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Abstract

Curvature measures are an important tool in geometric measure theory and other fields of mathematics for describing the geometry of sets in Euclidean space. But the 'classical' concepts of curvature are not directly applicable to fractal sets. We try to bridge this gap between geometric measure theory and fractal geometry by introducing a notion of curvature for fractals. For compact sets $F \subseteq \mathbb{R}^d$ (e.g. fractals), for which classical geometric characteristics such as curvatures or Euler characteristic are not available, we study these notions for their ε-parallel sets

$$F_\varepsilon := \{x \in \mathbb{R}^d : \inf_{y \in F} \|x - y\| \leq \varepsilon\}$$

instead, expecting that their limiting behaviour as $\varepsilon \to 0$ provides information about the structure of the initial set F. In particular, we investigate the limiting behaviour of the total curvatures (or intrinsic volumes) $C_k(F_\varepsilon), k = 0, \ldots, d$, as well as weak limits of the corresponding curvature measures $C_k(F_\varepsilon, \cdot)$ as $\varepsilon \to 0$. This leads to the notions of *fractal curvature* and *fractal curvature measure*, respectively. The well known Minkowski content appears in this context as one of the fractal curvatures.

For certain classes of self-similar sets, results on the existence of (averaged) fractal curvatures are presented. These limits can be calculated explicitly and are in a certain sense 'invariants' of the sets, which may help to distinguish and classify fractals. Based on these results also the fractal curvature measures of these sets are characterized. As a special case and a significant refinement of known results, a local characterization of the Minkowski content is given.

1. Introduction

Curvature measures are an important tool in the study of the geometry of sets. The purpose of this paper is to introduce the notions of *fractal curvatures* and *fractal curvature measures* for fractal sets. This is motivated on the one hand by the desire to find quantitative measures for fractals to characterize their geometry beyond dimension, and on the other hand by the longstanding quest in geometric measure theory to extend the concept of curvature measures for sets in Euclidean space as far as possible. Fractal curvatures are intended to contribute to both of these goals.

The paper *Curvature measures* [8], written by Herbert Federer in 1959, can be regarded as the birth of the notion of curvature measure, although he unified and generalized existing concepts from convex and differential geometry into one theory and built on the previous work of many people including A. D. Aleksandrov, W. Blaschke, W. Fenchel, H. Hadwiger, W. Minkowski, A. Santaló and H. Weyl to name just a few. The definition of curvature measures is based on parallel sets. For a set $K \subset \mathbb{R}^d$ and $\varepsilon > 0$, the *ε-parallel set* of K is defined as the set

$$K_\varepsilon := \{x \in \mathbb{R}^d : \inf_{y \in K} \|x - y\| \leq \varepsilon\},$$

of all points x with Euclidean distance at most ε from K. A closed set K is said to have *positive reach* if there exists some $\varepsilon > 0$ such that each point $x \in K_\varepsilon$ has a unique nearest point in K. The *reach* of K is the supremum over all such ε. Federer introduced curvature measures for sets with positive reach by means of the so-called local Steiner formula. It states that the (local) volume of the parallel set K_ε has a polynomial expansion in ε of degree at most d with coefficients, $C_0(K, \cdot), \ldots, C_d(K, \cdot)$, being locally finite signed measures. These are called the *curvature measures* of K. The elegance of this approach lies in the fact that it requires no differentiability or convexity assumption on the set K. Nevertheless the class of sets with positive reach includes all convex sets (sets with infinite reach) as well as differentiable manifolds (of class C^2, without boundary).

For compact sets K with positive reach, the curvature measures $C_k(K, \cdot)$ are finite and their total masses $C_k(K) := C_k(K, \mathbb{R}^d)$ are called *total curvatures*. In convex geometry they are better known as *intrinsic volumes* or *Minkowski functionals*. They are of independent interest as geometric invariants of the sets and numerical characteristics quantifying certain properties such as volume, surface area or mean width. In differential geometry, total curvatures correspond to the *integrals of mean curvatures*. More precisely, if the boundary $M = \partial K$ of K is a $(d-1)$-dimensional C^2-manifold without boundary and $k_1(x), \ldots, k_{d-1}(x)$ are the principal curvatures of M at the point $x \in M$, then, for

$k = 0, \ldots, d - 1$, the kth curvature measure has the integral representation

$$C_k(K, A) = (d\kappa_{d-k})^{-1} \binom{d}{k} \int_{M \cap A} H_{d-1-k}(k_1(x), \ldots, k_{d-1}(x)) \, d\mathcal{H}^{d-1}(x),$$

for Borel sets $A \subseteq \mathbb{R}^d$. Here κ_i denotes the volume of the unit ball in $\mathbb{R}^i$ and H_j is the jth normalized symmetric function given by

$$H_j(k_1, \ldots, k_{d-1}) = \binom{d-1}{j}^{-1} \sum_{1 \le i_1 < \cdots < i_j \le d-1} k_{i_1} \cdot \ldots \cdot k_{i_j}.$$

In particular, $H_{d-1}(k_1, \ldots, k_{d-1}) = k_1 \cdot \ldots \cdot k_{d-1}$ is the *Gaussian curvature*, i.e. the determinant of the second fundamental form, and so $C_0(K)$ is the integral of Gaussian curvature. Similarly, $H_1(k_1, \ldots, k_{d-1}) = \frac{1}{d-1}(k_1 + \cdots + k_{d-1})$ is the *mean curvature*, i.e. (up to a constant) the trace of the second fundamental form. Thus $C_{d-2}(K)$ is the integral of mean curvature. For more details see Federer [8, Remark 5.21, p. 466].

Not only the total curvatures but also the curvature measures have nice properties, including additivity, continuity, motion invariance and homogeneity. They can be characterized axiomatically by those properties, which underlines their special significance; see Schneider [30] for the case of convex bodies and Zähle [37] for sets with positive reach. The measures $C_k(K, \cdot)$ are concentrated on the boundary of K, except for the case $k = d$, for which $C_d(K, \cdot)$ is the volume or Lebesgue measure $\lambda_d(K \cap \cdot)$ restricted to K.

From the theoretical viewpoint as well as for applications, the class of sets with positive reach is not large enough. For example, simple sets such as the union of two intersecting balls or a non-convex polytope do not have positive reach. Therefore, many efforts have been made to extend the notion of curvature measure to more general sets. One approach in this direction uses additive extensions. Here the additivity of the curvature measures for sets with positive reach is used to define them for sets which can be represented as (locally finite) unions of these sets. Such an extension has first been considered by Groemer [12] for the subclass $\mathcal{K}^d$ of compact, convex sets, introducing curvature measures for *polyconvex sets*, i.e. finite unions of convex sets, in this way. The additive extension of the total curvatures (or intrinsic volumes) to this class of sets (called the convex ring $\mathcal{R}^d$) even goes back to Hadwiger [13]. Unions of sets with positive reach and the additive extension of curvature measures to this class of sets have been considered by Rataj and Zähle [26, 36]. Another approach is to characterize curvature measures directly by some type of Steiner formula. By introducing an index function counting multiplicities in the parallel set, the curvature measures for polyconvex sets can be obtained as coefficients of the polynomial expansion of some weighted parallel volume (cf. Schneider [31]). In a very recent paper by Hug, Last and Weil [14] *support measures* for general closed sets have been defined by means of some Steiner type formula.

Curvature measures can also be introduced using the approximation of K by its parallel sets K_ε, an approach we will also follow here when defining fractal curvatures. For sets with positive reach, curvature measures have an integral representation very similar to the one mentioned above for smooth manifolds where the principal curvatures are replaced by the so-called *generalized principal curvatures*, which are given as limits of corresponding principal curvatures in the parallel sets K_ε (cf. [35]).

Moreover, for a set K with reach $r > 0$, the parallel sets K_ε with $\varepsilon < r$ have positive reach as well, and the curvature measures $C_k(K_\varepsilon, \cdot)$ of these parallel sets converge weakly to the curvature measure $C_k(K, \cdot)$ of K,

$$C_k(K_\varepsilon, \cdot) \overset{\mathrm{w}}{\to} C_k(K, \cdot) \quad \text{as } \varepsilon \to 0.$$

This can be derived directly from the Steiner formula. Such a convergence also occurs for more general sets K, e.g. polyconvex sets or certain finite unions of sets with positive reach, and was the base for further generalizations of curvature measures, for instance to Lipschitz manifolds (cf. [27, 28]). Also compare the work of Fu [10]. There are numerous other attempts for generalizations of curvature measures to different classes of sets, some of which use completely different methods, for instance from algebraic geometry. For an overview and further references we recommend the survey of Bernig [2].

However, none of the available notions of curvature measures provides tools for describing the geometry of fractal sets. Except for the support measures of Hug, Last and Weil [14], most fractal sets are not included in any of the set classes for which curvature measures can be defined. In fractal geometry on the other hand, apart from the different concepts of dimension, there have been up to now very few generally accepted quantitative measures, which are able to provide additional information on the geometric structure of fractal sets. One of them is the *Minkowski content*. For any compact set $F \subset \mathbb{R}^d$, it is defined as the number

$$M(F) := \lim_{\varepsilon \to 0} \varepsilon^{m-d} \lambda_d(F_\varepsilon),$$

provided that the limit on the right hand side exists. Here $m = m(F)$ denotes the *Minkowski* or *box dimension* of F (and λ_d the Lebesgue measure in $\mathbb{R}^d$). The Minkowski content is a rather old notion in fractal geometry, and has proved to be an important tool. It was proposed by Mandelbrot in [23] as a *measure of lacunarity*, i.e. an index characterizing the *texture* of fractal sets. Moreover, in the theory of fractal strings, the *Minkowski measurability* of a fractal string, i.e. the existence of its Minkowski content, is related to certain properties of its spectrum (cf. Lapidus and van Frankenhuysen [20]).

We introduce *fractal curvatures* in much the same way as the Minkowski content replacing the volume by total curvatures. Suppose that, for a given compact set F, curvature measures $C_0(F_\varepsilon, \cdot), \ldots, C_d(F_\varepsilon, \cdot)$ and hence total curvatures $C_k(F_\varepsilon) = C_k(F_\varepsilon, \mathbb{R}^d)$, $k \in \{0, \ldots, d\}$, are defined for the parallel sets F_ε. Then the *kth fractal curvature* of F is introduced as the limit

$$C_k^f(F) = \lim_{\varepsilon \to 0} \varepsilon^{s_k} C_k(F_\varepsilon),$$

where the scaling exponent s_k has to be chosen appropriately. In order for this notion to make sense, the following three questions have to be answered: 1. Given a set F, are curvature measures $C_k(F_\varepsilon, \cdot)$ of its parallel sets defined? 2. What is the appropriate scaling exponent s_k for F? and 3. Does the limit exist? To none of these questions there is a simple general answer and so it is the primary and most important task to explore under which conditions fractal curvatures exist.

Since $C_d(F_\varepsilon, \cdot) = \lambda_d(F_\varepsilon \cap \cdot)$, the Minkowski content is included in this concept as the fractal curvature of order d. In [22], *fractal Euler numbers* were studied by Marta Llorente

and the author, which are derived in a similar manner from the Euler characteristic of the parallel sets. These numbers fit into the presented framework too. They are closely related to the 0th fractal curvature.

Together with the limiting behaviour of the total curvatures $C_k(F_\varepsilon)$, the question of the convergence of the corresponding curvature measures $C_k(F_\varepsilon, \cdot)$ as $\varepsilon \to 0$ arises. In case the total curvatures converge, the measures have to be rescaled with the same exponent s_k, i.e. the measures $\varepsilon^{s_k} C_k(F_\varepsilon, \cdot)$ are considered. If some weak limit of these rescaled curvature measures exists, it will be regarded as the *kth fractal curvature measure* of F.

The concept of fractal curvatures and fractal curvature measures arises naturally not only because of the direct analogy with the Minkowski content. Also the convergence behaviour of curvature measures discussed above suggests such an approach. The new feature in the fractal case is the need to rescale the curvatures.

We apply the concept of fractal curvatures to study self-similar sets F (which we always assume to satisfy the open set condition here). Regarding the first of the three questions above, in the present paper we will assume that the parallel sets of F are polyconvex. On the one hand, this assumption is easy to check, since the polyconvexity of a single parallel set F_ε implies the polyconvexity of all parallel sets of F. On the other hand, this assumption implies that the curvature measures are well defined for the parallel sets of F. For many popular self-similar sets, e.g. Sierpiński gasket or Sierpiński carpet, the assumption is satisfied. Nevertheless, we want to point out that the polyconvexity of the parallel sets is a rather strong assumption which narrows the class of sets to be included in the discussion significantly. But additional technical difficulties occur in any more general setting. We hope that this restriction can be overcome in future work.

Turning to the remaining two questions, we first characterize the scaling exponents s_k of the total curvatures, which, except for some degenerate cases, turn out to be directly related to the so-called similarity dimension of the self-similar set. Concerning the existence of the limits, one has to distinguish arithmetic and non-arithmetic self-similar sets. For sets that are not too regular (i.e. non-arithmetic sets), fractal curvatures are shown to exist, while for arithmetic sets—including for instance sets where all contraction ratios are the same, like Sierpiński gasket or Sierpiński carpet—some additional averaging is necessary in order to have convergence. Instead of considering the limit of $\varepsilon^{s_k} C_k(F_\varepsilon)$ as $\varepsilon \to 0$, the averaged limit

$$\overline{C}_k^f(F) := \lim_{\delta \to 0} \frac{1}{|\ln \delta|} \int_\delta^1 \varepsilon^{s_k} C_k(F_\varepsilon) \, \frac{d\varepsilon}{\varepsilon}$$

is studied, which can be shown to exist in general, i.e. in the arithmetic as well as in the non-arithmetic case. These results are analogous to known results for the Minkowski content (cf. in particular Gatzouras [11]) and are derived in a similar way by application of the Renewal Theorem to some appropriate renewal equation. We propose these quantities as geometric characteristics for fractal sets, which could play a similar role for fractal sets as the total curvatures (or equivalently volume, surface area, Euler characteristic and so on) in classical geometry. Therefore we will demonstrate in several examples how fractal curvatures are determined and interpreted.

For non-arithmetic self-similar sets F, the rescaled curvature measures $\varepsilon^{s_k} C_k(F_\varepsilon, \cdot)$ are shown to converge weakly to some limit measure, i.e. the kth fractal curvature measure exists. For arithmetic self-similar sets, again the convergence behaviour has to be improved by some averaging in order to obtain a weak limit. In both cases, the limit measure turns out to be some multiple of the Hausdorff measure on F. Its total mass is $C_k^f(F)$, the corresponding fractal curvature, or its averaged counterpart $\overline{C}_k^f(F)$, respectively. At first glance, the result that all fractal curvature measures are multiples of each other is surprising and somehow disappointing, since the original idea was to find some new geometric measures on these fractals. But the result means that, from the point of view of curvature measures, self-similar sets have a very simple structure. They are characterized by $d+1$ numbers and one measure, instead of $d+1$ different measures. Due to the self-similarity, the curvature is more or less spread uniformly over the set. In general, i.e. for non self-similar sets, one should expect the fractal curvature measures to be different.

In the case $k = d$, this convergence result extends and refines the known results for the Minkowski content. It provides a local characterization of the limiting behaviour of the volume of the parallel sets. For this result, the assumption on the parallel sets of being polyconvex is not needed. It holds for arbitrary self-similar sets satisfying the open set condition. The limit measure is $\overline{M}(F)\mu_F$, where μ_F denotes the normalized Hausdorff measure on F. The geometric interpretation of this convergence is as follows: If B is a 'nice' set, i.e. if $\mu_F(\partial B) = 0$, then the rescaled parallel volume $\varepsilon^{s_d}\lambda_d(F_\varepsilon \cap B)$ of F_ε in the set B converges to the corresponding 'local Minkowski content' $M(F)\mu_F(B)$ of F in B, in the non-arithmetic case, and the averaged rescaled parallel volume to the 'local average Minkowski content' $\overline{M}(F)\mu_F(B)$, respectively.

The paper is organized as follows. In Chapter 2 the concept of fractal curvatures is introduced and all the main results are presented. Most of the proofs are postponed to later chapters. First we recall in Section 2.1 the definition of curvature measures in the convex ring and their variation measures and discuss some of their properties. Section 2.2 provides the setting and the main definitions, while in Section 2.3 self-similar sets are introduced and fractal curvatures are studied for those sets. In Section 2.4, we discuss several examples to illustrate the results of the preceding section and show how fractal curvatures are practically computed and interpreted. Then we turn to the localization and study in Section 2.5 weak limits of rescaled curvature measures.

Chapters 3 and 4 provide several auxiliary results preparing the proofs of the main theorems. While in Chapter 3 some statements on curvature measures and their variation measures are discussed, Chapter 4 recalls some version of the Renewal Theorem and reformulates it in a way which is most convenient for our purposes. Chapter 5 is devoted to the proofs of the results of Section 2.3 on fractal curvatures and the associated scaling exponents, while in Chapter 6 the results of Section 2.5 on fractal curvature measures are proved. In the Appendix some facts about signed measures are recalled, especially the notion of weak convergence of signed measures, which has seldom been used in the literature.

2. Main results and examples

2.1. Curvature measures. We start by recalling the definition and basic properties of curvature measures of polyconvex sets. Main references for the mostly well known statements in this section are the books of Schneider [32] and Schneider and Weil [33]. Also compare the monograph by Klain and Rota [17].

The convex ring. A set $K \subseteq \mathbb{R}^d$ is said to be *convex* if for any two points $x, y \in K$ the line segment connecting them is a subset of K. Denote by $\mathcal{K}^d$ the family of all *convex bodies*, i.e. of all nonempty compact convex sets in $\mathbb{R}^d$. A set K is called *polyconvex* if it can be represented as a finite union of convex bodies. The class $\mathcal{R}^d$ of all polyconvex sets in $\mathbb{R}^d$ is called the *convex ring*. It is stable with respect to finite unions and intersections (provided the empty set is included).

Curvature measures. For each set $K \in \mathcal{R}^d$ there exist $d + 1$ totally finite signed measures $C_0(K, \cdot)$, $C_1(K, \cdot)$, ..., $C_d(K, \cdot)$, called the *curvature measures* of K, which describe the local geometry of K. For convex bodies K, they are characterized by the so-called *local Steiner formula*:

THEOREM 2.1.1. *For each $K \in \mathcal{K}^d$, there exist unique finite Borel measures $C_0(K, \cdot)$, ..., $C_d(K, \cdot)$ on $\mathbb{R}^d$ such that*

$$\lambda_d(K_\varepsilon \cap \pi_K^{-1}(B)) = \sum_{k=0}^{d} \varepsilon^{d-k} \kappa_{d-k} C_k(K, B)$$

for each Borel set $B \subseteq \mathbb{R}^d$ and $\varepsilon > 0$.

Here π_K denotes the *metric projection* onto the convex set $K \in \mathcal{K}^d$, mapping a point $x \in \mathbb{R}^d$ to its (uniquely determined) nearest point in K. For fixed $K \in \mathcal{K}^d$ and $\varepsilon > 0$, the set $K_\varepsilon \cap \pi_K^{-1}(B)$ is regarded as the *local parallel set* of K with respect to the Borel set B. λ_d denotes the Lebesgue measure in $\mathbb{R}^d$ and κ_i the i-dimensional volume (Lebesgue measure) of the unit ball in $\mathbb{R}^i$.

Additivity and additive extension. Curvature measures of convex bodies have the property of being *additive*. If $K, L \in \mathcal{K}^d$ with $K \cup L \in \mathcal{K}^d$, then

$$C_k(K \cup L, B) = C_k(K, B) + C_k(L, B) - C_k(K \cap L, B) \tag{2.1.1}$$

for each Borel set $B \subseteq \mathbb{R}^d$. Repeated application of this relation leads to the so-called *inclusion-exclusion principle*: If $K^1, \ldots, K^m$ and $K = \bigcup_{i=1}^{m} K^i$ are in $\mathcal{K}^d$, then for all Borel sets $B \subseteq \mathbb{R}^d$

$$C_k(K, B) = \sum_{I \in N_m} (-1)^{\#I - 1} C_k\left(\bigcap_{i \in I} K^i, B\right). \tag{2.1.2}$$

Here N_m denotes the family of all nonempty subsets I of $\{1,\ldots,m\}$ so that the sum runs through all intersections of the K^i, and $\#I$ is the cardinality of the set I.

In case the set K on the left hand side of (2.1.2) is not convex, the measure $C_k(K,\,\cdot\,)$ is not defined by the local Steiner formula. But then the right hand side could be regarded as its definition. This leads to the additive extension of curvature measures to the convex ring. Groemer showed in [12] that such an extension is indeed possible and unique, i.e. the so defined measures $C_k(K,\,\cdot\,)$ do not depend on the chosen representation of $K \in \mathcal{R}^d$ by convex sets K^i.

Curvature measures of polyconvex sets are in general signed measures, in contrast to the convex case. However, for $k = d$ and $d-1$, $C_k(K,\,\cdot\,)$ is a non-negative measure for each $K \in \mathcal{R}^d$. By definition, curvature measures are additive and so the inclusion-exclusion principle (2.1.2) is valid for all $K, K^i \in \mathcal{R}^d$.

Further properties. The total mass $C_k(K) := C_k(K, \mathbb{R}^d)$ of the measure $C_k(K,\,\cdot\,)$ is called the *kth total curvature* of K. It is also known as the *kth intrinsic volume* of K. Below we collect some important properties of curvature measures.

PROPOSITION 2.1.2. *Let $K, L \in \mathcal{R}^d$ and $B \subseteq \mathbb{R}^d$ an arbitrary Borel set. For each $k \in \{0,\ldots,d\}$ we have:*

  (i) Motion invariance: *If g is a Euclidean motion, then $C_k(gK, gB) = C_k(K, B)$.*
 (ii) Homogeneity of degree k: *For $\lambda > 0$, $C_k(\lambda K, \lambda B) = \lambda^k C_k(K, B)$.*
(iii) Locality: *If $K \cap A = L \cap A$ for some open set $A \subseteq \mathbb{R}^d$, then $C_k(K, B) = C_k(L, B)$ for all Borel sets $B \subseteq A$.*
(iv) Continuity in the first argument: *If $K, K^1, K^2, \ldots \in \mathcal{K}^d$ with $K^i \to K$ as $i \to \infty$ (with respect to the Hausdorff metric) then the measures $C_k(K^i,\,\cdot\,)$ converge weakly to $C_k(K,\,\cdot\,)$, $C_k(K^i,\,\cdot\,) \xrightarrow{\mathrm{w}} C_k(K,\,\cdot\,)$. In particular, $C_k(K^i) \to C_k(K)$.*
 (v) Monotonicity of the total curvatures: *If $K, L \in \mathcal{K}^d$ and $K \subseteq L$, then $C_k(K) \leq C_k(L)$.*

Note that the properties (i)–(iii) hold for arbitrary polyconvex sets, whereas (iv) and (v) are restricted to convex bodies.

We add a few words on the geometric meaning of curvature measures. $C_d(K,\,\cdot\,)$ is nothing but the volume restricted to K, i.e. the d-dimensional Lebesgue measure $\lambda_d(K \cap \,\cdot\,)$, while $C_{d-1}(K,\,\cdot\,)$ is half the surface area of K, provided that K is the closure of its interior. Except for $k = d$, $C_k(K,\,\cdot\,)$ is concentrated on the boundary ∂K of K. For convex bodies, $C_k(K,\,\cdot\,)$ has an interpretation in terms of the k-dimensional volumes of the projections of K to k-dimensional linear subspaces. More precisely, $C_k(K)$ is the average of these volumes over all linear subspaces, known as *projection formula*. In general, the numbers $C_k(K, B)$ describe the different aspects of the 'curvedness' of ∂K inside the set B. Finally, by the *Gauss–Bonnet formula*, the 0th total curvature $C_0(K)$ coincides with the *Euler characteristic* of K, the topological invariant defined in algebraic topology. For convex bodies $K \in \mathcal{K}^d$, always $C_0(K) = 1$.

Variation measures. The *positive, negative* and *total variation measures* $C_k^+(K,\,\cdot\,)$, $C_k^-(K,\,\cdot\,)$ and $C_k^{\mathrm{var}}(K,\,\cdot\,)$ of the signed measure $C_k(K,\,\cdot\,)$ are defined by setting for each

Borel set $A \subseteq \mathbb{R}^d$,

$$C_k^+(K, A) := \sup_{B \subseteq A} C_k(K, B), \quad C_k^-(K, A) := - \inf_{B \subseteq A} C_k(K, B)$$

and

$$C_k^{\mathrm{var}}(K, A) := C_k^+(K, A) + C_k^-(K, A).$$

The variations are non-negative measures (since $C_k(K, \emptyset) = 0$, the supremum above is non-negative and the infimum non-positive) and satisfy the relation

$$C_k(K, \cdot) = C_k^+(K, \cdot) - C_k^-(K, \cdot), \tag{2.1.3}$$

called the *Jordan decomposition* of $C_k(K, \cdot)$. Positive and negative variation measures are useful for localizing 'positive' and 'negative' curvature or, more figuratively, to distinguish locally convexity from concavity. Some of the properties of curvature measures carry over to their variation measures. In particular, the motion invariance, the homogeneity of degree k and the locality of $C_k^+(K, \cdot)$, $C_k^-(K, \cdot)$ and $C_k^{\mathrm{var}}(K, \cdot)$ follow immediately from the corresponding properties of $C_k(K, \cdot)$ in Proposition 2.1.2.

PROPOSITION 2.1.3. *For $k \in \{0, \ldots, d\}$ and $K \in \mathcal{R}^d$ the measures $C_k^+(K, \cdot)$, $C_k^-(K, \cdot)$ and $C_k^{\mathrm{var}}(K, \cdot)$ are motion invariant, homogeneous of degree k and have the locality property.*

Finally, we comment on the behaviour with respect to similarities. Since each similarity $S : \mathbb{R}^d \to \mathbb{R}^d$ is the composition of a Euclidean motion and a homothety with some ratio $r > 0$, the kth curvature measures have the following *scaling property* with respect to S:

$$C_k(SK, SB) = r^k C_k(K, B) \tag{2.1.4}$$

for $K \in \mathcal{R}^d$ and any Borel set $B \in \mathbb{R}^d$. Note that this scaling property is also valid for the variation measures: For $k \in \{0, \ldots, d\}$ and $\bullet \in \{+, -, \mathrm{var}\}$,

$$C_k^\bullet(SK, SB) = r^k C_k^\bullet(K, B). \tag{2.1.5}$$

Further properties and some estimates for the variation measures are discussed in Chapter 3. Also compare the Appendix for some general remarks on signed measures.

2.2. The concept of fractal curvatures

Central assumption. For fractal sets F curvature measures are typically not defined in any classical sense. To investigate their geometry, we study the curvature measures of their ε-parallel sets F_ε. In general, curvature measures need not be defined for the parallel sets, either. Therefore, unless indicated otherwise, throughout the paper we assume that $F_\varepsilon \in \mathcal{R}^d$ for all $\varepsilon > 0$. The assumption on F of having polyconvex parallel sets ensures the existence of the $d + 1$ curvature measures $C_0(F_\varepsilon, \cdot), C_1(F_\varepsilon, \cdot), \ldots, C_d(F_\varepsilon, \cdot)$ for each parallel set F_ε.

