Contents

Abstract

Real Legendrian subvarieties are classical objects of differential geometry and classical mechanics and they have been studied since antiquity (see [Arn74], [Sła91] and references therein). However, complex Legendrian subvarieties are much more rigid and have more exceptional properties. The most remarkable case is the Legendrian subvarieties of projective space; prior to the author's research only few smooth examples of these were known (see [Bry82], [LM07]). Strong restrictions on the topology of such varieties have been found and studied by Landsberg and Manivel ([LM07]).

This dissertation reviews the subject of Legendrian varieties and extends some of recent results.

The first series of results is related to the automorphism group of any Legendrian subvariety in any projective contact manifold. The connected component of this group (under suitable minor assumptions) is completely determined by the sections of the distinguished line bundle on the contact manifold vanishing on the Legendrian variety. Moreover, its action preserves the contact structure. The relation between the Lie algebra tangent to automorphisms and the sections is given by an explicit formula (see also [LeB95], [Bea07]). This summarises and extends some earlier results of the author.

The second series of results is devoted to finding new examples of smooth Legendrian subvarieties of projective space. The examples found by other researchers were some homogeneous spaces, many examples of curves and a family of surfaces birational to some K3 surfaces. Further the author found a couple of other examples including a smooth toric surface and a smooth quasihomogeneous Fano 8-fold. Finally, the author proved that both of these are special cases of a very general construction: a general hyperplane section of a smooth Legendrian variety, after a suitable projection, is a smooth Legendrian variety of smaller dimension. We review all of those examples and also add infinitely many new examples in every dimension, with various Picard rank, canonical degree, Kodaira dimension and other invariants.

The original motivation for studying complex Legendrian varieties comes from a 50 years old problem of giving compact examples of quaternion-Kähler manifolds (see [Ber55], [LS94], [LeB95] and references therein). Also Legendrian varieties are related to some algebraic structures (see [Muk98], [LM01], [LM02]). A new potential application to classification of smooth varieties with smooth dual arises in this dissertation.

Acknowledgements. The monograph builds on the author's PhD thesis [Buc08a].

The author was supported by the research project N20103331/2715 funded by the Polish Ministry of Science (2006–2008).

The author would like to thank especially his patient advisor Jarosław Wiśniewski for his comments, support and for answering numerous questions. Parts of the thesis were written while the author enjoyed the hospitality of Insong Choe at KIAS (Korea Institute for Advanced Study), Mark Gross at University of California, San Diego and Joseph M. Landsberg at Texas A&M University. The author is grateful for the invitations, financial support and for creating a stimulating atmosphere. He also acknowledges many enlightening discussions with Michel Brion, Stephen Coughlan, Zbigniew Jelonek, Grzegorz Kapustka, Michał Kapustka, Michał Krych, Joseph M. Landsberg, Adrian Langer, Laurent Manivel, Sung Ho Wang and Andrzej Weber. Finally, the author is grateful to the referees of the thesis: Sławomir Cynk, Laurent Manivel, Henryk Żołądek and the anonymous referee of this monograph—their comments have helped to improve the presentation.

2010 *Mathematics Subject Classification*: Primary 14M99; Secondary 53D10, 14L30, 53D20.
Key words and phrases: Legendrian variety, complex contact manifold, automorphism group.
Received 8.9.2008; revised version 9.11.2009.

A. Introduction

A.1. State of the art. In this monograph we study algebraic and geometric properties of complex Legendrian subvarieties. The main motivation for our research comes from the classification problem of contact Fano manifolds ([1]).

A.1.1. Contact manifolds and quaternion-Kähler manifolds. There are two known families of complex projective contact manifolds: projectivisations of cotangent bundles to projective manifolds (see Example E.8) and certain homogeneous spaces, namely the unique closed orbit of the adjoint action of a simple Lie group G on $\mathbb{P}(\mathfrak{g})$, where $\mathfrak{g}$ is the Lie algebra of G (see §D.2.2 and Example E.7 for more details). These orbits are also called *adjoint varieties*. The following theorem summarises work of Demailly [Dem02] and Kebekus, Peternell, Sommese and Wiśniewski [KPSW00]:

THEOREM A.1. *If Y is a complex projective contact manifold, then either Y is a projectivisation of the cotangent bundle to some projective manifold M, or Y is a Fano manifold with second Betti number $b_2 = 1$.*

The following conjecture would be an important classification result in algebraic geometry and it claims that known examples are all existing examples:

CONJECTURE A.2 (LeBrun, Salamon). *If Y^{2n+1} is a Fano complex contact manifold, then Y is an adjoint variety.*

This conjecture originated with a famous problem in Riemannian geometry. In 1955 Berger [Ber55] gave a list of all possible holonomy groups ([2]) of simply connected Riemannian manifolds. The existence problem for all cases has been solved locally. Compact non-homogeneous examples with most of the possible holonomy groups were constructed, for instance the two exceptional cases G_2 and $Spin_7$ were constructed by D. Joyce—see an excellent review [Joy00]. Since then all the cases from Berger's list have been illustrated with compact non-homogeneous examples, with the unique exception of quaternion-Kähler manifolds ([3]). Although there exist non-compact, non-homogeneous examples, it is con-

---

([1]) A complex manifold Y^{2n+1} is called a *contact manifold* if there exists a rank $2n$ vector subbundle $F \subset TY$ of the tangent bundle, such that the map $F \otimes F \to TY/F$ determined by the Lie bracket is nowhere degenerate (see Chapter E for more details). A projective manifold is *Fano* if its anticanonical line bundle is ample.

([2]) Given an m-dimensional Riemannian manifold M, the *holonomy group* of M is the subgroup of the orthogonal group $\mathbf{O}(T_x M)$ generated by the parallel translations along loops through x.

([3]) A Riemannian $4n$-dimensional manifold M is called *quaternion-Kähler* if its holonomy group is a subgroup of $\mathbf{Sp}(1) \times \mathbf{Sp}(n)/\mathbb{Z}_2$.

jectured that compact quaternion-Kähler manifolds must be homogeneous (see [LeB95] and references therein).

Conjecture A.3 (LeBrun, Salamon). *Let M^{4n} be a compact quaternion-Kähler manifold. Then M is a homogeneous symmetric space* (*more precisely, it is one of the Wolf spaces—see* [Wol65]).

The relation between the two conjectures is given by the construction of a twistor space Y, an S^2-bundle of complex structures on tangent spaces to a quaternion-Kähler manifold M. If M is compact, it has positive scalar curvature, and then Y has a natural complex structure and is a contact Fano manifold with a Kähler–Einstein metric. In particular, the twistor space of a Wolf space is an adjoint variety. Hence Conjecture A.2 implies Conjecture A.3. Conversely, LeBrun [LeB95] observed that if Y is a contact Fano manifold with Kähler–Einstein metric, then it is a twistor space of a quaternion-Kähler manifold.

A number of attempts have been undertaken to prove the above conjectures. They were proved in low dimension: for $n = 1$ by N. Hitchin [Hit81] and Y. Ye [Ye94], $n = 2$ by Y. S. Poon and S. M. Salamon [PS91] and S. Druel [Dru98] and Conjecture A.3 for $n = 3$ by H.&R. Herrera [HH02]. Moreover, A. Beauville, J. Wiśniewski, S. Kebekus, T. Peternell, A. Sommese, J.-P. Demailly, C. LeBrun, J.-M. Hwang and many other researchers have worked on this problem.

A.1.2. Lines on contact Fano manifold. If Y^{2n+1} is a contact complex manifold, then a subvariety $X \subset Y$ of pure dimension n is *Legendrian* if it is maximally F-integrable—see Chapter E for the details.

Let Y^{2n+1} be a contact Fano manifold not isomorphic to a projective space. A rational curve $C \subset Y$ is a *contact line* if its intersection with the anticanonical line bundle is minimal possible, i.e. equal to $n + 1$. Contact lines on Y are an instance of minimal rational curves on uniruled manifolds and they are extensively studied by numerous researchers. Wiśniewski [Wiś00] and Kebekus [Keb01], [Keb05] have studied geometric properties of contact lines. The following theorem is due to Kebekus [Keb05], but some parts of it were known before:

Theorem A.4. *With Y as above choose a point $y \in Y$. Let*

$$H_y \subset \mathrm{RatCurves}^n(Y)$$

be the subscheme parametrising contact lines through y, let $C_y \subset Y$ be the locus of these lines (i.e. the subset swept out by contact lines through y) and let $X_y \subset \mathbb{P}(T_yY)$ be the projectivised tangent cone to C_y at y (so that X_y is the set of tangent directions at y to lines through y). Then:

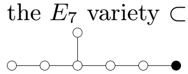
(i) *H_y is projective and C_y is closed in Y.*
(ii) *$C_y \subset Y$ is a Legendrian subvariety, $X_y \subset \mathbb{P}(F_y) \subset \mathbb{P}(T_yY)$ and X_y is a non-degenerate Legendrian subvariety in $\mathbb{P}(F_y)$.*
(iii) *If y is a general point, then H_y is isomorphic to X_y (in other words, every line through y is determined by its tangent direction) and H_y is smooth.*

(iv) *If y is a general point, then C_y is isomorphic to the projective cone $\widetilde{X}_y \subset \mathbb{P}(F_y \oplus \mathbb{C})$ over $X_y \subset \mathbb{P}(F_y)$. This isomorphism maps the contact line through y tangent to $x \in X_y$ onto the line $\mathbb{P}^1 = \mathbb{P}(x \oplus \mathbb{C}) \subset \mathbb{P}(F_y \oplus \mathbb{C})$. In particular, two distinct lines through y do not intersect except at y. Moreover, the distinguished line bundle $L = TY/F$ restricted to C_y is identified with $\mathcal{O}_{\widetilde{X}_y}(1)$ via this isomorphism.*

Note that in [Keb05, Thm. 1.1] Kebekus also claims that a certain irreducibility holds for H_y. However it was observed by Kebekus himself together with the author that there is a gap in the proof ([4]).

Thus contact lines through a general point behave very much like ordinary lines in a projective space. Moreover, their geometry is described by a nondegenerate smooth Legendrian subvariety $X := X_y$ in $\mathbb{P}^{2n-1}$. If Y is one of the adjoint varieties, then X will be a homogeneous Legendrian subvariety called a *subadjoint variety* (see [LM07], [Muk98]). Proving that there is an embedding of Y into a projective space which maps contact lines to ordinary lines would imply Conjecture A.2. Moreover, it is proved by Hong [Hon00] that if X is homogeneous, then so is Y. Therefore contact lines and particularly the Legendrian varieties determined by them are important objects, useful in the study of Conjecture A.2.

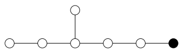
Table 1. Simple Lie groups together with the corresponding adjoint variety Y and its variety of tangent directions to contact lines: the subadjoint variety X (listed in detail also in §A.1.3).

Lie group	Type	Contact manifold Y^{2n+1}	Legendrian variety X^{n-1}	Remarks
SL_{n+2}	A_{n+1}	$\mathbb{P}(T\mathbb{P}^{n+1})$	$\mathbb{P}^{n-1} \sqcup \mathbb{P}^{n-1} \subset \mathbb{P}^{2n-1}$	$b_2(Y) = 2$
Sp_{2n+2}	C_{n+1}	$\mathbb{P}^{2n+1}$	$\emptyset \subset \mathbb{P}^{2n-1}$	Y does not have any contact lines
SO_{n+4}	$B_{(n+3)/2}$ or $D_{(n+4)/2}$	$\mathrm{Gr}_O(2, n+4)$	$\mathbb{P}^1 \times Q^{n-2} \subset \mathbb{P}^{2n-1}$	Y is the Grassmannian of projective lines on a quadric Q^{n+2}
	G_2	Grassmannian of special lines on Q^5	$\mathbb{P}^1 \subset \mathbb{P}^3$	X is the twisted cubic curve
	F_4	an F_4 variety	$\mathrm{Gr}_L(3, 6) \subset \mathbb{P}^{13}$	
	E_6	an E_6 variety	$\mathrm{Gr}(3, 6) \subset \mathbb{P}^{19}$	
	E_7	an E_7 variety	$\mathbb{S}_6 \subset \mathbb{P}^{31}$	X is the spinor variety
	E_8	an E_8 variety	the E_7 variety $\subset \mathbb{P}^{55}$ 	

([4]) This gap is on page 234 in Step 2 of the proof of Proposition 3.2 where Kebekus claims to have constructed "a well defined family of cycles". This is not necessarily a well defined family of cycles: condition (3.10.4) in [Kol96, §I.3.10] is not necessarily satisfied. As a consequence the map Φ is not necessarily regular at non-normal points of D^0.

A.1.3. Legendrian subvarieties of projective space. Prior to the author's research the following were the only known examples of smooth Legendrian subvarieties of projective space (see [Bry82], [LM07]):

1) linear subspaces;
2) some homogeneous spaces called *subadjoint varieties* (see Table A.1.2): the product of a line and a quadric $\mathbb{P}^1 \times Q^{n-2}$ and five exceptional cases (see below for discussion and references for this series of examples; general properties of homogeneous spaces, in particular definitions and some more detailed properties of $\mathbb{S}_6$ and E_7, as well as other homogeneous spaces, are contained in standard representation theory textbooks; see for instance [FH91, §23.3 and earlier]):

   - twisted cubic curve $\mathbb{P}^1 \subset \mathbb{P}^3$,
   - Grassmannian $\mathrm{Gr}_L(3,6) \subset \mathbb{P}^{13}$ of Lagrangian subspaces in $\mathbb{C}^6$,
   - full Grassmannian $\mathrm{Gr}(3,6) \subset \mathbb{P}^{19}$,
   - spinor variety $\mathbb{S}_6 \subset \mathbb{P}^{31}$ (i.e. the homogeneous $\mathbf{SO}(12)$-space parametrising the vector subspaces of dimension 6 contained in a non-degenerate quadratic cone in $\mathbb{C}^{12}$; see [FH91, §20] for the spinor representation they arise from),
   - the 27-dimensional E_7-variety in $\mathbb{P}^{55}$ corresponding to the marked root:
      (see [FH91, §22.4] for exceptional groups and their representations);

3) every smooth projective curve admits a Legendrian embedding in $\mathbb{P}^3$ (see [Bry82]);
4) a family of smooth surfaces birational to Kummer $K3$-surfaces (see [LM07]).

The subadjoint varieties are expected to be the only homogeneous Legendrian subvarieties in $\mathbb{P}^{2n-1}$ (for $n > 2$). Many partial results are known. For instance, the following theorem is known:

THEOREM A.5. *If $X \subset \mathbb{P}(V)$ is homogeneous and Legendrian and the Legendrian embedding is equivariant (i.e. automorphisms of X extend to linear automorphisms of $\mathbb{P}(V)$), then X is one of the subadjoint varieties.*

To the author's best knowledge the statement of the theorem has not been explicitly written down before, but according to J. M. Landsberg, already É. Cartan knew which homogeneous spaces were Legendrian. Also the theorem immediately follows from various sources. As an instance, subadjoint varieties are the only smooth irreducible Legendrian varieties whose ideals are generated by quadratic polynomials (see [Buc06]). But it is a well known theorem of Kostant that a homogeneous space in its equivariant embedding has ideal generated by quadrics (see for instance [Pro07, §10.6.6]).

Also Landsberg and Manivel proved [LM07, Thm. 11]:

THEOREM A.6. *Let $X = G/P \neq \mathbb{P}^1$ be a homogeneous space with Picard number one. Suppose that X admits a Legendrian embedding into $\mathbb{P}^{2n-1}$, not necessarily equivariant a priori. Then X is subadjoint. In particular, the Legendrian embedding is the equivariant subadjoint embedding.*

Subadjoint varieties are strongly related to the groups they arise from. Landsberg and Manivel [LM02] use subadjoint varieties to reconstruct the adjoint variety they arise from. Thus subadjoint varieties are one of the essential ingredients of their proof of the classification of simple Lie groups by means of projective geometry only. Also Mukai [Muk98] relates subadjoint Legendrian varieties to another algebraic structure: simple Jordan algebras. In [LM01] the authors give a uniform description of the exceptional cases (arising from F_4, E_6, E_7 and E_8).

For an arbitrary Legendrian subvariety in projective space, Landsberg and Manivel proved certain restrictions on its topology, the simplest of which is:

THEOREM A.7. *Let $X \subset \mathbb{P}^{2n-1}$ be a smooth Legendrian variety, let $h \in H^2(X, \mathbb{Q})$ be the hyperplane class and $c_i \in H^2(X, \mathbb{Q})$ be the ith Chern class of X. Then*

$$c_1^2 - 2c_2 = 2hc_1 - nh^2.$$

This gives in particular the following restrictions on X:

- *X cannot be an abelian variety or parallelisable variety.*
- *If $X \simeq \mathbb{P}^{n-1}$ for $n > 2$, then $X \subset \mathbb{P}^{2n-1}$ is a linear subspace.*
- *Suppose that $X \simeq Y \times Z$. Then $X = \mathbb{P}^1 \times Q^{n-2}$ and the Legendrian embedding is the Segre embedding.*
- *If $X \simeq \mathbb{P}(E)$, the total space of a $\mathbb{P}^p$ bundle over some smooth variety Y, then $p = 1$.*

Note that the statement in [LM07, Prop. 3] suggests that the Chern class identities hold in $H^\bullet(X, \mathbb{Z})$, yet their proof uses cohomology with rational coefficients (they use e^h and the Chern character $\mathrm{ch}(TX)$).

Further restrictions on the topology of X were derived if $\dim X = 2$ (see [LM07] for details).

Also the following was proved [LM07, Prop. 17(2)]:

PROPOSITION A.8. *Let $X \subset \mathbb{P}(V)$ be a non-degenerate Legendrian variety. Then:*

- *The secant variety $\sigma(X)$ fills out $\mathbb{P}(V)$ (i.e. it is of expected dimension).*
- *If X is in addition smooth, then the tangent variety $\tau(X) \subset \mathbb{P}(V)$ and the dual variety $X^* \subset \mathbb{P}(V^*)$ are hypersurfaces isomorphic via $\tilde{\omega} : V \to V^*$.*

See §B.9 for the definitions of secant, tangent and dual varieties. See §B.2 for the definition of $\tilde{\omega}$. Here the symplectic form ω is the form associated with the contact structure on $\mathbb{P}(V)$ (see Example E.10). The statement about the secant variety is contained in the proof of [LM07, Prop. 17(2)].

For tangent and dual varieties, we note that the original formulation in [LM07] omits the smoothness assumption. If, however, X is not smooth, their proof does not work and decomposable Legendrian varieties (see §E.2.1) are counterexamples. In the proof the authors freely interchange the tangent variety $\tau(X)$ (which by definition is the union of the limits of secants through two points approaching a third fixed point) and the closure of the union of embedded tangent spaces at smooth points. These are the same for X smooth. The tangent variety $\tau(X)$ is indeed a hypersurface in the secant variety $\sigma(X)$. The closure of the embedded tangent spaces at smooth points is indeed isomorphic to X^*.

A.2. Topics of the dissertation. This dissertation addresses two complementary problems regarding Legendrian varieties:

- Write explicit restrictions on the properties of Legendrian varieties.
- Give examples of smooth Legendrian varieties.

We address the first problem by giving a very precise analysis of the embedded automorphism group of a Legendrian variety. The second problem is solved by proving that a general hyperplane section of a smooth Legendrian variety admits a Legendrian embedding.

In [Buc03], we prove that the quadratic part of the ideal of a Legendrian subvariety X of projective space $\mathbb{P}^{2n-1}$ produces a connected subgroup of projective automorphisms of X. In [Buc06] we improve this result by observing that this group is actually the maximal connected subgroup of automorphisms of the contact structure on $\mathbb{P}^{2n-1}$ preserving the Legendrian subvariety. The relation between quadrics and automorphisms is explicitly defined:

Theorem A.9. *Let $X \subset \mathbb{P}(V)$ be a Legendrian subvariety. Consider the following map ρ:*
$$H^0(\mathcal{O}_{\mathbb{P}(V)}(2)) \simeq \operatorname{Sym}^2 V^* \ni \; q = (x \mapsto x^T M(q) x) \overset{\rho}{\mapsto} 2J \cdot M(q) \in \mathfrak{sp}(V),$$
where $M(q)$ is the matrix of q and $J = M(\omega)$ is the matrix of the symplectic form on V associated with the contact structure on $\mathbb{P}(V)$. Let $\widetilde{\mathcal{I}}(X)_2 \subset \operatorname{Sym}^2 V^*$ be the vector space of quadrics containing X. Then:

- *$\rho(\widetilde{\mathcal{I}}(X)_2)$ is a Lie subalgebra of $\mathfrak{sp}(V)$ tangent to the closed subgroup*
$$\overline{\exp(\rho(\widetilde{\mathcal{I}}(X)_2))} < \mathbf{Sp}(V).$$
- *We have a natural action of $\mathbf{Sp}(V)$ on $\mathbb{P}(V)$. The group $\overline{\exp(\rho(\widetilde{\mathcal{I}}(X)_2))}$ is the maximal connected subgroup in $\mathbf{Sp}(V)$ which under this action preserves $X \subset \mathbb{P}(V)$.*

For the proof see [Buc06, Cor. 4.4, Cor. 5.5, Lem. 5.6].

In the present dissertation we extend this result further. Firstly, we replace projective space $\mathbb{P}(V)$ with an arbitrary contact manifold Y:

Theorem A.10. *Let Y be a compact contact manifold and let $X \subset Y$ be a Legendrian subvariety. Then the connected component of the subgroup of $\operatorname{Aut}(Y)$ that preserves both the contact structure and $X \subset Y$ is completely determined by those sections of a distinguished line bundle L on Y that vanish on X.*

See Corollary E.23 for the proof. In Theorem E.13 (quoted from [Bea98]) we also make it explicit how to obtain an infinitesimal automorphism of Y from a given section of L (the analogue of the map ρ in Theorem A.9).

Secondly, we try to remove the assumption that the automorphisms preserve the contact structure. By applying the results of [LeB95] and [Keb01] on the uniqueness of contact structures we can deal with this problem for most projective contact Fano manifolds (see Corollary E.23). The remaining cases are the projectivised cotangent bundles and the projective space. The first case is not very interesting, as all the Legendrian subvarieties are classified for these contact manifolds (see Corollary E.17). On the other hand, the case of projective space is the most important and interesting. It is described precisely in Chapter F. We present there the following theorem originally proved in [Buc07]:

Theorem A.11. *If $X \subset \mathbb{P}^{2n-1}$ is a smooth irreducible Legendrian subvariety which is not a linear subspace and $G < \mathbb{P}\mathbf{GL}_{2n}$ is a connected subgroup preserving X, then the action of G on $\mathbb{P}^{2n-1}$ necessarily preserves the contact structure. Thus in this case the group $\exp(\rho(\widetilde{\mathcal{I}}(X)_2))$ from Theorem* A.9 *is also the maximal connected subgroup in $\mathbf{SL}(V)$ which preserves $X \subset \mathbb{P}(V)$.*

Our methodology for finding new examples of smooth Legendrian subvarieties is the following. We pose questions of classification of smooth Legendrian varieties satisfying certain additional conditions. For instance, we assume that the variety is toric (see Chapter G, which follows [Buc07]):

Theorem A.12. *Every smooth toric Legendrian variety of dimension at least* 2 *in a projective space, whose embedding is torus equivariant, is isomorphic to one of the following:*

- *a linear subspace,*
- $\mathbb{P}^1 \times Q_1 \subset \mathbb{P}^5$,
- $\mathbb{P}^1 \times Q_2 \simeq \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^7$,
- $\mathbb{P}^2$ *blown up in three non-colinear points.*

For the proofs see Corollaries G.6 and G.10. The linear subspace is not really interesting, the products $\mathbb{P}^1 \times Q_1$ and $\mathbb{P}^1 \times Q_2$ are well known (see §A.1.3). The last case of blow up was an original example of [Buc07].

We also classify those smooth Legendrian varieties which are contained in a specific F-cointegrable variety (see Chapter H, which follows [Buc09]). In this way we produce a few new smooth examples including a quasihomogeneous Fano 8-fold:

Theorem A.13. *A general hyperplane section of* $\mathrm{Gr}(3,6)$ *admits a Legendrian embedding into $\mathbb{P}^{17}$.*

Finally, following [Buc08b] we generalise this and prove that a general hyperplane section of a smooth Legendrian variety admits a Legendrian embedding into a smaller projective space:

Theorem A.14. *Let $X \subset \mathbb{P}(V)$ be an irreducible Legendrian subvariety which is smooth or has only isolated singularities. Then a general hyperplane section of X admits a Legendrian embedding into a projective space of appropriate dimension via a specific subsystem of the linear system $\mathcal{O}(1)$.*

More generally, assume $X \subset \mathbb{P}(V)$ is an irreducible Legendrian subvariety with singular locus of dimension k and $H \subset \mathbb{P}(V)$ is a general hyperplane. Then there exists a variety $\widetilde{X}_H$ whose singular locus has dimension at most $k-1$ and which has an open subset isomorphic to the smooth locus of $X \cap H$ such that $\widetilde{X}_H$ admits a Legendrian embedding.

The specific linear system and the construction of $\widetilde{X}_H$ are described in §I.1.1 and there we prove that the resulting variety is Legendrian. The proof that for a general section the result has the required smoothness is presented in §I.1.2.

This simple observation has quite strong consequences. Many researchers, including Landsberg, Manivel, Wiśniewski, Hwang and the author of this dissertation, believed that the structure of smooth Legendrian subvarieties in projective space had to be somehow rigid at least in higher dimensions. So far the only non-rational examples known were in dimensions 1 and 2 (see §A.1.3) and these were also the only known to come in families. Already by a naive application of the theorem to the subadjoint varieties we get many more examples with various properties:

Example A.15. The following smooth varieties and families of smooth varieties admit Legendrian embedding:

(a) a family of $K3$ surfaces of genus 9;
(b) three different types of surfaces of general type;
(c) some Calabi–Yau 3-folds, some Calabi–Yau 5-folds and some Calabi–Yau 9-folds;
(d) some varieties of general type in dimensions 3, 4 (two families for every dimension), 5, 6, 7 and 8 (one family per dimension);
(e) some Fano varieties, like the blow up of a quadric Q^n in a codimension 2 hyperplane section Q^{n-2}, a family of Del Pezzo surfaces of degree 4 and others;
(f) infinitely many non-isomorphic, non-homogeneous Legendrian varieties in every dimension arising as a codimension k linear section of $\mathbb{P}^1 \times Q^{n+k}$.

Example (a) agrees with the prediction of [LM07, §2.3]. Examples (b) and (d) give a partial answer to the question of a possible Kodaira dimension of a Legendrian variety (also see [LM07, §2.3]). Example (f) is a counterexample to the naive expectation that a Legendrian variety in a sufficiently high dimension must be homogeneous.

We also note that the previous non-homogeneous examples also arise in this way. Example (e) for $n = 2$ is the toric example described in Example G.4. Hyperplane sections of $\mathrm{Gr}(3, 6)$, $\mathrm{Gr}_L(3, 6)$, $\mathbb{S}_6$ are studied in more detail in Chapter H. Also non-homogeneous examples of the other authors (Bryant [Bry82], Landsberg and Manivel [LM07]) can be reconstructed by Theorem A.14 from some varieties with only isolated singularities (see §I.3).

In this monograph we also demonstrate a more refined construction, using decomposable Legendrian varieties, which also uses Theorem A.14 (see §E.2.1). This yields an even bigger list of examples, in particular we obtain varieties of general type in every dimension:

Theorem A.16. *Asume n and ρ are two integers such that either:*

- $n \geq 3$ *and* $\rho \geq n + 3$ *or*
- $3 \leq n \leq 27$ *and* $\rho \geq 1$.

Then there exists a smooth irreducible projective variety X of dimension n, Kodaira dimension n and rank of Picard group ρ which admits a Legendrian embedding into $\mathbb{P}^{2n+1}$.

With this method we are also able to give new examples from old ones without dropping the dimension. This is described in detail in §I.2.

All the varieties arising from Theorem A.14 and our construction in §I.1.1 are embedded by a non-complete linear system. Therefore a natural question arises: can the construction be inverted? So for a given Legendrian but not linearly normal embedding of some variety $\widetilde{X}$, can we find a bigger Legendrian variety X such that $\widetilde{X}$ is a projection of a hyperplane section of X?

Following [Buc08b] and building upon ideas of Bryant, Landsberg and Manivel we suggest a construction that provides some (but far from perfect) answer to this question in §I.3:

THEOREM A.17. *Let $\widetilde{X} \subset \mathbb{P}^{2n-1}$ be an irreducible Legendrian subvariety. Then there exist a Legendrian subvariety $X \subset \mathbb{P}^{2n+1}$, a hyperplane $H \subset \mathbb{P}^{2n+1}$ and an irreducible component of $X \cap H$ such that $\widetilde{X}$ is the image of this component under the projection described in* §I.1.1.

This theorem is a simplified version of Theorem I.9.

In particular, we represent the example of Landsberg and Manivel as a hyperplane section of a 3-fold with only isolated singularities and the examples of Bryant as hyperplane sections of surfaces with at most isolated singularities.

Chapter B is devoted to introducing our notation and presenting some elementary algebro-geometric facts.

Chapter C revises the differential geometric properties of infinitesimal automorphisms that are necessary for Chapter E, but can be expressed without any explicit reference to the contact structure.

Chapter D is a brief revision of symplectic geometry that will be used in our discussion of contact manifolds. Also some statements from [Buc06] are generalised to this context.

Chapter E contains an independent review of local geometry of contact manifolds, with emphasis on their infinitesimal automorphisms. There we compare (after [LeB95] and [Bea07]) two natural Lie algebra structures related to a contact manifold Y: the Lie bracket of vector fields and the Poisson bracket on the structure sheaf of the symplectisation of Y. We use this comparison to prove the first theorem on embedded automorphisms of Legendrian subvarieties (see Theorem A.10).

In Chapters F–I we turn our attention to Legendrian subvarieties of projective space.

In Chapter F we continue the topic of automorphisms of Legendrian varieties. We prove the second theorem on embedded automorphisms of Legendrian subvarieties in projective space (see Theorem A.11). The results of this chapter were published in [Buc07].

In Chapter G we illustrate, in the case of subvarieties of projective space, how to classify toric Legendrian subvarieties and we give the list of all smooth cases (see Theorem A.12). Also the results of that chapter were published in [Buc07].

Chapter H contains the classification of Legendrian varieties which are contained in a specific F-cointegrable variety. Another new example arises in this way: the smooth quasihomogeneous 8-fold (see Theorem A.13). Also we present two other variants of the construction, producing a smooth 5-fold and a smooth 14-fold. The content of that chapter was published in [Buc09].

Finally, Chapter I describes a Legendrian embedding of a hyperplane section of a Legendrian variety (see Theorem A.14). Also a variant of an inverse construction (i.e. to describe a bigger Legendrian variety from a given one, such that a hyperplane section of the big one is the original one) is presented and is applied to Bryant's, Landsberg's and

Manivel's examples of smooth Legendrian varieties (see Theorem A.17). Parts of that chapter were published in [Buc08b]. The new parts are §I.2, where we construct a new series of examples (see Theorem A.16), and §I.4, where we explain the relation of certain Legendrian varieties to smooth varieties with smooth dual.

A.3. Open problems. Keeping in mind the elegant results sketched in §A.1 and having many new examples of smooth Legendrian varieties (as well as families of such), several natural questions remain unanswered.

New contact manifolds? Can we construct a new example of a contact manifold, whose variety of tangent directions to contact lines is one of the new Legendrian varieties (or is in the given family)? If Conjecture A.2 is true, then the answer is negative. If the answer is negative, then what are the obstructions, i.e., what conditions should we require on the Legendrian variety to make the reconstruction of the contact manifold possible?

Further applications to algebra? Can the new Legendrian varieties be used in a similar manner to the subadjoint cases and will they prove themselves to be equally extraordinary varieties? The first tiny piece of evidence for this is explained in §H.2.1. On the other hand, it is unlikely that such a big variety of examples can have analogous special properties.

Self-dual varieties? Another problem we want to mention here is a classical question in projective geometry: what are the smooth subvarieties of projective space, whose dual variety ([5]) is also smooth? So far the only examples of these are the self-dual varieties. Thanks to L. Ein [Ein86], the classification of smooth self-dual varieties $Z \subset \mathbb{P}^m$ is known when $3 \operatorname{codim} Z \geq \dim Z$. In Corollary I.16 we prove that the problem of classifying smooth varieties with smooth dual can be expressed in terms of Legendrian varieties and possibly we can apply the techniques of Legendrian varieties to finish the classification.

Projectively and linearly normal Legendrian varieties? We dare to conjecture:

Conjecture A.18. *Let $X \subset \mathbb{P}(V)$ be a smooth linearly normal* ([6]) *Legendrian variety. Then X is one of the subadjoint varieties.*

In view of Theorems A.14 and A.17, the classification of linearly normal Legendrian varieties might be a necessary step towards a classification of Legendrian varieties.

Furthermore, the conjecture might also contribute to the proof of Conjecture A.2. For instance, assume Conjecture A.18 holds and Y is a contact Fano manifold for which the variety cut out by contact lines through a general point is normal. Then by applying Theorem A.4 we find that the associated Legendrian variety $X \subset \mathbb{P}^{2n-1}$ is projectively normal ([7]) and by the conjecture and results of [Hon00] the manifold Y is an adjoint variety.

([5]) Given a subvariety $Z \subset \mathbb{P}(W)$, the *dual variety* $Z^* \subset \mathbb{P}(W^*)$ is the closure of the set of hyperplanes tangent to Z (see §B.9 for details).

([6]) A subvariety $X \subset \mathbb{P}^m$ is *linearly normal* if it is embedded by a complete linear system.

([7]) A subvariety $X \subset \mathbb{P}^m$ is *projectively normal* if its affine cone is normal. If X is projectively normal, then it is also linearly normal by [Har77, Ex. II.5.14(d)].

The author is able to prove Conjecture A.18 if $\dim X = 1$, but this is not an elegant argument nor does it have important applications. We refrain from presenting the proof until we manage to improve the argument or to generalise it to higher dimensions.

B. Notation and elementary properties

In the present monograph we always work over the $\mathbb{C}$ field of complex numbers.

B.1. Vector spaces and projectivisation. Let V be a vector space over $\mathbb{C}$. By $\mathbb{P}(V)$ we mean the naive projectivisation of V, i.e. the quotient $(V\backslash\{0\})/\mathbb{C}^*$.

If $v \in V \setminus \{0\}$, then by $[v] \in \mathbb{P}(V)$ we denote the line spanned by v.

Analogously, if E is a vector bundle, by $\mathbb{P}(E)$ we denote the naive projectivisation of E. Let $s_0 \subset E$ be the zero section of E. If $v \in E \setminus s_0$, then by $[v] \in \mathbb{P}(E)$ we denote the line spanned by v in the appropriate fibre of E.

B.2. Bilinear forms and their matrices. Let V be a complex vector space of dimension m and f a bilinear form on V. Fix a basis $\mathcal{B}$ of V and let $M(f) = M(f, \mathcal{B})$ be the $m \times m$-matrix such that

$$f(v, w) = v^T M(f) w,$$

where v and w are arbitrary column vectors of V. We say that $M(f)$ is the *matrix* of f *in the basis* $\mathcal{B}$.

In particular, if ω is a symplectic form (see §D.1.1), $\dim V = 2n$ and $\mathcal{B}$ is a symplectic basis, then

$$J := M(\omega, \mathcal{B}) = \begin{bmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{bmatrix}.$$

Moreover, in that case $-J$ is also the matrix of the linear map

$$\tilde{\omega} : \ V \to V^*, \qquad v \mapsto \omega(v, \cdot),$$

in the basis $\mathcal{B}$ on V and the dual basis on V^*.

Similarly, if q is a quadratic form on V, then we denote by $M(q) = M(q, \mathcal{B})$ the *matrix of* q *in the basis* $\mathcal{B}$:

$$q(v) = v^T M(q) v.$$

B.3. Complex and algebraic manifolds. Our main concern is with complex projective manifolds and varieties. This is where two categories meet: complex algebraic varieties and analytic spaces (see [Gri74]). Since the author's interests lie in algebraic geometry, this monograph's intention is to study algebraic Legendrian varieties. However, for some statements there is no reason to limit oneself to the algebraic case, so we state them also for the analytic situation.

So Y will usually be an ambient manifold (for example contact or symplectic manifold), either a complex manifold or smooth algebraic variety. Some statements are local for Y (in the analytic topology), hence it is enough to prove them for $Y \simeq D^{2n}$, where $D^{2n} \subset \mathbb{C}^n$ is a complex disc.

