in L^p(\partial\Omega,d\omega)$. As in [WhWi], our sufficient conditions are valid for all p's and q's in the range $1<p\le q<\infty$, with $q\ge 2$. Unfortunately, our conditions also come with an extra hypothesis on v: we require that the measure $v^{1-p'}d\omega$ belong to the Muckenhoupt A^∞ class relative to the measure $d\omega$ (in symbols, $v^{1-p'}d\omega\in A^\infty(\omega)$). The precise (and standard) definition of this relation is given at the beginning of Section 2. For now it is probably enough for the reader to know that we are requiring $v^{1-p'}d\omega$ to be absolutely continuous with respect to $d\omega$ in a way that is uniform under changes of scale.

What is the right translation of (0.2) to Lipschitz domains? We see that (0.2) seems to have several "moving parts". On one side we have a term involving $\mu(T(Q))$ and one that is an integral of a weight against a "bump function" centered around Q (both raised to appropriate powers). On the other side we have the Lebesgue measure of Q, raised to a certain power.

When we work on a Lipschitz domain, the cube Q will be replaced by the projection of a certain "genuine" cube onto $\partial\Omega$; we shall denote such boundary cubes by Q_b. The set $T(Q)$ (actually $T(Q_b)$) will be a subset of Ω for which $d(T(Q_b),Q_b)$ is comparable to $\ell(Q)$. A fast and reasonably accurate way to visualize $T(Q_b)$ is to think of a ball inside Ω whose radius is comparable to its distance to $\partial\Omega$ (indeed, for our purposes, such a definition would work fine). This radius is essentially $\ell(Q)$. For ease of reading, we will denote this latter quantity by $\ell(Q_b)$. It will always be obvious from the context that $\ell(Q_b)$ is comparable to the diameter of Q_b, with comparability constants that only depend on d and M.

Corresponding to each boundary cube Q_b will be its dilates λQ_b (roughly, the cube concentric with Q_b, but with side length λ times as big). Since Ω is assumed to be bounded, these dilates will *not* keep getting bigger indefinitely as λ increases. Instead, we will set $\lambda Q_b \equiv \partial\Omega$ when $\lambda\ell(Q_b)$ exceeds some positive r_0 that depends on Ω but is otherwise unspecified. For $j = 0, 1, 2, \ldots$, we will define $R_0(Q_b) = Q_b$ for $j = 0$ and $R_j(Q_b) = 2^j Q_b \setminus 2^{j-1} Q_b$ for $j \geq 1$. So, $R_0(Q_b)$ is just Q_b and $R_j(Q_b)$ ($j \geq 1$) is (approximately) an annulus concentric with Q_b and having inner and outer radii comparable to $2^j\ell(Q_b)$. For our purposes there is nothing wrong with thinking of $R_j(Q_b)$ ($j \geq 1$) as those x's in $\partial\Omega$ whose distance to x_Q (the approximate center of Q_b) lies between two fixed positive multiples of $2^j\ell(Q_b)$. Our convention on λQ_b has the happy consequence that all the $2^j Q_b$'s are the same—and therefore all the $R_j(Q_b)$'s are *empty*—for j sufficiently large.

The Lipschitz analogue to the bump function $(1 + |x - x_Q|/\ell(Q))^{-M}$ is

$$\left[\omega(Q_b) \sum_{j=0}^{\infty} \frac{2^{-j\epsilon}}{\omega(2^j Q_b)} \chi_{R_j(Q_b)}(x)\right]^{p'/2}, \tag{0.5}$$

where $\epsilon > 0$ is a constant depending on the domain Ω. To see that this generalization is the natural one, think of (0.5) as a function of $x \in \mathbb{R}^d$ and replace ω with Lebesgue measure. Set $Q_b \equiv Q$. Then (0.5) is bounded above and below by positive constants times

$$[\ell(Q)^d (1 + |x - x_Q|/\ell(Q))^{-\epsilon} (\ell(Q) + |x - x_Q|)^{-d}]^{p'/2}; \tag{0.5M}$$

because, when $x \in R_j(Q)$, $(1 + |x - x_Q|/\ell(Q)) \sim 2^j$ and $(\ell(Q) + |x - x_Q|)^d \sim |2^j Q|$. But (0.5M) simplifies to

$$\begin{aligned}[(1 + |x - x_Q|/\ell(Q))^{-(d+\epsilon)}]^{p'/2} &= (1 + |x - x_Q|/\ell(Q))^{-(p'/2)(d+\epsilon)} \\ &\equiv (1 + |x - x_Q|/\ell(Q))^{-M},\end{aligned}$$

where we have set $M \equiv (p'/2)(d + \epsilon)$. It is important to note that, for any x, the sum in (0.5) contains essentially only one term.

Finally there is the right-hand term $\ell(Q)^{d+1}$. We will replace this with $\ell(Q_b)\omega(Q_b)$.

With the precise definitions still to follow, the rephrased version of (0.2) is

$$\mu(T(Q_b))^{1/q} \left(\int_{\partial\Omega} \left[\omega(Q_b) \sum_{j=0}^{\infty} \frac{2^{-j\epsilon}}{\omega(2^j Q_b)} \chi_{R_j(Q_b)}(x)\right]^{p'/2} \sigma(x)\, d\omega(x) \right)^{1/p'} \leq c\ell(Q_b)\omega(Q_b),$$

and our main theorem (Theorem 3.1), which we prove in Section 3, is

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^{d+1}$ be a bounded Lipschitz domain, and let ω be harmonic measure on $\partial\Omega$ for a fixed point $X_0 \in \Omega$. Suppose that $v \in L^1(\partial\Omega, d\omega)$ is a non-negative function and μ is a positive Borel measure on Ω. Define $\sigma \equiv v^{1-p'}$, and suppose that $\sigma d\omega \in A^\infty(\omega)$ on $\partial\Omega$. If $1 < p \leq q < \infty$ and $q \geq 2$, there is an $\epsilon = \epsilon(\Omega) > 0$, and there is a positive constant C such that*

$$\left(\int_{\Omega_\delta} |\nabla u(t, y)|^q \, d\mu(t, y) \right)^{1/q} \leq \left(\int_{\partial\Omega} |f|^p v \, d\omega \right)^{1/p}$$

will hold for all $f \in L^p(\partial\Omega, d\omega)$, for some positive δ, if, for all sufficiently small boundary *cubes* $Q_b \in \mathcal{G}$,

$$\mu(T(Q_b))^{1/q}\left(\int_{\partial\Omega}\left[\omega(Q_b)\sum_{j=0}^{\infty}\frac{2^{-j\epsilon}}{\omega(2^jQ_b)}\chi_{R_j(Q_b)}(s)\right]^{p'/2}\sigma(s)\,d\omega(s)\right)^{1/p'}\le C\ell(Q_b)\omega(Q_b),$$

where C *depends only on* p, q, Ω *and the choice of the point* X_0.

We will prove Theorem 3.1 by means of a general Littlewood–Paley inequality, valid on arbitrary doubling measure spaces. This inequality is a natural generalization of the main result from [Wi1]. This method works also to prove a slightly weaker form of Theorem 3.1 in the case where u is not assumed to be f's harmonic extension, but to satisfy a strictly elliptic equation in divergence form. In this case, as one might expect, ordinary harmonic measure is replaced by the corresponding elliptic measure.

The chief problem arising in the elliptic case is that the solution u is only guaranteed to be Hölder continuous, and ∇u may not be defined pointwise. There are several ways around this obstacle: we can replace ∇u by a discretized version of the original gradient, or by a local Hölder coefficient. These two methods are presented in Section 4. Ironically, it is by the discretization method that we will prove the sufficient condition for *harmonic* measure.

The paper is organized as follows. In Section 1 we introduce the setting and background by stating and sketching the proof of a theorem for harmonic u on $\mathbb{R}^{d+1}_+$. In Section 2 we prove a Littlewood–Paley inequality for a general doubling measure which is valid on homogeneous spaces; we also prove, for a family of functions having the properties of geometric decay and minimal smoothness, that, if each member of the family has mean value zero with respect to the doubling measure, then the family satisfies an almost-orthogonality estimate. In Section 3 we review some facts about Lipschitz domains and particularly about harmonic measure on Lipschitz domains; we then state the precise form of Theorem 3.1 and its proof. In Section 4, the last section of the paper, we prove two results for elliptic functions on Lipschitz domains. One result relates very closely to the theorem for harmonic functions in Section 3, the other theorem is tailored to the setting that is natural for elliptic functions.

1. Euclidean space

We motivate our work in Lipschitz domains by first considering a model case, that of harmonic functions u defined on the upper half-space $\mathbb{R}^{d+1}_+ = \mathbb{R}^d\times(0,\infty)$. We suppose we are given a positive Borel measure μ, defined on $\mathbb{R}^{d+1}_+$, and a non-negative $v \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$. If $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ is such that $|f|(1+|x|)^{-d-1} \in L^1(\mathbb{R}^d)$, and $Z = (x, y) \in \mathbb{R}^{d+1}_+$, we set $u(Z) = \int_{\mathbb{R}^d} K(Z, s)f(s)\,ds$, where $K(Z, \cdot)$ is the usual Poisson kernel for the point Z. Given exponents p and q $(1 < p, q < \infty)$, we seek sufficient conditions on μ and v which ensure that

$$\left(\int_{\mathbb{R}^{d+1}_+}|\nabla u(t,y)|^q\,d\mu(t,y)\right)^{1/q}\le\left(\int_{\mathbb{R}^d}|f|^p v\,dx\right)^{1/p} \tag{1.1}$$

holds for all of these f, where ∇u is the full gradient in t and y. *For technical reasons, we are at present able to obtain these conditions only when $p \le q$ and $q \ge 2$. Henceforth we will always assume these extra restrictions on p and q.*

The expression on the left side of (1.1) equals

$$\int_{\mathbb{R}^{d+1}_+} |g(t,y)|\,|\nabla u(t,y)|\,d\mu(t,y) \tag{1.2}$$

for some $g : \mathbb{R}^{d+1}_+ \to \mathbb{R}$ such that

$$\int |g|^{q'}\,d\mu = 1,$$

where q' is q's dual exponent. We can conveniently break (1.2) into infinitely many pieces. For every $Q \in \mathcal{D}$, the dyadic cubes in $\mathbb{R}^d$, we set $T(Q) = Q \times [\ell(Q)/2, \ell(Q)) = \{(t,y) : t \in Q,\ \ell(Q)/2 \le y < \ell(Q)\}$, sometimes called the top half of the *Carleson box* $\widehat{Q} = Q \times (0, \ell(Q))$. Since the $T(Q)$'s tile $\mathbb{R}^{d+1}_+$, we can rewrite (1.2) as

$$\sum_{Q\in\mathcal{D}} \int_{T(Q)} |g(t,y)|\,|\nabla u(t,y)|\,d\mu(t,y). \tag{1.3}$$

Each integral in (1.3) can be replaced by a "discretized" term. Define $\tilde{T}(Q) = (1.1Q) \times [.45\ell(Q), 1.1\ell(Q))$, where $1.1Q$ is concentric with Q but has side length 1.1 times as big. Elementary estimates on the Poisson kernels for balls in $\mathbb{R}^{d+1}$ show that

$$\sup_{(t,y)\in T(Q)} |\nabla u(t,y)| \le C_d \ell(Q)^{-1} \sup_{Z,Z'\in\tilde{T}(Q)} |u(Z) - u(Z')|.$$

Therefore, for every Q, we can find Z_Q and Z'_Q in $\tilde{T}(Q)$ such that

$$\int_{T(Q)} |g(t,y)|\,|\nabla u(t,y)|\,d\mu(t,y) \le C\ell(Q)^{-1}|u(Z_Q) - u(Z'_Q)| \int_{T(Q)} |g(t,y)|\,d\mu(t,y).$$

Thus sufficient conditions for (1.1) will follow from sufficient conditions for

$$\sum_{Q\in\mathcal{D}} \ell(Q)^{-1}|u(Z_Q) - u(Z'_Q)| \int_{T(Q)} |g(t,y)|\,d\mu(t,y) \le C\biggl(\int_{\mathbb{R}^d} |f|^p\,v\,dx\biggr)^{1/p} \tag{1.4}$$

for a fixed constant C, holding for all g as we have described, and for all choices of Z_Q and Z'_Q in $\tilde{T}(Q)$. Clearly, it is enough to solve this problem under the hypothesis that $\int_{\mathbb{R}^d} |f|^p\,v\,dx \le 1$. With this assumption, inequality (1.4) reduces to

$$\sum_{Q\in\mathcal{D}} \ell(Q)^{-1}|u(Z_Q) - u(Z'_Q)| \int_{T(Q)} |g(t,y)|\,d\mu(t,y) \le C \tag{1.5}$$

for all such g, f, and appropriate choices of Z_Q and Z'_Q. Now, inequality (1.5) will hold for a fixed C if and only if

$$\sum_{Q\in\mathcal{F}} \ell(Q)^{-1}|u(Z_Q) - u(Z'_Q)| \int_{T(Q)} |g(t,y)|\,d\mu(t,y) \le C \tag{1.6}$$

holds for the same C, where $\mathcal{F}$ is an arbitrary *finite* subset of $\mathcal{D}$. This restriction to finite families $\mathcal{F}$ serves the same purpose as only considering compactly-supported (but otherwise arbitrary) g's: it ensures that all of our integrals make sense, that we can freely exchange the order of summation and integration, etc. Each $u(Z_Q)$ is expressible as

$$\int_{\mathbb{R}^d} K(Z_Q, s) f(s)\, ds,$$

where $K(Z_Q, \cdot)$ is the Poisson kernel we saw earlier (and likewise for $u(Z'_Q)$). Therefore a bound for the left-hand side of (1.6) will follow from the same bound for

$$\int_{\mathbb{R}^d} f(s) \left(\sum_{Q\in\mathcal{F}} \ell(Q)^{-1} (K(Z_Q, s) - K(Z'_Q, s)) \int_{T(Q)} g(t, y)\, d\mu(t, y) \right) ds, \tag{1.7}$$

valid for all f such that $\int_{\mathbb{R}^d} |f|^p\, v\, ds \le 1$ and all g satisfying $\int_{\mathbb{R}^{d+1}_+} |g|^{q'}\, d\mu = 1$. (Note that we have removed the absolute-value bars from g inside the integrals in (1.7).) By Hölder's inequality, with our normalization on f, (1.7) will have absolute value $\le C$ if

$$\int_{\mathbb{R}^d} \left| \sum_{Q\in\mathcal{F}} \ell(Q)^{-1} (K(Z_Q, s) - K(Z'_Q, s)) \int_{T(Q)} g(t, y)\, d\mu(t, y) \right|^{p'} v^{1-p'}\, ds \le C''. \tag{1.8}$$

For the rest of this section we will focus our attention on (1.8). Notice the "dual weight" $v^{1-p'}$. The weight will show up in various forms throughout our discussions.

We will handle the sum inside the big absolute-value bars via Littlewood–Paley theory. We can do this because each summand

$$\ell(Q)^{-1} (K(Z_Q, s) - K(Z'_Q, s)) \int_{T(Q)} g(t, y)\, d\mu(t, y) \tag{1.9}$$

can be written as $\lambda_Q b_{(Q)}(s)$, where the λ_Q's are real numbers and the $b_{(Q)}$'s are functions satisfying certain smoothness and decay conditions and possessing an additional property called "almost-orthogonality". We will say what our conditions on $b_{(Q)}$ are first. By "reverse engineering" it will then be easy to get the right bounds for λ_Q.

We ask that the functions $b_{(Q)}(s)$ satisfy three conditions:

1) (*Decay*) For some $\epsilon > 0$ (independent of Q) and all $s \in \mathbb{R}^d$,

$$|b_{(Q)}(s)| \le |Q|^{-1/2} \left(1 + \frac{|s - s_Q|}{\ell(Q)}\right)^{-d-\epsilon},$$

where s_Q is Q's center.

2) (*Smoothness*) For the same $\epsilon > 0$, some $\alpha > 0$ (independent of Q), and all s and s' in $\mathbb{R}^d$,

$$\begin{aligned} &|b_{(Q)}(s) - b_{(Q)}(s')| \\ &\quad \le \left(\frac{|s - s'|}{\ell(Q)}\right)^{\alpha} |Q|^{-1/2} \left(\left(1 + \frac{|s - s_Q|}{\ell(Q)}\right)^{-d-\epsilon} + \left(1 + \frac{|s' - s_Q|}{\ell(Q)}\right)^{-d-\epsilon} \right). \end{aligned}$$

3) (*Almost-orthogonality*) For all finite linear sums $\sum \gamma_Q b_{(Q)}$,

$$\int_{\mathbb{R}^d} \left|\sum \gamma_Q b_{(Q)}(s)\right|^2 ds \le \sum |\gamma_Q|^2.$$

It is well known (see [FrJaWeis]) that, if 1) and 2) hold, then 3) will follow (modulo a multiplicative constant) if the $b_{(Q)}$'s also satisfy $\int b_{(Q)} = 0$. In Section 2 we extend this fact to the case of a doubling measure replacing Lebesgue measure.

Our estimates follow from familiar facts about $K(Z, s)$.

Facts about $K(Z, s)$.

1k) If $Z \in \tilde{T}(Q)$ then

$$|K(Z,s)| \le C_d |Q|^{-1}\left(1+\frac{|s-s_Q|}{\ell(Q)}\right)^{-d-1}$$

for any $s \in \mathbb{R}^d$.

2k) If $Z \in \tilde{T}(Q)$ and s and s' are in $\mathbb{R}^d$ then

$$|K(Z,s)-K(Z,s')| \le \left(\frac{|s-s'|}{\ell(Q)}\right) C_d |Q|^{-1}\left(\left(1+\frac{|s-s_Q|}{\ell(Q)}\right)^{-d-1}+\left(1+\frac{|s'-s_Q|}{\ell(Q)}\right)^{-d-1}\right).$$

3k) $\int_{\mathbb{R}^d} K(Z,s)\,ds = 1$.

Remark. The decay exponent in 2k) is actually $-d-2$, but $-d-1$ works for us and it is, for the *illustrative* purposes of this section, perfectly adequate.

Property 3k) implies $\int_{\mathbb{R}^d} b_{(Q)}\,ds = 0$. For the others, we observe that

$$\left|\ell(Q)^{-1}(K(Z_Q,s)-K(Z'_Q,s))\int_{T(Q)} g(t,y)\,d\mu(t,y)\right| \le C_d \ell(Q)^{-1}|Q|^{-1}\left(1+\frac{|s-s_Q|}{\ell(Q)}\right)^{-d-1}\int_{T(Q)}|g(t,y)|\,d\mu(t,y)$$

and

$$\begin{aligned}|(K(Z_Q,s)-K(Z_Q,s'))-(K(Z'_Q,s)-K(Z'_Q,s'))| &\times \left|\ell(Q)^{-1}\int_{T(Q)} g(t,y)\,d\mu(t,y)\right| \\ &\le C_d \ell(Q)^{-1}|Q|^{-1}\left(\frac{|s-s'|}{\ell(Q)}\right)\left(\left(1+\frac{|s-s_Q|}{\ell(Q)}\right)^{-d-1}+\left(1+\frac{|s'-s_Q|}{\ell(Q)}\right)^{-d-1}\right) \\ &\times \int_{T(Q)}|g(t,y)|\,d\mu(t,y),\end{aligned}$$

from which it follows that we can set

$$\begin{aligned}\lambda_Q &= C_d \ell(Q)^{-1}|Q|^{-1/2}\int_{T(Q)}|g(t,y)|\,d\mu(t,y) \\ &\le C_d \ell(Q)^{-1}|Q|^{-1/2}\left(\int_{T(Q)}|g(t,y)|^{q'}\,d\mu(t,y)\right)^{1/q'}\mu(T(Q))^{1/q} \qquad (1.10)\end{aligned}$$

We will actually be taking λ_Q to be the *second* (potentially larger) quantity (1.10).

We can write

$$\sum_{Q\in\mathcal{F}} \ell(Q)^{-1}(K(Z_Q,s)-K(Z'_Q,s))\int_{T(Q)} g(t,y)\,d\mu(t,y)$$

as a finite linear sum $\sum_{Q\in\mathcal{F}}\lambda_Q b_{(Q)}(s)$, where the $b_{(Q)}$'s satisfy the decay, smoothness, and almost-orthogonality conditions given above.

The following theorem gives us a way to control such sums.

Theorem 1.1. *Suppose that $0<r<\infty$, $\sigma\in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ is a Muckenhoupt A^∞ weight, and the family $\{b_{(Q)}\}_{Q\in\boldsymbol{D}}$ satisfies the decay, smoothness and almost-orthogonality conditions given above. If $\rho>d$, there is a constant $C=C(r,\rho,d,\sigma,\epsilon,\alpha)$ such that, for all finite linear sums $f(s)=\sum_{Q\in\mathcal{F}\subset\mathcal{D}}\lambda_Q b_{(Q)}(s)$, indexed over finite families $\mathcal{F}$ of dyadic cubes,*

$$\begin{aligned}\int_{\mathbb{R}^d}\Big|\sum_{Q\in\mathcal{F}\subset\mathcal{D}}\lambda_Q b_{(Q)}(s)\Big|^r\sigma(s)\,ds&=\int_{\mathbb{R}^d}|f(s)|^r\sigma(s)\,ds\\ &\le C\int_{\mathbb{R}^d}\bigg(\sum_{Q\in\mathcal{F}}\frac{\lambda_Q^2}{|Q|}\Big(1+\frac{|s-s_Q|}{l(Q)}\Big)^{-2d-2\epsilon+\rho}\bigg)^{r/2}\sigma(s)\,ds=C\int_{\mathbb{R}^d}(g^*(f))^r\sigma\,ds.\end{aligned}$$

Remark. Theorem 1.1 is essentially proved in [Wi1]. In that paper the smoothness condition 2) is replaced by the hypothesis that $b_{(Q)}$ is C^1 and that

$$|\nabla b_{(Q)}(s)|\ \le |Q|^{-1/2}\ell(Q)^{-1}\Big(1+\frac{|s-s_Q|}{\ell(Q)}\Big)^{-d-\epsilon},$$

which implies our smoothness condition (modulo a constant) for $\alpha=1$. In the following section we will prove Theorem 1.1 along with a generalization of the main theorem from [Wi1]. For now the reader should concentrate on the application of Theorem 1.1 to the case at hand.