REMARK 2.2.1. In order to introduce the concepts below it is essential to have curvature measures defined for the parallel sets. These need not necessarily be the curvature measures from the convex ring setting. The concepts discussed here (in particular Definitions 2.2.3, 2.2.7 and 2.2.8 below) can easily be generalized to larger classes of sets,

for instance to sets whose parallel sets are unions of sets with positive reach. We restrict ourselves to the polyconvex setting here, since we do not have any results yet in a more general setting and since this will keep the presentation simpler. Further investigations are necessary to relax this restriction.

One particular advantage of the polyconvex setting is the property of polyconvex sets to have polyconvex parallel sets. If $F_\varepsilon \in \mathcal{R}^d$ for some $\varepsilon > 0$, then $F_{\varepsilon+\delta} \in \mathcal{R}^d$ for all $\delta > 0$. Therefore, the existence of arbitrary small $\varepsilon > 0$ such that $F_\varepsilon \in \mathcal{R}^d$ is already sufficient to ensure that all parallel sets are polyconvex. Conversely, if there exist some $\varepsilon_0 > 0$ such that F_{ε_0} is not polyconvex, then the same holds for all smaller parallel sets of F.

PROPOSITION 2.2.2. *Either $F_\varepsilon \in \mathcal{R}^d$ for all $\varepsilon > 0$ or there exists $\varepsilon_0 > 0$ such that $F_\varepsilon \notin \mathcal{R}^d$ for all $0 < \varepsilon \leq \varepsilon_0$.*

For self-similar sets we even have the dichotomy that either all or none of their parallel sets are polyconvex (cf. Proposition 2.3.1).

Scaling exponents. We first concentrate on the total curvatures $C_k(F_\varepsilon) = C_k(F_\varepsilon, \mathbb{R}^d)$ of the parallel sets F_ε and study the expressions $\varepsilon^t C_k(F_\varepsilon)$ where the exponent $t \in \mathbb{R}$ has to be chosen appropriately for each k (and F). If $t \in \mathbb{R}$ is chosen too small, $\varepsilon^t C_k(F_\varepsilon)$ will tend to $\pm\infty$, while $\varepsilon^t C_k(F_\varepsilon)$ tends to 0 if t is too large. So the exponent should be at the borderline between these two extremes. Taking the infimum over all t for which $\varepsilon^t C_k(F_\varepsilon)$ approaches zero or the supremum over those t for which $\varepsilon^t |C_k(F_\varepsilon)|$ is unbounded seems a reasonable choice for the exponent, especially if both numbers happen to coincide. But taking into account the local character of curvature measures and the fact that they are, in general, signed measures, it turns out to be more appropriate to use the total variation in the definition of the scaling exponent. The total mass $C_k(F_\varepsilon)$ can be zero, while at the same time locally the measure $C_k(F_\varepsilon, \cdot)$ is very large. The positive curvature in some part can equal out the negative curvature in some other part of the set to give total mass zero. A fractal set where this phenomenon occurs is discussed in Example 2.4.6.

DEFINITION 2.2.3. Let $F \subseteq \mathbb{R}^d$ a compact set with polyconvex parallel sets, and let $k \in \{0, 1, \ldots, d\}$. The *kth curvature scaling exponent* of F is the number

$$s_k = s_k(F) := \inf\{t : \varepsilon^t C_k^{\mathrm{var}}(F_\varepsilon) \to 0 \text{ as } \varepsilon \to 0\}.$$

s_k is well defined at least in the sense of being an element of $\mathbb{R} \cup \{-\infty, +\infty\}$. If $\liminf_{\varepsilon \to 0} \varepsilon^{s_k} C_k^{\mathrm{var}}(F_\varepsilon) > 0$, then, clearly, s_k is the only interesting exponent for this expression, since, for all $t > s_k$, $\varepsilon^t C_k^{\mathrm{var}}(F_\varepsilon) \to 0$ as $\varepsilon \to 0$ and for all $t < s_k$, $\varepsilon^t C_k^{\mathrm{var}}(F_\varepsilon) \to +\infty$. Then s_k is equivalently characterized by the number

$$\underline{s}_k := \sup\{t : \varepsilon^t C_k^{\mathrm{var}}(F_\varepsilon) \to \infty \text{ as } \varepsilon \to 0\}.$$

In general, $\underline{s}_k$ need not coincide with s_k (but always $\underline{s}_k \leq s_k$) and one should then better speak of lower and upper scaling exponents, respectively. However, here we will only bother about s_k. Observe that, due to the motion invariance and homogeneity of $C_k^{\mathrm{var}}(F_\varepsilon)$ (cf. Proposition 2.1.3), also $s_k(F)$ is invariant under Euclidean motions and scaling.

PROPOSITION 2.2.4. *Let F be a compact set with $F_\varepsilon \in \mathcal{R}^d$ for $\varepsilon > 0$ and $k \in \{0, \ldots, d\}$. Then $s_k(gF) = s_k(F)$ for any Euclidean motion g and $s_k(\lambda F) = s_k(F)$ for each $\lambda > 0$.*

REMARK 2.2.5. (i) Since $C_d^{\mathrm{var}}(F_\varepsilon) = C_d(F_\varepsilon) = \lambda_d(F_\varepsilon)$, the number $d - s_d(F)$ corresponds to the *upper Minkowski dimension* (or *box dimension*) of F, which is defined more generally, namely for arbitrary (compact) sets.

(ii) In [22], Marta Llorente and the author introduced the *Euler exponent* $\sigma = \sigma(F)$ of F as the infimum $\inf\{t \geq 0 : \varepsilon^t |\chi(F_\varepsilon)| \to 0 \text{ as } \varepsilon \to 0\}$, where χ denotes the Euler characteristic. $\sigma(F)$ is defined for more general sets F than those discussed here. For sets F with $F_\varepsilon \in \mathcal{R}^d$ for all $\varepsilon > 0$, however, the Euler exponent is closely related to $s_0(F)$, since, by the Gauss–Bonnet formula, $C_0(F_\varepsilon) = \chi(F_\varepsilon)$. The main difference is that in the definition of $\sigma(F)$ we work with absolute values $|C_0(F_\varepsilon)| = |\chi(F_\varepsilon)|$, while for $s_0(F)$ we used the total variations $C_0^{\mathrm{var}}(F_\varepsilon)$. Often both exponents coincide, but sometimes they differ. This corresponds very well to the different geometric meaning of $\chi(F_\varepsilon)$ as a topological invariant and of $C_0(F_\varepsilon, \cdot)$ as a curvature measure. For the different interpretations of σ and s_0 see Example 2.4.6. Note that in general $\sigma(F) \leq s_0(F)$.

REMARK 2.2.6. By replacing $C_k^{\mathrm{var}}(F_\varepsilon)$ in the definition of s_k with $C_k^+(F_\varepsilon)$ and $C_k^-(F_\varepsilon)$, scaling exponents s_k^+ and s_k^- can be defined. Since $C_k^{\mathrm{var}}(F_\varepsilon) = C_k^+(F_\varepsilon) + C_k^-(F_\varepsilon)$, they satisfy the relation $s_k = \max\{s_k^+, s_k^-\}$. Hence one of these two exponents must always coincide with s_k, while the second one can be smaller. This does in fact happen as we will see later (cf. Remark 2.4.7).

Fractal curvatures. Having defined the scaling exponents for total curvatures, we can now ask about the existence of rescaled limits and, in case they fail to exist, of average limits.

DEFINITION 2.2.7. Let $k \in \{0, 1, \ldots, d\}$. If the limit

$$C_k^f(F) := \lim_{\varepsilon \to 0} \varepsilon^{s_k} C_k(F_\varepsilon)$$

exists, then it is called the *kth fractal (total) curvature* of the set F.

DEFINITION 2.2.8. Let $k \in \{0, 1, \ldots, d\}$. If the limit

$$\overline{C}_k^f(F) := \lim_{\delta \to 0} \frac{1}{|\ln \delta|} \int_\delta^1 \varepsilon^{s_k} C_k(F_\varepsilon) \, \frac{d\varepsilon}{\varepsilon}$$

exists, then it is called the *kth average fractal (total) curvature* of the set F.

In both definitions one could also work with upper and lower limits. However, here we concentrate on the existence of the (average) limits. Note that whenever the limit in Definition 2.2.7 exists, then the corresponding average limit exists as well and coincides with the limit. Moreover, fractal curvatures are motion invariant and homogeneous, provided they exist.

PROPOSITION 2.2.9. *Let $F \subset \mathbb{R}^d$ be a compact set with polyconvex parallel sets and $k \in \{0, 1, \ldots, d\}$. Provided the limit $C_k^f(F)$ exists, the limits $C_k^f(gF)$ and $C_k^f(\lambda F)$ exist as well and satisfy $C_k^f(gF) = C_k^f(F)$ and $C_k^f(\lambda F) = \lambda^{s_k + k} C_k^f(F)$, respectively.*

Note that these properties hold likewise for the average fractal curvatures. They are immediate consequences of the corresponding properties of the total curvatures (cf. Proposition 2.1.2(i), (ii)).

Finally, we point out that there is a certain consistency of fractal curvatures with the classical theory. If the total curvatures of a set F are defined and do not vanish, then no rescaling is necessary (i.e. the scaling exponents are zero) and the fractal curvatures coincide with the total curvatures.

PROPOSITION 2.2.10. *Let $F \in \mathcal{R}^d$ and $k \in \{0, \ldots, d\}$. If $C_k(F) \neq 0$, then $s_k(F) = 0$ and so $C_k^f(F)$ exists and coincides with $C_k(F)$.*

For convex sets F, the statement follows immediately from the continuity (cf. Proposition 2.1.2(iv)) and the positivity of the curvature measures. For a proof of the statement for polyconvex sets, it is convenient to use Lemmas 3.1.1 and 3.1.4 and so the proof is postponed to Chapter 3 (see p. 35). However, Proposition 2.2.10 will not be used in the sequel. Note that, in case $C_k(F) = 0$, the fractal curvature $C_k^f(F)$ may be different from zero and may provide additional information about the set F.

2.3. Fractal curvatures of self-similar sets

Self-similar sets. Let $S_i : \mathbb{R}^d \to \mathbb{R}^d$, $i = 1, \ldots, N$, be contracting similarities. Denote the contraction ratio of S_i by $r_i \in (0, 1)$ and set $r_{\max} := \max_{i=1,\ldots,N} r_i$ and $r_{\min} := \min_{i=1,\ldots,N} r_i$. It is a well known fact in fractal geometry (cf. Hutchinson [15]) that for such a system $\{S_1, \ldots, S_N\}$ of similarities there is a unique, non-empty, compact subset F of $\mathbb{R}^d$ such that $\mathbf{S}(F) = F$, where $\mathbf{S}$ is the set mapping defined by

$$\mathbf{S}(A) = \bigcup_{i=1}^{N} S_i A, \quad A \subseteq \mathbb{R}^d.$$

F is called the *self-similar set* generated by the system $\{S_1, \ldots, S_N\}$. Moreover, the unique solution s of $\sum_{i=1}^{N} r_i^s = 1$ is called the *similarity dimension* of F. The system $\{S_1, \ldots, S_N\}$ is said to satisfy the *open set condition* (OSC) if there exists an open, non-empty, bounded subset $O \subset \mathbb{R}^d$ such that $\bigcup_i S_i O \subseteq O$ and $S_i O \cap S_j O = \emptyset$ for all $i \neq j$. In [1], O was called a *feasible open set* of the S_i, or of F, a terminology that we adopt here. For convenience, we also say that F satisfies OSC, always having in mind a particular system of similarities generating F. If additionally $O \cap F \neq \emptyset$ for some feasible open set O of F, then $\{S_1, \ldots, S_N\}$ (or F) is said to satisfy the *strong open set condition* (SOSC). It was shown by Schief [29] that in $\mathbb{R}^d$, SOSC is equivalent to OSC, i.e. if F satisfies OSC, then there always exists some feasible open set O such that $O \cap F \neq \emptyset$. It is easily seen that $F \subseteq \overline{O}$ for each feasible open sets O of F.

Let $h > 0$. A finite set $\{y_1, \ldots, y_N\}$ of positive real numbers is called *h-arithmetic* if h is the largest number such that $y_i \in h\mathbb{Z}$ for $i = 1, \ldots, N$. If no such h exists, the set is called *non-arithmetic*. We attribute these properties to the system $\{S_1, \ldots, S_N\}$ or to F if the set $\{-\ln r_1, \ldots, -\ln r_N\}$ has them. In this sense, each set F (generated by some $\{S_1, \ldots, S_N\}$) is either h-arithmetic for some $h > 0$ or non-arithmetic.
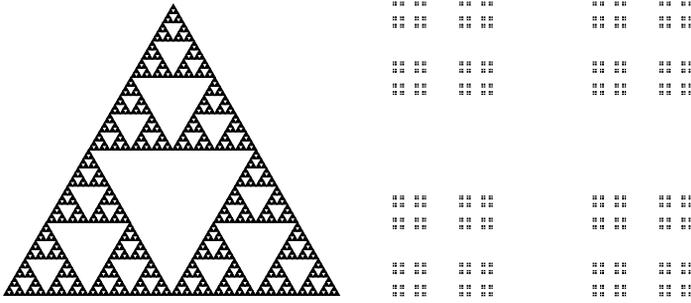
Fig. 1. The Sierpiński gasket has polyconvex parallel sets, while the Cantor set on the right does not.

Parallel sets of self-similar sets. Unfortunately, not all self-similar sets F have polyconvex parallel sets. But there is the dichotomy that either all or none of the parallel sets of F are polyconvex. This was already shown in [22].

PROPOSITION 2.3.1. *Let F a self-similar set. If $F_\varepsilon \in \mathcal{R}^d$ for some $\varepsilon > 0$, then $F_\varepsilon \in \mathcal{R}^d$ for all $\varepsilon > 0$.*

Therefore it suffices to check for an arbitrary parallel set F_ε, whether or not it is polyconvex, to know it for all parallel sets of F. For completeness, a simple proof of this statement is included in Section 5.1 (see p. 41). The *Sierpiński gasket* and *Sierpiński carpet* are self-similar sets with polyconvex parallel sets. Also *Rauzy's fractal* and the *Lévy curve* have this property as well as many Dragon tiles, while for instance the *von Koch curve* or the Cantor set in Figure 1 do not have polyconvex parallel sets. Indeed, it seems difficult to construct a Cantor set in $\mathbb{R}^d$, $d \geq 2$, with polyconvex parallel sets. (This might be possible if the similarities of the underlying IFS involve rotations.) In $\mathbb{R}$, on the other hand, all self-similar sets have polyconvex parallel sets, since for each $\varepsilon > 0$, $F_\varepsilon \subset \mathbb{R}$ consists of a finite number of intervals. In [22], some sufficient geometric conditions have been discussed which ensure that a self-similar set has polyconvex parallel sets. However, they are far from being necessary. The crucial point seems to be to understand the structure of the set $\partial \operatorname{conv} F \cap F$, where $\operatorname{conv} F$ is the convex hull of F. Giving conditions that are necessary *and* sufficient for a self-similar set to be included in the setting seems a challenging open problem. In the sequel assume that $F_\varepsilon \in \mathcal{R}^d$ for some (and thus all) $\varepsilon > 0$.

Scaling exponents of self-similar sets. The following result provides an upper bound for the kth scaling exponent $s_k(F)$ of a self-similar sets F.

THEOREM 2.3.2. *Let F be a self-similar set satisfying OSC and $F_\varepsilon \in \mathcal{R}^d$, and let $k \in \{0, \ldots, d\}$. The quantity $\varepsilon^{s-k} C_k^{\mathrm{var}}(F_\varepsilon)$ is uniformly bounded in $(0,1]$, i.e. there is a constant M such that for all $\varepsilon \in (0,1]$, $\varepsilon^{s-k} C_k^{\mathrm{var}}(F_\varepsilon) \leq M$.*

The proof is postponed to Section 5.5. The stated boundedness has the following immediate implications.

COROLLARY 2.3.3. $s_k \leq s - k$.

Proof. By Theorem 2.3.2, $\limsup_{\varepsilon \to 0} \varepsilon^{s-k} C_k^{\mathrm{var}}(F_\varepsilon) \leq M$ and thus for each $t > 0$,

$$\lim_{\varepsilon \to 0} \varepsilon^{s-k+t} C_k^{\mathrm{var}}(F_\varepsilon) \leq \lim_{\varepsilon \to 0} \varepsilon^t \limsup_{\varepsilon \to 0} \varepsilon^{s-k} C_k^{\mathrm{var}}(F_\varepsilon) = 0. \quad \blacksquare$$

COROLLARY 2.3.4. *The quantity $\varepsilon^{s-k}|C_k(F_\varepsilon)|$ is bounded in $(0,1]$ by M.*

Proof. Observe that $|C_k(F_\varepsilon)| \leq C_k^{\mathrm{var}}(F_\varepsilon)$ for each $\varepsilon > 0$. $\quad \blacksquare$

Corollary 2.3.3 provides the upper bound $s - k$ for the kth scaling exponent $s_k(F)$ and raises the question whether the equality $s_k = s - k$ holds. It will turn out that for most self-similar sets (and most k) indeed $s_k = s - k$. Unfortunately, this is not always the case as the following example shows.

EXAMPLE 2.3.5. The unit cube $Q = [0,1]^d \subset \mathbb{R}^d$ is a self-similar set generated for instance by a system of 2^d similarities each with contraction ratio $1/2$, which has similarity dimension $s = d$. For the curvature measures of its parallel sets no rescaling is necessary. Since Q is convex, its parallel sets Q_ε are and so the continuity implies that, for $k = 0, \ldots, d$, $C_k(Q_\varepsilon, \cdot) \to C_k(Q, \cdot)$ as $\varepsilon \to 0$. Therefore, $s_k(Q) = 0$, which, for $k < d$, is certainly different to $d - k$.

Now we investigate the limiting behaviour of the expression $\varepsilon^{s-k} C_k(F_\varepsilon)$ as $\varepsilon \to 0$. Since degenerate examples like the cubes exist, we cannot expect that this will always be the right expression to study in order to get the fractal curvatures.

Scaling functions. For a self-similar set F and $k \in \{0, \ldots, d\}$, define the *kth curvature scaling function* $R_k : (0, \infty) \to \mathbb{R}$ by

$$R_k(\varepsilon) = C_k(F_\varepsilon) - \sum_{i=1}^{N} \mathbf{1}_{(0,r_i]}(\varepsilon) C_k((S_i F)_\varepsilon). \tag{2.3.1}$$

The function R_k allows us to formulate a renewal equation so that the required information on the limiting behaviour of the expression $\varepsilon^{s-k} C_k(F_\varepsilon)$ can be obtained from the Renewal Theorem. Therefore it is essential to understand its properties. Geometrically the meaning of $R_k(\varepsilon)$ is the following: Since $F_\varepsilon = \bigcup_{i=1}^{N} (S_i F)_\varepsilon$, the inclusion-exclusion formula (2.1.2) implies that, for small ε (i.e. $\varepsilon \leq r_{\min}$), R_k describes the curvature of the intersections of the sets $(S_i F)_\varepsilon$:

$$R_k(\varepsilon) = \sum_{\#I \geq 2} (-1)^{\#I-1} C_k\Big(\bigcap_{i \in I} (S_i F)_\varepsilon\Big), \tag{2.3.2}$$

where the sum is taken over all subsets I of $\{1, \ldots, N\}$ with at least two elements. Hence the function R_k is related to the kth curvature of the set $\bigcup_{i \neq j} (S_i F)_\varepsilon \cap (S_j F)_\varepsilon$, the *overlap* of F_ε (cf. Fig. 3).

Existence of fractal curvatures. If the scaling function R_k is well behaved, which is the case for sets satisfying OSC, then usually the kth (average) fractal curvatures exist. The precise statement is derived from the following result, which characterizes the limiting behaviour of $\varepsilon^{s-k} C_k(F_\varepsilon)$.

THEOREM 2.3.6. *Let F be a self-similar set satisfying OSC and $F_\varepsilon \in \mathcal{R}^d$. Then for $k \in \{0, \ldots, d\}$ the following holds:*

(i) *The limit $\lim_{\delta \to 0} \frac{1}{|\ln \delta|} \int_\delta^1 \varepsilon^{s-k} C_k(F_\varepsilon) \frac{d\varepsilon}{\varepsilon}$ exists and equals the finite number*

$$X_k = \frac{1}{\eta} \int_0^1 \varepsilon^{s-k-1} R_k(\varepsilon)\, d\varepsilon, \tag{2.3.3}$$

where $\eta = -\sum_{i=1}^N r_i^s \ln r_i$.

(ii) *If F is non-arithmetic, then the limit $\lim_{\varepsilon \to 0} \varepsilon^{s-k} C_k(F_\varepsilon)$ exists and equals X_k.*

The proof is given in Section 5.2. The number X_k defined in (2.3.3) as an integral of the function R_k determines the average limit—and in the non-arithmetic case also the limit—of the expression $\varepsilon^{s-k} C_k(F_\varepsilon)$. If for F we had $s_k = s - k$, then, by definition, the average fractal curvature $\overline{C}_k^f(F)$ would coincide with X_k, and in the case of a non-arithmetic set F also with the fractal curvature $C_k^f(F)$. But this is not always true as we have seen in Example 2.3.5, where for some k, s_k was strictly smaller than $s - k$. Additional assumptions are required. A sufficient condition for $s_k = s - k$ is that X_k is different from 0.

COROLLARY 2.3.7. *If $X_k \neq 0$, then $s_k = s - k$. Consequently, $\overline{C}_k^f(F) = X_k$ and, in case F is non-arithmetic, also $C_k^f(F) = X_k$.*

Proof. The assumption $X_k \neq 0$ and Theorem 2.3.6(i) imply that

$$0 < |X_k| = \left| \lim_{\delta \to 0} \frac{1}{|\ln \delta|} \int_\delta^1 \varepsilon^{s-k} C_k(F_\varepsilon) \frac{d\varepsilon}{\varepsilon} \right| \leq \lim_{\delta \to 0} \frac{1}{|\ln \delta|} \int_\delta^1 \varepsilon^{s-k} |C_k(F_\varepsilon)| \frac{d\varepsilon}{\varepsilon}$$

$$\leq \limsup_{\varepsilon \to 0} \varepsilon^{s-k} |C_k(F_\varepsilon)| \leq \limsup_{\varepsilon \to 0} \varepsilon^{s-k} C_k^{\mathrm{var}}(F_\varepsilon)$$

and thus, for all $t > 0$,

$$\limsup_{\varepsilon \to 0} \varepsilon^{s-k-t} C_k^{\mathrm{var}}(F_\varepsilon) \geq |X_k| \lim_{\varepsilon \to 0} \varepsilon^{-t} = \infty.$$

Hence there is no $t > 0$ such that $\varepsilon^{s-k-t} C_k^{\mathrm{var}}(F_\varepsilon) \to 0$, implying $s_k = s - k$. ∎

Since the curvatures involved in (2.3.2) are usually much easier to determine than the curvatures of the whole parallel sets F_ε (compare the examples in Section 2.4 and, in particular, Remark 2.4.2), formula (2.3.3) provides an explicit way to calculate X_k and therefore (average) fractal curvatures, at least in case $X_k \neq 0$. For $k = d$ it can be shown that always $X_d > 0$ and thus $s_d = s - d$. This case is related to the Minkowski content and will be discussed separately below. So assume for the moment that $k \in \{0, \dots, d-1\}$. For those k it remains to clarify the situation when $X_k = 0$. First note that the condition $X_k \neq 0$ is not necessary for s_k to be $s - k$. For $X_k = 0$ both situations are possible: either $s_k < s - k$ or $s_k = s - k$. In Example 2.4.6 we will discuss a set of the latter type, while the cubes in Example 2.3.5 above are of the former type. Note that in the latter case, Theorem 2.3.6 provides the right values for the (average) kth fractal curvature, namely $\overline{C}_k^f(F) = X_k = 0$ and in the non-arithmetic case $C_k^f(F) = 0$ as well.

The following theorem provides a tool to detect sets of the latter type, i.e. with $s_k = s - k$ (with or without $X_k = 0$). Here it is necessary to work locally rather than just with the total curvatures. For $\varepsilon > 0$, define the *inner ε-parallel set* of a set A by

$$A_{-\varepsilon} := \{x \in A : d(x, A^c) > \varepsilon\} \tag{2.3.4}$$

or, equivalently, as the complement of the (outer) ε-parallel set of the complement of A, i.e. $A_{-\varepsilon} = ((A^c)_\varepsilon)^c$. Inner parallel sets only make sense for sets with non-empty interior, otherwise they are empty.

THEOREM 2.3.8. *Let F be a self-similar set satisfying OSC and $F_\varepsilon \in \mathcal{R}^d$, O some feasible open set of F, and $k \in \{0, \ldots, d\}$. Suppose there exist some constants $\varepsilon_0, \beta > 0$ and some Borel set $B \subset O_{-\varepsilon_0}$ such that*

$$C_k^{\mathrm{var}}(F_\varepsilon, B) \geq \beta \quad \text{for each } \varepsilon \in (r_{\min}\varepsilon_0, \varepsilon_0].$$

Then for all $\varepsilon < \varepsilon_0$,

$$\varepsilon^{s-k} C_k^{\mathrm{var}}(F_\varepsilon) \geq c, \quad \text{where} \quad c := \beta \varepsilon_0^{s-k} r_{\min}^s > 0.$$

The rough idea is that curvature in some advantageous location B in a large parallel set F_ε is exponentiated and spread by the self-similarity as ε tends to zero. A complete proof of this statement is provided in Section 5.5. An immediate consequence is that, under the hypotheses of Theorem 2.3.8, $s - k$ is a lower bound for s_k and thus

COROLLARY 2.3.9. $s_k = s - k$.

Proof. Theorem 2.3.8 implies that $\liminf_{\varepsilon \to 0} \varepsilon^{s-k} C_k^{\mathrm{var}}(F_\varepsilon) \geq c$ and thus for each $t > 0$ we have $\liminf_{\varepsilon \to 0} \varepsilon^{s-k-t} C_k^{\mathrm{var}}(F_\varepsilon) \geq c \lim_{\varepsilon \to 0} \varepsilon^{-t} = \infty$. Hence $s_k \geq s - k$. The reverse inequality was stated in Corollary 2.3.3. ∎

Theorem 2.3.8 can be seen as a counterpart to Theorem 2.3.2. While the latter provides an upper bound for the scaling exponent s_k which is in a sense universal, the former gives the corresponding lower bound, though only under additional assumptions. It is a useful supplement to Corollary 2.3.7 for the treatment of self-similar sets with $X_k = 0$. Its power is revealed in Example 2.4.6 below.

