Contents

Abstract

Always when a numerical method gives exact results an interesting functional equation arises. And, since no regularity is assumed, some unexpected solutions may appear. Here we deal with equations constructed in this spirit. The vast majority of this paper is devoted to the equation

$$\sum_{i=0}^{l}(y-x)^i[f_{1,i}(\alpha_{1,i}x+\beta_{1,i}y)+\cdots+f_{k_i,i}(\alpha_{k_i,i}x+\beta_{k_i,i}y)]=0 \tag{1}$$

and its particular cases.

We use Sablik's lemma to prove that all solutions of (1) are polynomial functions. Since a continuous polynomial function is an ordinary polynomial, the crucial problem throughout the whole paper will be the continuity of solutions of (1).

The first of the particular forms of (1) which we consider is

$$F(y)-F(x)=(y-x)[a_1f(\alpha_1x+\beta_1y)+\cdots+a_nf(\alpha_nx+\beta_ny)] \tag{2}$$

and is motivated by the quadrature formulas of numerical integration. Quadrature rules give exact results for polynomials, and therefore the following problem becomes interesting: do equations of the type (2) characterize polynomials? We present new results concerning this equation, in particular, we obtain a general solution of (2) in the case of rational α_i,β_i, $i=1,\dots,n$, and we show that if (2) has discontinuous solutions then the equation

$$a_1f(\alpha_1x+\beta_1y)+\cdots+a_nf(\alpha_nx+\beta_ny)=0$$

has nontrivial solutions. This result allows us to solve functional equations motivated by all classical quadrature rules such as the rule of Simpson (this equation was already solved earlier), Radau, Lobatto and Gauss.

Further we also consider the following equation:

$$F(y)-F(x)=(y-x)[a_1f(\alpha_1x+\beta_1y)+\cdots+a_nf(\alpha_nx+\beta_ny)]+(y-x)^2[g(y)-g(x)], \tag{3}$$

which is connected with Hermite quadrature formulas where on the right-hand side derivatives of f are used;

$$\begin{aligned}F(y)-F(x)=(y-x)[a_1f(x)+b_1f(\alpha_1x+\beta_1y)+\cdots+b_nf(\alpha_nx+\beta_ny)+a_1f(y)]\\+(y-x)^3[c_1g(\alpha_1x+\beta_1y)+\cdots+c_ng(\alpha_nx+\beta_ny)],\end{aligned} \tag{4}$$

which stems from Birkhoff quadrature rules where f'' is involved; and

$$g(\alpha x+\beta y)(y-x)^k=a_1f(\alpha_1x+\beta_1y)+\cdots+a_nf(\alpha_nx+\beta_ny), \tag{5}$$

which is motivated by formulas used in numerical differentiation. Results concerning (5) are used to obtain new facts about the well known equation

$$f[x_1,\dots,x_n]=g(x_1+\cdots+x_n)$$

($f[x_1,\dots,x_n]$ is the nth divided difference of f).

We also present a direct method which may be used to show that solutions of (2) must be polynomial functions and, motivated by this method, we obtain a generalization of the Aczél

equation

$$F(y) - F(x) = (y - x)g\left(\frac{x + y}{2}\right).$$

At the end of the paper we present a list of open problems.

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1. Introduction

1.1. The origin of the problem. This paper is devoted mainly to the functional equation

$$\sum_{i=0}^{l}(y-x)^i[f_{1,i}(\alpha_{1,i}x+\beta_{1,i}y)+\cdots+f_{k_i,i}(\alpha_{k_i,i}x+\beta_{k_i,i}y)]=0 \tag{1}$$

and to its diverse particular cases.

We shall deal with this equation for functions defined on $\mathbb{R}$ and taking values in $\mathbb{R}$. It is possible to prove numerous results on integral domains but then the assumptions (on the domain) would expand and the results would become unclear. Therefore, for the sake of simplicity, we restrict ourselves to the real case.

The idea to study this equation was motivated by the growing number of equations of this form stemming from numerical analysis. The first of them is the equation

$$F(y)-F(x)=(y-x)[f_1(\alpha_1x+\beta_1y)+\cdots+f_n(\alpha_nx+\beta_ny)], \tag{1.1}$$

which is connected with the classical quadrature rules used in numerical integration. Particular cases of this equation were already studied by several authors.

The first author to be mentioned here is J. Aczél [1] who observed that the function $F(x)=x^2$ satisfies the equation

$$F(x)-F(y)=(y-x)F'\left(\frac{x+y}{2}\right).$$

Then he introduced the functional equation

$$F(y)-F(x)=(y-x)f\left(\frac{x+y}{2}\right) \tag{1.2}$$

with two unknown functions f and F. In [1] it was proved that functions f,F satisfy this equation if and only if $f(x)=ax+b$ and F is its antiderivative (which was quite surprising because no regularity of functions f and F was assumed). The problem of continuity of solutions of equations which we consider will play a crucial role throughout this paper. Aczél's equation (1.2) was then generalized by W. Rudin [32] who asked for the solutions of the equation

$$F(y)-F(x)=(y-x)f(sx+ty), \tag{1.3}$$

which was solved by M. S. Jacobson, Pl. Kannappan and P. K. Sahoo [13].

The next equation of the form (1.1), considered by Sh. Haruki [12], has the form

$$F(y)-F(x)=(y-x)[f(x)+f(y)]. \tag{1.4}$$

Although Aczél and Haruki were not motivated by quadrature rules, we easily see that equations (1.2), (1.3) and (1.4) are of the form (1.1).

Now we would like to give some examples of quadrature rules and construct the related functional equations in the spirit of Aczél. His idea was to replace the derivative by some unknown function and to find the solutions of the resulting functional equation. As a consequence, he obtained a functional equation which may be considered without assuming any regularity of the functions involved, and the continuity of solutions is a consequence of the equation itself.

Quadrature rules are used in numerical analysis to approximate the definite integral in the following way:

$$\int_x^y f(t)\,dt \approx (y-x)[a_1 f(\alpha_1 x+\beta_1 y)+\cdots+a_n f(\alpha_n x+\beta_n y)]. \tag{1.5}$$

Therefore we shall consider functional equations (1.1) and the equation

$$F(y)-F(x)=(y-x)[a_1 f(\alpha_1 x+\beta_1 y)+\cdots+a_n f(\alpha_n x+\beta_n y)], \tag{2}$$

which is a simplified (depexiderized) version of (1.1). We want to find the solutions of these equations, which means that we want to determine for which functions the equations connected with quadrature rules are exact. It will be shown that (2) is much more general than (1.2) and it does not imply the continuity of f, in general.

It is worth noticing that the error we make when we calculate the integral using formula (1.5) is just the derivative of certain order (calculated at some point) multiplied by a constant. For this reason (1.5) is exact for polynomials of the degree which depends on the number n, on the form of nodes $\alpha_i x+\beta_i y$ and coefficients a_i and, consequently, equation (2) is satisfied by polynomials. Thus polynomials may be called "model solutions", and clearly we are interested in answering the question in which case functional equations of the form (1.1) or (2) may have solutions other than polynomials.

The first quadrature rule which was studied from the point of view of functional equations was the Simpson quadrature rule

$$\int_x^y f(t)\,dt \approx (y-x)\left[\frac{1}{6}f(x)+\frac{2}{3}f\left(\frac{x+y}{2}\right)+\frac{1}{6}f(y)\right]. \tag{1.6}$$

Then the equation

$$F(y)-F(x)=(y-x)\left[\frac{1}{6}f(x)+\frac{2}{3}f\left(\frac{x+y}{2}\right)+\frac{1}{6}f(y)\right] \tag{1.7}$$

and its pexiderized version

$$F(y)-F(x)=(y-x)\big[f(x)+g(x+y)+h(y)\big] \tag{1.8}$$

were considered in [16], [31], [10], [19] and others.

The next example is given by the Simpson 3/8 rule

$$\int_x^y f(t)\,dt \approx (y-x)\left[\frac{1}{8}f(x)+\frac{3}{8}f\left(\frac{1}{3}x+\frac{2}{3}y\right)+\frac{3}{8}f\left(\frac{2}{3}x+\frac{1}{3}y\right)+\frac{1}{8}f(y)\right].$$

T. Riedel and P. K. Sahoo [31] asked about solutions of the related functional equation

$$F(y)-F(x)=(y-x)[f(x)+g(x+2y)+g(2x+y)+f(y)],$$

which was then treated by Sahoo [35], [36]. Further M. Sablik and A. Lisak solved a pexiderized version of [26] and a result on an integral domain with some properties was obtained in [23].

Now we would like to consider more general quadrature rules. The most important examples of quadrature rules are those of Gauss, Lobatto and Radau. In the Gaussian quadrature rule all (suitably chosen) nodes lie in the interior of the interval $[x, y]$, and this rule is exact for polynomials of degree at most $2n-1$. For example if $n = 2$ then we get

$$\int_x^y f(t)\,dt \approx (y-x)\left[f\left(\frac{3-\sqrt{3}}{6}x + \frac{3+\sqrt{3}}{6}y\right) + f\left(\frac{3+\sqrt{3}}{6}x + \frac{3-\sqrt{3}}{6}y\right)\right].$$

The functional equation connected with this quadrature rule is

$$F(y) - F(x) = (y-x)[f(\alpha x + \beta y) + f(\beta x + \alpha y)]$$

and was treated in [20]. This was the first paper where nodes with irrational coefficients were admitted.

For the exact values of the node weights occurring in Gauss quadrature rules for $n > 2$ see e.g. [30, 45].

In the quadrature rules of Lobatto and Radau two (resp. one) of the endpoints are used as nodes. Then of course the degree of polynomials for which the formula fits exactly will get smaller. For example the two-node Radau rule

$$\int_x^y f(t)\,dt \approx (y-x)\left[f(x) + f\left(\frac{1}{3}x + \frac{2}{3}y\right)\right]$$

is exact for polynomials of degree at most 2. For further details see [30, 46, 47].

The most general result (until now) concerning equations of the form (2) was proved in [21] where it was shown (under some assumptions) that F must be continuous and if $\alpha_i + \beta_i = 1$, $a_1 + \cdots + a_n \neq 0$ then f must also be continuous.

Another formula used in numerical analysis to approximate the definite integral is provided by the Hermite quadrature rule (we assume here that $x < y$)

$$\int_x^y f(t)\,dt \approx \frac{y-x}{n}\left[\frac{f(x)+f(y)}{2} + f\left(x + \frac{y-x}{n}\right) + \cdots + f\left(x + (n-1)\frac{y-x}{n}\right)\right] + \frac{(y-x)^2}{12}[f'(x) - f'(y)]. \tag{1.9}$$

As we can see, in this case not only values of the function are used but also values of its derivative. Thus this quadrature rule is exact for polynomials of higher degree than for quadrature rules of the form (1.5) (if we use the same number of nodes). A functional equation stemming from this quadrature rule (with $n = 2$)

$$F(y) - F(x) = (y-x)[f(x) + af(\alpha x + \beta y) + f(y)] + (y-x)^2[g(y) - g(x)]$$

was solved in [22]. There are no results concerning solutions of the equation connected with (1.9) in its full generality.

Moreover, sometimes also higher order derivatives are used to approximate the definite integral. For example the Birkhoff quadrature rule

$$\int_x^y f(t)\,dt \approx (y-x)\left[\frac{1}{10}f(x) + \frac{4}{5}f\left(\frac{x+y}{2}\right)\frac{1}{10}f(y)\right] + \frac{(y-x)^3}{60}f''\left(\frac{x+y}{2}\right)$$

is exact for polynomials of degree at most 5. However, functional equations of the form

$$F(y) - F(x) = \sum_{i=1}^{l} (y-x)^i [f_{1,i}(\alpha_{1,i}x + \beta_{1,i}y) + \cdots + f_{k_i,i}(\alpha_{k_i,i}x + \beta_{k_i,i}y)] \tag{1.10}$$

with $l > 2$ have not been studied yet.

Observe that if we deal with functional equations stemming from quadrature rules, it is enough to consider the functional equation (1.10). The reason that we deal with (1) is that we want to cover the case of functional equations connected with numerical differentiation. Thus we shall now write a few words about numerical differentiation. The simplest way of approximating the derivative is by the formula

$$f'(x) \approx \frac{1}{2h}[f(x-h) - f(x+h)],$$

which is exact for polynomials of degree at most 2. The related functional equation

$$g\left(\frac{x+y}{2}\right)(y-x) = f(x) - f(y)$$

is still of the form (1.10). However, to obtain equations satisfied by polynomials of higher degrees, more complicated formulas must be used, for example

$$f'(x) \approx \frac{1}{12h}[-f(x+2h) + 8f(x+h) - 8f(x-h) + f(x-2h)]$$

or

$$f''(x) \approx \frac{1}{12h^2}[-f(x+2h) + 16f(x+h) - 30f(x) + 16f(x-h) - f(x-2h)].$$

The corresponding functional equations are

$$3f\left(\frac{x+y}{2}\right)(y-x) = F(x) - 8F\left(\frac{3x+y}{4}\right) + 8F\left(\frac{x+3y}{4}\right) - F(y)$$

and

$$\frac{3}{4}f\left(\frac{x+y}{2}\right)(y-x)^2 = -F(x) + 16F\left(\frac{3x+y}{4}\right) - 30F\left(\frac{x+y}{2}\right) + 16F\left(\frac{x+3y}{4}\right) - F(y).$$

Note that here $h = y-x$ and the constants on the left-hand sides are introduced to ensure that $F' = f$ (in the case of regular solutions). Thus now we shall work with equations of the form

$$g(\alpha x + \beta y)(y-x)^k = a_1 f(\alpha_1 x + \beta_1 y) + \cdots + a_n f(\alpha_n x + \beta_n y), \tag{5}$$

which are not of the form (1.10). To cover this case we introduce a more general equation (1). We shall prove that functions satisfying equations of this kind are polynomial functions. However (as is easily seen) low order summands of f will not have to be continuous.

1.2. Known methods. We speak here about equations (2) and (1.1) only, because equation (1) is introduced in the current paper and has not been studied yet.

There are several ways of solving functional equations of the type (2) and (1.1). The first of them may be called "direct". In this method no tools are required and solutions are obtained by substitutions and other simple operations.

Another method was used in [28], where the following theorem can be found.

Theorem 1.1. *Let $I \subset \mathbb{R}$ be an open interval and let $\Omega \subset I \times I$ be an open subset containing the diagonal $D(I) := \{(x,x) : x \in I\}$ of I. Then every continuous solution of*

$$\int_0^1 f(x + t(y-x))\, d\mu(t) = 0, \qquad (x,y) \in \Omega, \tag{1.11}$$

is a polynomial of degree at most $n-1$ where n is the smallest nonnegative integer such that

$$\mu_n := \int_0^1 t^n\, d\mu(t) \neq 0.$$

To see that from this theorem we can obtain solutions of functional equations of the type (2) let us consider the equation

$$\frac{k+1}{y-x}\int_x^y f(t)\,dt = \frac{f(x)+f(y)}{2} + kf\left(\frac{x+y}{2}\right),$$

which may be rewritten in the form

$$\frac{f(x)+f(y)}{2} + kf\left(\frac{x+y}{2}\right) - (k+1)\int_0^1 f(t)\,dt = 0.$$

Now it suffices to consider the measure

$$\mu := \frac{\delta_0 + \delta_1}{2} + k\delta_{1/2} - (k+1)\lambda,$$

where δ_t is the Dirac measure concentrated at t and λ stands for the Lebesgue measure. Then we obtain an equation of the type (1.11), and after some further calculations it is easy to show that f must be a polynomial of degree at most 3. For further details see [28]

However, this method works only if functions occurring in (1.1) are continuous.

The third method is based on a lemma proved by M. Sablik [33]. First, using this lemma, we show that functions f_i and F satisfying these equations must be polynomial functions (continuous or not). In the next step we show that if some polynomial functions satisfy our equation then their monomial summands are also solutions, and then we work with these monomial functions. This method was used in [20], [21] in the case of equations of the form (2). In the current paper we shall also use this method.

Remark 1.2. In general it is impossible to use Sablik's lemma in order to show that the F occurring in (1.1) and in (1.10) is a polynomial function. However, if we know that other functions occurring in this equation are polynomial then it is easy to find the exact form of F by taking $x = 0$ in this equation.

The diversity of sets of solutions in particular cases of equation (1) is so vast that it is impossible to completely describe the set of its solutions. However, if we know that solutions of some particular case of (1) are continuous then these functions must be polynomials.

Therefore the continuity of solutions plays a crucial role here.

1.3. A short description of new results. In the first part of the paper we prove that all solutions of (1) must be polynomial functions. This is a generalization of the result from [23] where the same assertion was proved for (1.1). If we know that solutions of this equation are polynomial functions then the problem we face is their continuity.

In [21] it was proved (under some assumptions) that if f and F satisfy (2) then F must be continuous, and if α_i, β_i, a_i satisfy some additional assumptions then f is also continuous. In the current paper we extend results of this kind, i.e. we shall prove the continuity of F for the much more general equation (1.10). As a consequence, we shall answer a question posed by M. Sablik. More precisely, it will be proved that if the functional equation (1.1) has a discontinuous solution then the equation

$$a_1 f(\alpha_1 x + \beta_1 y) + \cdots + a_n f(\alpha_n x + \beta_n y) = 0$$

has a nontrivial solution. We shall even prove a more general result which may be called a monomial version of the conjecture of Sablik. It is worth noticing that the main result of [21] may be easily obtained from this theorem. Moreover, using this theorem, we shall obtain a complete solution of (2) with rational α_i, β_i.

Further the result that F satisfying (1.10) (with some $f_{j,i}$) must be continuous will be used to deal with equations

$$F(y) - F(x) = (y - x)[a_1 f(\alpha_1 x + \beta_1 y) + \cdots + a_n f(\alpha_n x + \beta_n y)] + (y - x)^2[g(y) - g(x)],$$

motivated by the Hermite quadrature rule, and

$$\begin{aligned} F(y) - F(x) = (y - x)&[a_1 f(\alpha_1 x + \beta_1 y) + \cdots + a_n f(\alpha_n x + \beta_n y)] \\ &+ (y - x)^3[a_1 g(\alpha_1 x + \beta_1 y) + \cdots + a_n g(\alpha_n x + \beta_n y)], \end{aligned}$$

connected with the Birkhoff quadrature rules. In both cases we shall be able to prove the continuity of the functions involved (under some reasonable assumptions).

Then we consider equations stemming from numerical differentiation, i.e. (5). This equation seems more difficult to deal with than the previous ones. Obviously (5) is a particular case of (1) but it is harder to obtain a continuity result similar to those which are valid in the case of (1.10). We shall prove a result of this kind but the assumptions needed will be much stronger than in the case of (1.1). Then we shall apply the result obtained to get the solutions of the equation

$$g(x_1 + \cdots + x_n) = f[x_1, \ldots, x_n]$$

assuming that $x_1, \ldots, x_n$ belong to some subset of $\mathbb{R}^n$ ($f[x_1, \ldots, x_n]$ is the nth divided difference of f). This means that the result we obtain is, in fact, stronger than known results concerning this equation. We shall also present a new and simple way to prove that functions f_i satisfying (1.1) must be polynomial functions (without use of Sablik's lemma). Surprisingly, this method also works if the equation considered is satisfied on an interval. Then (motivated by this method) we shall present a far-reaching generalization of equation (1.2).

As discussed in the last part of the paper, this method can also be used to prove the superstability of equation (1.2) (which was done in [40]) and of the equation considered by Sh. Haruki,

$$F(y) - F(x) = (y - x)[f(x) + f(y)]. \tag{1.4}$$

2. Preliminaries

2.1. Polynomial functions. First we shall introduce a notion of polynomial functions which will play an important role throughout. Our approach is based on [25] where also further details may be found. Let $I \subset \mathbb{R}$ be an interval, let $f : I \to \mathbb{R}$ be an arbitrary function and let $x, h \in \mathbb{R}$ be such that $x, x+h \in I$. The difference operator with span h is given by

$$\Delta_h f(x) = f(x+h) - f(x).$$

The iterates Δ_h^n are defined recursively,

$$\Delta_h^0 f := f, \qquad \Delta_h^{n+1} f := \Delta_h(\Delta_h^n f), \qquad n = 1, 2, \ldots.$$

Using this operator, we define polynomial functions in the following way:

Definition 2.1. A function $f : \mathbb{R} \to \mathbb{R}$ is called a *polynomial function* of order n if

$$\Delta_h^{n+1} f(x) = 0 \quad \text{ for all } x \in \mathbb{R}.$$

The shape of solutions of this equation was obtained in [27]. To present this result we shall need the notion of multiadditive functions. A function $f : \mathbb{R}^n \to \mathbb{R}$ is *n-additive* if for every $i \in \{1, \ldots, n\}$ and all $x_1, \ldots, x_n, y_i \in \mathbb{R}$ we have

$$f(x_1, \ldots, x_{i-1}, x_i + y_i, x_{i+1}, \ldots, x_n) \\ = f(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) + f(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n).$$

Further, having a function $F : \mathbb{R}^n \to \mathbb{R}$, by the *diagonalization* of F we mean the function f defined by

$$f(x) := F(x, \ldots, x).$$

Now we can present the characterization of polynomial functions.