Our main interest is in $X \subset Y$, which will be either an analytic subspace (if Y is a complex manifold), or an algebraic subvariety (if Y is algebraic). For short, we will always say $X \subset Y$ is a subvariety.

B.4. Vector bundles, sheaves and sections. Given an analytic space or algebraic variety Y, we denote by $\mathcal{O}_Y$ both the structure sheaf (consisting of either holomorphic or algebraic functions on Y in the appropriate analytic or Zariski topology) and the trivial line bundle. If $X \subset Y$ is a subvariety, then by $\mathcal{I}(X)$ we denote the sheaf of ideals in $\mathcal{O}_Y$ defining X.

Given a vector bundle E on Y we will use the same letter E for the sheaf of sections of E. To avoid confusion and too many brackets (for example $\mathcal{I}(X)(U)$), given an open subset $U \subset Y$ and a sheaf (or vector bundle) $\mathcal{F}$, we will write $H^0(U, \mathcal{F})$ rather than $\mathcal{F}(U)$ for the value of the sheaf at the open subset U (or sections of a vector bundle). By $\mathcal{F}|_U$ we denote the sheaf (or vector bundle) restriction of $\mathcal{F}$ to the open subset U.

When there can be no confusion, given a sheaf $\mathcal{F}$ which does not have any natural vector bundle structure we will write $s \in \mathcal{F}$ to mean that there is an open $U \subset Y$ with $s \in H^0(U, \mathcal{F})$. On the other hand, if E is a vector bundle, then by writing $v \in E$ we mean that v is a vector in the bundle space.

Given a vector bundle E, we denote by E^* the dual vector bundle:

$$E^* := \mathcal{H}om(E, \mathcal{O}).$$

If $\theta\colon \mathcal{F} \to \mathcal{G}$ is a map of sheaves or vector bundles and $s \in H^0(U, \mathcal{F})$, then by $\theta(s)$ we denote the image section of $\mathcal{G}$.

B.5. Derivatives. Given a complex manifold or smooth algebraic variety Y and a holomorphic or algebraic k-form $\theta \in H^0(U, \Omega^k Y)$, by $\mathrm{d}\theta$ we denote the exterior derivative of θ. This convention is also valid for 0-forms

$$f \in H^0(U, \mathcal{O}_Y) = H^0(U, \Omega^0 Y).$$

By TY we denote the tangent vector bundle. Nevertheless keep in mind that a vector field $\mu \in H^0(U, TY)$ can also be interpreted as a derivation $\mu : \mathcal{O}_Y \to \mathcal{O}_Y$. In particular, we can define the Lie bracket of two vector fields $\mu, \nu \in H^0(U, TY)$ as

$$[\mu, \nu] = \nu\mu - \mu\nu.$$

This convention is in agreement with [Arn74].

Given a holomorphic or algebraic map $\phi : Y \to Y'$, by $\mathrm{D}\phi$ we denote the derivative map

$$\mathrm{D}\phi : TY \to \phi^* TY'.$$

If $\theta \in H^0(U, \Omega^k Y)$ and $\mu \in H^0(U, TY)$, then by $\theta(\mu)$ we denote the contracted $(k-1)$-form. For example, if $\theta = \theta_1 \wedge \theta_2$ for 1-forms θ_i, then

$$\theta(\mu) = \theta_1(\mu)\theta_2 - \theta_2(\mu)\theta_1.$$

B.6. Homogeneous differential forms and vector fields. Let Y, Y' be two complex manifolds and let $\phi : Y' \to Y$ be a holomorphic map. For a k-form $\omega \in H^0(Y, \Omega^k Y)$, by $\phi^*\omega \in H^0(Y', \Omega^k Y')$ we denote the pull-back of ω:

$$(\phi^*\omega)_y(v_1, \ldots, v_k) := \omega_{\phi(y)}(\mathrm{D}_y\phi(v_1), \ldots, \mathrm{D}_y\phi(v_k)).$$

Now assume we have a $\mathbb{C}^*$-action on Y:

$$(t, y) \mapsto \lambda_t(y).$$

We say that $\omega \in H^0(Y, \Omega^k Y)$ is *homogeneous of weight* $\mathrm{wt}(\omega)$ if

$$\forall t \in \mathbb{C}^* \quad \lambda_t^*\omega = t^{\mathrm{wt}(\omega)}\omega.$$

For example, assume $Y = \mathbb{A}^n = \mathrm{Spec}(\mathbb{C}[y_1, \ldots, y_n])$ and $\mathbb{C}^*$ acts via homotheties. We say $\omega \in \Omega^k\mathbb{A}^n$ is *constant* if it is a $\mathbb{C}$-linear combination of $\mathrm{d}y_{i_1} \wedge \cdots \wedge \mathrm{d}y_{i_k}$. Constant k forms are homogeneous of weight k (not of weight 0 as one could possibly expect). Conversely, if $\omega \in H^0(\mathbb{A}^n, \Omega^k\mathbb{A}^n)$ is homogeneous of weight k, then it is constant, because every global form can be written as $\sum f_{i_1,\ldots,i_k}\mathrm{d}y_{i_1} \wedge \cdots \wedge \mathrm{d}y_{i_k}$. Since $\mathrm{d}y_{i_1} \wedge \ldots \wedge \mathrm{d}y_{i_k}$ are already of weight k, it follows that $f_{i_1,\ldots,i_k}$ are constant functions.

Let $\mu \in H^0(Y, TY)$ be a vector field. We say μ is *homogeneous of weight* $\mathrm{wt}(\mu)$ if

$$\mathrm{D}\lambda_{t^{-1}}\mu = t^{\mathrm{wt}(\mu)}\mu.$$

Lemma B.1. *Let Y, Y' be complex manifolds, both with a $\mathbb{C}^*$-action. Moreover, assume $\phi : Y' \to Y$ is a $\mathbb{C}^*$-equivariant map, $\omega \in H^0(Y, \Omega^k Y)$ is a homogeneous k-form for some $k \in \{0, 1, \ldots, \dim Y\}$ and $\mu \in H^0(Y, TY)$, $\nu \in H^0(Y', TY')$ are two homogeneous vector fields.*

(i) *$\omega(\mu)$ is homogeneous and* $\mathrm{wt}(\omega(\mu)) = \mathrm{wt}(\omega) + \mathrm{wt}(\mu)$;
(ii) *$\phi^*\omega$ is homogeneous of weight* $\mathrm{wt}(\omega)$ *and $\mathrm{D}\phi(\nu)$ is homogeneous of weight* $\mathrm{wt}(\nu)$;
(iii) *$\mathrm{d}\omega$ is homogeneous of weight* $\mathrm{wt}(\omega)$.

Proof. This is an immediate calculation. For instance (i):

$$\begin{aligned}\lambda_t^*(\omega(\mu))_x(v_1, \ldots, v_{k-1}) &= \omega_{\lambda_t(x)}(\mu, \mathrm{D}\lambda_t(v_1), \ldots, \mathrm{D}\lambda_t(v_{k-1}))\\ &= (\lambda_t^*\omega)_x(\mathrm{D}\lambda_{t^{-1}}(\mu), v_1, \ldots, v_{k-1}) = t^{\mathrm{wt}(\omega)}t^{\mathrm{wt}(\mu)}(\omega(\mu))_x(v_1, \ldots, v_{k-1}). \ \blacksquare\end{aligned}$$

B.7. Submersion onto image. We recall the standard fact that every algebraic map is generically a submersion on the closure of the image.

Lemma B.2. *Let Y and Y' be two algebraic varieties over an algebraically closed field of characteristic 0 and let $\pi : Y \to Y'$ be a map such that $Y' = \overline{\pi(Y)}$. Then for a general $y \in Y$, the derivative $D_y\pi : T_yY \to T_{\pi(y)}Y'$ is surjective.*

Proof. See [Har77, Cor. III.10.7]. ■

As a corollary, we prove an easy proposition about subvarieties of product manifolds.

Proposition B.3. *Let Y_1 and Y_2 be two smooth algebraic varieties and suppose that $X \subset Y_1 \times Y_2$ is a closed irreducible subvariety. Let $X_i \subset Y_i$ be the closure of the image of X under the projection π_i onto Y_i. Assume that for a Zariski open dense subset of* *smooth points $U \subset X$ the tangent bundle to X decomposes as*

$$TX|_U = (TX \cap \pi_1^* TY_1)|_U \oplus (TX \cap \pi_2^* TY_2)|_U,$$

a sum of two vector bundles. Then $X = X_1 \times X_2$.

Proof. Since X is irreducible, so are X_1 and X_2, and clearly $X \subset X_1 \times X_2$. So it is enough to prove that

$$\dim X_1 + \dim X_2 = \dim X = \dim U.$$

However, the maps $\mathrm{D}(\pi_i|_U)$ are surjective onto $TX \cap \pi_i^* TY_i$ and hence applying Lemma B.2 yields

$$\dim X_1 + \dim X_2 = \operatorname{rk}(TX \cap \pi_1^* TY_1)|_U + \operatorname{rk}(TX \cap \pi_2^* TY_2)|_U = \operatorname{rk} TX|_U = \dim X. \blacksquare$$

B.8. Tangent cone. We recall the notion of tangent cone and a few of its properties. For more details and proofs we refer to [Har95, Lecture 20] and [Mum99, III.§3,§4].

For an irreducible Noetherian scheme X over $\mathbb{C}$ and a closed point $x \in X$ we consider the local ring $\mathcal{O}_{X,x}$ and we let $\mathfrak{m}_x$ be the maximal ideal in $\mathcal{O}_{X,x}$. Let

$$R := \bigoplus_{i=0}^{\infty} (\mathfrak{m}_x^i / \mathfrak{m}_x^{i+1}),$$

where $\mathfrak{m}_x^0$ is just the whole of $\mathcal{O}_{X,x}$. Now we define the *tangent cone $TC_x X$ at x to X* to be $\operatorname{Spec} R$.

If X is a subscheme of an affine space $\mathbb{A}^m$ (which we will usually assume to be an affine piece of a projective space), the tangent cone at x to X can be understood as a subscheme of $\mathbb{A}^m$. Its equations can be derived from the ideal of X. For simplicity assume $x = 0 \in \mathbb{A}^m$ and then the polynomials defining $TC_0 X$ are the lowest degree homogeneous parts of the polynomials in the ideal of X.

Another interesting pointwise definition is that $v \in TC_0 X$ is a closed point if and only if there exists a holomorphic map φ_v from the disc $D_t := \{t \in \mathbb{C} : |t| < \delta\}$ to X such that $\varphi_v(0) = 0$ and the first non-zero coefficient in the Taylor expansion in t of $\varphi_v(t)$ is v, i.e.

$$\varphi_v : D_t \to X, \quad t \mapsto t^k v + t^{k+1} v_{k+1} + \cdots,$$

We list some properties of the tangent cone which will be used freely in the proofs:

(1) The dimension of every component of $TC_x X$ is equal to the dimension of X.
(2) $TC_x X$ is naturally embedded in the Zariski tangent space to X at x and $TC_x X$ spans (as a scheme) the tangent space.
(3) X is regular at x if and only if $TC_x X$ is equal (as a scheme) to the tangent space.

B.9. Secant, tangent and dual varieties. Let W be a vector space of dimension $n+1$. Let $Z \subset \mathbb{P}^n = \mathbb{P}(W)$ be any subvariety.

We denote by $\sigma(Z) \subset \mathbb{P}^n$ its *secant variety*, i.e., the closure of the union of all projective lines through z_1 and z_2, where (z_1, z_2) vary through all pairs of different points of Z.

Let $\tau(Z)$ denote the *tangent variety* of Z, i.e. the closure of the union of limits of secant lines as z_1 converges to z_2.

Also:

Definition. We let $Z^* \subset \check{\mathbb{P}}^n := \mathbb{P}(W^*)$ be the closure of the set of hyperplanes tangent to Z at some point:

$$Z^* := \overline{\{H \in \check{\mathbb{P}}^n \mid \exists z \in Z : T_z Z \subset H\}}.$$

We say Z^* is the *dual variety* to Z.

C. Vector fields, forms and automorphisms

In the course of this dissertation, particularly in Chapter E, we use some differential geometric facts, which we summarise in this chapter. Although all these facts are standard or follow easily from the standard material, we reproduce or at least sketch most of the proofs. We do this for the sake of completeness and also because various authors use different notation and combining them one can get rather confused (at least this has happened to the present author).

C.1. Vector fields, Lie bracket and distributions. Let Y be a complex manifold or a smooth algebraic variety, let $F \subset TY$ be a corank 1 subbundle ([1]) and let $\theta : TY \to TY/F =: L$ be the quotient map, so that the following sequence is exact:

$$0 \to F \to TY \xrightarrow{\theta} L \to 0.$$

Also assume U is an open subset. We say that a (possibly singular) subvariety $X \subset U$ with smooth locus X_0 is *F-integrable* if TX_0 is contained in F.

Proposition C.1. *With the assumptions as above:*

(i) $\mathrm{d}\theta$ *gives a well defined map of $\mathcal{O}_Y$-modules:*

$$\mathrm{d}\theta : \textstyle\bigwedge^2 F \to L.$$

We refer to this map as the twisted 2-form $\mathrm{d}\theta$.

(ii) *Assume μ and ν are two vector fields on U, both contained in F. Then $\theta([\mu,\nu])(y) = \mathrm{d}\theta_y(\mu(y),\nu(y))$. In particular, $\theta([\mu,\nu])(y)$ does not depend on the vector fields, but only on their values at y.*

(iii) *Again assume μ and ν are two vector fields on U, but now only ν is contained in F. Then again $\theta([\mu,\nu])(y)$ depends only on the value of ν at y, but not on the whole vector field. In other words, the map of sheaves $F \to L$ given by $\theta([\mu,\cdot])$ is $\mathcal{O}_Y$-linear and hence it determines a map of vector bundles $F \to L$.*

(iv) *If X is F-integrable, then $\mathrm{d}\theta|_{X_0} \equiv 0$. In particular,*

$$\dim X \le \operatorname{rk} F - \frac{1}{2}\min_{x\in X}(\operatorname{rk} \mathrm{d}\theta_x).$$

([1]) One could also consider F to be a corank r subbundle for any $r \in \{1,\dots,\dim Y\}$. Some of the statements below can be generalised to any r (not necessarily $r = 1$), but the proofs get more complicated, especially in notation. We restrict our considerations to the $r = 1$ case, as this is the only one used in the dissertation.

Proof. All the statements are analytically local, so it is enough to assume that Y is a disc $D^{2n} \subset \mathbb{C}^n$ with coordinates $y_1, \dots, y_m$, $U = Y$, $y = 0$ and θ is a nowhere vanishing section of $\Omega^1 Y \otimes L \simeq \Omega^1 Y$ (the choice of the trivialisation of L is of course not unique):

$$\theta = \sum_i A_i \mathrm{d}y_i = \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{y},$$

where the collection $(A_1, \dots, A_m)$ (respectively $(\mathrm{d}y_1, \dots, \mathrm{d}y_m)^T$) is denoted by $\boldsymbol{A}$ (respectively $\mathrm{d}\boldsymbol{y}$). Then

$$F := \Big\{ v \in TD^{2n} \;\Big|\; \sum_i A_i \mathrm{d}y_i(v) = 0 \Big\}.$$

To prove (i) note that

$$\mathrm{d}\theta = \sum_i \mathrm{d}A_i \wedge \mathrm{d}y_i = \mathrm{d}\boldsymbol{A} \wedge \mathrm{d}\boldsymbol{y}.$$

We must check that this does not depend on the choice of the trivialisation $\boldsymbol{A}$ of L. So assume $\boldsymbol{B}$ is a different trivialisation, so there exists $g : Y \to \mathbf{GL}(1) \simeq \mathbb{C}^*$ such that

$$\boldsymbol{B} = g \cdot \boldsymbol{A}.$$

We must prove that $\mathrm{d}\boldsymbol{B} \wedge \mathrm{d}\boldsymbol{y}$ restricted to F transforms in the same manner:

$$\mathrm{d}\boldsymbol{B} \wedge \mathrm{d}\boldsymbol{y} = \mathrm{d}(g \cdot \boldsymbol{A}) \wedge \mathrm{d}\boldsymbol{y} = (\mathrm{d}g \cdot \boldsymbol{A} + g \cdot \mathrm{d}\boldsymbol{A}) \wedge \mathrm{d}\boldsymbol{y} \overset{\boldsymbol{A}=0 \text{ on } F}{=} (g \cdot \mathrm{d}\boldsymbol{A}) \wedge \mathrm{d}\boldsymbol{y}.$$

To prove (ii) let

$$\mu = \sum_k \mu_k \frac{\partial}{\partial y_k}, \qquad \nu = \sum_k \nu_k \frac{\partial}{\partial y_k}$$

for some holomorphic functions μ_k and ν_k. Since μ and ν are contained in F we have

$$\sum_k A_k \mu_k = 0 \quad \text{and} \quad \sum_l A_l \nu_l = 0.$$

Therefore for every k or l we have

$$\sum_k \frac{\partial A_k}{\partial y_l} \mu_k = -\sum_k A_k \frac{\partial \mu_k}{\partial y_l}, \tag{C.2a}$$

$$\sum_l \frac{\partial A_l}{\partial y_k} \nu_l = -\sum_l A_l \frac{\partial \nu_l}{\partial y_k}. \tag{C.2b}$$

Since

$$[\mu, \nu] = \sum_{k,l} \left(\nu_k \frac{\partial \mu_l}{\partial y_k} \frac{\partial}{\partial y_l} - \mu_l \frac{\partial \nu_k}{\partial y_l} \frac{\partial}{\partial y_k} \right),$$

it follows that

$$\begin{aligned} \theta([\mu, \nu]) &= \sum_{k,l} \left(A_l \nu_k \frac{\partial \mu_l}{\partial y_k} - A_k \mu_l \frac{\partial \nu_k}{\partial y_l} \right) \overset{\text{(C.2)}}{=} \sum_{k,l} \left(-\frac{\partial A_l}{\partial y_k} \mu_l \nu_k + \frac{\partial A_k}{\partial y_l} \mu_l \nu_k \right) \\ &= \sum_{k,l} \left(\frac{\partial A_l}{\partial y_k} (\mu_k \nu_l - \mu_l \nu_k) \right) = \sum_{k,l} \left(\frac{\partial A_l}{\partial y_k} (\mathrm{d}y_k \wedge \mathrm{d}y_l)(\mu, \nu) \right) = \mathrm{d}\theta(\mu, \nu). \end{aligned}$$

We note that the above calculation is a special case of [KN96, Prop. I.3.11], though the reader should be careful, as the notation in [KN96] is different and as a consequence a constant factor -2 is "missing" in our formula.

The proof of (iii) is identical to the beginning of the proof of (ii).

Finally, to prove (iv) just use (ii) and the fact that the Lie bracket of two vector fields tangent to X is tangent to X. ∎

C.2. Automorphisms. Here we introduce the notation for several types of automorphisms of a manifold Y and its subvariety X. Also we recall some standard properties and relations between them.

Let Y be a complex manifold (or respectively, smooth algebraic variety) and let $U \subset Y$ be an open subset in analytic (or respectively, Zariski) topology. By $\mathrm{Aut}^{\mathrm{hol}}(U)$ (respectively, $\mathrm{Aut}^{\mathrm{alg}}(U)$) we denote the group of holomorphic (respectively, algebraic) automorphisms of U. By $\mathrm{Aut}^{\bullet}(U)$ we mean either $\mathrm{Aut}^{\mathrm{hol}}(U)$ or $\mathrm{Aut}^{\mathrm{alg}}(U)$, whenever specifying is not necessary.

Assume that a complex Lie group (respectively, an algebraic group) G acts on U, i.e. we have a group homomorphism $G \to \mathrm{Aut}^{\bullet}(U)$. Also let $\mathfrak{g}$ be the Lie algebra of G. By G^0 we denote the connected component of identity in G.

An *infinitesimal automorphism* of U is a vector field $\mu \in H^0(U, TY)$. Differentiating the action map $G \times U \to U$ with respect to the first coordinate we get the induced map $\mathfrak{g} \times Y \to TY$ or more precisely $\mathfrak{g} \to H^0(U, TY)$. This map preserves the Lie bracket (see [Akh95, Thm. in §1.7]) and if the action is faithful, then it is injective (see [Akh95, Thm. in §1.5]).

A particular case is when $G = \mathbb{C}^*$. Then we get a map $\mathbb{C} \to H^0(U, TY)$ and we set $\mu_{\mathbb{C}^*}$ to be the image of $1 \in \mathbb{C}$ under this map. We say $\mu_{\mathbb{C}^*}$ is the *vector field related to the* $\mathbb{C}^*$*-action.* Note that $\mu_{\mathbb{C}^*}$ is homogeneous of weight 0.

The infinitesimal automorphisms form a sheaf TY of Lie algebras, which at the same time is an $\mathcal{O}_Y$-module. The two multiplication structures are related by the following Leibniz rule:

$$\forall f \in H^0(U, \mathcal{O}_Y),\ \forall \mu, \nu \in H^0(U, TY) \quad [f\mu, \nu] = f[\mu, \nu] + \mathrm{d}f(\nu)\mu. \tag{C.3}$$

The following theorem comparing infinitesimal, algebraic and holomorphic automorphisms for a projective variety is well known and standard:

THEOREM C.4. *Let Y be a projective variety. Then:*

(i) $\mathrm{Aut}^{\mathrm{hol}}(Y)$ *is a complex Lie group.*
(ii) *Every holomorphic automorphism of Y is algebraic and hence*

$$\mathrm{Aut}(Y) := \mathrm{Aut}^{\mathrm{hol}}(Y) = \mathrm{Aut}^{\mathrm{alg}}(Y).$$

(iii) *Let $\mathfrak{aut}(Y)$ be the tangent Lie algebra to* $\mathrm{Aut}(Y)$. *Every infinitesimal automorphism is tangent to some* 1*-parameter subgroup of* $\mathrm{Aut}^{\mathrm{hol}}(Y)$, *so that* $\mathfrak{aut}(Y) = H^0(Y, TY)$.

Proof. Part (i) is proved in [Akh95, §2.3]. Part (ii) is a consequence of [Gri74, Thm. IV.A]. Part (iii) is explained in [Akh95, Prop. in §1.5 & Cor. 1 in §1.8]. ∎

Clearly $H^0(U, \mathcal{O}_Y)$ is a representation of G and hence also of $\mathfrak{g}$. We also have the following Lie algebra action of the sheaf of infinitesimal automorphisms:

$$TY \times \mathcal{O}_Y \to \mathcal{O}_Y, \qquad (\mu, f) \mapsto \mathrm{d}f(\mu),$$

which is given by derivation in the direction of the vector field.

The action of $\mathfrak{g}$ on $H^0(U, \mathcal{O}_Y)$ is the composition

$$\mathfrak{g} \to H^0(U, TY) \to \mathfrak{gl}(H^0(U, \mathcal{O}_Y)).$$

Let $X \subset Y$ be a subvariety. By $\mathrm{Aut}^\bullet(U, X)$ we denote the subgroup of $\mathrm{Aut}^\bullet(U)$ preserving $U \cap X$. If Y is projective, then $\mathfrak{aut}(Y, X)$ is the Lie algebra tangent to $\mathrm{Aut}^\bullet(Y, X)$. By $\mathfrak{aut}^{\mathrm{inf}}(U, X)$ we denote the Lie algebra of infinitesimal automorphisms of U preserving X, i.e.

$$\mathfrak{aut}^{\mathrm{inf}}(U, X) := \{\mu \in H^0(U, TY) \mid \forall f \in \mathcal{I}(X)|_U \ (\mathrm{d}f)(\mu) \in \mathcal{I}(X)|_U\},$$

where $\mathcal{I}(X) \lhd \mathcal{O}_Y$ is the sheaf of ideals of X.

Clearly, if G preserves X, then the image of $\mathfrak{g} \to H^0(U, TY)$ is contained in $\mathfrak{aut}^{\mathrm{inf}}(U, X)$. Conversely, if the image is contained in $\mathfrak{aut}^{\mathrm{inf}}(U, X)$, then the action of the connected component G^0 preserves X.

Corollary C.5. *If Y is projective, then* $\mathfrak{aut}^{\mathrm{inf}}(Y, X) = \mathfrak{aut}(Y, X)$. ∎

Moreover, $\mathfrak{aut}^{\mathrm{inf}}(\cdot, X)$ forms in TY a subsheaf of Lie algebras and $\mathcal{O}_Y$-modules.

C.3. Line bundles and $\mathbb{C}^*$-bundles. Let Y be a complex manifold or a smooth algebraic variety and let L be a line bundle on Y. By $\mathbf{L}^\bullet$ we denote the principal $\mathbb{C}^*$-bundle over Y obtained as the line bundle L^* with the zero section removed. Let π be the projection $\mathbf{L}^\bullet \to Y$.

Let $\mathcal{R}_L$ be the sheaf of graded $\mathcal{O}_Y$-algebras $\bigoplus_{m \in \mathbb{Z}} L^m$ on Y. Given an open subset $U \subset Y$ the ring $\mathcal{R}_L(U)$ consists of all the algebraic functions on $\pi^{-1}(U)$, i.e. $\mathcal{R}_L = \pi_* \mathcal{O}_{\mathbf{L}^\bullet}$. Therefore

$$\mathbf{L}^\bullet = \mathcal{S}pec_Y \mathcal{R}_L.$$

Moreover, $H^0(U, L^m) \subset H^0(\pi^{-1}(U), \mathcal{O}_{\mathbf{L}^\bullet})$ is the set of homogeneous functions of weight m (see §B.6).

Lemma C.6. *Let Y be a smooth algebraic variety and let L be a line bundle on Y. Then* $\mathrm{Pic}(\mathbf{L}^\bullet) \simeq \mathrm{Pic}(Y)/\langle L \rangle$ *and the map* $\mathrm{Pic}(Y) \twoheadrightarrow \mathrm{Pic}(\mathbf{L}^\bullet)$ *is induced by the projection* $\pi : \mathbf{L}^\bullet \to Y$.

Proof. The Picard group of the total space of L^* is isomorphic to $\mathrm{Pic}\, Y$ and the isomorphisms are given by the projection and the zero section $s_0 : Y \to L^*$. Further, $s_0(Y)$ is a Cartier divisor linearly equivalent to any other rational section $s : Y \dashrightarrow L^*$. Therefore $s_0^*(s_0(Y)) = L^*$ and hence by [Har77, Prop. 6.5(c)] the following sequence is exact:

$$\begin{aligned} &\mathbb{Z} \to \mathrm{Pic}\, Y \xrightarrow{\pi^*} \mathrm{Pic}\, \mathbf{L}^\bullet \to 0 \\ &1 \mapsto [L^*] \end{aligned}$$ ∎

The relative tangent bundle, i.e. $\ker(\mathrm{D}\pi : T\mathbf{L}^\bullet \to \pi^* TY)$, is trivialised by the vector field $\mu_{\mathbb{C}^*}$ related to the action of $\mathbb{C}^*$ (see §C.2) and hence we have the short exact sequence

$$0 \to \mathcal{O}_{\mathbf{L}^\bullet} \to T\mathbf{L}^\bullet \to \pi^* TY \to 0.$$

In particular, $K_{\mathbf{L}^\bullet} = \pi^* K_Y$.

C.4. Distributions and automorphisms preserving them. Let $F \subset TY$ be a corank 1 vector subbundle. Our particular interest will be in the case where F is a contact distribution—see §E.3 for the definition. But what follows holds true for any F. By $\mathrm{Aut}_F^\bullet(U)$, $\mathfrak{aut}_F(Y)$, $\mathfrak{aut}_F^{\inf}(U)$, $\mathrm{Aut}_F^\bullet(U, X)$, $\mathfrak{aut}_F(Y, X)$ and $\mathfrak{aut}_F^{\inf}(U, X)$ we denote the appropriate automorphisms or infinitesimal automorphisms preserving F and possibly the subvariety X.

For instance,

$$\mathfrak{aut}_F^{\inf}(U) = \{\mu \in H^0(U, TY) \mid [\mu, F] \subset F\}. \tag{C.7}$$

Also $\mathfrak{aut}_F^{\inf}$ is a sheaf of Lie algebras, but usually it is not an $\mathcal{O}_Y$-submodule of TY. To see that, take any $\mu \in \mathfrak{aut}_F^{\inf}(U)$ for U small enough. Assume that for all $f \in \mathcal{O}_Y(U)$ we have $f\mu \in \mathfrak{aut}_F^{\inf}(U)$. Then by the Leibniz rule (see (C.3))

$$\forall \nu \in H^0(U, F) \quad \mathrm{d}f(\nu) \cdot \mu \in H^0(U, F).$$

This can only happen if either:

- $\mu \in H^0(U, F)$ or
- $F = 0$, i.e. F is a rank 0 bundle.

We will see that the first case does not happen if F is a contact distribution (unless $\mu = 0$, see Theorem E.13(1)). In fact, one can prove that it never happens for any $\mu \in H^0(U, F)$ (remember that U is small enough), unless $F = 0$.

If G acts on U and preserves the distribution F, then the map $\mathfrak{g} \to H^0(U, TY)$ factors through $\mathfrak{aut}_F^{\inf}(U)$. Conversely, if G is connected, acts on U and the map $\mathfrak{g} \to H^0(U, TY)$ factors through $\mathfrak{aut}_F^{\inf}(U)$, then the action of G preserves F. As a consequence we get:

Corollary C.8. *If Y is projective and $X \subset Y$ is a subvariety, then:*

(i) $\mathfrak{aut}_F(Y) = \mathfrak{aut}_F^{\inf}(Y)$,
(ii) $\mathfrak{aut}_F(Y, X) = \mathfrak{aut}_F^{\inf}(Y, X)$.

Proof. This follows from the above considerations and from Theorem C.4. ∎

Further, let L be the quotient bundle and θ be the quotient map:

$$0 \to F \to TY \xrightarrow{\theta} L \to 0.$$

If the action of G on U extended to $TY|_U$ preserves F, then in an obvious way we get the *induced action* of G *on the total spaces* of $L|_U$ and $L^*|_U$. These actions preserve the zero sections.

Let $\mathbf{L}^\bullet$ and $\mathcal{R}_L$ be as in §C.3.

By analogy with the above we want to define an action of $\mathfrak{aut}_F^{\inf}$ on $\mathbf{L}^\bullet$. In other words, we define a special lifting of the vector fields from $\mathfrak{aut}_F^{\inf} \subset TY$ to vector fields on $\mathbf{L}^\bullet$.

First observe that the sheaf of Lie algebras $\mathfrak{aut}_F^{\inf}$ acts on the sheaf L: if $s \in H^0(U, L)$, then choose an open subset $V \subset U$ small enough and any lifting $s_{TY} \in H^0(V, TY)$, $\theta(s_{TY}) = s|_V$ and let $\mu \in \mathfrak{aut}_F^{\inf}(U)$ act on $H^0(U, L)$ locally by

$$s|_V \mapsto (\mu.s)|_V := \theta([s_{TY}, \mu|_V]). \tag{C.9}$$

By equation (C.7) defining $\mathfrak{aut}_F^{\inf}$, this does not depend on the choice of s_{TY} and hence, by elementary properties of sheaves, it glues uniquely to an element of $H^0(U, L)$. Hence we get a Lie algebra representation $\mathfrak{aut}_F^{\inf}(U) \to \mathfrak{gl}(H^0(U, L))$.

Secondly, we can extend the action of $\mathfrak{aut}_F^{\inf}$ on the locally free sheaf L defined in (C.9) to an action on $\mathcal{R}_L$, by requiring that the action satisfies the Leibniz rule:

$$t, s \in \mathcal{R}_L,\ \mu \in \mathfrak{aut}_F^{\inf} \ \Rightarrow\ \mu.(ts) = (\mu.t)s + t(\mu.s) \tag{C.10}$$

—locally every section of L^m can be written as a sum of products of sections of L (or their inverses, if $m < 0$).

Finally, we can extend this action to $\mathcal{O}_{\mathbf{L}^\bullet}$, again requiring the Leibniz rule. Eventually, we get an action which we will call the *induced action* of $\mathfrak{aut}_F^{\inf}$ on $\mathbf{L}^\bullet$. The following property justifies the name:

Proposition C.11. *If the action of G preserves F, then the tangent action to the induced action of G on $\mathbf{L}^\bullet|_U := \pi^{-1}(U)$ is the composition of $\mathfrak{g} \to \mathfrak{aut}_F^{\inf}(U)$ and the induced action of $\mathfrak{aut}_F^{\inf}$ on $\mathbf{L}^\bullet$.* ■

For a fixed $\mu \in \mathfrak{aut}_F^{\inf}(U)$, the induced map $\mathcal{O}_{\mathbf{L}^\bullet}|_{\pi^{-1}(U)} \to \mathcal{O}_{\mathbf{L}^\bullet}|_{\pi^{-1}(U)}$ is a derivation, so it corresponds to a vector field $\breve{\mu} \in H^0(\pi^{-1}(U), T\mathbf{L}^\bullet)$ such that

$$\forall f \in \mathcal{O}_{\mathbf{L}^\bullet} \quad \mu.f = \mathrm{d}f(\breve{\mu}). \tag{C.12}$$

By construction we also have $\mathrm{D}\pi(\breve{\mu}) = \mu$.

C.5. Lifting and descending twisted forms. In this section we explain how to lift a twisted form $\theta : TY \to L$ to a form $\theta^\bullet$ on $\mathbf{L}^\bullet$. Later we study the derivative $\mathrm{d}\theta^\bullet$ and show its relation to $\mathrm{d}\theta : \bigwedge^2 F \to L$. Finally, we prove the homogeneity of $\theta^\bullet$ and $\mathrm{d}\theta^\bullet$ and thus explicitly set conditions when a $\mathbb{C}^*$-bundle together with a 2-form ω arises as $(\mathbf{L}^\bullet, \mathrm{d}\theta^\bullet)$ from some distribution $F \subset TY$.

With the notation and assumptions as in the previous sections, we have a canonical isomorphism of line bundles $\tau : \pi^* L \xrightarrow{\simeq} \mathcal{O}_{\mathbf{L}^\bullet}$: if $y \in Y$, $\lambda \in \mathbf{L}^\bullet_y = \pi^{-1}(y)$, $l \in L_y$, then we set

$$\tau(y, \lambda, l) := (y, \lambda, \lambda(l)).$$

We let $\theta^\bullet := \tau \circ \pi^*\theta \circ \mathrm{D}\pi$ (2):

$$T\mathbf{L}^\bullet \xrightarrow{\mathrm{D}\pi} \pi^* TY \xrightarrow{\pi^*\theta} \pi^* L \xrightarrow{\tau} \mathcal{O}_{\mathbf{L}^\bullet}.$$

(2) In [Bea98], [LeB95] the authors denote $\theta^\bullet$ simply as $\pi^*\theta$, since the other maps are natural. This is a bit confusing to some people (including the present author, but see also a comment in [SCW04] about a small mistake in [KPSW00]) and therefore we underline that $\theta^\bullet$ is a composition of three maps.

LEMMA C.13. *For every $\mu \in \mathfrak{aut}^{\mathrm{inf}}_F(U)$ the induced infinitesimal automorphism $\breve{\mu}$ preserves $\theta^\bullet$, i.e.*

$$L_{\breve{\mu}}(\theta^\bullet) := \lim_{t\to 0} \frac{\gamma_{\breve{\mu}}(t)^*\theta^\bullet - \theta^\bullet}{t} = 0,$$

where $L_{\breve{\mu}}$ is the Lie derivative operator and $\gamma_{\breve{\mu}}(t)$ is the local 1-parameter group of transformations of $\mathbf{L}^\bullet$ determined by $\breve{\mu}$.