Minkowski content. For the case $k = d$ the above results hold in a more general setting, namely without the assumption of polyconvex parallel sets. For this case the results are known. For general $d > 1$, they are due to Dimitris Gatzouras [11]. We want to recall Gatzouras's results and discuss more carefully how they fit into our setting.

First recall that $C_d(F_\varepsilon, \cdot) = \lambda_d(F_\varepsilon \cap \cdot)$, whenever $C_d(F_\varepsilon, \cdot)$ is defined. Therefore it is straightforward to generalize the definitions of the dth scaling exponent and the dth (average) fractal curvature to arbitrary compact sets $F \subset \mathbb{R}^d$. The resulting quantities are

$$s_d = s_d(F) := \inf\{t : \varepsilon^t \lambda_d(F_\varepsilon) \to 0 \text{ as } \varepsilon \to 0\}, \quad M(F) := \lim_{\varepsilon \to 0} \varepsilon^{s_d} \lambda_d(F_\varepsilon)$$

and

$$\overline{M}(F) := \lim_{\delta \to 0} \frac{1}{|\ln \delta|} \int_\delta^1 \varepsilon^{s_d} \lambda_d(F_\varepsilon) \frac{d\varepsilon}{\varepsilon}.$$

Obviously, $d - s_d$ coincides with the (*upper*) *Minkowski dimension*, while $M(F)$ and $\overline{M}(F)$ are well known as *Minkowski content* and *average Minkowski content* of F, respectively, provided they are defined.

For self-similar sets F satisfying OSC, it is well known that the Minkowski dimension coincides with the similarity dimension s of F. Hence $s_d = s - d$, which answers completely the question for the scaling exponent for the volume of the parallel sets $\lambda_d(F_\varepsilon)$. However, for a long time it had been an open problem whether self-similar sets are *Minkowski*

measurable, i.e. whether their Minkowski content exists, although this question aroused considerable interest. After partial answers for sets in $\mathbb{R}$ by Lapidus [19] and Falconer [6], Gatzouras gave the following classification of Minkowski measurability of self-similar sets in $\mathbb{R}^d$ (cf. [11, Theorems 2.3 and 2.4]).

THEOREM 2.3.10 (Gatzouras's theorem). *Let F be a self-similar set satisfying OSC. The average Minkowski content of F always exists and coincides with the strictly positive value*

$$X_d = \frac{1}{\eta} \int_0^1 \varepsilon^{s-d-1} R_d(\varepsilon) \, d\varepsilon.$$

If F is non-arithmetic, then also the Minkowski content $M(F)$ of F exists and equals X_d.

Here the function R_d is the dth scaling function generalized in the obvious way:

$$R_d(\varepsilon) = \lambda_d(F_\varepsilon) - \sum_{i=1}^N \mathbf{1}_{(0,r_i]}(\varepsilon) \lambda_d((S_i F)_\varepsilon), \tag{2.3.5}$$

and η is the same constant as in Theorem 2.3.6. This statement includes all the results discussed before for the case $k = d$ and extends them to arbitrary self-similar sets satisfying OSC. Note, in particular, that the case $X_k = 0$, which causes a lot of trouble in the general discussion, does not occur for $k = d$, since it is possible to show explicitly that always $X_d > 0$.

With only little extra work we derive in Section 5.6 a proof of Gatzouras's theorem from the proof of Theorem 2.3.6, which differs in some parts from the one provided by Gatzouras in [11] and shows more clearly the close connection to curvature measures. Moreover, this proof prepares a strengthening of Gatzouras's theorem which is presented in Theorem 2.5.4 below. It characterizes the limiting behaviour of the parallel volume not only globally but also locally.

Fractal Euler numbers. In [22], the *fractal Euler number* of a set F was introduced as

$$\chi_f(F) := \lim_{\varepsilon \to 0} \left(\frac{\varepsilon}{b}\right)^\sigma \chi(F_\varepsilon),$$

where b is the diameter of F and σ the Euler exponent (cf. Remark 2.2.5). Similarly, the *average fractal Euler number* of F was defined as

$$\overline{\chi}_f(F) := \lim_{\delta \to 0} \frac{1}{|\ln \delta|} \int_\delta^1 \left(\frac{\varepsilon}{b}\right)^\sigma \chi(F_\varepsilon) \frac{d\varepsilon}{\varepsilon}.$$

The normalizing factor $b^{-\sigma}$ was inserted to ensure the scaling invariance of these numbers. It does not affect the limiting behaviour.

We want to work out the relation between $C_0^f(F)$ and $\chi_f(F)$ more clearly and compare the results obtained here to those in [22]. Assume that the parallel sets of F are polyconvex. Then always $\sigma(F) \leq s_0(F)$. If equality holds, then also the numbers $C_0^f(F)$ and $\chi_f(F)$ coincide up to the factor $b^{-\sigma}$, provided both are defined. The same is true for their averaged counterparts: $\overline{\chi}_f(F) = b^{-\sigma} \overline{C}_0^f(F)$. Therefore, the results obtained for the 0th fractal curvatures of self-similar sets can be carried over to the fractal Euler numbers of these sets. From Theorem 2.3.6 and Corollary 2.3.7 we immediately deduce the following.

COROLLARY 2.3.11. *Let F be a self-similar set satisfying OSC and $F_\varepsilon \in \mathcal{R}^d$. If $X_0 \neq 0$ then $\sigma = s$. Moreover, $\overline{\chi}_f(F)$ exists and equals $b^{-s}X_0$. If F is non-arithmetic, then also $\chi_f(F) = b^{-s}X_0$.*

For the above class of sets this statement is a significant improvement of the results obtained in [22]. In Corollary 2.3.11, we have no additional assumptions on F apart from the OSC. In fact, it can be shown that the additional conditions in Theorem 2.1 in [22] are always satisfied in the situation of Corollary 2.3.11. It should be noted that on the contrary the results in [22] apply to a larger class of self-similar sets. We do not require their parallel sets to be polyconvex, since the Euler characteristic is defined more generally.

REMARK 2.3.12. For sets F in $\mathbb{R}$ exactly two fractal curvatures are available, $C_1^f(F)$ and $C_0^f(F)$. Since in $\mathbb{R}$, for each F and $\varepsilon > 0$, F_ε is a finite union of closed intervals, $C_1^f(F)$ is defined if and only if the Minkowski content $M(F)$ exists, and both numbers coincide. Similarly, since $C_0(F_\varepsilon, \cdot)$ is in this case a positive measure, we always have $s_0 = \sigma$, and $C_0^f(F) = b^{-\sigma}\chi_f(F)$ whenever one of these numbers exists. Corresponding relations hold for the average counterparts.

Fractal Euler numbers in $\mathbb{R}$ have been discussed in detail in [22]. Also the close relation to the gap counting function was outlined there. The limiting behaviour of the gap counting function and the Minkowski content have been studied extensively for sets in $\mathbb{R}$, not only for self-similar sets. We refer in particular to the book by Lapidus and van Frankenhuysen [20]. In this book some kind of Steiner formula was obtained for general sets F in $\mathbb{R}$, where Minkowski content and gap counting limit, i.e. in fact $C_0^f(F)$ and $C_1^f(F)$, appear as coefficients among others. This suggests that there are interesting relations between fractal curvatures and the theory of complex dimensions. These connections are still waiting for being studied in detail.

Open questions and conjectures. In all the examples considered, in particular in all the examples presented here and in the next section, one can observe that always either $s_k = s - k$ or $s_k = 0$. It is a very interesting question whether other values are possible for s_k. We conjecture that this is not the case. For $k = d$ the situation is clear. We always have $s_d = s - d$. Hence $s_d = 0$ only occurs when $s = d$, i.e. when the set considered is full-dimensional like the cubes in Example 2.3.5. For $k < d$, we conjecture that the case $s_k = 0$ occurs if and only if the set is a 'classical' set, i.e. in the class of self-similar sets F satisfying OSC and $F_\varepsilon \in \mathcal{R}^d$, exactly those have scaling exponent $s_k = 0$ which are themselves polyconvex. All the other sets in this class have scaling exponents $s_k = s - k$, and should be regarded as 'true' fractals. Note that this classification would be independent of $k \in \{0, \ldots, d-1\}$.

2.4. Examples. We illustrate the results of the previous section with some examples and determine the (average) fractal curvatures for some self-similar sets F in $\mathbb{R}^2$. Roughly speaking, the three functionals available in $\mathbb{R}^2$, $\overline{C}_2^f(F)$, $\overline{C}_1^f(F)$ and $\overline{C}_0^f(F)$, can be regarded as *fractal volume*, *fractal boundary length* and *fractal curvature*, respectively.

EXAMPLE 2.4.1 (Sierpiński gasket). Let F be the Sierpiński gasket generated as usual by three similarities S_1, S_2 and S_3 with contraction ratios $1/2$ such that the diame-
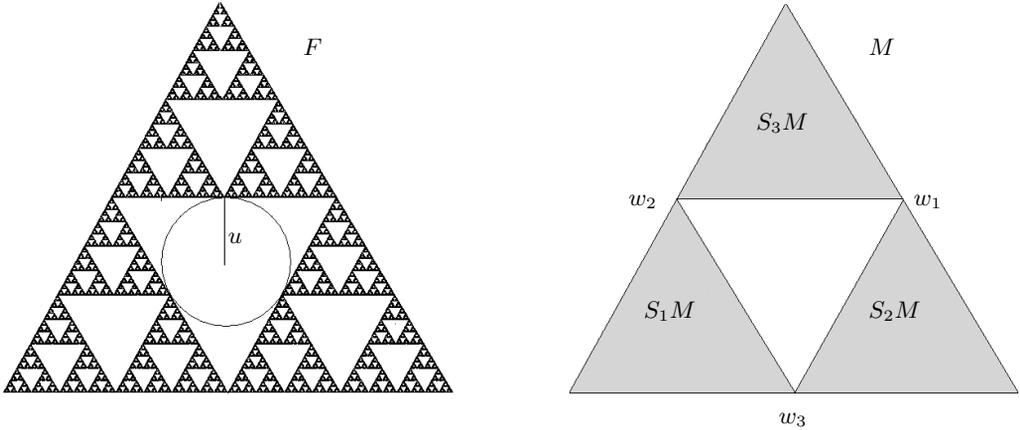
Fig. 2. The Sierpiński gasket F and a picture showing how the three similarities S_1, S_2 and S_3 generating F act on its convex hull M.

ter of F is 1 and the convex hull M of F is an equilateral triangle (cf. Figure 2). F satisfies the OSC and has polyconvex parallel sets F_ε. Since F is $\ln 2$-arithmetic, the above results ensure only the existence of average fractal curvatures. (It is not difficult to see that fractal curvatures do not exist for the Sierpiński gasket: Choose two appropriate null sequences, for instance $\varepsilon_n = u2^{-n}$ and $\delta_n = \frac{3}{2}\varepsilon_n$. Then the sequences $\varepsilon_n^{s-k} C_k(F_{\varepsilon_n})$ and $\delta_n^{s-k} C_k(F_{\delta_n})$ converge to different values as $n \to \infty$. Hence the limit $\lim_{\varepsilon \to 0} \varepsilon^{s-k} C_k(F_\varepsilon)$ cannot exist. For instance, since $C_0(F_{\varepsilon_n}) = C_0(F_{\delta_n}) = \frac{1}{2}(3 - 3^n)$ for each n, we have $\lim_{n \to \infty} \varepsilon_n^{s-k} C_0(F_{\varepsilon_n}) = -u^s/2$ and $\lim_{n \to \infty} \delta_n^{s-k} C_0(F_{\delta_n}) = -(3/2)^s u^s/2$, respectively.)

We determine the scaling functions R_k. It turns out that they have at most two discontinuities, namely at $1/2$, where the indicator functions in R_k switch from 0 to 1, and at $u = \sqrt{3}/12$, the radius of the incircle of the middle triangle (cf. Figure 2), where the intersection structure of the sets $(S_i F)_\varepsilon$ changes. For the case $k = 0$, recall that $C_0(K)$ is the Euler characteristic of the set K, i.e. in $\mathbb{R}^2$ the number of connected components minus the number of 'holes' of K. From Figure 3 it is easily seen that

$$R_0(\varepsilon) = \begin{cases} C_0(F_\varepsilon) & = 1 & \text{for } 1/2 \le \varepsilon, \\ C_0(F_\varepsilon) - \sum_i C_0((S_i F)_\varepsilon) & = -2 & \text{for } u \le \varepsilon < 1/2, \\ -\sum_{i \ne j} C_0((S_i F)_\varepsilon \cap (S_j F)_\varepsilon) & = -3 & \text{for } \varepsilon < u. \end{cases} \quad (2.4.1)$$

Since here $s = \ln 3 / \ln 2$ and $\eta = \ln 2$, integration according to formula (2.3.3) yields

$$X_0 = -\frac{u^s}{\eta s} = -\frac{u^s}{\ln 3} \approx -0.042.$$

Being different from zero, X_0 is, by Corollary 2.3.7, the value of the 0th average fractal curvature $\overline{C}_0^f(F)$ of F.

For the case $k = 1$, we use the interpretation that $C_1(K)$ is half the boundary length of K. Looking at Figure 3, it is easily seen that
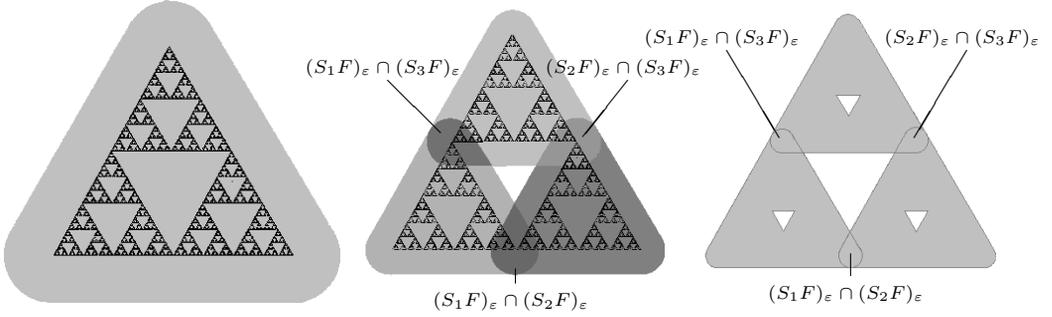
Fig. 3. Some ε-parallel sets of the Sierpiński gasket F for $\varepsilon \geq u$ (left) and $\varepsilon < u$ (middle and right). For $\varepsilon < u$, the sets $(S_iF)_\varepsilon \cap (S_jF)_\varepsilon$, $i \neq j$, remain convex and mutually disjoint as $\varepsilon \to 0$.

$$R_1(\varepsilon) = \begin{cases} C_1(F_\varepsilon) & = 3/2 + \pi\varepsilon & \text{for } 1/2 \leq \varepsilon, \\ C_1(F_\varepsilon) - \sum_i C_1((S_iF)_\varepsilon) & = -3/4 - 2\pi\varepsilon & \text{for } u \leq \varepsilon < 1/2, \\ -\sum_{i \neq j} C_1((S_iF)_\varepsilon \cap (S_jF)_\varepsilon) & = -(2\pi + 3\sqrt{3})\varepsilon & \text{for } \varepsilon < u. \end{cases} \quad (2.4.2)$$

Therefore, by formula (2.3.3),

$$X_1 = \frac{1}{\eta}\left(\frac{3}{4}\frac{u^{s-1}}{s-1} - 3\sqrt{3}\frac{u^s}{s}\right) = \frac{3}{4\ln\frac{3}{2}}u^{s-1} - \frac{3\sqrt{3}}{\ln 3}u^s \approx 0.38,$$

which is obviously non-zero and thus $\overline{C}_1^f(F) = X_1$. Similarly for $k = 2$,

$$R_2(\varepsilon) = \begin{cases} C_2(F_\varepsilon) & = \sqrt{3}/4 + 3\varepsilon + \pi\varepsilon^2 & \text{for } 1/2 \leq \varepsilon, \\ C_2(F_\varepsilon) - \sum_i C_2((S_iF)_\varepsilon) & = \sqrt{3}/16 - 3\varepsilon/2 - 2\pi\varepsilon^2 & \text{for } u \leq \varepsilon < 1/2, \\ -\sum_{i \neq j} C_2((S_iF)_\varepsilon \cap (S_jF)_\varepsilon) & = -(2\pi + 3\sqrt{3})\varepsilon^2 & \text{for } \varepsilon < u, \end{cases}$$
$$(2.4.3)$$

and so the average Minkowski content of F (which always exists) is

$$\overline{C}_2^f(F) = X_2 = -\frac{\sqrt{3}}{16\ln\frac{3}{4}}u^{s-2} + \frac{3}{2\ln\frac{3}{2}}u^{s-1} - \frac{3\sqrt{3}}{\ln 3}u^s \approx 1.81.$$

REMARK 2.4.2. Observe that the functions R_k are much easier to determine and handle than the corresponding total curvatures $C_k(F_\varepsilon)$. R_k is piecewise polynomial (with three pieces) in $(0, \infty)$, while in the same interval, the expression for $C_k(F_\varepsilon)$ changes infinitely many times, namely at each point $\varepsilon_n = 2^{-n}u$ for $n = 0, 1, 2, \ldots$. Therefore, it is not easy to compute the fractal curvatures directly by taking the limit (or average limit), while formula (2.3.3) gives them by a simple integration. The same is true for all the examples below. Geometrically, this is explained by the observation that the *overlap* of F_ε, i.e. the set $\bigcup_{i \neq j}(S_iF)_\varepsilon \cap (S_jF)_\varepsilon$, has a much simpler geometric structure than the whole parallel set F_ε. While F_ε becomes more and more complicated as $\varepsilon \to 0$ (and converges to the fractal F in the limit), typically the overlap does not change its shape very much (and converges to a set which typically is not a fractal). In the above example of the Sierpiński gasket, the overlap consists of three convex sets ('drops') for all sufficiently small ε (cf. Figure 3).

A more explicit formula for X_k. Before we continue with further examples, we provide a more convenient formula for X_k, which reduces the amount of calculation to be carried out. As in the previous example of the Sierpiński gasket the scaling functions R_k are often, though not always, piecewise polynomials of degree $\leq k$.

LEMMA 2.4.3. *Let F be a self-similar set with similarity dimension s and polyconvex parallel sets. Let $k \in \{0, \ldots, d\}$ and, in case s is an integer, assume $k < s$. Suppose there are numbers $J \in \mathbb{N}$ and $0 = u_0 < u_1 < \cdots < u_J < u_{J+1} = 1$ such that the function R_k has a polynomial expansion of degree at most k in the interval (u_j, u_{j+1}) for each $j = 0, \ldots, J$, i.e. there are coefficients $a_{j,l} \in \mathbb{R}$ such that*

$$R_k(\varepsilon) = \sum_{l=0}^{k} a_{j,l} \varepsilon^l \quad for \ \varepsilon \in (u_j, u_{j+1}).$$

Then, setting $a_{J+1,l} := 0$ for each $l = 0, \ldots, k$, the following holds:

$$X_k = \frac{1}{\eta} \sum_{l=0}^{k} \frac{1}{s-k+l} \sum_{j=0}^{J} (a_{j,l} - a_{j+1,l}) u_{j+1}^{s-k+l}. \tag{2.4.4}$$

Proof. For a proof of (2.4.4), write the integral in (2.3.3) as a sum of integrals over (u_{j+1}, u_j) and then plug in the polynomial expansions of R_k:

$$\eta X_k = \sum_{j=0}^{J} \int_{u_j}^{u_{j+1}} \varepsilon^{s-k-1} \sum_{l=0}^{k} a_{j,l} \varepsilon^l \, d\varepsilon = \sum_{j=0}^{J} \sum_{l=0}^{k} a_{k,j,l} \int_{u_j}^{u_{j+1}} \varepsilon^{s-k-1+l} \, d\varepsilon.$$

Integration yields $\frac{1}{s-k+l}(u_{j+1}^{s-k+l} - u_j^{s-k+l})$ for the term with indices j and l (provided that $s - k + l \neq 0$, which is the case since we assumed s is non-integer or $k < s$) and so, by exchanging the order of summation,

$$\eta X_k = \sum_{l=0}^{k} \frac{1}{s-k+l} \Big(\sum_{j=0}^{J} a_{j,l} u_{j+1}^{s-k+l} - \sum_{j=0}^{J} a_{j,l} u_j^{s-k+l} \Big).$$

By rearranging the index j in the second sum and summarizing the terms with equal j, formula (2.4.4) easily follows. ∎

EXAMPLE 2.4.4 (Sierpiński carpet Q). The Sierpiński carpet Q is the well known self-similar set in Figure 4 generated by 8 similarities S_i each mapping the unit square $[0, 1]^2$ with contraction ratio $1/3$ to one of the smaller outer squares. Q has similarity dimension $s = \ln 8 / \ln 3$ and is $\ln 3$-arithmetic. Thus we can only expect the average fractal curvatures to exist. Indeed, all three of them exist and are different from zero as the computations below show.

In $(0, 1)$ the scaling functions have discontinuities at $1/3$, the switching point of the indicator functions, and $1/6$, the inradius of the middle cut-out square. Between these points, the intersection structure of the sets $Q_\varepsilon^i := (S_i Q)_\varepsilon$ and the symmetries suggest computing R_k as follows:

$$R_k(\varepsilon) = \begin{cases} C_k(Q_\varepsilon) & \text{for } 1/3 \leq \varepsilon, \\ C_k(Q_\varepsilon) - \sum_i C_k(Q_\varepsilon^i) & \text{for } 1/6 \leq \varepsilon < 1/3, \\ -\sum_{i \neq j} C_k(Q_\varepsilon^i \cap Q_\varepsilon^j) + \sum_{i \neq j \neq l} C_k(Q_\varepsilon^i \cap Q_\varepsilon^j \cap Q_\varepsilon^l) & \text{for } \varepsilon < 1/6. \end{cases}$$
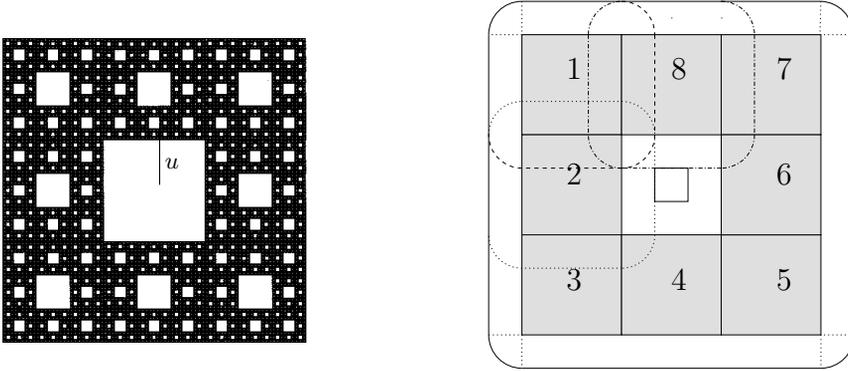
Fig. 4. Sierpiński carpet Q and a parallel set of Q for $\varepsilon = 1/9$

By symmetry, R_k simplifies for $\varepsilon < 1/6$ to

$$R_k(\varepsilon) = -8C_k(Q_\varepsilon^1 \cap Q_\varepsilon^2) - 4C_k(Q_\varepsilon^8 \cap Q_\varepsilon^2) + 4C_k(Q_\varepsilon^8 \cap Q_\varepsilon^1 \cap Q_\varepsilon^2).$$

Now for each scaling function the polynomials for each interval are easily determined (cf. Figure 4) and we obtain

$$R_0(\varepsilon) = \begin{cases} 1, \\ -7, \\ -8, \end{cases} \quad R_1(\varepsilon) = \begin{cases} 2 + \pi\varepsilon, \\ -10/3 - 7\pi\varepsilon, \\ -8/3 - (7\pi + 4)\varepsilon, \end{cases} \quad R_2(\varepsilon) = \begin{cases} 1 + 4\varepsilon + \pi\varepsilon^2, \\ 1/9 - 20\varepsilon/3 - 7\pi\varepsilon^2, \\ -16\varepsilon/3 - (7\pi + 4)\varepsilon^2, \end{cases}$$

for

$$\begin{cases} 1/3 \le \varepsilon \\ 1/6 \le \varepsilon < 1/3 \qquad \text{respectively.} \\ \varepsilon < 1/6 \end{cases}$$

Using formula (2.4.4) we can now compute X_0, X_1 and X_2 (note that $\eta = \ln 3$):

$$X_0 = -\frac{1}{\ln 3}\frac{1}{s}\left(\frac{1}{6}\right)^s \approx -0.0162,$$

$$X_1 = \frac{4}{\ln 3}\left(\frac{1}{s-1} - \frac{1}{s}\right)\left(\frac{1}{6}\right)^s \approx 0.0725,$$

$$X_2 = \frac{4}{\ln 3}\left(\frac{1}{s-2} + \frac{2}{s-1} - \frac{1}{s}\right)\left(\frac{1}{6}\right)^s \approx 1.352.$$

In the following example we modify the Sierpiński carpet to obtain a self-similar set with the same dimension but a different geometric and topological structure.