Theorem 2.2. *Let $f : \mathbb{R} \to \mathbb{R}$ be a polynomial function of order n. Then there exist unique k-additive functions $F_k : \mathbb{R}^k \to \mathbb{R}$, $k = 1, \ldots, n$, and a constant a_0 such that*

$$f(x) = a_0 + f_1(x) + \cdots + f_n(x), \tag{2.1}$$

where f_k is the diagonalization of F_k. Conversely, every function of the shape (2.1) *is a polynomial function of order n.*

One should also mention a result of L. Székelyhidi [39, Theorem 9.5]:

Theorem 2.3. *Let G be an Abelian semigroup, S an Abelian group, n a nonnegative integer, φ_i, ψ_i additive functions from G to G and let $\varphi_i(G) \subset \psi_i(G)$, $i = 1, \ldots, n$. If functions $f, f_i : G \to S$ satisfy the equation*

$$f(x) + \sum_{i=1}^{n} f_i(\varphi_i(x) + \psi_i(y)) = 0 \tag{2.2}$$

then $\Delta_{h_1,\ldots,h_n} f(x) = 0$.

2.2. Sablik's lemma. If we want to solve the functional equation (1.1) then one of the main steps in the proof is to show that functions satisfying this equation are polynomial functions. In [23], [20] this was done using a lemma proved in [33]. To quote this profound generalization of Theorem 2.3 we introduce some notation.

Let G, H be Abelian groups and $SA^0(G,H) := H$, $SA^1(G,H) := \mathrm{Hom}(G,H)$ (i.e. the group of all homomorphisms from G into H), and for $i \in \mathbb{N}$, $i \geq 2$, let $SA^i(G,H)$ be the group of all i-additive and symmetric mappings from G^i into H. Furthermore, let

$$\mathscr{P} := \{(\alpha,\beta) \in \mathrm{Hom}(G,G)^2 : \alpha(G) \subset \beta(G)\}.$$

Finally, for $x \in G$ let $x^{\langle i \rangle} = \underbrace{(x,\ldots,x)}_{i}$, $i \in \mathbb{N}$.

Lemma 2.4. *Fix $N \in \mathbb{N} \cup \{0\}$ and let $I_0,\ldots,I_N$ be finite subsets of $\mathscr{P}$. Suppose that H is uniquely divisible by $N!$ and that $\varphi_i : G \to SA^i(G,H)$ and $\psi_{i,(\alpha,\beta)} : G \to SA^i(G,H)$ $((\alpha,\beta) \in I_i$, $i = 0,\ldots,N)$ satisfy*

$$\varphi_N(x)(y^{\langle N \rangle}) + \sum_{i=0}^{N-1} \varphi_i(x)(y^{\langle i \rangle}) = \sum_{i=0}^{N} \sum_{(\alpha,\beta)\in I_i} \psi_{i,(\alpha,\beta)}\big(\alpha(x)+\beta(y)\big)(y^{\langle i \rangle}) \tag{2.3}$$

for all $x, y \in G$. Then φ_N is a polynomial function of order at most $k-1$, where

$$k = \sum_{i=0}^{N} \mathrm{card}\Big(\bigcup_{s=i}^{N} I_s\Big).$$

In Sablik's original result the order of φ_N was not greater than k, while I. Pawlikowska [29] noticed that it can be lowered by 1.

Using this lemma, it can be proved (under some mild assumptions) that all solutions f_i of (1.1) must be polynomial functions (see [23]).

Now we shall prove a much more general result concerning the equation

$$\sum_{i=0}^{l} (y-x)^i [f_{1,i}(\alpha_{1,i}x + \beta_{1,i}y) + \cdots + f_{k_i,i}(\alpha_{k_i,i}x + \beta_{k_i,i}y)] = 0. \tag{1}$$

To do this we shall need a generalized version of Lemma 2.4 which can be found in [26].

Lemma 2.5. *Let $N, M, K \in \mathbb{N} \cup \{0\}$ and let $I_0,\ldots,I_{M+K}$ be finite subsets of $\mathscr{P}$. Suppose further that H is uniquely divisible by $N!$ and that functions $\varphi_i : G \to SA^i(G;H)$, $i = 0,\ldots,N$, and functions $\psi_{i,(\alpha,\beta)} : G \to SA^i(G;H)$ $((\alpha,\beta) \in I_i$, $i = 0,\ldots,M+K)$ satisfy*

$$\begin{aligned}\varphi_N(x)(y^{\langle N \rangle}) + \sum_{i=0}^{N-1} \varphi_i(x)(y^{\langle i \rangle}) &= \sum_{i=0}^{M} \sum_{(\alpha,\beta)\in I_i} \psi_{i,(\alpha,\beta)}(\alpha(x)+\beta(y))(y^{\langle i \rangle}) \\ &\quad + \sum_{i=M+1}^{M+K} \sum_{(\alpha,\beta)\in I_i} \psi_{i,(\alpha,\beta)}(\alpha(x)+\beta(y))(x^{\langle i \rangle}) \end{aligned} \tag{2.4}$$

for all $x, y \in G$. Then φ_n is a polynomial function of order not greater than

$$\sum_{i=0}^{M+K} \mathrm{card}\Big(\bigcup_{s=i}^{M+K} I_s\Big) - 1.$$

Using this lemma, we shall prove that functions $f_{j,i}$ satisfying (1) are polynomial.

THEOREM 2.6. *Let l and k_i, $i = 0,\dots,l$, be positive integers and let*

$$\alpha_{1,i},\dots,\alpha_{k_i,i},\beta_{1,i},\dots,\beta_{k_i,i} \in \mathbb{R} \tag{2.5}$$

for $i \in \{0,\dots,l\}$. Assume that functions $f_{j,i} : \mathbb{R} \to \mathbb{R}$, $i = 0,\dots,l$, $j = 1,\dots,k_i$, fulfill for any $x, y \in \mathbb{R}$ the functional equation

$$\sum_{i=0}^{l} (y-x)^i [f_{1,i}(\alpha_{1,i}x + \beta_{1,i}y) + \dots + f_{k_i,i}(\alpha_{k_i,i}x + \beta_{k_i,i}y)] = 0. \tag{1}$$

Let $i_0 \in \{0,\dots,l\}$ and $j_0 \in \{1,\dots,k_{i_0}\}$ be such that

$$\alpha_{j_0,i_0} + \beta_{j_0,i_0} \neq 0, \tag{2.6}$$

and set

$$J_i := \left\{(\alpha_{j,i},\beta_{j,i}) : j \in \{1,\dots,k_i\}, \left|\begin{smallmatrix} \alpha_{j_0,i_0} & \beta_{j_0,i_0} \\ \alpha_{j,i} & \beta_{j,i} \end{smallmatrix}\right| = 0\right\}.$$

If

$$J_{i_0} = \{(\alpha_{j_0,i_0},\beta_{j_0,i_0})\} \quad \textit{and} \quad J_i = \emptyset \quad \textit{for all } i \in \{i_0+1,\dots,l\} \tag{2.7}$$

then f_{j_0,i_0} is a polynomial function of order at most

$$\sum_{i=0}^{l} \operatorname{card}\Big(\bigcup_{s=i}^{l} (\{(\alpha_{1,s},\beta_{1,s}),\dots,(\alpha_{k_s,s},\beta_{k_s,s})\} \setminus J_s)\Big) - 1.$$

Proof. First we take $\tilde{x} = \frac{x-\beta_{j_0,i_0}y}{\alpha_{j_0,i_0}+\beta_{j_0,i_0}}$, $\tilde{y} = \frac{x+\alpha_{j_0,i_0}y}{\alpha_{j_0,i_0}+\beta_{j_0,i_0}}$ in place of x and y in (1). We obtain $\tilde{y} - \tilde{x} = y$. Thus (1) turns into

$$\begin{aligned} \sum_{i=0}^{l} y^i \Bigg[f_{1,i}\bigg(& \frac{\alpha_{1,i}+\beta_{1,i}}{\alpha_{j_0,i_0}+\beta_{j_0,i_0}}x + \frac{\alpha_{j_0,i_0}\beta_{1,i} - \alpha_{1,i}\beta_{j_0,i_0}}{\alpha_{j_0,i_0}+\beta_{j_0,i_0}}y \bigg) \\ & + \dots + f_{k_i,i}\bigg(\frac{\alpha_{k_i,i}+\beta_{k_i,i}}{\alpha_{j_0,i_0}+\beta_{j_0,i_0}}x + \frac{\alpha_{j_0,i_0}\beta_{k_i,i} - \alpha_{k_i,i}\beta_{j_0,i_0}}{\alpha_{j_0,i_0}+\beta_{j_0,i_0}}y \bigg) \Bigg] = 0. \end{aligned} \tag{2.8}$$

Now we denote

$$\gamma_{j,i} := \frac{\alpha_{j,i}+\beta_{j,i}}{\alpha_{j_0,i_0}+\beta_{j_0,i_0}} \quad \text{and} \quad \delta_{j,i} := \frac{\alpha_{j_0,i_0}\beta_{j,i} - \alpha_{j,i}\beta_{j_0,i_0}}{\alpha_{j_0,i_0}+\beta_{j_0,i_0}}$$

which, in view of (2.8), gives

$$\sum_{i=0}^{l} y^i [f_{1,i}(\gamma_{1,i}x + \delta_{1,i}y) + \dots + f_{k_i,i}(\gamma_{k_i,i}x + \delta_{k_i,i}y)] = 0.$$

However,

$$\gamma_{j_0,i_0} = 1 \quad \text{and} \quad \delta_{j_0,i_0} = 0,$$

thus,

$$f_{j_0,i_0}(\gamma_{j_0,i_0}x + \delta_{j_0,i_0}y) = f_{j_0,i_0}(x),$$

i.e.

$$
\begin{aligned}
-y^{i_0} f_{j_0,i_0}(x) = & \sum_{i\in\{0,\dots,l\}\setminus\{i_0\}} y^i [f_{1,i}(\gamma_{1,i}x + \delta_{1.i}y) + \dots + f_{k_i,i}(\gamma_{k_i,i}x + \delta_{k_i,i}y)] \\
& + y^{i_0} \big[f_{1,i_0}(\gamma_{1,i_0}x + \delta_{1,i_0}y) + \dots + f_{j_0-1,i_0}(\gamma_{j_0-1,i_0}x + \delta_{j_0-1,i_0}y) \\
& + f_{j_0+1,i_0}(\gamma_{j_0+1,i_0}x + \delta_{j_0+1,i_0}y) + \dots + f_{k_{i_0},i_0}(\gamma_{k_{i_0},i_0}x + \delta_{k_{i_0},i_0}y)\big]. \quad (2.9)
\end{aligned}
$$

Observe that for (j,i) such that $(\alpha_{j,i},\beta_{j,i}) \in J_i$ we have $\delta_{j,i} = 0$, which means that

$$
f_{j,i}(\gamma_{j,i}x + \delta_{j,i}y) = f_{j,i}(\gamma_{j,i}x) \quad \text{if } (\alpha_{j,i},\beta_{j,i}) \in J_i.
$$

Using this equality in (2.9), we arrive at

$$
\begin{aligned}
& - y^{i_0} f_{j_0,i_0}(x) - \sum_{i=0}^{i_0-1} y^i \sum_{j\in\{1,\dots,k_i\}\cap\{J_i\}} f_{j,i}(\gamma_{j,i}x) \\
& \qquad = \sum_{i=0}^{l} y^i \sum_{j\in\{1,\dots,k_i\}\setminus\{J_i\}} f_{j,i}(\gamma_{j,i}x + \delta_{j,i}y). \quad (2.10)
\end{aligned}
$$

Note that, in view of (2.7), no summand on the left-hand side contains y^i with $i > i_0$. Moreover, all the $\delta_{j,i}$ on the right-hand side are different from zero. This means that (2.10) is of the form (2.4) with

$$
\begin{aligned}
& N = i_0, \quad M = l, \quad K = 0, \\
& \varphi_N = -f_{j_0,i_0}, \quad \varphi_i(x) = -\sum_{j\in\{1,\dots,k_i\}\cap\{J_i\}} f_{j,i}(\gamma_{j,i}x), \\
& I_i = \{(\alpha_{1,i},\beta_{1,i}),\dots,(\alpha_{k_i,i},\beta_{k_i,i})\} \setminus J_i
\end{aligned}
$$

(we identify the number α with the mapping $x \mapsto \alpha x$) and

$$
\psi_{i,(\alpha_{j,i},\beta_{j,i})} = f_{j,i}.
$$

This means that all assumptions of Lemma 2.5 are satisfied, so f_{j_0,i_0} is a polynomial function of the desired order. ■

Now we shall present some simple examples showing how this theorem may be used in concrete situations.

Proposition 2.7. *If functions F and f satisfy the equation*

$$
F(y) - F(x) = (y-x)\left[\frac{1}{6}f(x) + \frac{2}{3}f\left(\frac{x+y}{2}\right) + \frac{1}{6}f(y)\right] \tag{1.7}
$$

then f is a polynomial function of degree at most 3.

Proof. We write (1.7) in the form

$$
F(y) - F(x) - (y-x)\left[\frac{1}{6}f(x) + \frac{2}{3}f\left(\frac{x+y}{2}\right) + \frac{1}{6}f(y)\right] = 0
$$

and we take $l = 1$,

$$k_0 = 2, \quad f_{1,0} = F, \quad f_{2,0} = -F, \quad (\alpha_{1,0}, \beta_{1,0}) = (0,1), \quad (\alpha_{2,0}, \beta_{2,0}) = (1,0),$$

$$k_1 = 3, \quad f_1^1 = -\frac{1}{6}f, \quad f_2^1 = -\frac{2}{3}f, \quad f_3^1 = -\frac{1}{6}f,$$

$$(\alpha_{1,1}, \beta_{1,1}) = (1,0), \quad (\alpha_{2,1}, \beta_{2,1}) = \left(\frac{1}{2}, \frac{1}{2}\right), \quad (\alpha_{3,1}, \beta_{3,1}) = (0,1).$$

Further we take $i_0 = 1$, $j_0 = 2$; it is easy to see that all assumptions of Theorem 2.6 are satisfied, thus $\frac{2}{3}f = f_{j_0,i_0}$ is a polynomial function of order at most

$$\operatorname{card}\big(\{(\alpha_{1,0}, \beta_{1,0}), (\alpha_{2,0}, \beta_{2,0})\} \cup \{(\alpha_{1,1}, \beta_{1,1}), (\alpha_{3,1}, \beta_{3,1})\}\big) \\ + \operatorname{card}\{(\alpha_{1,1}, \beta_{1,1}), (\alpha_{3,1}, \beta_{3,1})\} - 1 = 3. \ \blacksquare$$

Remark 2.8. To give one more example of applications of Theorem 2.6 note that, as in Proposition 2.7, it may be proved that if functions F and f satisfy

$$F(y) - F(x) = (y-x)\left[\frac{1}{8}f(x) + \frac{3}{8}f\left(\frac{3x+y}{4}\right) + \frac{3}{8}f\left(\frac{x+3y}{4}\right) + \frac{1}{8}f(y)\right] \tag{2.11}$$

then f is a polynomial function of degree at most

$$\operatorname{card}\left(\left\{(1,0), \left(\frac{1}{4}, \frac{3}{4}\right), (0,1)\right\} \cup \{(1,0), (0,1)\}\right) + \operatorname{card}\left\{(1,0), \left(\frac{1}{4}, \frac{3}{4}\right), (0,1)\right\} - 1 = 5.$$

Remark 2.9. Notice that we could choose $\frac{1}{6}f$ as f_{j_0,i_0} in the proof of Proposition 2.7, but it is impossible to use Theorem 2.6 to show that F satisfying (1.7) is a polynomial. Indeed, if $f_{j_0,i_0} = F$ then assumption (2.7) is not satisfied. A similar situation occurs for the equation

$$F(y) - F(x) = (y-x)[f(x) + f(y)] + (y-x)^2[g(x) - g(y)]$$

(see [22]) where it is impossible to use Theorem 2.6 to show that f is a polynomial.

The following example will show that assumption (2.7) of Theorem 2.6 is essential.

Example 2.10. The functional equation

$$F(y) - F(x) = (y-x)[f(x+y) + g(2x+2y)]$$

is satisfied by $F(x) = \text{const}$ and $g(x) = -f(x/2)$ where f is any function. This means that assumption (2.7) of Theorem 2.6 is essential.

2.3. A new method. Now we shall present a simple and direct method which may be used to prove that solutions of functional equations of quadrature type are polynomial functions. This method is not as universal as Theorem 2.6 but there are several reasons to present it. First of all, this method will later bring us an idea of an interesting generalization of (1.2). Further, in some cases we shall prove that functions satisfying particular versions of equations of the type (1.1) are polynomial of lower order than it would be possible with the use of Theorem 2.6 and, moreover, this method may be used on an interval. Finally this method (after a suitable modification) may be used to obtain stability results.

We shall prove

Proposition 2.11. *If functions f, F satisfy the equation*

$$F(y) - F(x) = (y - x)[a_1 f(\alpha_1 x + \beta_1 y) + \cdots + a_n f(\alpha_n x + \beta_n y)] \tag{2}$$

then f satisfies the equation

$$\begin{aligned} a_1 f((\alpha_1 + \beta_1)x + \beta_1 h) + \cdots + a_n f((\alpha_n + \beta_n)x + \beta_n h) \\ + a_1 f((\alpha_1 + \beta_1)x + (\alpha_1 + 2\beta_1)h) + \cdots + a_n f((\alpha_n + \beta_n)x + (\alpha_n + 2\beta_n)h) \\ = 2[a_1 f((\alpha_1 + \beta_1)x + 2\beta_1 h) + \cdots + a_n f((\alpha_n + \beta_n)x + 2\beta_n h)] \end{aligned} \tag{2.12}$$

for all $x, h \in \mathbb{R}$.

Proof. Substituting $x + h$ in place of y in (2), we get

$$F(x + h) - F(x) = h[a_1 f((\alpha_1 + \beta_1)x + \beta_1 h) + \cdots + a_n f((\alpha_n + \beta_n)x + \beta_n h)].$$

Taking $x + h$ instead of x and $x + 2h$ in place of y, we obtain

$$\begin{aligned} &F(x + 2h) - F(x + h) \\ &\quad = h\big[a_1 f((\alpha_1 + \beta_1)x + (\alpha_1 + 2\beta_1)h) + \cdots + a_n f((\alpha_n + \beta_n)x + (\alpha_n + 2\beta_n)h)\big]; \end{aligned}$$

adding these two equations, we arrive at

$$\begin{aligned} F(x + 2h) - F(x) = h\big[&a_1 f((\alpha_1 + \beta_1)x + \beta_1 h) + \cdots + a_n f((\alpha_n + \beta_n)x + \beta_n h) \\ &+ a_1 f((\alpha_1 + \beta_1)x + (\alpha_1 + 2\beta_1 h) + \cdots + a_n f((\alpha_n + \beta_n)x + (\alpha_n + 2\beta_n h)\big]. \end{aligned} \tag{2.13}$$

But, on the other hand, taking $y = x + 2h$ in (2), we get

$$F(x + 2h) - F(x) = 2h[a_1 f((\alpha_1 + \beta_1)x + 2\beta_1 h) + \cdots + a_n f((\alpha_n + \beta_n)x + 2\beta_n h)],$$

which together with (2.13) yields (2.12). ■

Remark 2.12. Using this proposition, we can show that if f, F satisfy (1.2) then f must be an affine function (additive plus a constant). Indeed, in this case, $\alpha = \beta = 1/2$ and from (2.12) we get

$$f\left(x + \frac{h}{2}\right) + f\left(x + \frac{3h}{2}\right) = 2f(x + h),$$

which after substituting $x + h$ in place of x and $2h$ instead of h, yields $\Delta_h^2 f(x) = 0$, thus $f(x) = a(x) + b$ for some additive $a : \mathbb{R} \to \mathbb{R}$.