We will apply Theorem 1.1 to our sum $\sum_{Q\in\mathcal{F}}\lambda_Q b_{(Q)}(s)$ when $r=p'$ and $\sigma=$ the "dual weight" $v^{1-p'}$ from (1.8). Therefore, *for the rest of this section, we will assume that $v^{1-p'}$ belongs to A^∞*. To make our "model case" discussion specific (and easier to follow), we will set $\rho=d+1$.

Theorem 1.1 says that (1.8) will hold if

$$\int_{\mathbb{R}^d}\bigg(\sum_{Q\in\mathcal{F}}\frac{|\lambda_Q|^2}{|Q|}\Big(1+\frac{|s-s_Q|}{\ell(Q)}\Big)^{-d-1}\bigg)^{p'/2}\sigma(s)\,ds\le C, \tag{1.11}$$

because, in our model case, $\epsilon=1$ and we have put $\rho=d+1$. We will now show what conditions on μ and σ (hence, indirectly, on v) imply this. It is here that our special restrictions on p and q will come into play.

We first treat the case where $p\ge 2$. This implies $p'\le 2$, which makes the left-hand side of (1.11) less than or equal to

$$\sum_{Q\in\mathcal{F}}\frac{|\lambda_Q|^{p'}}{|Q|^{p'/2}}\int_{\mathbb{R}^d}\Big(1+\frac{|s-s_Q|}{\ell(Q)}\Big)^{-(p'/2)(d+1)}\sigma(s)\,ds,$$

which is less than or equal to a constant times

$$\sum_{Q\in\mathcal{F}} \ell(Q)^{-p'}|Q|^{-p'}\Big(\int_{T(Q)} |g(t,y)|^{q'}\,d\mu(t,y)\Big)^{p'/q'} \mu(T(Q))^{p'/q} \times \int_{\mathbb{R}^d}\Big(1+\frac{|s-s_Q|}{\ell(Q)}\Big)^{-(p'/2)(d+1)}\sigma(s)\,ds. \tag{1.12}$$

Having $p\le q$ forces $q'/p'\le 1$, which implies

$$\sum_{Q\in\mathcal{F}}\Big(\int_{T(Q)} |g(t,y)|^{q'}\,d\mu(t,y)\Big)^{p'/q'} \le \Big(\sum_{Q\in\mathcal{F}}\int_{T(Q)} |g(t,y)|^{q'}\,d\mu(t,y)\Big)^{p'/q'} = 1,$$

by our normalization of g. Therefore (1.12) will be less than or equal to an absolute constant if

$$\ell(Q)^{-p'}|Q|^{-p'}\mu(T(Q))^{p'/q}\int_{\mathbb{R}^d}\Big(1+\frac{|s-s_Q|}{\ell(Q)}\Big)^{-(p'/2)(d+1)}\sigma\,ds \le C$$

for all $Q\in\mathcal{D}$; and that will hold if

$$\mu(T(Q))^{1/q}\Big(\int_{\mathbb{R}^d}\Big(1+\frac{|s-s_Q|}{\ell(Q)}\Big)^{-(p'/2)(d+1)}\sigma\,ds\Big)^{1/p'} \le C\ell(Q)|Q| \tag{1.13}$$

for all $Q\in\mathcal{D}$.

Now we suppose $p<2$ (but, as always, q is ≥ 2). Then $p'/2>1$ and $q'/2\le 1$. Let $\tau>1$ be the dual exponent to $p'/2$ and let h be a non-negative function in $L^\tau(\sigma)$ such that $\int h^\tau\sigma\,ds=1$ and

$$\begin{aligned}
&\int_{\mathbb{R}^d}\Big(\sum_{Q\in\mathcal{F}}\frac{|\lambda_Q|^2}{|Q|}\Big(1+\frac{|s-s_Q|}{\ell(Q)}\Big)^{-d-1}\Big)^{p'/2}\sigma(s)\,ds\\
&\quad=\Big(\int_{\mathbb{R}^d}\Big(\sum_{Q\in\mathcal{F}}\frac{|\lambda_Q|^2}{|Q|}\Big(1+\frac{|s-s_Q|}{\ell(Q)}\Big)^{-d-1}\Big)h(s)\sigma(s)\,ds\Big)^{p'/2}\\
&\quad=\Big(\sum_{Q\in\mathcal{F}}\frac{|\lambda_Q|^2}{|Q|}\int_{\mathbb{R}^d}\Big(1+\frac{|s-s_Q|}{\ell(Q)}\Big)^{-d-1}h(s)\sigma(s)\,ds\Big)^{p'/2}.
\end{aligned}\tag{1.14}$$

Because of our normalization on h,

$$\int_{\mathbb{R}^d}\Big(1+\frac{|s-s_Q|}{\ell(Q)}\Big)^{-d-1}h(s)\sigma(s)\,ds \le \Big(\int_{\mathbb{R}^d}\Big(1+\frac{|s-s_Q|}{\ell(Q)}\Big)^{-(p'/2)(d+1)}\sigma(s)\,ds\Big)^{2/p'}.$$

Therefore (1.14) will be less than or equal to an absolute constant if

$$\sum_{Q\in\mathcal{F}}\frac{|\lambda_Q|^2}{|Q|}\Big(\int_{\mathbb{R}^d}\Big(1+\frac{|s-s_Q|}{\ell(Q)}\Big)^{-(p'/2)(d+1)}\sigma(s)\,ds\Big)^{2/p'} \le C. \tag{1.15}$$

When we plug (1.10), our bound for λ_Q, into (1.15), we see that (1.15) will follow if

$$\sum_{Q\in\mathcal{F}} \ell(Q)^{-2}|Q|^{-2}\Big(\int_{T(Q)} |g(t,y)|^{q'}\, d\mu(t,y)\Big)^{2/q'} \mu(T(Q))^{2/q} \times \Big(\int_{\mathbb{R}^d} \Big(1+\frac{|s-s_Q|}{\ell(Q)}\Big)^{-(p'/2)(d+1)} \sigma(s)\, ds\Big)^{2/p'} \le C.$$

Since $2/q' \ge 1$,

$$\sum_{Q\in\mathcal{F}} \Big(\int_{T(Q)} |g(t,y)|^{q'}\, d\mu(t,y)\Big)^{2/q'} \le \Big(\sum_{Q\in\mathcal{F}} \int_{T(Q)} |g(t,y)|^{q'}\, d\mu(t,y)\Big)^{2/q'} = 1,$$

because of g's normalization. Therefore the bound we seek will hold if

$$\ell(Q)^{-2}|Q|^{-2}\mu(T(Q))^{2/q}\Big(\int_{\mathbb{R}^d} \Big(1+\frac{|s-s_Q|}{\ell(Q)}\Big)^{-(p'/2)(d+1)} \sigma\, ds\Big)^{2/p'} \le C$$

for all $Q \in \mathcal{D}$. As in the preceding case, this will hold if (1.13) holds for all Q.

2. Littlewood–Paley theory on homogeneous spaces

2.1. The Littlewood–Paley inequality. In this section we continue to work only on $\mathbb{R}^d$. We begin by defining a few basic terms. Most of these are standard.

If Q is a cube in $\mathbb{R}^d$, with sides parallel to the coordinate axes, and $A \ge 1$, then AQ denotes the concentric A-fold dilate of Q. We will have occasion to refer to certain "annuli" around cubes Q. We set $R_0(Q) \equiv Q$, and, if $j \ge 1$, then $R_j(Q) \equiv 2^jQ \setminus 2^{j-1}Q$.

A positive measure ω on $\mathbb{R}^d$ is called *doubling* if there is a constant C such that, for all cubes Q,

$$\omega(2Q) \le C'\omega(Q).$$

In this section, ω will denote a fixed but arbitrary, non-trivial doubling measure on $\mathbb{R}^d$.

A cube $Q \subset \mathbb{R}^d$ is called *dyadic* if it has the form

$$Q = \Big[\frac{j_1}{2^k}, \frac{j_1+1}{2^k}\Big) \times \cdots \times \Big[\frac{j_d}{2^k}, \frac{j_d+1}{2^k}\Big),$$

where k and $j_1, \ldots, j_d$ are integers. Such a cube is said to have a side length of 2^{-k}, which we denote by $\ell(Q)$. We recall that, given any two dyadic cubes Q and Q', either $Q \cap Q' = \emptyset$ or one of them is contained in the other. We denote the family of dyadic cubes by $\mathcal{D}$.

A measure v on $\mathbb{R}^d$ is said to be A^∞ relative to ω (written $v \in A^\infty(\omega)$) if there are positive constants a and b such that, for all cubes Q and measurable subsets $E \subset Q$,

$$\frac{v(E)}{v(Q)} \le a\Big(\frac{\omega(E)}{\omega(Q)}\Big)^b. \tag{2a.1}$$

It is an easy exercise to show that, since ω is doubling and non-trivial, $\omega(Q) > 0$ for all Q.

If the reader is worried by the possibility that $v(Q) = 0$, he is free to rewrite (2a.1) as

$$v(E) \leq a\left(\frac{\omega(E)}{\omega(Q)}\right)^b v(Q), \tag{2a.2}$$

since that is the form we will be using anyway.

These definitions have been standard. Our next one is something special.

We shall say that a family of functions, $b_{(Q)} : \mathbb{R}^d \to \mathbb{R}$, indexed over $\mathcal{D}$, is a *standard family* if it satisfies the following size, smoothness, and (weak) cancellation conditions. The [positive] numbers α and β depend only on the family $\{b_{(Q)}\}_Q$.

1) (*Size*) If $x \in R_j(Q)$, then

$$|b_{(Q)}(x)| \leq \sqrt{\omega(Q)}\frac{2^{-j\alpha}}{\omega(2^j Q)};$$

or, more succinctly:

$$|b_{(Q)}(x)| \leq \sqrt{\omega(Q)}\sum_{j=0}^{\infty}\frac{2^{-j\alpha}}{\omega(2^j Q)}\chi_{R_j(Q)}(x).$$

2) (*Smoothness*) For any x and y in $\mathbb{R}^d$,

$$|b_{(Q)}(x) - b_{(Q)}(y)| \leq \left(\frac{|x-y|}{\ell(Q)}\right)^\beta \sqrt{\omega(Q)}\sum_{j=0}^{\infty}\frac{2^{-j\alpha}}{\omega(2^j Q)}(\chi_{R_j(Q)}(x) + \chi_{R_j(Q)}(y)).$$

Note that, given the size condition, the smoothness condition is only meaningful when $|x - y| \leq \ell(Q)$.

3) (*Cancellation*) For every finite linear combination $\sum_Q \gamma_Q b_{(Q)}$,

$$\int_{\mathbb{R}^d}\left|\sum_Q \gamma_Q b_{(Q)}\right|^2 d\omega \leq \sum_Q |\gamma_Q|^2.$$

All of our results depend on the next theorem which is a rephrasing of Theorem 1.1.

THEOREM 2.1. *Let $\{b_{(Q)}\}_{Q\in\mathcal{D}}$ be a standard family of functions, and let $v \in A^\infty(\omega)$. If $0 < p < \infty$ and $0 < \tau < 2\alpha$, there is a constant $C = C(v, \omega, \alpha, \beta, \tau, p, d)$ such that, for all finite linear sums $f = \sum_{Q\in\mathcal{F}} \lambda_Q b_{(Q)}$,*

$$\int_{\mathbb{R}^d} |f|^p\, dv \leq C \int_{\mathbb{R}^d} (g^*(f))^p\, dv,$$

where

$$g^*(f)(s) \equiv \left(\sum_{Q\in\mathcal{F}} |\lambda_Q|^2 \left[\sum_{j=0}^{\infty}\frac{2^{-j(2\alpha-\tau)}}{\omega(2^j Q)}\chi_{R_j(Q)}(s)\right]\right)^{1/2}.$$

REMARK. The proof will show that C's dependence on v is really a dependence on a and b (in the definition of "$v \in A^\infty(\omega)$"). The dependence on ω is really a dependence on d and on ω's doubling constant C'.

REMARK. This generalizes the main theorem proved in [Wi1], and the proof given here closely follows the earlier proof. The reader might want to refer to [Wi1] now and then to understand what is going on.

The chief virtue of Theorem 2.1 for our purposes is that it does not ask too much of the functions $b_{(Q)}$, while yielding a fairly good Littlewood–Paley estimate. In particular, we do not require the functions to decay especially fast, either in size or in their moduli of Hölder continuity, and our cancellation condition is simply "almost-orthogonality". In our present application, the mild decay and smoothness conditions—and nothing better—follow from classical estimates on kernel functions for the Laplacian (or even general elliptic operators), while the almost-orthogonality is a consequence of Green's Theorem (or the argument in the proof of Theorem 4.2).

The key to our argument lies in defining the right maximal function. Let us assume that we have a fixed finite linear combination $f = \sum_Q \lambda_Q b_{(Q)}$. If $I \in \mathcal{D}$, we define $S(I) \equiv \{Q \in \mathcal{D} : Q \not\subset I\}$. It is useful to think of $S(I)$ as the family of dyadic cubes that "surround" I. If $x \in I$, we define

$$F(I,x) \equiv \sum_{Q\,:\,Q\in S(I)} \lambda_Q b_{(Q)}(x),$$

and we do not define $F(I,x)$ for $x \notin I$. If x_I is the center of I, then we set $F(I) \equiv F(I,x_I)$. The right maximal function for the Littlewood–Paley function $g^*(f)$ turns out to be

$$F^*(x) \equiv \sup_{I\,:\,x\in I} |F(I)|.$$

Corresponding to $F^*(x)$ is a "maximal square function" adapted to $g^*(f)$. For $x \in I$, we define

$$G(I,x) \equiv \Bigg(\sum_{Q\in S(I)} |\lambda_Q|^2 \Bigg[\sum_{j=0}^{\infty} \frac{2^{-j(2\alpha-\tau)}}{\omega(2^j Q)} \chi_{R_j(Q)}(x)\Bigg]\Bigg)^{1/2},$$

and we do not define $G(I,x)$ for $x \notin I$. We similarly define $G(I) \equiv G(I,x_I)$ and

$$G^*(x) \equiv \sup_{I:x\in I} G(I).$$

In order to prove Theorem 2.1, we shall prove seven fairly elementary lemmas, followed by a difficult lemma, which is really the heart of the proof of Theorem 2.1. These lemmas are directly analogous to, respectively, Lemmas 1–7 and the Main Lemma in [Wi1]. Our more general formulation of the $b_{(Q)}$'s requires us to surmount some non-trivial technical obstacles.

Lemma 2.2. *For ω-a.e. x, $|f(x)| \le F^*(x)$.*

Proof. The inequality is obviously true Lebesgue almost everywhere. However, the only exceptional points lie on the faces of dyadic cubes, and these have ω-measure 0, because ω is doubling. ■

Lemma 2.3. *There is a constant C such that $G^*(x) \le C g^*(f)(x)$ ω-almost everywhere.*

Proof. Let $I \in \mathcal{D}$ and $x \in I$. We need to show that $G(I) \le C g^*(f)(x)$, for which it is clearly sufficient to show that

$$(G(I))^2 \le C \sum_{Q\,:\,Q\in S(I)} |\lambda_Q|^2 \Bigg[\sum_{j=0}^{\infty} \frac{2^{-j(2\alpha-\tau)}}{\omega(2^j Q)} \chi_{R_j(Q)}(x)\Bigg],$$

where (recall the definition above)

$$(G(I))^2 = \sum_{Q\,:\,Q\in S(I)} |\lambda_Q|^2 \Big[\sum_{j=0}^{\infty} \frac{2^{-j(2\alpha-\tau)}}{\omega(2^j Q)} \chi_{R_j(Q)}(x_I)\Big].$$

Comparing the sums termwise, we see that our inequality amounts to having

$$\sum_{j=0}^{\infty} \frac{2^{-j(2\alpha-\tau)}}{\omega(2^j Q)} \chi_{R_j(Q)}(x_I) \le C \sum_{j=0}^{\infty} \frac{2^{-j(2\alpha-\tau)}}{\omega(2^j Q)} \chi_{R_j(Q)}(x)$$

for any $x \in I$ and any $Q \in S(I)$. Let us now consider two cases: $\ell(Q) \ge \ell(I)$ and $\ell(Q) < \ell(I)$. In the former case, if $x_I \in R_j(Q)$, then x must belong to $R_{j'}(Q)$ for some $j' \le j + C(d)$, and the result follows because $\omega(2^{j'}Q) \le C''\omega(2^j Q)$. In the latter case, observe that, if we let x_Q denote the center of Q, then $|x - x_Q| \le C(d)|x_I - x_Q|$ (which also holds in the former case, but is easier to see for small Q), and the same inequality holds, for the same reason.

Lemma 2.4. *Let $0 < \eta < 0.1$. There is a $C > 0$ such that, if $x \in I$ and $d(x, \partial I) \ge \eta l(I)$, then $C^{-1}G(I, x) \le G(I) \le CG(I, x)$.*

Proof. The result depends on two simple facts. First: If $x \in I$ and $d(x, \partial I) \ge \eta\ell(I)$, then, for any $Q \in S(I)$, x will belong to $R_j(Q)$ and x_I will belong to $R_{j'}(Q)$ for some j and j' such that $|j - j'| \le C(\eta, d)$. Second: For such j and j', the ratio of $\omega(2^j Q)$ and $\omega(2^{j'}Q)$ is bounded above and below by positive constants depending only on η, d, and the doubling constant of ω. The inequality now follows by termwise comparison of the two sums defining $(G(I, x))^2$ and $(G(I))^2$. ∎

Lemma 2.5. *Let $0 < \eta < 0.1$. There is a $C > 0$ such that, if $x \in I$ and $d(x, \partial I) \ge \eta l(I)$, then $|F(I) - F(I, x)| \le CG(I)$.*

Proof. Write

$$\begin{aligned} |F(I) - F(I, x)| &= \Big| \sum_{Q\,:\,Q\in S(I)} \lambda_Q (b_{(Q)}(x_I) - b_{(Q)}(x)) \Big| \\ &\le \Big| \sum_{\substack{Q\,:\,Q\in S(I)\\ \ell(Q)\le\ell(I)}} \lambda_Q (b_{(Q)}(x_I) - b_{(Q)}(x)) \Big| \\ &\quad + \Big| \sum_{\substack{Q\,:\,Q\in S(I)\\ \ell(Q)>\ell(I)}} \lambda_Q (b_{(Q)}(x_I) - b_{(Q)}(x)) \Big| \equiv (I) + (II). \end{aligned}$$

We will control (I) by using only the size condition 1). The sum (II) will be controlled via the smoothness condition 2).

Handling (I). It is enough to bound

$$\sum_{\substack{Q\,:\,Q\in S(I)\\ \ell(Q)\le\ell(I)}} |\lambda_Q|\,|b_{(Q)}(x)|,$$

uniformly for $x \in I$, $d(x, \partial I) \ge \eta\ell(I)$, since (I) is less than or equal to a sum of two such terms.

By the Cauchy–Schwarz inequality and our size estimate 1),

$$
\begin{aligned}
(I) &\leq \Bigg(\sum_{\substack{Q\,:\,Q\in S(I)\\ \ell(Q)\leq\ell(I)}} |\lambda_Q|^2 \Bigg[\sum_{j=0}^{\infty} \frac{2^{-j(2\alpha-\tau)}}{\omega(2^jQ)} \chi_{R_j(Q)}(x)\Bigg]\Bigg)^{1/2} \\
&\quad \times \Bigg(\sum_{\substack{Q\,:\,Q\in S(I)\\ \ell(Q)\leq\ell(I)}} \omega(Q) \Bigg[\sum_{j=0}^{\infty} \frac{2^{-j\tau}}{\omega(2^jQ)} \chi_{R_j(Q)}(x)\Bigg]\Bigg)^{1/2} \\
&= G(I,x)\Bigg(\sum_{\substack{Q\,:\,Q\in S(I)\\ \ell(Q)\leq\ell(I)}} \omega(Q) \Bigg[\sum_{j=0}^{\infty} \frac{2^{-j\tau}}{\omega(2^jQ)} \chi_{R_j(Q)}(x)\Bigg]\Bigg)^{1/2} \\
&\leq CG(I)\Bigg(\sum_{\substack{Q\,:\,Q\in S(I)\\ \ell(Q)\leq\ell(I)}} \omega(Q) \Bigg[\sum_{j=0}^{\infty} \frac{2^{-j\tau}}{\omega(2^jQ)} \chi_{R_j(Q)}(x)\Bigg]\Bigg)^{1/2},
\end{aligned}
$$

where the last line follows from Lemma 2.4. We need to show that the *second* factor in the last line is bounded by an absolute constant.