Proof. For the simplicity of notation assume $\gamma_{\breve{\mu}}(t)$ is a global transformation. The following diagram of vector bundles is commutative:

$$\begin{tikzcd}
T\mathbf{L}^\bullet \arrow[r, "\mathrm{D}\pi"] \arrow[d, "\mathrm{D}\gamma_{\breve{\mu}}(t)"] & \pi^*TY \arrow[r, "\pi^*\theta"] \arrow[d, "\mathrm{D}^\pi\gamma_\mu(t)"] & \pi^*L \arrow[r, "\tau"] \arrow[d, "\gamma_\mu^{\pi^*L}(t)"] & \mathcal{O}_{\mathbf{L}^\bullet} \arrow[d, "\gamma_{\breve{\mu}}(t)\times \mathrm{id}_{\mathbb{C}}"] \\
T\mathbf{L}^\bullet \arrow[r, "\mathrm{D}\pi"] & \pi^*TY \arrow[r, "\pi^*\theta"] & \pi^*L \arrow[r, "\tau"] & \mathcal{O}_{\mathbf{L}^\bullet}
\end{tikzcd}$$

where $\mathrm{D}^\pi\gamma_\mu(t)$ is the automorphism of π^*TY determined by $\mathrm{D}\gamma_\mu(t) : TY \to TY$ and $\gamma_{\breve{\mu}}(t) : \mathbf{L}^\bullet \to \mathbf{L}^\bullet$; similarly $\gamma_\mu^{\pi^*L}(t)$ is determined by $\mathrm{D}\gamma_\mu(t) : TY/F \to TY/F$ and $\gamma_{\breve{\mu}}(t) : \mathbf{L}^\bullet \to \mathbf{L}^\bullet$. The composition of the whole upper row is equal to $\theta^\bullet$. The composition of the leftmost vertical arrow and the whole lower row is equal to $\gamma_{\breve{\mu}}(t)^*\theta^\bullet$. Since the rightmost arrow is the identity on the second component of $\mathcal{O}_{\mathbf{L}^\bullet} = \mathbf{L}^\bullet \times \mathbb{C}$ and since the diagram is commutative, both forms take the same values at every vector $v \in T\mathbf{L}^\bullet$, hence are equal and the claim follows. ∎

We also give a local description of $\theta^\bullet$ and $\mathrm{d}\theta^\bullet$. So now assume $Y \simeq D^{2m}$ and let $y_1, \dots, y_m$ be some coordinates on Y. Let z be a linear coordinate on the fibre of $\mathbf{L}^\bullet \simeq Y \times \mathbb{C}^*$. This means that z determines a section of L which trivialises L over D^{2m}. So we can think of θ as a holomorphic 1-form on $\mathbf{L}^\bullet$ depending only on y_i's and $\mathrm{d}y_i$'s. Let (y, z_0) be any point of $\mathbf{L}^\bullet$ and let $\bar{v}$ be any vector tangent to $\mathbf{L}^\bullet$ at (y, z_0). We write $\bar{v} = v + w$, where v is the component tangent to Y, while w is tangent to $\mathbb{C}^*$. Then

$$\theta^\bullet_{(y,z_0)}(\bar{v}) = (\tau \circ \pi^*\theta \circ \mathrm{D}\pi)_{y,z_0}(\bar{v}) = (\tau \circ \pi^*\theta)_{y,z_0}(v) = z_0(\theta_y(v)) = z_0 \cdot \theta_y(v)$$

or more concisely (in local coordinates)

$$\theta^\bullet = z\theta, \tag{C.14}$$

and therefore

$$\mathrm{d}\theta^\bullet = \mathrm{d}(z\theta) = z\mathrm{d}\theta + \mathrm{d}z \wedge \theta. \tag{C.15}$$

Since in this notation θ is a homogeneous 1-form of weight 0 and $\mathrm{wt}(z) = 1$, $\theta^\bullet$ and $\mathrm{d}\theta^\bullet$ are homogeneous forms of weight 1 (see §B.6).

In the above coordinates, the vector field $\mu_{\mathbb{C}^*}$ related to the $\mathbb{C}^*$-action can be expressed as follows:

$$\mu_{\mathbb{C}^*} = z\frac{\partial}{\partial z}.$$

PROPOSITION C.16. *Let Y be a complex manifold or smooth algebraic variety and let L be a line bundle on Y. Also let $\mathbf{L}^\bullet$ be the principal $\mathbb{C}^*$-bundle over Y as in §C.3 and let $\mu_{\mathbb{C}^*}$ be the vector field on $\mathbf{L}^\bullet$ associated to the action of $\mathbb{C}^*$. Finally, let ω be a homogeneous closed 2-form on $\mathbf{L}^\bullet$ of weight 1. Then:*

(i) $\omega = \mathrm{d}(\omega(\mu_{\mathbb{C}^*}))$.
(ii) *There exists a unique twisted* 1*-form* $\theta : TY \to L$ *such that* $\omega(\mu_{\mathbb{C}^*}) = \theta^\bullet$*, where* $\theta^\bullet$ *is defined from* θ *as above.*
(iii) *Moreover,* $\omega(\mu_{\mathbb{C}^*})$ *is nowhere vanishing if and only if* θ *is nowhere vanishing. If this is the case, then* ω *is non-degenerate if and only if* $\mathrm{d}\theta|_F$ *is non-degenerate.*

Proof. To prove (i) let z be a local coordinate linear on the fibres of $\pi : \mathbf{L}^\bullet \to Y$. Since ω is closed, locally it is exact, so

$$\omega = \mathrm{d}(z\phi' + g\mathrm{d}z)$$

for some function g and 1-form ϕ', both homogeneous of weight 0. However,

$$\mathrm{d}(z\phi' + g\mathrm{d}z) = \mathrm{d}(z(\phi' - \mathrm{d}g)).$$

Set $\phi := \phi' - \mathrm{d}g$, so that $\omega = \mathrm{d}(z\phi)$. Note that although ϕ' and g are not uniquely determined, ϕ is the unique homogeneous 1-form of weight 0 such that $\omega = \mathrm{d}(z\phi)$. Then

$$\omega(\mu_{\mathbb{C}^*}) = (\mathrm{d}z \wedge \phi)\left(z\frac{\partial}{\partial z}\right) + z\mathrm{d}\phi\left(z\frac{\partial}{\partial z}\right) = \mathrm{d}z\left(z\frac{\partial}{\partial z}\right) \cdot \phi = z\phi.$$

Hence $\mathrm{d}(\omega(\mu_{\mathbb{C}^*})) = \omega$, as claimed in (i).

To prove (ii), define θ to be locally the form ϕ from the above argument. One must verify that ϕ glues uniquely to a twisted 1-form $\theta : TY \to L$.

Part (iii) follows from the local descriptions of $\theta^\bullet$ and $\mathrm{d}\theta^\bullet$ (see (C.14) and (C.15)). For instance, if $n = \frac{1}{2}(\dim Y - 1)$, then

$$(\mathrm{d}\theta^\bullet)^{\wedge^{n+1}} = (n+1)\mathrm{d}z \wedge \theta \wedge (\mathrm{d}\theta)^{\wedge^n}.$$

Therefore $\mathrm{d}\theta^\bullet$ is non-degenerate at a given point if and only if θ does not vanish at that point and $\mathrm{d}\theta$ is non-degenerate on the kernel of θ. ∎

Lemma C.17. *Let* $X \subset Y$ *be any subvariety and* X_0 *its smooth locus. Then* X *is* F*-integrable if and only if* $\mathrm{d}\theta^\bullet$ *vanishes identically on the tangent space to* $\pi^{-1}(X_0)$*.*

Proof. First assume X is F-integrable. Then $\mathrm{d}\theta$ vanishes on $T(\pi^{-1}(X_0))$ by Proposition C.1(iv) and θ vanishes by definition. Hence from the local description of $\mathrm{d}\theta^\bullet$ (see (C.15)) we get the result.

On the other hand, if $\mathrm{d}\theta^\bullet|_{T(\pi^{-1}(X_0))} \equiv 0$, since

$$\mu_{\mathbb{C}^*}|_{\pi^{-1}(X_0)} \in H^0(\pi^{-1}(X_0), T(\pi^{-1}(X_0))),$$

we have in particular

$$\mathrm{d}\theta^\bullet(\mu_{\mathbb{C}^*}, T(\pi^{-1}(X_0))) \equiv 0.$$

But $\mathrm{d}\theta^\bullet(\mu_{\mathbb{C}^*}) = \theta^\bullet$ (see Proposition C.16(ii)), hence $\pi^{-1}X$ is (π^*F)-integrable and therefore X is F-integrable. ∎

For $s \in \mathcal{R}_L = \pi_*\mathcal{O}_{\mathbf{L}^\bullet}$, by $\tilde{s} \in \mathcal{O}_{\mathbf{L}^\bullet}$ we denote the lifting of s, i.e. $\tilde{s} := \tau \circ \pi^* s$.

Hence we have two different possibilities of lifting an infinitesimal automorphism $\mu \in \mathfrak{aut}_F^{\mathrm{inf}}$ to an object on $\mathbf{L}^\bullet$: either we lift it to a vector field $\breve{\mu}$ (see (C.12)), or we lift $\theta(\mu)$ to a function $\widetilde{\theta(\mu)}$. We will compare these two liftings and see how they behave with respect to the Lie bracket of vector fields.

Lemma C.18. *We have*

$$\forall \nu \in \mathfrak{aut}^{\inf}_F(U), \mu \in H^0(U, TY) \qquad \widetilde{\theta([\mu,\nu])} = \mathrm{d}(\widetilde{\theta(\mu)})(\breve{\nu}).$$

Proof. By (C.9),

$$\theta([\mu,\nu]) = \nu.\theta(\mu)$$

and hence $\widetilde{\theta([\mu,\nu])} = \nu.\widetilde{\theta(\mu)}$. By (C.12), this is equal to $\mathrm{d}(\widetilde{\theta(\mu)})(\breve{\nu})$. ∎

Proposition C.19. *If* $\mu \in \mathfrak{aut}^{\inf}_F(U)$, *then*

$$\mathrm{d}(\widetilde{\theta(\mu)}) = -(\mathrm{d}\theta^\bullet)(\breve{\mu}).$$

Proof. The following proof is quoted from [Bea98, Prop. 1.6]. Since $L_{\breve{\mu}}(\theta^\bullet) = 0$ (see Lemma C.13), by [KN96, Prop. I.3.10(a)] we have

$$(\mathrm{d}\theta^\bullet)(\breve{\mu}) = -\mathrm{d}(\theta^\bullet(\breve{\mu})).$$

On the other hand,

$$\theta^\bullet(\breve{\mu}) = \tau \circ \pi^*\theta \circ \mathrm{D}\pi(\breve{\mu}) = \tau \circ \pi^*(\theta(\mu)) = \widetilde{\theta(\mu)}.$$

Combining the two equalities, we get the result. ∎

Corollary C.20. *If* $\mu, \nu \in \mathfrak{aut}^{\inf}_F(U)$, *then*

$$\widetilde{\theta([\mu,\nu])} = -(\mathrm{d}\theta^\bullet)(\breve{\mu}, \breve{\nu}).$$

Proof. This combines Lemma C.18 and Proposition C.19. ∎

D. Elementary symplectic geometry

We introduce some elementary facts from symplectic geometry, having in mind the needs of subsequent chapters. Most of this material is contained in (or can be easily deduced from) classical textbooks on symplectic geometry, such as [MS98], although we rewrite this over the ground field $\mathbb{C}$ rather than $\mathbb{R}$.

D.1. Linear symplectic geometry. In this section we study the linear algebra of a vector space which has a symplectic form. Although it is elementary, it is very important for our considerations as it has threefold application: Firstly, the content of this section describes the local behaviour of symplectic manifolds (see §D.2), particularly the symplectisations of contact manifolds (see §E.3.1). Secondly, it describes much of the global geometry of projective space as a contact manifold (see Example E.10, but also look through Chapters F–I). Finally, it explains the fibrewise behaviour of contact distributions (see §E.3).

D.1.1. Symplectic vector space. A symplectic form on a vector space V is a non-degenerate skew-symmetric bilinear form. So $\omega \in \bigwedge^2 V^*$ is a symplectic form if and only if

$$\forall v \in V\ \exists w \in V \quad \text{such that} \quad \omega(v,w) \neq 0$$

or equivalently the map

$$\tilde{\omega} : V \to V^*, \qquad v \mapsto \omega(v, \cdot),$$

is an isomorphism.

If a vector space V has a symplectic form ω, we say that V (or (V,ω) if specifying the form is important) is a *symplectic vector space*. In that case the dimension of V is even and there exists a basis $v_1,\dots,v_n,w_1,\dots,w_n$ (where $n=\frac{1}{2}\dim V$) of V such that $\omega(v_i,w_i)=1$, $\omega(v_i,v_j)=0$ and $\omega(v_i,w_j)=0$ for $i\neq j$. Such a basis is called a *symplectic basis*.

By $\omega^\vee$ we denote the corresponding symplectic form on V^*:

$$\omega^\vee := (\tilde{\omega}^{-1})^*\omega.$$

If $v_1,\dots,v_n$, $w_1,\dots,w_n$ is a symplectic basis of V and $x_1,\dots,x_n$, $y_1,\dots,y_n$ is the dual basis of V^*, then $x_1,\dots,x_n,y_1,\dots,y_n$ is a symplectic basis of V^*. In that case $x_1,\dots,x_n,y_1,\dots,y_n$ are also called *symplectic coordinates* on V.

D.1.2. Subspaces in a symplectic vector space. Assume V is a vector space of dimension $2n$ and ω is a symplectic form on V. Suppose $W\subset V$ is a linear subspace. By $W^{\perp_\omega}$ we denote the ω-perpendicular complement of W:

$$W^{\perp_\omega} := \{v\in V \mid \forall w\in W\ \ \omega(v,w)=0\}.$$

Denote by π the natural projection $V^*\to W^*$. We say that the subspace W is:

isotropic	$\Leftrightarrow$	$\omega\vert_W\equiv 0$	$\Leftrightarrow$	$W\subset W^{\perp_\omega}$	$\Leftrightarrow$	$\ker\pi$ is coisotropic;
coisotropic (or *involutive*)	$\Leftrightarrow$	$\omega^\vee\vert_{\ker\pi}\equiv 0$	$\Leftrightarrow$	$W\supset W^{\perp_\omega}$	$\Leftrightarrow$	$\ker\pi$ is isotropic;
Lagrangian	$\Leftrightarrow$	W is isotropic or coisotropic and $\dim W=n=\frac{1}{2}\dim V$	$\Leftrightarrow$	$W=W^{\perp_\omega}$	$\Leftrightarrow$	$\ker\pi$ is Lagrangian;
symplectic	$\Leftrightarrow$	$\omega\vert_W$ is a symplectic form on W	$\Leftrightarrow$	$W\cap W^{\perp_\omega}=0$	$\Leftrightarrow$	$\ker\pi$ is symplectic.

D.1.3. Symplectic reduction of a vector space. With the assumptions as above let $W\subset V$ be any linear subspace and let $W':=W\cap W^{\perp_\omega}$. Define ω' to be the following bilinear form on $V':=W/W'$: for $w_1,w_2\in W$ let

$$\omega'([w_1],[w_2]) := \omega(w_1,w_2).$$

Then (V',ω') is a symplectic vector space.

The particular case we are mostly interested in is when W is a hyperplane or more generally a coisotropic subspace.

Note the following elementary properties of this construction:

Proposition D.1. *For a subspace $L\subset V$ let L' be the image of $L\cap W$ in V'.*

(a) *If L is isotropic (resp. coisotropic or Lagrangian) in V, then L' is isotropic (resp. coisotropic or Lagrangian) in V'.*
(b) *Conversely, if W is coisotropic, $L\subset W$ and L' is isotropic (resp. coisotropic or Lagrangian) in V', then L is isotropic (resp. coisotropic or Lagrangian) in V.* ∎

D.1.4. Symplectic automorphisms and weks-symplectic matrices. A linear automorphism ψ of a symplectic vector space (V,ω) is called a *symplectomorphism* if $\psi^*\omega = \omega$, i.e.

$$\forall u,v \in V \quad \omega(\psi(u),\psi(v)) = \omega(u,v).$$

We denote by $\mathbf{Sp}(V)$ the group of all symplectomorphisms of V and by $\mathfrak{sp}(V)$ its Lie algebra:

$$\mathfrak{sp}(V) = \{g \in \mathrm{End}(V) \mid \forall u,v \in V \ \ \omega(u,g(v)) + \omega(g(u),v) = 0\}.$$

A linear automorphism ψ of V is called a *conformal symplectomorphism* if $\psi^*\omega = c\omega$ for some constant $c \in \mathbb{C}^*$. We denote by $\mathbf{cSp}(V)$ the group of all conformal symplectomorphisms of V and by $\mathfrak{csp}(V)$ the tangent Lie algebra.

Fix a basis $\mathcal{B}$ of V and note that a matrix $g \in \mathfrak{gl}(V)$ is in the symplectic algebra $\mathfrak{sp}(V)$ if and only if

$$g^T J + Jg = 0$$

where $J := M(\omega,\mathcal{B})$. For the sake of Chapter F we also need to define a complementary linear subspace to $\mathfrak{sp}(V)$:

Definition. A matrix $g \in \mathfrak{gl}(V)$ is *weks-symplectic* ([1]) if and only if it satisfies the equation

$$g^T J - Jg = 0.$$

The vector space of all weks-symplectic matrices will be denoted by $\mathfrak{wsp}(V)$ (even though it is not a Lie subalgebra of $\mathfrak{gl}(V)$).

We immediately see that a matrix is weks-symplectic if and only if it corresponds to a linear endomorphism g such that for every $u,v \in V$,

$$\omega(gu,v) - \omega(u,gv) = 0. \tag{D.2}$$

This is a coordinate free way to describe $\mathfrak{wsp}(V)$.

Assume that our basis $\mathcal{B}$ is symplectic. In particular, $J^2 = M(\omega,\mathcal{B})^2 = -\mathrm{Id}_{2n}$.

Remark D.3. For a matrix $g \in \mathfrak{gl}(V)$ we have:

(a) $g \in \mathfrak{sp}(V) \Leftrightarrow Jg$ is a symmetric matrix;
(b) $g \in \mathfrak{wsp}(V) \Leftrightarrow Jg$ is a skew-symmetric matrix.

Note that if $g \in \mathfrak{gl}(V)$, then we can write:

$$g = \frac{1}{2}(g + Jg^T J) + \frac{1}{2}(g - Jg^T J)$$

and the first component $g_+ := \frac{1}{2}(g + Jg^T J)$ is in $\mathfrak{sp}(V)$, while the second $g_- := \frac{1}{2}(g - Jg^T J)$ is in $\mathfrak{wsp}(V)$. Obviously, this decomposition corresponds to expressing the matrix Jg as a sum of symmetric and skew-symmetric matrices.

We list some properties of $\mathfrak{wsp}(V)$:

([1]) A better name would be *skew-symplectic* or *anti-symplectic*, but these are reserved for other notions.

Proposition D.4. *Let $g, h \in \mathfrak{wsp}(V)$. The following properties are satisfied:*

(i) *Write the additive Jordan decomposition for g:*

$$g = g_s + g_n$$

where g_s is semisimple and g_n is nilpotent. Then both g_s and g_n are in $\mathfrak{wsp}(V)$.

(ii) *For $\lambda \in \mathbb{C}$, denote by V_λ the λ-eigenspace of g. For any two different eigenvalues $\lambda_1, \lambda_2 \in \mathbb{C}$, V_{λ_1} is ω-perpendicular to V_{λ_2}.*

(iii) *If g is semisimple, then each space V_λ is symplectic.* ■

D.1.5. Standard symplectic structure on $W \oplus W^*$. Let W be any finite-dimensional vector space. Set $V := W \oplus W^*$. There is a canonical symplectic form on V,

$$\omega((v,\alpha),(w,\beta)) := \beta(v) - \alpha(w).$$

If $a_1, \dots, a_n$ is any basis of W and $\lambda_1, \dots, \lambda_n$ is the dual basis of W^*, then

$$a_1, \dots, a_n, \lambda_1, \dots, \lambda_n$$

is a symplectic basis of V. In particular, we have the natural embedding

$$\mathbf{GL}(W) \hookrightarrow \mathbf{Sp}(V), \qquad A \mapsto A \oplus (A^{-1})^T.$$

We note the following elementary lemma:

Lemma D.5. *Let $L \subset W$ be any linear subspace. Then $L \oplus \ker(W^* \to L^*) \subset V$ is a Lagrangian subspace.* ■

D.2. Symplectic manifolds and their subvarieties. Symplectic manifolds will serve us to understand some geometric and algebraic structures of the symplectisations of contact manifolds (see §E.3.1).

A complex manifold or a smooth complex algebraic variety Y is a *symplectic manifold* if there exists a global closed holomorphic 2-form $\omega \in H^0(\Omega^2 Y)$ with $\mathrm{d}\omega = 0$ which restricted to every fibre is a symplectic form on the tangent space. In other words, $\omega^{\wedge^n}$ is a nowhere vanishing section of $K_Y = \Omega^{2n} Y$. The form ω is called a *symplectic form on Y*.

As in the case of a vector space, the symplectic form determines an isomorphism

$$\tilde{\omega}: \ TY \xrightarrow{\simeq} T^*Y, \qquad v \mapsto \omega(v, \cdot).$$

The theory of compact (or projective) complex symplectic manifolds is well developed and has a lot of beautiful results (see for example [Leh04], [Huy03] and references therein). Yet here we will only use some non-compact examples as a tool for studying contact manifolds and we will only need a few of their basic properties. Also some extensions of the symplectic structure to the singularities of Y are studied, but we are interested only in the case where Y is smooth.

D.2.1. Subvarieties of a symplectic manifold. Let (Y, ω) be a symplectic manifold. For a subvariety $X \subset Y$ we say X is respectively

(i) *isotropic*, (ii) *coisotropic*, (iii) *Lagrangian*, if and only if there exists an open dense subset U (equivalently, for any open dense subset U) of smooth points of X such that for every $x \in U$ the tangent space $T_x X \subset T_x Y$ is respectively

(i) isotropic, (ii) coisotropic, (iii) Lagrangian.

Or equivalently, for every $x \in U$ the conormal space $N^*_x X \subset T^*_x Y$ is respectively

(i) coisotropic, (ii) isotropic, (iii) Lagrangian.

Note that a subvariety is Lagrangian if and only if it is isotropic (or coisotropic) and the dimension is equal to n.

D.2.2. Examples. The following examples are important for our considerations, as they will appear as symplectisations of projective contact manifolds (see §E.3.1).

The affine space. Our key example is the simplest possible: an affine space of even dimension. So assume (V, ω) is a symplectic vector space of dimension $2n$. Then take the affine space $\mathbb{A}^{2n}$ of the same dimension, whose tangent space at every point is V and globally $T\mathbb{A}^{2n} = \mathbb{A}^{2n} \times V$. Then ω trivially extends to the product and it is a symplectic form on $\mathbb{A}^{2n}$.

By an abuse of notation, we will denote the affine space by V as well (so in particular a 0 is fixed in the affine space and the action of $\mathbb{C}^*$ by homotheties is chosen). In this setup, the form ω is homogeneous of weight 2 (see §B.6).

Products. Assume Y_1 and Y_2 are two symplectic manifolds with symplectic forms ω_1 and ω_2 respectively. Clearly $Y_1 \times Y_2$ is a symplectic manifold with the symplectic form $p_1^*\omega_1 + p_2^*\omega_2$, where the p_i's are the appropriate projections.

Next, let $X_i \subset Y_i$ be two subvarieties. Both the X_i's are respectively

(i) isotropic, (ii) coisotropic, (iii) Lagrangian,

if and only if the product $X_1 \times X_2 \subset Y_1 \times Y_2$ is respectively

(i) isotropic, (ii) coisotropic, (iii) Lagrangian.

Cotangent bundle. Let M be a complex manifold or a smooth algebraic variety of dimension n. Set Y to be the total space of the cotangent vector bundle T^*M and let $p : Y \to M$ be the projection map. If $x_1, \dots, x_n$ are local coordinates on $U \subset M$, then $x_1, \dots, x_n, y_1 = \mathrm{d}x_1, \dots, y_n = \mathrm{d}x_n$ form the local coordinates on $Y|_U$. Then we can set

$$\omega|_U := \mathrm{d}x_1 \wedge \mathrm{d}y_1 + \dots + \mathrm{d}x_n \wedge \mathrm{d}y_n \in H^0(U, \Omega^2 Y),$$

and these glue to a well defined symplectic form $\omega \in H^0(Y, \Omega^2 Y)$. This symplectic form is homogeneous of weight 1 with respect to the usual $\mathbb{C}^*$-action on cotangent spaces.

Since for $m \in M$, $x \in T^*_m M$ we have $T_{(m,x)} Y = T_m M \oplus T^*_m M$ this example of symplectic manifold generalises the standard symplectic structure on $W \oplus W^*$ (see §D.1.5). The following example generalises Lemma D.5:

Example D.6. *Let $Z \subset M$ be any subvariety. Define $\hat{Z}^\# \subset Y$ to be the* conormal variety *to Z, i.e. the closure of the union of the conormal spaces to smooth points of Z:*

$$\hat{Z}^\# := \overline{N^* Z_0 / M}.$$

Then $\hat{Z}^\#$ is a Lagrangian subvariety in Y.

Proof. Let $z \in Z$ be a smooth point and let $x \in N_z^* Z_0/M$. Then one can choose local coordinates on M around z and a local trivialisation of the cotangent bundle T^*M such that

$$T_x \hat{Z}^{\#} = T_z Z \oplus N_z^* Z_0/M \subset T_z M \oplus T_z^* M.$$

This is a Lagrangian subspace by Lemma D.5. ∎

LEMMA D.7. *Conversely, assume M is a smooth algebraic variety and Y is the total space of T^*M. Moreover, assume $X \subset Y$ is an irreducible closed Lagrangian subvariety invariant under the $\mathbb{C}^*$-action on Y. If $Z = \overline{p(X)}$, then $X = \hat{Z}^{\#}$.*

Proof. Let $x \in X$ be a general point and let $z := p(x)$. So x is a point in $T_z^* M$ and

$$T_x Y = T_z M \oplus T_z^* M.$$

Since X is $\mathbb{C}^*$-invariant, under the above identification

$$(0, x) \in T_x X \subset T_x Y.$$

We want to prove that $(0, x) \in N_z^* Z/M$ and this will follow if we prove

$$T_x X \cap T_z^* M = N_z^* Z/M.$$

By Lemma B.2 the map $\mathrm{D}p : T_x X \to T_z Z$ is surjective, so

$$T_x X + T_z^* M = T_z Z \oplus T_z^* M.$$

Since X is Lagrangian, we also have the dual equality

$$T_x X \cap T_z^* M = (T_x X)^{\perp_\omega} \cap (T_z^* M)^{\perp_\omega} = (T_x X + T_z^* M)^{\perp_\omega} = (T_z Z \oplus T_z^* M)^{\perp_\omega} = N_z^* Z/M.$$

Hence $T_x X \cap T_z^* M = N_z^* Z/M$ as claimed and therefore $x \in N_z^* Z/M$. Since x was a general point of X and both X and Z are irreducible, we have $X \subset \hat{Z}^{\#}$ and by dimension count, $X = \hat{Z}^{\#}$. ∎

Adjoint and coadjoint orbits. Let G be a semisimple complex Lie group and consider the coadjoint action on the dual of its Lie algebra $\mathfrak{g}^*$. Let Y be an orbit of this action. The tangent space at $\xi \in Y$ is naturally isomorphic to $\mathfrak{g}/Z(\xi)$, where

$$Z(\xi) = \{v \in \mathfrak{g} \mid \forall w \in \mathfrak{g} \ \ \xi([v, w]) = 0\}.$$

Here $[v, w]$ denotes the Lie bracket in $\mathfrak{g}$. For $v, w \in \mathfrak{g}$ let $[v]$ and $[w]$ be the corresponding vector fields on Y determined by v and w. We define

$$\omega_\xi([v], [w]) := \xi([v, w]).$$

Then ω is a symplectic form on Y, which is called the Kostant–Kirillov form (see for instance [Bea98, (2.1)]).

Now assume G is simple and Y is invariant under homotheties (for instance Y is the unique minimal nonzero orbit—see [Bea98, Props. 2.2 and 2.6]). Then the actions of G and $\mathbb{C}^*$ commute (because G acts on $\mathfrak{g}^*$ by linear automorphisms, $\mathbb{C}^*$ via homotheties and every linear map commutes with a homothety). Therefore vector fields of the form $[v]$ for some $v \in \mathfrak{g}$ are homogeneous of weight 0 and hence

$$(\lambda_t^* \omega)_\xi([v], [w]) = \omega_{\lambda_t(\xi)}([v], [w]) = t\xi([v, w]) = t\omega_\xi([v], [w]),$$

i.e. ω is homogeneous of weight 1.

We can identify $\mathfrak{g}^*$ and $\mathfrak{g}$ by the Killing form (see [Hum75]), so equally well we can consider adjoint orbits. Therefore if Y is as above, then it is isomorphic to a $\mathbb{C}^*$-bundle over an adjoint variety (see §A.1.1). More precisely, Y is a symplectisation (see §E.3.1) of the adjoint variety.

Open subsets. Let (Y, ω) be a symplectic manifold and let U be an open subset. Then $(U, \omega|_U)$ is again a symplectic manifold.

D.3. The Poisson bracket. The Poisson bracket is an important algebraic structure of a symplectic manifold. In Corollary E.12 we observe that given a contact manifold and its symplectisation, the Poisson bracket descends from the symplectisation to a bracket on a specific sheaf of rings on the contact manifold. Moreover, this descended structure is strictly related to the automorphisms of the contact manifold (see Theorem E.13).

Let (Y, ω) be a symplectic manifold and let $\mathcal{O}_Y$ be the sheaf of holomorphic (or algebraic) functions on Y. Given $f, g \in H^0(U, \mathcal{O}_Y)$ let $\xi_g \in H^0(U, TY)$ be the unique vector field such that $\omega(\xi_g) = \mathrm{d}g$. Then we set

$$\{f, g\} := \mathrm{d}f(\xi_g),$$

or equivalently

$$\{f, g\}(x) := \omega_x^\vee(\mathrm{d}g_x, \mathrm{d}f_x).$$

The bilinear skew-symmetric map $\{\cdot,\cdot\} : \mathcal{O}_Y \times \mathcal{O}_Y \to \mathcal{O}_Y$ is called the *Poisson bracket*.

Lemma D.8. *The Poisson bracket satisfies the Jacobi identity and therefore makes $\mathcal{O}_Y$ into a sheaf of Lie algebras. The Poisson bracket and the standard ring multiplication on $\mathcal{O}_Y(U)$ are related by the following Leibniz rule:*

$$\{fg, h\} = f\{g, h\} + g\{f, h\}.$$

Proof. See for example [Arn74, §40]—the proof is identical to the real case. ■

The Poisson bracket is determined by the symplectic form and moreover, it is defined locally. Hence we have the following property:

Proposition D.9. *Assume (Y, ω) and (Y', ω') are two symplectic manifolds of dimension $2n$. Assume moreover that we have a finite covering map $\psi : Y \to Y'$ such that $\psi^*\omega' = \omega$. Then the Poisson structures are compatible: for $f, g \in \mathcal{O}_{Y'}$ we have $\psi^*\{f, g\} = \{\psi^* f, \psi^* g\}$.* ■

Theorem D.10. *Assume Y is a symplectic manifold.*

(i) *Suppose $X \subset Y$ is a coisotropic subvariety. Then the sheaf of ideals $\mathcal{I}(X) \subset \mathcal{O}_Y$ is a subalgebra with respect to the Poisson bracket.*
(ii) *Conversely, suppose $X \subset Y$ is a closed, generically reduced subscheme and that $\mathcal{I}(X)$ is preserved by the Poisson bracket. Then the corresponding variety X_{red} is coisotropic.*

Versions of the theorem can be found in [Cou95, Prop. 11.2.4] and [Buc06, Thm. 4.2]. We follow more or less the proof from [Buc06]:

Proof. Let X_0 be the locus of smooth points of X. We must show that $\omega^\vee|_{N^*X_0/Y} \equiv 0$ if and only if $\mathcal{I}(X)$ is a Lie subalgebra sheaf in $\mathcal{O}_Y$.

Suppose that $x \in X_0$ is any point, $U \subset Y$ is an open neighbourhood of x and that $f, g \in H^0(U, \mathcal{I}(X))$ are some functions vanishing on X. Then

$$\mathrm{d}f_x, \mathrm{d}g_x \in N^*X_0/Y.$$

If $\omega^\vee|_{N^*X_0/Y} \equiv 0$, then

$$\{f, g\}(x) = \omega_x^\vee(\mathrm{d}g_x, \mathrm{d}f_x) = 0,$$

i.e. $\{f, g\}|_{X_0} = 0$, so extending the equality to the closure of X_0 we get

$$\{f, g\} \in H^0(U, \mathcal{I}(X)).$$

Hence $\mathcal{I}(X)$ is a Lie subalgebra.

Conversely, if $\mathcal{I}(X)$ is a Lie subalgebra, then

$$\omega^\vee(\mathrm{d}g_x, \mathrm{d}f_x) = \{f, g\}(x) = 0.$$

Since the map

$$H^0(U, \mathcal{I}(X)) \to N_x^* X_0/Y, \quad f \mapsto \mathrm{d}f_x,$$

is an epimorphism of vector spaces for each $x \in X_0$ and for U sufficiently small, we have $\omega|_{N^*X_0/Y} \equiv 0$. ∎

D.3.1. Properties of Poisson bracket. In our considerations on contact manifolds and their various subvarieties we will need three lemmas of this subsection. These lemmas refer to Proposition D.10—we have seen that there is a relation between coisotropic varieties and Lie subalgebras of $\mathcal{O}_Y$ that are ideals under the standard ring multiplication.

The first lemma claims that to test if an ideal is a subalgebra it is enough to test it on an appropriate open cover of Y.

Lemma D.11. *Let Y be a symplectic manifold and let $\mathcal{I} \lhd \mathcal{O}_Y$ be a coherent sheaf of ideals. Then $\mathcal{I}$ is preserved by the Poisson bracket if and only if there exists an open cover $\{U_i\}$ of Y such that for each i:*

- *if $V \subset U_i$ is another open subset, then the functions in $H^0(V, \mathcal{O}_Y)$ are determined by the functions in $H^0(U_i, \mathcal{O}_Y)$—this means that if Y is an algebraic variety (respectively, analytic space), then the elements of $H^0(V, \mathcal{O}_Y)$ can all be written as quotients (respectively, Taylor series) of elements of $H^0(U_i, \mathcal{O}_Y)$; this holds for instance if U_i is affine or if U_i is biholomorphic to a disk $D^{4n} \subset \mathbb{C}^{2n}$ or to $D^{4n-2} \times \mathbb{C}^*$;*
- *the ideal $H^0(U_i, \mathcal{I}) \lhd H^0(U_i, \mathcal{O}_Y)$ is preserved by the Poisson bracket.*

Proof. One implication is obvious, while the other follows from the Leibniz rule (see Lemma D.8) and from elementary properties of coherent sheaves. ∎

The second lemma asserts that for an isotropic subvariety X, only functions constant on X can preserve $\mathcal{I}(X)$ under Poisson multiplication.

Lemma D.12. *Assume Y is a symplectic manifold and X is a closed irreducible isotropic subvariety. Let $h \in H^0(Y, \mathcal{O}_Y)$ be any function such that*

$$\{h|_U, H^0(U, \mathcal{I}(X))\} \subset H^0(U, \mathcal{I}(X)) \quad \text{for any open subset } U \subset Y.$$

Then h is constant on X.

Proof. Choose an arbitrary $x \in X_0$, a small enough open neighbourhood $U \subset Y$ of x, and take any $f \in H^0(U, \mathcal{I}(X))$. Since $\{h|_U, f\} \in H^0(U, \mathcal{I}(X))$,

$$0 = \{h|_U, f\}(x) = \omega(\mathrm{d}f_x, \mathrm{d}h_x),$$

and since U can be taken so small that $\{\mathrm{d}f_x \mid f \in H^0(U, \mathcal{I}(X))\}$ span the conormal space we have

$$\mathrm{d}h_x \in (N^*_x X/Y)^{\perp_\omega} \overset{X \text{ isotropic}}{\subset} N^*_x X/Y.$$

So $\mathrm{d}h$ vanishes on TX_0 and hence h is constant on X. ■

Finally, the third lemma states that for isotropic X, very few subvarieties $S \subset X$ can have the property that $\{\mathcal{I}(S), \mathcal{I}(X)\} \subset \mathcal{I}(S)$.