Remark 2.13. In view of Theorem 2.3, functions satisfying (1.1) are polynomial of order at most $3n - 2$.

Note that, in general, the order obtained in Remark 2.13 is higher than the one provided by Sablik's result but in some concrete situations we obtain the same or even better estimate of this order. First we prove the following proposition.

Proposition 2.14. *Let functions $f, F : \mathbb{R} \to \mathbb{R}$ satisfy the equation*

$$F(y) - F(x) = (y - x)\left[a_1 f(x) + a_2 f\left(x + \frac{y - x}{n - 1}\right) + a_3 f\left(x + 2\frac{y - x}{n - 1}\right) + \cdots + a_n f(y)\right] \tag{2.14}$$

for some $n \in \mathbb{N} \setminus \{1\}$ and $a_i \in \mathbb{R}$, $i = 1, \ldots, n$. Then f is a polynomial function of order at most $2n - 3$.

Proof. Equation (2.14) is of the form (2) with

$$\alpha_i = 1 - \frac{i-1}{n-1} \quad \text{and} \quad \beta_i = \frac{i-1}{n-1}, \qquad i = 1, \ldots, n. \tag{2.15}$$

Therefore, we know from Proposition 2.11 that (2.12) is satisfied. Now, using (2.15) in (2.12), we get

$$\begin{aligned} a_1 f(x) + a_2 f\left(x + \frac{h}{n-1}\right) + \cdots + a_n f\left(x + (n-1)\frac{h}{n-1}\right) \\ + a_1 f(x+h) + a_2 f\left(x + \frac{nh}{n-1}\right) + \cdots + a_n f(x+2h) \\ = 2\left[a_1 f(x) + a_2 f\left(x + 2\frac{h}{n-1}\right) + \cdots + a_n f(x+2h)\right]. \end{aligned} \tag{2.16}$$

Observe that each point occurring on the right-hand side is also present on the left-hand side. Moreover, the last summand of the first sum involves the same point as the first summand of the second sum. This means that there are only $2n-1$ distinct points in this equation, and in view of Theorem 2.3 we conclude that f is a polynomial function of order at most $2n-1-2 = 2n-3$. ■

REMARK 2.15. The above proposition may be used for the equations

$$F(y) - F(x) = (y-x)\left[\frac{1}{6}f(x) + \frac{2}{3}f\left(\frac{x+y}{2}\right) + \frac{1}{6}f(y)\right], \tag{1.7}$$

$$F(y) - F(x) = (y-x)\left[\frac{1}{8}f(x) + \frac{3}{8}f\left(\frac{3x+y}{4}\right) + \frac{3}{8}f\left(\frac{x+3y}{4}\right) + \frac{1}{8}f(y)\right] \tag{2.11}$$

and all other equations stemming from quadrature rules with equidistant nodes.

REMARK 2.16. In some papers (see for example [10] and [29]) the case of x, y lying in an interval is considered. It is worth noticing that the method presented here is based on Székelyhidi's result, and since there are versions of the Székelyhidi lemma on an interval (see [38]), we may also use this method for functions defined on an interval. However, we shall not go into the details.

3. Continuity of functions satisfying (1.1)

3.1. Introduction. The first functional equation stemming directly from a quadrature rule which may be found in papers devoted to functional equations is the equation

$$F(y) - F(x) = (y-x)\left[\frac{1}{6}f(x) + \frac{2}{3}f\left(\frac{x+y}{2}\right) + \frac{1}{6}f(y)\right] \tag{1.7}$$

connected with (1.6). Results concerning this equation may be found in [31] in the case of functions acting on $\mathbb{R}$, in [19] for functions acting on an integral domain and in [10] on the interval. The situation here is similar to that of equation (1.2), i.e. all solutions of (1.7) are continuous. More precisely, if f and F satisfy (1.7) then f is a polynomial of degree at most 3 and F is a primitive function of f. However if we consider the (partially) pexiderized version of (1.7)

$$F(y) - F(x) = (y-x)\left[f(x) + g\left(\frac{x+y}{2}\right) + h(y)\right] \tag{3.1}$$

then the situation becomes different. Let us quote [31, Theorem 3.8, p. 106]:

THEOREM 3.1. *The functions $f, g, h, k : \mathbb{R} \to \mathbb{R}$ satisfy the functional equation*

$$f(x) - g(y) = (x-y)[h(x+y) + k(x) + k(y)] \tag{3.2}$$

for all $x, y \in \mathbb{R}$ if and only if

$$\begin{aligned} f(x) &= 3ax^4 + 2bx^3 + cx^2 + dx + s, \\ g(x) &= 3ax^4 + 2bx^3 + cx^2 + dx + s, \\ h(x) &= ax^3 + bx^2 + A(x) + d - 2t, \\ k(x) &= 2ax^3 + bx^2 + cx - A(x) + t, \end{aligned}$$

where $A : \mathbb{R} \to \mathbb{R}$ is an additive function and $a, b, c, d, s, t \in \mathbb{R}$ are arbitrary constants.

In this theorem the left-hand side of (3.1) is slightly generalized but, since the substitution $x = y$ yields $f = g$, it does not make a big difference. In [31] a similar result is proved in the case where instead of two occurrences of k two different functions are used but this (more general) equation can easily be solved with the use of Theorem 3.1.

There are three interesting properties of solutions of (3.1) to observe here:

(i) f and g are continuous (although no regularity is assumed),
(ii) possibly discontinuous parts of h and k vanish at the right-hand side,
(iii) there are no discontinuous monomial summands of orders 2 and 3.

All these interesting properties of solutions of (3.2) will be explained by the main result of the current section.

Before we proceed to this result let us notice that one more problem occurs when we deal with equations of the form (1.1). As we can see, the set of solutions of (3.2) is already quite rich (it depends on six constants and one additive function). The situation becomes more complicated if we increase the length of the right-hand side of (1.1). Let us for example consider the equation

$$g(y) - f(x) = (y-x)[h(x) + k(sx+ty) + k(tx+sy) + h(y)], \tag{3.3}$$

which was solved (under some regularity assumptions) by P. K. Sahoo [36] and then a general solution was obtained by A. Lisak and M. Sablik [26] for rational s, t (the particular case $g = f$, $s = 2$, $t = 1$ for functions acting on an integral domain was treated in [24]). A pexiderized version of (3.3)

$$f_1(y) - g_1(x) = (y-x)[f_2(x) + f_3(sx+ty) + f_4(tx+sy) + f_5(y)] \tag{3.4}$$

was also solved in [26], and in this case solutions depend on six arbitrary constants, two additive functions and one biadditive and symmetric function. Moreover, it should be emphasized that only one of the four points occurring on the right-hand side of (3.4) is arbitrary (two points are chosen as the endpoints of the interval $[x, y]$ and the fourth one is symmetric to the first).

Therefore it is not possible to present a full set of solutions of (1.1), however if we prove that all solutions of (1.1) (in some case) are continuous then, in view of Lemma 2.6, we can see that these solutions are ordinary polynomials. This means that the continuity of solutions plays a crucial role here.

3.2. Main result. First notice that, using Theorem 2.6, it is possible to show that functions satisfying equation (1.1) are polynomial. Now we shall prove that it is possible to work with their monomial summands.

LEMMA 3.2. *Let $m, k \in \mathbb{N}$, $\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_k \in \mathbb{R}$ be some constants, let functions $F, f_i : \mathbb{R} \to \mathbb{R}$ be of the form*

$$f_i(x) := f_{m,i}(x) + \cdots + f_{1,i}(x) + t_{0,i}, \qquad i = 1, \ldots, n, \tag{3.5}$$

and

$$F(x) := F_{m+1}(x) + \cdots + F_1(x) + T_0, \tag{3.6}$$

where $T_0, t_{0,i} \in \mathbb{R}$ are some constants and functions $F_j, f_{j,i}$, $i = 1, \ldots, n$, satisfy

$$F_j(2x) = 2^j F_j(x), \qquad x \in \mathbb{R},\ j = 1, \ldots, m+1, \tag{3.7}$$

$$f_{j,i}(2x) = 2^j f_{j,i}(x), \qquad x \in \mathbb{R},\ i = 1, \ldots, n,\ j = 1, \ldots, m. \tag{3.8}$$

If the functions f_i, $i = 1, \ldots, n$, and F satisfy the equation

$$F(y) - F(x) = (y - x)[f_1(\alpha_1 x + \beta_1 y) + \cdots + f_n(\alpha_n x + \beta_n y)] \tag{1.1}$$

then for each $j \in \{1, \ldots, m\}$ the functions $F_{j+1}, f_{j,i}$, $i = 1, \ldots, n$, satisfy (1.1).

Proof. Denote $F_0(x) := T_0$, $f_{0,i}(x) := t_{0,i}$, $x \in \mathbb{R}$, $i = 1, \ldots, k$. Then we may write (1.1) in the form

$$\sum_{j=0}^{m+1} F_j(x) - \sum_{j=0}^{m+1} F_j(y) = (y - x) \sum_{i=1}^{n} \sum_{j=0}^{m} f_{j,i}(\alpha_i x + \beta_i y). \tag{3.9}$$

Then, substituting in (1.1) $2x$ in place of x and $2y$ in place of y, we may write

$$\sum_{j=1}^{m+1} 2^j F_j(x) - \sum_{j=1}^{m+1} 2^j F_j(y) = 2(y - x) \sum_{i=1}^{n} \sum_{j=0}^{m} 2^j f_{j,i}(\alpha_i x + \beta_i y) \tag{3.10}$$

(sums on the left-hand side begin with $j = 1$ since T_0 vanishes). Multiplying (3.9) by 2 and subtracting the resulting equation from (3.10), we get

$$\sum_{j=2}^{m+1} (2^j - 2) F_j(x) - \sum_{j=2}^{m+1} (2^j - 2) F_j(y) = 2(y - x) \sum_{i=1}^{n} \sum_{j=1}^{m} (2^j - 1) f_{j,i}(\alpha_i x + \beta_i y),$$

i.e.

$$\sum_{j=2}^{m+1} (2^j - 2) F_j(x) - \sum_{j=2}^{m+1} (2^j - 2) F_j(y) = (y - x) \sum_{i=1}^{n} \sum_{j=1}^{m} (2^{j+1} - 2) f_{j,i}(\alpha_i x + \beta_i y). \tag{3.11}$$

This means that we have eliminated F_1 and $f_{i,0},\ i = 1, \dots, n$. Now we substitute $2x$ and $2y$ in place of x and y, respectively, in (3.11), to get

$$\sum_{j=2}^{m+1} 2^j(2^j - 2)F_j(x) - \sum_{j=2}^{m+1} 2^j(2^j - 2)F_j(y) = 2(y-x)\sum_{i=1}^{n}\sum_{j=1}^{m} 2^j(2^{j+1} - 2)f_{j,i}(\alpha_i x + \beta_i y). \quad (3.12)$$

Further we multiply (3.11) by 4 and we subtract the result from (3.12). The functions F_2 and $f_{1,i},\ i = 1, \dots, n$, will not occur in the equation obtained. Repeating this procedure m times, we obtain a sequence of equations. The last of them is

$$bF_{m+1}(y) - bF_{m+1}(x) = b(y-x)[f_{m,1}(\alpha_1 x + \beta_1 y) + \cdots + f_{m,n}(\alpha_n x + \beta_n y)], \quad (3.13)$$

where $b \neq 0$ is some real constant. This means that $F_{m+1}, f_{1,m}, \dots, f_{n,m}$ satisfy (1.1). However, the last but one equation reads

$$\begin{aligned}[cF_{m+1}(y) + dF_m(y)] &- [cF_{m+1}(x) + dF_m(x)] \\ &= c(y-x)[f_{m,1}(\alpha_1 x + \beta_1 y) + \cdots + f_{m,n}(\alpha_n x + \beta_n y)] \\ &\quad + d(y-x)[f_{m-1,1}(\alpha_1 x + \beta_1 y) + \cdots + f_{m-1,n}(\alpha_n x + \beta_n y)].\end{aligned}$$

Together with (3.13) this means that $F_m, f_{m-1,1}, \dots, f_{m-1,n}$ satisfy (1.1). Proceeding further in the same manner we obtain our assertion. ∎

Remark 3.3. For the sake of simplicity in Lemma 3.2 we proved a result concerning equation (1.1). It is clear that the analogous assertion is true in the case of (1), however the proof in this case would become unnecessarily complicated.

In [21] continuity results concerning equation (2) were obtained in the following way: after substituting $x = 0$ a formula expressing F was obtained. Then this formula was used in (2). After some further observations, continuity of F was proved. Our approach will be different. We shall act in a more direct way, and our result will be both simpler and valid for a very broad class of functional equations. First we shall prove the following simple but useful lemma.

Lemma 3.4. *Let k be a positive integer, let $F : \mathbb{R} \to \mathbb{R}$ be a monomial function of order k, let c, d be constants and let $\varphi_i : \mathbb{R} \to \mathbb{R}$ be monomial functions of order i for $i = 1, \dots, k-1$. If these functions satisfy the equation*

$$F(y) - c = (y-1)[d + \varphi_1(y) + \cdots + \varphi_{k-1}(y)] \quad (3.14)$$

then the functions φ_i, $i = 1, \dots, k-1$, and F are continuous, and moreover

$$\begin{aligned}\varphi_1(y) &= dy, \\ \varphi_2(y) &= dy^2, \\ &\vdots \\ \varphi_{k-1}(y) &= dy^{k-1}, \\ F(y) &= dy^k.\end{aligned}$$

Proof. It is enough to equate terms of equal orders occurring in (3.14). Indeed, equating terms of order 1 we obtain $dy = \varphi_1(y)$, which means that φ_1 is continuous. But equating terms of order 2 we get $y\varphi_1(y) = \varphi_2(y)$, i.e.

$$\varphi_2(y) = dy^2.$$

Continuing, we obtain continuity of all functions φ_i. Finally equating functions of order k we get

$$F(y) = y\varphi_{k-1}(y) = dy^k. \ \blacksquare$$

We are ready to state the most important result of this section. We shall prove it for an equation which may be called a depexiderized version of (1.10) and which is more general than (2).

THEOREM 3.5. *Let l and k_i, $i = 0, \dots, l$, be given positive integers and let*

$$\alpha_{1,i}, \dots, \alpha_{k_i,i}, \beta_{1,i}, \dots, \beta_{k_i,i} \in \mathbb{R}, \qquad a_i \in \mathbb{R} \setminus \{0\} \tag{3.15}$$

for $i \in \{0, \dots, l\}$. Assume that $f_{j,i} : \mathbb{R} \to \mathbb{R}$ fulfill for any $x, y \in \mathbb{R}$ the functional equation

$$F(y) - F(x) = \sum_{i=1}^{l} (y-x)^i [a_{1,i} f_i(\alpha_{1,i}x + \beta_{1,i}y) + \cdots + a_{k_i,i} f_i(\alpha_{k_i,i}x + \beta_{k_i,i}y)]. \tag{3.16}$$

Suppose that for all $i \in \{1, \dots, l\}$ there exists $j \in \{1, \dots, k_i\}$ such that $\alpha_{j,i} + \beta_{j,i} \neq 0$ and

$$\begin{vmatrix} \alpha_{j,i} & \beta_{j,i} \\ \alpha_{m,n} & \beta_{m,n} \end{vmatrix} \neq 0 \tag{3.17}$$

for all $n \in \{i+1, \dots, l\}$, $m \in \{1, \dots, k_n\}$ and for all pairs (m, i) with $m \neq j$. Then $f_1, \dots, f_l$ are polynomial functions and F is a polynomial.

Proof. It is easy to see that all assumptions of Theorem 2.6 are satisfied, which means that all functions f_i are polynomial. Let us write

$$f_i(x) = A_0 + A_{1,i}(x) + A_{2,i}(x, x) + \cdots + A_{n_i,i}(x, \overset{(n_i)}{\dots}, x),$$

where $(x, \overset{(n)}{\dots}, x)$ denotes $(x, \dots, x)$ (n times), A_0 is a constant and $A_{m,i} : \mathbb{R}^m \to \mathbb{R}$ is m-additive and symmetric. This means that f_i are of the form (3.5) where diagonalizations of $A_{j,i}$ satisfy (3.8). Taking $x = 0$, we obtain

$$F(y) = \sum_{i=1}^{l} (y)^i [a_{1,i} f_i(\beta_{1,i}y) + \cdots + a_{k_i,i} f_i(\beta_{k_i,i}y)],$$

which means that F is also of the form (3.6) where F_i satisfy (3.7), and therefore we may use Lemma 3.2. This lemma was stated for a simpler equation (1.1) but, as was mentioned in Remark 3.3, this result is also valid in our case.

Thus, from now on we shall work with monomial functions, i.e. we shall assume that F is a monomial function of order m and that

$$f_i(x) = A_{m-i}(x, \overset{(m-i)}{\dots}, x), \quad x \in \mathbb{R}, \tag{3.18}$$

where A_j is j-additive and symmetric for $j = 1, \dots, m-1$.

Now we take $x = 1$ in (3.16) to get

$$F(y) - F(1) = \sum_{i=1}^{l} (y-1)^i [a_{1,i} f_i(\alpha_{1,i} + \beta_{1,i} y) + \cdots + a_{k_i,i} f_i(\alpha_{k_i,i} + \beta_{k_i,i} y)]. \tag{3.19}$$

Using (3.18), we may write, for any constants $\alpha, \beta \in \mathbb{R}$,

$$\begin{aligned} f_i(\alpha + \beta y) = {} & A_{m-i}(\alpha, \overset{(m-i)}{\ldots}, \alpha) + (m-i) A_{m-i}(\alpha, \overset{(m-i-1)}{\ldots}, \alpha, \beta y) \\ & + \cdots + (m-i) A_{m-i}(\alpha, \beta y, \overset{(m-i-1)}{\ldots}, \beta y) + A(\beta y, \overset{(m-i)}{\ldots}, \beta y). \end{aligned} \tag{3.20}$$

Now, let $i \in \{1, \ldots, l\}$ and let g_i be defined by the formula

$$g_i(y) := (y-1)^{i-1} [a_{1,i} f_i(\alpha_{1,i} + \beta_{1,i} y) + \cdots + a_{k_i,i} f_i(\alpha_{k_i,i} + \beta_{k_i,i} y)]. \tag{3.21}$$

Then from (3.19) we get

$$F(y) - F(1) = (y-1)[g_1(y) + \cdots + g_l(y)]. \tag{3.22}$$

Now, using (3.20), it is easy to see that

$$g_i(y) = g_{m-1,i}(y) + \cdots + g_{1,i}(y) + g_{0,i}, \tag{3.23}$$

where $g_{j,i}$ for $j = 1, \ldots, m-1$ are monomial functions of order j and $g_{0,i}$ is a constant. Indeed, let $g_{0,i}$ be the sum of all constants occurring in (3.21), further let $g_{1,i}$ contain all functions of order 1 occurring in (3.21), and so on. Using (3.23) in (3.22), we arrive at

$$F(y) - F(1) = (y-1)\left(\sum_{j=0}^{m-1} g_{j,1} + \cdots + \sum_{j=0}^{m-1} g_{j,l} \right).$$

Now we define $\varphi_j(y) := \sum_{i=1}^{l} g_{j,i}(y)$ to get

$$F(y) - F(1) = (y-1) \sum_{j=0}^{m-1} \varphi_j(y),$$

which, in view of Lemma 3.4, means that F is a monomial of degree m.

We have proved that each monomial summand of F is an ordinary monomial. Thus F is a polynomial. ∎

Remark 3.6. For simplicity we proved the above theorem for equation (3.16); it is clear that the continuity of F may also be obtained in the case of the more general equation (1.10).

3.3. Applications. Using Theorem 3.5, we shall answer a question asked by M. Sablik, we shall prove a result which was obtained in [21] and we shall show how these results may be used to solve some concrete equations. First we need a technical lemma.