Write $\mathbb{R}^d \setminus I = \bigcup_l I_l$, where the I_l's are congruent copies of I (thus, $\{I\} \cup \{I_l\}_l$ tiles $\mathbb{R}^d$). We set

$$
H(l) \equiv \sum_{Q\,:\,Q\subset I_l} \omega(Q) \Bigg[\sum_{j=0}^{\infty} \frac{2^{-j\tau}}{\omega(2^jQ)} \chi_{R_j(Q)}(x)\Bigg].
$$

For $k = 0, 1, 2, \ldots$, let S_k denote the set of I_l's such that $I_l \cap R_k(I) \neq \emptyset$. These cubes are at a distance of approximately $2^k\ell(I)$ from $x \in I$. Indeed, there is a constant C, independent of x, such that $x \in R_{k'}(I_l)$ for some k with $|k - k'| \leq C$.

Let us temporarily fix $I_l \in S_k$ and consider $Q \subset I_l$ with $\ell(Q) = 2^{-m}\ell(I)$. If $x \in I$ and $d(x, \partial I) \geq \eta\ell(I)$, then $x \in R_j(Q)$ for some j which is approximately equal to $m + k$; i.e., there is C such that $x \in R_j(Q)$ for $|m + k - j| \leq C$. This is simply another way of saying that $d(x, Q)$ is approximately $2^{m+k}\ell(Q) = 2^k\ell(I)$. Therefore, for such I_l,

$$
H(l) \leq C \sum_{m=0}^{\infty} \sum_{\substack{Q\,:\,Q\subset I_l\\ \ell(Q)=2^{-m}\ell(I)}} \omega(Q) \frac{2^{-(m+k)\tau}}{\omega(2^{m+k}Q)}.
$$

Notice that, because of ω's doubling property, all of the numbers $\omega(2^{m+k}Q)$ are comparable to $\omega(2^kI)$, with comparability constants which only depend on d and ω's doubling constant. Therefore,

$$
\begin{aligned}
H(l) &\leq C\omega(2^kI)^{-1} \sum_{m=0}^{\infty} \sum_{\substack{Q\,:\,Q\subset I_l\\ \ell(Q)=2^{-m}\ell(I)}} 2^{-(m+k)\tau}\omega(Q) \\
&= C2^{-k\tau}\omega(2^kI)^{-1} \sum_{m=0}^{\infty} 2^{-m\tau}\omega(I_l) \leq C2^{-k\tau}\omega(I_l)\omega(2^kI)^{-1}.
\end{aligned}
$$

If we now sum over the I_l's in a fixed S_k, we get

$$\sum_{l\,:\,I_l\in S_k} H(l) \le C2^{-k\tau}\omega(2^kI)^{-1}\sum_{l\,:\,I_l\in S_k}\omega(I_l) \le C2^{-k\tau}\omega(2^kI)^{-1}\omega(2^kI) = C2^{-k\tau},$$

where the second inequality follows from ω's doubling property. Summing over all k yields $\sum H(l) \le C$, and term (I) has been bounded.

Handling (II). This one is easier:

$$\begin{aligned}(II) &= \Big|\sum_{\substack{Q\,:\,Q\in S(I)\\ \ell(Q)>\ell(I)}} \lambda_Q(b_{(Q)}(x_I) - b_{(Q)}(x))\Big| \\ &\le \sum_{\substack{Q\,:\,Q\in S(I)\\ \ell(Q)>\ell(I)}} |\lambda_Q|\Big(\frac{|x_I - x|}{\ell(Q)}\Big)^{\beta}\sqrt{\omega(Q)}\sum_{j=0}^{\infty}\frac{2^{-j\alpha}}{\omega(2^jQ)}(\chi_{R_j(Q)}(x_I) + \chi_{R_j(Q)}(x)).\end{aligned}$$

Since we are considering cubes Q that are larger than I, if $x_I \in R_j(Q)$ and $x \in I$, then $x \in R_{j'}(Q)$ for some j' such that $|j - j'| \le C$. Therefore,

$$(II) \le C\sum_{\substack{Q\,:\,Q\in S(I)\\ \ell(Q)>\ell(I)}} |\lambda_Q|\Big(\frac{|x_I - x|}{\ell(Q)}\Big)^{\beta}\sqrt{\omega(Q)}\sum_{j=0}^{\infty}\frac{2^{-j\alpha}}{\omega(2^jQ)}\chi_{R_j(Q)}(x_I).$$

Continuing, this is bounded by

$$\begin{aligned} &C\sum_{\substack{Q\,:\,Q\in S(I)\\ \ell(Q)>\ell(I)}} |\lambda_Q|\Big(\frac{\ell(I)}{\ell(Q)}\Big)^{\beta}\sqrt{\omega(Q)}\sum_{j=0}^{\infty}\frac{2^{-j\alpha}}{\omega(2^jQ)}\chi_{R_j(Q)}(x_I) \\ &\qquad = C\sum_{k=1}^{\infty}\sum_{Q\,:\,\ell(Q)=2^k\ell(I)} |\lambda_Q|2^{-k\beta}\sqrt{\omega(Q)}\sum_{j=0}^{\infty}\frac{2^{-j\alpha}}{\omega(2^jQ)}\chi_{R_j(Q)}(x_I) \\ &\qquad \le CG(I)\Big(\sum_{k=1}^{\infty}\sum_{Q\,:\,\ell(Q)=2^k\ell(I)} 2^{-2k\beta}\omega(Q)\sum_{j=0}^{\infty}\frac{2^{-j\tau}}{\omega(2^jQ)}\chi_{R_j(Q)}(x_I)\Big)^{1/2},\end{aligned}$$

where the last line follows from the Cauchy–Schwarz inequality.

We now need to show that the second factor is bounded by a constant.

We temporarily fix k and consider

$$\sum_{Q\,:\,\ell(Q)=2^k\ell(I)} \omega(Q)\sum_{j=0}^{\infty}\frac{2^{-j\tau}}{\omega(2^jQ)}\chi_{R_j(Q)}(x_I). \tag{2.3}$$

If $\chi_{R_j(Q)}(x_I) \ne 0$ then $Q \subset 2^{k+j'}I$ for some j' such that $|j - j'| \le C$. Also, for such Q, $\omega(2^jQ)$ will be comparable to $\omega(2^{k+j'}I)$, because ω is doubling. Therefore, for each j,

$$\sum_{\substack{Q\,:\,\ell(Q)=2^k\ell(I)\\ \chi_{R_j(Q)}(x_I)\ne 0}} \omega(Q)\frac{2^{-j\tau}}{\omega(2^jQ)} \le C2^{-j\tau}.$$

Summing on j, we see that (2.3) is bounded by a constant. If we now multiply this by $2^{-2k\beta}$ and sum on k, we get our result. Lemma 2.5 is proved. ■

If I is a dyadic cube, we define $N(I) \equiv \{I^* \in \mathcal{D} : I^* \subset I,\ \ell(I^*) = \ell(I)/2\}$, the "next generation" of cubes "begotten" by I.

Lemma 2.6. *If $I^* \in N(I)$, then $G(I) \le CG(I^*)$.*

Proof. By Lemma 2.4, $G(I) = G(I, x_I) \le CG(I, x_{I^*})$. But $G(I^*) = G(I^*, x_{I^*}) \ge G(I, x_{I^*})$. ■

Lemma 2.7. *There is a positive constant C such that, for all $I^* \in N(I)$ and all $x \in I$, $G^*(x) \ge CG(I^*)$.*

Proof. If $x \in I^*$, there is nothing to prove. So, let $x \in I \setminus I^*$, and let $L \subset I \setminus I^*$ be a dyadic cube such that $x \in L$ and L is smaller than any of the cubes Q in the sum $\sum \lambda_Q b_{(Q)}$ defining f. Then

$$G(L)^2 = \sum_{J\,:\,J \in S(L)} |\lambda_J|^2 \Bigg[\sum_{j=0}^{\infty} \frac{2^{-j(2\alpha-\tau)}}{\omega(2^j J)} \chi_{R_j(J)}(x_L)\Bigg]$$
$$\ge \sum_{J\,:\,J \in S(I^*)} |\lambda_J|^2 \Bigg[\sum_{j=0}^{\infty} \frac{2^{-j(2\alpha-\tau)}}{\omega(2^j J)} \chi_{R_j(J)}(x_L)\Bigg],$$

because the second sum *excludes* the J's contained in I^*. (Technically, the first sum excludes the J's contained in L, but, because L is so small, these contribute nothing to the sum.)

The lemma will be proved once we show that, for $J \in S(I^*)$,

$$\sum_{j=0}^{\infty} \frac{2^{-j(2\alpha-\tau)}}{\omega(2^j J)} \chi_{R_j(J)}(x_L) \ge C \sum_{j=0}^{\infty} \frac{2^{-j(2\alpha-\tau)}}{\omega(2^j J)} \chi_{R_j(J)}(x_{I^*}).$$

But this follows from what are (by now) "the usual reasons". Simply note that, if $J \in S(I^*)$, then $|x_L - x_J| \le C(d)|x_{I^*} - x_J|$. Thus, if $\frac{2^{-j(2\alpha-\tau)}}{\omega(2^j J)} \chi_{R_j(J)}(x_{I^*}) \ne 0$, then $\frac{2^{-j'(2\alpha-\tau)}}{\omega(2^{j'} J)} \chi_{R_{j'}(J)}(x_L)$ will also be non-zero for some j' such that $2^{j'} \le C2^j$. The doubling property of ω ensures that $\omega(2^{j'} J) \le C\omega(2^j J)$, which finishes the proof of Lemma 2.7. ■

Lemma 2.8. *There is a positive constant C such that, if $I^* \in N(I)$, it follows that $|F(I) - F(I^*)| \le CG(I^*)$.*

Proof. We have

$$\begin{aligned}|F(I) - F(I^*)| &= |F(I, x_I) - F(I^*, x_{I^*})| \\ &\le |F(I, x_I) - F(I, x_{I^*})| + |F(I, x_{I^*}) - F(I^*, x_{I^*})|.\end{aligned}$$

The first difference is $\le CG(I) \le G(I^*)$, by Lemmas 2.5 and 2.6. The second is less than or equal to

$$\sum_{\substack{Q\,:\,Q \subset I \\ Q \not\subset I^*}} |\lambda_Q|\, |b_{(Q)}(x_{I^*})|;$$

which, by the Cauchy–Schwarz inequality (see the proof of Lemma 2.5) is less than or equal to

$$G(I^*)\Big(\sum_{\substack{Q\,:\,Q\subset I\\ Q\not\subset I^*}}\omega(Q)\Big[\sum_{j=0}^{\infty}\frac{2^{-j\tau}}{\omega(2^jQ)}\chi_{R_j(Q)}(x_{I^*})\Big]\Big)^{1/2}.$$

But the first part of the proof of Lemma 2.5 shows that the second factor is bounded by a constant. ∎

We are now ready to prove Lemma 2.9, from which Theorem 2.1 will follow as a corollary.

Lemma 2.9. *Let $Q_0 \in \mathcal{D}$ be the dyadic unit cube, and let $\{b_{(Q)}\}_{Q\in\mathcal{D}}$ be a family of functions satisfying* 1), 2), *and* 3). *Let $f = \sum_Q \lambda_Q b_{(Q)}$ be a finite linear combination such that $\lambda_Q = 0$ for all Q not contained in Q_0. Then for each $\delta > 0$, there is a $\gamma > 0$ such that*

$$\omega(\{x \in Q_0 : F^*(x) > 1,\, G^*(x) \le \gamma\}) \le \delta\omega(Q_0). \tag{2.4}$$

Remark. If $\upsilon \in A^\infty(\omega)$, then Lemma 2.9 immediately implies the same conclusion for υ; i.e., for all $\delta > 0$ there is a $\gamma > 0$ such that

$$\upsilon(\{x \in Q_0 : F^*(x) > 1,\, G^*(x) \le \gamma\}) \le \delta\upsilon(Q_0).$$

The corollary is what we will use to obtain Theorem 2.1.

Proof of Lemma 2.9. Let $A > 1$ be a large number, to be chosen presently. Let $\{I_k\}_k$ be the family of maximal dyadic subcubes of Q_0 having the property that, for some $I^* \in N(I_k)$,

$$G(I^*) > A\gamma.$$

By Lemma 2.7, if A is chosen large enough, the set $\{x \in Q_0 : G^*(x) > \gamma\}$ will contain $\bigcup_k I_k$. We henceforth assume that A has been chosen "large enough". On the other hand, notice that, if $x \notin \bigcup_k I_k$, then $G^*(x) \le A\gamma$: this will be important.

With $\{I_k\}_k$ now fixed, let $\{J_l\}_l$ be the maximal subcubes of Q_0 such that, first, no J_l is contained in any I_k, and second, $|F(J_l)| > 1$. Denote the union $\{I_k\} \cup \{J_l\}$ by $\mathcal{P}$, and let $\{P_i\}_i$ be the family of maximal cubes from $\mathcal{P}$.

We claim that

$$\{x \in Q_0 : F^*(x) > 1,\, G^*(x) \le \gamma\} \subset \bigcup_i P_i. \tag{2.5}$$

To see this, suppose that x belongs to the left-hand side of (2.5). Since $F^*(x) > 1$, x must belong to some cube J such that $|F(J)| > 1$. If this cube J were contained in some I_k, then we would have $G^*(x) > \gamma$, a contradiction. Therefore, x belongs to one of the special cubes J_l. But $\bigcup_l J_l \subset \bigcup_i P_i$.

Thus, our problem has now reduced to *controlling the size of*

$$\sum_{i\,:\,|F(P_i)|>1}\omega(P_i).$$

The reader may wonder why we throw the cubes I_k into $\mathcal{P}$ at all, since only the cubes J_l are needed to cover $\{x \in Q_0 : F^*(x) > 1,\, G^*(x) \le \gamma\}$. The reason will soon become apparent. But, essentially: we use the family $\{I_k\}_k$ to control the size of $G^*(x)$ *globally* on Q_0 (i.e., even at points where $F^*(x) \le 1$). This is very much in the spirit of the proof of the classical good-λ inequality for the dyadic square function.

Define $\mathcal{F}_1 = \{Q \subset Q_0 : \forall i\,(Q \not\subset P_i)\}$ and $\mathcal{F}_2 = \{Q \subset Q_0 : Q \subset P_i$ for some $i\}$; and set $f_i = \sum_{Q\in\mathcal{F}_i} \lambda_Q b_{(Q)}$ for $i = 1, 2$. It is obvious that $f = f_1 + f_2$. Corresponding to f_1 and f_2, we define

$$F_i(I,x) = \sum_{\substack{Q\,:\,Q\in S(I)\\ Q\in\mathcal{F}_i}} \lambda_Q b_{(Q)}(x), \qquad F_i(I) = F_i(I, x_I), \qquad F_i^*(x) = \sup_{I\,:\,x\in I} |F_i(I)|,$$

$$G_i(I,x) = \Bigg(\sum_{\substack{Q\in S(I)\\ Q\in\mathcal{F}_i}} |\lambda_Q|^2 \Bigg[\sum_{j=0}^{\infty} \frac{2^{-j(2\alpha-\tau)}}{\omega(2^j Q)} \chi_{R_j(Q)}(x)\Bigg]\Bigg)^{1/2},$$

$$G_i(I) = G_i(I, x_I), \qquad G_i^*(x) = \sup_{I\,:\,x\in I} G_i(I);$$

where, as before, we do not define $F_i(I,x)$ or $G_i(I,x)$ for $x \notin I$.

Before going on, let us note—what is easy to see—that $F(I,x) = F_1(I,x) + F_2(I,x)$ and $G(I,x) \le G_1(I,x) + G_2(I,x)$. It is also easy to see that each $G_i(I,x) \le G(I,x)$.

For any cube Q, define $C(Q) \equiv \{x \in Q : |x - x_Q| < .1\ell(Q)\}$. Because ω is doubling, we have $\omega(Q) \le C_\omega \omega(C(Q))$ for any Q, and so our problem reduces to bounding

$$\sum_{i\,:\,|F(P_i)|>1} \omega(C(P_i)).$$

Clearly,

$$\sum_{i\,:\,|F(P_i)|>1} \omega(C(P_i)) \le \sum_{i\,:\,|F_1(P_i)|>1/2} \omega(C(P_i)) + \sum_{i\,:\,|F_2(P_i)|>1/2} \omega(C(P_i)) \equiv (I) + (II).$$

Let us consider (I) first. Each P_i satisfies $G(P_i) \le A\gamma$. Therefore, if $x \in C(P_i)$, we have (by Lemma 2.5) $|F(P_i) - F(P_i, x)| \le C\gamma$. If we take γ small enough, then this difference will be less than $1/4$, and having $|F(P_i)| > 1/2$ will force $|F(P_i, x)| > 1/4$ on all of $C(P_i)$. Let us assume that γ is so chosen. We get

$$\sum_{i\,:\,|F_1(P_i)|>1/2} \omega(C(P_i)) \le \sum_i \omega(\{x \in C(P_i) : |F(P_i,x)| > 1/4\}).$$

It is this last sum which we will now control. Recall that to this point we have used the decay and smoothness properties of the functions $b_{(Q)}$, but not their almost-orthogonality. Now is the time to apply 3).

Our argument relies on a **TRICK**: *If $x \in P_i$, then $F_1(P_i, x) = f_1(x)$.* This is true because (see also page 41 in [Wi1])

$$F_1(P_i, x) = \sum_{\substack{Q\,:\,Q\in\mathcal{F}_1\\ Q\not\subset P_i}} \lambda_Q b_{(Q)}(x) = \sum_{Q\,:\,Q\in\mathcal{F}_1} \lambda_Q b_{(Q)}(x) = f_1(x),$$

where the second equality follows because having $Q \in \mathcal{F}_1$ automatically implies $Q \not\subset P_i$.

Because of property 3),

$$\int |f_1|^2\, d\omega \le \sum_{Q\,:\,Q\in\mathcal{F}_1} |\lambda_Q|^2.$$

We rewrite and bound the second sum as

$$\sum_{Q\,:\,Q\in\mathcal{F}_1} |\lambda_Q|^2 = \int \Big(\sum_{Q\,:\,x\in Q\in\mathcal{F}_1} \frac{|\lambda_Q|^2}{\omega(Q)} \Big)\, d\omega.$$

We claim that

$$\sum_{Q\,:\,x\in Q\in\mathcal{F}_1} \frac{|\lambda_Q|^2}{\omega(Q)} \leq C\gamma^2 \qquad \omega\text{-a.e.}$$

Proof of claim. If $x \in P_i$, then

$$\begin{aligned}\sum_{Q\,:\,x\in Q\in\mathcal{F}_1} \frac{|\lambda_Q|^2}{\omega(Q)} = \sum_{Q\,:\,P_i\subset Q\in\mathcal{F}_1} \frac{|\lambda_Q|^2}{\omega(Q)} &\leq \sum_{Q\,:\,P_i\subset Q\in\mathcal{F}_1} |\lambda_Q|^2 \sum_{j=0}^{\infty} \frac{2^{-j(2\alpha-\tau)}}{\omega(2^jQ)} \chi_{R_j(Q)}(x) \\ &\leq C(G(P_i))^2 \leq C(A\gamma)^2.\end{aligned}$$

If $x \notin \bigcup_i P_i$, then

$$\sum_{Q\,:\,x\in Q\in\mathcal{F}_1} \frac{|\lambda_Q|^2}{\omega(Q)} \leq \sum_{Q} |\lambda_Q|^2 \sum_{j=0}^{\infty} \frac{2^{-j(2\alpha-\tau)}}{\omega(2^jQ)} \chi_{R_j(Q)}(x) \leq (G^*(x))^2 \leq (A\gamma)^2,$$

as noted above.

Putting it all together, we get

$$\int |f_1|^2\, d\omega \leq C\gamma^2\omega(Q_0),$$

implying $\omega(\{x \in Q_0 : |f_1(x)| > 1/4\}) \leq C\gamma^2$. Therefore,

$$(I) \leq C\gamma^2\omega(Q_0) \leq (\delta/2)\omega(Q_0),$$

if γ is taken sufficiently small.

Now we look at (II). Reasoning precisely as we did for (I), we only need to control

$$\sum_i \omega(\{x \in C(P_i) : |F_2(P_i, x)| > 1/4\}).$$

We will handle this last sum via a pure brute-force argument, using only the size condition 1). An estimate reminiscent of Carleson measures comes in at the end.

Let κ be a number greater than 1, and chosen so that, for any cube Q, $\omega(\kappa Q \setminus Q) < (\delta/4)\omega(Q)$. Such a κ exists because ω is doubling. We define

$$D \equiv \bigcup_i (\kappa P_i \setminus P_i).$$

We call this the "zone of death". It consists of a union of thin bands (or shells) around the cubes P_i, inside which we may encounter bad edge effects when estimating f_2 and its associated functionals.

By our choice of κ, $\omega(D) \leq (\delta/4)\omega(Q_0)$. Thus, it is sufficient to bound

$$\sum_i \omega(\{x \in C(P_i) \setminus D : |F_2(P_i, x)| > 1/4\})$$

by $C\delta$, where C depends on ω's doubling constant.