EXAMPLE 2.4.5 (Modified carpet). The self-similar set M in Figure 5 is generated by eight similarities S_i each mapping the unit square with contraction ratio $1/3$ to one of the eight small squares leaving out the upper middle one. This time some of the similarities include some rotation by $\pm\pi/2$ or π as indicated. Like the Sierpiński carpet, M has similarity dimension $s = \ln 8/\ln 3$ and is $\ln 3$-arithmetic. We compute the average fractal curvatures of M and compare them to those of the Sierpiński carpet.
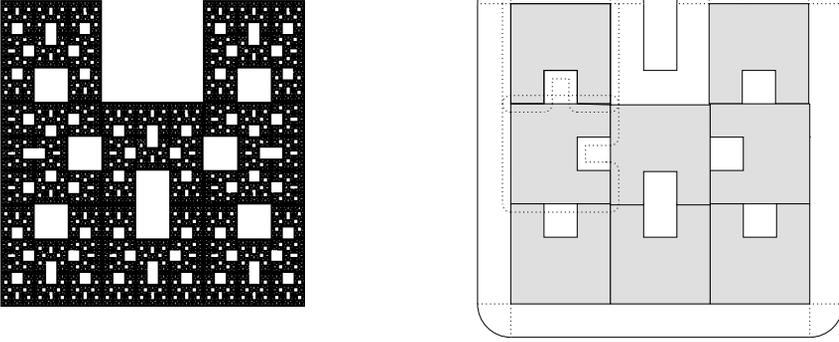
Fig. 5. Modified Sierpiński carpet M and some parallel sets

In $(0,1)$ the scaling function R_0 has a discontinuity at $1/3$ and one at $1/18$, since for $\varepsilon < 1/18$ the first holes appear in M. With similar arguments as for the Sierpiński carpet we obtain

$$R_0(\varepsilon) = \begin{cases} 1 & \text{for } 1/3 \le \varepsilon, \\ -7 & \text{for } 1/18 \le \varepsilon < 1/3, \\ -14 & \text{for } \varepsilon < 1/18, \end{cases}$$

and so integration according to formula (2.3.3) yields

$$\overline{C}_0^f(M) = X_0 = -\frac{1}{s\ln 3}\,\frac{7}{8}\left(\frac{1}{6}\right)^s \approx -0.014.$$

For $k = 1$ and $k = 2$, the situation is more complicated. First observe that

$$C_1(F_\varepsilon) = \begin{cases} \dfrac{11}{6} + \left(\pi + \arcsin\dfrac{1}{6\varepsilon}\right)\varepsilon & \text{for } 1/6 \le \varepsilon, \\[2ex] \dfrac{7}{3} + \left(\dfrac{3}{2}\pi - 2\right)\varepsilon & \text{for } 1/18 \le \varepsilon < 1/6, \end{cases}$$

and similarly for $i = 1, \ldots, 8$,

$$C_1((S_i M)_\varepsilon) = \frac{11}{18} + \left(\pi + \arcsin\frac{1}{18\varepsilon}\right)\varepsilon \quad \text{for } 1/18 \le \varepsilon.$$

From these two equations $R_1(\varepsilon)$ can be determined in the interval $[1/18, 1)$ by means of the relation

$$R_1(\varepsilon) = C_1(F_\varepsilon) - 8\,C_1((S_i M)_\varepsilon)\,\mathbf{1}_{(0,1/3]}(\varepsilon).$$

Obviously, this time R_1 is not piecewise a polynomial as in the previous examples. For $\varepsilon < 1/18$, we derive R_1 from the intersections of the $(S_i M)_\varepsilon$ and obtain

$$R_1(\varepsilon) = -\frac{20}{9} - 7\left(\frac{3}{2}\pi + 2\right)\varepsilon.$$

Integrating R_1 according to (2.3.3) yields

$$\overline{C}_1^f(M) = X_1 \approx 0.0720.$$

Similarly, for $k = 2$ we determine the area of F_ε:

$$C_2(F_\varepsilon) = \begin{cases} 1 + \dfrac{11}{3}\varepsilon + \left(\pi + \arcsin \dfrac{1}{6\varepsilon}\right)\varepsilon^2 + \dfrac{1}{6}\sqrt{\varepsilon^2 - \left(\dfrac{1}{6}\right)^2} & \text{for } 1/6 \le \varepsilon, \\[3mm] \dfrac{8}{9} + \dfrac{14}{3}\varepsilon + \left(\dfrac{3}{2}\pi - 2\right)\varepsilon^2 & \text{for } 1/18 \le \varepsilon < 1/6, \end{cases}$$

and for $i = 1, \ldots, 8$,

$$C_2((S_i M)_\varepsilon) = \dfrac{1}{9} + \dfrac{11}{9}\varepsilon + \left(\pi + \arcsin \dfrac{1}{18\varepsilon}\right)\varepsilon^2 + \dfrac{1}{18}\sqrt{\varepsilon^2 - \left(\dfrac{1}{18}\right)^2} \quad \text{for } 1/18 \le \varepsilon.$$

Now $R_2(\varepsilon)$ can be derived for $\varepsilon \in [1/18, 1)$ using that

$$R_2(\varepsilon) = C_2(F_\varepsilon) - 8\, C_2((S_i M)_\varepsilon) \, \mathbf{1}_{(0,1/3]}(\varepsilon).$$

For $\varepsilon < 1/18$, we look again at the intersections of the $(S_i M)_\varepsilon$ and obtain

$$R_2(\varepsilon) = -\dfrac{40}{9}\varepsilon - 7\left(\dfrac{3}{2}\pi + 2\right)\varepsilon^2.$$

Integrating R_2 according to (2.3.3) yields

$$\overline{C}_2^f(M) = X_2 \approx 1.3439.$$

Comparison of the carpets. We summarize the approximate values determined above for the fractal curvatures of the two carpets:

	C_0^f	C_1^f	C_2^f
Sierpiński carpet	−0.016	0.0725	1.352
Modified carpet	−0.014	0.0720	1.344

The corresponding fractal curvatures are different. Hence they can be used to distinguish both sets. On the other hand, the values are rather close to each other, which corresponds to the impression that geometrically the sets are not very different. More investigations are necessary to understand whether fractal curvatures are useful characteristics for the distinction and classification of fractal sets, and whether they have some more explicit geometric interpretation. In particular, it would be interesting to see whether sets with very similar geometric structure also have fractal curvatures which are very close to each other, i.e. whether there is some kind of continuity.

In contrast to the cube Q in Example 2.3.5 it can happen that for a self-similar set, X_k equals zero but nevertheless $s_k = s - k$. The following example is a set for which $X_0 = 0$ but $s_0 = s$. It also clarifies the difference between s_0 and the Euler exponent σ, defined in [22].

EXAMPLE 2.4.6 (Sierpiński tree). Let the set F be generated by three similarities S_1, S_2 and S_3. S_1 has contraction ratio $4/5$ and shrinks an equilateral triangle of diameter 1 towards one of its corners, while the other two similarities S_2 and S_3 map the triangle with ratio $1/5$ to the remaining two corners, including a rotation by $2\pi/3$ and $-2\pi/3$, respectively (cf. Figure 6). F has similarity dimension s given by $5^s - 4^s = 2$ and is
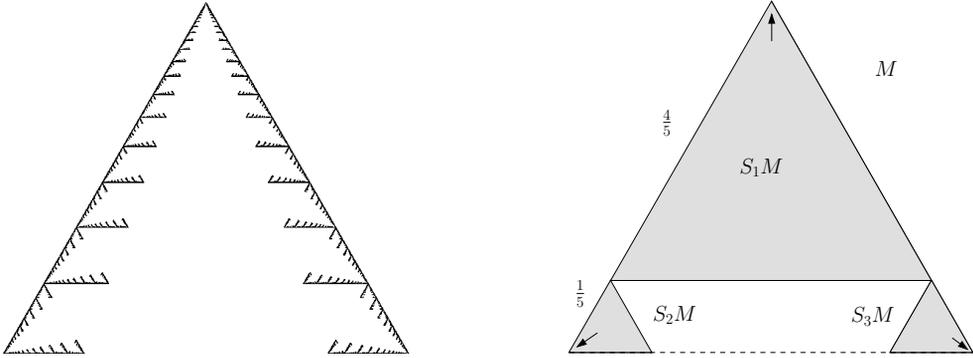
Fig. 6. The Sierpiński tree F and how it is generated. The right picture indicates how the similarities generating F act on its convex hull M. The arrows indicate to which points the upper corner of M is mapped.
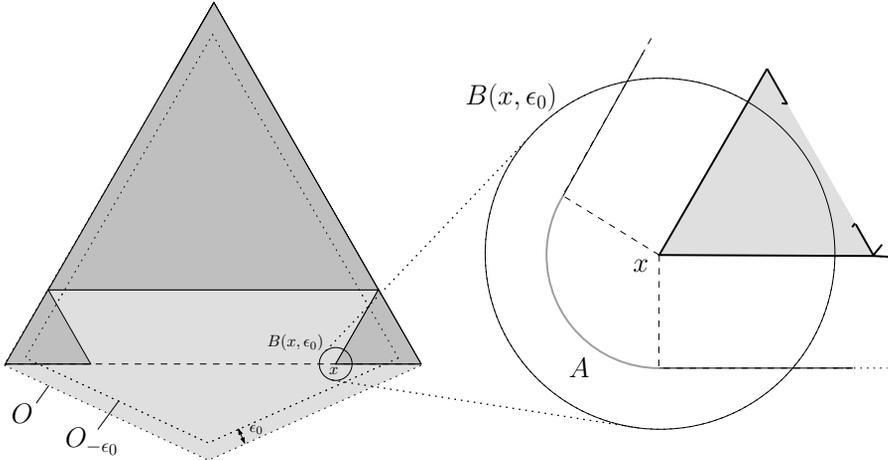
Fig. 7. Feasible open set O of the Sierpiński tree F and some inner parallel set $O_{-\varepsilon_0}$. The enlarged part on the right shows some detail of the ε-parallel set of F near x for some $\varepsilon < \varepsilon_0$. The arc $A \subset \partial F_\varepsilon$ contributes $1/3$ to the 0th curvature of F_ε.

non-arithmetic, since $(\ln 5 - \ln 4)/\ln 5$ is not rational. Therefore, this time we can try to determine the fractal curvatures rather than just the averaged counterparts. We only compute $C_0^f(F)$, for which we first determine the scaling function R_0. ($C_1^f(F)$ and $C_2^f(F)$ can be determined explicitly as well, but their computation is omitted since it would not provide any further insights.)

The only discontinuity points of $R_0(\varepsilon)$ in $(0,1)$ are $4/5$ and $1/5$ and so

$$R_0(\varepsilon) = \begin{cases} C_0(F_\varepsilon) & = 1 & \text{for } 4/5 \le \varepsilon, \\ C_0(F_\varepsilon) - \sum_i C_0((S_i F)_\varepsilon) & = 0 & \text{for } 1/5 \le \varepsilon < 4/5, \\ -C_0((S_1 F)_\varepsilon \cap (S_2 F)_\varepsilon) - C_0((S_1 F)_\varepsilon \cap (S_3 F)_\varepsilon) & = -2 & \text{for } \varepsilon < 1/5. \end{cases}$$

By formula (2.4.4),

$$X_0 = \frac{1}{\eta s}\left(1 - \left(\frac{4}{5}\right)^s - 2\left(\frac{1}{5}\right)^s\right) = 0.$$

Unfortunately, Corollary 2.3.7 does not allow concluding directly that $s_0 = s$ and thus $C_0^f(F) = 0$. But this is in fact true and will be derived from Theorem 2.3.8. Let x and O be as indicated in Figure 6. Note that O is a feasible open set of F. Choose some $\varepsilon_0 < 1/20$ and let $B = B(x, \varepsilon_0)$, the ball with center x and radius ε_0. It is not difficult to see that $B \subset O_{-\varepsilon_0}$, since $d(x, \partial O) = 1/10$. Moreover, for $\varepsilon \leq \varepsilon_0$, $C_0^{\mathrm{var}}(F_\varepsilon, B) \geq 1/3$, since the set $\partial F_\varepsilon \cap B$ does always contain the arc A whose length is $1/3$ of the perimeter of the circle with center x and radius ε. This contributes the amount of $1/3$ to the mass of $C_0^+(F_\varepsilon, B)$ and thus to $C_0^{\mathrm{var}}(F_\varepsilon, B)$. Now Theorem 2.3.8 implies that $\varepsilon^s C_0^{\mathrm{var}}(F_\varepsilon) \geq c$ for $\varepsilon < \varepsilon_0$ (where $c = \frac{1}{3}(\varepsilon_0/5)^s$) and so, by Corollary 2.3.9, $s_0 = s$. Hence $C_0^f(F) = 0$ as claimed.

The above example contrasts with Example 2.3.5, where we also had $X_0 = 0$ but $s_0 < s$. While for the parallel sets of the cube Q not only the 0th total curvature $C_0(Q_\varepsilon)$ remains bounded as $\varepsilon \to 0$ but also the local 0th curvature, here locally the curvature grows as $\varepsilon \to 0$ and only the total curvature $C_0(F_\varepsilon)$ remains bounded (in fact, constant). For all ε, the positive curvature of F_ε equals its negative curvature. In [22] we considered the Euler characteristic, which equals the 0th total curvature. It does not 'see' the local behaviour of the curvature and therefore we obtain $\sigma = 0$ and $\chi^f(F) = 1$, which reflects somehow its topological structure (connected and simply connected) but not its 'fractality' which is better revealed from $s_0 = s$ and $C_0^f(F) = 0$ (0th curvature scales locally with ε^s but vanishes globally).

REMARK 2.4.7. In Remark 2.2.6 we introduced the scaling exponents s_k^+ and s_k^-, by taking in Definition 2.2.3 only the positive or negative curvature, respectively, into account. In our examples in $\mathbb{R}^2$ this distinction only makes sense for $k = 0$. It is not difficult to see that for the Sierpiński gasket or the carpets only the negative curvature $C_0^-(F_\varepsilon)$ increases as $\varepsilon \to 0$, while the positive curvature $C_0^+(F_\varepsilon)$ remains bounded. Therefore $s_0^- = s_0 = s$ and $s_0^+ = 0 < s_0$ in those examples. The situation is different for the Sierpiński tree. Here the negative and positive curvature grow with the same speed. Hence $s_0^+ = s_0^- = s_0 = s$.

2.5. Fractal curvature measures. With the convergence of rescaled total curvatures as discussed in Section 2.3, the natural question arises how the corresponding (rescaled) curvature measures behave.

Rescaled curvature measures. As before, let $F \subset \mathbb{R}^d$ be a compact set such that for all $\varepsilon > 0$, $F_\varepsilon \in \mathcal{R}^d$. This ensures that the curvature measures $C_k(F_\varepsilon, \cdot)$ are defined. We study the limiting behaviour of these measures as $\varepsilon \to 0$. The weak convergence of signed measures seems to be the appropriate notion of convergence here. It is the straightforward generalization of the usual weak convergence of positive measures. For each $\varepsilon > 0$, let μ_ε be a totally finite signed measure on $\mathbb{R}^d$. The measures μ_ε are said to *converge weakly* to a totally finite signed measure μ as $\varepsilon \to 0$, $\mu_\varepsilon \overset{\mathrm{w}}{\to} \mu$, if and only if $\int_{\mathbb{R}^d} f \, d\mu_\varepsilon \to \int_{\mathbb{R}^d} f \, d\mu$ for all bounded continuous functions f on $\mathbb{R}^d$. We refer to the Appendix for more details.

Since weak convergence is always accompanied by the convergence of the total masses of the measures, it is clear that the measures $C_k(F_\varepsilon, \cdot)$ have to be rescaled by the factor ε^{s_k}, where s_k is the kth scaling exponent as defined above. Therefore, for each $\varepsilon > 0$ and $k = 0, \ldots, d$, we define the *kth rescaled curvature measure* $\nu_{k,\varepsilon}$ of F_ε by

$$\nu_{k,\varepsilon}(\,\cdot\,) := \varepsilon^{s_k} C_k(F_\varepsilon, \cdot). \tag{2.5.1}$$

In general, these measures need not converge weakly as $\varepsilon \to 0$. We have seen above that often the *total mass* $\nu_{k,\varepsilon}(\mathbb{R}^d) = \varepsilon^{s_k} C_k(F_\varepsilon)$ already fails to converge, which makes weak convergence impossible. Therefore, in analogy with Definition 2.2.8, we also define averaged versions $\overline{\nu}_{k,\varepsilon}$ of the rescaled curvature measures $\nu_{k,\varepsilon}$ by

$$\overline{\nu}_{k,\varepsilon}(\,\cdot\,) := \frac{1}{|\ln \varepsilon|} \int_\varepsilon^1 \tilde{\varepsilon}^{s_k} C_k(F_{\tilde{\varepsilon}}, \cdot) \frac{d\tilde{\varepsilon}}{\tilde{\varepsilon}}. \tag{2.5.2}$$

For $k = d$ and $k = d - 1$, the measure $C_k(F_\varepsilon, \cdot)$ is known to be positive, hence so are $\nu_{k,\varepsilon}$ and $\overline{\nu}_{k,\varepsilon}$. In general, the measures $\nu_{k,\varepsilon}$ and $\overline{\nu}_{k,\varepsilon}$ are totally finite signed measures.

Weak convergence for self-similar sets. Let F be a self-similar set satisfying OSC and s its similarity dimension. Denote by μ_F the normalized s-dimensional Hausdorff measure on F, i.e.

$$\mu_F = \frac{\mathcal{H}^s|_F(\,\cdot\,)}{\mathcal{H}^s(F)}. \tag{2.5.3}$$

It is well known that the OSC implies $0 < \mathcal{H}^s(F) < \infty$. Hence μ_F is well defined. As before, we additionally assume that the parallel sets of F are polyconvex.

The question of weak convergence of the curvature measures has to be restricted to sets for which $s_k = s - k$, since for $s_k < s - k$, we do not even have information on the convergence behaviour of the total masses of these measures. It is again necessary to distinguish between arithmetic and non-arithmetic self-similar sets F. The weak convergence of the measures $\nu_{k,\varepsilon}$ as $\varepsilon \to 0$ is only ensured in the non-arithmetic case, while the measures $\overline{\nu}_{k,\varepsilon}$ converge in general. This is not very surprising, since the convergence of the total masses $\nu_{k,\varepsilon}(\mathbb{R}^d)$ was ensured by Theorem 2.3.6 only for non-arithmetic sets, while the total masses $\overline{\nu}_{k,\varepsilon}(\mathbb{R}^d)$ converge in general. So the value of the total mass of the limit measure (if it exists) must be $C_k^f(F)$ or $\overline{C}_k^f(F)$, respectively. The following statement gives an answer to when weak limits exist and what the limit measures are.

THEOREM 2.5.1. *Let F be a self-similar set satisfying OSC and $F_\varepsilon \in \mathcal{R}^d$. Let $k \in \{0, \ldots, d\}$ and assume $s_k = s - k$. Then always*

$$\overline{\nu}_{k,\varepsilon} \xrightarrow{\mathrm{w}} \overline{C}_k^f(F)\mu_F \quad \text{as } \varepsilon \to 0.$$

If $\{-\ln r_1, \ldots, -\ln r_N\}$ is non-arithmetic, then

$$\nu_{k,\varepsilon} \xrightarrow{\mathrm{w}} C_k^f(F)\mu_F \quad \text{as } \varepsilon \to 0.$$

For each k, the limit measure is some multiple of the measure μ_F. Note that the case $\overline{C}_k^f(F) = 0$ is included in this formulation. For this case the limit is the zero measure. Otherwise it is either a positve or a purely negative measure, depending on the signum of the factor $C_k^f(F)$ or $\overline{C}_k^f(F)$, respectively. The limit measure $C_k^f(F)\mu_F$ (or $\overline{C}_k^f(F)\mu_F$, respectively) should be regarded as the *kth fractal curvature measure* of F. The theorem

states that the $d + 1$ fractal curvature measures of F all coincide up to some constant factors. Taking into account the self-similarity of the relevant sets, it is not very surprising that all fractal curvature measures essentially coincide with μ_F. Any measure on a self-similar set F describing its geometry should respect the self-similar structure of F. But the self-similar measures on F are well known and so it is not surprising to rediscover them here. The proof of this theorem is given in Section 6.2.

Weak limits of the parallel volume. For $k = d$, Theorem 2.5.1 can be generalized to arbitrary self-similar sets satisfying OSC. By replacing $C_d(F_\varepsilon, \cdot)$ with the Lebesgue measure, we can again drop the assumption of polyconvexity for the parallel sets F_ε. The definition of the rescaled measure $\nu_{d,\varepsilon}$ in (2.5.1) generalizes to arbitrary compact sets $F \subset \mathbb{R}^d$ by setting

$$\nu_{d,\varepsilon}(\,\cdot\,) := \varepsilon^{s_d} \lambda_d(F_\varepsilon \cap \,\cdot\,). \tag{2.5.4}$$

Similarly, (2.5.2) generalizes to

$$\overline{\nu}_{d,\varepsilon}(\,\cdot\,) := \frac{1}{|\ln \varepsilon|} \int_\varepsilon^1 \tilde{\varepsilon}^{s_d} \lambda_d(F_{\tilde{\varepsilon}} \cap \,\cdot\,) \frac{d\tilde{\varepsilon}}{\tilde{\varepsilon}}. \tag{2.5.5}$$

We call $\nu_{d,\varepsilon}$ the *rescaled ε-parallel volume* and $\overline{\nu}_{d,\varepsilon}$ the *average rescaled ε-parallel volume* of F, respectively. If some weak limit of these measures exists, as $\varepsilon \to 0$, then the total mass of the limit measure must coincide with the Minkowski content $M(F)$ or its average counterpart $\overline{M}(F)$, respectively. Indeed, such limit measures exist as the following statement shows.

THEOREM 2.5.2. *Let F be a self-similar set satisfying OSC. Then always*

$$\overline{\nu}_{d,\varepsilon} \xrightarrow{\text{w}} \overline{M}(F)\mu_F \quad \text{as } \varepsilon \to 0.$$

If F is non-arithmetic, then also

$$\nu_{d,\varepsilon} \xrightarrow{\text{w}} M(F)\mu_F \quad \text{as } \varepsilon \to 0.$$

This result extends Gatzouras's theorem. Not only does the total (average) ε-parallel volume of self-similar sets converge as $\varepsilon \to 0$, but also the convergence holds locally in every 'nice' subset of $\mathbb{R}^d$. This is the meaning of weak convergence. More precisely, if $B \subset \mathbb{R}^d$ is a μ_F-*continuity set*, i.e. $\mu_F(\partial B) = 0$, then $\nu_{d,\varepsilon}(B) \to M(F)\mu_F(B)$ as $\varepsilon \to 0$ for F non-arithmetic and $\overline{\nu}_{d,\varepsilon}(B) \to \overline{M}(F)\mu_F(B)$ for general F.

Normalized curvature measures. We also want to explore another type of limit for the curvature measures which avoids the averaging and yields convergence nevertheless. Although in our situation averaging is a natural procedure to improve the convergence behaviour, it is not the only possible one. Another way to overcome the problem of oscillations, which prevent the convergence, is to normalize the measures. Since normalization is only possible for positive and finite measures and since the rescaled curvature measures $\nu_{k,\varepsilon}$ are in general signed measures for $k \in \{0, \ldots, d-2\}$, this only makes sense for $k = d - 1$ and $k = d$. We discuss both cases separately, since for $k = d$ we can again obtain more general results.

The case $k = d - 1$. Define the $(d-1)th$ *normalized curvature measure* of F_ε by

$$\nu^1_{d-1,\varepsilon}(\,\cdot\,) := \frac{\nu_{d-1,\varepsilon}(\,\cdot\,)}{\nu_{d-1,\varepsilon}(\mathbb{R}^d)} = \frac{C_{d-1}(F_\varepsilon, \,\cdot\,)}{C_{d-1}(F_\varepsilon)}.$$

THEOREM 2.5.3. *Let F be a self-similar set satisfying OSC and $F_\varepsilon \in \mathcal{R}^d$. Assume that $\underline{X}_{d-1} := \liminf_{\varepsilon \to 0} \varepsilon^{s-d+1} C_{d-1}(F_\varepsilon) > 0$. Then*

$$\nu^1_{d-1,\varepsilon} \xrightarrow{\mathrm{w}} \mu_F \quad as \ \varepsilon \to 0.$$

Observe that the measures $\nu^1_{d-1,\varepsilon}$ converge weakly even in the arithmetic case. No distinction is necessary between arithmetic and non-arithmetic self-similar sets. The normalization has a similar effect as the averaging. The additional assumption $\underline{X}_{d-1} > 0$ implies that in particular $X_{d-1} > 0$, since $X_{d-1} \geq \underline{X}_{d-1}$. Hence $s_{d-1} = s - d + 1$. In the non-arithmetic case this assumption is equivalent to $X_{d-1} > 0$. For arithmetic sets it is slightly stronger. The proof of Theorem 2.5.3 is given in Section 6.3.

In general, this result does not carry over to the non-normalized counterparts $\nu_{d-1,\varepsilon}$. Under the conditons of Theorem 2.5.3, we obviously have the relation

$$\nu_{d-1,\varepsilon} = \varepsilon^{s_{d-1}} C_{d-1}(F_\varepsilon) \nu^1_{d-1,\varepsilon}.$$

In the arithmetic case, the convergence of the prefactor $\varepsilon^{s_{d-1}} C_{d-1}(F_\varepsilon)$ is not ensured and so in general the existence of a weak limit of these measures cannot be derived from this result.

The case $k = d$. Define the *normalized parallel volume* $\nu_{d,\varepsilon}$ of F_ε by

$$\nu^1_{d,\varepsilon}(\,\cdot\,) = \frac{\nu_{d,\varepsilon}(\,\cdot\,)}{\nu_{d,\varepsilon}(\mathbb{R}^d)} = \frac{\lambda_d(F_\varepsilon \cap \,\cdot\,)}{\lambda_d(F_\varepsilon)}.$$

The measures $\nu_{d,\varepsilon}$ are well defined for each compact set $F \subseteq \mathbb{R}^d$ and $\varepsilon > 0$. For self-similar sets F satisfying OSC the normalized parallel volume converges weakly to μ_F as the following theorem states.

THEOREM 2.5.4. *Let F be a self-similar set satisfying OSC and $F_\varepsilon \in \mathcal{R}^d$. Then*

$$\nu^1_{d,\varepsilon} \xrightarrow{\mathrm{w}} \mu_F \quad as \ \varepsilon \to 0.$$

Again it is not necessary to distinguish between arithmetic and non-arithmetic self-similar sets. Here no additional assumptions are required, since, by Gatzouras's theorem, always $\underline{X}_d > 0$. The proof of Theorem 2.5.4 can be found in Section 6.3.

3. The variations of curvature measures

In this chapter we discuss some further properties of curvature measures and their variation measures. In particular, we derive some useful estimates for the total variations, which we require later on. Recall that the positive, negative and total variation of the measure $C_k(K, \,\cdot\,)$ are given by

$$C_k^+(K, B) = \sup_{B' \subseteq B} C_k(K, B'), \quad C_k^-(K, B) = - \inf_{B' \subseteq B} C_k(K, B')$$

and $C_k^{\mathrm{var}}(K, B) = C_k^+(K, B) + C_k^-(K, B)$ respectively, for each Borel set $B \subseteq \mathbb{R}^d$. Some of their properties have already been discussed in the last paragraph of Section 2.1.

Curvature of parallel sets. Since any parallel set of a convex set is again convex, the parallel sets of polyconvex sets are polyconvex as well (cf. Proposition 2.2.2) and their curvature measures are defined. For sets $K \in \mathcal{R}^d$, we are particularly interested in the continuity properties of the total curvatures $C_k(K_\varepsilon)$ as a function of ε. The statement below is a consequence of the continuity of the total curvatures for convex sets (cf. Proposition 2.1.2(iv)).