Lemma 3.7. *If $f : \mathbb{R} \to \mathbb{R}$ is a monomial function, $\alpha, \beta, x, y \in \mathbb{R}$ and q_n is a sequence of rational numbers that tends to* 1 *then*

$$\lim_{n \to \infty} f(\alpha q_n x + \beta y) = f(\alpha x + \beta y).$$

Proof. Let f be a monomial function of order k. Then $f(x) = A(x, \overset{(k)}{\dots}, x)$, and consequently

$$\begin{aligned} &f(\alpha q_n x + \beta y) \\ &\quad = f(q_n\alpha x) + kA(q_n\alpha x, \overset{(k-1)}{\dots}, q_n\alpha x, \beta y) + \dots + kA(q_n\alpha x, q_n\alpha x, \overset{(k-1)}{\dots}, q_n\alpha x) + f(\beta y) \\ &\quad = q_n^k f(\alpha x) + q_n^{k-1} kA(\alpha x, \overset{(k-1)}{\dots}, \alpha x, \beta y) + \dots + q_n kA(\alpha x, \beta y, \overset{(k-1)}{\dots}, \beta y) + f(\beta y). \end{aligned}$$

Now, since $q_n \to 1$, we obtain our assertion. ∎

THEOREM 3.8. *Let $f_1, \dots, f_n : \mathbb{R} \to \mathbb{R}$ be monomial functions of order k and let F be a monomial function of order $k+1$. If $f_1, \dots, f_n, F$ satisfy* (1.1) *with some $\alpha_i, \beta_i \in \mathbb{R}$, $i = 1, \dots, n$, and there exists $i_0 \in \{1, \dots, n\}$ such that f_{i_0} is discontinuous then there exist monomial functions $g_1, \dots, g_n : \mathbb{R} \to \mathbb{R}$ of order k, not all zero, satisfying*

$$g_1(\alpha_1 x + \beta_1 y) + \dots + g_n(\alpha_n x + \beta_n y) = 0. \tag{3.24}$$

Proof. Assume that f_{i_0} is discontinuous. Then there exists $a \in \mathbb{R}$ such that

$$f_{i_0}(ax) \neq a^k f_{i_0}(x). \tag{3.25}$$

Using this a, we define

$$g_i(x) := a^k f_i(x) - f_i(a^k x), \qquad k = 1, \dots, n. \tag{3.26}$$

We have

$$F(y) - F(x) = (y - x)[f_1(\alpha_1 x + \beta_1 y) + \dots + f_n(\alpha_n x + \beta_n y)], \tag{1.1}$$

but also

$$F(ay) - F(ax) = a(y - x)[f_1(a(\alpha_1 x + \beta_1 y)) + \dots + f_n(a(\alpha_n x + \beta_n y))]. \tag{3.27}$$

Now, since from Theorem 3.5 (or more precisely from Remark 3.6) we know that F is continuous, we deduce from (3.27) the equation

$$a^k[F(y) - F(x)] = (y - x)[f_1(a(\alpha_1 x + \beta_1 y)) + \dots + f_n(a(\alpha_n x + \beta_n y))]. \tag{3.28}$$

On the other hand, multiplying (1.1) by a^k, we obtain

$$a^k[F(y) - F(x)] = a^k(y - x)[f_1(\alpha_1 x + \beta_1 y) + \dots + f_n(\alpha_n x + \beta_n y)]. \tag{3.29}$$

Subtracting (3.28) and (3.29) we arrive at

$$0 = \left[f_1(a(\alpha_1 x + \beta_1 y)) - a^k f_1(\alpha_1 x + \beta_1 y) + \dots + f_n(a(\alpha_n x + \beta_n y)) - a^k f_n(\alpha_n x + \beta_n y)\right]$$

for all $x, y \in \mathbb{R}$, $x \neq y$. To show that g_i, $i = 1, \dots, n$, satisfy (3.24) for all $x, y \in \mathbb{R}$ we take a sequence of rational numbers $q_n \neq 1$ tending to 1 and we substitute $x = q_n y$ in (3.24). Using Lemma 3.7, we obtain (3.24) for $x = y$.

This means that g_i, $i = 1, \dots, n$, satisfy (3.24), and from (3.25) we have $g_{i_0} \neq 0$. ∎

The first result obtained from Theorem 3.8 will be a positive answer to a question asked by M. Sablik.

COROLLARY 3.9. *Let functions F, f_i, $i = 1, \dots, n$, satisfy* (1.1) *with some α_i, β_i, $i = 1, \dots, n$, such that*

$$\alpha_i + \beta_i \neq 0, \qquad i = 1, \dots, n,$$

and

$$\begin{vmatrix} \alpha_i & \beta_i \\ \alpha_j & \beta_j \end{vmatrix} \neq 0 \quad \textit{for } i \neq j.$$

If at least one f_i is discontinuous then equation (3.24) *has a discontinuous solution.*

Proof. From Theorem 2.6 we know that the f_i are polynomial functions. Let f_{i_0} be discontinuous. Then f_{i_0} must have a discontinuous summand of order j_0. From Lemma 3.2 we know that all summands of f_i of order j_0 together with the summand of F of order $j_0 + 1$ satisfy equation (2). Thus we may use Theorem 3.8 which states that (3.24) has a discontinuous solution. ■

The main result of [21] states that under some assumptions all solutions of (1.1) must be polynomials. We shall show how this result may be obtained from Theorem 3.8.

THEOREM 3.10. *Let $f, F : \mathbb{R} \to \mathbb{R}$ satisfy* (2) *with some $a_1, \dots, a_n, \lambda_1, \dots, \lambda_n \in \mathbb{R}$ such that*

$$\sum_{i=1}^{n} a_i \neq 0, \qquad \alpha_i + \beta_i = 1, \; i = 1, \dots, n,$$

and

$$\begin{vmatrix} \alpha_i & \beta_i \\ \alpha_j & \beta_j \end{vmatrix} \neq 0 \qquad \textit{for } i \neq j.$$

Then f is a polynomial of degree at most $2n - 1$ and F is a polynomial of degree at most $2n$. Moreover, F is differentiable and $F' = (\sum_{i=1}^{n} a_i) f$.

Proof. We know from Theorem 2.6 that f is a polynomial function,

$$f(x) = f_m(x) + \dots + f_1(x) + c,$$

where f_i is a monomial function of order i. We shall show that all f_i are continuous. Assume that there exists i such that f_i is discontinuous. Then, by Theorem 3.8, the equation

$$a_1 g(\alpha_1 x + \beta_1 y) + \dots + a_n g(\alpha_n x + \beta_n y) = 0$$

has a nontrivial solution g of order i. However taking $x = y$ we obtain

$$(a_1 + \dots + a_n) g(x) = 0,$$

which means that $g = 0$, a contradiction which shows that in this case all solutions of (2) must be continuous. Using the continuity of f, it is easy to show the differentiability of F. ■

Note that it is possible to use Theorem 3.8 to work also with equations which have discontinuous solutions. To give an example we shall solve an equation which has been solved in [31] but we shall provide a much shorter proof.

THEOREM 3.11. *The functions $f, g, h, k : \mathbb{R} \to \mathbb{R}$ satisfy the functional equation*

$$f(x) - g(y) = (x - y)[h(x + y) + k(x) + k(y)] \tag{3.2}$$

for all $x, y \in \mathbb{R}$ *if and only if*

$$\begin{aligned} f(x) &= 3ax^4 + 2bx^3 + cx^2 + 2dx + s, \\ g(x) &= 3ax^4 + 2bx^3 + cx^2 + 2dx + s, \\ h(x) &= ax^3 + bx^2 + A(x) + d - 2t, \\ k(x) &= 2ax^3 + bx^2 + cx - A(x) + t, \end{aligned} \tag{3.30}$$

where $a, b, c, d, s, t \in \mathbb{R}$ *are constants and* $A : \mathbb{R} \to \mathbb{R}$ *is an additive function.*

Proof. First we set $x = y$ in (3.2) to see that $f = g$. Using Theorem 2.6, we can see that h and k are polynomial functions of order at most 3. It is easy to see that the equation

$$k(x) + h(x + y) + k(y) = 0$$

has no monomial solutions of order higher than 1. In view of Theorem 3.8, this means that the monomial summands of h and k of orders 2 and 3 are continuous. Thus

$$\begin{aligned} h(x) &= a_h x^3 + b_h x^2 + A_h(x) + d_h, \\ k(x) &= a_k x^3 + b_k x^2 + A_k(x) + d_k; \end{aligned}$$

substituting $y = 0$ in (3.2) and taking $s := f(0)$, we get

$$f(x) = x[a_h x^3 + b_h x^2 + A_h(x) + d_h + a_k x^3 + b_k x^2 + A_k(x) + d_k + d_k] + s.$$

Now it suffices to use Lemma 3.2 to obtain all connections between the constants occurring in f, h and k. For example the functions $f_4(x) := (a_h + a_k)x^4$, $h_3(x) = a_h x^3$ and $k_3(x) = a_k x^3$ satisfy (3.2), thus

$$(a_h + a_k)x^4 - (a_h + a_k)y^4 = (x - y)[a_h(x + y)^3 + a_k x^3 + a_k y^3],$$

i.e.

$$(a_h + a_k)[x^3 + x^2 y + xy^2 + y^2] = a_h(x + y)^3 + a_k x^3 + a_k y^3;$$

taking here $y = 1$, we obtain

$$(a_h + a_k)[x^3 + x^2 + x + 1] = a_h(x + 1)^3 + a_k x^3 + a_k.$$

Now equating terms of degree 2, we may write

$$a_h + a_k = 3a_h,$$

which means that f_4, h_3 and k_3 have the desired forms.

Now we shall obtain the exact forms of f_2, h_1 and k_1. We know that f is continuous, so $f(x) = cx^2$ for some $c \in \mathbb{R}$; on the other hand from the form of f we get

$$f_2(x) = x(A_h(x) + A_k(x)).$$

This means that $A_h(x) + A_k(x) = cx$, and therefore we may write $A_h = A$ and $A_k(x) = c - A(x)$ as claimed. Equating terms of the remaining orders, we obtain all other equalities from (3.30).

Clearly, the functions given by (3.30) satisfy (3.2). ■

Remark 3.12. The above theorem shows that Theorem 3.5 (together with Remark 3.6) may be used to solve some concrete equations of the shape (1.1). Note that in the proof of Theorem 3.11 we also used the Sablik conjecture. Moreover, we used it in a stronger "monomial" form which is given by Theorem 3.8. Thus this example justifies our effort to state it in this setting.

Remark 3.13. A careful inspection of the proof of Theorem 3.11 shows that the procedure of solving particular cases of (1.1) becomes now automated. Thus it is probably possible to write a computer program which will solve such kind of equations, just as it was done in [11] for the equation

$$\sum_{i=1}^{n+1} f_i(p_i x + q_i y) = 0.$$

3.4. Solution of equation (2) **with rational coefficients.** In [23] it has been shown that if the weights α_i, β_i occurring in (2) satisfy some equalities then monomial solutions of (2) of a certain order must be continuous. Let us quote the simplest result of this kind contained in [23].

Lemma 3.14. *Let P be an integral domain and let $a_1 : P \to P$ be an additive function satisfying equation* (2) *for all $x, y \in P$ and some $b_1, \dots, b_n, \alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n \in P$ such that for all $i = 1, \dots, n$ and $x \in P$ we have*

$$a_1(\alpha_i x) = \alpha_i a_1(x), \qquad a_1(\beta_i x) = \beta_i a_1(x).$$

If $\gamma, \delta \in P$ are defined by

$$\gamma := b_1\alpha_1 + \dots + b_n\alpha_n, \qquad \delta := b_1\beta_1 + \dots + b_n\beta_n, \tag{3.31}$$

then

(i) *if $\gamma \neq \delta$ then $a_1 = 0$;*
(ii) *if $\gamma = \delta \neq 0$ then $a_1(x) = ax$ for some $a \in P$ and all $x \in P$.*
(iii) *if $\gamma = \delta = 0$ then a_1 may be any function.*

Conversely, in each of the above cases the function a_1 with

$$F(x) = x[b_1 a_1(\alpha_1 x) + \dots + b_n a_1(\alpha_n x)] + c, \tag{3.32}$$

where $c \in P$ is some constant, is a solution of (2).

However, results of this kind obtained in [23] were valid only for additive, biadditive and 3-additive functions. This means that these theorems may only be used to solve very simple equations connected with quadrature rules. Now, using Theorem 3.8, we shall prove a result valid for monomial functions of any order.

Recall that in the whole paper we work with functions acting on $\mathbb{R}$, so this result cannot be called a generalization of results obtained in [23].

In this theorem we use the convention that $0^0 = 1$.

Theorem 3.15. *Let $f, F : \mathbb{R} \to \mathbb{R}$ be monomial functions of orders $k, k+1$, respectively, satisfying*

$$F(y) - F(x) = (y - x)[a_1 f(\alpha_1 x + \beta_1 y) + \dots + a_n f(\alpha_n x + \beta_n y)]. \tag{2}$$

Denote

$$f(x) = A(x, \dots, x)$$

for some k-additive and symmetric function $A : \mathbb{R} \to \mathbb{R}$, and let α_i, β_i satisfy

$$A(\alpha_i x_1, x_2, \ldots, x_k) = \alpha_i A(x_1, \ldots, x_k), \qquad A(\beta_i x_1, x_2, \ldots, x_k) = \beta_i A(x_1, \ldots, x_k) \tag{3.33}$$

for $i = 1, \ldots, n$. If f is discontinuous then

$$\sum_{i=1}^{n} a_i \alpha_i^k = \sum_{i=1}^{n} a_i \alpha_i^{k-1} \beta_i = \cdots = \sum_{i=1}^{n} a_i \beta_i^k = 0. \tag{3.34}$$

On the other hand, if (3.34) *is satisfied then every monomial function f of order k is a solution of* (2) (with $F = \text{const}$). *If f is continuous and $f \neq 0$ then for all $l, m \in \{0, \ldots, k\}$ we have*

$$\binom{n}{l} \sum_{i=1}^{n} a_i \alpha_i^{k-l} \beta_i^l = \binom{n}{m} \sum_{i=1}^{n} a_i \alpha_i^{k-m} \beta_i^m. \tag{3.35}$$

Conversely, if (3.35) *is satisfied then every monomial f of degree k with F given by*

$$F(x) = x[a_1 f(\alpha_1 x) + \cdots + a_n f(\alpha_n x)] + c, \tag{3.36}$$

where $c \in \mathbb{R}$ is some constant, is a solution of (2)*.*

Proof. Let $A : \mathbb{R} \to \mathbb{R}$ be a k-additive and symmetric function, and let

$$f(x) = A(x, \overset{(k)}{\ldots}, x)$$

be a discontinuous solution of (2). We shall show that all expressions from (3.34) are zero. From Theorem 3.8 we know that the equation

$$a_1 g(\alpha_1 x + \beta_1 y) + \cdots + a_n g(\alpha_n x + \beta_n y) = 0$$

has a nontrivial solution g which is a monomial function of order k. Write

$$g(x) = B(x, \overset{(k)}{\ldots}, x).$$

Then we may write

$$\sum_{i=1}^{n} a_i g(\alpha_i x) + k \sum_{i=1}^{n} a_i B(\alpha_i x, \ldots, \alpha_i x, \beta_i y) + \cdots + k \sum_{i=1}^{n} a_i B(\alpha_i x, \beta_i y, \ldots, \beta_i y) + \sum_{i=1}^{n} a_i g(\beta_i x) = 0.$$

Since each of the sums occurring in the above expression has a different order with respect to x, it is clear that all these sums must be zero, and thus we obtain

$$\sum_{i=1}^{n} a_i g(\alpha_i x) = 0,$$

$$\sum_{i=1}^{n} a_i B(\alpha_i x, \ldots, \alpha_i x, \beta_i y) = 0,$$

$$\vdots$$

$$\sum_{i=1}^{n} a_i B(\alpha_i x, \beta_i y, \ldots, \beta_i y) = 0,$$
$$\sum_{i=1}^{n} a_i g(\alpha_i x) = 0.$$

Note that g may be obtained from (3.26) and we have (3.33). Thus

$$\sum_{i=1}^{n} a_i \alpha_i^k g(x) = 0,$$
$$\sum_{i=1}^{n} a_i \alpha_i^{k-1} \beta_i B(x, x, \ldots, y) = 0,$$
$$\vdots$$
$$\sum_{i=1}^{n} a_i \alpha_i \beta_i^{k-1} B(x, y, \ldots, y) = 0,$$
$$\sum_{i=1}^{n} a_i \beta_i^k g(y) = 0.$$

Now, since $g \neq 0$, all sums from (3.34) must be zero.

It is easy to see that if (3.34) is satisfied and f is a monomial function then the right-hand side of (2) vanishes, thus f is a solution of (2) with constant F (as claimed).

Suppose that $f(x) = x^k$. Then, taking $x = 0$ in (2), we obtain

$$F(y) = \sum_{i=1}^{n} a_i \alpha_i^k y^{k+1}.$$

On the other hand, taking $y = 0$ in (2), we get

$$F(x) = \sum_{i=1}^{n} a_i \beta_i^k x^{k+1}.$$

Using the forms of f and F in (2), we get

$$\sum_{i=1}^{n} a_i \alpha_i^k y^{k+1} - \sum_{i=1}^{n} a_i \beta_i^k x^{k+1} = (y - x) \sum_{i=1}^{n} a_i (\alpha_i x + \beta_i y)^k.$$

Canceling the same expressions on both sides, we arrive at

$$\begin{aligned} x \sum_{i=1}^{n} a_i \alpha_i^k y^k &- y \sum_{i=1}^{n} a_i \beta_i^k x^k \\ &= (y - x) \sum_{i=1}^{n} a_i \left[\binom{n}{1} \alpha_i^{k-1} \beta_i x^{k-1} y + \cdots + \alpha_i \beta_i^{k-1} x y^{k-1} \binom{n}{n-1} \right]. \end{aligned} \tag{3.37}$$

Now we equate the coefficients of equal powers of x. We begin with x^k:

$$\sum_{i=1}^{n} a_i \beta_i^k = \binom{n}{1} \sum_{i=1}^{n} a_i \alpha_i^{k-1} \beta_i,$$

which may be written in the form

$$\binom{n}{0}\sum_{i=1}^{n} a_i\alpha_i^k\beta_i^0 = \binom{n}{1}\sum_{i=1}^{n} a_i\alpha_i^{k-1}\beta_i.$$

Next, equating the coefficients of x^{k-1}, we get

$$\binom{n}{1}\sum_{i=1}^{n} a_i\alpha_i^{k-1}\beta_i = \binom{n}{2}\sum_{i=1}^{n} a_i\alpha_i^{k-2}\beta_i^2.$$

In this way we obtain all equalities from (3.35).

On the other hand, if (3.35) is satisfied then we clearly have (3.37), which means that the functions $f(x) = x^k$ and F given by (3.36) satisfy (2). ∎

The next corollary will show how this theorem may be used to solve come concrete equations.