Lemma D.13. *Assume Y is a symplectic manifold, X is a closed irreducible isotropic subvariety and $S \subset X$ is a closed subvariety. If $\{\mathcal{I}(S), \mathcal{I}(X)\} \subset \mathcal{I}(S)$, then either S is contained in the singular locus of X, or X is Lagrangian and $S = X$.*

Proof. The proof goes along the lines of the proof of [Buc06, Thm. 5.8]. Suppose S is not contained in the singular locus of X, so that a general point $s \in S$ is a smooth point of both X and S. Let $U \subset Y$ be an open neighbourhood of s. Then for all $f \in H^0(U, \mathcal{I}(S))$ and $g \in H^0(U, \mathcal{I}(X))$,

$$0 = \{f, g\}(s) = \omega(\mathrm{d}f_s, \mathrm{d}g_s), \tag{D.14}$$

so

$$\begin{aligned} N^*_s X/Y &= \operatorname{span}\{(\mathrm{d}g)_s \mid g \in H^0(U, \mathcal{I}(X))\} \subset (N^*_s S/Y)^{\perp_\omega} \quad \text{by (D.14)} \\ &\subset (N^*_s X/Y)^{\perp_\omega} \subset N^*_s X/Y \quad \text{because } X \text{ is isotropic.} \end{aligned}$$

Therefore all inclusions are equalities and in particular

$$\operatorname{codim} S = \operatorname{codim} X$$

and hence $S = X$. Moreover, $(N^*_s X/Y)^{\perp_\omega} = N^*_s X/Y$, where s is a general point of X, so X is Lagrangian. ■

D.3.2. Homogeneous symplectic form

Lemma D.15. *Assume (Y, ω) is a symplectic manifold with a $\mathbb{C}^*$-action and ω is homogeneous. Let $U \subset Y$ be a $\mathbb{C}^*$-invariant open subset and let $f, g \in H^0(U, \mathcal{O}_Y)$ be homogeneous functions. Then $\{f, g\}$ is homogeneous of weight* $\mathrm{wt}(f) + \mathrm{wt}(g) - \mathrm{wt}(\omega)$.

Proof. Let $\xi_g \in H^0(U, TY)$ be a vector field with $\omega(\xi_g) = \mathrm{d}g$. By Lemma B.1(i) we have $\mathrm{wt}(\xi_g) = \mathrm{wt}(g) - \mathrm{wt}(\omega)$ and since $\{f, g\} = (\mathrm{d}f)(\xi_g)$, the claim follows from Lemma B.1(i), (iii). ■

D.3.3. Example: Veronese map of degree 2. The following example is important for our considerations, as it proves that for the contact manifold $\mathbb{P}^{2n-1}$, we can equally well consider the Poisson structure on $\bigoplus_{i \in \mathbb{N}} \operatorname{Sym}^i \mathbb{C}^{2n}$ (as we do in [Buc03] and [Buc06]) and the Poisson structure on $\bigoplus_{i \in 2\mathbb{N}} \operatorname{Sym}^i \mathbb{C}^{2n}$ (as naturally will arise from the point of view of contact manifolds—see §E.3.1). Also this example will be used to illustrate that every contact structure on $\mathbb{P}^{2n-1}$ comes from a symplectic structure on $\mathbb{C}^{2n}$. Moreover,

Y' defined below is the minimal adjoint orbit (see §D.2.2) for the simple group $\mathbf{Sp}_{2n}$. This simple Lie group and its minimal adjoint orbit have quite exceptional behaviour (see Table A.1.2) and thus we find it worthwhile to explain this example in more detail.

Let (V,ω) be a symplectic vector space. We let

$$\mathbb{C}[V] = \mathbb{C}[x_1,\dots x_{2n}] = \bigoplus_{i\in\mathbb{N}} \operatorname{Sym}^i V^*$$

be the coordinate ring of V. Also consider

$$\mathcal{S} := \mathbb{C}[V]^{\text{even}} = \bigoplus_{i\in 2\mathbb{N}} \operatorname{Sym}^i V^*$$

and let $Y' := \operatorname{Spec}\mathcal{S} \setminus \{0\}$. Then we have the following $\mathbb{Z}_2$-covering map:

$$\psi : V\setminus\{0\} \to Y',$$

which is the restriction of the map induced by $\mathcal{S} \hookrightarrow \mathbb{C}[V]$. This is the underlying map of the second Veronese embedding of $\mathbb{P}(V)$. In the language of §C.3, we have $Y' = (\mathcal{O}_{\mathbb{P}(V)}(2))^{\bullet}$ and $V\setminus\{0\} = (\mathcal{O}_{\mathbb{P}(V)}(1))^{\bullet}$.

The symplectic form ω is $\mathbb{Z}_2$-invariant:

$$\omega(-v,-w) = \omega(v,w),$$

hence it descends to a symplectic form ω' on Y', making Y' a symplectic manifold, such that

$$\psi^*\omega' = \omega.$$

The natural gradings on $\mathbb{C}[V]$ and on $\mathcal{S}$ induce actions of $\mathbb{C}^*$ on $V\setminus\{0\}$ and on Y' (note that the action on Y' is not faithful, its kernel is $\mathbb{Z}_2$) and ψ is equivariant with respect to these actions.

Corollary D.16. *With the setup as above, the form ω' is homogeneous of weight 2 with respect to the $\mathbb{C}^*$-action described above, so it is of weight 1 with respect to the faithful action of $\mathbb{C}^*/\mathbb{Z}_2 \simeq \mathbb{C}^*$. Conversely, if ω' is a homogeneous symplectic form on Y' of weight 2, then $\psi^*\omega'$ is a constant symplectic form on $V\setminus\{0\}$.*

Proof. This follows from Lemma B.1(ii) and the characterisation of constant forms on an affine space in §B.6. ∎

Corollary D.17. *The Poisson bracket on $\mathcal{S}$ induced by ω' is the restriction of the Poisson bracket on $\mathbb{C}[V]$ induced by ω.*

Proof. This follows immediately from Proposition D.9.

E. Contact geometry

Projective space seems to be the most standard example of a projective variety. Yet, as a contact manifold, the projective space of odd dimension is the most exceptional among exceptional examples. As a consequence, the study of its Legendrian subvarieties is quite complicated and very interesting. We start our considerations from this case. Further we generalise to other contact manifolds.

E.1. Projective space as a contact manifold. Let (V,ω) be a symplectic vector space and let $\mathbb{P}(V)$ be its naive projectivisation. Then for every $[v]\in\mathbb{P}(V)$ the tangent space $T_{[v]}\mathbb{P}(V)$ is naturally isomorphic to the quotient $V/[v]$. Let $F=F_{\mathbb{P}(V)}\subset T\mathbb{P}(V)$ be a corank 1 vector subbundle defined fibrewise:

$$F_{[v]} := ([v]^{\perp_\omega})/[v].$$

Also let L be the quotient line bundle, so that we have the following short exact sequence:

$$0\to F\to T\mathbb{P}(V)\xrightarrow{\theta} L\to 0.$$

We say that F (respectively θ) is the *contact distribution* (respectively the *contact form*) *associated with the symplectic form* ω.

By §D.1.3 the vector space F_p carries a natural symplectic structure ω_{F_p}. By Proposition C.1(i), $\mathrm{d}\theta$ gives a well defined twisted 2-form on F:

$$\mathrm{d}\theta := \textstyle\bigwedge^2 F\to L.$$

Proposition E.1. *With an appropriate choice of local trivialisation of L, for every $p\in\mathbb{P}(V)$ one has $\omega_{F_p}=(\mathrm{d}\theta)_p$. In particular, $\mathrm{d}\theta$ is nowhere degenerate and it determines an isomorphism*

$$F\simeq F^*\otimes L.$$

Moreover, $L\simeq\mathcal{O}_{\mathbb{P}(V)}(2)$.

Proof. See also [LeB95, Ex. 2.1]. Let $x_1,\dots,x_n,y_1,\dots,y_n$ be some symplectic coordinates on V. Then the ω-perpendicular space to $(a_1,\dots,a_n,b_1,\dots,b_n)$ is given by the equation

$$b_1x_1+\dots+b_nx_n-a_1y_1-\dots-a_ny_n=0.$$

We look for a twisted 1-form θ on $\mathbb{P}(V)$ whose kernel at each point is exactly as above. This is for instance satisfied by

$$\theta=\tfrac{1}{2}(-y_1\mathrm{d}x_1-\dots-y_n\mathrm{d}x_n+x_1\mathrm{d}y_1+\dots+x_n\mathrm{d}y_n).$$

The ambiguity is only in the choice of the scalar coefficient—we choose $\frac12$ in order to acquire the right formula for $\mathrm{d}\theta$. Choose an affine piece $U\subset\mathbb{P}(V)$, say where $x_1=1$. On U we have

$$\theta|_U=\tfrac{1}{2}(-y_2\mathrm{d}x_2-\dots-y_n\mathrm{d}x_n+\mathrm{d}y_1+x_2\mathrm{d}y_2+\dots+x_n\mathrm{d}y_n)$$

and then

$$\mathrm{d}\theta|_U=\mathrm{d}x_2\wedge\mathrm{d}y_2+\dots+\mathrm{d}x_n\wedge\mathrm{d}y_n.$$

On the other hand, fixing $p\in U$, $p=[1,a_2,\dots,a_n,b_1,\dots,b_n]$, we have

$$\begin{aligned}F_p=\{(x_1,\dots,x_n,y_1,\dots,y_n)\in V\mid b_1x_1+b_2x_2+\dots+b_nx_n&\\ -y_1-a_2y_2-\dots-a_ny_n=0\}/[p].&\end{aligned}$$

Therefore F is the image under the projection $V\to V/[p]$ of

$$\hat{F}_p:=\{(0,x_2,\dots,x_n,\ a_2y_2+\dots+a_ny_n-b_2x_2-\dots-b_nx_n,y_2,\dots,y_n)\in V\}$$

and

$$\omega|_{\hat{F}_p} = \mathrm{d}x_2 \wedge \mathrm{d}y_2 + \cdots + \mathrm{d}x_n \wedge \mathrm{d}y_n.$$

To see that $L \simeq \mathcal{O}_{\mathbb{P}(V)}(2)$ take a section of $T\mathbb{P}(V)$, for instance $x_1 \frac{\partial}{\partial x_1}$. Then

$$\theta\left(x_1 \frac{\partial}{\partial x_1}\right) = -\frac{1}{2} x_1 y_1$$

is a section of L and hence $L \simeq \mathcal{O}_{\mathbb{P}(V)}(2)$. ∎

E.2. Legendrian subvarieties of projective space. Assume (V, ω) is a symplectic vector space of dimension $2n$.

In [Buc03], [Buc06], [Buc07], [Buc09] and [Buc08b] the author found it convenient to use the following definition:

Definition. We say that a subvariety $X \subset \mathbb{P}(V)$ is *Legendrian* if the affine cone $\hat{X} \subset V$ is a Lagrangian subvariety (see §D.2.1).

Yet the original definition is formulated in a slightly different, but equivalent manner:

Proposition E.2. *Let $X \subset \mathbb{P}(V)$ be a subvariety. The following conditions are equivalent:*

- *X is Legendrian;*
- *X is $F_{\mathbb{P}(V)}$-integrable and it is of pure dimension $n-1$.*

Proof. If X is $F_{\mathbb{P}(V)}$-integrable, then X is Legendrian by Propositions E.1 and C.1(iv). The other implication follows from the definition of $F_{\mathbb{P}(V)}$. ∎

E.2.1. Decomposable and degenerate Legendrian subvarieties

Definition. Let V_1 and V_2 be two symplectic vector spaces and let $X_1 \subset \mathbb{P}(V_1)$ and $X_2 \subset \mathbb{P}(V_2)$ be two Legendrian subvarieties. Now assume $V := V_1 \oplus V_2$ and $X := X_1 * X_2 \subset \mathbb{P}(V)$, i.e. X is the join of X_1 and X_2, that is, the union of all lines from X_1 to X_2. Now, clearly, the affine cone $\hat{X}$ is the product $\hat{X}_1 \times \hat{X}_2$ (where $\hat{X}_i$ is the affine cone of X_i), so by §D.2.2, X is Legendrian. In such a situation we say that X is a *decomposable Legendrian subvariety*. We say that a Legendrian subvariety in $\mathbb{P}(V)$ is *indecomposable* if it is not of the form for any non-trivial symplectic decomposition $V = V_1 \oplus V_2$.

Indecomposable Legendrian subvarieties have a more consistent description of their projective automorphism group (see Chapter F). On the other hand, decomposable Legendrian varieties (which usually themselves are badly singular) will be used to provide some very interesting families of examples of smooth Legendrian varieties (see Chapter I).

We say a subvariety of projective space is *degenerate* if it is contained in some hyperplane. Otherwise, we say it is *non-degenerate*. The following easy proposition in some versions is well known. The presented version comes from [Buc06, Thm. 3.4] but see also [LM07, Prop. 17(1)] or [Buc03, Tw. 3.16].

Proposition E.3. *Let $X \subset \mathbb{P}(V)$ be a Legendrian subvariety. Then the following conditions are equivalent:*

(i) *X is degenerate.*
(ii) *There exists a symplectic linear subspace $W' \subset V$ of codimension 2 such that $X' = \mathbb{P}(W') \cap X$ is a Legendrian subvariety in $\mathbb{P}(W')$ and X is a cone over X'.*
(iii) *X is a cone over some variety X'.*

In particular, degenerate Legendrian subvarieties are decomposable. ∎

E.3. Contact manifolds

Definition. Let Y be a complex manifold or smooth algebraic variety and fix a short exact sequence

$$0 \to F \to TY \xrightarrow{\theta} L \to 0$$

where $F \subset TY$ is a corank 1 subbundle of the tangent bundle. We say that Y is a *contact manifold* if the twisted 2-form

$$\mathrm{d}\theta : \bigwedge^2 F \to L$$

(see Proposition C.1(i)) is nowhere degenerate, so that $\mathrm{d}\theta_y$ is a symplectic form on F_y for every $y \in Y$. In such a case F is called the *contact distribution* on Y and θ is the *contact form* on Y.

Example E.4. By Proposition E.1, the projective space with the contact distribution associated with a symplectic form is a contact manifold.

The following properties are standard (see for instance [Bea98]):

Proposition E.5. *Every contact manifold Y has the following properties:*

(i) *The dimension of Y is odd.*
(ii) *Let $U \subset Y$ be an open subset, let $\mu_F \in H^0(U, F)$ be any section and let $\phi_{\mu_F} : F|_U \to L|_U$ be a map of sheaves:*
$$\forall \nu \in H^0(U, F) \quad \phi_{\mu_F}(\nu) := \theta([\mu_F, \nu]).$$
Then ϕ_{μ_F} is a map of $\mathcal{O}_U$-modules and the assignment $\mu_F \mapsto \phi_{\mu_F}$ is an isomorphism of $\mathcal{O}_Y$-modules:
$$F \simeq F^* \otimes L.$$
(iii) *The canonical divisor K_Y is isomorphic to $L^{\otimes(-n-1)}$. In particular, Y is a Fano variety if and only if L is ample.*

Proof. We only prove (ii), the other parts follow easily. The map ϕ_{μ_F} is a map of $\mathcal{O}_U$-modules by Proposition C.1(iii). By Proposition C.1(ii) we have

$$\phi_{\mu_F}(\nu) = \mathrm{d}\theta(\mu_F, \nu).$$

Since $\mathrm{d}\theta$ is non-degenerate, it follows that $\mu_F \mapsto \phi_{\mu_F}$ is indeed an isomorphism. ∎

E.3.1. Symplectisation. The following construction is standard—see e.g. [Arn74], [KPSW00], [Bea98].

Let $\mathbf{L}^\bullet$ be the principal $\mathbb{C}^*$-bundle as in §C.3. In §C.4 and §C.5 we studied in detail the properties of $\mathbf{L}^\bullet$ and an extension of the twisted form θ to $\mathbf{L}^\bullet$. We have an equivalence between contact structures on Y and symplectic homogeneous weight 1 structures on $\mathbf{L}^\bullet$:

Theorem E.6. *Let Y be a complex manifold or smooth algebraic variety with a line bundle L and the principal $\mathbb{C}^*$-bundle $\mathbf{L}^\bullet$ as in* §C.3.

- *If $\theta : TY \to L$ is a contact form, then* $\mathrm{d}\theta^\bullet$ (see §C.5) *is a homogeneous symplectic form on $\mathbf{L}^\bullet$ of weight* 1.
- *Conversely, assume ω is a symplectic form on $\mathbf{L}^\bullet$, which is homogeneous of weight* 1. *Then there exists a unique contact form $\theta : TY \to L$ on Y, such that $\omega = \mathrm{d}\theta^\bullet$.*

Proof. See Proposition C.16. ∎

If (Y, F) is a contact manifold, then the symplectic manifold $(\mathbf{L}^\bullet, \mathrm{d}\theta^\bullet)$ from the theorem is called the *symplectisation* of (Y, F).

Using the theorem and §D.2.2 we have the following examples of contact manifolds:

Example E.7. Let G be a simple group and let Y be the closed orbit in $\mathbb{P}(\mathfrak{g})$. Then Y is a contact manifold (compare with Conjecture A.2).

Example E.8. If $Y \simeq \mathbb{P}(T^*M)$, then let $L = \mathcal{O}_{\mathbb{P}(T^*M)}(1)$ and hence $\mathbf{L}^\bullet \simeq T^*M \backslash s_0$, where s_0 is the zero section and Y is a contact manifold.

Example E.9. If Y is a contact Fano manifold, then

$$Y \simeq \operatorname{Proj}\Big(\bigoplus_{m\in\mathbb{N}} H^0(Y, L^m)\Big), \quad \mathbf{L}^\bullet \simeq \operatorname{Spec}\Big(\bigoplus_{m\in\mathbb{N}} H^0(Y, L^m)\Big) \setminus \{0\},$$

where 0 is the point corresponding to the maximal ideal $\bigoplus_{m\geq 1} H^0(Y, L^m)$ (see [Gro61, §2.3]).

Example E.10. If $Y \simeq \mathbb{P}(V)$, then by Proposition E.5(b) we have

$$L \simeq \mathcal{O}_{\mathbb{P}(V)}(2).$$

Therefore $V\backslash\{0\}$ is a 2-to-1 unramified cover of $\mathbf{L}^\bullet$ (see §D.3.3). In particular, from Theorem E.6 and Corollary D.16 we conclude that every contact structure on $\mathbb{P}(V)$ is associated to some constant symplectic form ω on V (see §E.1).

Recall (Theorem A.1) that every contact projective manifold Y is either isomorphic to $\mathbb{P}(T^*M)$ or it is Fano with $b_2 = 1$. In the second case by Proposition E.5(iii) and the Kobayashi–Ochiai characterisation of projective space [KO73], either $Y \simeq \mathbb{P}(V)$ or $\operatorname{Pic} Y = \mathbb{Z} \cdot [L]$.

E.3.2. Contact automorphisms. Automorphisms of contact manifolds preserving the contact structure were also studied by LeBrun [LeB95] and Beauville [Bea98]. We use their methods to state slightly more general results about infinitesimal automorphisms. Finally, we globalise the automorphisms for projective contact manifolds.

Let Y be a contact manifold and let $\pi : \mathbf{L}^\bullet \to Y$ be the symplectisation as in §E.3.1. Also let $\mathcal{R}_L$ be as in §C.3.

Example E.11. If Y is a contact Fano manifold, then

$$H^0(Y, \mathcal{R}_L) = H^0(\mathbf{L}^\bullet, \mathcal{O}_{\mathbf{L}^\bullet}) = \Big(\bigoplus_{m\in\mathbb{N}} H^0(Y, L^m)\Big).$$

Since $Y = \operatorname{Proj}(H^0(Y, \mathcal{R}_L))$ (see Example E.9), the whole structure of Y as well as its global and local behaviour are determined by this ring of global sections. Hence in this case whatever is stated below for the sheaf $\mathcal{R}_L$ can be deduced from the analogous statement about $H^0(Y, \mathcal{R}_L)$ only.

Corollary E.12.

(i) *Let $f, g \in \mathcal{O}_{\mathbf{L}^\bullet}$ be two functions on $\mathbf{L}^\bullet$ homogeneous with respect to the action of $\mathbb{C}^*$. Then $\{f, g\}$ is also homogeneous and* $\operatorname{wt}\{f, g\} = \operatorname{wt} f + \operatorname{wt} g - 1$.

(ii) *The Poisson bracket descends to $\mathcal{R}_L$ and determines a bilinear map:*

$$H^0(U, L^{m_1}) \times H^0(U, L^{m_2}) \to H^0(U, L^{m_1+m_2-1}).$$

Proof. This follows from Corollary D.15. See also [LeB95, Rem. 2.3]. ■

We will refer to the Lie algebra structure on $\mathcal{R}_L$ defined above also as Poisson structure and denote the bracket by $\{\cdot, \cdot\}$. For $s \in H^0(U, L)$ let $\tilde{s}$ be the corresponding element in $H^0(\pi^{-1}(U), \mathbf{L}^\bullet) = \mathcal{R}_L(U)$.

By Corollary E.12 the invertible sheaf L has a Lie algebra structure, and it is crucial for our considerations that it is isomorphic to the sheaf $\mathfrak{aut}_F^{\inf}$ of infinitesimal automorphisms of Y preserving F (see §C.4 for more details):

$$\mathfrak{aut}_F^{\inf}(U) := \{\mu \in H^0(U, TY) \mid [\mu, F] \subset F\}.$$

Theorem E.13. *Let Y be a contact manifold, F be the contact distribution, θ be the contact form and let $U \subset Y$ be an open subset. Using the notation of* §C.2 *and* §C.4 *we have:*

(1) *$TY = \mathfrak{aut}_F^{\inf} \oplus F$ as sheaves of Abelian groups.*

(2) *The map of sheaves $\theta|_{\mathfrak{aut}_F^{\inf}} : \mathfrak{aut}_F^{\inf} \to L$ maps isomorphically the Lie algebra structure of $\mathfrak{aut}_F^{\inf}$ onto the Lie algebra structure of L given by the Poisson bracket.*

(3) *The following two Lie algebra representations of $\mathfrak{aut}_F^{\inf}$ on $\mathcal{O}_{\mathbf{L}^\bullet}$ are equal:*
- *The induced representation of $\mathfrak{aut}_F^{\inf}$ on $\mathbf{L}^\bullet$* (see §C.4).
- *The representation induced by the adjoint representation:*

$$\mu \in \mathfrak{aut}_F^{\inf}(U), f \in H^0(U, \mathcal{O}_{\mathbf{L}^\bullet}) \ \Rightarrow\ \mu.f := \{\widetilde{\theta(\mu)}, f\}.$$

Proof. The following proof of (1) is taken from [Bea98, Prop. 1.1], but see also [LeB95, Prop. 2.1].

To prove (1), take any $\mu \in H^0(U, TY)$ and consider the map of sheaves

$$F|_U \to L|_U, \qquad \nu \mapsto \theta([\mu, \nu]).$$

By Proposition C.1(iii) it is a map of $\mathcal{O}_Y|_U$-modules, hence an element of $H^0(U, F^* \otimes L)$. Let μ_F be the corresponding element of $H^0(U, F)$ (see Proposition E.5(ii)). By the definition of the isomorphism $F^* \otimes L \simeq F$, we have

$$\theta([\mu_F, \nu]) = \theta([\mu, \nu])$$

for every $\nu \in F|_U$, hence $[\mu - \mu_F, \nu] \in F|_U$. Therefore $\mu - \mu_F \in \mathfrak{aut}_F^{\inf}(U)$, so

$$\mu = \mu_F + (\mu - \mu_F)$$

gives the required splitting.

For (2) see also [Bea98, Prop. 1.6] and [LeB95, Rem. 2.3]. By (1), the map $\theta|_{\mathfrak{aut}_F^{\inf}}$ is an isomorphism of sheaves of Abelian groups. So it is enough to prove that $\theta|_{\mathfrak{aut}_F^{\inf}}$ preserves the Lie algebra structures. For every $\mu, \nu \in \mathfrak{aut}_F^{\inf}(U)$ denote by $\breve{\mu}$ and $\breve{\nu}$ the induced infinitesimal automorphisms of $\mathbf{L}^\bullet$ (see §C.4). We have

$$\begin{aligned}\{\widetilde{\theta(\mu)}, \widetilde{\theta(\nu)}\} &= (\mathrm{d}\theta^\bullet)^\vee(\mathrm{d}(\widetilde{\theta(\nu)}), \mathrm{d}(\widetilde{\theta(\mu)})) \\ &= (\mathrm{d}\theta^\bullet)^\vee(\mathrm{d}\theta^\bullet(\breve{\nu}), \mathrm{d}\theta^\bullet(\breve{\mu})) \quad \text{by Prop. C.19} \\ &= \mathrm{d}\theta^\bullet(\breve{\nu}, \breve{\mu}) \\ &= \widetilde{\theta([\mu,\nu])} \qquad\qquad\qquad\qquad \text{by Cor. C.20.}\end{aligned}$$

Hence $\theta|_{\mathfrak{aut}_F^{\inf}}$ preserves the Lie algebra structures.

Part (3) is local and since both representations satisfy the Leibniz rule (see (C.12) and Lemma D.8), it is enough to check the equality for multiplicative generators of $\mathcal{O}_{\mathbf{L}^\bullet}$. Locally, these might be taken for instance as sections of L and so (3) follows from (2). ∎

We underline that $\mathfrak{aut}_F^{\inf}$, as a subsheaf of TY, is not an $\mathcal{O}_Y$-submodule (see §C.4). So in particular the resulting splitting of the short exact sequence of sheaves of Abelian groups

$$0 \to F \to TY \xrightarrow{\theta} L \to 0$$

is not a splitting of vector bundles.

Turning to the global situation, assume Y is projective and let $\mathrm{Aut}(Y)$, $\mathrm{Aut}_F(Y)$ and $\mathfrak{aut}(Y)$, $\mathfrak{aut}_F(Y)$ denote, respectively, the group of automorphisms of Y, the group of automorphisms of Y preserving the contact structure, and their Lie algebras.

LeBrun [LeB95] and Kebekus [Keb01] observed that in the case of projective contact Fano manifolds with Picard group generated by L, the global sections of L are isomorphic to $\mathfrak{aut}(Y)$:

Corollary E.14. *Let Y be a projective contact manifold with contact distribution F.*

(i) *Then θ maps isomorphically $\mathfrak{aut}_F(Y)$ onto $H^0(Y, L)$.*
(ii) *If moreover Y is Fano with $\mathrm{Pic}(Y) = \mathbb{Z}[L]$, then $\mathrm{Aut}(Y) = \mathrm{Aut}_F(Y)$ and hence the Lie algebra $H^0(Y, L)$ is naturally isomorphic to $\mathfrak{aut}(Y)$.*

Proof. By Corollary C.8 we have $\mathfrak{aut}_F(Y) = \mathfrak{aut}_F^{\inf}(Y)$, so (i) follows from Theorem E.13(2).

On the other hand, (ii) follows from [Keb01, Cor. 4.5]. ∎

E.4. Legendrian subvarieties in contact manifolds

Definition. Let Y be a complex contact manifold with a contact distribution F. A subvariety $X \subset Y$ is *Legendrian* if X is F-integrable (i.e., $TX \subset F$) and $2\dim X + 1 = \dim Y$ (i.e., X has maximal possible dimension).

If $Y \simeq \mathbb{P}^{2n+1}$, then the above definition agrees with the definition in §E.2 by Proposition E.2. In general, we have analogous properties with V replaced by $\mathbf{L}^\bullet$:

Proposition E.15. *Let Y be a contact manifold with a contact distribution $F \subset TY$ and with symplectisation $\pi : \mathbf{L}^\bullet \to Y$. Assume $X \subset Y$ is a subvariety. Then:*

(a) *X is F-integrable if and only if $\pi^{-1}(X) \subset \mathbf{L}^\bullet$ is isotropic.*
(b) *X is Legendrian if and only if $\pi^{-1}(X) \subset \mathbf{L}^\bullet$ is Lagrangian.*

Proof. Part (a) follows from Lemma C.17, and (b) follows from (a). ∎

In the case of subvarieties of a symplectic manifold, we have three important types of subvarieties (isotropic, Legendrian and coisotropic). Also for subvarieties of contact manifold, in addition to F-integrable and Legendrian subvarieties, it is useful to consider the subvarieties corresponding to the coisotropic case:

Definition. In the setup of Proposition E.15, we say that X is *F-cointegrable* if $\pi^{-1}(X) \subset \mathbf{L}^\bullet$ is coisotropic.

Example E.16. Assume that $\widetilde{X} \subset \mathbf{L}^\bullet$ is irreducible and Lagrangian, and let X be the closure of $\pi(X) \subset Y$. Then X is F-cointegrable. If moreover $\widetilde{X}$ is not $\mathbb{C}^*$-invariant, then $\dim X = \frac{1}{2}(\dim Y + 1)$.

Corollary E.17. *If $Y = \mathbb{P}(T^*M)$ for some smooth algebraic variety M, and X is an algebraic Legendrian subvariety, then X is the conormal variety $Z^\#$ to some algebraic subvariety $Z \subset M$.*

Proof. This follows from Proposition E.15, Example E.8 and Lemma D.7. ∎

Let $\mathcal{R}_L = \pi_* \mathcal{O}_{\mathbf{L}^\bullet}$ be the sheaf of rings on Y defined in §C.3. For a subvariety $X \subset Y$, let $\widetilde{\mathcal{I}}(X) \lhd \mathcal{R}_L$ be the sheaf of ideals generated by those local sections of L^m that vanish on X. Then

$$\pi_* \mathcal{I}(\pi^{-1}(X)) = \widetilde{\mathcal{I}}(X) \tag{E.18}$$

where $\mathcal{I}(\pi^{-1}(X)) \lhd \mathcal{O}_{\mathbf{L}^\bullet}$ is the ideal sheaf of $\pi^{-1}(X)$. In this context, the meaning of Lemma D.11 is the following:

Lemma E.19. *With the notation as above, let $\mathcal{I} \lhd \mathcal{O}_{\mathbf{L}^\bullet}$ be a coherent sheaf of ideals. Then $\mathcal{I}$ is preserved by the Poisson bracket on $\mathcal{O}_{\mathbf{L}^\bullet}$ if and only if $\pi_*\mathcal{I}$ is preserved by the Poisson bracket on $\mathcal{R}_L$.* ∎

Hence we get the description of F-cointegrable subvarieties in terms of the Poisson bracket on $\mathcal{R}_L$:

Proposition E.20. *With the assumptions as above, a subvariety $X \subset Y$ is F-cointegrable if and only if $\widetilde{\mathcal{I}}(X)$ is preserved by the Poisson bracket on $\mathcal{R}_L$.*

Proof. The proposition combines (E.18), Theorem D.10 and Lemma E.19. ∎

Given a subvariety $X \subset Y$, we define $\mathfrak{aut}_F^{\inf}(\cdot, X)$ to be the sheaf of Lie algebras of those infinitesimal automorphisms of Y which preserve X and the contact distribution F (see also §C.4):

$$\mathfrak{aut}_F^{\inf}(U, X) := \{\mu \in H^0(U, TY) \mid [\mu, F] \subset F \text{ and } \forall f \in \mathcal{I}(X)|_U \ \ (\mathrm{d}f)(\mu) \in \mathcal{I}(X)|_U\}.$$

Further, let $\widetilde{\mathcal{I}}(X)_1 \subset L$ be the degree 1 part of the sheaf of homogeneous ideals $\widetilde{\mathcal{I}}(X)$. Since L is a line bundle with the action of $\mathfrak{aut}_F^{\inf}$ (see §C.4), choosing a local trivialisation and using the gluing property of sheaves we can replace $\mathcal{I}(X)$ in the definition of

$\mathfrak{aut}_F^{\inf}(\cdot, X)$ with $\widetilde{\mathcal{I}}(X)_1$:

$$\mathfrak{aut}_F^{\inf}(U,X) = \{\mu \in H^0(U,TY) \mid [\mu, F] \subset F \text{ and } \mu.\widetilde{\mathcal{I}}(X)_1|_U \subset \widetilde{\mathcal{I}}(X)_1|_U\} \tag{E.21}$$

where . denotes the induced action of $\mathfrak{aut}_F^{\inf}$ on L described in §C.4.

The following theorem establishes a connection between the infinitesimal automorphisms of a Legendrian variety and its ideal:

Theorem E.22. *Let Y be a contact manifold with a contact distribution F and let $\theta : TY \to L$ be the quotient map. Also let $U \subset Y$ be an open subset. Assume $X \subset Y$ is an irreducible subvariety.*

A. *If X is F-integrable, then $\theta(\mathfrak{aut}_F^{\inf}(U,X)) \subset H^0(U, \widetilde{\mathcal{I}}(X)_1)$.*
B. *If X is F-cointegrable, then $\theta(\mathfrak{aut}_F^{\inf}(U,X)) \supset H^0(U, \widetilde{\mathcal{I}}(X)_1)$.*
C. *If X is Legendrian, then $\theta(\mathfrak{aut}_F^{\inf}(U,X)) = H^0(U, \widetilde{\mathcal{I}}(X)_1)$.*

Proof. In the case A, choose any $\mu \in \mathfrak{aut}_F^{\inf}(U,X)$. We must prove that $\theta(\mu) \in H^0(U, \widetilde{\mathcal{I}}_1(X)$ or, equivalently, that

$$\widetilde{\theta(\mu)} \in H^0(\pi^{-1}(U), \mathcal{I}(\pi^{-1}(X)))$$

(recall that for a section $s \in H^0(U,L)$, we denote by $\tilde{s}$ the corresponding element in $H^0(\pi^{-1}(U), \mathcal{O}_{\mathbf{L}^\bullet}))$.

By (E.21) the action of μ preserves $\widetilde{\mathcal{I}}(X)|_U$ and hence also $\mathcal{I}(\pi^{-1}(X))|_{\pi^{-1}(U)}$. By Theorem E.13(3) this means that

$$\{\widetilde{\theta(\mu)}, \mathcal{I}(\pi^{-1}(X))|_{\pi^{-1}(U)}\} \subset \mathcal{I}(\pi^{-1}(X))|_{\pi^{-1}(U)}.$$

Moreover, $\pi^{-1}(X)$ is isotropic by Proposition E.15.

By Lemma D.12 the function $\widetilde{\theta(\mu)}$ is constant on $\pi^{-1}(X)$. But it is also $\mathbb{C}^*$-homogeneous of weight 1, so it must vanish on $\pi^{-1}(X)$. Therefore $\widetilde{\theta(\mu)} \in H^0(\pi^{-1}(U), \mathcal{I}(\pi^{-1}(X)))$ as claimed.

To prove B let $\mu \in \mathfrak{aut}_F^{\inf}(U)$ be an infinitesimal automorphism such that $\theta(\mu) \in \widetilde{\mathcal{I}}(X)_1$. By Proposition E.20,

$$\{\theta(\mu), \widetilde{\mathcal{I}}(X)\} \subset \widetilde{\mathcal{I}}(X),$$

so by Theorem E.13 (3) we have

$$\mu.\widetilde{\mathcal{I}}(X) \subset \widetilde{\mathcal{I}}(X)$$

(where . denotes the induced representation of $\mathfrak{aut}_F^{\inf}$ on $\mathbf{L}^\bullet$, see §C.4). Hence by (E.21) the infinitesimal automorphism μ is contained in $\mathfrak{aut}_F^{\inf}(U,X)$ and $H^0(U, \widetilde{\mathcal{I}}(X)_1) \subset \theta(\mathfrak{aut}_F^{\inf}(U,X))$ as claimed.

Part C is an immediate consequence of A and B. ∎

The following corollary says that when Y is projective, also the global automorphisms of a Legendrian subvariety can be understood in terms of the ideal of the variety. In particular, in (i) below, we generalise Theorem A.9.

Corollary E.23. *Let Y be a projective contact manifold, let F be the contact distribution and let X be a Legendrian subvariety. Let $\mathfrak{aut}(Y,X)$ (resp. $\mathfrak{aut}_F(Y,X)$) be the Lie algebra of group of automorphisms of Y preserving X (resp. preserving X and F). Then:*

(i) $\theta(\mathfrak{aut}_F(Y,X)) = H^0(Y, \widetilde{\mathcal{I}}(X)_1)$.
(ii) *If in addition* $\operatorname{Pic} Y = \mathbb{Z}[L]$*, then* $\theta(\mathfrak{aut}(Y,X)) = H^0(Y, \widetilde{\mathcal{I}}(X)_1)$.

Proof. This follows from Corollary E.14 and Theorem E.22C. ∎

In Chapter F we discuss the extension of Corollary E.23(ii) to $Y \simeq \mathbb{P}^{2n+1}$.