LEMMA 3.1.1. *For $K \in \mathcal{R}^d$ and $k \in \{0, \dots, d\}$, $C_k(K_\varepsilon)$, as a function of ε, has a finite set of discontinuities in $(0, \infty)$ and $\lim_{\varepsilon \to 0} C_k(K_\varepsilon) = C_k(K)$.*

Proof. Let $K^1, \dots, K^m \in \mathcal{K}^d$ be sets such that $K = \bigcup_{i=1}^m K^i$. Then by the inclusion-exclusion principle,

$$C_k(K_\varepsilon) = \sum_{I \in N_m} (-1)^{\#I-1} C_k\Big(\bigcap_{i \in I} K_\varepsilon^i\Big), \qquad (3.1.1)$$

where the sets $\bigcap_{i \in I} K_\varepsilon^i$ are convex (possibly empty) for all $\varepsilon \geq 0$. (Here $K_0 = K$.) More precisely, for each I there exists $\varepsilon_I \geq 0$ such that $\bigcap_{i \in I} K_\varepsilon^i = \emptyset$ for all $0 \leq \varepsilon < \varepsilon_I$ and $\bigcap_{i \in I} K_\varepsilon^i \neq \emptyset$ for all $\varepsilon \geq \varepsilon_I$. Now the continuity property implies that $C_k(\bigcap_{i \in I} K_\varepsilon^i)$ is continuous in (ε_I, ∞) and continuous from the right at ε_I. Moreover, $C_k(\bigcap_{i \in I} K_\varepsilon^i) \equiv 0$ in $[0, \varepsilon_I)$ and thus the only possible discontinuity point in $(0, \infty)$ is ε_I. Since this holds for every $I \in N_m$, by (3.1.1), $C_k(K_\varepsilon)$ has finitely many discontinuities in $(0, \infty)$ (at most $\#N_m$). In particular, since always $C_k(\bigcap_{i \in I} K_\varepsilon^i) \to C_k(\bigcap_{i \in I} K^i)$ as $\varepsilon \to 0$, we conclude that $C_k(K_\varepsilon) \to C_k(K)$. ∎

A generalization of Lemma 3.1.1 to the variation measures would be very useful and it was conjectured in [34, p. 38, Conj. 2.2.2] that a corresponding result holds for the variation measures. Fortunately, this problem has recently been solved by Jan Rataj [25]. From his results the following is easily derived:

PROPOSITION 3.1.2. *For $K \in \mathcal{R}^d$, $k \in \{0, \dots, d-2\}$ and $\bullet \in \{+, -, \mathrm{var}\}$, $C_k^\bullet(K_\varepsilon)$, as a function of ε, has a finite set of discontinuities in $(0, \infty)$.*

For $k = d$ and $k = d - 1$, the corresponding statement is trivial, since $C_k(K_\varepsilon, \cdot)$ is a nonnegative measure in this case. The convergence $\lim_{\varepsilon \to 0} C_k^\bullet(K_\varepsilon) = C_k^\bullet(K)$, which had also been conjectured in [34], does not hold in general, as is seen from Example 2 in [25] where a corresponding set in $\mathbb{R}^3$ is given. It seems still open whether such an example also exists in $\mathbb{R}^2$ or whether $\lim_{\varepsilon \to 0} C_k^\bullet(K_\varepsilon) = C_k^\bullet(K)$ holds for all polyconvex sets $K \subset \mathbb{R}^2$. However, if the set K itself is a parallel set of some polyconvex set, then this convergence takes place (cf. [25, Theorem 2]).

Proof of Proposition 3.1.2. Let $K = \bigcup_{i=1}^m K^i$ be a representation of K with convex sets K^i. Recall that convex bodies $K^1, \dots, K^m$ *osculate* if there exists a nonempty subset $I \subset \{1, \dots, m\}$, a point $x \in \bigcap_{i \in I} K^i$ and nonzero outer normal vectors $n_i \in \mathrm{Nor}(K^i, x)$ with $\sum_{i \in I} n_i = 0$. Here $\mathrm{Nor}(X, x)$ denotes the *normal cone* of the set X at $x \in X$. From the arguments in the proof of Lemma 4 in [24] it follows that the number of values $\varepsilon \in (0, \infty)$ such that $K_\varepsilon^1, \dots, K_\varepsilon^m$ osculate is finite. Denote those values by $\varepsilon_0 < \varepsilon_1 < \cdots < \varepsilon_l = \infty$.

Theorem 2 in [25] states that there is r_0 such that $C_k^{\mathrm{var}}(K_\varepsilon)$ is continuous in $\varepsilon \in (0, r_0)$. Moreover, it follows from the proof of this theorem that r_0 can be chosen to be ε_0, i.e. the smallest number such that $K_{r_0}^1, \dots, K_{r_0}^m$ osculate. Applying the same arguments to the set K_{ε_i}, we find that $C_k^{\mathrm{var}}(K_\varepsilon)$ is continuous in $\varepsilon \in (\varepsilon_i, \varepsilon_{i+1})$ and so the assertion follows. ∎

REMARK 3.1.3. Proposition 3.1.2 allows one to extend some of the main results to the variation measures (cf. Remark 5.5.2).

Estimates for the total variation measure. The properties discussed above are more or less the same for the curvature measures and their variation measures. Unfortunately, this is not true for additivity. The variation measures fail to be additive in general. Therefore, we now derive some inequalities for the variation measures, which, in a way, take over the role the inclusion-exclusion principle plays for curvature measures.

Let $K^j \in \mathcal{R}^d$ for $j = 1, \dots, m$ and $K = \bigcup_{j=1}^m K^j$. Recall that N_m was the family of all nonempty subsets of $\{1, \dots, m\}$. For each $I \in N_m$ write $K(I) := \bigcap_{j \in I} K^j$.

LEMMA 3.1.4. *Let $K^j \in \mathcal{R}^d$ for $j = 1, \dots, m$ and $K = \bigcup_{j=1}^m K^j$. Then*

$$C_k^{\mathrm{var}}(K, B) \leq \sum_{I \in N_m} C_k^{\mathrm{var}}(K(I), B) \tag{3.1.2}$$

for each Borel set B. If the sets K^j are convex then

$$C_k^{\mathrm{var}}(K, B) \leq (2^m - 1) \max_j C_k(K^j). \tag{3.1.3}$$

Note that since $C_k^{\pm}(K, \cdot) \leq C_k^{\mathrm{var}}(K, \cdot)$, estimate (3.1.3) remains valid if $C_k^{\mathrm{var}}(K, B)$ on the left hand side is replaced with $C_k^+(K, B)$ or $C_k^-(K, B)$. In general, inequality (3.1.2) does not remain valid when the total variation is replaced with the positive or negative variation measure.

Proof. Let $\mathbb{R}^d = K^+ \cup K^-$ be a Hahn decomposition of the signed measure $C_k(K, \cdot)$, i.e. K^+ and K^- are disjoint sets satisfying $C_k^+(K, K^-) = C_k^-(K, K^+) = 0$ (cf. Appendix, Theorem A.1.1). By the inclusion-exclusion formula, the equality

$$C_k^{\pm}(K, B) = (\pm 1) C_k(K, B \cap K^{\pm}) = (\pm 1) \sum_{I \in N_m} (-1)^{\#I - 1} C_k(K(I), B \cap K^{\pm})$$

holds, and since $|C_k(K, \cdot)| \leq C_k^{\mathrm{var}}(K, \cdot)$,

$$C_k^{\pm}(K, B) \leq \sum_{I \in N_m} C_k^{\mathrm{var}}(K(I), B \cap K^{\pm}).$$

Hence we obtain

$$\begin{aligned}
C_k^{\mathrm{var}}(K, B) &= C_k^+(K, B) + C_k^-(K, B) \\
&\leq \sum_{I \in N_m} C_k^{\mathrm{var}}(K(I), B \cap K^+) + \sum_{I \in N_m} C_k^{\mathrm{var}}(K(I), B \cap K^-) \\
&= \sum_{I \in N_m} C_k^{\mathrm{var}}(K(I), B)
\end{aligned}$$

for each Borel set $B \subseteq \mathbb{R}^d$, as stated in (3.1.2). The second assertion follows from the first one by noting that for each $I \in N_m$, the set $K(I)$ is convex and thus $C_k^{\mathrm{var}}(K(I), \cdot) =$

$C_k(K(I), \cdot)$. Since $K(I) \subseteq K^j$ for some $j \in \{1, \ldots, m\}$, the monotonicity of the total curvatures in $\mathcal{K}^d$ yields $C_k(K(I), B) \leq C_k(K(I)) \leq \max_j C_k(K^j)$. If we now observe that the number of summands, i.e. the number of sets in N_m, is $2^m - 1$, the second assertion follows. ∎

Using the above Lemma 3.1.4, we now give a proof of the consistency stated in Proposition 2.2.10.

Proof of Proposition 2.2.10. Let $F \in \mathcal{R}^d$ and $F = \bigcup_{i=1}^m K^i$ a representation of F with convex bodies K^i. For proving $s_k(F) = 0$, it suffices to show that $\varepsilon^t C_k^{\mathrm{var}}(F_\varepsilon) \to 0$ as $\varepsilon \to 0$ for all $t > 0$ and $\varepsilon^t C_k^{\mathrm{var}}(F_\varepsilon) \to \infty$ as $\varepsilon \to 0$ for all $t < 0$.

By Lemma 3.1.4, for each $\varepsilon \geq 0$ we have $C_k^{\mathrm{var}}(F_\varepsilon) \leq (2^m - 1) \max_i K_\varepsilon^i$. The monotonicity (Proposition 2.1.2(v)) implies that, for $\varepsilon_0 > 0$ and all $0 \leq \varepsilon \leq \varepsilon_0$, $C_k^{\mathrm{var}}(F_\varepsilon) \leq (2^m - 1) \max_i K_{\varepsilon_0}^i$. Thus, for $t > 0$, $\varepsilon^t C_k^{\mathrm{var}}(F_\varepsilon) \to 0$ as $\varepsilon \to 0$.

On the other hand, we have $C_k^{\mathrm{var}}(F_\varepsilon) \geq |C_k(F_\varepsilon)|$ for each $\varepsilon > 0$. Since, by Lemma 3.1.1 and the assumption $C_k(F) \neq 0$, $|C_k(F_\varepsilon)| \to |C_k(F)| > 0$, we conclude that, for each $t < 0$, $\varepsilon^t C_k^{\mathrm{var}}(F_\varepsilon) \to \infty$ as $\varepsilon \to 0$. Hence, $s_k(F) = 0$ and the assertion $C_k^f(F) = C_k(F)$ follows immediately from Lemma 3.1.1. ∎

The estimates obtained in Lemma 3.1.4 are not satisfactory in case we have a large number m of sets K^j with comparably few mutual intersections. But in this situation it can be improved easily. Define the *intersection number* $\Gamma = \Gamma(\mathcal{X})$ of a finite family $\mathcal{X} = \{K^1, \ldots, K^m\}$ of sets as the maximum over all $l \in \{1, \ldots, m\}$ of the number of nonempty intersections $K^l \cap K^j$ with $K^j \in \mathcal{X}$, i.e.

$$\Gamma(\mathcal{X}) = \max_l \#\{j : K^j \cap K^l \neq \emptyset\}. \tag{3.1.4}$$

If Γ is small compared to m, then the following estimate is useful.

COROLLARY 3.1.5. *Let $K^j \in \mathcal{R}^d$ for $j = 1, \ldots, m$ and $K = \bigcup_{j=1}^m K^j$, Γ the intersection number of the family $\{K^1, \ldots, K^m\}$ and $b > 0$ such that for all $I \in N_m$*

$$C_k^{\mathrm{var}}(K(I), B) \leq b.$$

Then

$$C_k^{\mathrm{var}}(K, B) \leq m 2^\Gamma b.$$

Proof. Let I_l be the set of all indices j such that $K^l \cap K^j \neq \emptyset$. For each K^l it suffices to consider its intersections with sets K^j with $j \in I_l$, all other intersections being empty. Therefore the sum on the right hand side of (3.1.2) is contained in

$$\sum_{l=1}^m \sum_{I \subseteq I_l} C_k^{\mathrm{var}}(K^l \cap K(I), B),$$

where we set $K(I) := \mathbb{R}^d$ in case $I = \emptyset$. By assumption, each term is bounded from above by b. Moreover, $\#I_l \leq \Gamma$ and so the number of subsets of I_l is not greater than 2^Γ. Hence the asserted estimate follows. ∎

Now assume that $L = \bigcap_{j=1}^m K^j$ is the intersection of a finite number of sets $K^j \in \mathcal{R}^d$. Again we ask for an upper bound of $C_k^{\mathrm{var}}(L, \cdot)$ in terms of the curvatures of the sets K^j.

For convex sets K^j there is an obvious bound for the total curvature. In this case the set L is convex as well and the monotonicity implies

$$C_k(L) \leq \min_{j=1,\ldots,m} C_k(K^j).$$

If the K^j are polyconvex, we have at least some bound in terms of representations of K^j with convex sets.

LEMMA 3.1.6. *Let* $L = \bigcap_{j=1}^m K^j$. *Assume that each* K^j *has a representation* $K^j = \bigcup_{i=1}^P K^{j,i}$ *as a union of (at most) P convex sets* $K^{j,i}$. *Then*

$$C_k^{\mathrm{var}}(L) \leq (2^{(P^m)} - 1) \max_{j,i} C_k(K^{j,i}).$$

Note that, since $C_k^{\pm}(L, \cdot) \leq C_k^{\mathrm{var}}(L, \cdot)$, the assertion remains true when $C_k^{\mathrm{var}}(L)$ is replaced with $C_k^+(L)$ or $C_k^-(L)$.

Proof. We have

$$L = \bigcap_{j=1}^m K^j = \bigcap_{j=1}^m \bigcup_{i_j=1}^P K^{j,i_j} = \bigcup_{i_1,\ldots,i_m=1}^P \bigcap_{j=1}^m K^{j,i_j},$$

i.e. L is the union of the P^m convex sets $\bigcap_j K^{j,i_j}$. Therefore, the assertion follows immediately from the second statement in Lemma 3.1.4. ∎

4. Adapted Renewal Theorem

For the proofs of Theorem 2.3.6 and Theorem 2.3.10 we require the Renewal Theorem which we recall and discuss now. Afterwards we will reformulate it in a way which is most convenient for our purposes. Later on we will only use this variant of the Renewal Theorem. It is stated in Theorem 4.1.4.

The Renewal Theorem. Let P be a Borel probability measure with support contained in $[0, \infty)$ and $\eta := \int_0^\infty t\, P(dt) < \infty$. Let $z : \mathbb{R} \to \mathbb{R}$ be a function with a discrete set of discontinuities satisfying

$$|z(t)| \leq c_1 e^{-c_2|t|} \quad \text{ for all } t \in \mathbb{R} \tag{4.1.1}$$

for some constants $0 < c_1, c_2 < \infty$. It is well known in probability theory that under these conditions on z the equation

$$Z(t) = z(t) + \int_0^\infty Z(t - \tau)\, P(d\tau) \tag{4.1.2}$$

has a unique solution $Z(t)$ in the class of functions satisfying $\lim_{t \to -\infty} Z(t) = 0$. Equation (4.1.2) is called a *renewal equation* and the asymptotic behaviour of its solution as $t \to \infty$ is given by the so-called Renewal Theorem. The Renewal Theorem is a standard tool in probability theory (cf. e.g. Feller [9]). In the last years, it has also been discovered as a tool in fractal geometry. Therefore, versions are available which are adapted to the fractal

setting. In fractal applications, P is usually a probability measure supported by a finite set of points $y_1, \ldots, y_N \in [0, \infty)$ such that $P(\{y_i\}) = p_i$ for $i = 1, \ldots, N$ and therefore

$$\eta = \sum_{i=1}^{N} y_i p_i. \tag{4.1.3}$$

Discrete versions of the Renewal Theorem are for instance provided by Falconer [7, Corollary 7.3, p. 122]) or Levitin and Vassiliev [21].

A function $g : \mathbb{R} \to \mathbb{R}$ is said to be *asymptotic* to a function $f : \mathbb{R} \to \mathbb{R}$, $g \sim f$, if for all $\varepsilon > 0$ there exists a number $D = D(\varepsilon)$ such that

$$(1 - \varepsilon)f(t) \leq g(t) \leq (1 + \varepsilon)f(t) \quad \text{for all } t > D. \tag{4.1.4}$$

Recall from Section 2.3 that the set $\{y_1, \ldots, y_N\}$ is called *h-arithmetic* if h is the largest number such that $y_i \in h\mathbb{Z}$ for $i = 1, \ldots, N$ and *non-arithmetic* if no such number h exists.

THEOREM 4.1.1 (Renewal Theorem). *Let* $0 < y_1 \leq \cdots \leq y_N$ *and* $p_1, \ldots, p_N$ *be positive real numbers such that* $\sum_{i=1}^{N} p_i = 1$. *For a function* z *as defined in (4.1.1), let* $Z : \mathbb{R} \to \mathbb{R}$ *be the unique solution of the renewal equation*

$$Z(t) = z(t) + \sum_{i=1}^{N} p_i Z(t - y_i) \tag{4.1.5}$$

satisfying $\lim_{t \to -\infty} Z(t) = 0$. *Then the following holds:*

(i) *If the set* $\{y_1, \ldots, y_N\}$ *is non-arithmetic, then*

$$\lim_{t \to \infty} Z(t) = \frac{1}{\eta} \int_{-\infty}^{\infty} z(\tau) \, d\tau.$$

(ii) *If* $\{y_1, \ldots, y_N\}$ *is h-arithmetic for some* $h > 0$, *then*

$$Z(t) \sim \frac{h}{\eta} \sum_{k=-\infty}^{\infty} z(t - kh).$$

Moreover, Z is uniformly bounded in $\mathbb{R}$.

Theorem 4.1.1 implies that in the non-arithmetic case the limit $\lim_{t \to \infty} Z(t)$ exists, while in the *h*-arithmetic case Z is asymptotic to some periodic function of period $h > 0$ (i.e. to some function f with $f(t + h) = f(t)$ for all $t \in \mathbb{R}$). The latter is sufficient for the limit $\lim_{T \to \infty} T^{-1} \int_0^T Z(t) \, dt$ to exist, which is easily derived from the following observation.

LEMMA 4.1.2. *Let* f *be a locally integrable periodic function with period* $h > 0$ *and let* $L := \int_0^h f(t) \, dt$.

(i) *The limit* $\lim_{T \to \infty} T^{-1} \int_0^T f(t) \, dt$ *exists and equals* $h^{-1}L$.
(ii) *If* $g : \mathbb{R} \to \mathbb{R}$ *is a function such that* $g \sim f$, *then also the limit* $\lim_{T \to \infty} T^{-1} \int_0^T g(t) \, dt$ *exists and equals* $h^{-1}L$.

As a direct consequence of the Renewal Theorem and Lemma 4.1.2 we obtain

COROLLARY 4.1.3. *Under the assumptions of Theorem 4.1.1 the following limit always exists and is equal to the expression on the right hand side:*

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T Z(t)\, dt = \frac{1}{\eta} \int_{-\infty}^{\infty} z(\tau)\, d\tau.$$

Proof. If $\{y_1, \ldots, y_N\}$ is h-arithmetic, just note that $f(t) = (h/\eta) \sum_{k=-\infty}^{\infty} z(t - kh)$ in Theorem 4.1.1(ii) is uniformly bounded and periodic, and apply Lemma 4.1.2(ii) to $g(t) = Z(t)$. In the non-arithmetic case the limit $\lim_{t \to \infty} Z(t)$ exists and the assertion follows by applying Lemma 4.1.2 to $g(t) = Z(t)$, which is asymptotic to the constant function $f \equiv \lim_{t \to \infty} Z(t)$. ∎

Reformulation of the Renewal Theorem. Now we are ready to restate the Renewal Theorem in a more convenient way. Here we will always consider some self-similar set F with contraction ratios r_i and similarity dimension s. Therefore we fix $p_i = r_i^s$ and $y_i = -\ln r_i$. We substitute $t = -\ln \varepsilon$, since we are interested in the limiting behaviour of functions $f : (0, \infty) \to \mathbb{R}$ as the argument ε tends to zero. Moreover, by taking into account Corollary 4.1.3, we conclude the existence of average limits from the asymptotic periodicity.

THEOREM 4.1.4 (Adapted Renewal Theorem). *Let F be a self-similar set with ratios $r_1, \ldots, r_N$ and similarity dimension s. For a function $f : (0, \infty) \to \mathbb{R}$, suppose that for some $k \in \mathbb{R}$ the function φ_k defined by*

$$\varphi_k(\varepsilon) = f(\varepsilon) - \sum_{i=1}^{N} r_i^k \mathbf{1}_{(0, r_i]}(\varepsilon) f(\varepsilon/r_i) \tag{4.1.6}$$

has a discrete set of discontinuities and satisfies

$$|\varphi_k(\varepsilon)| \le c \varepsilon^{k-s+\gamma} \tag{4.1.7}$$

for some constants $c, \gamma > 0$ and all $\varepsilon > 0$. Then $\varepsilon^{s-k} f(\varepsilon)$ is uniformly bounded in $(0, \infty)$ and the following holds:

(i) *The limit $\lim_{\delta \to 0} \frac{1}{|\ln \delta|} \int_\delta^1 \varepsilon^{s-k} f(\varepsilon) \frac{d\varepsilon}{\varepsilon}$ exists and equals*

$$\frac{1}{\eta} \int_0^1 \varepsilon^{s-k-1} \varphi_k(\varepsilon)\, d\varepsilon, \tag{4.1.8}$$

   where $\eta = -\sum_{i=1}^{N} r_i^s \ln r_i$.
(ii) *If $\{-\ln r_1, \ldots, -\ln r_N\}$ is non-arithmetic, then the limit of $\varepsilon^{s-k} f(\varepsilon)$ as $\varepsilon \to 0$ exists and equals the average limit.*

Proof. The definition (4.1.6) of φ_k implies that

$$f(\varepsilon) = \sum_{i=1}^{N} r_i^k \mathbf{1}_{(0, r_i]}(\varepsilon) f(\varepsilon/r_i) + \varphi_k(\varepsilon). \tag{4.1.9}$$

Define

$$Z(t) = \begin{cases} e^{(k-s)t} f(e^{-t}) & \text{for } t \ge 0, \\ 0 & \text{for } t < 0. \end{cases} \tag{4.1.10}$$

Taking into account (4.1.9), for all $t \geq 0$ we have

$$Z(t) = e^{(k-s)t}\left(\sum_{i=1}^{N} r_i^k \mathbf{1}_{(0,r_i]}(e^{-t})f(e^{-(t+\ln r_i)}) + \varphi_k(e^{-t})\right)$$

$$= \sum_{i=1}^{N} r_i^s e^{(k-s)(t+\ln r_i)} \mathbf{1}_{[-\ln r_i,\infty)}(t)f(e^{-(t+\ln r_i)}) + e^{(k-s)t}\varphi_k(e^{-t})$$

The ith term of the sum can be replaced by $r_i^s Z(t+\ln r_i)$. (For $t < -\ln r_i$ this expression is zero as is the corresponding ith term in the sum.) Thus we have the renewal equation

$$Z(t) = \sum_{i=1}^{N} r_i^s Z(t+\ln r_i) + z(t), \tag{4.1.11}$$

where the function $z : \mathbb{R} \to \mathbb{R}$ is defined by

$$z(t) = \begin{cases} e^{(k-s)t}\varphi_k(e^{-t}) & \text{for } t \geq 0, \\ 0 & \text{for } t < 0. \end{cases} \tag{4.1.12}$$

Observing that the assumptions on φ_k ensure z has a discrete set of discontinuities and satisfies

$$|z(t)| = e^{(k-s)t}|\varphi_k(e^{-t})| \leq ce^{-t\gamma}$$

for some constants $c, \gamma > 0$ and all $t \geq 0$, we can apply the Theorem 4.1.1 with $p_i = r_i^s$ and $y_i = -\ln r_i$. There are two cases to discuss.

The non-arithmetic case. If $\{-\ln r_1, \ldots, -\ln r_N\}$ is non-arithmetic, then the limit

$$\lim_{t\to\infty} Z(t) = \lim_{t\to\infty} e^{-t(s-k)}f(e^{-t}) = \lim_{\varepsilon\to 0} \varepsilon^{s-k}f(\varepsilon)$$

exists and is equal to the integral

$$\frac{1}{\eta}\int_0^\infty z(\tau)\,d\tau. \tag{4.1.13}$$

By (4.1.12) and with the substitution $r = e^{-\tau}$ we obtain

$$\lim_{\varepsilon\to 0} \varepsilon^{s-k}f(\varepsilon) = \frac{1}{\eta}\int_0^1 r^{s-k-1}\varphi_k(r)\,dr.$$

This completes the proof of (ii) of Theorem 4.1.4. For the non-arithmetic case, (i) follows immediately from (ii), since the average limit exists whenever the limit exists and they coincide.

The arithmetic case. If $\{-\ln r_1, \ldots, -\ln r_N\}$ is h-arithmetic for some $h > 0$, Corollary 4.1.3 states that the limit

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T Z(t)\,dt = \lim_{\delta\to 0} \frac{1}{|\ln\delta|}\int_\delta^1 \varepsilon^{s-k}f(\varepsilon)\,\frac{d\varepsilon}{\varepsilon}$$

exists and equals the integral in (4.1.13). Therefore we obtain formula (4.1.8) also for the h-arithmetic case. This completes the proof of Theorem 4.1.4. ∎

REMARK 4.1.5. In the h-arithmetic case, Theorem 4.1.4 concentrates on the existence of average limits. However, some additional information on the limiting behaviour can

be derived from the original Renewal Theorem, which stated the existence of some (additively) periodic function of period h, to which $Z(t)$ is asymptotic as $t \to \infty$. In the situation of Theorem 4.1.4 this is translated into the existence of a (multiplicatively) periodic function $G(\varepsilon)$ of period $\zeta = e^{-h}$, i.e. $G(\zeta\varepsilon) = G(\varepsilon)$ for all $\varepsilon > 0$, to which the function $g(\varepsilon) = \varepsilon^{s-k}f(\varepsilon)$ is asymptotic as $\varepsilon \to 0$.

5. Proofs: Fractal curvatures

In this chapter we prove the results presented in Section 2.3. After some preparations and the proof of Proposition 2.3.1 in the first section, we state the key estimate (Lemma 5.2.1) which will enable us to prove Theorem 2.3.6. The problem left is to verify this estimate, which is done in Sections 5.3 and 5.4. In Section 5.5, we provide proofs of Theorem 2.3.2 and Theorem 2.3.8. Finally, in Section 5.6 we reprove Gatzouras's theorem on the existence of the (average) Minkowski content.

Throughout the chapter we assume F to be a self-similar set in $\mathbb{R}^d$ satisfying OSC. Moreover, O will always denote some feasible open set of F such that the SOSC is satisfied, i.e. in particular $F \cap O \neq \emptyset$.

5.1. Preparations

Code space and level sets. Set $\Sigma := \{1, \dots, N\}$ and for $n = 0, 1, 2, \dots$ let Σ^n denote the set of all sequences $w_1 w_2 \dots w_n$ such that $w_i \in \Sigma$; n is called the *length* of $w = w_1 \dots w_n$. We set

$$\Sigma^* := \bigcup_{n=0}^{\infty} \Sigma^n.$$

Observe that Σ^* contains all finite sequences including the *empty word* $w \in \Sigma^0$.