Fix P_i. If $x \in C(P_i) \setminus D$, then

$$\begin{aligned}
|F_2(P_i,x)| &= \Big| \sum_{j\,:\,j\neq i} \sum_{Q\,:\,Q\subset P_j} \lambda_Q b_{(Q)}(x) \Big| \\
&\le G_2(P_i,x) \Big(\sum_{j\,:\,j\neq i} \sum_{Q\,:\,Q\subset P_j} \omega(Q) \Big[\sum_{k=0}^{\infty} \frac{2^{-k\tau}}{\omega(2^k Q)} \chi_{R_k(Q)}(x) \Big] \Big)^{1/2} \\
&\le C G(P_i) \Big(\sum_{j\,:\,j\neq i} \sum_{Q\,:\,Q\subset P_j} \omega(Q) \Big[\sum_{k=0}^{\infty} \frac{2^{-k\tau}}{\omega(2^k Q)} \chi_{R_k(Q)}(x) \Big] \Big)^{1/2} \\
&\le C\gamma \Big(\sum_{j\,:\,j\neq i} \sum_{Q\,:\,Q\subset P_j} \omega(Q) \Big[\sum_{k=0}^{\infty} \frac{2^{-k\tau}}{\omega(2^k Q)} \chi_{R_k(Q)}(x) \Big] \Big)^{1/2} \equiv C\gamma H_i(x).
\end{aligned}$$

We claim that

$$H_i(x) \le C \Big(\sum_j \omega(P_j) \Big[\sum_{k=0}^{\infty} \frac{2^{-k\tau}}{\omega(2^k P_j)} \chi_{R_k(P_j)}(x) \Big] \Big)^{1/2}. \tag{2.6}$$

Note that the right-hand side of (2.6) does not depend on i. Once we have (2.6), we will obtain

$$\begin{aligned}
\sum_i \int_{C(P_i)\setminus D} |F_2(P_i,x)|^2 \, d\omega(x) &\le C\gamma^2 \int \Big(\sum_j \omega(P_j) \Big[\sum_{k=0}^{\infty} \frac{2^{-k\tau}}{\omega(2^k P_i)} \chi_{R_k(P_i)}(x) \Big] \Big) \, d\omega(x) \\
&\le C\gamma^2 \sum_j \omega(P_j) \int \Big[\sum_{k=0}^{\infty} \frac{2^{-k\tau}}{\omega(2^k P_i)} \chi_{R_k(P_i)}(x) \Big] \, d\omega(x) \\
&\le C\gamma^2 \sum_j \omega(P_j) \le C\gamma^2 \omega(Q_0),
\end{aligned}$$

and the bound for (II) will follow from Chebyshev's inequality.

Let us now fix a $j \neq i$ and consider the sum

$$\sum_{Q\,:\,Q\subset P_j} \omega(Q) \Big[\sum_{k=0}^{\infty} \frac{2^{-k\tau}}{\omega(2^k Q)} \chi_{R_k(Q)}(x) \Big]$$

for $x \in C(P_i) \setminus D$. We rewrite the sum as

$$\sum_{l=0}^{\infty} \sum_{\substack{Q\,:\,Q\subset P_j \\ \ell(Q)=2^{-l}\ell(P_j)}} \omega(Q) \Big[\sum_{k=0}^{\infty} \frac{2^{-k\tau}}{\omega(2^k Q)} \chi_{R_k(Q)}(x) \Big].$$

Now let us fix l. Let k' be such that $x_{P_i} \in R_{k'}(P_j)$. If $Q \subset P_j$, $\ell(Q) = 2^{-l}\ell(P_j)$, and $x \in C(P_i) \setminus D$, then $x \in R_k(Q)$ for some k such that $|k' + l - k| \le C$, where C only depends on the dimension d. Conversely, if $x \in C(P_i) \setminus D$ and belongs to $R_k(Q)$, then $x \in R_{\tilde{k}}(P_j)$ for some $\tilde{k}$ satisfying $|\tilde{k} + l - k| \le C'$. The reason for these inequalities is that the distance between x_Q and x is comparable to the distance between x_{P_i} and x_{P_j}, with comparability constants depending only on d and κ. For all such Q, the ω-measure of $2^k Q$ will be comparable to $\omega(2^{k-l} P_j)$, and thus comparable to $\omega(2^{k'} P_j)$, because ω is doubling. Therefore, for each fixed $l \geq 0$, and all $x \in C(P_i) \setminus D$,

$$\begin{aligned}
\sum_{\substack{Q\,:\,Q\subset P_j\\ \ell(Q)=2^{-l}\ell(P_j)}} &\omega(Q)\Big[\sum_{k=0}^{\infty}\frac{2^{-k\tau}}{\omega(2^kQ)}\chi_{R_k(Q)}(x)\Big]\\
&\leq C\sum_{Q\,:\,\ell(Q)=2^{-l}\ell(P_j)}\frac{\omega(Q)}{\omega(2^kQ)}\sum_{k'+l-C}^{k'+l+C}2^{-k\tau}\chi_{R_k(Q)}(x)\\
&\leq C\Big[\frac{2^{-k'\tau}}{\omega(2^{k'}P_j)}\sum_{\tilde{k}=k'-C-C'}^{k'+C+C'}\chi_{R_{\tilde{k}}(P_j)}(x)\Big]\Big[2^{-l\tau}\sum_{\substack{Q\,:\,Q\subset P_j\\ \ell(Q)=2^{-l}\ell(P_j)}}\omega(Q)\Big]\\
&= C2^{-l\tau}\omega(P_j)\Big[\frac{2^{-k'\tau}}{\omega(2^{k'}P_j)}\sum_{\tilde{k}=k'-C-C'}^{k'+C+C'}\chi_{R_{\tilde{k}}(P_j)}(x)\Big]\\
&\leq C2^{-l\tau}\omega(P_j)\sum_{\tilde{k}=0}^{\infty}\frac{2^{-k'\tau}}{\omega(2^{k'}P_j)}\chi_{R_{\tilde{k}}(P_j)}(x).
\end{aligned}$$

This holds for every $l \geq 0$. Summing over l we get

$$\sum_{Q\,:\,Q\subset P_j}\omega(Q)\Big[\sum_{k=0}^{\infty}\frac{2^{-k\tau}}{\omega(2^kQ)}\chi_{R_k(Q)}(x)\Big]\leq C\omega(P_j)\sum_{\tilde{k}=0}^{\infty}\frac{2^{-k'\tau}}{\omega(2^{k'}P_j)}\chi_{R_{\tilde{k}}(P_j)}(x).$$

When we sum this over $j \neq i$, we get (2.6). Lemma 2.9 is proved. ∎

Proof of Theorem 2.1. Recall that $f = \sum \lambda_Q b_{(Q)}$ is a finite sum, from which it follows that, for large x, $|f(x)| \leq Cg^*(f)(x)$. This is because, if we take a single term in the sum defining f, we get

$$|\lambda_Q b_{(Q)}| \leq |\lambda_Q|\frac{\sqrt{\omega(Q)}\,2^{-k\alpha}}{\omega(2^kQ)}$$

when $x \in R_k(Q)$. However, for $x \in R_k(Q)$,

$$g^*(f)(x) \geq |\lambda_Q|\frac{2^{-k(\alpha-\tau/2)}}{\sqrt{\omega(2^kQ)}} \geq |\lambda_Q|\frac{2^{-k\alpha}}{\sqrt{\omega(2^kQ)}}.$$

After some canceling on both sides, our desired inequality

$$|\lambda_Q b_{(Q)}| \ \leq g^*(f)$$

reduces to $\sqrt{\omega(Q)} \leq 2^{\tau k/2}\sqrt{\omega(2^kQ)}$ which is true for all $k \geq 0$. (We note that the C in our inequality $|f(x)| \leq Cg^*(f)(x)$ depends *strongly* on f, and on the fact that f is a finite sum.)

Therefore, without loss of generality, we may assume that $g^*(f)$ belongs to $L^p(dv)$, and thus that G^* does too (because $G^* \leq cg^*(f)$). This said, the Main Theorem will follow from a good-λ inequality: *For every* $\epsilon > 0$ *there is a* $\gamma > 0$ *so that, for all* $\lambda > 0$,

$$v(\{x : F^*(x) > 2\lambda,\, G^*(x) \leq \gamma\lambda\}) \leq \epsilon v(\{x : F^*(x) > \lambda\}).$$

Let $\{I_i^\lambda\}$ be the maximal dyadic cubes such that $|F(I_i^\lambda)| > \lambda$. It is enough to prove that, for all $\epsilon > 0$, there is a γ so that, for all i and all λ,

$$v(\{x \in I_i^\lambda : F^*(x) > 2\lambda,\, G^*(x) \le \gamma\lambda\}) \le \epsilon v(I_i^\lambda). \tag{2.7}$$

Since $v \in A^\infty(\omega)$, it is enough to prove (2.7) with the measure v replaced by ω; and that is what we shall do.

From this point the proof is essentially identical to the (very short) proof of Theorem A in [Wi1] (on page 45). Fix I_i^λ. We can take $G(I_i^\lambda) \le \gamma\lambda$, or there is nothing to prove. Let I be the unique dyadic cube such that $I_i^\lambda \in N(I)$. Because of I_i^λ's maximality, $|F(I)| \le \lambda$. By Lemma 2.8, we may therefore take $|F(I_i^\lambda)| \le 1.1\lambda$, if we choose γ small enough—and of course we do.

Now let $\eta > 0$ be so small that $\omega(\{x \in I_i^\lambda : d(x, \partial I_i^\lambda) < \eta\ell(I_i^\lambda)\}) \le (\epsilon/2)\omega(I_i^\lambda)$. The number η only depends on ϵ, d, and ω's doubling constant. With η chosen, we can now take γ so small that if $x \in I_i^\lambda$ and $d(x, \partial I_i^\lambda) \ge \eta\ell(I_i^\lambda)$, then (Lemma 2.5) $|F(I_i^\lambda) - F(I_i^\lambda, x)| \le .1\lambda$. Therefore, we can neglect the contribution of $F(I_i^\lambda, x)$ on the part of I_i^λ that stays away from the boundary.

Define

$$h(x) \equiv \sum_{Q \subset I_i^\lambda} \lambda_Q b_{(Q)}(x).$$

Let the functions $H(I, x)$, $H(I)$, and $H^*(x)$ be defined for h just as $F(I, x)$, $F(I)$, and $F^*(x)$ were for f. Our problem reduces to showing that

$$\omega(\{x \in I_i^\lambda : H^*(x) > .8\lambda,\, G^*(x) \le \gamma\lambda\}) \le (\epsilon/2)\omega(I_i^\lambda).$$

But this is just a rescaled version of Lemma 2.9 (divide h by $.8\lambda$). Theorem 2.1 is proved. ∎

2.2. Almost-orthogonality for a doubling measure. In this section we will prove that certain families of functions $\{\phi_{(Q)}\}_{Q\in\mathcal{D}}$, indexed over the dyadic cubes $Q \in \mathcal{D}$, are "almost-orthogonal" in $L^2(\mathbb{R}^d, \omega)$, where ω is any doubling measure.

The functions $\phi_{(Q)}$ will be asked to satisfy:

1. For all $x \in \mathbb{R}^d$,
$$|\phi_{(Q)}(x)| \le \omega(Q)^{1/2} \sum_{k=0}^{\infty} \frac{2^{-k\epsilon}}{\omega(2^k Q)} \chi_{R_k(Q)}(x),$$
where $\epsilon > 0$ does not depend on Q.
2. For all x and x' in $\mathbb{R}^d$,
$$|\phi_{(Q)}(x) - \phi_{(Q)}(x')| \le \left(\frac{|x - x'|}{\ell(Q)}\right)^\alpha \left[\omega(Q)^{1/2} \sum_{k=0}^{\infty} \frac{2^{-k\epsilon}}{\omega(2^k Q)} (\chi_{R_k(Q)}(x) + \chi_{R_k(Q)}(x'))\right],$$
where $\alpha > 0$ also does not depend on Q.
3. $\int_{\mathbb{R}^d} \phi_{(Q)}(x)\, d\omega(x) = 0$.

Our almost-orthogonality result is:

THEOREM 2.2. *For every $\epsilon > 0$ and $\alpha > 0$ there is a constant $C = C(\omega, \alpha, \epsilon)$ such that, if the family $\{\phi_{(Q)}\}_{Q\in\mathcal{D}}$ satisfies* 1, 2, *and* 3, *and $\sum \gamma_Q \phi_{(Q)}$ is any finite linear sum from* *the family, then*

$$\int_{\mathbb{R}^d} \Big|\sum \gamma_Q \phi_{(Q)}\Big|^2 d\omega \le C \sum |\gamma_Q|^2.$$

This was proved in [FrJaWeis] for $\omega =$ Lebesgue measure. The proof is quite involved and technical—enough to make a result like Theorem 2.2 seem intractable.

It becomes less formidable if we approach it indirectly. We will use an ω-adapted version of a Littlewood–Paley type function called the *intrinsic square function* (ISF). The ISF was first described in [Wi2], and further developed in [Wi3], but always in the Euclidean (Lebesgue measure-adapted) setting.

If $\alpha > 0$ and $(t, y) \in \mathbb{R}^{d+1}_+$, we define $\mathcal{C}_\alpha(B(t; y))$ to be the family of functions $\phi : \mathbb{R}^d \to \mathbb{R}$ such that:

(i) $\operatorname{supp} \phi \subset B(t; y)$.
(ii) For all x and x' in $\mathbb{R}^d$,

$$|\phi(x) - \phi(x')| \le \left(\frac{|x - x'|}{y}\right)^\alpha \omega(B(t; y))^{-1}.$$

(iii) $\int \phi \, d\omega = 0$.

If f is locally integrable with respect to ω, we define

$$A_\alpha(f)(t, y) = \sup\left\{\left|\int f\,\phi\, d\omega\right| : \phi \in \mathcal{C}_\alpha(B(t; y))\right\}.$$

For $x \in \mathbb{R}^d$, the *intrinsic square function of order* α of the function f at the point x is defined to be

$$G_\alpha(f)(x) \equiv \left(\int_{\Gamma(x)} (A_\alpha(f)(t, y))^2 \, \frac{d\omega(t)\, dy}{\omega(B(t; y))y}\right)^{1/2};$$

here and in the future, $\Gamma(x) \equiv \{(t, y) \in \mathbb{R}^{d+1}_+ : |x - t| < y\}$, the "standard cone" with vertex at x.

The functional $A_\alpha(f)(t, y)$ is defined by inner products with compactly-supported functions ϕ. It has a companion that is defined by inner products with non-compactly-supported functions.

If $\alpha > 0$ and $\epsilon > 0$, and $(t, y) \in \mathbb{R}^{d+1}_+$, we define $\mathcal{C}_{(\alpha,\epsilon)}(B(t; y))$ to be the family of functions $\phi : \mathbb{R}^d \to \mathbb{R}$ such that:

(i′) For all $x \in \mathbb{R}^d$,

$$|\phi(x)| \le \sum_{k=0}^\infty \frac{2^{-k\epsilon}}{\omega(B(t; 2^k y))} \chi_{R_k(B(t;y))}(x),$$

where $R_0(B(t; y)) = B(t; y)$ and, for $k \ge 1$, $R_k(B(t; y)) = B(t; 2^k y) \setminus B(t; 2^{k-1} y)$.

(ii′) For all x and x' in $\mathbb{R}^d$,

$$|\phi(x) - \phi(x')| \le \left(\frac{|x - x'|}{y}\right)^\alpha \sum_{k=0}^\infty \frac{2^{-k\epsilon}}{\omega(B(t; 2^k y))} (\chi_{R_k(B(t;y))}(x) + \chi_{R_k(B(t;y))}(x')).$$

(iii′) $\int_{\mathbb{R}^d} \phi(x)\, d\omega = 0$.

If

$$|f(x)|\sum_{k=0}^{\infty}\frac{2^{-k\epsilon}}{\omega(B(t;2^k y))}\chi_{R_k(B(t;y))}(x)\in L^1(\omega), \tag{A}$$

we define

$$\tilde{A}_{(\alpha,\epsilon)}(f)(t,y)=\sup\left\{\left|\int f\,\phi\,d\omega\right|:\phi\in\mathcal{C}_{(\alpha,\epsilon)}(B(t;y))\right\}.$$

For $x\in\mathbb{R}^d$, the *intrinsic square function of order* (α,ϵ) of the function f at the point x is defined to be

$$\tilde{G}_{(\alpha,\epsilon)}(f)(x)\equiv\left(\int_{\Gamma(x)}(A_{(\alpha,\epsilon)}(f)(t,y))^2\,\frac{d\omega(t)\,dy}{\omega(B(t;y))y}\right)^{1/2}.$$

We will show that $\|\tilde{G}_{(\alpha,\epsilon)}(f)\|_{L^2(\omega)}\le C\|f\|_{L^2(\omega)}$, *from which Theorem* 2.2 *will follow easily.*

The definitions of these two intrinsic square functions give them a certain flexibility, which makes them easy to manipulate. Define

$$\sigma_\alpha(f)(x)\equiv\Big(\sum_{k=-\infty}^{\infty}(A_\alpha(f)(x,2^k))^2\Big)^{1/2},\quad\tilde{\sigma}_{(\alpha,\epsilon)}(f)(x)\equiv\Big(\sum_{k=-\infty}^{\infty}(\tilde{A}_{(\alpha,\epsilon)}(f)(x,2^k))^2\Big)^{1/2}.$$

The functionals $\sigma_\alpha(f)$ and $\tilde{\sigma}_{(\alpha,\epsilon)}(f)$ are "discretized" analogues of (respectively) $G_\alpha(f)$ and $\tilde{G}_{(\alpha,\epsilon)}(f)$. The useful fact to observe is that $\sigma_\alpha(f)$ and $G_\alpha(f)$ are pointwise comparable (written $\sigma_\alpha(f)\sim G_\alpha(f)$), with comparability constants only depending on α and ω; and similarly $\tilde{\sigma}_{(\alpha,\epsilon)}(f)\sim\tilde{G}_{(\alpha,\epsilon)}(f)$, with comparability constants only depending on α, ϵ, and ω. We will show comparability for $\sigma_\alpha(f)$ and $G_\alpha(f)$; the proof for the other pair is even easier.

Comparability of $\sigma_\alpha(f)$ and $G_\alpha(f)$ follows from

$$(A_\alpha(f)(x,2^k))^2\le C_1\int_{(t,y)\in\Gamma(x)\,:\,2^{k+2}\le y\le 2^{k+3}}(A_\alpha(f)(t,y))^2\,\frac{d\omega(t)\,dy}{\omega(B(t;y))y} \tag{2b.1}$$

and

$$\int_{(t,y)\in\Gamma(x)\,:\,2^{k}\le y\le 2^{k+1}}(A_\alpha(f)(t,y))^2\,\frac{d\omega(t)\,dy}{\omega(B(t;y))y}\le C_2(A_\alpha(f)(x,2^{k+3}))^2, \tag{2b.2}$$

for some positive constants C_1 and C_2 that only depend on α and ω (and *not* on k), because summing both sides of (2b.1) over all k will yield

$$(\sigma_\alpha(f)(x))^2\le C_1(G_\alpha(f)(x))^2,$$

and summing both sides of (2b.2) over all k will yield

$$(G_\alpha(f)(x))^2\le C_2(\sigma_\alpha(f)(x))^2.$$

Inequalities (2b.1) and (2b.2) follow from two easy geometric facts: 1) There is an absolute constant $C=C(\alpha,\omega)$ such that if $\phi\in\mathcal{C}_\alpha(B(x;2^k))$, then $C\phi\in\mathcal{C}_\alpha(B(t;y))$ for *all* (t,y) such that $|t-x|<2^k$ and $2^{k+2}\le y\le 2^{k+3}$. 2) There is an absolute constant $C'=C'(\alpha,\omega)$ such that if $\phi\in\mathcal{C}_\alpha(B(t;y))$ for some $(t,y)\in\Gamma(x)$ such that $2^k\le y\le 2^{k+1}$, then $C'\phi\in\mathcal{C}_\alpha(B(x;2^{k+3}))$. We will sketch the proof of 2) and leave the proof of 1) to the reader. Let $\phi\in\mathcal{C}_\alpha(B(t;y))$, with $|x-t|<y$ and $2^k\le y\le 2^{k+1}$. Then ϕ's support

is contained in $B(t;y) \subset B(x;2^{k+3})$, and $\int \phi\, d\omega = 0$. The only thing left to check is ϕ's Hölder modulus. We have

$$|\phi(s) - \phi(s')| \le \left(\frac{|s-s'|}{y}\right)^{\alpha} \omega(B(t;y))^{-1}$$

for any s and s'. But

$$\left(\frac{|s-s'|}{y}\right)^{\alpha} \le C\left(\frac{|s-s'|}{2^{k+3}}\right)^{\alpha},$$

because $y \ge 2^k$; and $\omega(B(t;y)) \ge c\omega(B(x;2^{k+3}))$, because $y \ge 2^k$, $|x-t| < y \le 2^{k+1}$, and ω is doubling.