The following corollary generalises [Buc06, Thm. 5.8]:

Corollary E.24. *If Y is a projective contact manifold and $X \subset Y$ is an irreducible Legendrian subvariety such that $\widetilde{\mathcal{I}}(X)$ is generated by $H^0(Y, \widetilde{\mathcal{I}}(X)_1)$, then* $\operatorname{Aut}_F(Y,X)$ *acts transitively on the smooth locus of X. In particular, if X is in addition smooth, then X is a homogeneous space.*

Proof. If $S \subset X$, $S \neq X$, is a closed subvariety invariant under the action of $\operatorname{Aut}_F(Y,X)$, then by Theorem E.13(3) and by Corollary E.23(i),

$$\forall f \in H^0(Y, \widetilde{\mathcal{I}}(X)_1) \quad \{\widetilde{\mathcal{I}}(S), f\} \subset \widetilde{\mathcal{I}}(S).$$

Hence by the Leibniz rule and since $\widetilde{\mathcal{I}}(X)$ is generated by $H^0(Y, \widetilde{\mathcal{I}}(X)_1)$, we have

$$\{\mathcal{I}(\pi^{-1}(S)), \mathcal{I}(\pi^{-1}(X))\} \subset \mathcal{I}(\pi^{-1}(S)).$$

So by Lemma D.13, the variety S is contained in the singular locus of X.

Now let $O \subset X$ be the orbit of a smooth point under the action of $\operatorname{Aut}_F(Y,X)$. Then the closure $\bar{O}$ is not contained in the singular locus so by the above it must be all of X. Moreover, $\overline{O} \setminus O$ is a closed subset invariant under the action and not equal to X, so it is contained in the singular locus. So O is the whole smooth locus of X. ∎

We conclude this chapter by underlining that, unfortunately, the above results are proved only for automorphisms of Y that preserve the Legendrian subvariety X, and not simply for automorphisms of X.

F. Projective automorphisms of a Legendrian variety

The content of this chapter appeared in [Buc07].

We are interested in the following conjecture:

Conjecture F.1. *Let $X \subset \mathbb{P}^{2n-1}$ be an irreducible indecomposable Legendrian subvariety and let $G < \mathbb{P}\mathbf{GL}_{2n}$ be a connected subgroup of linear automorphisms preserving X. Then G is contained in the image of the natural map $\mathbf{Sp}_{2n} \to \mathbb{P}\mathbf{GL}_{2n}$.*

It is quite natural to believe that if a linear map preserves a form on a large number of linear subspaces, then it actually preserves the form (at least up to scalar). With this approach, Janeczko and Jelonek proved the conjecture in the case where the image of X under the Gauss map is non-degenerate in the Grassmannian of Lagrangian subspaces in $\mathbb{C}^{2n}$—see [JJ04, Cor. 6.4]. Unfortunately, this is not enough—for example $\mathbb{P}^1 \times Q_1 \subset \mathbb{P}^5$ has a degenerate image under the Gauss map and this is one of the simplest examples of smooth Legendrian subvarieties.

In §F.2 we prove Theorem A.11, i.e. that the conjecture holds for smooth X. This theorem, combined with Corollary E.23, gives us a good understanding of the group of projective automorphisms of a smooth Legendrian subvariety in $\mathbb{P}^{2n-1}$.

F.1. Discussion of assumptions. One obvious remark is that homotheties act trivially on $\mathbb{P}(V)$, but in general are not symplectic on V. Therefore, it is more convenient to think of conformal symplectomorphisms (see §D.1.4).

It is clear that if we hope for a positive answer to the question whether a projective automorphism of a Legendrian subvariety necessarily preserves the contact structure, then we must assume that our Legendrian variety is non-degenerate.

Another natural assumption is that X is irreducible—one can also easily produce a counterexample if we skip this assumption. Yet this still is not enough.

Let $X = X_1 * X_2 \subset \mathbb{P}(V_1 \oplus V_2)$ be a decomposable Legendrian variety. Then we can act via $\lambda_1 \operatorname{Id}_{V_1}$ on V_1 and via $\lambda_2 \operatorname{Id}_{V_2}$ on V_2—such an action will preserve X and in general it is not conformal symplectic. This explains why the assumptions of our Conjecture F.1 are necessary.

F.2. Preservation of contact structure. Let $X' \subset \mathbb{P}(V)$ be an irreducible, indecomposable Legendrian subvariety, let X be the affine cone over X' and X_0 be the smooth locus of X. Assume that G is the maximal connected subgroup in $\mathbf{GL}_{2n}$ preserving X. Let $\mathfrak{g} < \mathfrak{gl}_{2n}$ be the Lie algebra tangent to G. To prove the conjecture it would be enough to show that $\mathfrak{g}$ is contained in the Lie algebra $\mathfrak{csp}_{2n}$ tangent to conformal symplectomorphisms, i.e. the Lie algebra spanned by $\mathfrak{sp}_{2n}$ and the identity matrix Id_{2n}.

Recall from §D.1.4 the notion of weks-symplectic matrices.

Theorem F.2. *With the above notation the following properties hold:*

I. *The underlying vector space of $\mathfrak{g}$ decomposes into symplectic and weks-symplectic part:*
$$\mathfrak{g} = (\mathfrak{g} \cap \mathfrak{sp}(V)) \oplus (\mathfrak{g} \cap \mathfrak{wsp}(V)).$$

II. *If $g \in \mathfrak{g} \cap \mathfrak{wsp}(V)$, then g preserves every tangent space to X:*
$$\forall x \in X_0 \quad g(T_x X) \subset T_x X$$
and hence also
$$\forall t \in \mathbb{C}\ \forall x \in X_0 \quad T_{\exp(tg)(x)} X = \exp(tg)(T_x X) = T_x X.$$

III. *If $g \in \mathfrak{g} \cap \mathfrak{wsp}(V)$ is semisimple, then $g = \lambda \operatorname{Id}$ for some $\lambda \in \mathbb{C}$.*

IV. *Assume $0 \neq g \in \mathfrak{g} \cap \mathfrak{wsp}(V)$ is nilpotent and let $m \geq 1$ be an integer such that $g^{m+1} = 0$ and $g^m \neq 0$. Then $g^m(X)$ is always non-zero and is contained in the singular locus of X. In particular, X' is singular.*

In what follows we prove the four parts of Theorem F.2.

I. Decomposition into symplectic and weks-symplectic part. *Proof.* Take $g \in \mathfrak{g}$. Then for every $x \in X_0$ one has $g(x) \in T_x X$ and therefore
$$0 = \omega(g(x), x) = x^T g^T J x = \frac{1}{2} x^T (g^T J - Jg) x.$$

Hence the quadratic polynomial $f(x) := x^T(g^T J - Jg)x$ is identically zero on X and hence it is in the ideal of X. Therefore by maximality of G and Theorem A.9 the map $J(g^T J - Jg)$ is also in $\mathfrak{g}$. However,

$$J(g^T J - Jg) = Jg^T J + g,$$

so $Jg^T J \in \mathfrak{g}$ and both the symplectic and weks-symplectic components g_+ and g_- are in $\mathfrak{g}$. ■

From the point of view of the conjecture, the symplectic part is fine. We would only need to prove that $g_- = \lambda \operatorname{Id}$. So from now on we assume $g = g_- \in \mathfrak{wsp}(V)$.

II. Action on tangent space. *Proof.* Let $\gamma_t := \exp(tg)$ for $t \in \mathbb{C}$. Then $\gamma_t \in G$ and hence it acts on X. Choose a point $x \in X_0$ and two tangent vectors in the same tangent space $u, v \in T_x X$. Then clearly also $\gamma_t(u)$ and $\gamma_t(v)$ are contained in one tangent space, namely $T_{\gamma_t(x)}X$. Hence

$$\begin{aligned} 0 = \omega(\gamma_t(u), \gamma_t(v)) &= \omega((\operatorname{Id}_{2n} + tg + \cdots)u, (\operatorname{Id}_{2n} + tg + \cdots)v) \\ &= \omega(u, v) + t(\omega(gu, v) + \omega(u, gv)) + t^2(\ldots). \end{aligned}$$

In particular, the part of the expression linear in t vanishes, hence $\omega(gu, v) + \omega(u, gv) = 0$. Combining this with (D.2) we get

$$\omega(gu, v) = \omega(u, gv) = 0.$$

However, this implies that $gu \in (T_x X)^{\perp_\omega} = T_x X$. Therefore g preserves the tangent space at every smooth point of X and hence also γ_t preserves that space. ■

III. Semisimple part. Since G is an algebraic subgroup in $\mathbf{GL}(V)$, $\mathfrak{g}$ has the natural Jordan decomposition inherited from $\mathfrak{gl}(V)$, i.e. if we write the Jordan decomposition $g = g_s + g_n$, then $g_s, g_n \in \mathfrak{g}$ (see [Hum75, Thm. 15.3(b)]). Therefore by Proposition D.4(i), proving that for $g \in \mathfrak{g} \cap \mathfrak{wsp}(V)$ we have $g_s = \lambda \operatorname{Id}_{2n}$ and $g_n = 0$ would be enough to establish the conjecture.

Here we deal with the semisimple part.

Proof. Argue by contradiction. Let V_1 be an arbitrary eigenspace of g and let V_2 be the sum of the other eigenspaces. If $g \neq \lambda \operatorname{Id}_{2n}$, then both V_1 and V_2 are non-zero and by Proposition D.4(ii), (iii) they are ω-perpendicular, complementary symplectic subspaces of V. Let $x \in X_0$ be any point. Since g preserves $T_x X$ by part II it follows that $T_x X = (T_x X \cap V_1) \oplus (T_x X \cap V_2)$. But then both $T_x X \cap V_i \subset V_i$ are Lagrangian subspaces, hence have constant (independent of x) dimensions. Hence $T_x X_0 = (T_x X_0 \cap V_1) \oplus (T_x X_0 \cap V_2)$ is a sum of two vector bundles and Proposition B.3 implies that X is a product of two Lagrangian subvarieties, contradicting our assumption that X' is indecomposable. ■

IV. Nilpotent part—X' is singular

Lemma F.3. *Assume $X' \subset \mathbb{P}(V)$ is any closed subvariety preserved by the action of $\exp(tg)$ for some nilpotent endomorphism $g \in \mathfrak{gl}(V)$. If v is a point of the affine cone over X' and m is an integer such that $g^{m+1}(v) = 0$ and $g^m(v) \neq 0$, then $[g^m(v)] \in \mathbb{P}(V)$ is in X'.*

Proof. The point $[g^m(v)] \in \mathbb{P}(V)$ is just the limit of $[\exp(tg)(v)]$ as t goes to ∞. ∎

LEMMA F.4. *Assume $g \in \mathfrak{gl}(V)$ is nilpotent and $g^{m+1} = 0$, $g^m \neq 0$ for an integer $m \geq 1$. Let $X \subset V$ be an affine cone over some irreducible projective subvariety in $\mathbb{P}(V)$, which is preserved by the action of $\exp(tg)$, but is not contained in the set of fixed points. Assume that this action preserves the tangent space $T_x X$ at every smooth point x of X. If there exists a non-zero vector in V which is a smooth point of X contained in $g^m(X)$, then X is a linear subspace.*

Proof. Step 0: notation. We let Y be the closure of $g^m(X)$, so in particular Y is irreducible. By Lemma F.3, we know that $Y \subset X$. Let y be a general point of Y. Then by our assumptions y is a smooth point of both X and Y.

Next denote
$$W_y := (g^m)^{-1}(\mathbb{C}^* y).$$
You can think of W_y as the union of those lines in V (or points in the projective space $\mathbb{P}(V)$) which under the action of $\exp(tg)$ converge to the line spanned by y (or $[y]$) ([1]) as t goes to ∞. We also note that the closure $\overline{W_y}$ is a linear subspace spanned by an arbitrary element $v \in W_y$ and $\ker g^m$.

Also we let $F_y := W_y \cap X$, so that
$$F_y := (g^m|_X)^{-1}(\mathbb{C}^* y).$$
Finally, v from now on will always denote an arbitrary point of F_y.

Step 1: tangent space to X at points of F_y. Since y is a smooth point of X, also F_y consists of smooth points of X. This is because the set of singular points is closed and $\exp(tg)$ is invariant. By our assumptions $\exp(tg)$ preserves every tangent space to X and thus for every $v \in F_y$ we have
$$T_v X = T_{t^{-m}\exp(tg)(v)} X = T_{\lim_{t\to\infty}(t^{-m}\exp(tg)(v))} = T_y X.$$
So the tangent space to X is constant over the F_y and in particular $F_y \subset T_y X$.

Step 2: dimensions of Y and F_y. From the definitions of Y and y and by Step 1 we see that for any point $v \in F_y$,
$$T_y Y = \operatorname{im}(g^m|_{T_v X}) = \operatorname{im}(g^m|_{T_y X}).$$
Hence $\dim Y = \dim T_y Y = \operatorname{rk}(g^m|_{T_y X})$.

Since y was a general point of Y, we have
$$\dim Y + \dim F_y = \dim X + 1.$$
So $\dim F_y = \dim \ker(g^m|_{T_y X}) + 1$.

Step 3: the closure of F_y is a linear subspace. From the definition of F_y and Step 1 we know that $F_y \subset T_y X \cap W_y$ and
$$T_y X \cap \overline{W_y} = T_y X \cap \operatorname{span}\{v, \ker g^m\} = \operatorname{span}\{v, \ker(g^m|_{T_y X})\}.$$
Hence $\dim F_y = \dim T_y X \cap W_y$, so the closure of F_y is exactly $T_y X \cap \overline{W_y}$ and clearly this closure is contained in X. In particular, $\ker(g^m|_{T_y X}) \subset X$.

([1]) This statement is not perfectly precise, though it is true on an open dense subset. There are some other lines, which converge to $[y]$, namely those generated by $v \in \ker g^m$, but $g^k(v) = \lambda y$ for some $k < m$. We are not interested in those points.

Step 4: Y is contained in $\ker(g^m|_{T_yX})$. Let Z be $X \cap \ker g^m$. By Step 3 we know that $\ker(g^m|_{T_yX}) \subset Z$. Now we calculate the local dimension of Z at y:

$$\dim \ker(g^m|_{T_yX}) \le \dim_y Z \le \dim T_y Z \le \dim(T_y X \cap \ker g^m) = \dim \ker(g^m|_{T_yX}).$$

Since the first and the last entries are identical, we must have all equalities. In particular, the local dimension of Z at y is equal to the dimension of the tangent space to Z at y. So y is a smooth point of Z and therefore there is a unique component of Z passing through y, namely the linear space $\ker(g^m|_{T_yX})$. Since Y is contained in Z (because $\operatorname{im} g^m \subset \ker g^m$) and $y \in Y$, we must have $Y \subset \ker(g^m|_{T_yX})$.

Step 5: vary y. Recall that by Step 1 the tangent space to X is the same all over F_y. So it is also the same at every smooth point of X which falls into the closure of F_y. But by Step 4, Y is a subset of $\ker(g^m|_{T_yX})$, which is in the closure of F_y by Step 3. So the tangent space to X is the same for an open subset of points in Y. Now apply again Step 1 for different y's in this open subset to deduce that X has constant tangent space on a dense open subset of X. This is possible if and only if X is a linear subspace, which completes the proof of the lemma. ∎

Now part IV of the theorem follows easily:

Proof. By the assumptions of the theorem X is not contained in any hyperplane, so in particular X is not contained in $\ker g^m$. So by Lemma F.3 the image $g^m(X)$ contains points other than 0. Next by Lemma F.4 and part II of the theorem, since X cannot be a linear subspace, there can be no smooth points of X in $g^m(X)$. ∎

Smooth case. We conclude that parts I, III and IV of Theorem F.2 together with Proposition D.4(i) and [Hum75, Thm. 15.3(b)] imply Theorem A.11. We only note that a smooth Legendrian subvariety either is a linear subspace or is indecomposable.

F.3. Some comments. Conjecture F.1 is now reduced to the following special case not covered by Theorem F.2:

CONJECTURE F.5. *Let $X' \subset \mathbb{P}(V)$ be an irreducible Legendrian subvariety. Let g in $\mathfrak{wsp}(V)$ be a nilpotent endomorphism and m be an integer such that $g^m \neq 0$ and $g^{m+1} = 0$. Assume that the action of* $\exp(tg)$ *preserves X'. Assume moreover that X' is singular at all points of the image of the rational map $g^m(X')$. Then X' is decomposable.*

We also note the improved relation between projective automorphisms of a Legendrian subvariety and quadratic equations satisfied by its points:

COROLLARY F.6. *Let $X \subset \mathbb{P}(V)$ be an irreducible Legendrian subvariety for which Conjecture* F.1 *holds (for example, X is smooth). If $G < \mathbb{P}\mathbf{GL}(V)$ is the maximal subgroup preserving X, then* $\dim G = \dim \mathcal{I}_2(X)$, *where $\mathcal{I}_2(X)$ is the space of homogeneous quadratic polynomials vanishing on X.*

Proof. This follows immediately from the statement of the conjecture and Theorem A.9. ∎

Finally, it is important to note that part III of Theorem F.2 does not imply that every torus acting on an indecomposable, but singular Legendrian variety X' is contained in the image of $\mathbf{Sp}(V)$. It only says that the intersection of such a torus with the weks-symplectic part is always finite. Therefore if there is a non-trivial torus acting on X', there is also some non-trivial connected subgroup of $\mathbf{Sp}(V)$ acting on X' and also some quadratic equations in the ideal of X'.

G. Toric Legendrian subvarieties in projective space

The content of this chapter comes from [Buc07].

We apply Theorem A.11 to classify smooth toric Legendrian subvarieties. We choose appropriate coordinates to reduce this problem to some combinatorics (for surface case—see §G.2) and some elementary geometry of convex bodies (for higher dimensions—see §G.3). Eventually the proof of Theorem A.12 is obtained in Corollaries G.6 and G.10.

G.1. Classification of toric Legendrian varieties. Within this chapter, X is a toric subvariety of dimension $n-1$ in projective space of dimension $2n-1$. We assume it is embedded torically, so that the action of $T := (\mathbb{C}^*)^{n-1}$ on X extends to an action on the whole $\mathbb{P}^{2n-1}$, but we do not assume that the embedding is projectively normal. The notation is based on [Stu97] though we also use techniques of [Oda88]. We would like to understand when X can be Legendrian with respect to some contact structure on $\mathbb{P}^{2n-1}$, and in particular, when it can be a smooth toric Legendrian variety.

There are two reasons for considering non-projectively normal toric varieties here. The first one is that the new example we have found is not projectively normal. The second one is the conjecture [Stu97, Conj. 2.9], which says that a smooth, toric, projectively normal variety is defined by quadrics. We do not expect to produce a counterexample to this conjecture and on the other hand all smooth Legendrian varieties defined by quadrics are known to be just subadjoint varieties (see [Buc06, Thm. 5.11]).

In addition we assume that either X is smooth or at least the following condition is satisfied:

($\star$) The action of the torus T on $\mathbb{P}^{2n-1}$ preserves the contact structure on $\mathbb{P}^{2n-1}$. In other words, the image of $T \to \mathbb{P}\mathbf{GL}_{2n}$ is contained in the image of $\mathbf{Sp}_{2n} \to \mathbb{P}\mathbf{GL}_{2n}$.

In the case where X is smooth, the ($\star$) condition is always satisfied by Theorem A.11. But for some statements below we do not need non-singularity, so we only assume ($\star$).

Theorem G.1. *Let $X \subset \mathbb{P}^{2n-1}$ be a toric (in the above sense) non-degenerate Legendrian subvariety satisfying ($\star$). Then there exists a choice of symplectic coordinates on $\mathbb{C}^{2n}$ and coprime integers $a_0 \geq a_1 \geq \cdots \geq a_{n-1} > 0$ such that X is the closure of the image of the map*

$$T \ni (t_1, \ldots, t_{n-1}) \mapsto [-a_0 t_1^{a_1} t_2^{a_2} \ldots t_{n-1}^{a_{n-1}}, \quad a_1 t_1^{a_0},\ a_2 t_2^{a_0},\ \ldots,\ a_{n-1} t_{n-1}^{a_0},$$
$$t_1^{-a_1} t_2^{-a_2} \ldots t_{n-1}^{-a_{n-1}}, \quad t_1^{-a_0},\ t_2^{-a_0},\ \ldots,\ t_{n-1}^{-a_0}] \in \mathbb{P}^{2n-1}.$$

In other words, X is the closure of the orbit of a point

$$[-a_0, a_1, a_2, \ldots, a_{n-1}, 1, 1, \ldots, 1] \in \mathbb{P}^{2n-1}$$

under the torus action with weights

$$w_0 := (a_1, a_2, \ldots, a_{n-1}),$$
$$w_1 := (a_0, 0, \ldots, 0),\ \ w_2 := (0, a_0, 0, \ldots, 0),\ \ \ldots,\ \ w_{n-1} := (0, \ldots, 0, a_0)$$
$$\text{and } -w_0, -w_1, \ldots, -w_{n-1}.$$

Moreover, every such X is a non-degenerate toric Legendrian subvariety.

We are aware that for many choices of the a_i's from the theorem, the action of the torus on X (and on $\mathbb{P}^{2n-1}$) is not faithful, so that for such examples a better choice of coordinates could be made. However, we are willing to pay the price of taking a quotient of T to get a uniform description. One advantage of the description given in the theorem is that a part of it is almost independent of the choice of the a_i's. This part is the $(n-1)$-dimensional "octahedron"

$$\operatorname{conv}\{w_1, \ldots, w_{n-1}, -w_1, \ldots, -w_{n-1}\} \subset \mathbb{Z}^{n-1} \otimes \mathbb{R}.$$

Proof. Assume X is Legendrian with respect to a symplectic form ω, that X is non-degenerate, that the torus T acts on $\mathbb{P}^{2n-1}$ preserving X and satisfies $(\star)$. Replacing T by some covering if necessary, we may assume that $T \to \mathbb{P}\mathbf{GL}_{2n}$ factorises through a maximal torus $T_{\mathbf{Sp}_{2n}} \subset \mathbf{Sp}_{2n}$:

$$T \to T_{\mathbf{Sp}_{2n}} \subset \mathbf{Sp}_{2n} \to \mathbb{P}\mathbf{GL}_{2n}.$$

This implies that for an appropriate symplectic basis the variety X is the closure of the image of the map $T \to \mathbb{P}^{2n-1}$ given by

$$T \ni t \mapsto [x_0 t^{w_0}, x_1 t^{w_1}, \ldots, x_{n-1} t^{w_{n-1}}, t^{-w_0}, t^{-w_1}, \ldots, t^{-w_{n-1}}] \in \mathbb{P}^{2n-1}$$

where $x_i \in \mathbb{C}$, $w_i \in \mathbb{Z}^{n-1}$ and for $v = (v_1, \ldots, v_{n-1}) \in \mathbb{Z}^{n-1}$ we let $t^v := t_1^{v_1} \ldots t_{n-1}^{v_{n-1}}$. This means that X is the closure of the T-orbit of the point [1] $[x_0, \ldots, x_{n-1}, 1, \ldots, 1]$ where T acts with weights $w_0, \ldots, w_{n-1}, -w_0, \ldots, -w_{n-1}$.

Since X is non-degenerate, the weights are pairwise different. Also the weights are not contained in any hyperplane in $\mathbb{Z}^{n-1} \otimes \mathbb{R}$, because the dimension of T is equal to the dimension of X and we assume X has an open orbit of the T-action. So there exists exactly one (up to a scalar) linear relation

$$-a_0 w_0 + a_1 w_1 + \cdots + a_{n-1} w_{n-1} = 0.$$

We assume that the a_i's are coprime integers. Permuting coordinates appropriately we can assume that $|a_0| \geq |a_1| \geq \cdots \geq |a_{n-1}| \geq 0$. After a symplectic change of coordinates, we can assume without loss of generality that all the a_i's are non-negative by exchanging w_i with $-w_i$ (and x_i with $-1/x_i$) if necessary. Clearly not all the a_i's are zero, so in particular $a_0 > 0$ and hence

$$w_0 = \frac{a_1 w_1 + \cdots + a_{n-1} w_{n-1}}{a_0}.$$

[1] Note that usually one assumes that this point is just $[1, \ldots, 1]$. In our case we would have to consider non-symplectic coordinates. We prefer to deal with a point with more complicated coordinates.

Therefore, if we set $e_i := w_i/a_0$ for $i \in \{1, \dots, n-1\}$, the points e_i form a basis of a lattice M containing all w_i's. The lattice M might be finer than the one generated by the w_i's. Replacing again T by a finite cover, we can assume that the action of T is expressible in terms of weights in M. Then

$$\begin{aligned} w_0 &= a_1 e_1 + \dots + a_{n-1} e_{n-1}, \\ w_1 &= a_0 e_1, \\ &\vdots \\ w_{n-1} &= a_0 e_{n-1}. \end{aligned}$$

It remains to prove three things: that $a_{n-1} > 0$, that the x_i's may be chosen as in the statement of the theorem and finally that every such variety is actually Legendrian. We will do all three together.

The torus acts symplectically on the projective space, thus the tangent spaces to the affine cone are Lagrangian if and only if just one tangent space at a point of the open orbit is Lagrangian. So take the point $[x_0, \dots, x_{n-1}, 1, \dots, 1]$. The affine tangent space is spanned by the following vectors:

$$\begin{aligned} v &:= (x_0, x_1, x_2, \dots, x_{n-1}, 1, 1, 1, \dots, 1), \\ u_1 &:= (x_0 a_1, x_1 a_0, 0, \dots, 0, -a_1, -a_0, 0, \dots, 0), \\ u_2 &:= (x_0 a_2, 0, x_2 a_0, \dots, 0, -a_2, 0, -a_0, \dots, 0), \\ &\vdots \\ u_{n-1} &:= (x_0 a_{n-1}, 0, 0, \dots, x_{n-1} a_0, -a_{n-1}, 0, 0, \dots, -a_0). \end{aligned}$$

Now the products are the following:

$$\omega(u_i, u_j) = 0, \qquad \omega(u_i, v) = 2(x_0 a_i + x_i a_0).$$

Therefore the linear space spanned by v and the u_i's is Lagrangian if and only if

$$x_i = -x_0 \frac{a_i}{a_0}.$$

In particular, since $x_i \neq 0$, the a_i cannot be zero either.

After another conformal symplectic base change, we can assume that $x_0 = -a_0$ and then $x_i = a_i$. On the other hand, the above equation is satisfied for the variety in the theorem. Hence the theorem is proved. ∎

Our next goal is to determine for which values of the a_i's the variety X is smooth. The curve case is not interesting at all and also very easy, so we start from $n = 3$, i.e. Legendrian surfaces.

G.2. Smooth toric Legendrian surfaces. We are interested in when the toric projective surface with weights of torus action

$$\begin{aligned} w_0 &:= (a_1, a_2), & w_1 &:= (a_0, 0), & w_2 &:= (0, a_0), \\ -w_0 &= (-a_1, -a_2), & -w_1 &= (-a_0, 0), & -w_2 &= (0, -a_0) \end{aligned}$$

is smooth. Our assumptions on the a_i's are the following:

$$a_0 \geq a_1 \geq a_2 > 0 \tag{G.2}$$

and a_0, a_1, a_2 are coprime integers.

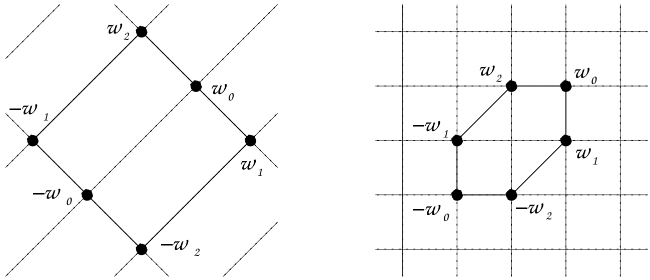


Fig. 1. The two examples of weights giving smooth toric Legendrian surfaces.

Example G.3. Let $a_0 = 2$ and $a_1 = a_2 = 1$ (see Figure 1). Then X is the product of $\mathbb{P}^1$ and a quadric plane curve Q_1.

Example G.4. Let $a_0 = a_1 = a_2 = 1$ (see Figure 1). Although the embedding is not projectively normal (we lack the weight $(0,0)$ in the middle), the image is smooth anyway. Then X is the blow up of $\mathbb{P}^2$ in three non-colinear points.

We will prove there is no other smooth example.

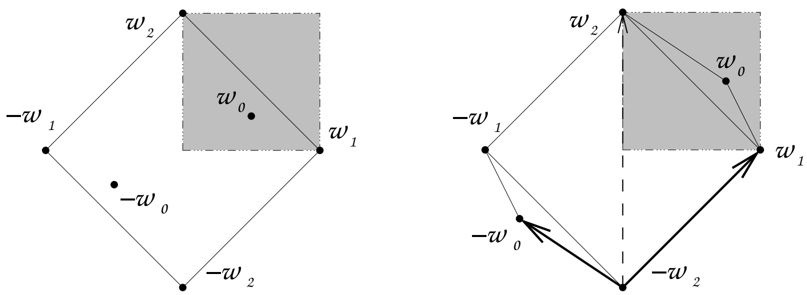


Fig. 2. Due to the inequalities $a_0 \geq a_1 > 0$ and $a_0 \geq a_2 > 0$, the weight w_0 is located somewhere in the grey square. The two cases we consider are that w_0 is also inside the square $\mathrm{conv}\{w_1, w_2, -w_1, -w_2\}$ (left) or outside (right). In the second case, a necessary condition to get a smooth variety is that the two bold vectors generate a lattice containing all the weights. In particular, the dashed vector can be obtained as an integer combination of the bold ones.

We must consider two cases (see Figure 2): either $a_0 > a_1 + a_2$ (which means that w_0 is in the interior of the square $\mathrm{conv}\{w_1, w_2, -w_1, -w_2\}$) or $a_0 \leq a_1 + a_2$ (so that w_0 is outside or on the border of the square).

Geometrically, the case $a_0 > a_1 + a_2$ means that the normalisation of X is $\mathbb{P}^1 \times \mathbb{P}^1$. It is just an easy explicit verification that X is not smooth with these additional weights in the interior.

In the other case, for a vertex v of the polytope

$$\operatorname{conv}\{w_0, w_1, w_2, -w_0, -w_1, -w_2\},$$

we define the sublattice M_v to have the origin at v and to be generated by

$$\{w_0 - v, w_1 - v, w_2 - v, -w_0 - v, -w_1 - v, -w_2 - v\}.$$

Since X is smooth, for every vertex v the vectors of the edges meeting at v must form a basis of M_v (cf. [Stu97, Prop. 2.4 & Lemma 2.2]). In particular, if $v = -w_2$ (it is immediate from (G.2) that v is indeed a vertex), then $w_2 - (-w_2) = (0, 2a_0)$ can be expressed as an integer combination of $w_1 + w_2 = (a_0, a_0)$ and $-w_0 + w_2 = (-a_1, a_0 - a_2)$ (see Figure 2, right). So write

$$(0, 2a_0) = k(a_0, a_0) + l(-a_1, a_0 - a_2) \tag{G.5}$$

for some integers k and l. It is obvious that k and l must be strictly positive, since w_2 is in the cone generated by $w_1 + w_2$ and $-w_0 + w_2$ with vertex at $-w_2$. But then (since $a_0 - a_2 \geq 0$) from equation (G.5) on the second coordinate we find that either $k = 1$ or $k = 2$.

If $k = 1$, then we easily get $a_0 = la_1$, $a_0 = a_1 + a_2$. Hence $(l-1)a_1 = a_2$ and by (G.2) we get $l = 2$ and therefore (since the a_i's are coprime) $(a_0, a_1, a_2) = (2, 1, 1)$, which is Example G.3.

On the other hand, if $k = 2$, then $a_0 = a_2$ and hence by (G.2) and since the a_i's are coprime, we get $(a_0, a_1, a_2) = (1, 1, 1)$, which is Example G.4.

Corollary G.6. *If $X \subset \mathbb{P}^5$ is a smooth toric Legendrian surface, then it is either $\mathbb{P}^1 \times Q_1$, or $\mathbb{P}^2$ blown up in three non-colinear points, or a plane $\mathbb{P}^2 \subset \mathbb{P}^5$.* ∎

G.3. Higher dimensional toric Legendrian varieties. In this section we assume that $n \geq 4$. By means of the geometry of convex bodies we will prove there is only one smooth toric non-degenerate Legendrian variety in dimension $n - 1 = 3$ and no more in higher dimensions. We use Theorem G.1 so that we have a toric variety with weights

$$\begin{aligned}
&w_0 := (a_1, a_2, \ldots, a_{n-1}),\\
&w_1 := (a_0, 0, \ldots, 0),\\
&\vdots\\
&w_{n-1} := (0, \ldots, 0, a_0),\\
&-w_0, -w_1, \ldots, -w_{n-1}
\end{aligned}$$

where the a_i's are coprime positive integers with $a_0 \geq a_1 \geq \cdots \geq a_{n-1}$.

Example G.7. Let $n = 4$ and $(a_0, a_1, a_2, a_3) = (1, 1, 1, 1)$. Then the related toric variety is $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ (see Figure G.3).

Further, let A be the polytope defined by the weights:

$$A := \operatorname{conv}\{w_0, w_1, \ldots, w_{n-1}, -w_0, -w_1, \ldots, -w_{n-1}\} \subset \mathbb{Z}^{n-1} \otimes \mathbb{R}.$$

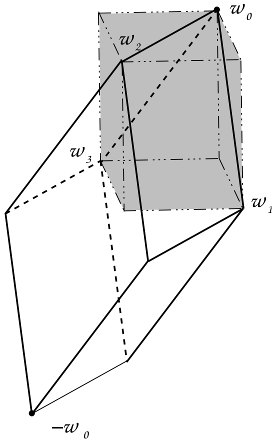


Fig. 3. The smooth example in dimension 3: $(a_0, a_1, a_2, a_3) = (1, 1, 1, 1)$.

LEMMA G.8. *Let $I, J \subset \{1, \dots, n-1\}$ be two complementary subsets of indices.*

(a) *Assume $i_1, i_2 \in I$ and $i_1 \neq i_2$. If*

$$\Big|\sum_{i\in I} a_i - \sum_{j\in J} a_j\Big| < a_0,$$

then the interval (w_{i_1}, w_{i_2}) is an edge of A.

(b) *Assume $k \in I$ and $l \in J$. If*

$$\sum_{i\in I} a_i - \sum_{j\in J} a_j > a_0,$$

then both intervals (w_0, w_k) and $(w_0, -w_l)$ are edges of A.

(c) *If $k, l \in \{1, \dots, n-1\}$ and $k \neq l$, then $(w_k, -w_l)$ is an edge of A.*

Proof. Fix $\epsilon > 0$ small enough, set

$$\alpha := \sum_{i\in I} a_i - \sum_{j\in J} a_j$$

and define the following hyperplanes in $\mathbb{Z}^{n-1} \otimes \mathbb{R}$:

$$H_a := \Big\{\sum_{i\in I} x_i - (1-\epsilon)\sum_{j\in J} x_j = a_0\Big\},$$

$$H_b := \Big\{(a_0 - a_k)\Big(\sum_{i\in I} x_i - \sum_{j\in J} x_j - \alpha\Big) + (\alpha - a_0)(x_k - a_k) = 0\Big\},$$

$$H_b' := \Big\{(a_0 + a_l)\Big(\sum_{i\in I} x_i - \sum_{j\in J} x_j - \alpha\Big) + (\alpha - a_0)(x_l + a_l) = 0\Big\}$$

$$H_c := \{x_k - x_l = a_0\}.$$

Assuming the inequality of (a), $H_a \cap A$ is equal to $\operatorname{conv}\{w_i \mid i \in I\}$ and the rest of A lies on one side of H_a. So H_a is a supporting hyperplane for the face $\operatorname{conv}\{w_i \mid i \in I\}$, which is a simplex of dimension $\#I - 1$ and therefore all its edges are also edges of A as claimed in (a).

Next assume that the inequality of (b) holds. Then H_b (respectively H_b') is a supporting hyperplane for the edge (w_0, w_k) (respectively $(w_0, -w_l)$).

Similarly, in the case of (c), H_c is a supporting hyperplane for $\{w_k, -w_l\}$. ∎

THEOREM G.9. *Let $X \subset \mathbb{P}^{2n-1}$ be a toric non-degenerate Legendrian variety of dimension $n-1$ satisfying $(\star)$* (see page 52). *If $n \geq 4$ and normalisation of X has at most quotient singularities, then $n = 4$ and $X = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.*

Proof. Since the normalisation of X has at most quotient singularities, it follows that the polytope A is simple, i.e. every vertex has exactly $n-1$ edges (see [Ful93] or [Oda88, §2.4, p. 102]). We will prove this is impossible, unless $n = 4$ and $(a_0, a_1, a_2, a_3) = (1, 1, 1, 1)$.