Finite sequences $w \in \Sigma^*$ are also called *words* over the alphabet Σ. If $v = v_1 \dots v_m$ and $w = w_1 \dots w_n$ are words in Σ^*, then vw simply denotes the word $v_1 \dots v_m w_1 \dots w_n$. Moreover, let Σ^∞ denote the family of all infinite sequences $w_1 w_2 \dots$ such that $w_i \in \Sigma$. For $w = w_1 \dots w_n \in \Sigma^*$, we introduce the abbreviations

$$r_w := r_{w_1} \dots r_{w_n} \quad \text{and} \quad S_w := S_{w_1} \circ \dots \circ S_{w_n}.$$

The sets $S_w F$ are called the *level sets* of F. Since the S_i are contractions and since $S_{wi}F \subset S_w F$ for each $w \in \Sigma^*$ and $i \in \Sigma$, the mapping $\pi : \Sigma^\infty \to F$ given by

$$w_1 w_2 \dots \mapsto x := \bigcap_{n=1}^{\infty} S_{w_1 \dots w_n} F$$

is well defined. Observe that π is surjective but not necessarily injective, i.e. for each $x \in F$ there exists some sequence $w \in \Sigma^\infty$ such that $x = \pi(w)$ but it might not be unique.

The families $\Sigma(r)$. For $0 < r \le 1$, let $\Sigma(r)$ be the family of all finite words $w = w_1 \dots w_n \in \Sigma^*$ such that

$$r_w < r \le r_w r_{w_n}^{-1}. \tag{5.1.1}$$

For convenience, we define $\Sigma(r)$, for $r > 1$, to be the set containing only the empty word. It is clear that for fixed $r \leq 1$ for each sequence $w_1 w_2 w_3 \ldots \in \Sigma^\infty$ there is exactly one n such that $w = w_1 \ldots w_n$ satisfies (5.1.1) (and, for $r > 1$, $n = 0$, correspondingly). Therefore, on the one hand

$$F = \bigcup_{w \in \Sigma(r)} S_w F \tag{5.1.2}$$

for each $r > 0$. On the other hand the words in $\Sigma(r)$ are mutually *incompatible*, i.e. there is no pair of words $v, w \in \Sigma(r)$ such that $v = ww'$ for some nonempty $w' \in \Sigma^*$. Moreover, for each $r > 0$,

$$\sum_{w \in \Sigma(r)} r_w^s = 1, \tag{5.1.3}$$

which is easily seen from the definition of the similarity dimension s. Due to (5.1.1), $\Sigma(r)$ consists of words w for which the corresponding level sets $S_w F$ are approximately of the same size $r \cdot \operatorname{diam} F$. Note that (5.1.1) implies in particular

$$r_w < r \leq r_w r_{\min}^{-1} \tag{5.1.4}$$

for each $w \in \Sigma(r)$, where $r_{\min} = \min_i r_i$.

The cardinalities $\#\Sigma(r)$ of these finite families of words are bounded as follows. By (5.1.4), $r r_{\min} \leq r_w < r$ for each $w \in \Sigma(r)$. Hence, by (5.1.3), on the one hand

$$1 = \sum_{w \in \Sigma(r)} r_w^s < \sum_{w \in \Sigma(r)} r^s = r^s \#\Sigma(r)$$

and on the other hand

$$1 = \sum_{w \in \Sigma(r)} r_w^s \geq \sum_{w \in \Sigma(r)} (r r_{\min})^s = r_{\min}^s r^s \#\Sigma(r).$$

Therefore,

$$r^{-s} < \#\Sigma(r) \leq r_{\min}^{-s} r^{-s}. \tag{5.1.5}$$

The families $\Sigma(r)$ will play an important role in the proofs later on. As a first application of these families, we present a proof of Proposition 2.3.1.

Proof of Proposition 2.3.1. Assume $F_\varepsilon \in \mathcal{R}^d$. Then, by Proposition 2.2.2, $F_\delta \in \mathcal{R}^d$ for all $\delta \geq \varepsilon$. Let now $\delta < \varepsilon$ and set $r = \varepsilon^{-1}\delta$. By (5.1.2), we have

$$F_\delta = \bigcup_{w \in \Sigma(r)} (S_w F)_\delta = \bigcup_{w \in \Sigma(r)} S_w F_{\delta/r_w}.$$

Since $\delta/r_w > \delta/r = \varepsilon$, F_{δ/r_w} is a parallel set of F_ε and thus, again by Proposition 2.2.2, polyconvex. Hence each set $S_w F_{\delta/r_w}$ in the finite union above is polyconvex implying the same for F_δ. ∎

Definition of u, ρ and γ. Above we fixed some feasible open set O of F such that the SOSC is satisfied. The condition $F \cap O \neq \emptyset$ implies that there exists a sequence $u = u_1 \ldots u_p \in \Sigma^*$ such that

$$S_u F \subset O \tag{5.1.6}$$

and, since $S_u F$ is compact, some constant $\alpha > 0$ such that

$$d(x, \partial O) > \alpha \quad \text{for all } x \in S_u F.$$

Applying the similarity S_w, $w \in \Sigma^*$, the above inequality yields

$$d(x, \partial S_w O) > \alpha r_w \quad \text{for all } x \in S_{wu} F. \tag{5.1.7}$$

Define

$$\rho := r_{\min} \alpha / 2. \tag{5.1.8}$$

Moreover, for each $\varepsilon > 0$ we set $\varepsilon^* = \rho^{-1} \varepsilon$. As will become clear later, it is very convenient to look at the level sets with $w \in \Sigma(\varepsilon^*)$ when investigating ε-parallel sets.

Finally, we introduce the following numbers. Choose some r such that $u \in \Sigma(r)$ and let $\overline{s}$ be the unique solution of

$$\sum_{v \in \Sigma(r),\, v \neq u} r_v^{\overline{s}} = 1 \tag{5.1.9}$$

and $\gamma := s - \overline{s}$. By (5.1.3), $\overline{s} < s$ and so $\gamma > 0$. Obviously, γ and $\overline{s}$ depend on the word u we fixed above. Throughout Chapters 5 and 6 we consider the word u and the constant γ together with the set O as being once and for all fixed for the self-similar set F.

5.2. Proof of Theorem 2.3.6. Throughout this section we assume that the self-similar set F has polyconvex parallel sets, as demanded in the hypothesis of Theorem 2.3.6. Moreover, let $k \in \{0, 1, \ldots, d\}$ be fixed.

The key estimate. For each $r > 0$, we define the set

$$O(r) := \bigcup_{v \in \Sigma(r)} S_v O, \tag{5.2.1}$$

where O is the feasible open set for F we fixed above. Observe that $O(r)$ is again a feasible open set of F for each $r > 0$. In particular, $O = O(r)$ for any $r > 1$ and $O(1) = \mathbf{S}O = \bigcup_i S_i O$. For the complement $O(r)^c$ of these sets the following estimate holds.

LEMMA 5.2.1. *For each $r > 0$, there exists a constant $c > 0$ such that for all $\varepsilon \leq \delta \leq \rho r$*

$$C_k^{\mathrm{var}}(F_\varepsilon, (O(r)^c)_\delta) \leq c \varepsilon^{k-s} \delta^\gamma.$$

This estimate roughly means that, as δ and ε approach 0, the kth curvature concentrates more and more in the set $O(r)_{-\delta}$, the inner parallel set of $O(r)$, while the curvature in the complement $(O(r)_{-\delta})^c = (O(r)^c)_\delta$ vanishes. The constants ρ and γ are those we fixed in (5.1.8) and (5.1.9); c depends on r (and the k fixed above) but is independent of ε and δ. Since $C_k^{\pm}(F_\varepsilon, (O(r)^c)_\delta) \leq C_k^{\mathrm{var}}(F_\varepsilon, (O(r)^c)_\varepsilon)$ and $|C_k(F_\varepsilon, (O(r)^c)_\varepsilon)| \leq C_k^{\mathrm{var}}(F_\varepsilon, (O(r)^c)_\varepsilon)$, the above lemma provides also upper bounds for these expressions.

We require the key estimate in this full generality in the proofs on the weak convergence of curvature measures in Chapter 6. For the moment the following special version is sufficient, where we set $\varepsilon = \delta$ and also fix $r = 1$.

COROLLARY 5.2.2. *There exist some constant $c > 0$ such that for all $0 < \varepsilon \leq 1$,*

$$C_k^{\mathrm{var}}(F_\varepsilon, ((\mathbf{S}O)^c)_\varepsilon) \leq c \varepsilon^{k-s+\gamma}.$$

Proof. Setting in Lemma 5.2.1, $r = 1$, i.e. $O(r) = \mathbf{SO}$, and $\varepsilon = \delta$, the validity of the stated inequality follows immediately for all $\varepsilon \leq \rho$. If necessary, the constant c can be enlarged so that the inequality also holds for $\rho < \varepsilon \leq 1$. ∎

Note that, for ε fixed, the estimate remains valid with $(\mathbf{SO})^c)_\varepsilon$ replaced by any of its subsets. In particular, since $\mathbf{SO} \subseteq O$ and thus $(O^c)_\varepsilon \subseteq ((\mathbf{SO})^c)_\varepsilon$, the estimate holds as well for the sets $(O^c)_\varepsilon$.

We postpone the proof of Lemma 5.2.1 for the moment and first discuss how it can be used to prove Theorem 2.3.6. The first step is the investigation of the scaling functions.

Scaling functions. Recall from (2.3.1) that the kth scaling function R_k is defined by

$$R_k(\varepsilon) = C_k(F_\varepsilon) - \sum_{i=1}^{N} \mathbf{1}_{(0,r_i]}(\varepsilon) C_k((S_i F)_\varepsilon)$$

for $\varepsilon > 0$. We investigate the properties of R_k to see that the Renewal Theorem can be applied. On the one hand, we require an upper bound for the growth of $|R_k|$ as $\varepsilon \to 0$, which will be derived from Corollary 5.2.2, and on the other hand a statement on the continuity of R_k.

LEMMA 5.2.3. *There is a constant $c > 0$ such that for all $0 < \varepsilon \leq 1$,*

$$|R_k(\varepsilon)| \leq c\varepsilon^{k-s+\gamma}. \tag{5.2.2}$$

Proof. For $\varepsilon > 0$, let $U(\varepsilon) = \bigcup_{i \neq j}(S_i F)_\varepsilon \cap (S_j F)_\varepsilon$ and $B^j(\varepsilon) = (S_j F)_\varepsilon \setminus U(\varepsilon)$. Then $F_\varepsilon = \bigcup_j B^j(\varepsilon) \cup U(\varepsilon)$ is a disjoint union and so

$$C_k(F_\varepsilon) = \sum_{j=1}^{N} C_k(F_\varepsilon, B^j(\varepsilon)) + C_k(F_\varepsilon, U(\varepsilon)).$$

Similarly,

$$C_k((S_j F)_\varepsilon) = C_k((S_j F)_\varepsilon, B^j(\varepsilon)) + C_k((S_j F)_\varepsilon, U(\varepsilon)),$$

since $B^j(\varepsilon) \cap (S_i F)_\varepsilon = \emptyset$ for $j \neq i$. Thus the function R_k can be written as

$$R_k(\varepsilon) = \sum_{j=1}^{N} \left(C_k(F_\varepsilon, B^j(\varepsilon)) - C_k((S_j F)_\varepsilon, B^j(\varepsilon)) \right) + C_k(F_\varepsilon, U(\varepsilon)) - \sum_{j=1}^{N} C_k((S_j F)_\varepsilon, U(\varepsilon)).$$

Observe that the set $A^j(\varepsilon) = (\bigcup_{i \neq j}(S_i F)_\varepsilon)^c$ is open and that $F_\varepsilon \cap A^j(\varepsilon) = (S_j F)_\varepsilon \cap A^j(\varepsilon)$. Since $B^j(\varepsilon) \subseteq A^j(\varepsilon)$, the locality property of C_k implies

$$C_k(F_\varepsilon, B^j(\varepsilon)) = C_k((S_j F)_\varepsilon, B^j(\varepsilon)).$$

Hence all terms of the first sum on the right hand side equal zero and can be omitted. Taking absolute values, we infer that

$$|R_k(\varepsilon)| \leq |C_k(F_\varepsilon, U(\varepsilon))| + \sum_{j=1}^{N} |C_k((S_j F)_\varepsilon, U(\varepsilon))|. \tag{5.2.3}$$

For the first term on the right hand side we claim that $U(\varepsilon) \subseteq ((\mathbf{SO})^c)_\varepsilon$ and conclude from Corollary 5.2.2 the existence of $c > 0$ such that for all $\varepsilon > 0$,

$$|C_k(F_\varepsilon, U(\varepsilon))| \leq C_k^{\mathrm{var}}(F_\varepsilon, U(\varepsilon)) \leq c\varepsilon^{k-s+\gamma}. \tag{5.2.4}$$

To prove $U(\varepsilon) \subseteq ((\mathbf{SO})^c)_\varepsilon$, let $x \in U(\varepsilon)$. We show that $d(x, (\mathbf{SO})^c) \leq \varepsilon$ and thus $x \in ((\mathbf{SO})^c)_\varepsilon$. Assume $d(x, (\mathbf{SO})^c) > \varepsilon$. Since the union $\mathbf{SO} = \bigcup_i S_i O$ is disjoint, there is a unique j such that $x \in S_j O$. Moreover, $d(x, \partial S_j O) > \varepsilon$. Since $x \in U(\varepsilon)$, there is at least one index $i \neq j$ such that $x \in (S_i F)_\varepsilon$ and consequently a point $y \in S_i F$ with $d(x, y) \leq \varepsilon$. But then $y \in S_i F \cap S_j O$, a contradiction to OSC. Hence, $d(x, (\mathbf{SO})^c) \leq \varepsilon$.

For the remaining terms in (5.2.3) observe that for each j,

$$|C_k((S_j F)_\varepsilon, U(\varepsilon))| = r_j^k |C_k(F_{\varepsilon/r_j}, S_j^{-1} U(\varepsilon))| \leq r_j^k C_k^{\mathrm{var}}(F_{\varepsilon/r_j}, S_j^{-1} U(\varepsilon)).$$

We show that $S_j^{-1} U(\varepsilon) \cap F_{\varepsilon/r_j} \subseteq (O^c)_{\varepsilon/r_j}$. Let $x \in S_j^{-1} U(\varepsilon) \cap F_{\varepsilon/r_j}$. Then $S_j x \in U(\varepsilon)$ and so there exists at least one index $i \neq j$ with $S_j x \in (S_i F)_\varepsilon$. Hence $d(S_j x, \partial S_j O) \leq \varepsilon$ since otherwise there would exist a point $y \in S_i F \cap S_j O$, a contradiction to OSC. Therefore, $d(x, \partial O) \leq \varepsilon/r_j$, i.e. $x \in (O^c)_{\varepsilon/r_j}$.

By the set inclusion just proved and Corollary 5.2.2, there exists a constant $c > 0$ such that for all ε, $C_k^{\mathrm{var}}(F_{\varepsilon/r_j}, S_j^{-1} U(\varepsilon))$ is bounded from above by $c(\varepsilon/r_j)^{k-s+\gamma}$ and thus

$$|C_k((S_j F)_\varepsilon, U(\varepsilon))| \leq c_j \varepsilon^{k-s+\gamma}$$

where $c_j := c r_j^{s-\gamma}$.

Since each of the terms in (5.2.3) is bounded from above by $c\varepsilon^{k-s+\gamma}$ for some constant $c > 0$, we can also find such a constant for $|R_k(\varepsilon)|$ and so the assertion follows. ∎

The last missing ingredient for the application of Theorem 4.1.4 is a statement on the continuity properties of R_k. We require that R_k is continuous except on a discrete set, i.e. the discontinuities are well separated from each other and do not accumulate inside the domain $(0, \infty)$ of R_k.

LEMMA 5.2.4. *The function R_k has a discrete set of discontinuities in $(0, \infty)$.*

Proof. From Lemma 3.1.1 it is easily seen that $C_k(F_\varepsilon)$ and $C_k((S_i F)_\varepsilon)$ have this property, since they have at most finitely many discontinuities in each interval $[\varepsilon_0, \infty)$, $\varepsilon_0 > 0$. Thus R_k has at most finitely many discontinuities in each interval $[\varepsilon_0, \infty)$ and the assertion follows. ∎

Note that the discontinuities of R_k possibly accumulate at 0.

Proof of Theorem 2.3.6. Since

$$C_k((S_i F)_\varepsilon) = r_i^k C_k(F_{\varepsilon/r_i}),$$

the functions $f(\varepsilon) := C_k(F_\varepsilon)$ and $\varphi_k(\varepsilon) := R_k(\varepsilon)$ satisfy a renewal equation

$$\varphi_k(\varepsilon) = f(\varepsilon) - \sum_{i=1}^{N} r_i^k \mathbf{1}_{(0, r_i]}(\varepsilon) f(\varepsilon/r_i)$$

as in (4.1.6). Now Lemmas 5.2.3 and 5.2.4 ensure that the hypotheses of Theorem 4.1.4 are satisfied. Therefore the average limit of the expression $\varepsilon^{s-k} C_k(F_\varepsilon)$ exists and in case of a non-arithmetic set F so also does the limit. ∎

To complete the proof of Theorem 2.3.6, it remains to verify the key estimate Lemma 5.2.1. This is done in the next two sections.

5.3. Convex representations of F_ε. In this section we translate the problem of proving the key estimate into the problem of estimating the cardinalities of certain families of level sets. To get control over the behaviour of the measure $C_k^{\mathrm{var}}(F_\varepsilon, \cdot)$ as $\varepsilon \to 0$, we decompose F_ε into convex sets for each $\varepsilon > 0$. The idea is to use small copies from a fixed collection of convex sets K^i for the decomposition, so that only the number of convex sets used in the decomposition increases as $\varepsilon \to 0$ while their curvatures are 'fixed' (up to scaling). This allows us to reduce the problem to estimating the number of sets involved in such representations.

First we fix the collection of convex sets that will be used. It is convenient to use a representation of F_ρ by convex sets, where ρ is the constant we defined in (5.1.8).

Decomposition of parallel sets. Let K^i, $i = 1, \ldots, P$, be convex sets such that

$$F_\rho = \bigcup_{i=1}^{P} K^i.$$

Note that for each $\varepsilon > \rho$ this provides a decomposition of F_ε into convex sets, namely into parallel sets of the K^i. If $\varepsilon = \rho + \delta$ then

$$F_\varepsilon = \bigcup_{i=1}^{P} K_\delta^i. \tag{5.3.1}$$

For $\varepsilon < \rho$ the decomposition is done in two steps. First we decompose F_ε into small copies of F_ρ:

$$F_\varepsilon = \bigcup_{w \in \Sigma(\varepsilon^*)} (S_w F)_\varepsilon.$$

The choice of w from the family $\Sigma(\varepsilon^*)$ ensures that $\rho < \varepsilon/r_w \le \rho r_{\min}^{-1}$ (cf. (5.1.4)) and so $(S_w F)_\varepsilon = S_w F_{\varepsilon/r_w} = S_w (F_\rho)_\delta$ for some $0 \le \delta < \delta_{\max}$ where $\delta_{\max} := \rho(r_{\min}^{-1} - 1)$. Hence $(S_w F)_\varepsilon$ has a representation by small copies of δ-parallel sets of K^i. For each $w \in \Sigma(\varepsilon^*)$,

$$(S_w F)_\varepsilon = \bigcup_{i=1}^{P} S_w K_\delta^i \quad \text{for some } 0 \le \delta < \delta_{\max}. \tag{5.3.2}$$

This also provides a representation of F_ε as a union of convex sets.

Intersection numbers. Now the first task is to investigate finite intersections of decomposition sets $(S_w F)_\varepsilon$. While for the existence of a representation (5.3.2) it was only important that ε is not too small compared to r_w ($r_w < \varepsilon^*$), the reverse relation that ε is also not too large compared to r_w ($\varepsilon^* \le r_w r_{\min}^{-1}$) will now be essential for the control of intersection numbers. Recall the definition of the intersection number of a finite family of sets from (3.1.4).

The first statement says that the intersection number of the family of sets $(S_w F)_\varepsilon$ with $w \in \Sigma(\varepsilon^*)$ is uniformly bounded (independent of ε).

LEMMA 5.3.1. *There exists a constant $\Gamma_{\max}$ such that for each $\varepsilon > 0$ and $r \ge \varepsilon^*$,*

$$\Gamma(\{(S_w F)_\varepsilon : w \in \Sigma(r)\}) \le \Gamma_{\max}.$$

Proof. Note that it suffices to prove the assertion for $r = \varepsilon^*$, since choosing for fixed $r > 0$ some $\varepsilon < \rho r$ (i.e. $r > \varepsilon^*$) does not increase the intersection number compared to the choice $\varepsilon = \rho r$. Fix $\varepsilon > 0$ and recall the definition of the word u from (5.1.6). First we show that

(i) *The sets* $(S_{wu}F)_\varepsilon, w \in \Sigma(\varepsilon^*)$, *are pairwise disjoint.* By (5.1.7), we have

$$d(x, \partial S_w O) > \alpha r_w \geq \alpha r_{\min} \varepsilon^* \geq \varepsilon$$

for each $w \in \Sigma(\varepsilon^*)$ and $x \in S_{wu}F$. This implies

$$(S_{wu}F)_\varepsilon \subseteq S_w O.$$

Since, by OSC, the sets $S_w O$, $w \in \Sigma(\varepsilon^*)$, are pairwise disjoint, assertion (i) follows.

Fix some $v \in \Sigma(\varepsilon^*)$ and a point $x \in S_{vu}F$. Let $\Gamma(v)$ denote the number of sequences $w \in \Sigma(\varepsilon^*)$ with $(S_w F)_\varepsilon \cap (S_v F)_\varepsilon \neq \emptyset$. Then the following is true:

(ii) *For each sequence w counted in $\Gamma(v)$, the set $(S_{wu}F)_\varepsilon$ is contained in the ball* $B(x, c\varepsilon)$ *where* $c := 2\rho^{-1}\mathrm{diam}\, F + 3$. Note that c is independent of v or ε. Let $y \in (S_{wu}F)_\varepsilon$. Since $(S_w F)_\varepsilon \cap (S_v F)_\varepsilon \neq \emptyset$, there is a point x' in this intersection. Therefore

$$d(x', y) \leq \mathrm{diam}\,(S_w F)_\varepsilon = r_w \mathrm{diam}\, F + 2\varepsilon \leq \varepsilon^* \mathrm{diam}\, F + 2\varepsilon = (\rho^{-1}\mathrm{diam}\, F + 2)\varepsilon,$$

since $w \in \Sigma(\varepsilon^*)$, and similarly

$$d(x, x') \leq \mathrm{diam}\,(S_v F) + \varepsilon \leq (\rho^{-1}\mathrm{diam}\, F + 1)\varepsilon.$$

Thus $d(x, y) \leq (2\rho^{-1}\mathrm{diam}\, F + 3)\varepsilon = c\varepsilon$ for all $y \in (S_{wu}F)_\varepsilon$, and so $(S_{wu}F)_\varepsilon \subseteq B(x, c\varepsilon)$ as stated in (ii).

Observing that each ε-parallel set contains an ε-ball and has thus Lebesgue measure at least $\kappa_d \varepsilon^d$, where κ_j denotes the volume of the j-dimensional unit ball in $\mathbb{R}^j$, and taking into account (i) and (ii), we obtain

$$\Gamma(v)\kappa_d \varepsilon^d \leq \lambda_d(B(x, c\varepsilon)) = \kappa_d(c\varepsilon)^d.$$

Hence $\Gamma(v) \leq c^d =: \Gamma_{\max}$, where $\Gamma_{\max}$ is independent of ε, as desired. ∎

Curvature of intersections of level sets. Lemma 5.3.1 allows us to bound the variation measures of arbitrary intersections of sets $(S_w F)_\varepsilon$ with $w \in \Sigma(\varepsilon^*)$.

LEMMA 5.3.2. *There is a constant $c > 0$ such that for all $\varepsilon > 0$ and all Borel sets $B \subseteq \mathbb{R}^d$,*

$$C_k^{\mathrm{var}}((S_{w(1)}F)_\varepsilon \cap \ldots \cap (S_{w(m)}F)_\varepsilon, B) \leq c\varepsilon^k$$

whenever $m \in \mathbb{N}$ and $w(1), \ldots, w(m) \in \Sigma(\varepsilon^*)$.

Proof. It suffices to show that the total masses $C_k^{\mathrm{var}}((S_{w(1)}F)_\varepsilon \cap \ldots \cap (S_{w(m)}F)_\varepsilon)$ satisfy the inequality for some $c > 0$. Moreover, we can assume $m \leq \Gamma_{\max}$, since, by Lemma 5.3.1, $(S_{w(1)}F)_\varepsilon \cap \ldots \cap (S_{w(m)}F)_\varepsilon = \emptyset$ for $m > \Gamma_{\max}$.

Applying Lemma 3.1.6 to the sets $X^j := (S_{w(j)}F)_\varepsilon$, which have representations (5.3.2) by P convex sets $K^{j,i} := S_{w(j)}K^i_{\delta(j)}$ for some $\delta(j)$ with $0 \leq \delta(j) < \delta_{\max}$, we obtain for the set $X := \bigcup_{j=1}^m X^j$,

$$C_k^{\mathrm{var}}(X) \leq (2^{(P^m)} - 1) \max_{j,i} C_k(K^{j,i}). \tag{5.3.3}$$

Observe now that $K^i_{\delta(j)} \subset K^i_{\delta_{\max}}$ and so the monotonicity of C_k for convex sets and the scaling property imply

$$C_k(K^{j,i}) = r^k_{w(j)} C_k(K^i_{\delta(j)}) \leq r^k_{w(j)} C_k(K^i_{\delta_{\max}}).$$

Since $w(j) \in \Sigma(\varepsilon^*)$, $r_{w(j)} \leq \varepsilon^*$ and so

$$C_k(K^{j,i}) \leq \rho^{-k} \varepsilon^k C_k(K^i_{\delta_{\max}}).$$

Since the right hand side does not depend on j, the maximum in (5.3.3) is bounded from above by $\rho^{-k} \varepsilon^k \max_i C_k(K^i_{\delta_{\max}})$. Noting that $m \leq \Gamma_{\max}$ it follows that the asserted inequality is satisfied for the constant $c := (2^{(P^{\Gamma_{\max}})} - 1) \rho^{-k} \max_i C_k(K^i_{\delta_{\max}})$, which depends neither on ε nor on m nor on the choice of the sequences $w(j)$. This completes the proof. ∎

Curvature estimates via cardinalities. Using the above estimate for finite intersections of level sets and the intersection number $\Gamma_{\max}$ we can now reduce the task of estimating $C^{\mathrm{var}}_k(F_\varepsilon, \cdot)$ to the problem of determining the cardinalities of certain families of level sets. For a closed set $B \subseteq \mathbb{R}^d$ and $\varepsilon > 0$, let

$$\Sigma(B, \varepsilon) = \{w \in \Sigma(\varepsilon^*) : (S_w F)_\varepsilon \cap B \neq \emptyset\}. \tag{5.3.4}$$

LEMMA 5.3.3. *There is a constant $c' > 0$ such that for all closed sets $B \subseteq \mathbb{R}^d$ and all $\varepsilon > 0$,*

$$C^{\mathrm{var}}_k(F_\varepsilon, B) \leq c' \, \#\Sigma(B, \varepsilon) \varepsilon^k.$$

Proof. Fix $\varepsilon > 0$. Observe that each $(S_w F)_\varepsilon$ with $w \in \Sigma(\varepsilon^*)$ which does not intersect B has some positive distance to B. Hence there is an open set A containing B such that $F_\varepsilon \cap A = \bigcup_w (S_w F)_\varepsilon \cap A$ where the union is taken over all $w \in \Sigma(B, \varepsilon)$. Hence the locality of the curvature measure implies

$$C^{\mathrm{var}}_k(F_\varepsilon, B) = C^{\mathrm{var}}_k\Big(\bigcup_{w \in \Sigma(B,\varepsilon)} (S_w F)_\varepsilon, B \Big).$$

The union on the right hand side consists of $m = \#\Sigma(B, \varepsilon)$ polyconvex sets. It satisfies the conditions of Corollary 3.1.5, since, by Lemma 5.3.2, there exist upper bounds $b := c\varepsilon^k$ (ε is fixed) for the variation measures with respect to finite intersections. Moreover, by Lemma 5.3.1, $\Gamma_{\max}$ is an upper bound for the intersection number of the family $\{(S_w F)_\varepsilon : w \in \Sigma(B, \varepsilon)\}$. Thus, by Corollary 3.1.5, the assertion holds for the constant $c' = 2^{\Gamma_{\max}} c$, where c is the constant of Lemma 5.3.2. ∎

5.4. Cardinalities of level set families. In view of Lemma 5.3.3 it is evident that in order to prove Lemma 5.2.1 we require upper bounds for the cardinalities of the families $\Sigma(B, \varepsilon)$ for the sets $B = (O(r)^c)_\delta$. Such bounds will be discussed now. Note that in this section no curvature is involved. Therefore, here the assumption of polyconvex parallel sets for F is not required. The main result of this section is the following:

LEMMA 5.4.1. *For each $r > 0$, there is a constant $c > 0$ such that for all $0 < \varepsilon \leq \delta \leq \rho r$,*

$$\#\Sigma\left((O(r)^c)_\delta, \varepsilon\right) \leq c\varepsilon^{-s}\delta^\gamma.$$

Again the constant γ is as defined in (5.1.9). Note that the above estimate remains valid with the set $(O(r)^c)_\delta$ replaced by any of its subsets. Lemma 5.2.1 follows immediately.