With these facts in hand, the proofs of (2b.1) and (2b.2) are easy. We will only prove (2b.1); the proof of (2b.2) is similar. Because of fact 1), for any (t,y) such that $|t-x| < 2^k$ and $2^{k+2} \le y \le 2^{k+3}$, $A_\alpha(f)(x,2^k) \le CA_\alpha(f)(t,y)$. Therefore

$$(A_\alpha(f)(x,2^k))^2 \le C\left(\int_{(t,y)\,:\,|t-x|<2^k,\, 2^{k+2}\le y\le 2^{k+3}} (A_\alpha(f)(t,y))^2 \frac{d\omega(t)\,dy}{\omega(B(t;y))y}\right) \times\left(\int_{(t,y)\,:\,|t-x|<2^k,\, 2^{k+2}\le y\le 2^{k+3}} \frac{d\omega(t)\,dy}{\omega(B(t;y))y}\right)^{-1}.$$

But ω's doubling property ensures that

$$\int_{(t,y)\,:\,|t-x|<2^k,\, 2^{k+2}\le y\le 2^{k+3}} \frac{d\omega(t)\,dy}{\omega(B(t;y))y} \sim 1,$$

with comparability constants that only depend on ω. Therefore

$$\begin{aligned}(A_\alpha(f)(x,2^k))^2 &\le C\int_{(t,y)\,:\,|t-x|<2^k,\, 2^{k+2}\le y\le 2^{k+3}} (A_\alpha(f)(t,y))^2 \frac{d\omega(t)\,dy}{\omega(B(t;y))y}\\ &\le C\int_{(t,y)\in\Gamma(x)\,:\,2^{k+2}\le y\le 2^{k+3}} (A_\alpha(f)(t,y))^2 \frac{d\omega(t)\,dy}{\omega(B(t;y))y},\end{aligned}$$

which is (2b.1). An analogous argument, using fact 2), yields (2b.2).

We continue with a familiar definition and an easy lemma.

Definition 2.1. Let $\alpha > 0$. If $S \subset \mathbb{R}^d$ is a bounded convex set (which will always be a ball or a cube), we say that $\phi : S \to \mathbb{R}$ is *adapted* to S if: a) the support of ϕ is contained in S; b) for all x and x' in R^d,

$$|\phi(x) - \phi(x')| \le \left(\frac{|x-x'|}{\mathrm{diam}(S)}\right)^{\alpha} \omega(S)^{-1/2},$$

where $\mathrm{diam}(S)$ is S's diameter; c) $\int \phi\, d\omega = 0$.

The reader should notice that, because ω is doubling, if ϕ is adapted to a ball B, then $c\phi$ is adapted (for the same α) to some cube Q having the same center as B, where c is an absolute positive constant and $\mathrm{diam}(B) \sim \mathrm{diam}(Q)$; and vice versa. Similarly, if ϕ is adapted (for α) to $B(t;y)$, then $\omega(B(t;y))^{-1/2}\phi \in \mathcal{C}_\alpha(B(t;y))$, and the reverse implication holds modulo a constant factor.

Lemma 2.10. *Let $\mathcal{F} \subset \mathcal{D}$ be a finite set of dyadic cubes. If $\{\phi_{(Q)}\}_{Q\in\mathcal{F}}$ is a family of functions such that, for some fixed α, each $\phi_{(Q)}$ is adapted to $\widetilde{Q}$, where $\widetilde{Q}$ denotes the* *concentric triple of Q, then there is a constant $C = C(\alpha,\omega)$ such that, for all linear combinations $\sum \gamma_Q \phi_{(Q)}$,*

$$\int_{\mathbb{R}^d} \Big|\sum \gamma_Q \phi_{(Q)}\Big|^2 \, d\omega \le C \sum |\gamma_Q|^2. \tag{2.8}$$

Proof. By a lemma in [Wi4], the family of triples of dyadic cubes in $\mathbb{R}^d$ can be split into 3^d disjoint families $\mathcal{G}_k$ possessing the same exclusion/inclusion properties as the dyadic cubes: a) if $\tilde{Q}$ and $\tilde{Q}'$ belong to $\mathcal{G}_k$, either one cube is contained in the other or they are disjoint; b) if $\tilde{Q}$ and $\tilde{Q}'$ belong to $\mathcal{G}_k$, and Q is properly contained in Q', then $\ell(Q) = 2^{-j}\ell(Q')$ for some positive integer j.

Without loss of generality, we can assume that all of the $\tilde{Q}$'s for which $Q \in \mathcal{F}$ belong to the *same* $\mathcal{G}_k$.

If $\tilde{Q} \subset \tilde{Q}'$, then

$$\int \phi_{(Q)}(x)\, \phi_{(Q')}(x)\, d\omega = \int_{\tilde{Q}} \phi_{(Q)}(x)(\phi_{(Q')}(x) - \phi_{(Q')}(x_{\tilde{Q}}))\, d\omega, \tag{2.9}$$

where $x_{\tilde{Q}}$ is the center of $\tilde{Q}$. Our assumptions on the ϕ's imply that the right-hand side of (2.9) has absolute value no bigger than a constant times

$$\left(\frac{\omega(\tilde{Q})}{\omega(\tilde{Q}')}\right)^{1/2} \cdot \left(\frac{\ell(Q)}{\ell(Q')}\right)^{\alpha}.$$

This implies that the left-hand side of (2.8) is no bigger than a constant times

$$\sum_{Q'} |\gamma_{Q'}| \sum_{Q\,:\,\tilde{Q}\subset\tilde{Q}'} |\gamma_Q| \left(\frac{\omega(\tilde{Q})}{\omega(\tilde{Q}')}\right)^{1/2} \cdot \left(\frac{\ell(Q)}{\ell(Q')}\right)^{\alpha}. \tag{2.10}$$

For each fixed Q',

$$\sum_{Q\,:\,\tilde{Q}\subset\tilde{Q}'} |\gamma_Q| \left(\frac{\omega(\tilde{Q})}{\omega(\tilde{Q}')}\right)^{1/2} \cdot \left(\frac{\ell(Q)}{\ell(Q')}\right)^{\alpha}$$
$$\le \bigg(\sum_{Q\,:\,\tilde{Q}\subset\tilde{Q}'} |\gamma_Q|^2 \left(\frac{\ell(Q)}{\ell(Q')}\right)^{\alpha}\bigg)^{1/2} \bigg(\sum_{Q\,:\,\tilde{Q}\subset\tilde{Q}'} \left(\frac{\omega(\tilde{Q})}{\omega(\tilde{Q}')}\right) \cdot \left(\frac{\ell(Q)}{\ell(Q')}\right)^{\alpha}\bigg)^{1/2}.$$

But

$$\sum_{Q\,:\,\tilde{Q}\subset\tilde{Q}'} \left(\frac{\omega(\tilde{Q})}{\omega(\tilde{Q}')}\right) \cdot \left(\frac{\ell(Q)}{\ell(Q')}\right)^{\alpha} = \omega(\tilde{Q}')^{-1} \sum_{k=0}^{\infty} 2^{-k\alpha} \sum_{\substack{Q\,:\,\tilde{Q}\subset\tilde{Q}' \\ \ell(Q)=2^{-k}\ell(Q')}} \omega(\tilde{Q}) \le C_\alpha.$$

Therefore the quantity in (2.10) is less than or equal to a constant times

$$\sum_{Q'} |\gamma_{Q'}| \bigg(\sum_{Q\,:\,\tilde{Q}\subset\tilde{Q}'} |\gamma_Q|^2 \left(\frac{\ell(Q)}{\ell(Q')}\right)^{\alpha}\bigg)^{1/2},$$

which, by the Cauchy–Schwarz inequality, is less than or equal to

$$\Big(\sum_{Q'}|\gamma_{Q'}|^2\Big)^{1/2}\Big(\sum_{Q'}\sum_{Q\,:\,\tilde{Q}\subset\tilde{Q}'}|\gamma_Q|^2\Big(\frac{\ell(Q)}{\ell(Q')}\Big)^{\alpha}\Big)^{1/2}$$

$$=\Big(\sum_{Q'}|\gamma_{Q'}|^2\Big)^{1/2}\Big(\sum_{Q}|\gamma_Q|^2\sum_{Q':\tilde{Q}\subset\tilde{Q}'}\Big(\frac{\ell(Q)}{\ell(Q')}\Big)^{\alpha}\Big)^{1/2}. \tag{2.11}$$

But, for each Q,

$$\sum_{Q'\,:\,\tilde{Q}\subset\tilde{Q}'}\Big(\frac{\ell(Q)}{\ell(Q')}\Big)^{\alpha}\le C_\alpha,$$

because, if $\tilde{Q}\subset\tilde{Q}'$, $\ell(Q)/\ell(Q')=2^{-k}$ for some $k\ge 0$; and, for each $k\ge 0$, $\tilde{Q}$ is contained in at most one such $\tilde{Q}'$. Therefore (2.11) is less than or equal to a constant times

$$\Big(\sum_{Q'}|\gamma_{Q'}|^2\Big)^{1/2}\Big(\sum_{Q}|\gamma_Q|^2\Big)^{1/2}=\sum_Q|\gamma_Q|^2,$$

which was to be proved. ■

The next lemma asserts that the "compact support" ISF is bounded on $L^2(\omega)$.

Lemma 2.11. *Let ω be a doubling measure on $\mathbb{R}^d$ and $\alpha>0$. There is a constant $C=C(\alpha,\omega)$ such that, for all $f\in L^2(\omega)$,*

$$\int_{\mathbb{R}^d}(G_\alpha(f)(x))^2\,d\omega(x)\le C\int|f(x)|^2\,d\omega(x).$$

Proof. We normalize f so that $\|f\|_{L^2(\omega)}=1$. By Fubini–Tonelli,

$$\int_{\mathbb{R}^d}(G_\alpha(f)(x))^2\,d\omega(x)=\int_{\mathbb{R}^{d+1}_+}(A_\alpha(f)(t,y))^2\,\frac{d\omega(t)\,dy}{y}.$$

Let $g:\mathbb{R}^{d+1}_+\to\mathbb{R}$ be non-negative, bounded, measurable, have compact support, and satisfy

$$\int_{\mathbb{R}^{d+1}_+}(g(t,y))^2\,\frac{d\omega(t)\,dy}{y}\le 1.$$

It suffices to show that

$$\int_{\mathbb{R}^{d+1}_+}g(t,y)A_\alpha(f)(t,y)\,\frac{d\omega(t)\,dy}{y}\le C, \tag{2.12}$$

where C is an absolute constant. We can pick in a measurable fashion functions $\phi^{(t,y)}\in\mathcal{C}_\alpha(B(t;y))$ such that, for each $(t,y)\in\mathbb{R}^{d+1}_+$,

$$\int f(x)\phi^{(t,y)}(x)\,d\omega(x)\ge\frac{1}{2}A_\alpha(f)(t,y).$$

Therefore, (2.12) will follow if we can show that

$$\int_{\mathbb{R}^{d+1}_+}g(t,y)\Big(\int f(x)\phi^{(t,y)}(x)\,d\omega(x)\Big)\,\frac{d\omega(t)\,dy}{y}$$

is less than or equal to an absolute constant. Because g is such a nice function, we can rewrite the preceding integral as

$$\int_{\mathbb{R}^d} f(x)\bigg(\int_{\mathbb{R}^{d+1}_+} g(t,y)\phi^{(t,y)}(x)\,\frac{d\omega(t)\,dy}{y}\bigg)\,d\omega(x).$$

Let us now define

$$H(x) \equiv \int_{\mathbb{R}^{d+1}_+} g(t,y)\phi^{(t,y)}(x)\,\frac{d\omega(t)\,dy}{y}. \tag{2.13}$$

The lemma has been reduced to showing that $\|H\|_{L^2(\omega)} \le C$.

For every dyadic cube $Q \subset \mathbb{R}^d$, we set $T(Q) \equiv Q \times [\ell(Q)/2, \ell(Q))$, and we define

$$b_{(Q)}(x) \equiv \int_{T(Q)} g(t,y)\phi^{(t,y)}(x)\,\frac{d\omega(t)\,dy}{y}.$$

Then

$$H(x) = \sum_{Q\in\mathcal{D}} b_{(Q)}(x),$$

and the sum makes sense because it is *finite* (remember that g has compact support). Each $b_{(Q)}$ can be written as $\gamma_Q a_{(Q)}(x)$, where each γ_Q equals a fixed constant times

$$\bigg(\int_{T(Q)} (g(t,y))^2\,\frac{d\omega(t)\,dy}{y}\bigg)^{1/2}.$$

We claim that if this "fixed constant" is chosen large enough (in a way only depending on α and ω), then each $a_{(Q)}$ will be *adapted* to its $\tilde{Q}$. Lemma 2.10 will then imply that

$$\int_{\mathbb{R}^d} |H(x)|^2\,d\omega(x) \le C\sum_Q |\gamma_Q|^2 \le C\int_{\mathbb{R}^{d+1}_+} (g(t,y))^2\,\frac{d\omega(t)\,dy}{y} \le C,$$

as desired.

It is easy to see that each $a_{(Q)}$ inherits cancellation from the $\phi^{(t,y)}$'s. If $(t,y) \in T(Q)$ then $B(t;y) \subset \tilde{Q}$; therefore $a_{(Q)}$'s support condition is also no problem. All we need to show is $a_{(Q)}$'s Hölder continuity. The doubling property of ω will be crucial here, and in particular the following fact: if $(t,y) \in T(Q)$ then $\omega(B(t;y)) \sim \omega(\tilde{Q})$, with comparability constants only depending on ω.

If x and x' belong to $\mathbb{R}^d$ then

$$|b_{(Q)}(x) - b_{(Q)}(x')| = \bigg|\int_{T(Q)} g(t,y)(\phi^{(t,y)}(x) - \phi^{(t,y)}(x'))\,\frac{d\omega(t)\,dy}{y}\bigg|,$$

which is less than or equal to

$$\bigg(\int_{T(Q)} (g(t,y))^2\,\frac{d\omega(t)\,dy}{y}\bigg)^{1/2}\bigg(\int_{T(Q)} |\phi^{(t,y)}(x) - \phi^{(t,y)}(x')|^2\,\frac{d\omega(t)\,dy}{y}\bigg)^{1/2}.$$

We focus on the *second* factor. For each $(t,y) \in T(Q)$,

$$|\phi^{(t,y)}(x) - \phi^{(t,y)}(x')| \le C\bigg(\frac{|x-x'|}{\ell(\tilde{Q})}\bigg)^{\alpha}\omega(\tilde{Q})^{-1}.$$

Therefore the second factor is less than or equal to a constant times

$$\Big(\frac{|x-x'|}{\ell(\tilde Q)}\Big)^{\alpha}\omega(\tilde Q)^{-1}\Big(\int_{T(Q)}\frac{d\omega(t)\,dy}{y}\Big)^{1/2}\le C\Big(\frac{|x-x'|}{\ell(\tilde Q)}\Big)^{\alpha}\omega(\tilde Q)^{-1/2},$$

which is what we wanted. Lemma 2.11 is proved. ∎

Our next lemma will show that functions in $\mathcal{C}_{(\alpha,\epsilon)}(B(t;y))$ have a "self-improving" property which allows them to be decomposed in a convenient way. But first we will say what it means for a function in $\mathcal{C}_{(\alpha,\epsilon)}(B(t;y))$ to be "improved".

Definition 2.2. If α and ϵ are positive numbers, and ω is a doubling weight, a function $\phi:\mathbb{R}^d\to\mathbb{R}$ is said to belong to *the Uchiyama class* $\mathcal{U}_{(\alpha,\epsilon)}(B(t,y))$ if:

(a) for all $x\in\mathbb{R}^d$,

$$|\phi(x)|\le\sum_{k=0}^{\infty}\frac{2^{-k\epsilon}}{\omega(B(t;2^k y))}\chi_{R_k(B(t;y))}(x);$$

(b) for all x and x' in $\mathbb{R}^d$,

$$|\phi(x)-\phi(x')|\le\Big(\frac{|x-x'|}{y}\Big)^{\alpha}\sum_{k=0}^{\infty}\frac{2^{-k(\epsilon+\alpha)}}{\omega(B(t;2^k y))}(\chi_{R_k(B(t;y))}(x)+\chi_{R_k(B(t;y))}(x'));$$

(c) $\int_{\mathbb{R}^d}\phi(x)\,d\omega=0$.

The only difference between $\mathcal{C}_{(\alpha,\epsilon)}(B(t;y))$ and $\mathcal{U}_{(\alpha,\epsilon)}(B(t;y))$ is that functions in the second space have extra decay in their Hölder moduli (compare property (b) and (ii′)). This is a real, though illusory, difference.

Lemma 2.12. *If $0<\alpha'\le\alpha$, $\alpha<\epsilon$, and we define $\epsilon'\equiv\epsilon-\alpha'$, then*

$$\mathcal{C}_{(\alpha,\epsilon)}(B(t;y))\subset\mathcal{U}_{(\alpha',\epsilon')}(B(t;y)).$$

Remark. We call this the Free Lunch Lemma.

Proof of Lemma 2.12. Let $\phi\in\mathcal{C}_{(\alpha,\epsilon)}(B(t;y))$. It is trivial that

$$|\phi(x)|\le\sum_{k=0}^{\infty}\frac{2^{-k\epsilon'}}{\omega(B(t;2^k y))}\chi_{R_k(B(t;y))}(x),$$

because $\epsilon'\le\epsilon$. The function ϕ already has cancelation. We only need to check ϕ's Hölder smoothness. If $|x-x'|/y\le 1$ then

$$\begin{aligned}|\phi(x)-\phi(x')|&\le\Big(\frac{|x-x'|}{y}\Big)^{\alpha}\sum_{k=0}^{\infty}\frac{2^{-k\epsilon}}{\omega(B(t;2^k y))}(\chi_{R_k(B(t;y))}(x)+\chi_{R_k(B(t;y))}(x'))\\ &\le\Big(\frac{|x-x'|}{y}\Big)^{\alpha'}\sum_{k=0}^{\infty}\frac{2^{-k\epsilon}}{\omega(B(t;2^k y))}(\chi_{R_k(B(t;y))}(x)+\chi_{R_k(B(t;y))}(x')) &(2.14)\\ &=\Big(\frac{|x-x'|}{y}\Big)^{\alpha'}\sum_{k=0}^{\infty}\frac{2^{-k(\epsilon'+\alpha')}}{\omega(B(t;2^k y))}(\chi_{R_k(B(t;y))}(x)+\chi_{R_k(B(t;y))}(x')); &(2.15)\end{aligned}$$

because $\alpha' \le \alpha$ (in (2.14)) and $\epsilon = \epsilon' + \alpha'$ (in (2.15)). On the other hand, if $|x - x'|/y > 1$ then

$$|\phi(x) - \phi(x')| \le |\phi(x)| + |\phi(x')| \le \sum_{k=0}^{\infty} \frac{2^{-k\epsilon}}{\omega(B(t;2^k y))}(\chi_{R_k(B(t;y))}(x) + \chi_{R_k(B(t;y))}(x'))$$
$$\le \left(\frac{|x-x'|}{y}\right)^{\alpha'} \sum_{k=0}^{\infty} \frac{2^{-k(\epsilon'+\alpha')}}{\omega(B(t;2^k y))}(\chi_{R_k(B(t;y))}(x) + \chi_{R_k(B(t;y))}(x')),$$

where the last line is true because $|x - x'|/y > 1$ and $\epsilon = \epsilon' + \alpha'$. Lemma 2.12 is proved. ■

Functions in $\mathcal{U}_{(\alpha,\epsilon)}(B(t;y))$ can be decomposed in a very nice way.

LEMMA 2.13. *Suppose that α and ϵ are positive numbers and ω is a doubling weight. There is a constant $C = C(\alpha, \epsilon, \omega)$ such that, if $(t, y) \in \mathbb{R}^{d+1}_+$ and $\phi \in \mathcal{U}_{(\alpha,\epsilon)}(B(t;y))$, then*

$$\phi(x) = C \sum_{k=0}^{\infty} 2^{-k\epsilon} \phi_k(x),$$

where each ϕ_k is in $\mathcal{C}_\alpha(B(t, 2^k y))$.

Proof. Let $(t, y) \in \mathbb{R}^{d+1}_+$. There are non-negative, radial functions—call them h_0 and h_1—in $\mathcal{C}_0^\infty(\mathbb{R}^d)$ such that $\operatorname{supp} h_0 \subset B(t;y)$, $\operatorname{supp} h_1 \subset B(t;2y)$, $h_0 \equiv 1$ on $B(t;y/2)$, and

$$h_0(x) + \sum_{k=0}^{\infty} h_1(2^{-k}x) \equiv 1$$

on $\mathbb{R}^d$. These functions can be chosen so that $|\nabla h_0| + |\nabla h_1| \le C/y$, where C only depends on d. We use h_0 and h_1 to define a sequence of functions $\{\rho_k\}_{k=0}^\infty$, via

$$\rho_k(x) = \begin{cases} h_0(x) & \text{if } k = 0, \\ h_1(2^{-k+1}x) & \text{if } k > 0. \end{cases}$$

Then $\sum_{k=0}^\infty \rho_k \equiv 1$, each ρ_k has support contained in $B(t;2^k y)$, and, if $k > 0$, ρ_k's support is contained in the *annulus* $B(t;2^k y) \setminus B(t;2^{k-2}y)$. The functions ρ_k also satisfy $\nabla\rho_k \le C2^{-k}/y$, where C is an absolute constant.

We first decompose ϕ in a preliminary fashion:

$$\phi(x) = \sum_{k=0}^{\infty} \phi(x)\rho_k(x) \equiv \sum_{k=0}^{\infty} \psi_k(x).$$

Each ψ_k has support contained in $B(t;2^k y)$ and satisfies

$$|\psi_k(x) - \psi_k(x')| \le C2^{-k\epsilon}\left(\frac{|x-x'|}{2^k y}\right)^{\alpha} \omega(B(t;2^k y))^{-1} \tag{2.16}$$

for some absolute constant C. Inequality (2.16) follows from the fact that

$$\psi_k(x) - \psi_k(x') = (\phi(x) - \phi(x'))\rho_k(x) + \phi(x')(\rho_k(x) - \rho_k(x')).$$

The first term is controlled by the smoothness of ϕ and the uniform boundedness of ρ_k. The second term is controlled by the size of ϕ and the smoothness of ρ_k (where, when $k \ge 1$, it is useful to note that ψ_k has support contained in the *annulus* $B(t;2^k y) \setminus B(t;2^{k-2}y)$).