If $w_0 \in B := \operatorname{conv}\{w_1, \dots, w_{n-1}, -w_1, \dots, -w_{n-1}\}$, then A is just equal to B and clearly in such a case every vertex of A has $2(n-2)$ edges, hence more than $n-1$ for $n \geq 4$.

Thus from now on we can assume that

$$a_1 + \dots + a_{n-1} > a_0.$$

By Lemma G.8(b), (w_0, w_i) is an edge for every $i \in \{1, \dots, n-1\}$.

Choose any $j \in \{1, \dots, n-1\}$ and set $I := \{1, \dots, j-1, j+1, \dots, n-1\}$.

If either

$$\Big|\Big(\sum_{i \in I} a_i\Big) - a_j\Big| < a_0 \quad \text{or} \quad \Big(\sum_{i \in I} a_i\Big) - a_j > a_0,$$

then using Lemma G.8 we can count the edges at either w_i or w_0 and see that there always more than $n-1$ of them. We note that $a_j - (\sum_{i \in I} a_i) \geq a_0$ never happens due to our assumptions on the a_i's.

Therefore the remaining case to consider is

$$\Big(\sum_{i \in I} a_i\Big) - a_j = a_0$$

for every $j \in \{1, \dots, n-1\}$. This implies

$$a_1 = a_2 = \dots = a_{n-1} = \frac{1}{n-3} a_0.$$

Since the a_i's are positive integers and coprime, we must have

$$(a_0, a_1, \dots, a_{n-1}) = (n-3, 1, \dots, 1),$$

which is exactly Example G.7 for $n = 4$. Otherwise, if $n \geq 5$ we can take $J := \{j_1, j_2\}$ for any two different $j_1, j_2 \in \{1, \dots, n-1\}$ and set I to be the complement of J. Then $\#I \geq 2$ and by Lemma G.8(a), (c) there are too many edges at the w_i's. ∎

COROLLARY G.10. *If $X \subset \mathbb{P}^{2n-1}$ is a smooth toric Legendrian subvariety and $n \geq 4$, then either X is a linear subspace, or $n = 4$ and $X = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.* ∎

H. Examples of quasihomogeneous Legendrian varieties

The content of this chapter comes from [Buc09].

We construct a family of examples of Legendrian subvarieties in projective spaces. Although most of them are singular, a new example of a smooth Legendrian variety in dimension 8 is in this family. The 8-fold has interesting properties: it is a compactification of the special linear group, a Fano manifold of index 5 and Picard number 1 (see Theorem H.4(b)). Also we show how this construction generalises to give new smooth examples in dimensions 5 and 14 (see §H.2.1).

In §H.1 we introduce the notation for this chapter. In §H.2 we formulate the results and make some comments on possible generalisations. In §H.3 we study the structure of a group action related to the problem. In §H.4 we finally prove the results.

H.1. Notation and definitions. For this chapter we fix an integer $m \geq 2$.

Vector space V. Let V be a vector space over $\mathbb{C}$ of dimension $2m^2$, which we interpret as a space of pairs of $m\times m$ matrices. The coordinates are: a_{ij} and b_{ij} for $i, j \in \{1, \dots, m\}$. By A we denote the matrix (a_{ij}) and similarly for B and (b_{ij}).

Given two $m \times m$ matrices A and B, by (A, B) we denote the point of the vector space V, while by $[A, B]$ the corresponding point of the projective space $\mathbb{P}(V)$.

Sometimes, we will represent some linear maps $V \to V$ and some 2-linear forms $V \otimes V \to \mathbb{C}$ as $2m^2 \times 2m^2$ matrices. In such a case we will assume the coordinates on V are arranged in lexicographical order:

$$a_{11}, \dots, a_{1m}, a_{21}, \dots, a_{mm}, b_{11}, \dots, b_{1m}, b_{21}, \dots, b_{mm}.$$

Symplectic form ω. On V we consider the standard symplectic form

$$\omega((A, B), (A', B')) := \sum_{i,j} (a_{ij} b'_{ij} - a'_{ij} b_{ij}) = \operatorname{tr}(A(B')^T - A'B^T). \tag{H.1}$$

Further, we set J to be the matrix of ω:

$$J := M(\omega) = \begin{bmatrix} 0 & \mathrm{Id}_{m^2} \\ -\mathrm{Id}_{m^2} & 0 \end{bmatrix}.$$

Varieties Y, $X_{\mathrm{inv}}(m)$ and $X_{\mathrm{deg}}(m, k)$. We consider the following subvariety of $\mathbb{P}(V)$:

$$Y := \{[A, B] \in \mathbb{P}(V) \mid AB^T = B^T A = \lambda^2 \mathrm{Id}_m \text{ for some } \lambda \in \mathbb{C}\}. \tag{H.2}$$

The square at λ seems to be irrelevant here, but it slightly simplifies the notation in the proofs of Theorem H.4(b) and Proposition H.10(ii). Although it is not essential for the content of this chapter, we note that Y is F-cointegrable.

Further we define two types of subvarieties of Y:

$$X_{\mathrm{inv}}(m) := \overline{\{[g, (g^{-1})^T] \in \mathbb{P}(V) \mid \det g = 1\}},$$
$$X_{\mathrm{deg}}(m, k) := \{[A, B] \in \mathbb{P}(V) \mid AB^T = B^T A = 0,\ \operatorname{rk} A \leq k,\ \operatorname{rk} B \leq m - k\},$$

where $k \in 0, 1, \dots, m$. The varieties $X_{\mathrm{deg}}(m, k)$ have also been studied in [Str82] and [MT99]. $X_{\mathrm{inv}}(m)$ (especially $X_{\mathrm{inv}}(3)$) is the main object of this chapter.

Automorphisms ψ_μ. For any $\mu \in \mathbb{C}^*$ we let ψ_μ be the following linear automorphism of V:

$$\psi_\mu((A,B)) := (\mu A, \mu^{-1} B).$$

Also the induced automorphism of $\mathbb{P}(V)$ will be written in the same way:

$$\psi_\mu([A,B]) := [\mu A, \mu^{-1} B].$$

Groups G and $\widetilde{G}$, Lie algebra $\mathfrak{g}$ and their representation. We set $\widetilde{G} := \mathbf{GL}_m \times \mathbf{GL}_m$ and let it act on V by

$$(g,h)\cdot(A,B) := (g^T A h, g^{-1} B (h^{-1})^T), \qquad (g,h) \in \widetilde{G},\ g,h \in \mathbf{GL}_m,\ (A,B) \in V.$$

This action preserves the symplectic form ω.

We will mostly consider the restricted action of $G := \mathbf{SL}_m \times \mathbf{SL}_m < \widetilde{G}$.

We also set $\mathfrak{g} := \mathfrak{sl}_m \times \mathfrak{sl}_m$ to be the Lie algebra of G and we have the tangent action of $\mathfrak{g}$ on V:

$$(g,h)\cdot(A,B) = (g^T A + Ah, -gB - Bh^T).$$

Though we denote the action of the groups G, $\widetilde{G}$ and the Lie algebra $\mathfrak{g}$ by the same $\cdot$ we hope it will not lead to any confusion. Also the induced action of G and $\widetilde{G}$ on $\mathbb{P}(V)$ will be denoted by $\cdot$.

Orbits $\mathcal{INV}^m$ and $\mathcal{DEG}^m_{k,l}$. We define the following sets:

$$\begin{aligned}\mathcal{INV}^m &:= \{[g,(g^{-1})^T] \in \mathbb{P}(V) \mid \det g = 1\},\\ \mathcal{DEG}^m_{k,l} &:= \{[A,B] \in \mathbb{P}(V) \mid AB^T = B^T A = 0,\ \operatorname{rk} A = k,\ \operatorname{rk} B = l\},\end{aligned}$$

so that $X_{\mathrm{inv}}(m) = \overline{\mathcal{INV}^m}$ and $X_{\mathrm{deg}}(m,k) = \overline{\mathcal{DEG}^m_{k,m-k}}$.

Clearly, if $k+l > m$, then $\mathcal{DEG}^m_{k,l}$ is empty, so whenever we are considering $\mathcal{DEG}^m_{k,l}$ we will assume $k+l \le m$.

Elementary matrices E_{ij} and points p_1 and p_2. Let E_{ij} be the elementary $m\times m$ matrix with unit in the ith row and the jth column and zeros elsewhere.

We distinguish two points $p_1 \in \mathcal{DEG}^m_{1,0}$ and $p_2 \in \mathcal{DEG}^m_{0,1}$:

$$p_1 := [E_{mm}, 0] \quad \text{and} \quad p_2 := [0, E_{mm}].$$

These points will be usually chosen as nice representatives of the closed orbits $\mathcal{DEG}^m_{1,0}$ and $\mathcal{DEG}^m_{0,1}$.

Submatrices—extracting rows and columns. Assume A is an $m \times m$ matrix and I, J are two sets of indices of cardinality k and l respectively:

$$\begin{aligned}I &:= \{i_1,\dots,i_k \mid 1 \le i_1 < \dots < i_k \le m\},\\ J &:= \{j_1,\dots,j_l \mid 1 \le j_1 < \dots < j_l \le m\}.\end{aligned}$$

Then we denote by $A_{I,J}$ the $(m-k)\times(m-l)$ submatrix of A obtained by removing rows of indices I and columns of indices J. Also for a set I of indices we denote by I' the set of $m-k$ indices complementary to I.

We will also use a simplified version of the above notation when we remove only a single column and single row: A_{ij} denotes the $(m-1)\times(m-1)$ submatrix of A obtained by removing the ith row and jth column, i.e. $A_{ij} = A_{\{i\},\{j\}}$.

Also in the simplest situation where we remove only the last row and the last column, we write A_m, so that $A_m = A_{mm} = A_{\{m\},\{m\}}$.

H.2. Main results. In this chapter we give a classification ([1]) of Legendrian subvarieties in $\mathbb{P}(V)$ that are contained in Y.

THEOREM H.3. *Let the projective space $\mathbb{P}(V)$, varieties Y, $X_{\mathrm{inv}}(m)$, $X_{\mathrm{deg}}(m,k)$ and automorphisms ψ_μ be defined as in* §H.1. *Assume $X \subset \mathbb{P}(V)$ is an irreducible subvariety. Then X is Legendrian and contained in Y if and only if X is one of the following varieties:*

(1) *$X = \psi_\mu(X_{\mathrm{inv}}(m))$ for some $\mu \in \mathbb{C}^*$ or*
(2) *$X = X_{\mathrm{deg}}(m,k)$ for some $k \in \{0,1,\dots m\}$.*

The proof of Theorem H.3 is based on the observation that every Legendrian subvariety contained in Y must be invariant under the action of the group G. This is explained in §H.3. The proof of the theorem is presented in §H.4.1.

Also we analyse which of the above varieties appearing in (1) and (2) are smooth:

THEOREM H.4. *With the definition of $X_{\mathrm{inv}}(m)$ as in* §H.1, *the family $X_{\mathrm{inv}}(m)$ contains the following varieties:*

(a) *$X_{\mathrm{inv}}(2)$ is a linear subspace.*
(b) *$X_{\mathrm{inv}}(3)$ is smooth, its Picard group is generated by a hyperplane section. Moreover, $X_{\mathrm{inv}}(3)$ is a compactification of $\mathbf{SL}_3$ and it is isomorphic to a hyperplane section of the Grassmannian* $\mathrm{Gr}(3,6)$. *The connected component of* $\mathrm{Aut}(X_{\mathrm{inv}}(3))$ *is equal to $G = \mathbf{SL}_3 \times \mathbf{SL}_3$ and $X_{\mathrm{inv}}(3)$ is not a homogeneous space.*
(c) *$X_{\mathrm{inv}}(4)$ is the* 15*-dimensional spinor variety $\mathbb{S}_6$.*
(d) *For $m \geq 5$, the variety $X_{\mathrm{inv}}(m)$ is singular.*

The proof of the theorem is explained in §H.4.3.

The variety $X_{\mathrm{inv}}(3)$ is an original example of [Buc09]. Also it is the first example of a smooth non-homogeneous Legendrian variety of dimension greater than 2 (see §A.1.3). This example is very close to a homogeneous one, namely it is isomorphic to a hyperplane section of $\mathrm{Gr}(3,6)$, a well known subadjoint variety. So a natural question arises whether general hyperplane sections of other Legendrian varieties admit a Legendrian embedding. The answer is yes and we explain it (as well as many conclusions from this surprisingly simple observation) in Chapter I.

THEOREM H.5. *With the definition of $X_{\mathrm{deg}}(m)$ as in* §H.1, *the variety $X_{\mathrm{deg}}(m,k)$ is smooth if and only if $k=0$, $k=m$ or $(m,k)=(2,1)$. In the first two cases, $X_{\mathrm{deg}}(m,0)$ and $X_{\mathrm{deg}}(m,m)$ are linear spaces, while $X_{\mathrm{deg}}(2,1) \simeq \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^7$.*

The proof of the theorem is presented in §H.4.2.

H.2.1. Generalisation: Representation theory and further examples. The interpretation of Theorem H.4(b), (c) can be the following: We take the exceptional Legendrian

([1]) This problem was suggested by Sung Ho Wang.

variety $\mathrm{Gr}(3,6)$, slice it with a linear section and we get a description which, when generalised to matrices of larger size, gives the larger exceptional Legendrian variety $\mathbb{S}_6$. A similar connection can be established between other exceptional Legendrian varieties (see §A.1.3).

For instance, assume that V^{sym} is a vector space of dimension $2\binom{m+1}{2}$, which we interpret as the space of pairs of $m \times m$ symmetric matrices A, B. Now in $\mathbb{P}(V^{\mathrm{sym}})$ consider the subvariety $X^{\mathrm{sym}}_{\mathrm{inv}}(m)$, which is the closure of the set

$$\{[A, A^{-1}] \in \mathbb{P}(V^{\mathrm{sym}}) \mid A = A^T \text{ and } \det A = 1\}.$$

THEOREM H.6. *All the varieties $X^{\mathrm{sym}}_{\mathrm{inv}}(m)$ are Legendrian and we have:*

(a) $X^{\mathrm{sym}}_{\mathrm{inv}}(2)$ *is a linear subspace.*
(b) $X^{\mathrm{sym}}_{\mathrm{inv}}(3)$ *is smooth and it is isomorphic to a hyperplane section of the Lagrangian Grassmannian* $\mathrm{Gr}_L(3,6)$*.*
(c) $X^{\mathrm{sym}}_{\mathrm{inv}}(4)$ *is smooth and it is the Grassmannian variety* $\mathrm{Gr}(3,6)$*.*
(d) *For $m \geq 5$, the variety $X^{\mathrm{sym}}_{\mathrm{inv}}(m)$ is singular.*

The proof is exactly as the proof of Theorem H.4.

Similarly, we can take V^{skew} to be a vector space of dimension $2\binom{2m}{2}$, which we interpret as the space of pairs of $2m \times 2m$ skew-symmetric matrices A, B. Now in $\mathbb{P}(V^{\mathrm{skew}})$ consider a subvariety $X^{\mathrm{skew}}_{\mathrm{inv}}(m)$, which is the closure of

$$\{[A, -A^{-1}] \in \mathbb{P}(V^{\mathrm{skew}}) \mid A = -A^T \text{ and } \operatorname{Pfaff} A = 1\}.$$

THEOREM H.7. *All the varieties $X^{\mathrm{skew}}_{\mathrm{inv}}(m)$ are Legendrian and we have:*

(a) $X^{\mathrm{skew}}_{\mathrm{inv}}(2)$ *is a linear subspace.*
(b) $X^{\mathrm{skew}}_{\mathrm{inv}}(3)$ *is smooth and it is isomorphic to a hyperplane section of the spinor variety* $\mathbb{S}_6$*.*
(c) $X^{\mathrm{skew}}_{\mathrm{inv}}(4)$ *is smooth and it is the 27-dimensional E_7 variety.*
(d) *For $m \geq 5$, the variety $X^{\mathrm{skew}}_{\mathrm{inv}}(m)$ is singular.*

Here the only difference is that we replace the determinants by the Pfaffians of the appropriate submatrices and also for the symmetric cases we pick some diagonal matrices as nice representatives. Since there is no non-zero skew-symmetric diagonal matrix, we must modify our calculations a little, but there is essentially no difference in the technique.

Prior to [Buc09] neither $X^{\mathrm{sym}}_{\mathrm{inv}}(3)$ nor $X^{\mathrm{skew}}_{\mathrm{inv}}(3)$ have been identified as smooth Legendrian subvarieties.

Therefore we have established a connection between the subadjoint varieties of the four exceptional groups F_4, E_6, E_7 and E_8. A similar connection was obtained in [LM02].

We note that $m\times m$ symmetric matrices, $m\times m$ matrices and $2m\times 2m$ skew-symmetric matrices naturally correspond to $m \times m$ Hermitian matrices with coefficients in $\mathbb{F} \otimes_{\mathbb{R}} \mathbb{C}$, where $\mathbb{F}$ is the field of, respectively, real numbers $\mathbb{R}$, complex numbers $\mathbb{C}$ and quaternions $\mathbb{H}$ (see [LM01] and references therein). An algebraic relation (analogous to parts (c) of Theorems H.4, H.6 and H.7) between Lie algebras of types E_6, E_7 and E_8 and 4×4 Hermitian matrices with coefficients in $\mathbb{F} \otimes_{\mathbb{R}} \mathbb{C}$ is described in [BK94].

H.3. G-action and its orbits. Recall the definition of Y in §H.1.

The following polynomials are in the homogeneous ideal of Y (the indices i, j below run through $\{1, \dots, m\}$, k is a summation index):

$$\sum_{k=1}^{m} a_{ik} b_{ik} - \sum_{k=1}^{m} a_{1k} b_{1k}, \tag{H.8a}$$

$$\sum_{k=1}^{m} a_{ik} b_{jk} \quad \text{for } i \neq j, \tag{H.8b}$$

$$\sum_{k=1}^{m} a_{ki} b_{ki} - \sum_{k=1}^{m} a_{k1} b_{k1}, \tag{H.8c}$$

$$\sum_{k=1}^{m} a_{ki} b_{kj} \quad \text{for } i \neq j. \tag{H.8d}$$

These equations simply come from eliminating λ from the defining equation of Y—see (H.2).

For the statement and proof of the following proposition, recall our notation of §H.1.

PROPOSITION H.9. *Let $X \subset \mathbb{P}(V)$ be a Legendrian subvariety. If X is contained in Y, then X is preserved by the induced action of G on $\mathbb{P}(V)$.*

Proof. Let $\widetilde{\mathcal{I}}(X)_2$ be as in Theorem A.9 and define $\widetilde{\mathcal{I}}(Y)_2$ analogously. Clearly $\widetilde{\mathcal{I}}(Y)_2 \subset \widetilde{\mathcal{I}}(X)_2$. Also let ρ be the map described in Theorem A.9. By Theorem A.9 it is enough to show that $\mathfrak{g} \subset \rho(\widetilde{\mathcal{I}}(Y)_2)$ or that the images of the quadrics (H.8a)–(H.8d) under ρ generate $\mathfrak{g}$.

We write out the details of the proof only for $m = 2$. There is no difference between this case and the general one, except for the complexity of notation.

Let us take the quadric

$$q_{ij} := \sum_{k=1}^{m} a_{ik} b_{jk} = a_{i1} b_{j1} + a_{i2} b_{j2}$$

for any $i, j \in \{1, \dots, m\} = \{1, 2\}$. Also let Q_{ij} be the $2m^2 \times 2m^2$ symmetric matrix corresponding to q_{ij}. For instance,

$$Q_{12} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Choose an arbitrary $(A, B) \in V$; for the moment, think of it as a single vertical $2m^2$-vector: $(A, B) = [a_{11}, a_{12}, a_{21}, a_{22}, b_{11}, b_{12}, b_{21}, b_{22}]^T$, so that the following multiplication makes sense:

$$
\begin{aligned}
\rho(q_{12}) &= 2J \cdot Q_{12} \cdot (A,B) \\
&= \begin{bmatrix} 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1\\ -1&0&0&0&0&0&0&0\\ 0&-1&0&0&0&0&0&0\\ 0&0&-1&0&0&0&0&0\\ 0&0&0&-1&0&0&0&0 \end{bmatrix}
\begin{bmatrix} 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0 \end{bmatrix}
\begin{bmatrix} a_{11}\\ a_{12}\\ a_{21}\\ a_{22}\\ b_{11}\\ b_{12}\\ b_{21}\\ b_{22} \end{bmatrix} \\
&= \begin{bmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&-1&0\\ 0&0&0&0&0&0&0&-1\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{bmatrix}
\begin{bmatrix} a_{11}\\ a_{12}\\ a_{21}\\ a_{22}\\ b_{11}\\ b_{12}\\ b_{21}\\ b_{22} \end{bmatrix} \\
&= \begin{bmatrix} 0\\ 0\\ a_{11}\\ a_{12}\\ -b_{21}\\ -b_{22}\\ 0\\ 0 \end{bmatrix} \overset{\text{back to matrix notation}}{=} \left(\begin{bmatrix} 0 & 0\\ a_{11} & a_{12} \end{bmatrix}, \begin{bmatrix} -b_{21} & -b_{22}\\ 0 & 0 \end{bmatrix} \right) \\
&= \left(\begin{bmatrix} 0 & 1\\ 0 & 0 \end{bmatrix}^T \begin{bmatrix} a_{11} & a_{12}\\ a_{21} & a_{22} \end{bmatrix}, -\begin{bmatrix} 0 & 1\\ 0 & 0 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12}\\ b_{21} & b_{22} \end{bmatrix} \right) = (E_{12}^T A,\ -E_{12}B).
\end{aligned}
$$

Similar calculations show that

$$2J \cdot Q_{ij} \cdot (A,B) \ = \ (E_{ij}^T A,\ -E_{ij}B).$$

Next in the ideal of Y we have the following quadrics: q_{ij} for $i \neq j$ (see (H.8b)) and $q_{ii} - q_{11}$ (see (H.8a)). By taking images under ρ of the linear combinations of those quadrics we can get an arbitrary traceless matrix $g \in \mathfrak{sl}_m$ acting on V in the following way:

$$g \cdot (A,B) = (g^T A, -gB).$$

Exponentiate this action of $\mathfrak{sl}_m$ to get an action of $\mathbf{SL}_m$:

$$g \cdot (A,B) = (g^T A, g^{-1}B).$$

This proves that the action of the subgroup $\mathbf{SL}_m \times 0 < G = \mathbf{SL}_m \times \mathbf{SL}_m$ preserves X as claimed in the proposition. The action of the other component $0 \times \mathbf{SL}_m$ is calculated in the same way, but using the quadrics (H.8c)–(H.8d). ∎

H.3.1. Invariant subsets. Here we want to decompose Y into a union of some G-invariant subsets, most of which are orbits.

PROPOSITION H.10.

(i) *The sets* $\mathcal{INV}^m$, $\psi_\mu(\mathcal{INV}^m)$ *and* $\mathcal{DEG}^m_{k,l}$ *are* G*-invariant and they are all contained in* Y.
(ii) Y *is equal to the union of all* $\psi_\mu(\mathcal{INV}^m)$ *(for* $\mu \in \mathbb{C}^*$*) and all* $\mathcal{DEG}^m_{k,l}$ *(for integers* $k, l \geq 0$, $k + l \leq m$*).*
(iii) *Every* $\psi_\mu(\mathcal{INV}^m)$ *is an orbit of the action of* G*. If* m *is odd, then* $\mathcal{INV}^m$ *is isomorphic (as an algebraic variety) to* $\mathbf{SL}_m$*. If* m *is even, then* $\mathcal{INV}^m$ *is isomorphic to* $\mathbf{SL}_m/\mathbb{Z}_2$*. In both cases*

$$\dim \psi_\mu(\mathcal{INV}^m) = \dim \mathcal{INV}^m = m^2 - 1.$$

Proof. The proof of (i) is an explicit verification from the definitions in §H.1.

To prove (ii), assume $[A, B]$ is a point of Y, so $AB^T = B^T A = \lambda^2 \operatorname{Id}_m$. First assume that the ranks of both matrices are maximal:

$$\operatorname{rk} A = \operatorname{rk} B = m.$$

Then λ must be non-zero and $B = \lambda^2 (A^{-1})^T$. Let $d := (\det A)^{-1/m}$ so that $\det(dA) = 1$ and let $\mu := \frac{1}{d\lambda}$. Then we have

$$\begin{aligned}[A, B] = [A, \lambda^2 (A^{-1})^T] &= \left[\frac{dA}{d\lambda}, d\lambda((dA)^{-1})^T\right] \\ &= [\mu(dA), \mu^{-1}((dA)^{-1})^T] = \psi_\mu([(dA), ((dA)^{-1})^T]).\end{aligned}$$

Therefore $[A, B] \in \psi_\mu(\mathcal{INV}^m)$.

Next, if either of the ranks is not maximal:

$$\operatorname{rk} A < m \quad \text{or} \quad \operatorname{rk} B < m,$$

then by (H.2) we must have $AB^T = B^T A = 0$. So $[A, B] \in \mathcal{DEG}^m_{k,l}$ for $k = \operatorname{rk} A$ and $l = \operatorname{rk} B$.

Now we prove (iii). The action of G commutes with ψ_μ:

$$(g, h) \cdot \psi_\mu([A, B]) = \psi_\mu((g, h) \cdot [A, B]).$$

So to prove $\psi_\mu(\mathcal{INV}^m)$ is an orbit it is enough to prove that $\mathcal{INV}^m$ is an orbit, which follows from the definitions of the action and $\mathcal{INV}^m$.

We have the following epimorphic map:

$$\mathbf{SL}_m \to \mathcal{INV}^m, \qquad g \mapsto [g, (g^{-1})^T].$$

If $[g_1, (g_1^{-1})^T] = [g_2, (g_2^{-1})^T]$, then we must have $g_1 = \alpha g_2$ and $g_1 = \alpha^{-1} g_2$ for some $\alpha \in \mathbb{C}^*$. Hence $\alpha^2 = 1$ and $g_1 = \pm g_2$. If m is odd and $g_1 \in \mathbf{SL}_m$, then $-g_1 \notin \mathbf{SL}_m$ so $g_1 = g_2$. So $\mathcal{INV}^m$ is isomorphic to either $\mathbf{SL}_m$ or $\mathbf{SL}_m/\mathbb{Z}_2$ as stated. ∎

From Proposition H.10(ii) we conclude that $X_{\text{inv}}(m)$ is an equivariant compactification of $\mathbf{SL}_m$ (if m is odd) or $\mathbf{SL}_m/\mathbb{Z}_2$ (if m is even). See [Tim03] and references therein for the theory of equivariant compactifications. In the setup of [Tim03, §8], this is the compactification corresponding to the representation $W \oplus W^*$, where W is the standard representation of $\mathbf{SL}_m$. Therefore some properties of $X_{\mathrm{inv}}(m)$ could also be read off from the general description of group compactifications.

PROPOSITION H.11.

(i) *The dimension of* $\mathcal{DEG}^m_{k,l}$ *is* $(k+l)(2m-k-l)-1$. *In particular, if* $k+l=m$, *then the dimension is equal to* m^2-1.
(ii) $\mathcal{DEG}^m_{k,l}$ *is an orbit of the action of* G, *unless* m *is even and* $k=l=\frac{1}{2}m$.
(iii) *If* $m \geq 3$, *then there are exactly two closed orbits of the action of* G: $\mathcal{DEG}^m_{1,0}$ *and* $\mathcal{DEG}^m_{0,1}$.

Proof. Part (i) follows from [Str82, Prop. 2.10].

For (ii) let $[A,B] \in \mathcal{DEG}^m_{k,l}$ be any point. By Gaussian elimination and elementary linear algebra, we can prove that there exists $(g,h) \in G$ such that $[A',B'] := (g,h)\cdot[A,B]$ is a pair of diagonal matrices. Moreover, if $k+l<m$, then we can choose g and h such that

$$A' := \mathrm{diag}(\underbrace{1,\dots,1}_{k},\underbrace{0,\dots,0}_{l},\underbrace{0,\dots,0}_{m-k-l}), \qquad B' := \mathrm{diag}(\underbrace{0,\dots,0}_{k},\underbrace{1,\dots,1}_{l},\underbrace{0,\dots,0}_{m-k-l}).$$

Hence $\mathcal{DEG}^m_{k,l} = G\cdot[A',B']$ and this finishes the proof in the case $k+l<m$.

So assume $k+l=m$. Then we can choose (g,h) such that

$$A' := \mathrm{diag}(\underbrace{1,\dots,1}_{k},\underbrace{0,\dots,0}_{l}), \qquad B' := \mathrm{diag}(\underbrace{0,\dots,0}_{k},\underbrace{d,\dots,d}_{l}),$$

for some $d \in \mathbb{C}^*$. If $k \neq l$, then set $e := d^{1/(l-k)}$ and let

$$g' := \mathrm{diag}(\underbrace{e^l,\dots,e^l}_{k},\underbrace{e^{-k},\dots,e^{-k}}_{l}).$$

Clearly $\det(g')=1$ and

$$(g',\mathrm{Id}_m)\cdot[A',B'] = [\mathrm{diag}(\underbrace{e^l,\dots,e^l}_{k},\underbrace{0,\dots,0}_{l}),\mathrm{diag}(\underbrace{0,\dots,0}_{k},\underbrace{de^k,\dots,de^k}_{l})]$$

where

$$de^k = d^{1+k/(l-k)} = d^{l/(l-k)} = e^l.$$

So rescaling we get

$$(g',\mathrm{Id}_m)\cdot[A',B'] = [\mathrm{diag}(\underbrace{1,\dots,1}_{k},\underbrace{0,\dots,0}_{l}),\mathrm{diag}(\underbrace{0,\dots,0}_{k},\underbrace{1,\dots,1}_{l})]$$

and this finishes the proof of (ii).

For (iii), denote by W_1 (respectively, W_2) the standard representation of the first (respectively, second) component of $G=\mathbf{SL}_m\times\mathbf{SL}_m$ with the trivial action of the other component. Then V as a representation of G is isomorphic to $(W_1\otimes W_2)\oplus(W_1^*\otimes W_2^*)$. For $m\geq 3$ the representation W_i is not isomorphic to W_i^* and therefore V is a union of two irreducible non-isomorphic representations. It is a standard fact from representation theory that the closed orbits of the action of a semisimple group on a projectivised representation are in one-to-one correspondence with the irreducible subrepresentations.

So there are exactly two closed orbits of the action of G on $\mathbb{P}(V)$. These orbits are simply $\mathcal{DEG}^m_{1,0}$ and $\mathcal{DEG}^m_{0,1}$. ■

H.3.2. Action of $\widetilde{G}$. The action of $\widetilde{G}$ extends the action of G, but it does not preserve $X_{\mathrm{inv}}(m)$. So we will only consider the action of $\widetilde{G}$ when speaking of $X_{\mathrm{deg}}(m,k)$.

We have properties analogous to Proposition H.11(ii), (iii) but with no exceptional cases:

PROPOSITION H.12.

(i) *Every $\mathcal{DEG}^m_{k,l}$ is an orbit of the action of $\widetilde{G}$.*
(ii) *For every m there are exactly two closed orbits of the action of $\widetilde{G}$: $\mathcal{DEG}^m_{1,0}$ and $\mathcal{DEG}^m_{0,1}$.*

Proof. This is exactly as the proof of Proposition H.11(ii),(iii). ■

H.4. Legendrian varieties in Y. In this section we prove the main results of the chapter.

H.4.1. Classification. We start by proving Theorem H.3.

Proof. First assume X is Legendrian and contained in Y. If X contains a point $[A,B]$ where both A and B are invertible, then by Proposition H.9 it must contain the orbit of $[A,B]$, which by Proposition H.10(ii), (iii) is equal to $\psi_\mu(\mathcal{INV}^m)$ for some $\mu \in \mathbb{C}^*$. But the dimension of X is m^2-1, which is exactly the dimension of $\psi_\mu(\mathcal{INV}^m)$ (see Proposition H.10(iii)), so

$$X = \overline{\psi_\mu(\mathcal{INV}^m)} = \psi_\mu(X_{\mathrm{inv}}(m)).$$

On the other hand, if X does not contain any point $[A,B]$ where both A and B are invertible, then in fact X is contained in $Y_0 := \{[A,B] \mid AB^T = B^TA = 0\}$. This is just the union of all $\mathcal{DEG}^m_{k,l}$ and its irreducible components are the closures of $\mathcal{DEG}^m_{k,m-k}$, which are exactly $X_{\mathrm{deg}}(m,k)$. So in particular every irreducible component has dimension m^2-1 (see Proposition H.11(i)) and hence X must be one of these components.

Therefore it remains to show that all these varieties are Legendrian.

The fact that $X_{\mathrm{deg}}(m,k)$ is a Legendrian variety follows from [Str82, pp. 524–525]. Strickland proves there that the affine cone over $X_{\mathrm{deg}}(m,k)$ (or $W(k,m-k)$ in the notation of [Str82]) is the closure of a conormal bundle. Conormal bundles are classical examples of Lagrangian varieties (see Example D.6).

Since ψ_μ preserves the symplectic form ω, it is enough to prove that $X_{\mathrm{inv}}(m)$ is Legendrian.

The group G acts symplectically on V and the action has an open orbit on $X_{\mathrm{inv}}(m)$—see Proposition H.10(iii). Thus the tangent spaces to the affine cone over $X_{\mathrm{inv}}(m)$ are Lagrangian if and only if just one tangent space at a point of the open orbit is Lagrangian.

So we take $[A,B] := [\mathrm{Id}_m, \mathrm{Id}_m]$. Now the affine tangent space to $X_{\mathrm{inv}}(m)$ at $[\mathrm{Id}_m, \mathrm{Id}_m]$ is the linear subspace of V spanned by $(\mathrm{Id}_m, \mathrm{Id}_m)$ and the image of the tangent action of the Lie algebra $\mathfrak{g}$. We must prove that for any four traceless matrices g, h, g', h' we have

$$\omega((g,h)\cdot(\mathrm{Id}_m,\mathrm{Id}_m),\ (g',h')\cdot(\mathrm{Id}_m,\mathrm{Id}_m)) = 0, \tag{H.13a}$$

$$\omega((\mathrm{Id}_m,\mathrm{Id}_m),\ (g,h)\cdot(\mathrm{Id}_m,\mathrm{Id}_m)) = 0. \tag{H.13b}$$

Equality (H.13a) is true without the assumption on the trace of the matrices:

$$\begin{aligned}\omega((g,h)&\cdot(\mathrm{Id}_m,\mathrm{Id}_m),\ (g',h')\cdot(\mathrm{Id}_m,\mathrm{Id}_m))\\ &= \omega\big((g^T+h,\ -(g+h^T)),\ ((g')^T+h',\ -(g'+(h')^T))\big)\\ &\overset{(\mathrm{H}.1)}{=} \operatorname{tr}\big(-(g^T+h)((g')^T+h') + (g+h^T)(g'+(h')^T)\big) = 0.\end{aligned}$$

For (H.13b) we calculate

$$\begin{aligned}\omega((\mathrm{Id}_m,\mathrm{Id}_m),\ (g,h)\cdot(\mathrm{Id}_m,\mathrm{Id}_m)) &= \omega((\mathrm{Id}_m,\mathrm{Id}_m),\ (g^T+h,\ -(g+h^T)))\\ &\overset{(\mathrm{H}.1)}{=} -\operatorname{tr}(g^T+h) - \operatorname{tr}(g+h^T) = 0.\end{aligned}$$

Hence we have proved that the closure of $\mathcal{INV}^m$ is Legendrian. ■

H.4.2. Degenerate matrices. By [Str82, Prop. 1.3] the ideal of $X_{\deg}(m,k)$ is generated by the coefficients of AB^T, the coefficients of B^TA, the $(k+1)\times(k+1)$-minors of A and the $(m-k+1)\times(m-k+1)$-minors of B. In short, we will say that the equations of $X_{\deg}(m,k)$ are given by

$$AB^T = 0,\ B^TA = 0,\quad \mathrm{rk}(A) \le k,\ \mathrm{rk}(B) \le m-k. \tag{H.14}$$

Lemma H.15. *Assume $m \ge 2$ and $1 \le k \le m-1$. Then:*

- (i) *The tangent cone to $X_{\deg}(m,k)$ at p_1 is the product of a linear space of dimension $2m-2$ and the affine cone over $X_{\deg}(m-1,k-1)$.*
- (i′) *The tangent cone to $X_{\deg}(m,k)$ at p_2 is the product of a linear space of dimension $2m-2$ and the affine cone of $X_{\deg}(m-1,k)$.*
- (ii) *$X_{\deg}(m,k)$ is smooth at p_1 if and only if $k=1$.*
- (ii′) *$X_{\deg}(m,k)$ is smooth at p_2 if and only if $k=m-1$.*

Proof. We only prove (i) and (ii), while (i′) and (ii′) follow in the same way by exchanging a_{ij} and b_{ij}. Consider equations (H.14) of $X_{\deg}(m,k)$ restricted to the affine neighbourhood of p_1 obtained by substituting $a_{mm}=1$. Taking the lowest degree part of these equations we get some of the equations of the tangent cone at p_1 (recall our convention on the notation of submatrices—see §H.1):

$$\begin{gathered} b_{im} = b_{mi} = 0,\qquad A_mB_m^T = 0,\ B_m^TA_m = 0,\\ \mathrm{rk}\,A_m \le k-1,\qquad \mathrm{rk}\,B_m \le m-k.\end{gathered}$$

These equations define the product of the linear subspace $A_m = B_m = 0, b_{im} = b_{mi} = 0$ and the affine cone over $X_{\deg}(m-1,k-1)$ embedded in the set of those pairs of matrices whose last row and column are zero: $a_{im} = a_{mi} = 0, b_{im} = b_{mi} = 0$. So the variety defined by those equations is irreducible and its dimension is equal to $(m-1)^2+2m-2 = m^2-1 = \dim X_{\deg}(m,k)$. Since this variety contains the tangent cone we are interested in and by §B.8(1), they must coincide as claimed in (i).