COROLLARY 3.16. *Functions $f, F : \mathbb{R} \to \mathbb{R}$ satisfy the equation*

$$F(y) - F(x) = (y-x)\left[\frac{1}{8}f(x) + \frac{3}{8}f\left(\frac{2x+y}{3}\right) + \frac{3}{8}f\left(\frac{x+2y}{3}\right) + \frac{1}{8}f(y)\right] \tag{2.11}$$

if and only if f is a polynomial of degree at most 3 *and $F' = f$.*

Proof. From Remark 2.8 we know that f is a polynomial function of order at most 5. Since all constants occurring in this equation are positive, (3.34) cannot be satisfied. Thus monomial summands of f are continuous. This means that f is a polynomial of degree at most 5. However, for $k = 5$, (3.35) is not satisfied. Indeed,

$$\begin{aligned}
&\binom{5}{1}\cdot\left[\frac{1}{8}\cdot 1^4\cdot 0^1 + \frac{3}{8}\cdot\left(\frac{2}{4}\right)^4\cdot\left(\frac{1}{3}\right)^1 + \frac{3}{8}\cdot\left(\frac{1}{3}\right)^4\cdot\left(\frac{2}{3}\right)^1 + \frac{1}{8}\cdot 0^4\cdot 1^1\right]\\
\neq &\binom{5}{2}\cdot\left[\frac{1}{8}\cdot 1^3\cdot 0^2 + \frac{3}{8}\cdot\left(\frac{2}{3}\right)^3\cdot\left(\frac{1}{3}\right)^2 + \frac{3}{8}\cdot\left(\frac{1}{3}\right)^3\cdot\left(\frac{2}{3}\right)^2 + \frac{1}{8}\cdot 0^3\cdot 0^2\right],
\end{aligned}$$

which means that f is a polynomial of degree at most 4. Further if we consider $k = 4$, the situation is similar. Now let us check that for $k = 3$, (3.35) will be satisfied. We recall that here $0^0 = 1$. We clearly have

$$\begin{aligned}
&\binom{3}{1}\cdot\left[\frac{1}{8}\cdot 1^2\cdot 0^1 + \frac{3}{8}\cdot\left(\frac{2}{3}\right)^2\cdot\left(\frac{1}{3}\right)^1 + \frac{3}{8}\cdot\left(\frac{1}{3}\right)^2\cdot\left(\frac{2}{3}\right)^1 + \frac{1}{8}\cdot 0^2\cdot 1^1\right]\\
= &\binom{3}{2}\cdot\left[\frac{1}{8}\cdot 1^1\cdot 0^2 + \frac{3}{8}\cdot\left(\frac{2}{3}\right)^1\cdot\left(\frac{1}{3}\right)^2 + \frac{3}{8}\cdot\left(\frac{1}{3}\right)^1\cdot\left(\frac{2}{3}\right)^2 + \frac{1}{8}\cdot 0^1\cdot 1^2\right]\\
= &\binom{3}{0}\cdot\left[\frac{1}{8}\cdot 1^3\cdot 0^0 + \frac{3}{8}\cdot\left(\frac{2}{3}\right)^3\cdot\left(\frac{1}{3}\right)^0 + \frac{3}{8}\cdot\left(\frac{1}{3}\right)^3\cdot\left(\frac{2}{3}\right)^0 + \frac{1}{8}\cdot 0^3\cdot 1^0\right]\\
= &\binom{3}{3}\cdot\left[\frac{1}{8}\cdot 1^0\cdot 0^3 + \frac{3}{8}\cdot\left(\frac{2}{3}\right)^0\cdot\left(\frac{1}{3}\right)^3 + \frac{3}{8}\cdot\left(\frac{1}{3}\right)^0\cdot\left(\frac{2}{3}\right)^3 + \frac{1}{8}\cdot 0^0\cdot 1^3\right],
\end{aligned}$$

thus the function $f(x) = x^3$ satisfies our equation (with F given by (3.36)). Similarly one can show that the monomials of degrees 2 and 1 satisfy this equation. ∎

REMARK 3.17. As was shown in the proof of Corollary 3.16, solutions of concrete examples of equation (2) with rational α_i and β_i may be obtained by checking if some equalities are satisfied. This means that Theorem 3.15 provides, in fact, a general solution of the functional equation (2) in this case.

It is clear that Theorem 3.15 may also be used in the opposite direction, i.e. we may assume that equation (2) is satisfied by monomials (or monomial functions) of some degree (order) and then, using equalities (3.34) or (3.35), we can find (using a computer algebra system) coefficients a_i, α_i and β_i which admit such solutions.

Example 3.18. If the functional equation

$$F(y) - F(x) = (y-x)\frac{f(\alpha x + (1-\alpha)y) + f(\gamma x + (1-\gamma)y)}{2}, \tag{3.38}$$

where $\alpha, \gamma \in \mathbb{R}$, is satisfied by polynomials f of the second degree (with some F) then

$$\left(\alpha = \frac{3+\sqrt{3}}{6} \quad \text{and} \quad \gamma = \frac{3-\sqrt{3}}{6}\right) \quad \text{or} \quad \left(\gamma = \frac{3+\sqrt{3}}{6} \quad \text{and} \quad \alpha = \frac{3-\sqrt{3}}{6}\right). \tag{3.39}$$

Conversely, each polynomial f of degree not greater than 3 with its primitive function F satisfies equation (3.38) if α and γ are given by (3.39).

Proof. In view of Theorem 3.15, it is enough to solve the following system of equations:

$$\tfrac{1}{2}[\alpha + \gamma] = \tfrac{1}{2}[(1-\alpha) + (1-\gamma)],$$

which ensures that $f(x) = x$ satisfies (3.38), and

$$\tfrac{1}{2}[\alpha^2 + \gamma^2] = \tfrac{1}{2}[(1-\alpha)^2 + (1-\gamma)^2],$$
$$\tfrac{1}{2}[\alpha^2 + \gamma^2] = 2 \cdot \tfrac{1}{2}[\alpha(1-\alpha) + \gamma(1-\gamma)],$$

which are satisfied since $f(x) = x^2$ satisfies our equation. This gives $\gamma = 1 - \alpha$ and

$$6\alpha^2 - 6\alpha + 1 = 0,$$

i.e. α and γ satisfy (3.39). ∎

The next example will show how a functional equation having discontinuous solutions may be solved.

Example 3.19. If the functional equation

$$F(y) - F(x) = (y-x)[f(x) + af(\alpha x + \beta y) + f(y)], \tag{3.40}$$

where $a, \alpha, \gamma \in \mathbb{Q}$, is satisfied by a nonzero discontinuous additive function f and some F then we have

$$\alpha = \beta \quad \text{and} \quad a = -1/\alpha \tag{3.41}$$

and F is constant. Conversely if a, α and β satisfy (3.41) then every additive function f with constant F satisfies (3.40).

Proof. Theorem 3.15 gives us $1 + a\alpha = 0$ and $1 + a\beta = 0$, which yields (3.41). All other assertions are obvious. ∎

Remark 3.20. Equation (3.38) contains weights with possibly irrational coefficients, but it may be solved with the use of Theorem 3.15. Indeed, we use Theorem 2.6 to see that f is a polynomial function of degree at most 3. Now, from Theorem 3.10 we know that f is continuous. This allows us to use Theorem 3.15 (equalities (3.33) are satisfied).

3.5. Functional equations connected with Hermite quadrature rules. In this part of the paper we shall deal with functional equations connected with Hermite quadrature rules. In this kind of quadrature rules, not only values of a given function f are used but also values of its derivative. Thus the integral is approximated in the following way:

$$\int_x^y f(t)\,dt \approx (y-x)\left[\frac{f(x)+f(y)}{2} + f(\alpha_1 x+\beta_1 y)+\cdots+f(\alpha_n x+\beta_n y)\right] + (y-x)^2[f'(x)-f'(y)], \tag{3.42}$$

where the nodes $\alpha_i x+\beta_i y$ are equally distributed in the interval $[x,y]$.

Therefore, we shall now work with the equation

$$F(y)-F(x) = (y-x)\big[a_1 f(\alpha_1 x+\beta_1 y)+\cdots+a_n f(\alpha_n x+\beta_n y)\big] + (y-x)^2[g(y)-g(x)]. \tag{3}$$

Equation (3) was solved in [22] in a particular case

$$F(y)-F(x) = (y-x)\big[f(x)+af(x+y)+f(y)\big] + (y-x)^2[g(y)-g(x)]; \tag{3.43}$$

now we shall deal with it in full generality and (under some assumptions) we shall show that functions satisfying this equation are polynomials.

Using Theorem 2.6, it is easy to show that both f and g are polynomial functions. We need the following notation.

Definition 3.21. We say that numbers $a_1,\dots,a_n$ and $(\alpha_1,\beta_1),\dots,(\alpha_n,\beta_n)$ satisfy the *SH-condition* if there exists $i_0\in\{1,\dots,n\}$ such that $a_{i_0}\neq 0$, $\alpha_{i_0}+\beta_{i_0}\neq 0$,

$$\alpha_{i_0},\beta_{i_0}\neq 0 \tag{3.44}$$

and

$$\begin{vmatrix}\alpha_i & \beta_i\\ \alpha_{i_0} & \beta_{i_0}\end{vmatrix}\neq 0, \qquad i\in\{1,\dots,n\},\ i\neq i_0. \tag{3.45}$$

Lemma 3.22. *Let $n\in\mathbb{N}$. If functions f,g,F satisfy* (3) *with some $a_i,\alpha_i,\beta_i\in\mathbb{R}$, $i=1,\dots,n$, then:*

(i) *g is a polynomial function of order at most $2n+2$.*
(ii) *If $a_1,\dots,a_n$ and $(\alpha_1,\beta_1),\dots,(\alpha_n,\beta_n)$ satisfy the SH-condition then f is a polynomial function of order at most $2n+3$.*

Proof. First we shall prove (ii). To this end observe that equation (3) is clearly of the form (1) with $l=2$, $k_0=2$, $k_1=n$, $k_2=2$ and

$$f_{1,0}=F,\quad f_{2,0}=-F,\quad \alpha_{1,0}=0,\quad \beta_{1,0}=1,\quad \alpha_{2,0}=1,\quad \beta_{2,0}=1,$$
$$f_{1,1}=a_1 f,\dots,f_{n,1}=a_n f,\quad \alpha_{1,1}=\alpha_1,\quad \beta_{1,1}=\beta_1,\dots,\alpha_{n,1}=\alpha_n,\quad \beta_{n,1}=\beta_n,$$
$$f_{1,2}=g,\quad f_{2,2}=-g,\quad \alpha_{1,2}=1,\quad \beta_{1,2}=0,\quad \alpha_{2,2}=0,\quad \beta_{2,2}=1.$$

Using (3.44) and (3.45), we can see that (2.7) is satisfied. Thus, using Theorem 2.6, we infer that $a_{i_0}f$ is a polynomial function of order at most

$$(2+(n-1))+(2+(n-1))+2-1=2n+3,$$

as claimed.

Similarly one can show that g is a polynomial function of order at most

$$(n+1)+(n+1)+1-1=2n+2. \ \blacksquare$$

Remark 3.23. If equation (3) takes the form

$$F(y)-F(x)=(y-x)[a_1f(x)+a_2f(\alpha_2x+\beta_2y)+\cdots+a_{n-1}f(\alpha_nx+\beta_ny)+a_nf(y)] \\ +(y-x)^2[g(y)-g(x)] \tag{3.46}$$

(in which the endpoints of the interval appear on the right-hand side) then $J_1\neq\emptyset$, and consequently the order of g is lower than what was proved in Lemma 3.22, namely this order is not greater than $(n-1)+(n-1)+1-1=2n-2$.

Remark 3.24. Equation (3.43) was solved in [21] in full generality. Observe that since we have to assume (3.44), Lemma 3.22 does not cover equation (3.43) in case $a=0$.

Remark 3.25. If f, g and F are polynomial functions satisfying (3) then, just as in Lemma 3.2, it can be proved that their monomial summands of respective orders k, $k-1$ and $k+1$ also satisfy (3).

We have proved that f and g are polynomial functions. However it is easy to obtain the exact form of F by taking $x=0$ in (3).

As a direct consequence of Theorem 3.5 we obtain the following theorem.

Theorem 3.26. *Let $f,g,F:\mathbb{R}\to\mathbb{R}$ satisfy* (3)*, where $a_i,\alpha_i,\beta_i,\ i=1,\dots,n$, satisfy the SH-condition. Then F is a polynomial.*

In the next theorem we shall also show (under some assumptions) the continuity of f and g.

Theorem 3.27. *Let functions f,g,F satisfy* (3)*. If numbers $\alpha_i,\beta_i,a_i\in\mathbb{R}$ satisfying SH-condition are such that $\alpha_i+\beta_i=1$ and $a_1+\cdots+a_n\neq 0$ then functions f,g,F are polynomials.*

Proof. In view of Lemma 3.22 and Remark 3.25, we may assume that F,f,g are monomial functions of orders $k,k-1,k-2$, respectively. Moreover, from Theorem 3.26 we know that F is continuous. This means that g and f satisfy the equation

$$dy^k-dx^k=(y-x)\big[a_1f(\alpha_1x+\beta_1y)+\cdots+a_nf(\alpha_nx+\beta_ny)\big]+(y-x)^2[g(y)-g(x)].$$

Canceling $y-x$ on both sides of this equation, we arrive at

$$d(y^{k-1}+y^{k-2}x+\cdots+x^{k-1}) \\ =a_1f(\alpha_1x+\beta_1y)+\cdots+a_nf(\alpha_nx+\beta_ny)+(y-x)[g(y)-g(x)] \tag{3.47}$$

for all $x,y\in\mathbb{R}$ such that $x\neq y$. We would like to show that this equation is satisfied also for $x=y$. To this end take q_n as a sequence of rational numbers such that $q_n\neq 1$ and $q_n\to 1$. We put q_ny instead of x in (3.47) and obtain

$$d(q_n^{k-1}y^{k-1}+q_n^{k-2}y^{k-1}+\cdots+y^{k-1}) \\ =a_1f(q_n\alpha_1y+\beta_1y)+\cdots+a_nf(q_n\alpha_ny+\beta_ny)+y(1-q_n)[(1-q_n^{k-1})g(y)]. \tag{3.48}$$

Letting $n\to\infty$, since f is a monomial function from Lemma 3.7, we get

$$dky^{k-1}=(a_1+\cdots+a_n)f(y).$$

Now it suffices to divide this equation by $a_1 + \cdots + a_n$ to obtain the continuity of f. It remains to show the continuity of g. To this end we substitute $y = 0$ in (3), which gives

$$F(x) = x[a_1 f(\alpha_1 x) + \cdots + a_n f(\alpha_n x)] + x^2 g(x).$$

Using the continuity of f and F, we easily get the continuity of g. ■

Remark 3.28. Now it is easy to obtain a solution of the equation stemming directly from the Hermite quadrature rule

$$\begin{aligned} F(y) - F(x) = \frac{y-x}{n}\left[\frac{f(x)+f(y)}{2} + f\left(\frac{(n-1)x+y}{n}\right) + \cdots + f\left(\frac{x+(n-1)y}{n}\right)\right] \\ + (y-x)^2[g(x) - g(y)]. \end{aligned} \tag{3.49}$$

Namely, we know that F, f and g are polynomials.

If we divide (3.49) by $y - x$ and let x tend to y, we obtain $F' = f$.

Taking $x = 0$ in (3.49), we arrive at

$$F(y) - F(0) = \frac{y}{n}\left[\frac{f(0)+f(y)}{2} + \sum_{i=1}^{n-1} f\left(\frac{iy}{n}\right)\right] + y^2[g(y) - g(0)]. \tag{3.50}$$

From Remark 3.25 we know that the monomial summands of F, f and g also satisfy this equation. Thus let $F(x) = ax^k$, $f(x) = kax^{k-1}$ and $g(x) = bx^{k-2}$. From numerical analysis we know that (3.49) is satisfied by every polynomial f of degree at most $2n - 1$, its derivative g and its primitive function F. However from (3.50) we see that for given f and F there is only one g satisfying (3.49). This means that $b = (k - 1)ka$.

Remark 3.29. From Remark 3.28 we can see that, as in previous sections, the crucial question is that of continuity of the functions involved. Once we know that all functions occurring in the equation considered are continuous, the solutions may be obtained using known facts from numerical analysis.

3.6. Functional equations connected with Birkhoff quadrature rules. In this section we shall deal with equations motivated by a quadrature rule in which the second derivative is involved. Such quadrature rules were studied in [43] and [44]. Namely, let P_{n-1} denote the Legendre polynomial of degree $n - 1$ and let

$$1 = x_{1,n} > x_{2,n} > \cdots > x_{n-1,n} > x_{n,n} = -1$$

be the zeros of the polynomial $(1 - x^2)P'_{n-1}(x)$. Then the quadrature formula

$$\begin{aligned} \int_{-1}^{1} f(x)\,dx \approx \frac{3}{n(2n-1)}[f(1) + f(-1)] \\ + \frac{2(2n-3)}{n(n-2)(2n-1)} \sum_{k=2}^{n-1} \frac{f(x_{k,n})}{(P_{n-1}(x_{k,n}))^2} \\ + \frac{1}{n(n-1)(n-2)(2n-1)} \sum_{k=2}^{n-1} \frac{f''(x_{k,n})}{(P_{n-1}(x_{k,n}))^2} \end{aligned} \tag{3.51}$$

is exact for polynomials of degree at most $2n - 1$ (see [44]).

Proceeding as previously, we obtain the functional equation

$$\begin{aligned} F(y)-F(x) &= (y-x)[a_1f(x)+b_1f(\alpha_1x+\beta_1y)+\cdots+b_nf(\alpha_nx+\beta_ny)+a_1f(y)] \\ &\quad + (y-x)^3[c_1g(\alpha_1x+\beta_1y)+\cdots+c_ng(\alpha_nx+\beta_ny)], \end{aligned} \tag{4}$$

where $a_1, b_i, c_i, \alpha_i, \beta_i$ are obtained from (3.51) taking into account that we are now on the interval $[x, y]$ instead of $[-1, 1]$. For example for $n = 3$ we get the equation

$$F(y)-F(x) = (y-x)\left[\frac{1}{10}f(x)+\frac{4}{5}f\left(\frac{x+y}{2}\right)+\frac{1}{10}f(y)\right]+\frac{(y-x)^3}{60}g\left(\frac{x+y}{2}\right), \tag{3.52}$$

which is satisfied by any polynomial function f of degree at most 5, its primitive function F and its second derivative g.

As in previous cases, we shall first prove that functions satisfying equations of this kind are polynomial.

Proposition 3.30. *If functions F, f, g satisfy* (4) *with $a_1, b_i, c_i, \alpha_i, \beta_i \in \mathbb{R}$, $i = 1, \dots, n$, such that $a_1 \neq 0$, $\beta_i \neq 0$, $i = 1, \dots, n$, and there exists $\tilde{i} \in \{1, \dots, n\}$ for which $\alpha_{\tilde{i}}+\beta_{\tilde{i}} \neq 0$ and $c_{\tilde{i}} \neq 0$ then F, f, g are polynomial functions of orders at most $4n+2$, $4n+1$, $4n-1$, respectively.*

Proof. First we shall use Theorem 2.6 to prove that f is a polynomial function. To this end we take $j_0 = 1$ and $f_{j_0,i_0} = a_1f$. Since $\beta_j \neq 0$, we get (2.7). Thus from Theorem 2.6 we deduce that a_1f (and also f) is a polynomial function of order at most

$$[(n+1)+(n+1)+n+n]-1 = 4n+1.$$

Similarly, it can be shown that g is a polynomial function of order at most

$$[(n+1)+(n+1)+(n-1)+(n-1)]-1 = 4n-1.$$

Taking $x = 0$ in (4), we get a formula for F which means that F is also a polynomial function of the desired order. ■

Remark 3.31. From the above proof it is clear that the assumptions on the constants occurring in (4) split in two parts: the first part implies that f is a polynomial and the second guarantees the polynomiality of g. For simplicity we have not done it.

Remark 3.32. Observe that the order obtained in Proposition 3.30 is higher than we could have expected, knowing that the quadrature rule (3.51) is exact for polynomials of degree at most $2n-1$. There may be two reasons for this: the nodes occurring in the Birkhoff quadrature formula are not optimal, or our procedure used to prove the polynomiality of F, f and g is not optimal.

Remark 3.33. As in Lemma 3.2, it may be shown that it is possible to work with monomial summands of F, f and g of orders, respectively, k, $k-1$ and $k-3$ separately.

Now we shall show (under some assumptions) that functions F, f, g satisfying (4) must be continuous.

Theorem 3.34. *If monomial functions F, f and g of orders $k+1$, k and $k-2$, respectively, satisfy* (4) *with some $a_1, b_i, c_i, \alpha_i, \beta_i \in \mathbb{R}$, $i = 1, \dots, n$, then F is continuous.*

Moreover, if

$$\alpha_i + \beta_i = 1, \quad i = 1, \dots, n, \quad \text{and} \quad 2a_1 + b_1 + \dots + b_n \neq 0$$

then f is continuous, and if additionally

$$c_1 + \dots + c_n \neq 0 \tag{3.53}$$

then g is continuous.

Proof. Observe that the continuity of F is a consequence of Theorem 3.5.

To prove the second assertion we shall use the form of F in (4) to get

$$\begin{aligned} dy^{k+1} - dx^{k+1} &= (y-x)[a_1 f(x) + b_1 f(\alpha_1 x + \beta_1 y) + \dots + b_n f(\alpha_n x + \beta_n y) + a_1 f(y)] \\ &\quad + (y-x)^3 [c_1 g(\alpha_1 x + \beta_1 y) + \dots + c_n g(\alpha_n x + \beta_n y)]. \end{aligned} \tag{3.54}$$

Now we divide (3.54) by $y - x$ arriving at

$$\begin{aligned} d(y^k &+ y^{k-1}x + \dots + yx^{k-1} + x^k) \\ &= a_1 f(x) + b_1 f(\alpha_1 x + \beta_1 y) + \dots + b_n f(\alpha_n x + \beta_n y) + a_1 f(y) \\ &\quad + (y-x)^2 [c_1 g(\alpha_1 x + \beta_1 y) + \dots + c_n g(\alpha_n x + \beta_n y)] \end{aligned} \tag{3.55}$$

for $x, y \in \mathbb{R}$, $x \neq y$.