Thus ψ_k would belong to $\mathcal{C}_\alpha(B(t;2^k y))$ if $\int \psi_k\, d\omega$ were equal to 0, but there is no reason to expect that to be true.

For $k \geq 0$ define

$$c_k \equiv \frac{\int \left(\sum_{j=0}^{k} \rho_j(t)\right) \phi(t)\, d\omega(t)}{\int \rho_k(t)\, d\omega(t)}. \tag{2.17}$$

Because ω is doubling, the denominator of (2.17) is bigger than or equal to a constant times $\omega(B(t;2^k y))$, which goes to infinity as $k \to \infty$. Because of ϕ's cancelation property, the numerator in (2.17) equals

$$-\int \Big(\sum_{j>k} \rho_j(t)\Big)\phi(t)\, d\omega(t),$$

which, because of our estimate on ϕ's size, has magnitude less than or equal to a constant times $\sum_{j>k} 2^{-j\epsilon} \leq C 2^{-k\epsilon}$. Therefore $c_k \to 0$ as $k \to \infty$. Define $g_k(x) \equiv c_k \rho_k(x)$. Then

$$0 = g_0(x) + \sum_{k=1}^{\infty} (g_k(x) - g_{k-1}(x)), \tag{2.18}$$

and the series converges uniformly. Define

$$\eta_k(x) = \begin{cases} g_0(x) & \text{if } k = 0, \\ g_k(x) - g_{k-1}(x) & \text{if } k > 0. \end{cases}$$

Then each η_k satisfies (modulo an absolute constant) the same size and smoothness bounds as the corresponding ψ_k. But also, for each k,

$$\int \psi_k(x)\, d\omega(x) = \int \eta_k(x)\, d\omega(x).$$

Therefore we get our desired decomposition by putting

$$\phi = \sum_{k=0}^{\infty} \psi_k = \sum_{k=0}^{\infty} (\psi_k - \eta_k),$$

where the last equation is true because of (2.18). That proves Lemma 2.13. ■

The next lemma shows that, although $\tilde{G}_{(\alpha,\epsilon)}(f)$ looks like a more complicated object, it is no harder to control than $G_{\alpha'}(f)$, as long as we choose α' small enough.

Lemma 2.14. *Let $0 \leq \alpha' \leq \alpha$ and $\alpha' < \epsilon$. There is a constant $C = C(\alpha, \epsilon, \alpha', \omega)$ so that, for all f such that $\widetilde{G}_{(\alpha,\epsilon)}(f)$ makes sense (i.e. those f for which* (A) *holds), $\widetilde{G}_{(\alpha,\epsilon)}(f) \leq C \widetilde{G}_{\alpha'}(f)$ pointwise.*

Proof. Since $\sigma_{\alpha'}(f) \sim G_{\alpha'}(f)$ and $\tilde{\sigma}_{(\alpha,\epsilon)}(f) \sim \tilde{G}_{(\alpha,\epsilon)}(f)$, it suffices to show

$$\tilde{\sigma}_{(\alpha,\epsilon)}(f) \leq C \sigma_{\alpha'}(f).$$

We shall do so at $x = 0$. Let $\phi \in \mathcal{C}_{(\alpha,\epsilon)}(B(0;1))$; then, because of Lemmas 2.12 and 2.13, we can write

$$\phi(x) = C \sum_{k=0}^{\infty} 2^{-k\epsilon'} \phi_k(x),$$

where each $\phi_k \in \mathcal{C}_{\alpha'}(B(0;2^k))$, where $\epsilon' = \epsilon - \alpha' > 0$. Therefore

$$\tilde{A}_{(\alpha,\epsilon)}(f)(0,1) \leq C \sum_{k=0}^{\infty} 2^{-k\epsilon'} A_{\alpha'}(f)(0,2^k) \leq \Big(C' \sum_{k=0}^{\infty} 2^{-k\epsilon'} (A_{\alpha'}(f)(0,2^k))^2\Big)^{1/2}.$$

Similarly, for any integer j,

$$\tilde{A}_{(\alpha,\epsilon)}(f)(0,2^j) \leq \Big(C' \sum_{k=0}^{\infty} 2^{-k\epsilon'} (A_{\alpha'}(f)(0,2^{k+j}))^2\Big)^{1/2}.$$

When we square both sides and sum on j we get

$$\begin{aligned}(\tilde{\sigma}_{(\alpha,\epsilon)}(f)(0))^2 &= \sum_{j=-\infty}^{\infty} (\tilde{A}_{(\alpha,\epsilon)}(f)(0,2^j))^2 \leq C' \sum_{j=-\infty}^{\infty}\sum_{k=0}^{\infty} 2^{-k\epsilon'}(A_{\alpha'}(f)(0,2^{k+j}))^2 \\ &= C' \sum_{l=-\infty}^{\infty} (A_{\alpha'}(f)(0,2^l))^2 \sum_{j=-\infty}^{l} 2^{-(l-j)\epsilon'} \leq C'' \sum_{l=-\infty}^{\infty} (A_{\alpha'}(f)(0,2^l))^2 \\ &= C''(\sigma_{\alpha'}(f)(0))^2,\end{aligned}$$

which proves Lemma 2.14. ∎

Proof of Theorem 2.2. Lemmas 2.14 and 2.11 taken together imply that $\|\tilde{G}_{(\alpha,\epsilon)}(f)\|_{L^2(\omega)} \leq C\|f\|_{L^2(\omega)}$. (The only thing we need to check is that $f \in L^2(\omega)$ implies that (A) holds, but that is trivial.) Suppose that the family $\{\phi_{(Q)}\}_{Q\in\mathcal{D}}$ satisfies 1, 2, and 3, and $\sum \gamma_Q \phi_{(Q)}$ is a finite linear sum from that family. Let $f \in L^2(\omega)$ satisfy $\|f\|_{L^2(\omega)} \leq 1$. Then

$$\Big|\int \Big(\sum \gamma_Q \phi_{(Q)}(x)\Big) f(x)\, d\omega(x)\Big| \leq \Big(\sum_Q |\gamma_Q|^2\Big)^{1/2} \Big(\sum_Q \Big|\int f(x)\phi_{(Q)}(x)\, d\omega(x)\Big|^2\Big)^{1/2},$$

so it will be enough to show that

$$\sum_Q \Big|\int f(x)\phi_{(Q)}(x)\, d\omega(x)\Big|^2 \leq C, \tag{2.19}$$

where C only depends on α, α', ϵ, and ω. However, for any $(t,y) \in T(Q)$,

$$\Big|\int f(x)\phi_{(Q)}(x)\, d\omega(x)\Big| \leq C\omega(Q)^{1/2}\tilde{A}_{(\alpha,\epsilon)}(f)(t,y),$$

and therefore

$$\Big|\int f(x)\phi_{(Q)}(x)\, d\omega(x)\Big|^2 \leq C \int_{T(Q)} (\tilde{A}_{(\alpha,\epsilon)}(f)(t,y))^2 \, \frac{d\omega(t)\, dy}{y},$$

since ω's doubling property implies

$$\omega(Q) \sim \int_{T(Q)} \frac{d\omega(t)\, dy}{y}.$$

Therefore the left-hand side of (2.19) is less than or equal to a constant times

$$\begin{aligned}\sum_Q \int_{T(Q)} (\tilde{A}_{(\alpha,\epsilon)}(f)(t,y))^2 \, \frac{d\omega(t)\, dy}{y} &\leq \int_{\mathbb{R}^{d+1}_+} (\tilde{A}_{(\alpha,\epsilon)}(f)(t,y))^2 \, \frac{d\omega(t)\, dy}{y} \\ &\leq C \int_{\mathbb{R}^d} (\tilde{G}_{(\alpha,\epsilon)}(f)(x))^2 \, d\omega(x) \leq C \int_{\mathbb{R}^d} |f(x)|^2 \, d\omega(x),\end{aligned}$$

and that finishes the proof of Theorem 2.2.

3. Harmonic functions on bounded Lipschitz domains

We are going to apply Theorem 2.1 to a family of functions $\{b_{(Q)}\}_Q$ defined on a part of the boundary of a bounded Lipschitz domain $\Omega \subset \mathbb{R}^{d+1}$. That is, $\partial\Omega$ is a finite union of translations and rotations of graphs Γ_i of functions $\psi_i : \mathbb{R}^d \to \mathbb{R}$, and there is a fixed M such that each ψ_i satisfies $|\psi_i(x) - \psi_i(y)| \le M|x-y|$ for all x and y.

As we said in the introduction, our attention will be directed at points near $\partial\Omega$, where we can pretend that we are working in a domain lying above a graph Γ_i. But first we wish to say a few words about the points in Ω that are (relatively) far away from the boundary.

Recall our definition of $\Omega_\delta \equiv \{x \in \Omega : d(x, \partial\Omega) \le \delta\}$. Given a measure μ defined on Ω, write $\mu = \mu_1 + \mu_2$ where $\mu_1(E) = \mu(E \setminus \Omega_\delta)$ and $\mu_2(E) = \mu(E \cap \Omega_\delta)$.

We claim that an inequality of the form

$$\left(\int_\Omega |\nabla u|^q \, d\mu_1\right)^{1/q} \le C\left(\int_{\partial\Omega} |f|^p \, v \, d\omega\right)^{1/p} \tag{3.1}$$

comes almost (but not quite) for free.

We have, for any $x \notin \Omega_\delta$,

$$|\nabla u(x)| \le C\int_{\partial\Omega} |f| \, d\omega,$$

where C depends on Ω, δ, and our choice of X_0 (which, without loss of generality, we may assume lies in $\Omega \setminus \Omega_\delta$). This inequality comes from the fact that

$$\sup_{x\notin\Omega_\delta} |\nabla u(x)| \le C\delta^{-1} \sup_{x\notin\Omega_{\delta/2}} |u(x)|$$

linked to the inequality:

$$\sup_{x\notin\Omega_{\delta/2}} |u(x)| \le C_{\Omega,\delta}\int_{\partial\Omega} |f| \, d\omega.$$

By Hölder's inequality,

$$\int_{\partial\Omega} |f| \, d\omega \le \left(\int_{\partial\Omega} |f|^p \, v \, d\omega\right)^{1/p}\left(\int_{\partial\Omega} \sigma \, d\omega\right)^{1/p'}.$$

Thus, to get (3.1), it is sufficient to have

$$\mu(\Omega\setminus\Omega_\delta)^{1/q}\left(\int_{\partial\Omega} \sigma \, d\omega\right)^{1/p'} \le c',$$

where c' is small positive constant depending on Ω, δ, and X_0.

A moment's thought shows that this is just a "global" version of the sufficient condition from Theorem 3.1, with $\partial\Omega$ (a bounded set, recall) playing the role of a cube Q_b, and $\Omega\setminus\Omega_\delta$ pretending to be $T(Q_b)$. The bump function being integrated against σ is just $\chi_{\partial\Omega}$. It has no "tail" because there is no room for one: the "cube" Q_b fills up all of $\partial\Omega$.

To fit the pattern of Theorem 3.1, the constant c' should really be

$$c \operatorname{diam}(\Omega)\omega(\partial\Omega) = c \operatorname{diam}(\Omega).$$

But, of course, it is—assuming we choose c properly!

That is (almost) all we will say about the parts of Ω lying far from the boundary.

3.1. Lipschitz domains. Our problem now consists in finding an appropriate condition on the measure μ_2 which ensures that

$$\left(\int_\Omega |\nabla u|^q\,d\mu_2\right)^{1/q} \le \left(\int_{\partial\Omega} |f|^p\,v\,d\omega\right)^{1/p}$$

holds for all $f \in L^p$, where we remember that μ_2 is supported in Ω_δ.

By cutting the integral up, taking δ small enough, rescaling, and doing a rotation, we can assume that the support of μ_2—which we will henceforth call μ—lies in a set of the form

$$\mathcal{R} \equiv \{(x,y) : \psi(x) < y < \psi(x)+\delta,\ |x| \le 1\},$$

where $\psi : \mathbb{R}^d \to \mathbb{R}$ is a Lipschitz function, with Lipschitz constant M. We can assume that $\mathcal{R}$ is scaled so small that the set $\Gamma \equiv \{(x,\psi(x)) : |x| \le 2\}$ is a subset of $\partial\Omega$.

Let $\mathcal{D}$ be the family of dyadic cubes $Q \subset \mathbb{R}^d$ such that $Q \subset \{x \in \mathbb{R}^d : |x| \le 3/2\}$. For each cube $Q \in \mathcal{D}$, we define $Q_b = \{(x,\psi(x)) : x \in Q\}$, the *boundary cube* corresponding to Q, and we define

$$\hat{Q}_b = \{(x,y) : x \in Q,\ \psi(x) < y \le \psi(x)+\ell(Q)\}$$
$$T(Q_b) = \{(x,y) : x \in Q,\ \psi(x)+\ell(Q)/2 < y \le \psi(x)+\ell(Q)\},$$

corresponding to the usual Carleson box and "top-half" of a Carleson box, familiar from Euclidean harmonic analysis.

Note that, if we take δ small enough, the support of our "reduced" μ is contained completely inside $\bigcup_{Q\in\mathcal{D}} T(Q_b)$.

Because ψ is Lipschitz, there is a constant $\eta = \eta(M,d)$ such that

$$d(T(Q_b),\partial\Omega) > \eta\ell(Q).$$

Let us define $\tilde{T}(Q_b) \equiv \{X \in \Omega : d(X,T(Q_b)) \le (\eta/4)\ell(Q)\}$ and $\widetilde{T}(\widetilde{Q}_b) \equiv \{X \in \Omega : d(X,T(Q_b)) \le (\eta/2)\ell(Q)\}$. The sets $T(Q_b)$ are pairwise disjoint. It is only a little harder to see that the sets $\tilde{T}(Q_b)$ (respectively, $\widetilde{T}(\widetilde{Q}_b)$) have the bounded overlap property, i.e., there is a $C = C(M,d,\eta)$ such that no $X \in \Omega$ belongs to more than C of the sets $\tilde{T}(Q_b)$ (respectively, $\widetilde{T}(\widetilde{Q}_b)$).

For each $Q \in \mathcal{D}$, we set

$$Z_Q = (x_Q, \psi(x_Q)+\ell(Q)),$$

the "top midpoint" of $\hat{Q}_b$.

We will have occasion to speak of dilates of boundary cubes Q_b, e.g., 2^jQ_b. What this notation means is $(2^jQ)_b$, i.e., the projection onto $\partial\Omega$ of the usual dilate 2^jQ. However, we will assume that these dilates do not extend too far. That is, we will sometimes use expressions like $\sum_{j=0}^{\infty} E_j(2^jQ_b)$, where $E_j(2^jQ_b)$ is some expression depending on 2^jQ_b. In such expressions, we will want to have $E_j = 0$ when $2^j\ell(Q)$ is bigger than some constant r_0. Fortunately, there is an easy way to do this. The expressions E_j which interest us will actually depend on the annuli $2^{j+1}Q_b \setminus 2^jQ_b$; indeed, they will be multiplied by the characteristic function of this set. We will *define* 2^jQ_b to be all of $\partial\Omega$ when $2^j\ell(Q)$ is bigger than our fixed (but unspecified) constant r_0. This automatically makes the annulus $2^{j+1}Q_b \setminus 2^jQ_b$ empty.

Let ω be harmonic measure on $\partial\Omega$ for the point X_0. In other words, if $f \in C(\partial\Omega)$, and u is the solution to Laplace's equation in Ω with boundary data f, then $u(X_0) = \int_{\partial\Omega} f\, d\omega$. For any $X \in \Omega$, the value of $u(X)$ is given by

$$u(X) = \int_{\partial\Omega} K(X,s) f(s)\, d\omega(s),$$

where K is the so-called "kernel function" for ω.

Our functions $b_{(Q)}(s)$ will have the following form. For any X_1^Q and X_2^Q, two arbitrary points in $\tilde{T}(Q_b)$, $Q \in \mathcal{D}$, we let

$$b_{(Q)}(s) = \sqrt{\omega(Q_b)}(K(X_1^Q, s) - K(X_2^Q, s)).$$

The size and smoothness conditions 1) and 2) in the definition of a standard family follow for these $b_{(Q)}(s)$ from classical estimates for the kernel function; to wit,

$$|K(X_i^Q, s)| \le C \sum_{j=0}^{\infty} \frac{2^{-j\alpha}}{\omega(2^j Q_b)} \chi_{R_j(Q_b)}(x),$$

where, of course, $R_j(Q_b) = 2^j Q_b \setminus 2^{j-1} Q_b$ when $j > 0$ and equals Q_b for $j = 0$ (see [JeK], [K]). And

$$|K(X_i^Q, s) - K(X_i^Q, s')| \le C \left(\frac{|s - s'|}{\ell(Q)} \right)^{\beta} \sum_{j=0}^{\infty} \frac{2^{-j\alpha}}{\omega(2^j Q_b)} \chi_{R_j(Q_b)}(x)$$

(see [K]).

All we need now is condition 3), the almost-orthogonality.

Let $G(X)$ be the Green's function for Ω, with a pole at X_0. By classical estimates for the Green's function and harmonic measure, if Q is one of our cubes, then $\omega(Q_b)$ is bounded above and below by positive constants times $G(Z_Q)\ell(Q)^{d-1}$. We shall refer to this fact as 'inequality (3a.2).' In symbols

$$c_1 \omega(Q_b) \le G(Z_Q)\ell(Q)^{d-1} \le c_2 \omega(Q_b). \tag{3a.2}$$

(Note: the exponent on $\ell(Q)$ is $d-1$ and not the usual $d-2$, because we are working in a subset of $\mathbb{R}^{d+1}$.)

Let $\{\lambda_Q\}_Q$ be an arbitrary finite collection of real numbers indexed over $\mathcal{F}$, and set

$$g(s) = \sum \lambda_Q b_{(Q)}(s).$$

We wish to show that

$$\int |g|^2\, d\omega(s) \le C \sum |\lambda_Q|^2.$$

To this purpose, let $f \in L^2(\partial\Omega, \omega)$ be continuous and satisfy $\int_{\partial\Omega} f\, d\omega = 0$. We consider the integral

$$\int_{\partial\Omega} gf\, d\omega. \tag{3a.3}$$

Our job now is to show that this integral is less than or equal to a constant times

$$\Big(\sum_Q |\lambda_Q|^2 \Big)^{1/2} \Big(\int_{\partial\Omega} |f|^2\, d\omega \Big)^{1/2}.$$

The integral (3a.3) equals

$$\sum \lambda_Q \sqrt{\omega(Q_b)}(u(X_1^Q) - u(X_2^Q)).$$

By Cauchy–Schwarz, this has magnitude less than or equal to

$$\Big(\sum |\lambda_Q|^2\Big)^{1/2}\Big(\sum \omega(Q_b)|u(X_1^Q) - u(X_2^Q)|^2\Big)^{1/2},$$

and so our problem reduces to showing

$$\sum \omega(Q_b)|u(X_1^Q) - u(X_2^Q)|^2 \le C \int |f|^2\, d\omega. \tag{3a.4}$$

By the ordinary, differential mean value theorem, and the sub-mean-value property for harmonic functions,

$$\begin{aligned} |u(X_1^Q) - u(X_2^Q)|^2 &\le C\ell(Q)^{-d-1} \int_{\widetilde{T}(\widetilde{Q}_b)} (\ell(Q)|\nabla u(X)|)^2\, dX \\ &= C\ell(Q)^{-d+1} \int_{\widetilde{T}(\widetilde{Q}_b)} |\nabla u(X)|^2\, dX, \end{aligned} \tag{3a.5}$$

where the constant C depends on the "usual" parameters.

By successively applying the estimate from the last inequality, inequality (3a.2), the Harnack property for $G(X)$, and the bounded overlap property of the sets $\widetilde{T}(\widetilde{Q}_b)$, we obtain

$$\begin{aligned} \sum \omega(Q_b)|u(X_1^Q) - u(X_2^Q)|^2 &\le C\sum_Q G(Z_Q) \int_{\widetilde{T}(\widetilde{Q}_b)} |\nabla u(X)|^2\, dX \\ &\le C\sum_Q \int_{\widetilde{T}(\widetilde{Q}_b)} G(X)|\nabla u(X)|^2\, dX \le C\int_\Omega G(X)|\nabla u(X)|^2\, dX. \end{aligned}$$

But by Green's Theorem and our normalization on f—i.e., $u(X_0) = \int_{\partial\Omega} f\, d\omega = 0$—the last quantity is less than or equal to $C\int_{\partial\Omega} |f|^2\, d\omega$. Therefore, modulo multiplication by a small positive constant, our family $\{b_{(Q)}\}$ satisfies 1), 2), and 3) on the homogeneous space $\partial\Omega$, with the Euclidean metric and measure ω.

3.2. The weighted-norm theorem. Let us briefly recap our situation. We have a bounded Lipschitz domain Ω, whose boundary can be written as an overlapping union of (pieces of) graphs of Lipschitz functions ψ_i (appropriately rotated, scaled, and translated). On each of these pieces we have a collection of dyadic boundary cubes that are near the origin. We can assume that we have enough pieces so that the union of these cubes covers all of $\partial\Omega$. Let us throw all of these cubes into a big family, which we will call $\mathcal{G}$. For each one of these cubes Q_b we can talk about $\hat{Q}_b$, $\ell(Q_b)$, and $T(Q_b)$. It is possible that a given Q_b will have more than one definition of $T(Q_b)$ (or $\ell(Q_b)$). This is okay. For a given Q_b, all of its possible values of $\ell(Q_b)$ will be comparable (with comparability constants depending on our domain's Lipschitz constant). There can be no more than C different $T(Q_b)$'s, where C is the number of pieces into which we have divided $\partial\Omega$. Since we are mainly interested in the size of $\mu(T(Q_b))$, in our statement of Theorem 3.1 below, we can take $\mu(T(Q_b))$ to be the *largest* of these numbers.