Proof of Lemma 5.2.1. Combine Lemmas 5.3.3 and 5.4.1 and note that the constant c' in Lemma 5.3.3 is independent of the choice of the set B. ∎

Splitting the proof of Lemma 5.4.1. It remains to provide a proof of Lemma 5.4.1, which will be divided into several steps. For this purpose we introduce some more notation. We say that a word $v \in \Sigma^*$ *occurs* in a word $w \in \Sigma^*$, in symbols $v \subset w$, if there are words $w', w'' \in \Sigma^*$ such that $w = w'vw''$. We write $v \not\subset w$ if v does not occur in w.

Recall the definition of the word $u = u_1 \ldots u_p$ in (5.1.6). The main idea of the proof is that some level set $(S_w F)_\varepsilon$ for which u occurs in w lies sufficiently far away from the boundary of $O(r)$ and is thus not counted in the family $\#\Sigma\left((O(r)^c)_\delta, \varepsilon\right)$ if ε, δ, r are arranged appropriately. The problem then reduces to counting the number of words w in certain families such that u does not occur in w.

For $\varepsilon > 0$ let $\Xi(\varepsilon)$ be the family of all words $w \in \Sigma(\varepsilon)$ such that u does not occur in w, i.e.

$$\Xi(\varepsilon) = \{w \in \Sigma(\varepsilon) : u \not\subset w\}.$$

For $0 < \varepsilon \le \delta$ and $w \in \Sigma(\varepsilon)$ there exists a subword $w' \in \Sigma(\delta)$ such that $w = w'w''$ for some $w'' \in \Sigma^*$. So, for $0 < \varepsilon \le \delta$, let

$$\Omega(\varepsilon, \delta) = \{w \in \Sigma(\varepsilon) : w = w'w'', \, w' \in \Sigma(\delta), \, u \not\subset w'\}. \tag{5.4.1}$$

Similarly, for $0 < \varepsilon \le \delta \le r$, let

$$\Lambda(\varepsilon, \delta, r) = \{w \in \Sigma(\varepsilon) : w = w^0 w' w'', \, w^0 \in \Sigma(r), \, w^0 w' \in \Sigma(\delta), \, u \not\subset w'\}. \tag{5.4.2}$$

Then the following relations hold for the cardinalities of these families (recall that $\varepsilon^* = \rho^{-1}\varepsilon$):

    I. For all $\varepsilon^* \le \delta^* \le r$, $\#\Sigma\left((O(r)^c)_\delta, \varepsilon\right) \le \#\Lambda(\varepsilon^*, \delta^*, r).$

    II. For all $\varepsilon \le \delta \le r$, $\#\Lambda(\varepsilon, \delta, r) \le \#\Sigma(r)\#\Omega(\varepsilon/r, \delta/rr_{\min}).$

    III. For all $\varepsilon \le \delta$, $\#\Omega(\varepsilon, \delta) \le r_{\min}^{-s}(\varepsilon/\delta)^{-s}\#\Xi(\delta).$

    IV. There exist $c_1, \gamma > 0$ such that $\#\Xi(\varepsilon) \le c_1 \varepsilon^{\gamma-s}.$

Combining these estimates, Lemma 5.4.1 is easily derived. Fix $r > 0$. Combining II–IV, we derive, for $\varepsilon \le \delta \le r$,

$$\#\Lambda(\varepsilon, \delta, r) \le \#\Sigma(r)\#\Omega\left(\frac{\varepsilon}{r}, \frac{\delta}{rr_{\min}}\right) \le \#\Sigma(r)r_{\min}^{-s}\left(\frac{\varepsilon r_{\min}}{\delta}\right)^{-s}\#\Xi\left(\frac{\delta}{rr_{\min}}\right)$$
$$\le \#\Sigma(r)r_{\min}^{-2s}\varepsilon^{-s}\delta^s c_1 (rr_{\min})^{s-\gamma}\delta^{\gamma-s} \le c_2 \varepsilon^{-s}\delta^\gamma,$$

where the constant c_2 only depends on r ($c_2 = c_1 r_{\min}^{-s-\gamma}\#\Sigma(r)r^{s-\gamma}$). Applying this to the right hand side of I, we obtain, for $\varepsilon^* \le \delta^* \le r$,

$$\#\Sigma\left((O(r)^c)_\delta, \varepsilon\right) \le \#\Lambda(\varepsilon^*, \delta^*, r) \le c_2 (\rho^{-1}\varepsilon)^{-s}(\rho^{-1}\delta)^\gamma = c\varepsilon^{-s}\delta^\gamma$$

where $c = c_2 \rho^{s-\gamma}$ is independent of δ and ε. Hence we have derived a constant c satisfying the assertion of Lemma 5.4.1.

It remains to provide proofs of the four inequalities I–IV.

Proof of I. Let $0 < \varepsilon^* \leq \delta^* \leq r$ and $w \in \Sigma(\varepsilon^*) \setminus \Lambda(\varepsilon^*, \delta^*, r)$, i.e $w \in \Sigma(\varepsilon^*)$ and $w = w^0 w' w''$ such that $w^0 \in \Sigma(r)$ and $w^0 w' \in \Sigma(\delta^*)$ but $u \subset w'$. We show that this implies $(S_w F)_\varepsilon \cap (O(r)^c)_\delta = \emptyset$ and thus $w \notin \Sigma((O(r)^c)_\delta, \varepsilon)$, proving the assertion.

Let $x \in (S_w F)_\varepsilon$. There exists $y \in S_w F$ with $d(x, y) \leq \varepsilon$. The assumption $u \subset w'$ implies that there exist $u', u'' \in \Sigma^*$ such that $w = u' u u''$. By definition of u, the set inclusions

$$S_w F \subseteq S_{w^0 u' u} F \subset S_{w^0 u'} O \subseteq S_{w^0} O \subseteq O(r)$$

hold and so $y \in S_w F$ is an interior point of the open set $O(r)$. We estimate its distance to the boundary and thus to the complement of $O(r)$. Taking into account (5.1.7) and $w^0 w' \in \Sigma(\delta^*)$, we infer

$$d(y, \partial O(r)) \geq d(y, \partial S_{w^0 u'} O) > r_{w^0 u'} \alpha \geq 2\rho r_{\min}^{-1} r_{w^0 w'} \geq 2\rho \delta^* = 2\delta.$$

Hence $d(x, O(r)^c) \geq d(y, O(r)^c) - d(x, y) > 2\delta - \varepsilon \geq \delta$, implying $x \notin (O(r)^c)_\delta$. ∎

Proof of II. From the definitions it is easily seen that

$$\#\Lambda(\varepsilon, \delta, r) = \sum_{w^0 \in \Sigma(r)} \#\Omega\left(\frac{\varepsilon}{r_{w^0}}, \frac{\delta}{r_{w^0}}\right).$$

Now observe that for fixed δ, $\#\Omega(\varepsilon, \delta)$ is a decreasing function of ε (provided $0 < \varepsilon \leq \delta$), and for fixed ε, $\#\Omega(\varepsilon, \delta)$ is increasing in δ (as long as $\varepsilon \leq \delta$). Therefore, in $\#\Omega(\varepsilon/r_{w^0}, \delta/r_{w^0})$, ε/r_{w^0} can be replaced by the smaller value ε/r, and independently δ/r_{w^0} by the larger value $\delta/r_{\min} r$ to provide the upper bound $\#\Omega(\varepsilon/r, \delta/rr_{\min})$, for each $w^0 \in \Sigma(r)$. ($w^0 \in \Sigma(r)$ implies $r_{w^0} < r \leq r_{w^0} r_{\min}^{-1}$ and so $1/r < 1/r_{w^0} \leq 1/rr_{\min}$.) Since the resulting upper bound is independent of $w^0 \in \Sigma(r)$, we conclude that $\#\Lambda(\varepsilon, \delta, r) \leq \#\Sigma(r) \#\Omega(\varepsilon/r, \delta/rr_{\min})$ as asserted above. ∎

Proof of III. Let $\varepsilon \leq \delta$. If $w = w' w'' \in \Omega(\varepsilon, \delta)$, then necessarily $w' \in \Xi(\delta)$ and $w'' \in \Sigma(\varepsilon/r_{w'})$. Therefore,

$$\#\Omega(\varepsilon, \delta) = \sum_{w' \in \Xi(\delta)} \#\Sigma\left(\frac{\varepsilon}{r_{w'}}\right).$$

Observing now that $\#\Sigma(\varepsilon)$ is a nonincreasing function of ε, we can replace $\varepsilon/r_{w'}$ by the smaller value ε/δ ($w' \in \Xi(\delta)$ implies $r_{w'} < \delta$) to see that each term in the above sum is bounded from above by $\#\Sigma(\varepsilon/\delta)$, which is independent of w' and can thus be taken out of the sum. Hence

$$\#\Omega(\varepsilon, \delta) \leq \#\Xi(\delta) \#\Sigma(\varepsilon/\delta).$$

Now we infer from (5.1.5), that $\#\Sigma(\varepsilon/\delta) \leq r_{\min}^{-s} (\varepsilon/\delta)^{-s}$, proving assertion III. ∎

Proof of IV. We have to show that the expression

$$\xi(\varepsilon) := \varepsilon^{\overline{s}} \#\Xi(\varepsilon)$$

is bounded as $\varepsilon \to 0$, where $\overline{s} = s - \gamma$ as in (5.1.9). Observe that $\#\Xi(\varepsilon)$ is a nonincreasing, nonnegative function of ε. Fix some $r > 0$ such that $u \in \Sigma(r)$. For $\varepsilon \leq r$, each word

$w \in \Xi(\varepsilon)$ begins with some word $v \in \Sigma(r)$ different from u and so

$$\#\Xi(\varepsilon) \leq \sum_{v \in \Sigma(r), \, v \neq u} \#\Xi\left(\frac{\varepsilon}{r_v}\right) \tag{5.4.3}$$

for all $\varepsilon \leq r$.

By (5.4.3), $\xi(\varepsilon)$ satisfies, for all $\varepsilon \leq r$,

$$\xi(\varepsilon) \leq \sum_{v \in \Sigma(r), \, v \neq u} r_v^{\overline{s}} \xi\left(\frac{\varepsilon}{r_v}\right),$$

and so, for all $\varepsilon' \leq r$,

$$\sup_{\varepsilon \geq \varepsilon'} \xi(\varepsilon) \leq \sup_{\varepsilon \geq \varepsilon'} \sum_{v \in \Sigma(r), \, v \neq u} r_v^{\overline{s}} \xi\left(\frac{\varepsilon}{r_v}\right) \leq \sum_{v \in \Sigma(r), \, v \neq u} r_v^{\overline{s}} \sup_{\varepsilon \geq \varepsilon'} \xi\left(\frac{\varepsilon}{r_v}\right) \leq \sup_{\varepsilon \geq r^{-1}\varepsilon'} \xi(\varepsilon).$$

Since $\#\Xi(\varepsilon)$ and thus $\xi(\varepsilon)$ are bounded on each interval $[\varepsilon', \infty)$, by the above inequality, $\xi(\varepsilon)$ is bounded as $\varepsilon \to 0$. This completes the proof of IV. ∎

REMARK 5.4.2. Families of finite sequences similar to $\Xi(\varepsilon)$ have been studied by Steven P. Lalley in [18]. In the above proofs, especially in the proof of IV, we adopted some of his ideas.

5.5. Bounds for the total variations. In this section we prove Theorems 2.3.2 and 2.3.8, which provide upper and lower estimates, respectively, for the total mass $C_k^{\mathrm{var}}(F_\varepsilon)$ of the total variation measure of F_ε. While the proof of Theorem 2.3.2 is based on the key estimate of Lemma 5.2.1, for the one of Theorem 2.3.8 we only require appropriate decompositions of the parallel sets F_ε. Throughout the section we assume that $F_\varepsilon \in \mathcal{R}^d$.

More scaling functions. In analogy with the kth scaling function R_k we define functions R_k^+, R_k^- and R_k^{var} by

$$R_k^\bullet(\varepsilon) = C_k^\bullet(F_\varepsilon) - \sum_{i=1}^N \mathbf{1}_{(0,r_i]}(\varepsilon) C_k^\bullet((S_i F)_\varepsilon) \tag{5.5.1}$$

for $\bullet \in \{+, -, \mathrm{var}\}$ and each $\varepsilon > 0$. With similar arguments to the proof of Lemma 5.2.3, which provided an upper bound for $|R_k(\varepsilon)|$, we can show the following estimate.

LEMMA 5.5.1. *There are constants $c, \gamma > 0$ such that for all $\varepsilon \in (0, 1]$,*

$$|R_k^{\mathrm{var}}(\varepsilon)| \leq c\varepsilon^{k-s+\gamma}.$$

Corresponding estimates hold for R_k^+ and R_k^-. Based on this lemma, we will now prove the boundedness of the quantity $\varepsilon^{s-k} C_k^{\mathrm{var}}(F_\varepsilon)$.

Proof of Theorem 2.3.2. Since, by the scaling property, $C_k^{\mathrm{var}}((S_i F)_\varepsilon) = r_i^k C_k^{\mathrm{var}}(F_{\varepsilon/r_i})$, by multiplying with ε^{s-k} we derive from (5.5.1) that

$$\varepsilon^{s-k} C_k^{\mathrm{var}}(F_\varepsilon) = \sum_{i=1}^N r_i^s \mathbf{1}_{(0,r_i]}(\varepsilon) \left(\frac{\varepsilon}{r_i}\right)^{s-k} C_k^{\mathrm{var}}(F_{\varepsilon/r_i}) + \varepsilon^{s-k} R_k^{\mathrm{var}}(\varepsilon).$$

Define $g_k : (0, \infty) \to \mathbb{R}$ by setting $g_k(\varepsilon) = \varepsilon^{s-k} C_k^{\mathrm{var}}(F_\varepsilon)$ for $\varepsilon \in (0, 1]$ and $g_k(\varepsilon) = 0$ for $\varepsilon > 1$. Then, for each $\varepsilon \in (0, 1]$, the above equation can be rewritten as

$$g_k(\varepsilon) = \sum_{i=1}^{N} r_i^s g_k\left(\frac{\varepsilon}{r_i}\right) + \varepsilon^{s-k} R_k^{\mathrm{var}}(\varepsilon),$$

and so, by Lemma 5.5.1, the following inequality holds for some $c > 0$:

$$g_k(\varepsilon) \leq \sum_{i=1}^{N} r_i^s g_k\left(\frac{\varepsilon}{r_i}\right) + c\varepsilon^\gamma.$$

Note that this inequality is also trivially satisfied for all $\varepsilon > 1$. Hence for each $\varepsilon_0 > 0$,

$$\sup_{\varepsilon \in (\varepsilon_0, \varepsilon_0 r_{\max}^{-1}]} g_k(\varepsilon) \leq \sum_{i=1}^{N} r_i^s \sup_{\varepsilon \in (\varepsilon_0, \varepsilon_0 r_{\max}^{-1}]} g_k\left(\frac{\varepsilon}{r_i}\right) + \sup_{\varepsilon \in (\varepsilon_0, \varepsilon_0 r_{\max}^{-1}]} c\varepsilon^\gamma$$
$$\leq \sup_{\varepsilon \in (\varepsilon_0 r_{\max}^{-1}, \infty]} g_k(\varepsilon) + c(\varepsilon_0 r_{\max}^{-1})^\gamma$$

and so

$$\sup_{\varepsilon \in (\varepsilon_0, \infty)} g_k(\varepsilon) \leq \sup_{\varepsilon \in (\varepsilon_0 r_{\max}^{-1}, \infty)} g_k(\varepsilon) + c r_{\max}^{-\gamma} \varepsilon_0^\gamma.$$

Iterating this inequality, i.e. applying it repeatedly to the first term on the right hand side, after n steps we arrive at

$$\sup_{\varepsilon \in (\varepsilon_0, \infty)} g_k(\varepsilon) \leq \sup_{\varepsilon \in (\varepsilon_0 r_{\max}^{-n}, \infty)} g_k(\varepsilon) + c \sum_{j=1}^{n} (r_{\max}^\gamma)^{-j} \varepsilon_0^\gamma.$$

Now set $\varepsilon_0 = r_{\max}^n$. Then the first term on the right hand side, $\sup_{\varepsilon \in (1, \infty)} g_k(\varepsilon)$, equals zero and so for each $n \in \mathbb{N}$,

$$\sup_{\varepsilon \in (r_{\max}^n, \infty)} g_k(\varepsilon) \leq c \sum_{j=1}^{n} (r_{\max}^\gamma)^{n-j} = c \sum_{j=1}^{n} (r_{\max}^\gamma)^j.$$

Letting $n \to \infty$, the right hand side converges to $M := c/(1 - r_{\max}^\gamma)$ and so $\sup_{\varepsilon \in (0, \infty)} g_k(\varepsilon) \leq M$. Hence the expression $\varepsilon^{s-k} C_k^{\mathrm{var}}(F_\varepsilon)$ is uniformly bounded by M in the interval $(0, 1]$, as asserted in Theorem 2.3.2. ∎

Proof of Theorem 2.3.8. Fix some $k \in \{0, \ldots, d\}$. Let B be a set as in the hypotheses of Theorem 2.3.8, i.e. assume that there are constants $\varepsilon_0, \beta > 0$ such that $B \subseteq O_{-\varepsilon_0}$ and $C_k^{\mathrm{var}}(F_\varepsilon, B) \geq \beta$ for $\varepsilon \in (r_{\min}\varepsilon_0, \varepsilon_0]$.

Since for each $r > 0$ the sets $S_w O$, $w \in \Sigma(r)$, are pairwise disjoint, the same holds for their subsets $S_w B$, and so for arbitrary $\varepsilon > 0$,

$$C_k^{\mathrm{var}}(F_\varepsilon) \geq \sum_{w \in \Sigma(r)} C_k^{\mathrm{var}}(F_\varepsilon, S_w B).$$

Fix some $\varepsilon < \varepsilon_0$ and choose $r = r_{\min}^{-1} \varepsilon_0^{-1} \varepsilon$. Then, for each $w \in \Sigma(r)$, $r_w < r_{\min}^{-1} \varepsilon_0^{-1} \varepsilon \leq r_w r_{\min}^{-1}$, i.e. in particular $\varepsilon \leq r_w \varepsilon_0$, and so, since $B \subseteq O_{-\varepsilon_0}$,

$$S_w B \subseteq S_w O_{-\varepsilon_0} = (S_w O)_{-r_w \varepsilon_0} \subseteq (S_w O)_{-\varepsilon}.$$

Hence, by the locality property of C_k^{var} in the open set $(S_w O)_{-\varepsilon}$ (where $F_\varepsilon \cap (S_w O)_{-\varepsilon} = (S_w F)_\varepsilon \cap (S_w O)_{-\varepsilon}$) and the scaling property,

$$C_k^{\mathrm{var}}(F_\varepsilon, S_w B) = C_k^{\mathrm{var}}(S_w(F_{\varepsilon r_w^{-1}}), S_w B) = r_w^k C_k^{\mathrm{var}}(F_{\varepsilon r_w^{-1}}, B).$$

Since $\varepsilon r_w^{-1} \in (r_{\min}\varepsilon_0, \varepsilon_0]$, the hypothesis implies that $C_k^{\mathrm{var}}(F_{\varepsilon r_w^{-1}}, B) \geq \beta$ and therefore,

$$C_k^{\mathrm{var}}(F_\varepsilon) \geq \sum_{w \in \Sigma(r)} r_w^k C_k^{\mathrm{var}}(F_{\varepsilon r_w^{-1}}, B) \geq \sum_{w \in \Sigma(r)} (\varepsilon_0^{-1}\varepsilon)^k \beta = \beta \varepsilon_0^{-k} \varepsilon^k \#\Sigma(r).$$

Recalling from (5.1.5) that $\#\Sigma(r) \geq r^{-s} = r_{\min}^s \varepsilon_0^s \varepsilon^{-s}$, we obtain

$$C_k^{\mathrm{var}}(F_\varepsilon) \geq \beta r_{\min}^s \varepsilon_0^{s-k} \varepsilon^{-s+k} = c\varepsilon^{-s+k}.$$

Since $\varepsilon < \varepsilon_0$ was arbitrary, the assertion of Theorem 2.3.8 follows immediately.

REMARK 5.5.2. In view of Proposition 3.1.2 and in complete analogy with Lemma 5.2.4, it is easily seen that the functions $R_k^\bullet$ have as well discrete sets of discontinuities in $(0, \infty)$. This, combined with Lemma 5.5.1, allows one to apply the Renewal Theorem directly to the functions $f^\bullet(\varepsilon) := C_k^\bullet(F_\varepsilon)$ and $\varphi_k^\bullet(\varepsilon) := R_k^\bullet(\varepsilon)$ and so a statement similar to Theorem 2.3.6 is obtained on the limiting behaviour of the total masses $C_k^\bullet(F_\varepsilon)$ of the variation measures. Setting

$$X_k^\bullet = \frac{1}{\eta} \int_0^1 \varepsilon^{s-k-1} R_k^\bullet(\varepsilon)\, d\varepsilon \tag{5.5.2}$$

for $\bullet \in \{+, -, \mathrm{var}\}$, where $\eta = -\sum_{i=1}^N r_i^s \ln r_i$ (cf. (2.3.3)), the following holds for each self-similar set F satisfying OSC and $F_\varepsilon \in \mathcal{R}^d$:

The limit

$$\lim_{\delta \to 0} \frac{1}{|\ln \delta|} \int_\delta^1 \varepsilon^{s-k} C_k^\bullet(F_\varepsilon)\, \frac{d\varepsilon}{\varepsilon}$$

exists and equals the finite number $X_k^\bullet$. If F is non-arithmetic, then also the limit $\lim_{\varepsilon \to 0} \varepsilon^{s-k} C_k^\bullet(F_\varepsilon)$ exists and equals $X_k^\bullet$.

Note that

$$X_k = X_k^+ - X_k^- \quad \text{and} \quad X_k^{\mathrm{var}} = X_k^+ + X_k^-. \tag{5.5.3}$$

Hence, in case $s_k = s - k$, the values X_k^+ and X_k^- are something like the positive and negative part of the kth fractal curvature. This also allows the results on fractal curvature measures to be clarified. The rescaled variation measures converge weakly in a similar way as stated in Theorem 2.5.1 for the rescaled curvature measures. However, in general the limit measures of variation measures are not the variation measures of the corresponding fractal curvature measure. The latter is always the difference of the limit measures of the positive and negative variation measures, but except for the case when one of the limit measures is the zero measure, they do not live on disjoint sets.

5.6. Proof of Gatzouras's theorem. At the end of Section 2.3 we discussed Gatzouras's results on Minkowski measurability, which we want to prove now. In the proof we will use the results we have already obtained, in particular Theorems 2.3.6 and 2.3.8. Going again through the proofs of these theorems for $k = d$, we will check that, when

replacing $C_d(F_\varepsilon, \cdot)$ with $\lambda_d(F_\varepsilon \cap \cdot)$, most arguments remain valid even if the assumption $F_\varepsilon \in \mathcal{R}^d$ is dropped. Only few arguments have to be modified.

Let F be a self-similar set satisfying OSC. We emphasize that now we do not assume the parallel sets F_ε to be polyconvex. The main step towards a proof of Gatzouras's theorem is a generalization of Lemma 5.3.3. Recall the definition of the family $\Sigma(B, \varepsilon)$ from (5.3.4).

LEMMA 5.6.1. *There is a constant $c > 0$ such that for each closed sets $B \subseteq \mathbb{R}^d$ and all $\varepsilon > 0$,*
$$\lambda_d(F_\varepsilon \cap B) \leq c \, \#\Sigma(B, \varepsilon) \, \varepsilon^d.$$

Proof. For $\varepsilon > 0$, the set inclusion $F_\varepsilon \cap B \subseteq \bigcup_{w \in \Sigma(B, \varepsilon)}(S_w F)_\varepsilon$ implies that

$$\lambda_d(F_\varepsilon \cap B) \leq \lambda_d \Big(\bigcup_{w \in \Sigma(B, \varepsilon)} (S_w F)_\varepsilon \Big) \leq \sum_{w \in \Sigma(B, \varepsilon)} \lambda_d((S_w F)_\varepsilon) \leq \sum_{w \in \Sigma(B, \varepsilon)} r_w^d \lambda_d(F_{\varepsilon/r_w})$$

$$\leq \sum_{w \in \Sigma(B, \varepsilon)} (\rho^{-1}\varepsilon)^d \lambda_d(F_{\rho r_{\min}^{-1}}) = \#\Sigma(B, \varepsilon)\rho^{-d}\lambda_d(F_{\rho r_{\min}^{-1}})\varepsilon^d,$$

where the third inequality is due to the scaling property and the last one to the fact that $r_w < \rho^{-1}\varepsilon \leq r_w r_{\min}^{-1}$ for each $w \in \Sigma(B, \varepsilon)$. Therefore the constant $c := \rho^{-d}\lambda_d(F_{\rho r_{\min}^{-1}})$ satisfies the assertion. ∎

Since in Lemma 5.4.1 (as in the whole Section 5.4) we did not assume F to have polyconvex parallel sets, we immediately obtain a generalization of the key estimate of Lemma 5.2.1, by combining Lemma 5.4.1 and the just derived Lemma 5.6.1.