To show that the above equation is also satisfied for $x = y$, we use the same method as in the case of the equation stemming from the Hermite quadrature rule. Namely, we take a sequence of rational numbers q_n such that $q_n \neq 1$ and $q_n \to 1$. We put $q_n y$ instead of x in (3.55), and then letting $n \to \infty$ and using Lemma 3.7, we obtain (3.55) for $x = y$. This means that now we are able to write

$$dkx^k = a_1 f(x) + b_1 f(\alpha_1 x + \beta_1 x) + \dots + b_n f(\alpha_n x + \beta_n x) + a_1 f(x) \tag{3.56}$$

and, since $\alpha_i + \beta_i = 1$ and $2a_1 + b_1 + \dots + b_n \neq 0$, we obtain the continuity of f.

To prove that g is continuous we use the continuity of f in (3.55). We get

$$(y-x)^2 [c_1 g(\alpha_1 x + \beta_1 y) + \dots + c_n g(\alpha_n x + \beta_n y)] = p(x, y),$$

where p is some polynomial. However, this means that

$$c_1 g(\alpha_1 x + \beta_1 y) + \dots + c_n g(\alpha_n x + \beta_n y) = p_1(x, y), \tag{3.57}$$

where $p_1(x, y) = p(x, y)/(y-x)^2$ is also a polynomial. As before, this equality is true for $x \neq y$, but it is easy to show that it remains true for $x = y$. Thus we take $x = y$ in (3.57) and since $\alpha_i + \beta_i = 1$, $i = 1, \dots, n$, we find that $(c_1 + \dots + c_n)g(x)$ is continuous. In view of (3.53), this means that g is continuous. ∎

Using this theorem, it is easy to obtain a complete solution of equations of the form (4). Let us for example prove the following result.

Theorem 3.35. *Functions $F, f, g : \mathbb{R} \to \mathbb{R}$ satisfy* (3.52) *if and only if f is a polynomial of degree at most* 5, *F is its antiderivative and $g = f''$.*

Proof. From Theorem 3.30 we know that F, f and g are polynomial functions of orders at most $6, 5$ and 3, respectively. From Remark 3.33 we know that their monomial summands of appropriate orders also satisfy (3.52). Using Theorem 3.34, we obtain the continuity of these summands, and hence of F, f, g. In the next step we divide (3.52) by $y - x$ and let $y \to x$ to get $F' = f$. Finally, we use Remark 3.33 once more and we take $F(x) = x^p$, $f(x) = px^{p-1}$ and $g(x) = dx^{p-3}$. These monomials satisfy (3.52), and thus we easily obtain the desired form of d. ■

4. Functional equations connected with numerical differentiation

4.1. Introduction. In this section we shall work with functional equations motivated by the formulas used to approximate the derivatives of a given function. This is done in the following way:

$$f^{(k)}(\alpha x + \beta y)(y - x)^k \approx a_1 f(\alpha_1 x + \beta_1 y) + \cdots + a_n f(\alpha_n x + \beta_n y), \tag{4.1}$$

and therefore we consider the functional equation

$$g(\alpha x + \beta y)(y - x)^k = a_1 f(\alpha_1 x + \beta_1 y) + \cdots + a_n f(\alpha_n x + \beta_n y). \tag{5}$$

As in the case of quadrature rules, (4.1) is exact for polynomials of some degree depending on the concrete form of (4.1).

4.2. Polynomiality of functions satisfying (5). The next lemma shows that f and g satisfying (5) must be polynomial functions.

LEMMA 4.1. *Let $f, g : \mathbb{R} \to \mathbb{R}$ satisfy equation (5) with some $\alpha, \beta, \alpha_i, \beta_i, a_i \in \mathbb{R}$, $i = 1, \ldots, n$. If there exists $i_0 \in \{1, \ldots, n\}$ such that $\alpha_{i_0} + \beta_{i_0} \neq 0$ and*

$$\begin{vmatrix} \alpha & \beta \\ \alpha_{i_0} & \beta_{i_0} \end{vmatrix} \neq 0 \tag{4.2}$$

then f is a polynomial function of order at most $n + k$.

Further, if $\alpha + \beta \neq 0$ then g is a polynomial function of order at most n.

Proof. Let $i_0 \in \{1, \ldots, n\}$ be such that $a_{i_0} \neq 0$, $\alpha_{i_0} + \beta_{i_0} \neq 0$ and (4.2) is satisfied.

In view of Theorem 2.6, f is a polynomial function of order at most

$$n + 1 + \underbrace{1 + \cdots + 1}_{k} - 1 = n + k.$$

Similarly, Theorem 2.6 implies that g is a polynomial function of order at most n. ■

REMARK 4.2. If $(\alpha, \beta) = (\alpha_i, \beta_i)$ for some $i \in \{1, \ldots, n\}$ then the orders occurring in Lemma 4.1 may be replaced by $n + k - 2$ and $n - 2$, respectively.

Just as for equations stemming from quadrature rules, one can prove

LEMMA 4.3. *Let $f, g : \mathbb{R} \to \mathbb{R}$ be polynomial functions satisfying equation (5). Then their monomial summands g_i and f_{i+k} of orders respectively i and $i + k$ also satisfy this equation.*

In the next remark we shall obtain some simple connections between f and g.

REMARK 4.4. Let $f, g : \mathbb{R} \to \mathbb{R}$ satisfy equation (5) with constants α_i, β_i, $i = 1, \ldots, n$. If we take $y = 0$ in (5), we get

$$g(\alpha x)(-x)^k = a_1 f(\alpha_1 x) + \cdots + a_n f(\alpha_n x). \tag{4.3}$$

On the other hand, taking $x=0$ we get

$$g(\beta y)(y)^k = a_1 f(\beta_1 y) + \cdots + a_n f(\beta_n y). \tag{4.4}$$

Now, let p be a positive integer, g be a monomial function of order p, and f be a monomial function of order $p+k$. If the constants $\alpha, \beta, \alpha_i, \beta_i$, $i = 1, \dots, n$, are rational then from (4.3) we get

$$(-1)^k \alpha^p g(x) x^k = [a_1 \alpha_1^{p+k} + \cdots + a_n \alpha_n^{p+k}] f(x) \tag{4.5}$$

and from (4.4) we obtain

$$\beta^p g(x) x^k = [a_1 \beta_1^{p+k} + \cdots + a_n \beta_n^{p+k}] f(x). \tag{4.6}$$

Remark 4.5. From the last remark we can see that if $f, g : \mathbb{R} \to \mathbb{R}$ satisfy (5), p is a positive integer, g is a monomial function of order p, f is a monomial function of order $p+k$, at least one of $\alpha, \beta \in \mathbb{Q}$ is different from zero, and $a_i \in \mathbb{R}$, $\alpha_i, \beta_i \in \mathbb{Q}$, $i = 1, \dots, n$, satisfy

$$a_1 \alpha_1^{p+k} + \cdots + a_n \alpha_n^{p+k} = 0, \qquad a_1 \beta_1^{p+k} + \cdots + a_n \beta_n^{p+k} = 0$$

then $g = 0$.

4.3. Continuity. As we can see from the previous section, functions satisfying (5) must be polynomial. Thus the main issue now will be the continuity of f and g. However, as we shall see, the results will not be as simple as in the case of equations stemming from quadrature rules.

Theorem 4.6. *Let $f, g : \mathbb{R} \to \mathbb{R}$ satisfy equation* (5), *let p be a positive integer, g be a monomial function of order p, f be a monomial function of order $p+k$ and let $\alpha, \beta \in \mathbb{Q}$ and $\alpha_i, \beta_i \in \mathbb{Q}$, $i = 1, \dots, n$, and $a_i \in \mathbb{R}$, $i = 1, \dots, n$, satisfy*

$$a_1 \alpha_1^{p+k} + \cdots + a_n \alpha_n^{p+k} \neq 0, \qquad a_1 \beta_1^{p+k} + \cdots + a_n \beta_n^{p+k} \neq 0. \tag{4.7}$$

If exactly one of α, β is zero then f and g are zero.

If $\alpha, \beta \neq 0$ then

$$\frac{(-1)^k \alpha^p}{a_1 \alpha_1^{p+k} + \cdots + a_n \alpha_n^{p+k}} = \frac{\beta^p}{a_1 \beta_1^{p+k} + \cdots + a_n \beta_n^{p+k}}. \tag{4.8}$$

Set

$$\mathfrak{a}_p := \frac{(-1)^k \alpha^p}{a_1 \alpha_1^{p+k} + \cdots + a_n \alpha_n^{p+k}}. \tag{4.9}$$

If

$$\begin{gathered}
\mathfrak{a}_p [a_1 \alpha_1 \beta_1^{p+k-1} + \cdots + a_n \alpha_n \beta_n^{p+k-1}] \neq \alpha \beta^{p-1}, \\
\mathfrak{a}_p [a_1 \alpha_1^2 \beta_1^{p+k-2} + \cdots + a_n \alpha_n^2 \beta_n^{p+k-2}] \neq \alpha^2 \beta^{p-2}, \\
\vdots \\
\mathfrak{a}_p [a_1 \alpha_1^{p-1} \beta_1^{k+1} + \cdots + a_1 \alpha_n^{p-1} \beta_n^{k+1}] \neq \alpha^{p-1} \beta, \\
\mathfrak{a}_p [a_1 \alpha_1^p \beta_1^k + \cdots + a_n \alpha_n^p \beta_n^k] \neq \alpha^p,
\end{gathered} \tag{4.10}$$

then f and g are continuous.

Proof. Let g be a monomial function of order p. Then

$$g(x) = A(x, \overset{(p)}{\ldots}, x) \tag{4.11}$$

for some p-additive and symmetric function $A : \mathbb{R}^n \to \mathbb{R}$. From (4.5) and (4.6) we know that if α or β is zero then, in view of (4.7), $f(x) = 0$, and hence also $g(x) = 0$.

Thus we shall assume that $\alpha, \beta \neq 0$. Now let $\mathfrak{a}_p$ be defined by (4.9). Then from (4.6) and (4.5) we obtain

$$\mathfrak{a}_p = \frac{(-1)^k \alpha^p}{a_1 \alpha_1^{p+k} + \cdots + a_n \alpha_n^{p+k}} = \frac{\beta^p}{a_1 \beta_1^{p+k} + \cdots + a_n \beta_n^{p+k}},$$

i.e. (4.8) is satisfied.

Further from (4.6) we get

$$f(x) = \mathfrak{a}_p g(x) x^k, \tag{4.12}$$

and, using this form of f on the right-hand side of (5), we obtain

$$\begin{aligned} a_1 f(\alpha_1 x + \beta_1 y) &+ \cdots + a_n f(\alpha_1 x + \beta_n y) \\ &= \mathfrak{a}_p [a_1 g(\alpha_1 x + \beta_1 y)(\alpha_1 x + \beta_1 y)^k + \cdots + a_n g(\alpha_n x + \beta_n y)(\alpha_n x + \beta_n y)^k] \\ &= \mathfrak{a}_p \sum_{i=1}^n a_i \Bigg[\alpha_i^p g(x) + \binom{p}{1} \alpha_i^{p-1} \beta_i A(x, \ldots, x, y) + \cdots \\ &\qquad + \binom{p}{p-1} \alpha_i \beta_i^{p-1} A(x, y, \ldots, y) + \beta_i^p g(y) \Bigg] (\alpha_i x + \beta_i y)^k. \end{aligned} \tag{4.13}$$

On the other hand, in view of (4.11), the left-hand side of (5) takes the form

$$\begin{aligned} g(\alpha x + \beta y)(y - x)^k = \Bigg[&\alpha^p g(x) + \binom{p}{1} \alpha^{p-1} \beta A(x, \ldots, x, y) + \cdots \\ &+ \binom{p}{p-1} \alpha \beta^{p-1} A(x, y, \ldots, y) + \beta^p g(y) \Bigg] (y - x)^k. \end{aligned} \tag{4.14}$$

Using (4.14) and (4.13) in (5), we arrive at

$$\begin{aligned} \Bigg[\alpha^p g(x) + &\binom{p}{1} \alpha^{p-1} \beta A(x, \ldots, x, y) + \cdots + \binom{p}{p-1} \alpha \beta^{p-1} A(x, y, \ldots, y) + \beta^p g(y) \Bigg] (y - x)^k \\ &= \mathfrak{a}_p \sum_{i=1}^n a_i \Bigg[\alpha_i^p g(x) + \binom{p}{1} \alpha_i^{p-1} \beta_i A(x, \ldots, x, y) + \cdots \\ &\qquad + \binom{p}{p-1} \alpha_i \beta_i^{p-1} A(x, y, \ldots, y) + \beta_i^p g(y) \Bigg] (\alpha_i x + \beta_i y)^k. \end{aligned} \tag{4.15}$$

Let $k \geq p$. Putting $y = 1$ in (4.15) and equating terms of order p, we get

$$\begin{aligned} \alpha^p g(x) + \binom{p}{1} \alpha^{p-1} \beta A(x, \ldots, x, 1) \binom{k}{1} (-x) + \binom{p}{2} \alpha^{p-2} \beta^2 A(x, \ldots, x, 1, 1) \binom{k}{2} (-x)^2 + \cdots \\ + \binom{p}{p-1} \alpha \beta^{p-1} A(x, 1, \ldots, 1, 1) \binom{k}{p-1} (-x)^{p-1} + \binom{k}{p} \beta^p g(1) (-x)^p \end{aligned}$$

$$= \mathfrak{a}_p \sum_{i=1}^{n} a_i \Big[\alpha_i^p \beta_i^k g(x) + \binom{p}{1}\binom{k}{1} \alpha_i^{p-1} \beta_i^{k+1} A(x, \overset{(p-1)}{\ldots}, x, 1)x + \cdots$$
$$+ \alpha_i^{k+1} \beta_i^{p-1} \binom{p}{p-1}\binom{k}{p-1} A(x, 1, \overset{(p-1)}{\ldots}, 1)x^{p-1} + \beta_i^p \alpha_i^k g(1) x^k \Big]. \tag{4.16}$$

The coefficient of $g(x)$ is equal to

$$\alpha^p - \mathfrak{a}_p \sum_{i=1}^{n} a_i \alpha_i^p \beta_i^k$$

and, as we know from (4.10), it is different from zero. Thus to prove the continuity of g we have to show that all functions of the form

$$x \mapsto A(x, \overset{(p-i)}{\ldots}, x, 1, \overset{(i)}{\ldots}, 1), \qquad i = 1, \ldots, p-1, \tag{4.17}$$

are continuous. In case $k < p$, equation (4.16) looks somewhat different but the above reasoning remains unchanged.

To this end we put again $y = 1$ in (4.15) and we compare terms of all orders smaller than p ending at order 1 where we obtain

$$p\alpha\beta^{p-1} A(x, 1, \overset{(p-1)}{\ldots}, 1) + k\beta^p g(1)(-x)$$
$$= \mathfrak{a}_p \Big(p \sum_{i=1}^{n} a_i \alpha_i \beta_i^{p+k-1} A(x, 1, \overset{(p-1)}{\ldots}, 1) + k \sum_{i=1}^{n} a_i \alpha_i \beta_i^{p+k-1} p g(1) x \Big),$$

i.e.

$$\Big(\alpha\beta^{p-1} - \mathfrak{a}_p \sum_{i=1}^{n} a_i \alpha_i \beta_i^{p+k-1} \Big) A(x, 1, \overset{(p-1)}{\ldots}, 1) = \frac{kg(1)}{p} \Big(\sum_{i=1}^{n} a_i \alpha_i \beta_i^{p+k-1} + \beta^p \Big) x.$$

However, from (4.10) we know that

$$\alpha\beta^{p-1} - \mathfrak{a}_p \sum_{i=1}^{n} a_i \alpha_i \beta_i^{p+k-1} \neq 0,$$

which means that

$$A(x, 1, \ldots, 1) = b_1 x$$

for some $b_1 \in \mathbb{R}$. Now we shall use this formula in the equation which contains terms of order 2. Let us write this equation:

$$\binom{p}{2} \alpha^2 \beta^{p-2} A(x, x, 1, \overset{(p-2)}{\ldots}, 1) + pk\alpha\beta^{p-1} A(x, 1, \overset{(p-1)}{\ldots}, 1)(-x) + \binom{k}{2} g(1) \beta^p x^2$$
$$= \mathfrak{a}_p \Big(\binom{p}{2} \sum_{i=1}^{n} a_i \alpha_i^2 \beta_i^{p+k-2} A(x, x, 1, \overset{(p-2)}{\ldots}, 1)$$
$$+ pk \sum_{i=1}^{n} a_i \alpha_i \beta_i^{p+k-1} A(x, 1, \overset{(p-1)}{\ldots}, 1)x + \binom{k}{2} \sum_{i=1}^{n} a_i \beta_i^{p+k} g(1) x^2 \Big).$$

Again, using (4.10), we get

$$\mathfrak{a}_p \big(a_1 \alpha_1^2 \beta_1^{p+k-2} + \cdots + a_n \alpha_n^2 \beta_n^{p+k-2} \big) \neq \alpha^2 \beta^{p-2},$$

which, together with the continuity of $x \mapsto A(x, 1, \ldots, 1)$, yields

$$A(x, x, 1, \ldots, 1) = b_2 x^2$$

for some $b_2 \in \mathbb{R}$.

Proceeding further in the same manner we obtain the continuity of all functions of the form (4.17), and hence the continuity of g. Continuity of f is a consequence of (4.12). ■

Now we shall show how this theorem may be used to solve some equation connected with formulas used in numerical analysis to approximate differentiation.

COROLLARY 4.7. *Functions $f, g : \mathbb{R} \to \mathbb{R}$ satisfy the equation*

$$3g\left(\frac{x+y}{2}\right)(y-x) = f(x) - 8f\left(\frac{3x+y}{4}\right) + 8f\left(\frac{x+3y}{4}\right) - f(y)$$

if and only if

$$f(x) = ax^3 + bx^2 + cx + d, \qquad g(x) = 3ax^2 + 2bx + c.$$

Proof. Denote $h := 3g$. From Lemma 4.1 we know that f is a polynomial function of order at most 5, and h is a polynomial function of order at most 4. Moreover, from Lemma 4.3 we know that monomial summands of h and f of orders i and $i+1$ also satisfy this equation. We are going to show that these monomial functions must be continuous. To this end we have to check that (4.7) and (4.10) are satisfied. Let for example f be a monomial function of order 2, and h be a monomial function of order 1. Then p and k from Theorem 4.6 are equal to 1. Thus (4.7) becomes

$$1^2 - 8 \cdot \left(\frac{3}{4}\right)^2 + 8 \cdot \left(\frac{1}{4}\right)^2 - 0^2 = -3 \neq 0.$$

This means that

$$\mathfrak{a}_1 = \frac{-\frac{1}{2}}{-3} = \frac{1}{6}.$$

Next we check (4.10). In this case (4.10) contains only one condition,

$$\mathfrak{a}_1(a_1\alpha_1\beta_1 + a_2\alpha_2\beta_2 + a_3\alpha_3\beta_3 + a_4\alpha_4\beta_4) \neq \alpha,$$

i.e.

$$\frac{1}{6} \cdot \left(1 \cdot 1 \cdot 0 - 8 \cdot \frac{3}{4} \cdot \frac{1}{4} + 8 \cdot \frac{1}{4} \cdot \frac{3}{4} - 1 \cdot 0 \cdot 1\right) \neq \frac{1}{2},$$

and is obviously satisfied. In the same way we check that all other monomial summands of f and h are continuous. Once we have obtained the continuity of f and $3g$, we use (4.6) to get the connections between their coefficients. ■

REMARK 4.8. Theorem 4.6 may be used to solve many other equations stemming from rules of numerical differentiation such as

$$\frac{1}{4}g\left(\frac{x+y}{2}\right)(y-x)^2 = f(x) - 2f\left(\frac{x+y}{2}\right) + f(y), \tag{4.18}$$

which is connected with a well known method of numerical calculation of the second derivative of f, or

$$\frac{3}{4}g\left(\frac{x+y}{2}\right)(y-x)^2 = -f(x) + 16f\left(\frac{3x+y}{4}\right) - 30f\left(\frac{x+y}{2}\right) + 16f\left(\frac{x+3y}{4}\right) - f(y).$$

However, sometimes Theorem 4.6 cannot be used to prove the continuity of solutions of equations of the form (5).