We can now state the precise form of Theorem 3.1, as it was originally stated in the introduction:

THEOREM 3.1. *Let $\Omega \subset \mathbb{R}^{d+1}$ be a bounded Lipschitz domain, and let ω be harmonic measure on $\partial\Omega$, for some fixed point $X_0 \in \Omega$. Suppose that $v \in L^1(\partial\Omega, d\omega)$ is a non-negative function and μ is a positive Borel measure on Ω. Define $\sigma \equiv v^{1-p'}$ and suppose that $\sigma d\omega \in A^\infty(\omega)$ on $\partial\Omega$. If $1 < p \leq q < \infty$ and $q \geq 2$, then there exists an $\epsilon = \epsilon(\Omega) > 0$ and a positive constant c such that*

$$\left(\int_{\Omega_\delta} |\nabla u(x)|^q \, d\mu(x)\right)^{1/q} \leq \left(\int_{\partial\Omega} |f(s)|^p v(s) \, d\omega(s)\right)^{1/p}$$

will hold for all $f \in L^p(\partial\Omega, d\omega)$, for some positive δ, if, for all sufficiently small boundary cubes $Q_b \in \mathcal{G}$,

$$\mu(T(Q_b))^{1/q}\left(\int_{\partial\Omega}\left[\omega(Q_b)\sum_{j=0}^{\infty}\frac{2^{-j\epsilon}}{\omega(2^jQ_b)}\chi_{R_j(Q_b)}(s)\right]^{p'/2}\sigma(s)\,d\omega(s)\right)^{1/p'} \leq c\ell(Q_b)\omega(Q_b),$$

where c depends only on p, q, Ω, and the choice of the point X_0.

Proof. We begin with the observation that there is an absolute constant C such that, for every $Q_b \in \mathcal{G}$,

$$\sup_{Z\in T(Q_b)} |\nabla u(Z)| \leq C\ell(Q_b)^{-1} \sup_{X_i^{Q_b}\in\tilde{T}(Q_b)} |u(X_1^{Q_b}) - u(X_2^{Q_b})|. \qquad (3b.1)$$

For each $Q_b \in \mathcal{G}$, let $X_1^{Q_b}$ and $X_2^{Q_b}$ in $\widetilde{T}(Q_b)$ be chosen so that

$$\frac{1}{2}\sup_{Z\in T(Q_b)} |\nabla u(Z)| \leq C\ell(Q_b)^{-1}|u(X_1^{Q_b}) - u(X_2^{Q_b})|.$$

We will apply Theorem 2.1 to the family of functions defined by

$$\phi_{(Q_b)}(s) \equiv \sqrt{\omega(Q_b)}(K(X_1^{Q_b}, s) - K(X_2^{Q_b}, s)).$$

Now, let μ be a positive Borel measure defined on $\mathcal{R}$. We wish to control

$$\int_{\mathcal{R}} |\nabla u(Z)|^q \, d\mu(Z).$$

By our choice of the points $X_i^{Q_b}$, this is less than or equal to a constant times

$$\sum_{Q_b\in\mathcal{G}} \ell(Q_b)^{-q}|u(X_1^{Q_b}) - u(X_2^{Q_b})|^q\mu(T(Q_b)).$$

Let $g : \mathcal{G} \to \mathbb{R}$ be a finite sequence (indexed over $\mathcal{F}$), and satisfying

$$\sum_{Q_b\in\mathcal{G}} |g(Q_b)|^{q'}\mu(T(Q_b)) \leq 1,$$

and chosen so that

$$\Big|\sum_{Q_b\in\mathcal{G}} g(Q_b)\ell(Q_b)^{-1}(u(X_1^{Q_b}) - u(X_2^{Q_b}))\mu(T(Q_b))\Big|$$

$$\geq \frac{1}{2}\Big(\sum_{Q_b\in\mathcal{G}} \ell(Q_b)^{-q}|u(X_1^{Q_b}) - u(X_2^{Q_b})|^q\mu(T(Q_b))\Big)^{1/q}. \qquad (3b.2)$$

We need to show that, for every such g, the left-hand side of (3b.2) is not too big; i.e., that it is less than or equal to a constant times

$$\left(\int_{\partial\Omega} |f(s)|^p\, v\, d\omega(s)\right)^{1/p}.$$

We define

$$T(g)(s) \equiv \sum_{Q_b\in\mathcal{G}} g(Q_b)\mu(T(Q_b))\ell(Q_b)^{-1}(K(X_1^{Q_b}) - K(X_2^{Q_b}, s)),$$

and notice that

$$\sum_{Q_b\in\mathcal{G}} g(Q_b)\ell(Q_b)^{-1}(u(X_1^{Q_b}) - u(X_2^{Q_b}))\mu(T(Q_b)) = \int_{\partial\Omega} f(s)\, T(g)(s)\, d\omega(s).$$

Recall that $\sigma = v^{1-p'}$. The left-hand side of (3b.2) will be

$$\leq C\left(\int_{\partial\Omega} |f(s)|^p\, v\, d\omega(s)\right)^{1/p}$$

for all g as we have defined, if

$$\left(\int_{\partial\Omega} |T(g)(s)|^{p'}\sigma\, d\omega\right)^{1/p'} \leq C\Big(\sum_{Q_b\in\mathcal{G}} |g(Q_b)|^{q'}\mu(T(Q_b))\Big)^{1/q'} \tag{3b.3}$$

for all such g. It is this last inequality that we shall prove.

Write

$$T(g)(s) = \sum_{Q_b\in\mathcal{G}} \lambda_{Q_b}\phi_{(Q_b)},$$

where

$$\phi_{(Q_b)}(s) \equiv \sqrt{\omega(Q_b)}(K(X_1^{Q_b}, s) - K(X_2^{Q_b}, s)),$$

and

$$|\lambda_{Q_b}| \leq C\frac{|g(Q_b)|\mu(T(Q_b))}{\ell(Q_b)\sqrt{\omega(Q_b)}}.$$

The integral we need to estimate naturally breaks into two pieces. Let us recall the region we denoted by $\mathcal{R}$ in Section 3.1:

$$\mathcal{R} \equiv \{(x, y) : \psi(x) < y < \psi(x) + \delta,\ |x| \leq 1\},$$

where ψ is a Lipschitz function, and our measure μ is supported entirely inside $\mathcal{R}$. We will handle the part near $\mathcal{R}$ with Theorem 2.1 from above. The "far" part can be bounded by a naive brute-force observation. Let $\kappa > 0$ and define $\aleph \equiv \{x \in \partial\Omega : d(x, \mathcal{R}) > \kappa\}$. By our estimates on the $\phi_{(Q)}$'s,

$$|Tg(x)| \leq C_{\kappa,\Omega}\sum_{Q_b\in\mathcal{G}} |\lambda_{Q_b}|\sqrt{\omega(Q_b)}\sum_{j=0}^{\infty}\frac{2^{-j\alpha}}{\omega(2^jQ_b)}\chi_{R_j(Q_b)}(x).$$

There is a C, independent of $x \in \aleph$, such that $\frac{2^{-j\alpha}}{\omega(2^jQ_b)}\chi_{R_j(Q_b)}(x)$ can be non-zero for at most C many j's. For each of these j's, 2^{-j} is essentially equal to $\ell(Q_b)$, with the comparability constants depending on κ and Ω. Also, for such j, $\omega(2^j Q_b)$ is comparable to $\omega(\partial\Omega)$, which equals 1. Therefore, for $x \in \aleph$,

$$|Tg(x)| \le C_{\kappa,\Omega} \sum_{Q_b \in \mathcal{G}} |\lambda_{Q_b}| \sqrt{\omega(Q_b)}\, \ell(Q_b)^\alpha \le C \sum_{Q_b \in \mathcal{G}} |g(Q_b)| \mu(T(Q_b)) \ell(Q_b)^{-1} \ell(Q_b)^\alpha;$$

which looks funny—but we have a good reason for not combining the exponents in the $\ell(Q_b)$'s.

We assume that

$$\mu(T(Q_b))^{1/q} \left(\int_{\partial\Omega} \left[\omega(Q_b) \sum_{j=0}^{\infty} \frac{2^{-j\epsilon}}{\omega(2^j Q_b)} \chi_{R_j(Q_b)}(x) \right]^{p'/2} \sigma(x)\, d\omega(x) \right)^{1/p'} \le c\ell(Q_b)\omega(Q_b)$$

for every $Q_b \in \mathcal{G}$. We may replace the integral on the left-hand side of this inequality by an integral over the smaller region $\aleph$. Doing so, we may rewrite the inequality (after a change in c) as

$$\mu(T(Q_b))^{1/q} \le c\ell(Q_b)^{-\epsilon/2} \left(\frac{\ell(Q_b)\sqrt{\omega(Q_b)}}{(\int_\aleph \sigma\, d\omega)^{1/p'}} \right),$$

where $\epsilon > 0$ is small. Thus, for $x \in \aleph$,

$$\begin{aligned} |Tg(x)| &\le c \sum_{Q_b \in \mathcal{G}} |g(Q_b)| \mu(T(Q_b)) \ell(Q_b)^{-1} \ell(Q_b)^\alpha \\ &= c \sum_{Q_b \in \mathcal{G}} |g(Q_b)| \mu(T(Q_b))^{1/q'} \mu(T(Q_b))^{1/q} \ell(Q_b)^{-1} \ell(Q_b)^\alpha \\ &\le c \Big[\sum_{Q_b \in \mathcal{G}} |g(Q_b)| \mu(T(Q_b))^{1/q'} \ell(Q_b)^{\alpha-\epsilon/2} \sqrt{\omega(Q_b)} \Big] \left(\int_\aleph \sigma\, d\omega \right)^{-1/p'}. \end{aligned}$$

The expression in the brackets is bounded by

$$\Big(\sum_{Q_b \in \mathcal{G}} |g(Q_b)|^{q'} \mu(T(Q_b)) \Big)^{1/q'} \Big(\sum_{Q_b \in \mathcal{G}} \ell(Q_b)^{\epsilon'} \omega(Q_b)^{q/2} \Big)^{1/q},$$

where $\epsilon' > 0$. By hypothesis, the first factor is ≤ 1. Since $q \ge 2$, the second factor is no bigger than

$$\Big(\sum_{Q_b \in \mathcal{G}} \ell(Q_b)^{2\epsilon'/q} \omega(Q_b) \Big)^{1/2},$$

which, since $\epsilon' > 0$, is bounded by a constant.

This means that, when $x \in \aleph$, $|Tg(x)|$ is no bigger than a constant times

$$\left(\int_\aleph \sigma\, d\omega \right)^{-1/p'},$$

which trivially implies that

$$\int_\aleph |Tg(x)|^{p'} \sigma\, d\omega \le C.$$

So, now we look at the x's *close* to $\mathcal{R}$.

To keep ideas clear, let us first consider the simple case $p = q = 2$. Having thrown out the points that are far from $\mathcal{R}$, inequality (3b.3) reduces to

$$\int_{\partial\Omega\setminus\aleph} |T(g)(s)|^2\sigma\,d\omega \le C \sum_{Q_b\in\mathcal{G}} |g(Q_b)|^2\mu(T(Q_b)), \tag{3b.4}$$

where all the cubes Q_b are small and touch $\mathcal{R}$. By Theorem 2.1, the left-hand side of (3b.4) is less than or equal to a constant times

$$\begin{aligned}\int_{\partial\Omega}\Big(\sum_{Q_b\in\mathcal{G}}|\lambda_Q|^2\Big[\sum_{j=0}^{\infty}\frac{2^{-j(2\alpha-\tau)}}{\omega(2^jQ_b)}\chi_{R_j(Q_b)}(x)\Big]\Big)\sigma\,d\omega\\ \le C\sum_{Q_b\in\mathcal{G}}\left|\frac{g(Q_b)\mu(T(Q_b))}{\ell(Q_b)\sqrt{\omega(Q_b)}}\right|^2\int_{\partial\Omega}\Big[\sum_{j=0}^{\infty}\frac{2^{-j(2\alpha-\tau)}}{\omega(2^jQ_b)}\chi_{R_j(Q_b)}(x)\Big]\sigma\,d\omega\\ = C\sum_{Q_b\in\mathcal{G}}\frac{|g(Q_b)|^2\mu(T(Q_b))^2}{\ell(Q_b)^2\omega(Q_b)}\int_{\partial\Omega}\Big[\sum_{j=0}^{\infty}\frac{2^{-j(2\alpha-\tau)}}{\omega(2^jQ_b)}\chi_{R_j(Q_b)}(x)\Big]\sigma\,d\omega.\end{aligned}$$

We want this last quantity to be less than or equal to

$$C\sum_{Q_b\in\mathcal{G}}|g(Q_b)|^2\mu(T(Q_b)).$$

Comparing the sums term-by-term, we see that this *will* happen if, for every $Q\in\mathcal{F}$,

$$\mu(T(Q_b))\int_{\partial\Omega}\Big[\sum_{j=0}^{\infty}\frac{2^{-j(2\alpha-\tau)}}{\omega(2^jQ_b)}\chi_{R_j(Q_b)}(x)\Big]\sigma\,d\omega \le C\ell(Q_b)^2\omega(Q_b);$$

or, taking square roots,

$$\mu(T(Q_b))^{1/2}\Big(\int_{\partial\Omega}\Big[\sum_{j=0}^{\infty}\frac{2^{-j(2\alpha-\tau)}}{\omega(2^jQ_b)}\chi_{R_j(Q_b)}(x)\Big]\sigma\,d\omega\Big)^{1/2} \le C\ell(Q_b)\omega(Q_b)^{1/2}. \tag{3b.5}$$

Thus, the appropriate sufficient condition, when $p = q = 2$, is the one given by Theorem 3.1.

The proofs for more general p and q follow this same pattern. The main difficulty encountered is that we can no longer freely exchange the order of integration and summation. Fortunately, this can be circumvented by some standard trickery.

We shall consider the more difficult case first: $1 < p \le 2 \le q < \infty$. In this case, $p' \ge 2$. Before continuing, we recall our assumption that $\sigma\,d\omega$ is an $A^\infty(\omega)$ measure.

We can apply Theorem 2.1 to obtain

$$\int_{\partial\Omega}|T(g)|^{p'}\sigma\,d\omega \le C\int_{\partial\Omega}(g^*(T(g)))^{p'}\sigma\,d\omega.$$

Let $h\ge 0$, $h\in L^r(\sigma d\omega)$, $r = (p'/2)'$, be such that $(\int_{\partial\Omega}h(s)^r\sigma(s)\,d\omega(s))^{1/r} = 1$, and

$$\Big(\int_{\partial\Omega}(g^*(T(g))^{p'}\sigma\,d\omega\Big)^{1/p'} = \Big(\int_{\partial\Omega}(g^*(T(g))^2h\sigma\,d\omega\Big)^{1/2},$$

by duality. The integral on the right hand side of the last equation is equal to

$$\begin{aligned}
\Big(\int_{\partial\Omega}\Big(\sum_{Q_b\in\mathcal{G}}\lambda_{Q_b}^2\sum_{j=0}^{\infty}\frac{2^{-\epsilon j}}{\omega(2^jQ_b)}\chi_{R_j(Q_b)}(s)\Big)h(s)\sigma(s)\,d\omega(s)\Big)^{1/2}&\\
=\Big(\sum_{Q_b\in\mathcal{G}}\lambda_{Q_b}^2\int_{\partial\Omega}\Big(\sum_{j=0}^{\infty}\frac{2^{-\epsilon j}}{\omega(2^jQ_b)}\chi_{R_j(Q_b)}(s)\Big)h(s)\sigma(s)\,d\omega(s)\Big)^{1/2}&\\
\le\Big\{\sum_{Q_b\in\mathcal{G}}\lambda_{Q_b}^2\Big(\int_{\partial\Omega}\Big(\sum_{j=0}^{\infty}\frac{2^{-\epsilon j}}{\omega(2^jQ_b)}\chi_{R_j(Q_b)}(s)\Big)^{p'/2}\sigma(s)\,d\omega(s)\Big)^{2/p'}&\\
\times\Big(\int_{\partial\Omega}h(s)^r\sigma(s)\,d\omega(s)\Big)^{1/r}\Big\}^{1/2}&\\
=\Big(\sum_{Q_b\in\mathcal{G}}\lambda_{Q_b}^2\Big(\int_{\partial\Omega}\Big(\sum_{j=0}^{\infty}\frac{2^{-\epsilon j}}{\omega(2^jQ_b)}\chi_{R_j(Q_b)}(s)\Big)^{p'/2}\sigma(s)\,d\omega(s)\Big)^{2/p'}\Big)^{1/2}.&
\end{aligned}$$

We want to show that this sum is bounded by $C(\sum_{Q_b\in\mathcal{G}}g(Q_b)^{q'}\mu(T(Q_b)))^{1/q'}$, or that

$$\Big(\sum_{Q_b\in\mathcal{G}}\lambda_{Q_b}^2\Big(\int_{\partial\Omega}\Big(\sum_{j=0}^{\infty}\frac{2^{-\epsilon j}}{\omega(2^jQ_b)}\chi_{R_j(Q_b)}(s)\Big)^{p'/2}\sigma(s)\,d\omega(s)\Big)^{2/p'}\Big)^{q'/2}\le C\sum_{Q_b\in\mathcal{G}}g(Q_b)^{q'}\mu(T(Q_b)).$$

Since $q'/2<1$ this will be true if

$$\sum_{Q_b\in\mathcal{G}}\lambda_{Q_b}^{q'}\Big(\int_{\partial\Omega}\Big(\sum_{j=0}^{\infty}\frac{2^{-\epsilon j}}{\omega(2^jQ_b)}\chi_{R_j(Q_b)}(s)\Big)^{p'/2}\sigma(s)\,d\omega(s)\Big)^{q'/p'}\le C\sum_{Q_b\in\mathcal{G}}g(Q_b)^{q'}\mu(T(Q_b)).$$

Comparing term-by-term and recalling that $\lambda_{Q_b}\le C\frac{g(Q_b)\mu(T(Q_b))}{\ell(Q_b)\sqrt{\omega(Q_b)}}$, we can see that the above is valid if

$$\frac{\mu(T(Q_b))^{q'-1}}{\ell(Q_b)^{q'}(\sqrt{\omega(Q_b)})^{q'}}\times\Big(\int_{\partial\Omega}\Big(\sum_{j=0}^{\infty}\frac{2^{-\epsilon j}}{\omega(2^jQ_b)}\chi_{R_j(Q_b)}(s)\Big)^{p'/2}\sigma(s)\,d\omega(s)\Big)^{q'/p'}\le C,$$

or

$$\mu(T(Q_b))^{1/q}\Big(\int_{\partial\Omega}\Big(\sum_{j=0}^{\infty}\frac{2^{-\epsilon j}}{\omega(2^jQ_b)}\chi_{R_j}(s)\Big)^{p'/2}\sigma(s)\,d\omega(s)\Big)^{1/p'}\le C\ell(Q_b)\sqrt{\omega(Q_b)}.$$

But this is exactly our condition.