Next, (ii) follows immediately, since for $k = 1$ the equations above reduce to

$$b_{im} = b_{mi} = 0, \qquad A_m = 0,$$

and hence the tangent cone is just the tangent space, so p_1 is a smooth point of $X_{\deg}(m, 1)$. Conversely, if $k > 1$, then $X_{\deg}(m-1, k-1)$ is not a linear space, so by (i) the tangent cone is not a linear space either and X is singular at p_1—see §B.8(3). ∎

Now we can prove Theorem H.5:

Proof. It is obvious from the definition of $X_{\deg}(m, k)$ that $X_{\deg}(m, 0) = \{A = 0\}$ and $X_{\deg}(m, m) = \{B = 0\}$, so these are indeed linear spaces.

Therefore assume $1 \leq k \leq m-1$. But $X_{\deg}(m, k)$ is $\widetilde{G}$-invariant (see Proposition H.12(i)) and so is its singular locus S. Hence $X_{\deg}(m, k)$ is singular if and only if S contains a closed orbit of $\widetilde{G}$.

So $X_{\deg}(m, k)$ is smooth if and only if it is smooth at both p_1 and p_2 (see Proposition H.12(ii)), which by Lemma H.15(ii), (ii′) holds if and only if $k = 1$ and $m = 2$.

To finish the proof, it remains to verify what kind of variety $X_{\deg}(2, 1)$ is. Consider the following map:

$$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}(V) \simeq \mathbb{P}^7,$$
$$[\mu_1, \mu_2], [\nu_1, \nu_2], [\xi_1, \xi_2] \mapsto \left[\xi_1 \begin{pmatrix} \mu_1\nu_1 & \mu_1\nu_2 \\ \mu_2\nu_1 & \mu_2\nu_2 \end{pmatrix}, \xi_2 \begin{pmatrix} \mu_2\nu_2 & -\mu_2\nu_1 \\ -\mu_1\nu_2 & \mu_1\nu_1 \end{pmatrix} \right].$$

Clearly this is the Segre embedding in appropriate coordinates. The image of this embedding is contained in $X_{\deg}(2, 1)$ (see (H.14)) and since the dimension of $X_{\deg}(2, 1)$ is equal to the dimension of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ we conclude that the above map gives an isomorphism of $X_{\deg}(2, 1)$ and $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. ∎

H.4.3. Invertible matrices. We wish to determine some of the equations of $X_{\mathrm{inv}}(m)$. Clearly the equations of Y (see (H.8)) are quadratic equations of $X_{\mathrm{inv}}(m)$. To find other equations, we recall that

$$X_{\mathrm{inv}}(m) := \overline{\{[g, (g^{-1})^T] \in \mathbb{P}(V) \mid \det g = 1\}}.$$

However, for a matrix g with determinant 1 we know that the entries of $(g^{-1})^T$ are appropriate minors (up to sign) of g. Therefore we get many inhomogeneous equations satisfied by every pair $(g, (g^{-1})^T) \in V$ (recall our convention on the notation of submatrices—see §H.1):

$$\det(A_{ij}) = (-1)^{i+j} b_{ij} \quad \text{and} \quad a_{kl} = (-1)^{k+l} \det(B_{kl}).$$

To make them homogeneous, multiply two such equations appropriately:

$$\det(A_{ij}) a_{kl} = (-1)^{i+j+k+l} b_{ij} \det(B_{kl}). \tag{H.16}$$

These are degree m equations, which are satisfied by the points of $X_{\mathrm{inv}}(m)$ and we state the following theorem:

THEOREM H.17. *Let $m = 3$. Then the quadratic polynomials* (H.8a)–(H.8d) *and the cubic polynomials* (H.16) *generate a homogeneous ideal $\mathcal{I}$ in $\mathbb{C}[V]$ which defines $X_{\mathrm{inv}}(3)$ as a subscheme of $\mathbb{P}(V)$. Moreover, $X_{\mathrm{inv}}(3)$ is smooth.*

Proof. It is enough to prove that the scheme X defined by $\mathcal{I}$ is smooth, because the reduced subscheme of X coincides with $X_{\mathrm{inv}}(3)$.

The scheme X is G-invariant, hence as in the proof of Theorem H.5 and by Proposition H.11(iii) it is enough to verify smoothness at p_1 and p_2. Since we have the additional symmetry here (exchanging a_{ij}'s with b_{ij}'s) it is enough to verify the smoothness at p_1.

Now we calculate the tangent space to X at p_1 by taking the linear parts of the polynomials evaluated at $a_{33} = 1$. From polynomials (H.8) we get

$$b_{31} = b_{32} = b_{33} = b_{23} = b_{13} = 0.$$

Now from (H.16) for $k = l = 3$ and $i, j \neq 3$ we get the following evaluated equations:

$$a_{i'j'} - a_{i'3}a_{3j'} = \pm b_{ij}B_{33}$$

(where i' is either 1 or 2, whichever is different from i, and analogously for j') so the linear part is just $a_{i'j'} = 0$. Hence by varying i and j we get

$$a_{11} = a_{12} = a_{21} = a_{22} = 0.$$

Therefore the tangent space has codimension at least 9, which is exactly the codimension of $X_{\mathrm{inv}}(3)$—see Proposition H.10(iii). Hence X is smooth (in particular reduced) and $X = X_{\mathrm{inv}}(3)$. ∎

Remark H.18. The ideal $\mathcal{I}$ from Theorem H.17 could possibly be unsaturated (in other words, the affine subscheme in V defined by this ideal could be not reduced at 0). Computer calculations show that this is not the case and $\mathcal{I}$ is saturated. We will only need that the saturated ideal of $X_{\mathrm{inv}}(3)$ has no other quadratic polynomials than $\mathcal{I}$ and we will prove it later.

To describe $X_{\mathrm{inv}}(m)$ for $m > 3$ we must find more equations.

There is a more general version of the above property of the inverse of a matrix with determinant 1, which is less known.

Proposition H.19.

(i) *Assume A is an $m \times m$ matrix of determinant* 1 *and I, J are two sets of indices, both of cardinality k (again recall our convention on indices and submatrices—see* §H.1). *Denote $B := (A^{-1})^T$. Then the appropriate minors are equal (up to sign):*

$$\det A_{I,J} = (-1)^{\Sigma I + \Sigma J} \det B_{I',J'}.$$

(ii) *A coordinate free way to express these equalities is the following: Assume W is a vector space of dimension m, f is a linear automorphism of W and $k \in \{0, \dots, m\}$. Let $\bigwedge^k f$ be the induced automorphism of $\bigwedge^k W$. If $\bigwedge^m f = \mathrm{Id}_{\bigwedge^m W}$, then*

$$\bigwedge\nolimits^{m-k} f = \bigwedge\nolimits^k \big(\bigwedge\nolimits^{m-1} f\big).$$

(iii) *Consider the induced action of G on the polynomials on V. Then the vector space spanned by the set of equations of* (i) *for a fixed k is G-invariant.*

Proof. Part (ii) follows immediately from (i), since if A is a matrix of f, then the entries of the matrices of the maps $\bigwedge^{m-k} f$ and $\bigwedge^k(\bigwedge^{m-1} f)$ are exactly the appropriate minors of A and B.

Part (iii) follows easily from (ii).

As for (i), we only sketch the proof, leaving the details to the reader. Firstly, reduce to the case when I and J are just $\{1,\dots,k\}$ and the determinant of A is possibly ± 1 (which is where the sign shows up in the equality). Secondly if both $\det A_{I,J}$ and $\det B_{I',J'}$ are zero, then the equality is clearly satisfied. Otherwise assume for example $\det A_{I,J} \neq 0$. Then performing appropriate row and column operations we can change $A_{I,J}$ into a diagonal matrix, $A_{I',J}$ and $A_{I,J'}$ into zero matrices and all these operations can be done without changing $B_{I',J'}$ or $\det A_{I,J}$. Then the statement follows easily. ∎

In particular, we get:

Corollary H.20. *Assume k, I and J are as in Proposition* H.19(i).

(a) *If m is even and $k = \frac{1}{2}m$, then the equation*

$$\det A_{I,J} = (-1)^{\Sigma I + \Sigma J} \det B_{I',J'}$$

is homogeneous of degree $\frac{1}{2}m$ and it is satisfied by points of $X_{\mathrm{inv}}(m)$.

(b) *If $0 \le k < \frac{1}{2}m$ and $l = m - 2k$, then*

$$(\det A_{I,J})^2 = (\det B_{I',J'})^2 \cdot (a_{11}b_{11} + \cdots + a_{1m}b_{1m})^l$$

is a homogeneous equation of degree $2(m-k)$ satisfied by points of $X_{\mathrm{inv}}(m)$.

Proof. Clearly both equations are homogeneous. If $\det A = 1$ and $B = (A^{-1})^T$, then

$$\det A_{I,J} = (-1)^{\Sigma I + \Sigma J} \det B_{I',J'}, \tag{H.21}$$

$$1 = (a_{11}b_{11} + \cdots + a_{1m}b_{1m})^l \tag{H.22}$$

((H.21) follows from Proposition H.19(i), and (H.22) follows from $AB^T = \mathrm{Id}_m$). The equation in (b) is just (H.21) squared and multiplied by (H.22).

So both equations in (a) and (b) are satisfied by every pair $(A, (A^{-1})^T)$ and by homogeneity also by $(\lambda A, \lambda (A^{-1})^T)$. Hence (a) and (b) hold on an open dense subset of $X_{\mathrm{inv}}(m)$, so also on the whole $X_{\mathrm{inv}}(m)$. ∎

We know enough equations of $X_{\mathrm{inv}}(m)$ to prove Theorem H.4:

Case $m = 2$—linear subspace

Proof. To prove (a) just take the linear equations from Corollary H.20(a) for $k = 1$:

$$a_{ij} = \pm b_{i'j'},$$

where $\{i, i'\} = \{j, j'\} = \{1, 2\}$. ∎

Case $m = 3$—hyperplane section of $\mathrm{Gr}(3,6)$

Proof. For (b), $X_{\mathrm{inv}}(3)$ is smooth by Theorem H.17 and it is a compactification of $\mathcal{INV}^3 \simeq \mathbf{SL}_3$ by Proposition H.10(i), (iii).

Picard group of $X_{\mathrm{inv}}(3)$. The complement of the open orbit

$$D := X_{\mathrm{inv}}(3) \setminus \mathcal{INV}^3$$

must be a union of some orbits of G, each of dimension smaller than $\dim \mathcal{INV}^3 = 8$. So by Propositions H.10(ii), (iii), H.11(i), (ii) the only candidates are $\mathcal{DEG}^3_{1,1}$, $\mathcal{DEG}^3_{0,1}$, $\mathcal{DEG}^3_{1,0}$,

$\mathcal{DEG}^3_{0,2}$ and $\mathcal{DEG}^3_{2,0}$. We claim that only $\mathcal{DEG}^3_{1,1}$, $\mathcal{DEG}^3_{0,1}$ and $\mathcal{DEG}^3_{1,0}$ are contained in $X_{\mathrm{inv}}(3)$. To exclude $\mathcal{DEG}^3_{2,0}$ consider the point

$$\left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, 0\right].$$

It is in $\mathcal{DEG}^3_{2,0}$ and does not satisfy (H.16) for $i = j = 1$ and $k = l = 3$ and therefore is not in $X_{\mathrm{inv}}(3)$. So $\mathcal{DEG}^3_{2,0}$ is disjoint from $X_{\mathrm{inv}}(3)$.

Analogously, $\mathcal{DEG}^3_{0,2}$ is disjoint from $X_{\mathrm{inv}}(3)$.

It only remains to prove that $\mathcal{DEG}^3_{1,1} \subset X_{\mathrm{inv}}(3)$, since the other orbits are in the closure of $\mathcal{DEG}^3_{1,1}$. Take the curve in $X_{\mathrm{inv}}(3)$ parametrised by

$$\left[\begin{pmatrix} t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & t^{-1} \end{pmatrix}, \begin{pmatrix} t^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & t \end{pmatrix}\right].$$

For $t = 0$ the curve meets $\mathcal{DEG}^3_{1,1}$, and therefore $\mathcal{DEG}^3_{1,1}$ is contained in $X_{\mathrm{inv}}(3)$, which finishes the proof of the claim.

Since $\dim \mathcal{DEG}^3_{1,1} = 7$ (see Proposition H.11(i)), D is a prime divisor. We have $\mathrm{Pic}(\mathbf{SL}_3) = 0$ and by [Har77, Prop. II.6.5(c)] the Picard group of $X_{\mathrm{inv}}(3)$ is isomorphic to $\mathbb{Z}$ with the ample generator $[D]$.

Next we check that D is linearly equivalent (as a divisor on $X_{\mathrm{inv}}(3)$) to a hyperplane section H of $X_{\mathrm{inv}}(3)$. Since we already know that $\mathrm{Pic}(X_{\mathrm{inv}}(3)) = \mathbb{Z}\cdot[D]$, we must have $H \overset{\mathrm{lin}}{\sim} kD$ for some positive integer k. But there are lines contained in $X_{\mathrm{inv}}(3)$ (for example those contained in $\mathcal{DEG}^3_{1,0} \simeq \mathbb{P}^2 \times \mathbb{P}^2$) (2). So let $L \subset X_{\mathrm{inv}}(3)$ be any line and consider the intersection

$$D \cdot L = \frac{1}{k} H \cdot L = \frac{1}{k}.$$

But the result must be an integer, so $k = 1$ as claimed.

Complete embedding. Since D itself is definitely not a hyperplane section of $X_{\mathrm{inv}}(3)$, the conclusion is that the Legendrian embedding of $X_{\mathrm{inv}}(3)$ is not given by a complete linear system. A natural guess for a better embedding is the following:

$$X' := \overline{\{[1, g, \textstyle\bigwedge^2 g] \in \mathbb{P}^{18} = \mathbb{P}(\mathbb{C} \oplus V) \mid \det g = 1\}}$$

(we note that $\bigwedge^2 g = (g^{-1})^T$ for g with $\det g = 1$) and one can verify that the projection from the point $[1, 0, 0] \in \mathbb{P}^{18}$ restricted to X' gives an isomorphism with $X_{\mathrm{inv}}(3)$.

The Grassmannian $\mathrm{Gr}(3, 6)$ in its Plücker embedding can be described as the closure of

$$\{[1, g, \textstyle\bigwedge^2 g, \bigwedge^3 g] \in \mathbb{P}^{19} = \mathbb{P}(\mathbb{C} \oplus V \oplus \mathbb{C}) \mid g \in M_{3\times 3}\}$$

and we immediately identify X' as the section $H := \{\bigwedge^3 g = 1\}$ of the Grassmannian.

Though it is not essential, we note that $H^1(\mathcal{O}_{\mathrm{Gr}(3,6)}) = 0$ (see Kodaira vanishing theorem [Laz04, Thm. 4.2.1]; alternatively, this follows from the fact that $b_1 = 0$ for

(2) Actually, the reader could also easily find explicitly some lines (or even planes) which intersect the open orbit, and conclude that $X_{\mathrm{inv}}(3)$ is covered by lines.

Grassmannians) and hence the above embedding of $X_{\mathrm{inv}}(3)$ is given by the complete linear system.

Quadratic equations of $X_{\mathrm{inv}}(3)$. In order to calculate the automorphism group of $X_{\mathrm{inv}}(3)$ we need to know that the quadratic part of its saturated ideal has no other polynomial than linear combinations of polynomials (H.8a)–(H.8d) (see also Remark H.18).

The ideal of the Grassmannian $\mathrm{Gr}(3,6) \subset \mathbb{P}^{19}$ is generated by 35 quadratic equations, which in the above coordinates take the form

$$[\lambda_0, A, B, \lambda_3] \in \mathbb{P}(\mathrm{End}(\textstyle\bigwedge^0\mathbb{C}^3) \oplus \mathrm{End}(\bigwedge^1\mathbb{C}^3) \oplus \mathrm{End}(\bigwedge^2\mathbb{C}^3) \oplus \mathrm{End}(\bigwedge^3\mathbb{C}^3)),$$
$$\textstyle\bigwedge^2 A = \lambda_0 B, \quad \bigwedge^2 B = \lambda_3 A, \quad AB^T = \lambda_0\lambda_3 \,\mathrm{Id}_m, \quad B^T A = \lambda_0\lambda_3 \,\mathrm{Id}_m \,.$$

Although at first glance there are 36 equations above, the equality $\mathrm{tr}(AB^T) = \mathrm{tr}(B^T A)$ makes one of them redundant.

The homogeneous ideal $\mathcal{I}(X')$ is the same but with $\lambda := \lambda_0 = \lambda_3$, and the ideal of $X_{\mathrm{inv}}(3)$ arises as the elimination ideal of $\mathcal{I}(X')$ with the λ eliminated. In particular, the quadratic polynomials in $\mathcal{I}(X_{\mathrm{inv}}(3))$ are exactly those quadrics in $\mathcal{I}(X')$ that do not contain any term with λ. No term in λ other than λ^2 appears more than once in the equations listed above. Therefore the quadratic part of the ideal $\mathcal{I}(X_{\mathrm{inv}}(3))$ arises by eliminating λ from

$$AB^T = \lambda^2 \,\mathrm{Id}_m, \quad B^T A = \lambda^2 \,\mathrm{Id}_m \,.$$

Therefore the polynomials (H.8) span all the quadratic polynomials in $X_{\mathrm{inv}}(3)$.

Automorphism group. It remains to calculate $\mathrm{Aut}(X_{\mathrm{inv}}(3))^0$, the connected component of the automorphism group.

The tangent Lie algebra of the group of automorphisms of a complex projective manifold is equal to the global sections of the tangent bundle (see Theorem C.4). A vector field on $X_{\mathrm{inv}}(3)$ is also a section of $T\,\mathrm{Gr}(3,6)|_{X_{\mathrm{inv}}(3)}$ and we have the short exact sequence

$$0 \to T\,\mathrm{Gr}(3,6)(-1) \to T\,\mathrm{Gr}(3,6) \to T\,\mathrm{Gr}(3,6)|_{X_{\mathrm{inv}}(3)} \to 0$$

The homogeneous vector bundle $T\,\mathrm{Gr}(3,6)(-1)$ is isomorphic to $U^* \otimes Q \otimes \bigwedge^3 U$, where U is the universal subbundle in $\mathrm{Gr}(3,6) \times \mathbb{C}^6$ and Q is the universal quotient bundle. This bundle corresponds to an irreducible module of the parabolic subgroup in $\mathbf{SL}_6$. Calculating explicitly its highest weight and applying the Bott formula [Ott95] we get $H^1(T\,\mathrm{Gr}(3,6)(-1)) = 0$. Hence every section of $TX_{\mathrm{inv}}(3)$ extends to a section of $T\,\mathrm{Gr}(3,6)$. In other words, if $P < \mathrm{Aut}(\mathrm{Gr}(3,6)) \simeq \mathbb{P}\mathbf{GL}_6$ is the subgroup preserving $X_{\mathrm{inv}}(3) \subset \mathrm{Gr}(3,6)$, then the restriction map

$$P \to \mathrm{Aut}(X_{\mathrm{inv}}(3))^0$$

is epimorphic.

The action of $\mathbf{SL}_6$ on $\bigwedge^3\mathbb{C}^6$ preserves the natural symplectic form

$$\omega' : \textstyle\bigwedge^2(\bigwedge^3\mathbb{C}^6) \to \bigwedge^6\mathbb{C}^6 \simeq \mathbb{C}.$$

Since the action of P on $\mathbb{P}(\bigwedge^3\mathbb{C}^6)$ preserves the hyperplane H containing $X_{\mathrm{inv}}(3)$, it must also preserve $H^{\perp_{\omega'}}$, i.e. P preserves $[1,0,0,1] \in \mathbb{P}^{19} = \mathbb{P}(\mathbb{C} \oplus V \oplus \mathbb{C})$. Therefore P acts on the quotient $H/(H^{\perp_{\omega'}}) = V$ and hence the restriction map factorises as

$$P \to \mathrm{Aut}(\mathbb{P}(V), X_{\mathrm{inv}}(3))^0 \twoheadrightarrow \mathrm{Aut}(X_{\mathrm{inv}}(3))^0.$$

By Theorem A.11, the group $\mathrm{Aut}(\mathbb{P}(V), X_{\mathrm{inv}}(3))^0$ is contained in the image of the group homomorphism $\mathbf{Sp}(V) \to \mathbb{P}\mathbf{GL}(V)$, so by Theorem A.9, Proposition H.9 and the above considerations about quadratic polynomials,

$$\mathrm{Aut}(\mathbb{P}(V), X_{\mathrm{inv}}(3))^0 = G.$$

In particular, $X_{\mathrm{inv}}(3)$ cannot be homogeneous as it contains more than one orbit of the connected component of the automorphism group. ∎

We note that the fact that $X_{\mathrm{inv}}(3)$ is not homogeneous can also be proved without calculating the automorphism group. Since $\mathrm{Pic}\, X_{\mathrm{inv}}(3) \simeq \mathbb{Z}$, it follows from Theorem A.6 that $X_{\mathrm{inv}}(3)$ could only be one of the subadjoint varieties. But none of them has Picard group $\mathbb{Z}$ and dimension 8.

Case $m = 4$—spinor variety $\mathbb{S}_6$

Proof. Similarly to the Grassmannian $\mathrm{Gr}(3, 6)$, the spinor variety can be described in terms of matrices. Let $\bigwedge^2 \mathbb{C}^6$ be the vector space of all 6×6 skew-symmetric matrices. Consider the map

$$\begin{aligned} \textstyle\bigwedge^2 \mathbb{C}^6 &\to \mathbb{P}(\textstyle\bigwedge^0 \mathbb{C}^6 \oplus \bigwedge^2 \mathbb{C}^6 \oplus \bigwedge^4 \mathbb{C}^6 \oplus \bigwedge^6 \mathbb{C}^6), \\ g &\mapsto [1,\ g,\ g \wedge g,\ g \wedge g \wedge g]. \end{aligned} \tag{H.23}$$

In coordinates, $g \wedge g$ is just the matrix of all 4×4 Pfaffians of g, and $g \wedge g \wedge g$ is the Pfaffian of g. The spinor variety $\mathbb{S}_6$ is the closure of the image of this map.

For a skew-symmetric matrix g, denote by Pf_{ij} the Pfaffian of g with ith and jth rows and columns removed, and denote by $Pf(g)$ the Pfaffian of g. Now consider the map $\bigwedge^2 \mathbb{C}^6 \to \mathbb{P}(V)$ given by

$$g := \begin{pmatrix} 0 & g_{12} & g_{13} & g_{14} & g_{15} & g_{16} \\ -g_{12} & 0 & g_{23} & g_{24} & g_{25} & g_{26} \\ -g_{13} & -g_{23} & 0 & g_{34} & g_{35} & g_{36} \\ -g_{14} & -g_{24} & -g_{34} & 0 & g_{45} & g_{46} \\ -g_{15} & -g_{25} & -g_{35} & -g_{45} & 0 & g_{56} \\ -g_{16} & -g_{26} & -g_{36} & -g_{46} & -g_{56} & 0 \end{pmatrix} \mapsto$$

$$\left[\begin{pmatrix} Pf_{14} & -Pf_{15} & Pf_{16} & g_{23} \\ -Pf_{24} & Pf_{25} & -Pf_{26} & g_{13} \\ Pf_{34} & -Pf_{35} & Pf_{36} & g_{12} \\ g_{56} & g_{46} & g_{45} & 1 \end{pmatrix}, \begin{pmatrix} g_{14} & g_{15} & g_{16} & Pf_{23} \\ g_{24} & g_{25} & g_{26} & -Pf_{13} \\ g_{34} & g_{35} & g_{36} & Pf_{12} \\ Pf_{56} & -Pf_{46} & Pf_{45} & -Pf(g) \end{pmatrix}, \right]. \tag{H.24}$$

Its image lies in an open neighbourhood of p_1 and satisfies the equations of $X_{\mathrm{inv}}(4)$: 30 quadratic equations Y as in (H.8) and 36 quadratic equations from Corollary H.20(a). So the image is contained in $X_{\mathrm{inv}}(4)$. Moreover, (H.24) is just a linear coordinate change different from (H.23). Therefore, since $\dim X_{\mathrm{inv}}(4) = \dim \mathbb{S}_6$ and both varieties are irreducible it follows that (up to a linear change of coordinates) $X_{\mathrm{inv}}(4) = \mathbb{S}_6$, as claimed in the theorem. ■

Case $m \geq 5$—singular varieties

Proof. Finally, we prove (d). We want to prove that for $m \geq 5$ the variety $X_{\mathrm{inv}}(m)$ is singular at p_1. To do that, we calculate the reduced tangent cone

$$T := (TC_{p_1} X_{\mathrm{inv}}(m))_{\mathrm{red}}.$$

From (H.8) we easily get the following linear and quadratic equations of T (again we suggest looking at §H.1):

$$b_{im} = b_{mi} = 0, \qquad A_m B_m^T = B_m^T A_m = \lambda^2 \,\mathrm{Id}_{m-1}$$

for every $i \in \{1, \ldots, m\}$ and some $\lambda \in \mathbb{C}^*$.

Next assume I and J are two sets of indices, both of cardinality $k = \lfloor \frac{1}{2} m \rfloor$ and such that neither I nor J contains m. Consider the equation of $X_{\mathrm{inv}}(m)$ as in Corollary H.20(b):

$$(\det A_{I,J})^2 = (\det B_{I',J'})^2 \cdot (a_{11} b_{11} + \cdots + a_{1m} b_{1m})^l.$$

To get an equation of T, we evaluate at $a_{mm} = 1$ and take the lowest degree part, which is simply $(\det((A_m)_{I,J}))^2 = 0$. Since T is reduced, by varying I and J we get

$$\mathrm{rk}\, A_m \leq m - 1 - k - 1 = \left\lceil \frac{1}{2} m \right\rceil - 2$$

and therefore also

$$A_m B_m^T = B_m^T A_m = 0.$$

Hence T is contained in the product of the linear space $W := \{A_m = 0, B = 0\}$ and the affine cone $\hat{U}$ over the union of $X_{\mathrm{deg}}(m-1, k)$ for $k \leq \lceil \frac{1}{2} m \rceil - 2$. We claim that $T = W \times \hat{U}$. By Proposition H.11(i), every component of $W \times \hat{U}$ has dimension $2m - 2 + (m-1)^2 = m^2 - 1 = \dim X_{\mathrm{inv}}(m)$, so by §B.8(1) the tangent cone must be a union of some of the components. Therefore to prove the claim it is enough to find for every $k \leq \lceil \frac{1}{2} m \rceil - 2$ a single element of $\mathcal{DEG}^{m-1}_{k,m-k-1}$ that is contained in the tangent cone.

So take α and β to be two strictly positive integers such that

$$\alpha = \left(\frac{1}{2} m - k - 1 \right) \beta$$

and consider the curve in $\mathbb{P}(V)$ with the following parametrisation:

$$[\mathrm{diag}\{\underbrace{t^{\alpha}, \ldots, t^{\alpha}}_{k}, \underbrace{t^{\alpha+\beta}, \ldots, t^{\alpha+\beta}}_{m-k-1}, 1\}, \mathrm{diag}\{\underbrace{t^{\alpha+\beta}, \ldots, t^{\alpha+\beta}}_{k}, \underbrace{t^{\alpha}, \ldots, t^{\alpha}}_{m-k-1}, t^{2\alpha+\beta}\}].$$

It is easy to verify that this family is contained in $\mathcal{INV}^m$ for $t \neq 0$ and as t converges to 0, it gives rise to a tangent vector (i.e. an element of the reduced tangent cone—see the pointwise definition in §B.8) that belongs to $\mathcal{DEG}^{m-1}_{k,m-k-1}$.

So indeed $T = W \times \hat{U}$, which for $m \geq 5$ contains more than one component, hence cannot be a linear space. Therefore by §B.8(3) the variety $X_{\mathrm{inv}}(m)$ is singular at p_1. ■

I. Hyperplane sections of Legendrian subvarieties

The content of §I.1 and §I.3 of this chapter appeared in [Buc08b].

The Legendrian variety $X_{\mathrm{inv}}(3)$ constructed in Chapter H is isomorphic to a hyperplane section of another Legendrian variety $\mathrm{Gr}(3,6)$. In this chapter we prove that general hyperplane sections of other Legendrian varieties also admit a Legendrian embedding. This gives numerous new examples of smooth Legendrian subvarieties.

I.1. Hyperplane section

I.1.1. Construction. The idea of the construction builds on the concept of symplectic reduction (see §D.1.3). Let $H \in \mathbb{P}(V^*)$ be a hyperplane in V. By

$$h := H^{\perp_\omega} \subset V$$

we denote the subspace of V ω-perpendicular to H, which in this case is a line contained in H. We think of h both as a point in the projective space $\mathbb{P}(V)$ and a line in V. We define

$$\pi : \mathbb{P}(H) \setminus \{h\} \to \mathbb{P}(H/h)$$

to be the projection map and for a given Legendrian subvariety $X \subset \mathbb{P}(V)$ we let $\widetilde{X}_H := \pi(X \cap H)$.

We have the natural symplectic structure ω' on H/h determined by ω (see §D.1.3). Also $\widetilde{X}_H$ is always Legendrian by Proposition D.1 and Lemma B.2.

Note that so far we have not used any smoothness condition on X.

I.1.2. Proof of smoothness. Hence to prove Theorem A.14 it is enough to prove that for a general $H \in \mathbb{P}(V^*)$, the map π gives an isomorphism of the smooth locus of $X \cap H$ onto its image, an open subset in $\widetilde{X}_H$.

Recall the definition of secant variety from §B.9.

Lemma I.1. *Let $Y \subset \mathbb{P}^m$, choose $y \in \mathbb{P}^m$ such that $y \notin \sigma(Y)$ and let $\pi : \mathbb{P}^m \backslash \{y\} \to \mathbb{P}^{m-1}$ be the projection map.*

(a) *If Y is smooth, then π gives an isomorphism of Y and $\pi(Y)$.*
(b) *In general, π is one-to-one and an isomorphism of the smooth part of Y onto its image. In particular, the dimension of the singular locus of Y is greater than or equal to the dimension of the singular locus of $\pi(Y)$.*

Proof. See [Har77, Prop. IV.3.4]—the proof is identical to that for the curve case. We only note that if Y is smooth, then the secant variety $\sigma(Y)$ contains all the embedded tangent spaces of Y. They arise when the limits y_2 approach y_1. ∎

Now we can prove Theorem A.14:

Proof. By the lemma and the construction in §I.1.1 it is enough to prove that there exists $h \in \mathbb{P}(V)$ such that $h \notin \sigma(X \cap h^{\perp_\omega})$.

Given two different points x_1 and x_2 in a projective space we denote by $\langle x_1, x_2\rangle$ the projective line through x_1 and x_2. Let

$$\tilde{\sigma}(X) \subset X \times X \times \mathbb{P}(V), \qquad \tilde{\sigma}(X) := \overline{\{(x_1, x_2, p) \mid p \in \langle x_1, x_2\rangle\}},$$

so that $\tilde{\sigma}(X)$ is the incidence variety for the secant variety of X. Obviously,

$$\dim \tilde{\sigma}(X) = 2\dim X + 1 = \dim \mathbb{P}(V)$$

and $\tilde{\sigma}(X)$ is irreducible. Also we let

$$\kappa(X) \subset \tilde{\sigma}(X), \qquad \kappa(X) := \overline{\{(x_1, x_2, h) \mid h \in \langle x_1, x_2\rangle \text{ and } x_1, x_2 \in h^{\perp_\omega}\}},$$

so that the image of the projection of $\kappa(X)$ onto the last coordinate is the locus of 'bad' points. More precisely, for a point $h \in \mathbb{P}(V)$ there exist (x_1, x_2) such that $(x_1, x_2, h) \in \kappa(X)$ if and only if $h \in \sigma(X \cap h^{\perp_\omega})$.

We claim that the image of $\kappa(X)$ under the projection is not the whole $\mathbb{P}(V)$. To see this, note that the condition defining $\kappa(X)$, i.e., $h \in \langle x_1, x_2\rangle$, $x_1, x_2 \in h^{\perp_\omega}$, is equivalent to $h \in \langle x_1, x_2\rangle$ and $\langle x_1, x_2\rangle$ is an isotropic subspace of V. Now either X is a linear subspace and then both the claim and the theorem are obvious, or there exist two points $x_1, x_2 \in X$ such that $\omega(\hat{x}_1, \hat{x}_2) \neq 0$, where by $\hat{x}_i$ we denote some non-zero point in the line $x_i \subset V$. Therefore $\kappa(X)$ is strictly contained in $\tilde{\sigma}(X)$ and

$$\dim \kappa(X) < \dim \tilde{\sigma}(X) = \dim \mathbb{P}(V),$$

so the image of $\kappa(X)$ under the projection cannot be equal to $\mathbb{P}(V)$ ([1]). ■

Corollary I.2. *Let $X \subset \mathbb{P}(V)$ be an irreducible Legendrian subvariety whose singular locus has dimension at most $k-1$. Let F be the contact distribution on $\mathbb{P}(V)$. If $H \subset \mathbb{P}(V)$ is a general F-cointegrable linear subspace of codimension k, then H does not intersect the singular locus of X and $\widetilde{X}_H := X \cap H$ is smooth and admits a Legendrian embedding via an appropriate subsystem of the linear system $\mathcal{O}_{\widetilde{X}_H}(1)$ into $\mathbb{P}(H/H^{\perp_\omega})$.* ■

We sketch some proofs of Examples A.15:

Proof. $K3$ surfaces of (a) arise as codimension 4 linear sections of the Lagrangian Grassmannian $\mathrm{Gr}_L(3,6)$. Since the canonical divisor $K_{\mathrm{Gr}_L(3,6)}$ equals $\mathcal{O}_{\mathrm{Gr}_L(3,6)}(-4)$ (in other words, $\mathrm{Gr}_L(3,6)$ is Fano of index 4), by the adjunction formula, the canonical divisor of the section is indeed trivial. On the other hand, by [LM07, Prop. 9] it must have genus 9. Although we take quite special (F-cointegrable) sections, they fall into the 19-dimensional family of Mukai's $K3$-surfaces of genus 9 [Muk88] and they form a 13-dimensional subfamily.

The other families of surfaces as in (b) arise as sections of the other exceptional subadjoint varieties: $\mathrm{Gr}(3,6)$, $\mathbb{S}_6$ and E_7. They are all Fano of index 6, 10 and 18 respectively and their dimensions are 9, 15 and 27, hence taking successive linear sections we get to Calabi–Yau manifolds as stated in (c). Further, the canonical divisor is very ample, so we have examples of general type as stated in (b) and (d).