EXAMPLE 4.9. In the case of the equation

$$g(x)(y-x) = -3f(x) + 4f\left(\frac{x+y}{2}\right) - f(y), \tag{4.19}$$

Theorem 4.6 cannot be applied to show that monomial functions f and g of orders, respectively, 2 and 1 are continuous. Indeed, in this case

$$a_1\beta_1^2 + a_2\beta_2^2 + a_3\beta_3^2 = 0,$$

and so conditions (4.7) are not satisfied. Notice that if we take $y = 0$ in (4.19) then we obtain the formula

$$f(x) = xg(x)/2,$$

which, used in (4.19), yields the continuity of g (and f). However, the version of Theorem 4.6 which would cover this case is more complicated than the original one and we shall not present it here.

REMARK 4.10. It is easy to see that equation (4.18) has discontinuous solutions. This means that assumption (4.7) in Theorem 4.6 is essential. It remains an open problem whether assumptions (4.10) are essential or not.

4.4. Functional equations connected with divided differences. Surprisingly, our results inspired by numerical analysis may be used to obtain solutions of a well known class of functional equations. To present this application we shall need the notion of divided difference $f[x_1,\dots,x_n]$ ($x_1,\dots,x_n$ are pairwise distinct numbers) which will be defined by recurrence. Let

$$f[x_1] := f(x_1), \qquad f[x_1,\dots,x_p] := \frac{f[x_2,\dots,x_p] - f[x_1,\dots,x_{p-1}]}{x_1 - x_n}.$$

Clearly,

$$f[x_1,x_2] = \frac{f(x_2)-f(x_1)}{x_2-x_1},$$
$$f[x_1,x_2,x_3] = \frac{(x_3-x_2)f(x_1) + (x_1-x_3)f(x_2) + (x_2-x_1)f(x_3)}{(x_1-x_2)(x_2-x_3)(x_3-x_1)}.$$

We shall also need the following formula for the nth divided difference (see for example [31]).

THEOREM 4.11. *For all positive integers n, the n-point divided difference can be expressed in the form*

$$f[x_1,\dots,x_n] = \sum_{j=1}^{n} \frac{f(x_j)}{\prod_{k=1,\,k\neq j}^{n}(x_j-x_k)}.$$

The following fact may be found for example in [25].

LEMMA 4.12. *Let $x_1,\dots,x_n \in \mathbb{R}$ be of the form*

$$x_i = x_1 + (i-1)d, \qquad i = 1,\dots,n,$$

for some $d \in \mathbb{R}$. *Then*

$$f[x_1, \ldots, x_n] = \frac{\Delta_d^{n-1} f(x_1)}{(n-1)! d^{n-1}}. \tag{4.20}$$

Using this notion, Aczél's equation (1.2) may be written as

$$g(x+y) = f[x, y]. \tag{4.21}$$

Therefore, as a natural generalization of (4.21), Bailey [4] considered the equation

$$g(x+y+z) = f[x, y, z],$$

which he solved under the assumption that g is differentiable. Then he asked about regular solutions of a more general equation

$$g(x_1 + \cdots + x_n) = f[x_1, \ldots, x_n]. \tag{4.22}$$

Notice that there is a well known mean value theorem for divided differences (see for example [31]).

THEOREM 4.13. *Let* $f : [a, b] \to \mathbb{R}$ *be a real valued function with continuous* n*th derivative and* $x_1, \ldots, x_n \in [a, b]$. *Then there exists a point* η *in the interval*

$$[\min\{x_1, \ldots, x_n\}, \max\{x_1, \ldots, x_n\}]$$

such that

$$f[x_1, \ldots, x_n] = \frac{f^{(n-1)}(\eta)}{(n-1)!}.$$

In the same monograph one can find the following result.

THEOREM 4.14. *Suppose that* $f(x) = x^l$ *for some nonnegative integer* l. *Then for all positive integers* n,

$$f[x_1, \ldots, x_n] = \begin{cases} 0 & \text{for } l < n-1, \\ 1 & \text{for } l = n-1, \\ x_1 + \cdots + x_n & \text{for } l = n. \end{cases}$$

REMARK 4.15. Theorem 4.13 states that $f[x_1, \ldots, x_n]$ is the value of the $(n-1)$th derivative of f at some point. Further, from Theorem 4.14 we know that for monomials this point is the arithmetic mean of $x_1, \ldots, x_n$. Thus (4.22) has nontrivial solutions and it is reasonable to ask if there are solutions of (4.22) other than polynomials.

Equation (4.22) was solved by Kannappan and Sahoo [17]. Then several authors dealt with (4.22) on more general structures than $\mathbb{R}$. In [37] and [7] this equation was solved in a field of characteristic different from 2, and in [8] the case of integral domains was treated. Let us also mention that in the paper of Kannappan [14] the pexiderized version of (4.22) was solved in a field of characteristic different from 2 containing sufficiently many distinct points. We shall cite here a result valid on $\mathbb{R}$.

THEOREM 4.16 ([31, Theorem 2.8]). *Let* $f, g : \mathbb{R} \to \mathbb{R}$ *satisfy the functional equation* (4.22) *for pairwise different* $x_1, \ldots, x_n \in \mathbb{R}$. *Then* g *is a polynomial of degree at most* n *and* f *is a polynomial of first degree.*

We shall prove a more general result but first we shall solve an equation which is strictly connected with (4.22). In numerical analysis the second derivative is often calculated with the use of the formula

$$f''(x)=\frac{f(x+h)-2f(x)+f(x-h)}{h^2}-\frac{1}{12}h^2f^{(4)}(\xi),$$

which is a motivation for the functional equation

$$g(x)=\frac{f(x+h)-2f(x)+f(x-h)}{h^2}.$$

We substitute here $(x+y)/2$ in place of x, and $y-x$ in place of h. We get

$$g\left(\frac{x+y}{2}\right)(y-x)^2=f(x)-2f\left(\frac{x+y}{2}\right)+f(y).$$

This is clearly a functional equation of the form (5). Moreover, from Remark 4.2 we know that if f and g satisfy this equation then f is a polynomial function of order at most 3 and g is a polynomial function of order at most 1 (i.e. an affine function). Later on we shall solve this equation, but first we observe that it may be written in the following way:

$$g(x+y)(y-x)^2=\Delta^2_{(y-x)/2}f(x); \tag{4.23}$$

for simplicity we have replaced here $g(\frac{x+y}{2})$ by $g(x+y)$. This equation may be generalized to

$$g(x+y)(y-x)^n=\Delta^n_{(y-x)/n}f(x). \tag{4.24}$$

Equation (4.24) is not only a natural generalization of (4.23) but is also strictly connected with (4.22).

Proposition 4.17. *If $f,g:\mathbb{R}\to\mathbb{R}$ satisfy equation* (4.22) *then they also satisfy*

$$\frac{(n-1)!}{(n-1)^{n-1}}g\left(\frac{n(x+y)}{2}\right)(y-x)^{n-1}=\Delta^{n-1}_{\frac{y-x}{n-1}}f(x). \tag{4.25}$$

Proof. Assume that f and g satisfy (4.22) and take

$$x_i:=x+(i-1)\frac{y-x}{n-1},\qquad i=1,\dots,n.$$

Then, in view of Lemma 4.12,

$$f[x_1,\dots,x_n]=\frac{\Delta^{n-1}_{\frac{y-x}{n-1}}f(x)}{(n-1)!\left(\frac{y-x}{n-1}\right)^{n-1}}.$$

On the other hand,

$$x_1+\dots+x_n=x+x+\frac{y-x}{n-1}+\dots+x+(n-1)\frac{y-x}{n-1}=\frac{n(x+y)}{2},$$

which yields (4.25). ■

Remark 4.18. In the next theorem we shall obtain a solution of equation (4.25). Knowing solutions of (4.25), it is easy to get solutions of (4.24).

We need the following lemma.

Lemma 4.19. *Let p be a positive integer, let $g \neq 0$ be a monomial function of order p and let f be a monomial function of order $p+k$. If g, f satisfy* (5) *with some $a_i \in \mathbb{R}$, $\alpha, \beta, \alpha_i, \beta_i \in \mathbb{Q}$, $i = 1, \dots, n$, and g is not equal to zero then*

$$a_1 \alpha_1^{p+k} + \cdots + a_n \alpha_n^{p+k} \neq 0,$$

and the functions

$$x \mapsto x^p \quad \text{and} \quad x \mapsto \frac{(-1)^k}{a_1 \alpha_1^{p+k} + \cdots + a_n \alpha_n^{p+k}} x^{p+k}$$

also satisfy (5).

Proof. From Remark 4.4 we know that

$$(-1)^k \alpha^p g(x) x^k = [a_1 \alpha_1^{p+k} + \cdots + a_n \alpha_n^{p+k}] f(x).$$

If g is not equal to zero then $a_1 \alpha_1^{p+k} + \cdots + a_n \alpha_n^{p+k} \neq 0$, thus

$$f(x) = \mathfrak{a}_p g(x) x^k \quad \text{where} \quad \mathfrak{a}_p = \frac{(-1)^k}{a_1 \alpha_1^{p+k} + \cdots + a_n \alpha_n^{p+k}}.$$

Using this in (5), we obtain

$$g(\alpha x + \beta y)(y - x)^k \\ = \mathfrak{a}_p [a_1 g(\alpha_1 x + \beta_1 y)^p (\alpha_1 x + \beta_1 y)^k + \cdots + a_n g(\alpha_n x + \beta_n y)^p (\alpha_n x + \beta_n y)^k]. \tag{4.26}$$

Now, let $t \in \mathbb{R}$ be such that $g(t) \neq 0$. Taking $x, y \in \mathbb{Q}$ and putting xt, yt instead of x, y in (4.26), we get

$$(\alpha x + \beta y)^p g(t)(y - x)^k \\ = \mathfrak{a}_p g(t) [a_1 (\alpha_1 x + \beta_1 y)^p (\alpha_1 x + \beta_1 y)^k + \cdots + a_n (\alpha_n x + \beta_n y)^p (\alpha_n x + \beta_n y)^k].$$

Thus equation (5) is satisfied by the functions $x \mapsto x^p$ and $x \mapsto \frac{(-1)^k}{a_1 \alpha_1^{p+k} + \cdots + a_n \alpha_n^{p+k}} x^{p+k}$ for all rational x and y. However both these functions are continuous and $\mathbb{Q}^2$ is dense in $\mathbb{R}^2$, which means that they satisfy (5) for all $x, y \in \mathbb{R}$. ∎

Now we are ready to prove the main result of this section.

Theorem 4.20. *Functions $f, g : \mathbb{R} \to \mathbb{R}$ satisfy equation* (4.25) *if and only if*

$$f(x) = ax^n + bx^{n-1} + f_{n-2}(x) + \cdots + f_1(x) + c, \qquad g(x) = ax + b,$$

where f_i, $i = 1, \dots, n-2$, are monomial functions of order i and $a, b, c \in \mathbb{R}$ are some constants.

Proof. First we write (4.25) in the form

$$g\left(\frac{n(x+y)}{2}\right)(y - x)^{n-1} = \frac{(n-1)^{n-1}}{(n-1)!} \Delta^{n-1}_{\frac{y-x}{n-1}} f(x) \tag{4.27}$$

to see that (4.25) is of the form (5). So, let constants $\alpha, \beta, \alpha_i, \beta_i$, $i = 1, \dots, n$, be such that (5) with these constants takes the form (4.27). From Lemma 4.1 and Remark 4.2 we know that f is a polynomial function of order at most $2n-1$ and g is a polynomial function of order at most $n-2$. In fact (Remark 4.2) for odd n these orders are smaller. Let us write

$$f(x) = f_{2n-1}(x) + \dots + f_1(x) + c, \qquad g(x) = g_n(x) + \dots + g_1(x) + d,$$

where f_i, $i = 1, \dots, 2n-1$, and g_j, $j = 1, \dots, n$, are monomial functions of orders i and j, respectively. Further, from Lemma 4.3 we know that the pairs

$$(f_1, 0), \dots, (f_{n-2}, 0), (f_{n-1}, c), (f_n, g_1), \dots, (f_{2n-1}, g_n)$$

also satisfy (4.25). We have to show that f_n, f_{n-1} and g_1 are continuous and that f_{i+n}, g_i vanish for $i > 1$.

First we show that f_n and g_1 are continuous. To this end we shall apply Theorem 4.6. It is easy to see that

$$a_1\alpha_1^n + \dots + a_n\alpha_n^n \neq 0. \tag{4.28}$$

Indeed, from Theorem 4.14 we know that (4.22) is satisfied by the pair $\varphi(x) = x^n$ and $\psi(x) = x$, and, in view of Proposition 4.17, this means that this pair satisfies (4.25). In view of Remark 4.5, this yields (4.28).

Thus we only have to check that

$$\mathfrak{a}_1[a_1\alpha_1\beta_1^{n-1} + \dots + a_n\alpha_n\beta_n^{n-1}] \neq \alpha. \tag{4.29}$$

To this end we recall that (x^n, x) satisfies (4.25); in particular, all terms containing xy^{n-1} vanish. This means that

$$\begin{aligned}\alpha xy^{n-1} &- (n-1)\beta yxy^{n-2} \\ &= \big(\mathfrak{a}_1[a_1\alpha_1\beta_1^n + \dots + a_n\alpha_n\beta_n^n] + (n-1)\mathfrak{a}_1[a_1\alpha_1\beta_1^n + \dots + a_n\alpha_n\beta_n^n]\big)xy^{n-1},\end{aligned}$$

and so $\mathfrak{a}_1[a_1\alpha_1\beta_1^n + \dots + a_n\alpha_n\beta_n^n]$ and α cannot have the same sign, which implies (4.29).

Now we show that f_{n-1} is also continuous. We may assume that $f \neq 0$. As before, (4.25) is satisfied by $(x^{n-1}, 1)$. In view of Remark 4.5, this means that

$$a_1\alpha_1^{n-1} + \dots + a_n\alpha_n^{n-1} \neq 0.$$

Since g is constant, Remark 4.4 again yields the continuity of f.

Now we shall show that $g_i = 0$ for $i > 1$. Suppose that $g_i \neq 0$ for some $i \in \{2, \dots, n\}$. Then, using Lemma 4.19, we find that (4.25) is satisfied by $(\gamma x^{i+n-1}, x^i)$ for some $\gamma \in \mathbb{R}$, $\gamma \neq 0$. On the other hand M. Floater [9] proved that

$$\frac{\Delta_h^{n-1} f(\bar{x})}{h^{n-1}} - f^{(n-1)}(x) = \frac{n-1}{24} f^{(n+1)}(\xi)$$

where $\bar{x} = x + nh/2$ and $\xi \in [x, x+nh]$ is some point. This means that the functions $\gamma x^{i+n-1}, x^i$ cannot satisfy (4.25) (the error on the right-hand side is different from zero).

It remains to show that $f_i = 0$ for $i \in \{n+1, \dots, 2n-1\}$. We already know that in this case $g_{i-n+1} = 0$, thus

$$\Delta_{\frac{y-x}{n-1}}^{n-1} f_i(x) = 0;$$

but this means that f_i is a polynomial function of order at most $n-2$, so $f_i = 0$. ∎

Now, we are able to prove a result which is, in fact, stronger than Theorem 4.16.

THEOREM 4.21. *If $f, g : \mathbb{R} \to \mathbb{R}$ satisfy equation* (4.22) *for all $x_1, \dots, x_n$ such that $x_{i+1} - x_i$ is constant with respect to i and for all $x_1, \dots, x_n$ of the form $c_1, \dots, c_{n-1}, x$ where c_i, $i = 1, \dots, n$, are some pairwise different constants then*

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0, \qquad g(x) = a_n x + a_{n-1}$$

for some $a_i \in \mathbb{R}$, $i = 0, \dots, n$.

Conversely, functions given by the above formulas satisfy (4.22) *for all $x_1, \dots, x_n \in \mathbb{R}$.*

Proof. Using Proposition 4.17 and Theorem 4.20, we can see that

$$f(x) = a_{n+1} x^{n+1} + a_n x^n + f_{n-1}(x) + \dots + f_1(x) + a_0$$

for some monomial functions f_i, $i = 1, \dots, n-1$, and

$$g(x) = a_{n+1} x + a_n.$$

Thus we only have to show that $f_{n-1}, \dots, f_1$ are continuous. As before, these functions satisfy (4.22) with $g = 0$. Thus we shall prove that every monomial function h satisfying

$$h[x_1, \dots, x_n] = 0$$

must be continuous. Theorem 4.11 yields

$$\sum_{j=1}^{n} \frac{h(x_j)}{\prod_{k=1,\, k \neq j}^{n} (x_j - x_k)} = 0. \tag{4.30}$$

Now we fix any distinct constants $c_1, \dots, c_{n-1}$ and we put

$$x_1 := c_1, \ \dots, \ x_{n-1} = c_{n-1}, \quad x_n := x$$

in (4.30) to obtain

$$\sum_{j=1}^{n-1} \frac{h(c_j)}{(c_j - x) \prod_{k=1,\, k \neq j}^{n-1} (c_j - c_k)} + \frac{h(x)}{\prod_{k=1,\, k \neq j}^{n} (x - c_k)} = 0.$$

This means that

$$h(x) = -\sum_{j=1}^{n-1} \frac{\prod_{k=1,\, k \neq j}^{n} (x - c_k) h(c_j)}{(c_j - x) \prod_{k=1,\, k \neq j}^{n-1} (c_j - c_k)},$$

which finishes the proof. ∎

REMARK 4.22. Note that one of the steps of the proof of Theorem 4.20 was based on the following procedure. We assume that the relevant equation has some nonzero solutions, and then we use Lemma 4.19 to show that it is also satisfied by some monomials. This allows us to use a result from numerical analysis (because monomials are highly regular). We often struggle to show that some solutions of all kinds of equations must vanish. Therefore this procedure looks promising.

5. Generalizations of quadrature type functional equations

5.1. A generalization of the Aczél equation. As mentioned in the Introduction, the equation

$$F(x) - F(y) = (x-y)f\left(\frac{x+y}{2}\right) \tag{1.2}$$

may be viewed as the starting point for all equations which we consider in this dissertation. Therefore, trying to generalize our equations, we shall have a closer look at this equation. It was proved in [1] that all solutions of this equation are given by

$$F(x) = ax^2 + bx + c, \qquad f(x) = 2ax + b$$

for some $a, b, c \in \mathbb{R}$. For some more general results see [31]. Further in [2] the arithmetic mean occurring in (1.2) is replaced by the geometric or harmonic mean, i.e. among others the following equation is considered:

$$F(x) - F(y) = (x-y)f(\sqrt{xy}),$$

and it is proved that solutions of this equation are of the form

$$F(x) = a\frac{1}{x} + b + cx, \qquad f(x) = c - \frac{a}{x^2}.$$

We now present a different approach in order to replace the arithmetic mean by other means. Namely, in the case of the geometric mean we consider the equation of the form

$$F(x) - F(y) = (\log x - \log y)f(xy).$$

As will be shown, if f satisfies this equation (with some F) then $f(x) = a\log x + b$, which is similar to (1.2) but with the identity replaced by log which is a function that generates the geometric mean. We also present some other equations constructed in the same spirit. The main result of this section is

Theorem 5.1. *Let $(S,*), (G,\circ), (H,+)$ be semigroups such that $*$ is commutative and let $F : S \to G$, $f : S \to G$, $\varphi : S \to H$, $T : G \times H \to G$ be functions such that*

$$\varphi(x * y) = \varphi(x) + \varphi(y) \tag{5.1}$$

and all values of F, φ are invertible with respect to $\circ$ and $+$, respectively.