Let us now consider the easier case, $2<p\le q<\infty$. We have $p'<2$ and $p'/2<1$. Defining $T(g)$ as before, we apply Theorem 2.1 to get

$$\int_{\partial\Omega}|T(g)|^{p'}\sigma\,d\omega\le C\int_{\partial\Omega}\Big(\sum_{Q_b\in\mathcal{G}}|\lambda_{Q_b}|^2\Big[\sum_{j=0}^{\infty}\frac{2^{-j(2\alpha-\tau)}}{\omega(2^jQ_b)}\chi_{R_j(Q_b)}(x)\Big]\Big)^{p'/2}\sigma\,d\omega.$$

Since $p'/2 < 1$, the right-hand quantity is less than or equal to

$$C \int_{\partial\Omega} \Big(\sum_{Q_b \in \mathcal{G}} |\lambda_{Q_b}|^{p'} \Big[\sum_{j=0}^{\infty} \frac{2^{-j(2\alpha-\tau)}}{\omega(2^j Q_b)} \chi_{R_j(Q_b)}(x) \Big]^{p'/2} \Big) \sigma \, d\omega$$

$$= C \sum_{Q_b \in \mathcal{G}} |\lambda_{Q_b}|^{p'} \Big(\int_{\partial\Omega} \Big[\sum_{j=0}^{\infty} \frac{2^{-j(2\alpha-\tau)}}{\omega(2^j Q_b)} \chi_{R_j(Q_b)}(x) \Big] \Big)^{p'/2} \sigma \, d\omega.$$

Using our bound on λ_{Q_b}, the last quantity is less than or equal to

$$C \sum_{Q_b \in \mathcal{G}} \Big(\frac{|g(Q_b)| \mu(T(Q_b))}{\ell(Q_b)\sqrt{\omega(Q_b)}} \Big)^{p'} \Big(\int_{\partial\Omega} \Big[\sum_{j=0}^{\infty} \frac{2^{-j(2\alpha-\tau)}}{\omega(2^j Q_b)} \chi_{R_j(Q_b)}(x) \Big]^{p'/2} \sigma \, d\omega \Big);$$

which, since $p' \geq q'$, is bounded by a constant times

$$\Big[\sum_{Q_b \in \mathcal{G}} \Big[\frac{|g(Q_b)| \mu(T(Q_b))}{\ell(Q_b)\sqrt{\omega(Q_b)}} \Big]^{q'} \Big[\int_{\partial\Omega} \Big[\sum_{j=0}^{\infty} \frac{2^{-j(2\alpha-\tau)}}{\omega(2^j Q_b)} \chi_{R_j(Q_b)}(x) \Big]^{p'/2} \sigma \, d\omega \Big]^{q'/p'} \Big]^{p'/q'}.$$

In order for our dual inequality to hold, the last quantity must be less than or equal to

$$C \Big(\sum_{Q_b \in \mathcal{G}} |g(Q_b)|^{q'} \mu(T(Q_b)) \Big)^{p'/q'},$$

which will be true for all g's if

$$\Big(\frac{\mu(T(Q_b))}{\ell(Q_b)\sqrt{\omega(Q_b)}} \Big)^{q'} \Big(\int_{\partial\Omega} \Big[\sum_{j=0}^{\infty} \frac{2^{-j(2\alpha-\tau)}}{\omega(2^j Q_b)} \chi_{R_j(Q)}(x) \Big]^{p'/2} \sigma \, d\omega \Big)^{q'/p'} \leq C \mu(T(Q_b))$$

for all $Q_b \in \mathcal{G}$; i.e., after some transposition,

$$\mu(T(Q_b))^{1/q} \Big(\int_{\partial\Omega} \Big[\sum_{j=0}^{\infty} \frac{2^{-j(2\alpha-\tau)}}{\omega(2^j Q_b)} \chi_{R_j(Q_b)}(x) \Big]^{p'/2} \sigma \, d\omega \Big)^{1/p'} \leq C \ell(Q_b) \sqrt{\omega(Q_b)},$$

which is our condition from Theorem 3.1. ■

4. Elliptic functions

The Dirichlet problem on a bounded Lipschitz domain is solvable for elliptic L whenever the problem is solvable for Δ, by a classical result of [LiStWein]. The solution can be written as the integral of the boundary data against the elliptic measure $d\omega_L^x$. Unfortunately, this measure is not necessarily A^∞ with respect to surface measure; however, elliptic measure does satisfy a doubling condition. There is also a Green function, G_L, which, along with ω_L, satisfies estimates similar to those for harmonic G and ω. For $u(x)$, the solution to a strictly elliptic divergence form equation on a bounded Lipschitz domain, Ω, although $\nabla u(x)$ exists a.e. as a weak function, the gradient may not exist pointwise. Consequently, we no longer have the pointwise estimates on ∇u that are valid for harmonic u. Since we cannot use the kind of estimate on the gradient of an elliptic function that was used in the proof of Theorem 3.1, we need to find another way to proceed. One way to overcome this obstacle is to start with the difference quotient $|u(x_{Q_b}) - u(y_{Q_b})|/\ell(Q_b)$ as we do in Theorem 4.1.

Specifically, suppose that $Lu(x) = 0$ for $x \in \Omega$, $u(s) = f(s)$, for f in an appropriate class of functions, $s \in \partial\Omega$. $L = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}(a_{i,j}(x)\frac{\partial}{\partial x_j})$, $a_{i,j}(x) = a_{j,i}(x)$, and there is a number $\lambda > 0$ such that $\lambda^{-1}|\xi|^2 \le \sum_{i,j=1}^{d} \xi_i a_{i,j}(x)\xi_j \le \lambda|\xi|^2$ for all $x \in \Omega \subset \mathbb{R}^d$. For Theorem 4.1, let us assume that $\mathcal{G}$ is the family of boundary cubes used in the proof of Theorem 3.1. We also take $\{x_{T(Q_b)}\}$ and $\{y_{T(Q_b)}\}$ to be any two sequences of points in $T(Q_b)$, indexed over $\mathcal{G}$. We call such a double sequence *hyperbolically close.* In proving Theorem 3.1 we used the fact that, for every $Q_b \in \mathcal{G}$, $\sup_{x\in T(Q_b)} |\nabla u|(x)$ was "morally equivalent" to

$$\sup_{X,Y\in T(Q_b)} \ell(Q_b)^{-1}\,|u(X) - u(Y)|$$

when u is harmonic. In Theorem 4.1 we will replace the absolute value of the gradient of u by this quantity. So Theorem 4.1 follows as a corollary to Theorem 3.1.

Another way to overcome the obstacle is to replace $|\nabla u|(x)$ by a local Hölder coefficient in the weighted inequality, as we do in Theorem 4.2. In both approaches we are relying on the fact that $u(x)$ has a representation as an integral of a kernel function against the boundary data with respect to elliptic measure ([CaFaMSa], [K]).

Using the first approach we have

THEOREM 4.1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$, and let $\omega_L^{x_0} = \omega_L$ be the elliptic measure generated by L on Ω, L as described above. Let $f \in L^p(\partial\Omega, d\omega_L)$ and take $\mathcal{G}$ to be the collection of boundary cubes as defined above. Suppose $\upsilon \ge 0$, and $\upsilon \in L^1_{\mathrm{loc}}(\partial\Omega, d\omega_L)$; μ is a non-negative Borel measure defined on Ω. Define $\sigma(s) = \upsilon(s)^{1-p'}$ and assume that $\sigma d\omega_L$ is an A^∞ measure with respect to ω_L. If, for every cube $Q_b \in \mathcal{G}$, μ and υ satisfy*

$$\mu(T(Q_b))^{1/q}\biggl(\int_{\partial\Omega}\biggl(\omega(Q_b)\sum_{j=0}^{\infty}\frac{2^{-j\epsilon}}{\omega(2^jQ_b)}\chi_{R_j(Q_b)}(s)\biggr)^{p'/2}\sigma(s)\,d\omega_L(s)\biggr)^{1/p'} \le c\ell(Q_b)\omega_L(Q_b),$$

then there is a constant $C > 0$ so that, for any pair of points $x_{T(Q_b)}$ and $y_{T(Q_b)}$ in $T(Q_b)$,

$$\Bigl(\sum_{Q_b\in\mathcal{G}} \ell(Q_b)^{-q}|u(x_{T(Q_b)}) - u(y_{T(Q_b)})|^q \mu(\widetilde{T}(Q_b))\Bigr)^{1/q} \le C\biggl(\int_{\partial\Omega}|f(s)|^p\upsilon(s)\,d\omega_L(s)\biggr)^{1/p}$$

for $1 < p \le q < \infty$, $q \ge 2$. Here $C = C(d, \Omega, \lambda, p, q, \epsilon, \alpha, c)$ is independent of Q_b, μ, υ, u, and f.

Proof sketch. Here we take $b_Q(s) = \sqrt{\omega_L(Q_b)}(K_L(x_{T(Q_b)}, s) - K_L(y_{T(Q_b)}, s))$ with $K_L(x, s)$ the elliptic kernel function generated by the operator L on Ω ([CaFaMSa]). Decay and smoothness for these functions follow from classical estimates ([K]), and the proof of almost-orthogonality follows using duality as above. We note that the estimate $|u(x_{T(Q_b)}) - u(y_{T(Q_b)}|^2 \lesssim \ell(Q_b)^{-d+2}\int_{\widetilde{T}(Q_b)}|\nabla u(x)|^2\,dx$, used in the proof of almost-orthogonality for harmonic $u(x)$, is also valid here ([DJeK]).

The second approach was originally suggested to the authors by R. L. Wheeden. We prove the weighted inequality for a local Hölder coefficient, $\|u\|_{H^\alpha}(x)$, instead of for $|\nabla u(x)|$. Since strictly elliptic functions such as $u(x)$ always have a Hölder continuous representative in their domain of definition, it is much easier to work with the Hölder coefficient than with the gradient. (The weighted norm inequality for elliptic $u(x)$, using $|\nabla u(x)|$, can be proved, see [Sw1], but extra conditions must be placed on both the measure μ and the range of exponents p and q.) The definition of the Hölder coefficient is

$$\|u\|_{H^\alpha}(x) = \sup_{y\in B_{\delta/50},\, y\neq x} \frac{|u(x)-u(y)|}{|x-y|^\alpha}$$

with $B_{\delta(x)/50}(x) = B_{\delta/50}$ a small disk of radius $\delta(x)/50$, center x; $\delta(x) = d(x,\partial\Omega)$.

With the same assumptions as stated in Theorem 4.1 we have (in particular, we still have $\sigma d\omega_L \in A^\infty(\omega_L)$ on $\partial\Omega$):

THEOREM 4.2. *If, for all boundary cubes $Q_b \in G$ with $\ell(Q_b) \le C(R,\Omega)$ a fixed constant, μ and υ satisfy*

$$\mu(T(Q_b))^{1/q}\Big(\int_{\partial\Omega}\Big(\omega(Q_b)\sum_{j=0}^{\infty}\frac{2^{-j\epsilon}}{\omega(2^jQ_b)}\chi_{R_j(Q_b)}(y')\Big)^{p'/2}\sigma(y')\,d\omega_L(y')\Big)^{1/p'}$$
$$\le c\ell(Q_b)^{(1+\alpha)/2}\omega_L(Q_b),$$

then there is a constant $C = C(d,\Omega,\lambda,p,q,\epsilon,\alpha,\eta,c)$ so that, for all $1<p\le q<\infty$ and $q\ge 2$, we have

$$\Big(\int_\Omega \|u\|^q_{H^\alpha}(x)\,d\mu(x)\Big)^{1/q} \le C\Big(\int_{\partial\Omega}|f(s)|^p\upsilon(s)\,d\omega_L(s)\Big)^{1/p}.$$

Proof sketch. As in the harmonic case we start by using duality to estimate $(\int_\Omega \|u\|^q_{H^\alpha}(x)\,d\mu(x))^{1/q}$. For $g(x)$ such that $\|g\|_{L^{q'}(\Omega,d\mu)} \le 1$, the following equation will hold:

$$\sup_{g\,:\,\|g\|_{L^{q'}(\Omega,d\mu)}\le 1}\int_\Omega \|u\|_{H_\alpha}(x)g(x)\,d\mu(x) = \Big(\int_\Omega \|u\|^q_{H^\alpha}(x)\,d\mu(x)\Big)^{1/q}.$$

Consequently, we want to bound all such integrals by $(\int_{\partial\Omega}|f(s)|^p\upsilon(s)\,d\omega_L(s))^{1/p}$.

We have

$$\begin{aligned}\int_\Omega \|u\|_{H_\alpha}(x)g(x)\,d\mu(x) &= \int_\Omega \sup_{y\in B_{\delta/50}(x),\,y\neq x}\frac{|u(x)-u(y)|}{|x-y|^\alpha}g(x)\,d\mu(x)\\ &\le \int_\Omega \sup_{y\in B_{\delta/50}(x),\,y\neq x}\frac{1}{|x-y|^\alpha}\int_{\partial\Omega}|K_L(x,s)-K_L(y,s)|\,|f(s)|\,d\omega_L(s)\,g(x)\,d\mu(x)\\ &\le \sum_{Q_b\in\mathcal{G}\cap\operatorname{supp} g}\int_{T(Q_b)} g(x)\int_{\partial\Omega}\ell(Q_b)^{-\alpha}K(x_{T(Q_b)},s)|f(s)|\,d\omega_L(s)\,d\mu(x).\end{aligned}$$

We have assumed $g\ge 0$, and that $\operatorname{supp} g$ is compact in Ω [1]. The last inequality follows from writing the integral over Ω as a sum over the regions $T(Q_b)$ and using the Hölder continuity, and Harnack's inequality, for the kernel function $K_L(x,s)$ in the first variable. Now from Fubini's theorem and interchanging sum and integral, the last expression is

(1) This method is equivalent to the assumptions on g in the proof of Theorem 1.1, Section 1.

less than or equal to

$$\int_{\partial\Omega} \sum_{Q_b \in \mathcal{G} \cap \operatorname{supp} g} \int_{T(Q_b)} g(x)\ell(Q_b)^{-\alpha} K_L(x_{T(Q_b)}, s)\, d\mu(x) |f(s)|\, d\omega_L(s)$$

$$\leq \int_{\partial\Omega} \Big\{ \sum_{Q_b \in \mathcal{G} \cap \operatorname{supp} g} \ell(Q_b)^{-\alpha} K_L(x_{T(Q_b)}, s) \Big(\int_{T(Q_b)} g(x)^{q'}\, d\mu(x) \Big)^{1/q'} \mu(T_{Q_b})^{1/q} \Big\} \times |f(s)|\, d\omega_L(s).$$

This last integral can be written as

$$\int_{\partial\Omega} T(s)|f(s)|\, d\omega_L(s)$$

where

$$T(s) = \sum_{Q_b \in \mathcal{G} \cap \operatorname{supp} g} \lambda_{Q_b} b_{(Q_b)}(s),$$

with $b_{(Q_b)}(s) = \ell(Q_b)^{(1-\alpha)/2} \sqrt{\omega(Q_b)} K_L(x_{T(Q_b)}, s)$.

From this point on the argument is very similar to the proof presented in previous sections of this paper and to the proof in [Sw1]. The Lipschitz domain is dealt with by dividing Ω into finitely many Lipschitz cylinders, V_j, $j = 1, \ldots, m$, in which the graph of $\partial\Omega \cap \partial V_j$ can be described as the graph of a Lipschitz function. The V_j overlap with each other, but only up to a fixed number of times, so the estimates on each V_j can be summed at the end. We use the Littlewood–Paley type inequality which says the following: Suppose $\sigma(s)\, d\omega_L(s) \in A^\infty(d\omega_L, \partial\Omega)$, and $T(s) = \sum_{Q_b \in \mathcal{F}} \lambda_{Q_b} c_{(Q_b)}(s)$, where $\mathcal{F}$ is a finite family of cubes from $\mathcal{G}$, with the $c_{(Q_b)}$ satisfying the following three conditions:

(i) $|c_{(Q_b)}(s)| \lesssim \sqrt{\omega_L(Q_b)} 2^{-\epsilon j}/\omega(2^j Q_b)$ *for all* $s \in R_j(Q_b)$.

(ii) *For all* $s, t \in \partial\Omega$ *such that* $|s - t| \lesssim \ell(Q_b)$,

$$|c_{(Q_b)}(s) - c_{(Q_b)}(t)| \lesssim \Big(\frac{|s-t|}{\ell(Q_b)} \Big)^{\alpha} \sqrt{\omega_L(Q_b)} \Big(\sum_{j=0}^{\infty} \frac{2^{-\epsilon j}}{\omega(2^j Q_b)} (\chi_{R_j(Q_b)}(s) + \chi_{R_j(Q_b)}(t)) \Big).$$

(iii) *For any finite linear combination* $\sum_{Q_b \in \mathcal{F}} \lambda_{Q_b} c_{(Q_b)}(s) = k(s)$,

$$\int_{\partial\Omega} |k(s)|^2 d\omega_L(s) \lesssim \sum_{Q_b \in \mathcal{F}} \lambda_{Q_b}^2.$$

Then for any $0 < r < \infty$, there is a constant $C = C(r, d, \Omega, L, m, \epsilon, \alpha, \eta$, the A^∞ constants of $\sigma d\omega_L$, constants appearing in the three conditions satisfied by the functions $c_{(Q_b)})$ so that

$$\Big(\int_{\partial\Omega} T(s)^r \sigma(s)\, d\omega_L(s) \Big)^{1/r} \leq C \Big(\int_{\partial\Omega} \Big(\sum_{Q_b \in \mathcal{F}} \lambda_{Q_b}^2 \sum_{j=0}^{\infty} \frac{2^{-(2\epsilon - \eta)j}}{\omega(2^j Q_b)} \chi_{R_j(Q_b)}(s) \Big)^{r/2} \sigma(s)\, d\omega_L(s) \Big)^{1/r}.$$

This inequality is proved by the method of Theorem 2.1 adapted to Lipschitz domains. The proof of almost-orthogonality for the functions

$$b_{(Q_b)}(s) = c_{(Q_b)}(s) = \ell(Q_b)^{(1-\alpha)/2}\sqrt{\omega_L(Q_b)}K_L(x_{T(Q_b)}, s)$$

differs from the proof in Section 3; here is an updated version of the proof from [Sw1]:

Let $h(s) = \sum_{Q_b\in\mathcal{F}} \lambda_{Q_b} b_{(Q_b)}(s)$ be a finite sum with the λ_{Q_b} real numbers and the $b_{(Q_b)}(s) = \ell(Q_b)^{(1-\alpha)/2}\sqrt{\omega_L(Q_b)}K_L(x_{T(Q_b)}, s)$ as above.

Then, assuming that $\lambda_{Q_b} \geq 0$ and $h(s) \geq 0$,

$$\begin{aligned}\int_{\partial\Omega} h(s)^2\, d\omega_L(s) &= \int_{\partial\Omega} h(s)\Big(\sum_{Q_b\in\mathcal{F}} \lambda_{Q_b} b_{(Q_b)}(s)\Big)\, d\omega_L(s)\\ &= \sum_{Q_b\in\mathcal{F}} \lambda_{Q_b}\ell(Q_b)^{(1-\alpha)/2}\sqrt{\omega_L(Q_b)}\int_{\partial\Omega} h(s)K_L(x_{T(Q_b)}, s)\, d\omega_L(s)\\ &= \sum_{Q_b\in\mathcal{F}} \lambda_{Q_b}\ell(Q_b)^{(1-\alpha)/2}\sqrt{\omega_L(Q_b)}\cdot v(x_{T(Q_b)})\end{aligned}$$

where $v(x)$ is the solution to $Lv = 0$ in Ω, and $v(s) = h(s)$ on $\partial\Omega$. Letting $N(u)(s)$ stand for the nontangential maximal function of u on $\partial\Omega$, it is clear that $v(x_{T(Q_b)}) \leq \inf_{s\in Q_b} N(v)(s)$, so the last expression is

$$\begin{aligned}&\leq \sum_{Q_b\in\mathcal{F}} \lambda_{Q_b}\ell(Q_b)^{(1-\alpha)/2}\sqrt{\omega_L(Q_b)}\inf_{s\in Q_b} N(v)(s)\\ &\leq \sum_{Q_b\in\mathcal{F}} \lambda_{Q_b}\ell(Q_b)^{(1-\alpha)/2}\sqrt{\omega_L(Q_b)}\Big(\frac{1}{\omega_L(Q_b)}\int_{Q_b} N(v)^2(s)\, d\omega_L(s)\Big)^{1/2}\\ &= \sum_{k=-N_0}^{\infty}\sum_{\substack{\ell(Q_b)=2^{-k}\\ Q_b\in\mathcal{F}}} \lambda_{Q_b}\ell(Q_b)^{(1-\alpha)/2}\Big(\int_{Q_b} N(v)^2(s)\, d\omega_L(s)\Big)^{1/2}\end{aligned}$$

with N_0 being a fixed finite index, dependent on the size and Lipschitz characterization of the boundary of the domain Ω. Now by the Cauchy–Schwarz inequality applied first to the inside sum and then to the outer sum, this expression is

$$\begin{aligned}&\leq \sum_{k=-N_0}^{\infty} 2^{-k((1-\alpha)/2)}\Big(\sum_{\substack{\ell(Q_b)=2^{-k}\\ Q_b\in\mathcal{F}}}\lambda_{Q_b}^2\Big)^{1/2}\Big(\sum_{\substack{\ell(Q_b)=2^{-k}\\ Q_b\in\mathcal{F}}}\int_{Q_b} N(v)^2(s)\, d\omega_L(s)\Big)^{1/2}\\ &\leq \Big(\int_{\partial\Omega} N(v)^2(s)\, d\omega_L(s)\Big)^{1/2}\Big(\sum_{Q_b\in\mathcal{F}}\lambda_{Q_b}^2\Big)^{1/2}\Big(\sum_{k=-N_0}^{\infty} 2^{-k(1-\alpha)}\Big)^{1/2}.\end{aligned}$$

The fact that $N(u)(s)$ is bounded above by the Hardy–Littlewood maximal function, taken with respect to the elliptic measure ω_L, [FeKP], means that

$$\Big(\int_{\partial\Omega} N(v)^2(s)\, d\omega_L(s)\Big)^{1/2} \leq C\Big(\int_{\partial\Omega} h(s)^2\, d\omega_L(s)\Big)^{1/2}.$$

Since $0 < \alpha < 1$, $(\sum_{k=-N_0}^{\infty} 2^{-k(1-\alpha)})^{1/2} = C(\alpha, N_0)$ and we have

$$\Big(\int_{\partial\Omega} h(s)^2\, d\omega_L(s)\Big) \le C(\alpha,\Omega)\Big(\sum_{Q_b\in\mathcal{F}} \lambda_{Q_b}^2\Big)^{1/2}\Big(\int_{\partial\Omega} h(s)^2\, d\omega_L(s)\Big)^{1/2}.$$

Dividing by $(\int_{\partial\Omega} h(s)^2\, d\omega_L(s))^{1/2}$ gives the almost-orthogonality.

There have been extensions of these results to solutions of the heat equation on a half-space, [WhWi], [SwWi1], to solutions of strictly parabolic second order operators on bounded domains with rough boundaries, [SwWi2], [Sw3], and to solutions of the inhomogeneous elliptic equation, $Lu(x) = \nabla\cdot\vec{f}(x)$ for $x \in \Omega$, $u(s) = 0$ for $s \in \partial\Omega$, [Sw2].

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