The Fano varieties arise as intermediate steps, before coming down to the level of Calabi–Yau manifolds. Also $\mathbb{P}^1 \times Q^n$ is a subadjoint variety and its hyperplane section

([1]) The inequality on the dimensions, although simple, is essential for the proof. An analogous construction for Lagrangian subvarieties in symplectic manifolds is known as symplectic reduction (see §D.1.3 for linear algebra baby version of this), but does not produce smooth Lagrangian subvarieties.

is the blow up of a quadric Q^n in a codimension 2 linear section. The Del Pezzo surfaces are the hyperplane sections of the blow up of Q^3 in a conic curve. ∎

I.2. Linear sections of decomposable Legendrian varieties. Assume m_1 and m_2 are two positive integers, $m_1 \geq m_2$. Let $V_1 \simeq \mathbb{C}^{2m_1+2}$ and $V_2 \simeq \mathbb{C}^{2m_2+2}$ be two symplectic vector spaces, and let $X_1 \subset \mathbb{P}(V_1)$ and $X_2 \subset \mathbb{P}(V_2)$ be two smooth, irreducible, non-degenerate, Legendrian subvarieties. In this setup $\dim X_i = m_i$. Consider the decomposable variety $X_1 * X_2 \subset \mathbb{P}(V_1 \oplus V_2)$. Clearly $\operatorname{Sing}(X_1 * X_2) = X_1 \sqcup X_2$, hence $\dim(\operatorname{Sing}(X_1 * X_2)) = m_1$, while

$$\dim(X_1 * X_2) = m_1 + m_2 + 1.$$

Therefore a general codimension $m_1 + 1$ F-cointegrable linear section of $X_1 * X_2$ will be smooth of dimension m_2 and admit a Legendrian embedding. The purpose of this section is to explain that these newly constructed varieties have essentially different properties than those of X_1 and X_2. Hence our method also allows one to produce new examples without dropping the dimension.

Let L be the following line bundle on $X_1 \times X_2$:

$$L := \mathcal{O}_{X_1}(1) \boxtimes \mathcal{O}_{X_2}(-1).$$

Also let $(X_1 * X_2)_0$ be the smooth locus of $X_1 * X_2$.

LEMMA I.3. *$(X_1 * X_2)_0$ is isomorphic to $\mathbf{L}^\bullet$, the total space of the $\mathbb{C}^*$-bundle associated to L (see §C.3).*

Proof. Let $\mathbb{C}^*$ act on $V_1 \oplus V_2$ with weight -1 on V_1 and weight 1 on V_2. Then

$$(\mathbb{P}(V_1 \oplus V_2) \setminus (\mathbb{P}(V_1) \sqcup \mathbb{P}(V_2)))/\mathbb{C}^* = \mathbb{P}(V_1) \times \mathbb{P}(V_2),$$

and the quotient map

$$(\mathbb{P}(V_1 \oplus V_2) \setminus (\mathbb{P}(V_1) \sqcup \mathbb{P}(V_2))) \xrightarrow{/\mathbb{C}^*} \mathbb{P}(V_1) \times \mathbb{P}(V_2)$$

is a principal $\mathbb{C}^*$-bundle obtained by removing the zero section from the total space of the line bundle $\mathcal{O}_{\mathbb{P}(V_1)\times\mathbb{P}(V_2)}(d_1, d_2)$ for some integers d_1 and d_2. We have

$$\operatorname{Pic}(\mathbb{P}(V_1 \oplus V_2) \setminus (\mathbb{P}(V_1) \sqcup \mathbb{P}(V_2))) = \operatorname{Pic}\mathbb{P}(V_1 \oplus V_2) = \mathbb{Z}[\mathcal{O}_{\mathbb{P}(V_1\oplus V_2)}(1)]$$

by [Har77, Prop. II.6.5(b)]). On the other hand,

$$\operatorname{Pic}(\mathbb{P}(V_1 \oplus V_2) \setminus (\mathbb{P}(V_1) \sqcup \mathbb{P}(V_2))) = \operatorname{Pic}(\mathbb{P}(V_1) \times \mathbb{P}(V_2))/\langle \mathcal{O}_{\mathbb{P}(V_1)\times\mathbb{P}(V_2)}(d_1, d_2)\rangle$$

(by Lemma C.6). Moreover, via the isomorphism

$$\operatorname{Pic}(\mathbb{P}(V_1) \times \mathbb{P}(V_2))/\langle \mathcal{O}_{\mathbb{P}(V_1)\times\mathbb{P}(V_2)}(d_1, d_2)\rangle \simeq \mathbb{Z}[\mathcal{O}_{\mathbb{P}(V_1\oplus V_2)}(1)]$$

the class of the line bundle $\mathcal{O}_{\mathbb{P}(V_1\oplus V_2)}(e_1, e_2)$ is mapped to $\mathcal{O}_{\mathbb{P}(V_1\oplus V_2)}(e_1 + e_2)$. Hence $(d_1, d_2) = (1, -1)$ or $(-1, 1)$. In both cases the total spaces of the line bundles are the same after removing the zero sections (the difference is only in the sign of the weights of the $\mathbb{C}^*$-action, which we ignore at this point).

To finish the proof just note that

$$(X_1 * X_2)_0 = (X_1 * X_2) \cap (\mathbb{P}(V_1 \oplus V_2) \setminus (\mathbb{P}(V_1) \sqcup \mathbb{P}(V_2)))$$

and the image of $(X_1 * X_2)_0$ under the quotient map is equal to $X_1 \times X_2$. ∎

Hence by Lemma C.6 we have

$$\operatorname{Pic}(X_1 \times X_2) \twoheadrightarrow \operatorname{Pic}(X_1 * X_2)_0 = \operatorname{Cl}(X_1 * X_2)$$

and the kernel of the epimorphic map is generated by L. If $L_1 \in \operatorname{Pic} X_1$ and $L_2 \in \operatorname{Pic} X_2$, by $[L_1 \boxtimes L_2]$ we will denote the line bundle on $(X_1 * X_2)_0$ which represents the image of $L_1 \boxtimes L_2$ under the epimorphic map.

Theorem I.4. *Let m_1, m_2, X_1, X_2 be as above. Let F be the contact distribution on $\mathbb{P}(V_1 \oplus V_2)$ and let $H \subset \mathbb{P}(V_1 \oplus V_2)$ be a general F-cointegrable linear subspace of codimension $m_1 + 1$. Then $X := (X_1 * X_2) \cap H$ is smooth, admits a Legendrian embedding and has the following properties:*

(a) $\deg X = \deg X_1 \cdot \deg X_2$.
(b) $K_X \simeq [K_{X_1} \boxtimes K_{X_2}]|_X \otimes \mathcal{O}_X(m_1 + 1)$.
(c) *We have the restriction map on the Picard groups:*

$$i^* : \operatorname{Pic}(X_1 \times X_2)/\langle L \rangle \to \operatorname{Pic} X.$$

If $m_2 \geq 3$, then i^* is an isomorphism. If $m_2 = 2$, then i^* is injective.

In particular, we have:

(d) *X is not projectively isomorphic to neither X_1 or X_2.*
(e) *If $K_{X_1} \simeq \mathcal{O}_{X_1}(d_1)$ and $K_{X_2} \simeq \mathcal{O}_{X_2}(d_2)$, then $K_X \simeq \mathcal{O}_X(d_1 + d_2 + m_1 + 1)$.*
(f) *If $K_{X_1} \simeq \mathcal{O}_{X_1}(d_1) \otimes E_1$ and $K_{X_2} \simeq \mathcal{O}_{X_2}(d_2) \otimes E_2$, where the E_i's are line bundles corresponding to some effective divisors, then*

$$K_X \simeq \mathcal{O}_X(d_1 + d_2 + m_1 + 1) \otimes E$$

for some E corresponding to an effective divisor.
(g) *If $m_2 \geq 3$, $\operatorname{Pic} X_1 = \mathbb{Z}[\mathcal{O}_{X_1}(1)]$, $\operatorname{Pic} X_2 = \mathbb{Z}[\mathcal{O}_{X_2}(1)]$ and either X_1 or X_2 is simply connected (for example Fano), then $\operatorname{Pic} X = \mathbb{Z}[\mathcal{O}_X(1)]$.*

Proof. Since X_1 and X_2 are smooth, the singular locus of $X_1 * X_2$ has dimension m_1. By Corollary I.2, a general coisotropic linear section of codimension $m_1 + 1$ (such as H) does not intersect the singular locus of $X_1 * X_2$. Therefore X is smooth and has natural Legendrian embedding into $\mathbb{P}(H/H^{\perp_\omega})$.

Part (a) is immediate, since $\deg(X_1 * X_2) = \deg X_1 \cdot \deg X_2$ and neither a linear section nor a projection changes the degree.

Part (b) follows from Lemma I.3, §C.3 and the adjunction formula (see [Har77, Prop. II.8.20]).

Part (c) follows from the following generalisation of the Grothendieck–Lefschetz theorem due to Ravindra and Srinivas [RS06]:

Theorem I.5. *Let Y be a subvariety of $\mathbb{P}^m$ with singular locus of codimension at least 2. Let Y' be its general hyperplane section and let Y_0 and Y'_0 be the smooth loci of Y and Y' respectively. Then the restriction map $\operatorname{Pic} Y_0 \to \operatorname{Pic} Y'_0$ is an isomorphism if the dimension of Y' is at least 3, and is injective if $\dim Y' = 2$.*

Since our variety X arises as a repeated hyperplane section and projection of $X_1 * X_2$ and the projection part is always an isomorphism of smooth parts, we repeatedly apply the theorem of Ravindra and Srinivas to deduce our claim.

Part (d) follows from (a): X_1 and X_2 are not linear spaces, hence $\deg X_i > 1$. Therefore $\deg X > \deg X_i$ for $i = 1, 2$ and the varieties cannot be projectively isomorphic.

Parts (e) and (f) are immediate consequences of (b) and (c), since $[\mathcal{O}_{X_1}(d_1) \boxtimes \mathcal{O}_{X_2}(d_2)]$ is isomorphic to $\mathcal{O}_{(X_1 * X_2)_0}(d_1 + d_2)$.

Finally, (g) follows from (c) and from [Har77, Ex. III.12.6]. ∎

To conclude we give a further series of examples:

Example I.6. Apply the theorem to both X_1 and X_2 equal to the E_7-variety. As a result we get X which we denote by $(E_7)^{*2}$, a smooth Legendrian Fano variety of dimension 27, with Picard group generated by a hyperplane section and of index 8. Now apply the theorem to X_1 being the E_7-variety again and $X_2 = (E_7)^{*2}$. The result, $(E_7)^{*3}$, again has Picard group generated by a hyperplane section and $K_{(E_7)^{*3}} = \mathcal{O}_{(E_7)^{*3}}(2)$, hence is very ample. Analogously we construct $(E_7)^{*k}$ and combining this result with Corollary I.2, we get infinitely many families of smooth Legendrian varieties of general type with Picard group generated by a very ample class in every dimension d, where $3 \le d \le 27$.

Example I.7. Let $X_1 = \mathbb{P}^1 \times Q^{m_1 - 1}$ and X_2 be arbitrary. If $m_1 \ge 3$ and $\dim X_2 \ge 3$, then X has Picard group isomorphic to $\operatorname{Pic} X_2 \oplus \mathbb{Z}$. Hence we can get a smooth Legendrian variety with arbitrarily large Picard rank.

Example I.8. Let $X_1 = X_2 = \mathbb{P}^1 \times Q^{m-1}$. Let the resulting X be called $(\mathbb{P}^1 \times Q^{m-1})^{*2}$. Then $K_{X_i} = \mathcal{O}_{X_i}(-m) \otimes E_i$, where E_i is effective. Hence

$$K_{(\mathbb{P}^1 \times Q^{m-1})^{*2}} = \mathcal{O}_{(\mathbb{P}^1 \times Q^{m-1})^{*2}}(-m+1) \otimes E$$

for an effective E. Construct analogously $(\mathbb{P}^1 \times Q^{m-1})^{*k}$ by taking the section of

$$((\mathbb{P}^1 \times Q^{m-1})^{*(k-1)}) * (\mathbb{P}^1 \times Q^{m-1}).$$

We get

$$K_{(\mathbb{P}^1 \times Q^{m-1})^{*k}} = \mathcal{O}_{(\mathbb{P}^1 \times Q^{m-1})^{*k}}(-m-1+k) \otimes E$$

and for $k > m+1$ the canonical divisor can be written as an ample plus an effective one, so it is big. Hence in every dimension, it is possible to construct many smooth Legendrian varieties with the maximal Kodaira dimension.

I.3. Extending Legendrian varieties. Our motivation is the example of Landsberg and Manivel [LM07, §4], a Legendrian embedding of a Kummer $K3$-surface blown up in 12 points. It can be seen that this embedding is given by a codimension 1 linear system. We want to find a Legendrian 3-fold in $\mathbb{P}^7$ whose hyperplane section is this example. Unfortunately, we are not able to find a smooth 3-fold with these properties, but we get one with only isolated singularities.

We recall the setup for the construction of the example. Let $Z^\sharp \subset \mathbb{P}(T^*\mathbb{P}^n) \subset \mathbb{P}^n \times \check{\mathbb{P}}^n$ be the *conormal variety*, i.e., the closure of the union of the projectivised conormal spaces over smooth points of Z. Landsberg and Manivel study in detail an explicit birational map

$$\varphi := \varphi_{H_0, p_0} : \mathbb{P}(T^*\mathbb{P}^n) \dashrightarrow \mathbb{P}^{2n-1}$$

which depends on a hyperplane H_0 in $\mathbb{P}^n$ and on a point $p_0 \in H_0$. After Bryant [Bry82] they observe that $\overline{\varphi(Z^\sharp)}$ (if only it makes sense) is always a Legendrian subvariety, but usually singular. Next they study conditions under which $\overline{\varphi(Z^\sharp)}$ is smooth. In particular, they prove that the conditions are satisfied when Z is a Kummer quartic surface in $\mathbb{P}^3$ in general position with respect to p_0 and H_0, and this gives rise to their example.

We want to modify the above construction a little to obtain our 3-fold. Instead of considering $Z^\sharp$ as a subvariety in

$$\mathbb{P}(W) \times \mathbb{P}(W^*) = (W \setminus \{0\}) \times (W^* \setminus \{0\})/\mathbb{C}^* \times \mathbb{C}^*,$$

we consider a subvariety X in

$$\mathbb{P}^{2n+1} = \mathbb{P}(W \oplus W^*) = (W \times W^*) \setminus \{0\}/\mathbb{C}^*$$

such that the underlying affine cone of X in $W \times W^*$ is the same as the underlying affine pencil of $Z^\sharp$. In other words, we take X to be the closure of the preimage of $Z^\sharp$ under the natural projection map

$$p : \mathbb{P}(W \oplus W^*) \dashrightarrow \mathbb{P}(W) \times \mathbb{P}(W^*).$$

Both $\mathbb{P}(W)$ and $\mathbb{P}(W^*)$ are naturally embedded in $\mathbb{P}(W \oplus W^*)$. Let H be a hyperplane in $\mathbb{P}(W \oplus W^*)$ which contains neither $\mathbb{P}(W)$ nor $\mathbb{P}(W^*)$. Set $H_0 := \mathbb{P}(W) \cap H$ and let p_0 be the point in $\mathbb{P}(W)$ dual to $\mathbb{P}(W^*) \cap H$. Assume H is chosen in such a way that $p_0 \in H_0$.

Theorem I.9. *Let $X \subset \mathbb{P}(W \oplus W^*) \simeq \mathbb{P}^{2n+1}$ be a subvariety constructed as above from any irreducible subvariety $Z \subset \mathbb{P}(W)$. On $W \oplus W^*$ consider the standard symplectic structure* (*see* §D.1.5) *and on $\mathbb{P}(W \oplus W^*)$ consider the associated contact structure. Also assume H, H_0 and p_0 are chosen as above. Then:*

(i) *X is a Legendrian subvariety contained in the quadric $\overline{p^{-1}(\mathbb{P}(T^*\mathbb{P}(W)))}$.*

(ii) *Let $\widetilde{X}_H$ be the Legendrian variety in $\mathbb{P}^{2n-1}$ constructed from X and H as in* §I.1.1. *Also consider the closure of $\varphi_{H_0,p_0}(Z^\sharp)$ as in the construction of* [LM07, §4]. *Then the two constructions agree, i.e., the closure $\varphi_{H_0,p_0}(Z^\sharp)$ is a component of $\widetilde{X}_H$.*

(iii) *The singular locus of X is equal to the union of the following:*
- *on $\mathbb{P}(W)$, the singular points of Z,*
- *on $\mathbb{P}(W^*)$, the singular points of Z^*, and*
- *outside $\mathbb{P}(W) \cup \mathbb{P}(W^*)$, the preimage under p of the singular locus of the conormal variety $Z^\sharp$.*

Proof. For (i) consider $\widehat{Z} \subset W$, the affine cone over $Z \subset \mathbb{P}(W)$. The cotangent bundle to W is equal to $W \oplus W^*$. Furthermore, by our definition $\widehat{X} \subset V$, the affine cone over $X \subset \mathbb{P}(W \oplus W^*)$ is the conormal variety of $\widehat{X}$, so a Lagrangian subvariety (see Example D.6).

For (ii), we choose coordinates $x_0, x_1, \ldots, x_n$ on W and dual coordinates $y^0, y^1, \ldots, y^n$ on W^* such that in the induced coordinates on V the hyperplane H has the equation $x_0 - y^n = 0$. Now restrict to the affine piece $x_0 = y^n = 1$ on both H and $\mathbb{P}(W) \times \mathbb{P}(W^*)$. We see explicitly that the projection map $H \to \mathbb{P}^{2n-1}$,

$$[1, x_1, \ldots, x_n, y^0, \ldots, y^{n-1}, 1] \mapsto [y^1, \ldots, y^{n-1}, y^0 - x_n, x_1, \ldots, x_{n-1}, 1],$$

agrees with the map φ from [LM07, §4].

To find the singularities of X on $X \cap \mathbb{P}(W)$, as in (iii) note that X is invariant under the following action of $\mathbb{C}^*$ on $\mathbb{P}(W \oplus W^*)$:

$$t \cdot [w, \alpha] := [tw, t^{-1}\alpha].$$

In particular, the points of $X \cap \mathbb{P}(W)$ are the fixed points of the action. So let $[w, 0] \in X$ and then $T_{[w,0]}X$ decomposes into the eigenspaces of the action:

$$T_{[w,0]}X = T_{[w,0]}(X \cap \mathbb{P}(W)) \oplus T_{[w,0]}(X \cap F_w) \tag{I.10}$$

where F_w is the fibre of the projection $\rho : \mathbb{P}(W \oplus W^*)\backslash\mathbb{P}(W^*) \to \mathbb{P}(W)$, $F_w := \rho^{-1}([w])$. Clearly the image of X under the projection ρ is Z, so the dimension of a general fibre of $\rho|_X : X \to Z$ is equal to $\dim X - \dim Z = \dim \mathbb{P}(W) - \dim Z = \operatorname{codim}_{\mathbb{P}(W)} Z$. Therefore, since the dimension of the fibre can only grow at special points, we have

$$\dim T_{[w,0]}(X \cap F_w) \geq \dim(X \cap F_w) \geq \operatorname{codim}_{\mathbb{P}(W)} Z. \tag{I.11}$$

Also $\mathrm{D}_{[w,0]}(\rho|_X) : T_{[w,0]}X \to T_{[w]}Z$ maps $T_{[w,0]}(X \cap F_w)$ to 0 and $T_{[w,0]}(X \cap \mathbb{P}(W))$ onto $T_{[w]}Z$. Therefore

$$\dim T_{[w,0]}(X \cap \mathbb{P}(W)) \geq \dim T_{[w]}Z \geq \dim Z. \tag{I.12}$$

Now assume $[w, 0]$ is a smooth point of X. Then adding (I.11) and (I.12) we get

$$\begin{aligned}\dim X = \dim T_{[w,0]}X &\overset{(\mathrm{I}.10)}{=} \dim T_{[w,0]}(X \cap F_w) + \dim T_{[w,0]}(X \cap \mathbb{P}(W))\\ &\geq \operatorname{codim}_{\mathbb{P}(W)} Z + \dim Z = \dim \mathbb{P}(W).\end{aligned}$$

By (i), $\dim X = \dim \mathbb{P}(W)$, so in (I.11) and (I.12) all the inequalities are in fact equalities. In particular, $\dim T_{[w]}Z = \dim Z$, so $[w]$ is a smooth point of Z.

Conversely, assume $[w]$ is a smooth point of Z. Then

$$T_{[w,0]}X = T_{[w]}Z \oplus N^*_{[w]}(Z \subset \mathbb{P}(W)),$$

therefore clearly $[w, 0]$ is a smooth point of X.

Exactly the same argument shows that Z^* is singular at $[\alpha]$ if and only if X is singular at $[0, \alpha] \in X \cap \mathbb{P}(W^*)$.

For the last part of (iii) it is enough to note that p is a locally trivial $\mathbb{C}^*$-bundle when restricted to $\mathbb{P}(W \oplus W^*) \setminus (\mathbb{P}(W) \cup \mathbb{P}(W^*))$. ∎

Corollary I.13. *Given a Legendrian subvariety $\widetilde{Z} \subset \mathbb{P}^{2n-1}$ we can take*

$$Z^\sharp := \varphi^{-1}_{H_0,p_0}(\widetilde{Z})$$

to construct a Legendrian subvariety in $\mathbb{P}(T^*\mathbb{P}^n)$. Such a variety must be the conormal variety to some variety $Z \subset \mathbb{P}^n$ (see Corollary E.17*). Let $X \subset \mathbb{P}^{2n+1}$ be the Legendrian variety constructed from Z as above. By Theorem* I.9(ii)*, a component of a hyperplane section of X can be projected onto $\widetilde{Z}$.*

Unfortunately, in the setup of the theorem, X is almost always singular (see §I.4).

Example I.14. If Z is a Kummer quartic surface in $\mathbb{P}^3$, then X is a 3-fold with 32 isolated singular points (this follows from Theorem I.9(iii) because the Kummer quartic surface has 16 singular points, it is isomorphic to its dual and it has smooth conormal variety in $\mathbb{P}(T^*\mathbb{P}^3)$). Therefore by Theorem A.14 a general hyperplane section of X is smooth and admits a Legendrian embedding. By Theorem I.9 the example of Landsberg and Manivel is a special case of this hyperplane section. Even though the condition $p_0 \in H_0$ is a closed condition, it satisfies the generality conditions of Theorem A.14 and therefore this hyperplane section consists of a unique smooth component that is projected isomorphically onto $\widetilde{Z}$.

EXAMPLE I.15. Similarly, if Z is a curve in $\mathbb{P}^2$ satisfying the generality conditions of Bryant [Bry82, Thm. G], then X is a surface with only isolated singularities and its hyperplane section projects isomorphically onto a Bryant's Legendrian curve.

I.4. Smooth varieties with smooth dual. Furthermore, we observe that a classical problem of classifying smooth varieties with smooth dual variety can be expressed in terms of Legendrian varieties:

COROLLARY I.16. *Using the notation of the previous section, let* $Q_W \subset \mathbb{P}(W \oplus W^*)$ *be the quadric* $\overline{p^{-1}(\mathbb{P}(T^*\mathbb{P}(W)))}$*—see Theorem* I.9(i). *On* $W \oplus W^*$ *consider the standard symplectic structure* (*see* §D.1.5) *and on* $\mathbb{P}(W \oplus W^*)$ *consider the associated contact structure* (*see* §E.1).

- (i) *Let* $Z \subset \mathbb{P}(W)$ *be a smooth subvariety such that* $Z^* \subset \mathbb{P}(W^*)$ *is smooth. Let* $X \subset \mathbb{P}(W \oplus W^*)$ *be as in the above construction. Then* X *is a smooth Legendrian variety contained in* Q_W.
- (ii) *Conversely, assume* $X \subset \mathbb{P}(W \oplus W^*)$ *is irreducible, Legendrian and contained in* Q_W. *Let* $Z = X \cap \mathbb{P}(W)$. *Then* $Z^* = X \cap \mathbb{P}(W^*)$ *and the variety arising from* Z *in the above construction is exactly* X. *Moreover, if* X *is smooth, then* Z *and* Z^* *are smooth.*

We underline that although all the smooth quadrics of a given dimension are projectively isomorphic, the classification of quadrics relative to the contact structure is more complicated. The quadric Q_W can therefore be written as $x_0y_0 + \cdots + x_ny_n = 0$ in some *symplectic* coordinates $x_0, \ldots, x_n, y_0, \ldots, y_n$ on $W \oplus W^*$. We note (without proof) that such a quadric Q_W determines uniquely the pair of Lagrangian subspaces W and W^*.

Proof. Part (i) follows immediately from Theorem I.9(i), (iii).

To prove (ii), consider $p(X) \subset \mathbb{P}(T^*\mathbb{P}(W))$. By Lemma B.2 and Proposition D.1, $p(X)$ is Legendrian. By Corollary E.17, $p(X)$ is a conormal variety to some subvariety $Z \subset \mathbb{P}(W)$. The next thing to prove is that X coincides with the variety constructed above from Z, i.e.

$$X = \overline{p^{-1}(p(X))}.$$

Equivalently, it is enough to prove that X is $\mathbb{C}^*$-invariant. This is provided by Theorem A.9 since the quadric Q_W produces exactly the required action. Finally, it follows that $Z = X \cap \mathbb{P}(W)$. Moreover, $p(X)$ is also the conormal variety to $Z^* \subset \mathbb{P}(W^*)$ and hence $Z^* = X \cap \mathbb{P}(W^*)$. If X is in addition smooth, then Z and Z^* are smooth by Theorem I.9(iii). ■

Therefore the classification of smooth varieties with smooth dual is equivalent to the classification of pairs (X, Q), where $Q \subset \mathbb{P}^{2n+1}$ is a quadric which can be written as

Table 1. The known self-dual varieties and their corresponding Legendrian varieties. Note that Q^{2m} and $\mathbb{P}^1 \times \mathbb{P}^m$ lead to isomorphic Legendrian varieties, yet their embeddings in the distinguished quadrics are not isomorphic.

Smooth self-dual variety $Z \subset \mathbb{P}^n$	The corresponding Legendrian variety $X \subset \mathbb{P}^{2n+1}$
Q^m	$\mathbb{P}^1 \times Q^m$
$\mathbb{P}^1 \times \mathbb{P}^m$	$\mathbb{P}^1 \times Q^{2m}$
$\mathrm{Gr}(2,5)$	$\mathrm{Gr}(3,6)$
$\mathbb{S}_5$	$\mathbb{S}_6$

$x_0y_0 + \cdots + x_ny_n = 0$ in some symplectic coordinates

$$x_0, \ldots, x_n, y_0, \ldots, y_n$$

on $\mathbb{C}^{2n+2}$, and $X \subset \mathbb{P}^{2n+1}$ is a smooth Legendrian variety contained in Q. So far the only known examples of smooth varieties with smooth dual are the smooth self-dual varieties (see [Ein86]). From these we get some of the homogeneous Legendrian varieties (see Table 1). Therefore we cannot hope to produce new examples of smooth Legendrian varieties in this way. What we hope for is to classify the pairs (X, Q) as above and hence finish the classification of smooth varieties with smooth dual.

References

[Akh95] D. N. Akhiezer, *Lie Group Actions in Complex Analysis*, Aspects Math. E27, Vieweg, Braunschweig, 1995.

[Arn74] V. I. Arnold, *Mathematical Methods of Classical Mechanics*, Nauka, Moscow, 1974 (in Russian).

[Bea98] A. Beauville, *Fano contact manifolds and nilpotent orbits*, Comment. Math. Helv. 73 (1998), 566–583.

[Bea07] —, *Riemannian holonomy and algebraic geometry*, Enseign. Math. (2) 53 (2007), 97–126.

[Ber55] M. Berger, *Sur les groupes d'holonomie homogène des variétés à connexion affine et des variétés riemanniennes*, Bull. Soc. Math. France 83 (1955), 279–330.

[Bry82] R. L. Bryant, *Conformal and minimal immersions of compact surfaces into the 4-sphere*, J. Differential Geom. 17 (1982), 455–473.

[BK94] R. Brylinski and B. Kostant, *Minimal representations of E_6, E_7, and E_8 and the generalized Capelli identity*, Proc. Nat. Acad. Sci. U.S.A. 91 (1994), 2469–2472.

[Buc03] J. Buczyński, *Legendrian subvarieties of a complex projective space*, Master's thesis, Inst. Math., Warsaw Univ., 2003 (in Polish).

[Buc06] —, *Legendrian subvarieties of projective space*, Geom. Dedicata 118 (2006), 87–103.

[Buc07] —, *Toric Legendrian subvarieties*, Transform. Groups 12 (2007), 631–646.

[Buc08a] —, *Algebraic Legendrian varieties*, PhD thesis, Inst. Math., Warsaw Univ., 2008.

[Buc08b] —, *Hyperplane sections of Legendrian subvarieties*, Math. Res. Lett. 15 (2008), 623–629.

[Buc09] —, *Some quasihomogeneous Legendrian varieties*, Adv. Geom. 9 (2009), 257–277.

[Cou95] S. C. Coutinho, *A Primer of Algebraic D-Modules*, London Math. Soc. Student Texts 33, Cambridge Univ. Press, Cambridge, 1995.

[Dem02] J.-P. Demailly, *On the Frobenius integrability of certain holomorphic p-forms*, in: Complex Geometry (Göttingen, 2000), Springer, Berlin, 2002, 93–98.

[Dru98] S. Druel, *Structures de contact sur les variétés algébriques de dimension 5*, C. R. Acad. Sci. Paris Sér. I Math. 327 (1998), 365–368.

[Ein86] L. Ein, *Varieties with small dual varieties. I*, Invent. Math. 86 (1986), 63–74.

[Ful93] W. Fulton, *Introduction to Toric Varieties*, Ann. of Math. Stud. 131, Princeton Univ. Press, Princeton, NJ, 1993.

[FH91] W. Fulton and J. Harris, *Representation Theory*, Grad. Texts in Math. 129, Springer, New York, 1991.

[Gri74] P. Griffiths, *Topics in Algebraic and Analytic Geometry*, Math. Notes 13, Princeton Univ. Press, Princeton, NJ, 1974.

[Gro61] A. Grothendieck, *Éléments de géométrie algébrique. II. Étude globale élémentaire de quelques classes de morphismes*, Inst. Hautes Études Sci. Publ. Math. 8 (1961), 222 pp.

[Har95] J. Harris, *Algebraic Geometry*, Grad. Texts in Math. 133, Springer, New York, 1995.

[Har77] R. Hartshorne, *Algebraic Geometry*, Grad. Texts in Math. 52, Springer, New York, 1977.

[HH02] H. Herrera and R. Herrera, *$\hat{A}$-genus on non-spin manifolds with S^1 actions and the classification of positive quaternion-Kähler 12-manifolds*, J. Differential Geom. 61 (2002), 341–364.

[Hit81] N. J. Hitchin, *Kählerian twistor spaces*, Proc. London Math. Soc. (3) 43 (1981), 133–150.

[Hon00] J. Hong, *Fano manifolds with geometric structures modeled after homogeneous contact manifolds*, Int. J. Math. 11 (2000), 1203–1230.

[Hum75] J. E. Humphreys, *Linear Algebraic Groups*, Grad. Texts in Math. 21, Springer, New York, 1975.

[Huy03] D. Huybrechts, *Compact hyperkähler manifolds*, in: Calabi–Yau Manifolds and Related Geometries (Nordfjordeid, 2001), Universitext, Springer, Berlin, 2003, 161–225.

[JJ04] S. Janeczko and Z. Jelonek, *Linear automorphisms that are symplectomorphisms*, J. London Math. Soc. (2) 69 (2004), 503–517.

[Joy00] D. D. Joyce, *Compact manifolds with special holonomy*, Oxford Math. Monogr., Oxford Univ. Press, Oxford, 2000.

[Keb01] S. Kebekus, *Lines on contact manifolds*, J. Reine Angew. Math. 539 (2001), 167–177.

[Keb05] —, *Lines on complex contact manifolds. II*, Compos. Math. 141 (2005), 227–252.

[KPSW00] S. Kebekus, T. Peternell, A. J. Sommese, and J. A. Wiśniewski, *Projective contact manifolds*, Invent. Math. 142 (2000), 1–15.

[KN96] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry. Vol. I*, Wiley Classics Library, Wiley, New York, 1996.

[KO73] S. Kobayashi and T. Ochiai, *Characterizations of complex projective spaces and hyperquadrics*, J. Math. Kyoto Univ. 13 (1973), 31–47.

[Kol96] J. Kollár, *Rational Curves on Algebraic Varieties*, Ergeb. Math. Grenzgeb. 32, Springer, Berlin, 1996.

[LM01] J. M. Landsberg and L. Manivel, *The projective geometry of Freudenthal's magic square*, J. Algebra 239 (2001), 477–512.

[LM02] —, —, *Construction and classification of complex simple Lie algebras via projective geometry*, Selecta Math. (N.S.) 8 (2002), 137–159.

[LM07] J. M. Landsberg and L. Manivel, *Legendrian varieties*, Asian J. Math. 11 (2007), 341–359.

[Laz04] R. Lazarsfeld, *Positivity in Algebraic Geometry. I*, Ergeb. Math. Grenzgeb. 48, Springer, Berlin, 2004.

[LeB95] C. LeBrun, *Fano manifolds, contact structures, and quaternionic geometry*, Int. J. Math. 6 (1995), 419–437.

[LS94] C. LeBrun and S. M. Salamon, *Strong rigidity of positive quaternion-Kähler manifolds*, Invent. Math. 118 (1994), 109–132.

[Leh04] M. Lehn, *Symplectic moduli spaces*, in: Intersection Theory and Moduli, ICTP Lect. Notes XIX, Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2004, 139–184.

[MS98] D. McDuff and D. Salamon, *Introduction to Symplectic Topology*, 2nd ed., Oxford Math. Monogr., Oxford Univ. Press, New York, 1998.

[MT99] V. B. Mehta and V. Trivedi, *The variety of circular complexes and F-splitting*, Invent. Math. 137 (1999), 449–460.

[Muk88] S. Mukai, *Curves, $K3$ surfaces and Fano 3-folds of genus ≤ 10*, in: Algebraic Geometry and Commutative Algebra, Vol. I, Kinokuniya, Tokyo, 1988, 357–377.

[Muk98] —, *Simple Lie algebra and Legendre variety*, http://www.math.nagoya-u.ac.jp/˜mukai/, 1998.

[Mum99] D. Mumford, *The Red Book of Varieties and Schemes*, Lecture Notes in Math. 1358, Springer, Berlin, 1999.

[Oda88] T. Oda, *Convex Bodies and Algebraic Geometry*, Ergeb. Math. Grenzgeb. 15, Springer, Berlin, 1988.

[Ott95] G. Ottaviani, *Rational homogeneous varieties*, http://web.math.unifi.it/users/ottavian/rathomo/rathomo.ps, 1995.

[PS91] Y. S. Poon and S. M. Salamon, *Quaternionic Kähler 8-manifolds with positive scalar curvature*, J. Differential Geom. 33 (1991), 363–378.

[Pro07] C. Procesi, *Lie Groups*, Universitext, Springer, New York, 2007.

[RS06] G. V. Ravindra and V. Srinivas, *The Grothendieck–Lefschetz theorem for normal projective varieties*, J. Algebraic Geom. 15 (2006), 563–590.

[Sła91] J. J. Sławianowski, *Geometry of Phase Spaces*, Wiley, Chichester, 1991.

[SCW04] L. E. Solá Conde and J. A. Wiśniewski, *On manifolds whose tangent bundle is big and 1-ample*, Proc. London Math. Soc. (3) 89 (2004), 273–290.

[Str82] E. Strickland, *On the conormal bundle of the determinantal variety*, J. Algebra 75 (1982), 523–537.

[Stu97] B. Sturmfels, *Equations defining toric varieties*, in: Algebraic Geometry—Santa Cruz 1995, Proc. Sympos. Pure Math. 62, Amer. Math. Soc., Providence, RI, 1997, 437–449.

[Tim03] D. A. Timashëv, *Equivariant compactifications of reductive groups*, Mat. Sb. 194 (2003), 119–146 (in Russian).

[Wiś00] J. A. Wiśniewski, *Lines and conics on Fano contact manifolds*, http://www.mimuw.edu.pl/˜jarekw/, 2000.

[Wol65] J. A. Wolf, *Complex homogeneous contact manifolds and quaternionic symmetric spaces*, J. Math. Mech. 14 (1965), 1033–1047.

[Ye94] Y.-G. Ye, *A note on complex projective threefolds admitting holomorphic contact structures*, Invent. Math. 115 (1994), 311–314.