If the equation

$$F(x) \circ F(y)^{-1} = T(f(x*y), \varphi(y) - \varphi(x)) \tag{5.2}$$

is satisfied then f, φ and T satisfy

$$T(f(x^2 * h^2, 2\varphi(h))^2 = T(f(x^2 * h, \varphi(h)) \circ T(x^2 * h^3, \varphi(h)).$$

Proof. Putting $x * h$ in place of y in (5.2), we obtain

$$F(x) \circ F(x*h)^{-1} = T(f(x^2 * h), \varphi(x*h) - \varphi(x)),$$

which, in view of (5.1), means that

$$F(x) \circ F(x*h)^{-1} = T(f(x^2 * h), \varphi(h)). \tag{5.3}$$

Now, putting $x * h$ in place of x and $x * h^2$ in place of y, we get

$$F(x*h) \circ F(x*h^2)^{-1} = T(f(x^2 * h^3), \varphi(h)). \tag{5.4}$$

From (5.3) and (5.4) we get

$$F(x) \circ F(x * h^2)^{-1} = T(f(x^2 * h), \varphi(h)) \circ T(f(x^2 * h^3), \varphi(h)). \tag{5.5}$$

On the other hand, taking in (5.2) $x * h^2$ in place of y, we arrive at

$$F(x) \circ F(x * h^2)^{-1} = T(f(x^2 * h^2), 2\varphi(h)),$$

which, together with (5.5), means that we have obtained the equation desired. ■

Now we give a short proof of Aczél's celebrated theorem with the use of Theorem 5.1.

Corollary 5.2. *Functions $f, F : \mathbb{R} \to \mathbb{R}$ satisfy the equation*

$$F(x) - F(y) = (x - y)f(x + y) \tag{5.6}$$

if and only if $f(x) = ax + b$ and $F(x) = ax^2 + bx + c$ for some $a, b, c \in \mathbb{R}$.

Proof. From Theorem 5.1 we know that f satisfies the equation

$$2hf(2x + 2h) = hf(2x + h) + hf(2x + 3h).$$

After suitable substitutions this means that $f(x) = A(x) + b$ for some additive function A and some constant b. On the other hand, taking $y = 0$ in (5.6) we get

$$F(x) = xf(x) + c, \tag{5.7}$$

where $c = F(0)$. Inserting the forms of f and F in (5.6), we get

$$y[A(y) + b] - x[A(x) + b] = (y - x)[A(x + y) + b],$$

which gives

$$yA(x) = xA(y),$$

thus $A(x) = xA(1) = ax$, which means that $f(x) = ax + b$, and in view of (5.7) also $F(x) = ax^2 + bx + c$. ■

Corollary 5.3. *Functions $f, F : (0, \infty) \to \mathbb{R}$ satisfy the equation*

$$F(x) - F(y) = (\log x - \log y)f(xy) \tag{5.8}$$

if and only if $f(x) = a \log x + b$ and $F(x) = a(\log x)^2 + b \log x + c$ for some $a, b, c \in \mathbb{R}$.

Proof. From Theorem 5.1 we get

$$2 \log hf(x^2h^2) = \log hf(x^2h) + \log hf(x^2h^3)$$

and, after some substitutions,

$$2f(xh) = f(x) + f(xh^2), \qquad x \in (0, \infty),$$

which means that $f(x) = A(\log x) + b$ for some additive function A. As before, from (5.8) we obtain

$$F(x) = \log xf(x) + c,$$

where $c = F(1)$. Substituting these forms of f and F into (5.8), we get

$$\log y[A(\log y) + b] - \log x[A(\log x) + b] = (\log y - \log x)[A(\log xy) + b],$$

i.e.

$$\log yA(\log x) = \log xA(\log y).$$

Finally, taking $y = e$, we obtain

$$A(\log x) = \log x A(1),$$

which, in view of the form of f, means that $f(x) = a\log x + b$ and $F(x) = a(\log x)^2 + b\log x + c$. ∎

COROLLARY 5.4. *Functions $f, F : \mathbb{R} \to (0, \infty)$ satisfy the equation*

$$\frac{F(x)}{F(y)} = f(x+y)^{x-y} \tag{5.9}$$

if and only if $f(x) = b\exp ax$ and $F(x) = c(b\exp ax)^x$ for some $a, b, c \in \mathbb{R}$.

Proof. From Theorem 5.1 we infer that f satisfies

$$f(2x+2h)^{2h} = f(2x+h)^h f(2x+3h)^h,$$

i.e.

$$f(2x+2h)^2 = f(2x+h)f(2x+3h),$$

which means that $f(x) = b\exp A(x)$ for some additive function A and constant b.

Taking $y = 0$ in (5.9), we get $F(x) = cf(x)^x$ where $c = F(0)$, and using the forms of f and F in (5.9), we obtain

$$\frac{(b\exp A(x))^x}{(b\exp A(y))^y} = (b\exp A(x+y))^{x-y},$$

which gives

$$\frac{(\exp A(x))^x}{(\exp A(y))^y} = \frac{(\exp A(x)\exp A(y))^x}{(\exp A(x)\exp A(y)y},$$

and consequently

$$\exp A(y)^x = \exp A(x)^y,$$

which means that $A(x) = ax$ for some constant $a \in \mathbb{R}$. ∎

REMARK 5.5. As we can see, if f and F satisfy the Aczél equation then Theorem 5.1 shows that $f(x) = A(x) + b$ where A satisfies the Cauchy equation

$$A(x+y) = A(x) + A(y).$$

However, there are three more versions of the Cauchy equation: exponential, logarithmic and multiplicative, which are given by

$$A(x+y) = A(x)A(y), \qquad A(xy) = A(x) + A(y), \qquad A(xy) = A(x)A(y).$$

Thus equations (5.8) and (5.9) may be called, respectively, the logarithmic and the exponential Aczél equation. The only missing case is the multiplicative version which is given by

$$\frac{F(x)}{F(y)} = f(xy)^{\log x - \log y}.$$

5.2. A generalization of equation (1). In fact, this section could be transfered to the "open problems" section but, since we consider here an equation which is not of the (very general) form (1), we decided to keep it here.

Recently, a functional equation of the form

$$F(x+y) - F(x) - F(y) = xf(y) + yf(x) \tag{5.10}$$

was considered by W. Fechner and E. Gselmann. In [34] solutions of this equation (and also of some more general equations) were obtained using similar methods to those used in Section 3. Observe that this equation is not a particular case of (1) but it is easy to see that solutions of (5.10) must be polynomial functions. Indeed, (5.10) is of the form (2.4), which allows us to use Lemma 2.5. However if we try to generalize (5.10), we cannot use this lemma any more. Observe that on the right-hand side of (2.4) in all occurrences of x^i the exponent i is greater than in terms containing y^i. Therefore it would be impossible to work with the equation

$$\begin{aligned} F(x+y) - F(x) - F(y) = x[a_1 f(\alpha_1 x + \beta_1 y) + a_2 f(\alpha_1 x + \beta_2 y)] \\ + y[b_1 f(\gamma_2 x + \delta_2 y) + b_2 f(\alpha_2 x + \beta_2 y)], \end{aligned} \tag{5.11}$$

which is only a slightly generalized version of (5.10). Therefore we formulate the following problem.

Problem 5.6. *Check if Lemma* 2.5 *may be generalized to cover the case of the equation*

$$\begin{aligned} \sum_{i=0}^{m} x^i [a_{1,i} f_i(\alpha_{1,i} x + \beta_{1,i} y) + \cdots + a_{k_i,i} f_i(\alpha_{k_i,i} x + \beta_{k_i,i} y)] \\ = \sum_{i=0}^{m} y^i [b_{1,i} g_i(\gamma_{1,i} x + \delta_{1,i} y) + \cdots + b_{j_i,i} g_i(\gamma_{j_i,i} x + \delta_{j_i,i} y)]. \end{aligned}$$

6. Some open problems

6.1. A common generalization of (1.1) **and** (5)**.** Observe that the equations

$$F(y) - F(x) = (y-x)[a_1 f(\alpha_1 x + \beta_1 y) + \cdots + a_n f(\alpha_n x + \beta_n y)] \tag{2}$$

and

$$g(\alpha x + \beta y)(y-x)^k = a_1 f(\alpha_1 x + \beta_1 y) + \cdots + a_n f(\alpha_n x + \beta_n y) \tag{5}$$

may be viewed as generalizations of

$$F(y) - F(x) = (y-x) f\left(\frac{x+y}{2}\right). \tag{1.2}$$

Indeed, in (2) the right-hand side of (1.2) is replaced by a more general expression, while in (5) the right-hand side of (1.2) is only slightly changed but the left-hand side is substantially generalized. This observation leads us to the functional equation

$$\begin{aligned} [a_1 f(\alpha_1 x + \beta_1 y) + \cdots + a_n f(\alpha_n x + \beta_n y)](y-x)^k \\ = b_1 g(\gamma_1 x + \delta_1 y) + \cdots + b_m g(\gamma_m x + \delta_m y). \end{aligned} \tag{6.1}$$

Remark 6.1. The functional equation (6.1) is of the form (1). Thus if we assume that there exists an $i_0 \in \{1, \dots, n\}$ such that $\alpha_{i_0} + \beta_{i_0} \neq 0$ and there is $j_0 \in \{1, \dots, m\}$ such that $\gamma_{j_0} + \delta_{j_0} \neq 0$ and

$$\begin{vmatrix} \alpha_i & \beta_i \\ \gamma_{j_0} & \delta_{j_0} \end{vmatrix} \neq 0$$

for all $i \in \{1, \dots, n\}$ then Theorem 2.6 implies that g and f are polynomial functions.

We omit the proof of this fact because it is similar to the cases considered previously.

Let us give a reasonable example of a particular case of (6.1):

$$\left[\frac{1}{6} f(x) + \frac{2}{3} f\left(\frac{x+y}{2}\right) + \frac{1}{6} f(y)\right] y^2 = g(x) - 2g\left(\frac{x+y}{2}\right) + g(y). \tag{6.2}$$

Knowing that f and g are polynomial functions, we may proceed as in the case of equations stemming from differentiation formulas. Namely, one can work with monomial functions of orders $p, p+k$. Then we take $y = 0$ to obtain a relation between f and g. Finally, we substitute the resulting formula to (6.2) and we get the continuity of g (for $p \geq 2$). However this procedure must be executed for each equation separately. Therefore the following problem arises.

Problem 6.2. *Find general conditions under which solutions of* (6.1) *must be continuous.*

6.2. A complete solution of equation (2). Note that we have obtained a general solution of

$$F(y) - F(x) = (y - x)[a_1 f(\alpha_1 x + \beta_1 y) + \dots + a_n f(\alpha_n x + \beta_n y)] \tag{2}$$

in the case of rational coefficients α_i, β_i. Further, we have proved (Theorem 3.10) that if $\alpha_i + \beta_i = 1$, $i = 1, \dots, n$, and $a_1 + \dots + a_n \neq 0$ then f must be continuous. In the case $a_1 + \dots + a_n = 0$ we may reduce (2) to the much simpler equation

$$a_1 f(\alpha_1 x + \beta_1 y) + \dots + a_n f(\alpha_n x + \beta_n y) = 0. \tag{6.3}$$

Remark 6.3. If monomial functions F and f of orders respectively p and $p + 1$ satisfy (2) where a_1, α_i, β_i are such that

$$\alpha_i + \beta_i = 1, \quad i = 1, \dots, n, \quad a_1 + \dots + a_n = 0 \tag{6.4}$$

then $F = 0$ and f satisfies (6.3). Indeed, from Theorem 3.5 we get $F(x) = cx^{p+1}$ for some $c \in \mathbb{R}$. Using this form of F in (2), we obtain

$$c(y^{p-1}x + \dots + xy^{p-1}) = [a_1 f(\alpha_1 x + \beta_1 y) + \dots + a_n f(\alpha_n x + \beta_n y)].$$

Taking here $x = y$ and using the assumption $a_1 + \dots + a_n = 0$, we get $c = 0$. This means that f satisfies (6.3).

Since there are known results concerning equation (6.3) in the case of (6.4) (see for example [42], [18], [41]), equation (2) may be considered to be solved in the case

$$\alpha_i + \beta_i = 1, \quad i = 1, \dots, n.$$

Nevertheless, we may formulate the following problem.

Problem 6.4. *Find the general solutions of* (2) *in the case of irrational* α_i, β_i *without the assumption that all sums* $\alpha_i + \beta_i$ *are equal to* 1.

6.3. A generalization of the conjecture of Sablik. Let us recall Sablik's conjecture, which turned out to be true: as stated in Theorem 3.5, if functions $f_1, \dots, f_n$ and F satisfy (1.1) then F must be continuous. This means that the discontinuous parts of $f_1, \dots, f_n$ vanish on the right-hand side of (1.1). Moreover, in all particular cases of (5) we have considered, the same situation was true, namely in each case g turned out to be continuous and discontinuous parts of f vanished on the right-hand side.

Therefore, as a possible generalization, we state the following conjecture.

Problem 6.5. *Let $p \in \mathbb{N}$ and let $f_{j,i} : \mathbb{R} \to \mathbb{R}$ be monomial functions of order $p+i$ which fulfill for all $x, y \in \mathbb{R}$ the functional equation*

$$\sum_{i=0}^{l} (y-x)^i [f_{1,i}(\alpha_{1,i}x + \beta_{1,i}y) + \dots + f_{k_i,i}(\alpha_{k_i,i}x + \beta_{k_i,i}y)] = 0. \tag{1}$$

Prove that if there exist $i_0 \in \{0, \dots, l\}$ and $j_0 \in \{1, \dots, k_{i_0}\}$ such that f_{j_0,i_0} is discontinuous then the equation

$$g_1(\alpha_{1,i_0}x + \beta_{1,i_0}y) + \dots + g_{k_{i_0}}(\alpha_{k_{i_0},i_0}x + \beta_{k_{i_0},i_0}y) = 0$$

has a nonzero solution which is a monomial function of order $p + i_0$.

Solution of this problem would help obtain solutions of equations of the type (1).

6.4. Equation (5). Note that for equation (5), in all cases we have considered the function g appeared to be continuous. Therefore we formulate the following problem.

Problem 6.6. *Let $p, k \in \mathbb{N}$ and let $g, f : \mathbb{R} \to \mathbb{R}$ be monomial functions of orders, respectively, $p, p+k$ which fulfill for all $x, y \in \mathbb{R}$ the functional equation*

$$g(\alpha x + \beta y)(y-x)^k = a_1 f(\alpha_1 x + \beta_1 y) + \dots + a_n f(\alpha_n x + \beta_n y). \tag{5}$$

Check if the assumptions (4.10) of Theorem 4.6 are essential.

An affirmative solution of this problem would help solve equations of the type (5).

6.5. Equations stemming from Runge–Kutta methods. The next interesting class of functional equations is motivated by methods used in numerical solutions of differential equations. For example the Euler method yields the equation

$$F(y) - F(x) = (y-x) f(x, F(x)), \tag{6.5}$$

which may easily be solved.

However, if we consider more complicated methods, then more complicated equations appear. For example the midpoint method gives

$$F(y) - F(x) = (y-x) f\left(\frac{x+y}{2}, F(x) + \frac{y-x}{2} f(x, F(x))\right), \tag{6.6}$$

and from the Runge–Kutta method we get the equation

$$F(y) - F(x) = (y-x)\left[\frac{1}{6}k_1 + \frac{1}{3}k_2 + \frac{1}{3}k_3 + \frac{1}{6}k_4\right], \tag{6.7}$$

where

$$\begin{aligned} k_1 &= f(x, F(x)),\\ k_2 &= f\left(\frac{x+y}{2}, F(x) + k_1\frac{y-x}{2}\right),\\ k_3 &= f\left(\frac{x+y}{2}, F(x) + k_2\frac{y-x}{2}\right),\\ k_4 &= f\left(y, F(x) + k_3\frac{y-x}{2}\right). \end{aligned}$$

Observe that if f is constant with respect to the second variable then (6.6) takes the form (1.2), and from (6.7) we get (1.7).

However, in general, (6.7) is much more complicated than all equations we have considered. Nevertheless, it would be interesting to find out for which functions the Runge–Kutta and other numerical methods give exact results.

Problem 6.7. *Find solutions of equations of the type* (6.6), (6.7).

6.6. Stability of (1.2) **and** (1.4)**.** Here we shall say a few words concerning the stability properties of equations of the type (1.1). Thus we shall consider the inequality

$$\left|F(y) - F(x) - (y-x)[f_1(\alpha_1 x + \beta_1 y) + \cdots + f_n(\alpha_n x + \beta_n y)]\right| \le \varepsilon. \tag{6.8}$$

In [40] it was proved that the equation

$$F(y) - F(x) = (y-x)f\left(\frac{x+y}{2}\right) \tag{1.2}$$

is superstable. However, using the method from Section 2.3, it is possible to eliminate F from (6.8). Namely, we can prove the following proposition.

Proposition 6.8. *If* $F, f : \mathbb{R} \to \mathbb{R}$ *satisfy* (6.8) *then* f *satisfies*

$$\begin{aligned} \Big|h\big[&f_1((\alpha_1+\beta_1)x + \beta_1 h) + \cdots + f_n((\alpha_n+\beta_n)x + \beta_n h)\\ &+ f_1((\alpha_1+\beta_1)x + (\alpha_1+2\beta_1)h) + \cdots + f_n((\alpha_n+\beta_n)x + (\alpha_n+2\beta_n)h)\\ &- 2[f_1((\alpha_1+\beta_1)x + 2\beta_1 h) + \cdots + f_n((\alpha_n+\beta_n)x + 2\beta_n h)]\big]\Big| \le \varepsilon. \end{aligned} \tag{6.9}$$

Proof. We take $x+h$ in place of y in (6.8). We get

$$\left|F(x+h) - F(x) - h\big[f_i((\alpha_1+\beta_1)x + \beta_1 h) + \cdots + f_n((\alpha_n+\beta_n)x + \beta_n h)\big]\right| \le \varepsilon.$$

Then we take $x+2h$ in place of y and $x+h$ in place of x in (6.8), which gives

$$\begin{aligned} \big|F(x+2h) &- F(x+h)\\ &- h[f_1((\alpha_1+\beta_1)x + (\alpha_1+2\beta_1)h) + \cdots + f_n((\alpha_n+\beta_n)x + (\alpha_n+2\beta_n)h)]\big| \le \varepsilon. \end{aligned}$$

From these two inequalities we obtain

$$\left|F(x+2h) - F(x) - h\big[f_1((\alpha_1+\beta_1)x + 2\beta_1 h) + \cdots + f_n((\alpha_n+\beta_n)x + 2\beta_n h)\big]\right| \le \varepsilon. \tag{6.10}$$

On the other hand, taking $x+2h$ instead of y in (6.8), we obtain

$$\left|F(x+2h) - F(x) - 2h\big[f_1((\alpha_1+\beta_1)x + 2\beta_1 h) + \cdots + f_n((\alpha_n+\beta_n)x + 2\beta_n h)\big]\right| \le \varepsilon.$$

This equation together with (6.10) yields (6.9). ∎

In view of this proposition, the question of stability of equation (1.1) may be reduced to the stability of the equation

$$\begin{aligned} h\big[&f_1((\alpha_1+\beta_1)x+\beta_1 h)+\cdots+f_n((\alpha_n+\beta_n)x+\beta_n h) \\ &+f_1((\alpha_1+\beta_1)x+(\alpha_1+2\beta_1)h)+\cdots+f_n((\alpha_n+\beta_n)x+(\alpha_n+2\beta_n)h) \\ &=2[f_1((\alpha_1+\beta_1)x+2\beta_1 h)+\cdots+f_n((\alpha_n+\beta_n)x+2\beta_n h)]\big]=0, \end{aligned}$$

and although this equation looks complicated, in concrete situation it is much simpler. In particular, if we consider (1.2) then from Proposition 6.8 (after suitable substitutions) we get

$$|h\Delta_h^2 f(x)| \le \varepsilon \tag{6.11}$$

(see [40]). Also in [40] it was proved that for every $\eta > 0$ the following implication holds:

$$|\Delta_h^2 f(x)| \le \varepsilon \text{ for } h \ge \eta \;\Rightarrow\; |\Delta_h^2 f(x)| \le 2\varepsilon \text{ for all } h > 0.$$

This result allowed to obtain the stability of (6.11) and, consequently of (1.2).

Remark 6.9. Using Proposition 6.8, from

$$\big|F(y)-F(x)-(y-x)[f(x)+f(y)]\big| \le \varepsilon$$

we get (6.11). This means that it is easy to prove the superstability of (1.4).

Therefore we formulate the following problem.

Problem 6.10. *Is it true that for every $n \in \mathbb{N}$ there exists $k > 0$ such that for all $\eta > 0$,*

$$|\Delta_h^n f(x)| \le \varepsilon \text{ for } h \ge \eta \;\Rightarrow\; |\Delta_h^n f(x)| \le k\varepsilon \text{ for all } h > 0?$$

The (affirmative) answer to this question would result in the superstability of many equations stemming from quadrature rules.

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