LEMMA 5.6.2. *For each $r > 0$, there exists $c > 0$ such that for all $\varepsilon \leq \delta \leq \rho r$,*
$$\lambda_d(F_\varepsilon \cap (O(r)^c)_\delta) \leq c\varepsilon^{d-s}\delta^\gamma.$$

Note that this estimate will also be the key to the proof of Theorem 2.5.4, the localized version of Gatzouras's theorem. Here, by setting $r = 1$ and $\delta = \varepsilon$, we immediately derive an analogue of Corollary 5.2.2.

COROLLARY 5.6.3. *There exist some constant $c > 0$ such that for all $0 < \varepsilon \leq 1$,*
$$\lambda_d(F_\varepsilon, ((\mathbf{S}O)^c)_\varepsilon) \leq c\varepsilon^{d-s+\gamma}.$$

It follows at once that Lemma 5.2.3 generalizes in a similar way. Just note that for the Lebesgue measure the locality property trivially holds for arbitrary Borel sets $K, L, A \subseteq \mathbb{R}^d$, i.e. if $A \cap K = A \cap L$, then $\lambda_d(K \cap B) = \lambda_d(L \cap B)$ for all Borel sets $B \subseteq A$. Recall from (2.3.5) that in the general case the scaling function R_d was given by

$$R_d(\varepsilon) = \lambda_d(F_\varepsilon) - \sum_{i=1}^N \mathbf{1}_{(0, r_i]}(\varepsilon)\lambda_d((S_i F)_\varepsilon).$$

LEMMA 5.6.4. *There exists $c > 0$ such that for all $0 < \varepsilon \leq 1$,*
$$|R_d(\varepsilon)| \leq c\varepsilon^{d-s+\gamma}.$$

Since $\lambda_d(F_\varepsilon)$ is continuous in ε for each closed set F, the function $R_d(\varepsilon)$ has at most finitely many discontinuities (at the points r_i). Therefore the Renewal Theorem 4.1.4

applies. Taking into account that $s_d = s - d$, it follows that $\overline{M}(F) = X_d$ and for non-arithmetic sets F also $M(F) = X_d$.

It remains to show $X_d > 0$. For this observe that for $k = d$ Theorem 2.3.8 generalizes as follows when the assumption $F_\varepsilon \in \mathcal{R}^d$ is dropped.

PROPOSITION 5.6.5. *Let F be a self-similar set satisfying OSC and O some feasible open set of F. Suppose there exist some constants $\varepsilon_0, \beta > 0$ and some Borel set $B \subset O_{-\varepsilon_0}$ such that*

$$\lambda_d(F_\varepsilon \cap B) \geq \beta$$

for each $\varepsilon \in (r_{\min}\varepsilon_0, \varepsilon_0]$. Then for all $\varepsilon < \varepsilon_0$,

$$\varepsilon^{s-d}\lambda_d(F_\varepsilon) \geq c,$$

where $c := \beta\varepsilon_0^{s-d}r_{\min}^s > 0$.

Proof. Observe that the arguments in the proof of Theorem 2.3.8 remain valid in the general case with $C_d^{\mathrm{var}}(F_\varepsilon, \cdot) = C_d(F_\varepsilon, \cdot)$ replaced by $\lambda_d(F_\varepsilon \cap \cdot)$. ∎

It should be noted that the bound $\varepsilon^{s-d}\lambda_d(F_\varepsilon) \geq c > 0$ in Proposition 5.6.5 immediately implies $X_d > 0$. Therefore it suffices to show that there is always some set B satisfying the hypothesis of this proposition. Let O be some feasible open set of F such that the SOSC holds. Then there exists a point x in $F \cap O \neq \emptyset$ and, since O is open, some constant $\alpha' > 0$ such that $d(x, \partial O) > \alpha'$. Let $\varepsilon_0 = \alpha'/2$ and $B := B(x, \varepsilon_0)$. Then $B \subset O_{-\varepsilon_0}$, since for each $y \in B(x, \varepsilon_0)$,

$$d(y, \partial O) \geq d(x, \partial O) - d(x, y) > \alpha' - \alpha'/2 = \varepsilon_0.$$

Moreover, for each $\varepsilon \in (r_{\min}\varepsilon_0, \varepsilon_0]$,

$$\lambda_d(F_\varepsilon \cap B) \geq \lambda_d(B(x, r_{\min}\varepsilon_0)) = \kappa_d(r_{\min}\varepsilon_0)^d =: \beta.$$

Hence the hypothesis of Proposition 5.6.5 is satisfied and $X_d \geq c > 0$ follows. This completes the proof of Gatzouras's theorem.

6. Proofs: Fractal curvature measures

In this chapter we prove the results of Section 2.5 regarding weak limits of rescaled curvature measures. In the first section we construct a separating class which is adapted to the structure of F. This set family is the key to the proofs of all weak convergence results discussed here. It allows one to determine the limit measures uniquely by computing their values for the sets of the family. Most of Section 6.2 is dedicated to the proof of Theorem 2.5.1. In the end we outline briefly how the arguments in the proof of this theorem have to be adapted to obtain a proof of Theorem 2.5.2. In the last section we turn our attention to normalized curvature measures and prove Theorems 2.5.3 and 2.5.4. As before, F is some self-similar set satisfying OSC. The set O, the word u and the constants ρ and γ are as defined in Section 5.1.

6.1. A separating class for F. Let $\mathfrak{B}^d$ denote the Borel σ-algebra of $\mathbb{R}^d$. A family $\mathcal{A}$ of Borel sets is called a *separating class* if two measures that agree on $\mathcal{A}$ necessarily agree on $\mathfrak{B}^d$. We introduce a separating class $\mathcal{A}_F$, which is adapted to the structure of F. Define the set family

$$\mathcal{C}_F := \{C \in \mathfrak{B}^d : \exists r > 0 \text{ such that } C \subseteq O(r)^c\},$$

where $O(r)$ is as defined in (5.2.1), and let

$$\mathcal{A}_F := \{S_w O : w \in \Sigma^*\} \cup \mathcal{C}_F.$$

LEMMA 6.1.1. *$\mathcal{A}_F$ is an intersection stable generator of $\mathfrak{B}^d$.*

Proof. The stability of $\mathcal{A}_F$ with respect to intersections is easily seen. Either the intersection of two sets $S_v O$ and $S_w O$, $v, w \in \Sigma^*$, is empty or one of the sets is contained in the other. Moreover, any intersection $A \cap C$ of a set $C \in \mathcal{C}_F$ and a set $A \in \mathcal{A}_F$ is again an element of $\mathcal{C}_F$.

Since $\mathcal{A}_F$ consists of Borel sets, the σ-algebra $\sigma(\mathcal{A}_F)$ generated by $\mathcal{A}_F$ is contained in $\mathfrak{B}^d$. It remains to prove the reverse inclusion: $\mathfrak{B}^d \subseteq \sigma(\mathcal{A}_F)$. This is done by showing that each open set is a countable union of sets of $\mathcal{A}_F$.

Let B be an open set and $x \in B$. There exists some $r > 0$ such that $B(x, r) \subset B$. Set $l := (\operatorname{diam} O)^{-1}$ and let

$$\Sigma_x = \{w \in \Sigma(lr) : x \in \overline{S_w O}\}.$$

By definition of $\Sigma(lr)$, for all $w \in \Sigma_x$, $\operatorname{diam} S_w O = r_w \operatorname{diam} O \leq r$ and thus $S_w O \subset B(x, r)$.

For each $w \in \Sigma(lr) \setminus \Sigma_x$, $S_w O$ has some positive distance to x. Therefore we can find some positive constant c such that $d(x, \overline{S_w O}) > c$ for all $w \in \Sigma(lr) \setminus \Sigma_x$.

Let $C_x = B(x, c) \setminus \bigcup_{w \in \Sigma_x} S_w O$, which is obviously a subset of $O(lr)^c$ and thus an element of $\mathcal{C}_F$. Moreover, let $A_x = C_x \cup \bigcup_{w \in \Sigma_x} S_w O$. By construction, A_x is a finite union of sets from $\mathcal{A}_F$ and $A_x \subseteq B(x, r) \subset B$. On the other hand the family $\{A_x : x \in B\}$ covers the set B and thus $B = \bigcup_{x \in B} A_x$. Since $x \in \operatorname{int} A_x$, the family $\{\operatorname{int} A_x : x \in B\}$ forms an open cover of B, which by the Lindelöf Theorem has a countable open subcover, i.e. there are $x_1, x_2, \ldots$ in B such that $B = \bigcup_i \operatorname{int} A_{x_i}$. But then also $B = \bigcup_i A_{x_i}$ and so, since each A_{x_i} is a finite union of sets from $\mathcal{A}_F$, we obtain a representation of B as a countable union of sets from $\mathcal{A}_F$, as desired. This completes the proof. ∎

The properties derived in Lemma 6.1.1 are sufficient for $\mathcal{A}_F$ to be a separating class for the family of totally finite (signed) measures. We recall the uniqueness theorem which is well known for positive measures and easily generalized to signed measures.

THEOREM 6.1.2. *Let μ and ν be totally finite signed measures on $\mathfrak{B}^d$, and $\mathcal{A}$ an intersection stable generator of $\mathfrak{B}^d$ such that $\mu(A) = \nu(A)$ for all $A \in \mathcal{A}$. Then $\mu = \nu$.*

Although we cannot give a direct reference for this theorem for signed measures, we believe that it is well known. For a proof for positive measures see for instance Elstrodt [5, p. 60] or Jacobs [16, p. 51]. The arguments of the proof in [5] extend to signed measures, once one verifies that for any increasing sequence of sets A_n converging to A and any signed measure μ, $\mu(A_i) \to \mu(A)$ as $n \to \infty$.

By Lemma 6.1.1, the family $\mathcal{A}_F$ satisfies the hypotheses of Theorem 6.1.2 and is thus a separating class. $\mathcal{A}_F$ is constructed in such a way that the measures considered can easily be computed for the sets of this family.

In the proof of Theorem 2.5.1 we will have to show that certain measures coincide with some multiple of the measure μ_F defined in (2.5.3). Since by the above uniqueness theorem it is sufficient to compare the measures for sets $A \in \mathcal{A}_F$, we collect the values $\mu_F(A)$ for those sets. Recall that μ_F is the self-similar measure with weights $\{r_1^s, \ldots, r_N^s\}$, i.e. the unique probability measure satisfying the invariance relation

$$\mu_F = \sum_{i=1}^{N} r_i^s \mu_F \circ S_i^{-1}.$$

It is well known that, provided the OSC is satisfied, $\mu_F(O) = \mu_F(F)$ and similarly $\mu_F(S_w O) = \mu_F(S_w F)$ for each $w \in \Sigma^*$. Since $\mu_F(F) = 1$, the invariance relation implies $\mu_F(S_w F) = r_w^s$. Therefore, we record:

$$\mu_F(S_w O) = r_w^s \quad \text{for all } w \in \Sigma^*, \tag{6.1.1}$$

$$\mu_F(C) = 0 \quad \text{for all } C \in \mathcal{C}_F. \tag{6.1.2}$$

6.2. Proof of Theorems 2.5.1 and 2.5.2. First we prove Theorem 2.5.1, for which we assume that F has polyconvex parallel sets. Fix some $k \in \{0, \ldots, d\}$ and assume that the kth scaling exponent of F satisfies $s_k = s - k$. The first step is to deduce some technical estimates, which provide bounds for the curvature measure of F_ε in the sets $(S_w O)_\delta$ and $S_w O$ for $w \in \Sigma^*$. They are based on the key lemma (Lemma 5.2.1). The exponent γ which will occur in all the estimates below is the one we defined in (5.1.9).

LEMMA 6.2.1. *Let $w \in \Sigma^*$. There exists $c > 0$ such that for all $0 < \varepsilon \le \delta \le \rho r_w$ and $\bullet \in \{+, -, \mathrm{var}\}$,*

$$C_k^\bullet(F_\varepsilon, (S_w O)_\delta) \le r_w^k C_k^\bullet(F_{\varepsilon r_w^{-1}}) + c \varepsilon^{k-s} \delta^\gamma, \tag{6.2.1}$$

$$C_k^\bullet(F_\varepsilon, S_w O) \ge r_w^k C_k^\bullet(F_{\varepsilon r_w^{-1}}) - c \varepsilon^{k-s} \delta^\gamma. \tag{6.2.2}$$

Proof. Fix $w \in \Sigma^*$. Observe that $(S_w O)_\delta = (S_w O)_{-\delta} \cup (\partial S_w O)_\delta$. Provided that $\varepsilon \le \delta$, for the first set in this union, the locality of $C_k^\bullet$ (here in the open set $(S_w O)_{-\delta}$ we have $(S_w F)_\varepsilon \cap (S_w O)_{-\delta} = F_\varepsilon \cap (S_w O)_{-\delta}$) and the scaling property imply

$$C_k^\bullet(F_\varepsilon, (S_w O)_{-\delta}) = C_k^\bullet((S_w F)_\varepsilon, (S_w O)_{-\delta}) = r_w^k C_k^\bullet(F_{\varepsilon r_w^{-1}}, O_{-\delta r_w^{-1}})$$

$$\le r_w^k C_k^\bullet(F_{\varepsilon r_w^{-1}}). \tag{6.2.3}$$

Choose r such that $w \in \Sigma(r)$, i.e. $r_w < r \le r_w r_{\min}^{-1}$. Then the second set is a subset of $(O(r)^c)_\delta$, since $\partial S_w O \subseteq O(r)^c$. Therefore, by Lemma 5.2.1, there are constants $c, \gamma > 0$ such that

$$C_k^\bullet(F_\varepsilon, (\partial S_w O)_\delta) \le C_k^\bullet(F_\varepsilon, (O(r)^c)_\delta) \le c \varepsilon^{k-s} \delta^\gamma$$

for all $\varepsilon \le \delta \le \rho r$ (and so in particular for $\delta \le \rho r_w$). Hence the first estimate (6.2.1) follows immediately from the relation

$$C_k^\bullet(F_\varepsilon, (S_w O)_\delta) \le C_k^\bullet(F_\varepsilon, (S_w O)_{-\delta}) + C_k^\bullet(F_\varepsilon, (\partial S_w O)_\delta).$$

For the second estimate choose r such that $O = O(r)$, i.e. $1 < r \leq r_{\min}^{-1}$. Then, again by Lemma 5.2.1, there are $c', \gamma > 0$ such that for all $\varepsilon \leq \delta \leq \rho r$

$$C_k^\bullet(F_\varepsilon) \leq C_k^\bullet(F_\varepsilon, O_{-\delta}) + C_k^\bullet(F_\varepsilon, (O^c)_\delta) \leq C_k^\bullet(F_\varepsilon, O_{-\delta}) + c' \varepsilon^{k-s} \delta^\gamma.$$

Bringing $c' \varepsilon^{k-s} \delta^\gamma$ to the other side of the inequality and taking into account (6.2.3) we infer that for all $\varepsilon \leq \delta \leq \rho r$,

$$\begin{aligned} C_k^\bullet(F_\varepsilon, S_w O) &\geq C_k^\bullet(F_\varepsilon, (S_w O)_{-\delta}) = r_w^k C_k^\bullet(F_{\varepsilon r_w^{-1}}, O_{-\delta r_w^{-1}}) \\ &\geq r_w^k (C_k^\bullet(F_{\varepsilon r_w^{-1}}) - c'(\varepsilon r_w^{-1})^{k-s}(\delta r_w^{-1})^\gamma). \end{aligned}$$

Hence the estimate (6.2.2) holds for the constant $c = c' r_w^{s-\gamma}$ for all $\varepsilon \leq \delta \leq \rho r$ and thus in particular for $\varepsilon \leq \delta \leq \rho r_w$. For the maximum of the two constants c derived for the two estimates, both inequalities are satisfied, completing the proof of Lemma 6.2.1. ∎

For convenience we introduce the abbreviation

$$\nu(f) := \int_{\mathbb{R}^d} f \, d\nu \tag{6.2.4}$$

for the integral of a function f with respect to a (signed) measure ν. For $w \in \Sigma^*$ and $\delta > 0$, let $f_\delta^w : \mathbb{R}^d \to [0,1]$ be a continuous function such that

$$f_\delta^w(x) = \begin{cases} 1 & \text{for } x \in S_w O, \\ 0 & \text{for } x \text{ outside } (S_w O)_\delta. \end{cases}$$

For simplicity, assume that $f_\delta^w \leq f_{\delta'}^w$ for all $\delta < \delta'$. Obviously, f_δ^w has compact support and satisfies $\mathbf{1}_{S_w O} \leq f_\delta^w \leq \mathbf{1}_{(S_w O)_\delta}$. Moreover, as $\delta \to 0$, the functions f_δ^w converge (pointwise) to $\mathbf{1}_{S_w O}$, implying in particular the convergence of the integrals $\nu(f_\delta^w) \to \nu(\mathbf{1}_{S_w O}) = \nu(S_w O)$ with respect to any (signed) Radon measure ν.

Now recall the definition of the rescaled curvature measures $\nu_{k,\varepsilon}$ from (2.5.1). Using the above estimates, we derive some bounds for the integrals $\nu_{k,\varepsilon}(f_\delta^w)$.

LEMMA 6.2.2. *Let $w \in \Sigma^*$. Then for all $0 < \varepsilon \leq \delta \leq \rho r_w$,*

$$|\nu_{k,\varepsilon}(f_\delta^w) - r_w^s \nu_{k,\varepsilon r_w^{-1}}(\mathbb{R}^d)| \leq 2c\delta^\gamma,$$

where $c = c(w)$ is the constant in Lemma 6.2.1.

Proof. Fix $w \in \Sigma^*$. Since $\mathbf{1}_{S_w O} \leq f_w^\delta \leq \mathbf{1}_{(S_w O)_\delta}$, Lemma 6.2.1 implies that there exist $c, \gamma > 0$ such that for all $\varepsilon \leq \delta \leq \rho r$,

$$\nu_{k,\varepsilon}^\pm(f_\delta^w) \leq \nu_{k,\varepsilon}^\pm((S_w O)_\delta) \leq r_w^s \nu_{k,\varepsilon r_w^{-1}}^\pm(\mathbb{R}^d) + c\delta^\gamma \tag{6.2.5}$$

and similarly

$$\nu_{k,\varepsilon}^\pm(f_\delta^w) \geq \nu_{k,\varepsilon}^\pm(S_w O) \geq r_w^s \nu_{k,\varepsilon r_w^{-1}}^\pm(\mathbb{R}^d) - c\delta^\gamma. \tag{6.2.6}$$

Applying these inequalities to $\nu_{k,\varepsilon}(f_\delta^w) = \nu_{k,\varepsilon}^+(f_\delta^w) - \nu_{k,\varepsilon}^-(f_\delta^w)$, we find that on the one hand,

$$\nu_{k,\varepsilon}(f_\delta^w) \leq r_w^s \nu_{k,\varepsilon r_w^{-1}}^+(\mathbb{R}^d) + c\delta^\gamma - (r_w^s \nu_{k,\varepsilon r_w^{-1}}^-(\mathbb{R}^d) - c\delta^\gamma) = r_w^s \nu_{k,\varepsilon r_w^{-1}}(\mathbb{R}^d) + 2c\delta^\gamma,$$

and on the other hand,

$$\nu_{k,\varepsilon}(f_\delta^w) \geq r_w^s \nu_{k,\varepsilon r_w^{-1}}^+(\mathbb{R}^d) - c\delta^\gamma - (r_w^s \nu_{k,\varepsilon r_w^{-1}}^-(\mathbb{R}^d) + c\delta^\gamma) = r_w^s \nu_{k,\varepsilon r_w^{-1}}(\mathbb{R}^d) - 2c\delta^\gamma.$$

Combining both estimates the assertion of Lemma 6.2.2 immediately yields. ∎

Proof of the convergence $\nu_{k,\varepsilon} \xrightarrow{w} C_k^f(F)\mu_F$. Assume that F is non-arithmetic. The total masses of the variation measures $\nu_{k,\varepsilon}^+$ and $\nu_{k,\varepsilon}^-$ of $\nu_{k,\varepsilon}$ are uniformly bounded. Moreover, since for each $\varepsilon \leq 1$ the support of $\nu_{k,\varepsilon}^\pm$ is contained in the 1-parallel set of F, the families $\{\nu_{k,\varepsilon}^+\}_{\varepsilon \in (0,1]}$ and $\{\nu_{k,\varepsilon}^-\}_{\varepsilon \in (0,1]}$ are tight. Therefore, by Prokhorov's Theorem, they are relatively compact, i.e. every sequence has a weakly convergent subsequence. In particular, every null sequence has a subsequence $\{\varepsilon_n\}$ such that the measures ν_{k,ε_n}^+ converge weakly, and this subsequence has a further subsequence, again denoted $\{\varepsilon_n\}$, such that also the measures ν_{k,ε_n}^- converge weakly.

Now let $\{\varepsilon_n\}$ be such a sequence, i.e. assume that

$$\nu_{k,\varepsilon_n}^+ \xrightarrow{w} \nu_k^+ \quad \text{and} \quad \nu_{k,\varepsilon_n}^- \xrightarrow{w} \nu_k^- \quad \text{as } n \to \infty,$$

for some limit measures ν_k^+ and ν_k^-. The weak convergence of the variation measures $\nu_{k,\varepsilon_n}^\pm$ implies the weak convergence of the signed measures $\nu_{k,\varepsilon_n} = \nu_{k,\varepsilon_n}^+ - \nu_{k,\varepsilon_n}^-$ and the limit measure is given by $\nu_k = \nu_k^+ - \nu_k^-$. Note that the measures ν_k^+ and ν_k^- are not necessarily the positive and negative variations of ν_k. They are just some representation of ν_k as a difference of two positive measures and do not necessarily live on two disjoint sets. In general, the limit measures ν_k might depend on the chosen sequence $\{\varepsilon_n\}$. However, it is our aim to show that here this is not the case, i.e. we want to prove that for any such sequence $\{\varepsilon_n\}$, the limit measure ν_k is the same, namely that ν_k coincides with $\mu_k := C_k^f(F)\mu_F$, which implies at once that the weak limit $\nu_{k,\varepsilon}$ as $\varepsilon \to 0$ exists and coincides with μ_k, as stated in Theorem 2.5.1 for the non-arithmetic case.

In Section 6.1 we constructed the family $\mathcal{A}_F$ and showed that it is an intersection stable generator of $\mathcal{B}^d$. By Theorem 6.1.2, the measures ν_k and μ_k coincide if they coincide for all sets $A \in \mathcal{A}_F$. The measure μ_k is known. For sets $C \in \mathcal{C}_F$, by (6.1.2), $\mu_F(C) = 0$ and so $\mu_k(C) = C_k^f(F)\mu_F(C) = 0$. For sets $S_w O$, $w \in \Sigma^*$, by (6.1.1), $\mu_F(S_w O) = r_w^s$ and thus $\mu_k(S_w O) = C_k^f(F)r_w^s$.

Therefore, we have to show that for all $w \in \Sigma^*$,

$$\nu_k(S_w O) = C_k^f(F)r_w^s, \tag{6.2.7}$$

and for all $C \in \mathcal{C}_F$,

$$\nu_k(C) = 0. \tag{6.2.8}$$

Proof of (6.2.7). We approximate the measure of the sets $S_w O$ with the integrals of the functions f_δ^w defined in (6.2.4) and use Lemma 6.2.2. Fix $w \in \Sigma^*$ and let $r = r_w$. By Lemma 6.2.2, for all n and δ such that $\varepsilon_n \leq \delta \leq \rho r$ we have

$$|\nu_{k,\varepsilon_n}(f_\delta^w) - r_w^s \nu_{k,\varepsilon_n r_w^{-1}}(\mathbb{R}^d)| \leq 2c\delta^\gamma. \tag{6.2.9}$$

If we keep δ fixed and let $n \to \infty$, the weak convergence implies $\nu_{k,\varepsilon_n}(f_\delta^w) \to \nu_k(f_\delta^w)$, since f_δ^w is continuous. Moreover, $\nu_{k,\varepsilon_n r_w^{-1}}(\mathbb{R}^d) = (\varepsilon r_w^{-1})^{s-k}C_k(F_{\varepsilon r_w^{-1}}) \to C_k^f(F)$, by Theorem 2.3.6. Hence the above inequality yields

$$|\nu_k(f_\delta^w) - r_w^s C_k^f(F)| \leq 2c\delta^\gamma \tag{6.2.10}$$

for each $\delta \leq \rho r$. If we now let $\delta \to 0$, the integrals $\nu_k(f_\delta^w)$ converge to $\nu_k(\mathbf{1}_{S_w O}) = \nu_k(S_w O)$, while the right hand side of the inequality vanishes. Therefore, $|\nu_k(S_w O) - r_w^s C_k^f(F)| \leq 0$, which implies $\nu_k(S_w O) = r_w^s C_k^f(F)$, as claimed in (6.2.7). ∎

Proof of (6.2.8). Fix $r > 0$. It suffices to show $\nu_k^\pm(O(r)^c) = 0$, since this immediately implies that $\nu_k(C) = \nu_k^+(C) - \nu_k^-(C) = 0$ for all $C \subseteq O(r)^c$. As before, we approximate the indicator function of $O(r)^c$ by continuous functions. For $\delta > 0$, let $g_\delta : \mathbb{R}^d \to [0,1]$ be a continuous function such that

$$g_\delta(x) = \begin{cases} 1 & \text{for } x \in O(r)^c, \\ g_\delta(x) = 0 & \text{for } x \in (O(r))_{-\delta}. \end{cases} \tag{6.2.11}$$

Since $g_\delta \leq \mathbf{1}_{(O(r)^c)_\delta}$, by Lemma 5.2.1, for all $\varepsilon_n \leq \delta \leq \rho r$,

$$\nu_{k,\varepsilon_n}^\pm(g_\delta) \leq c\delta^\gamma.$$

If we keep δ fixed and let $n \to \infty$, the weak convergence implies that $\nu_{k,\varepsilon_n}^\pm(g_\delta) \to \nu_k^\pm(g_\delta)$ while the right hand side remains unchanged. If we now let $\delta \to 0$, the functions g_δ converge pointwise to $\mathbf{1}_{O(r)^c}$ and thus $\nu_k^\pm(g_\delta) \to \nu_k^\pm(O(r)^c)$, while $c\delta^\gamma$ vanishes. Hence $\nu_k^\pm(O(r)^c) = 0$, completing the proof of (6.2.8). ∎

This completes the proof of the weak convergence of $\nu_{k,\varepsilon}$ to μ_k as $\varepsilon \to 0$ for the non-arithmetic case.

Now we want to discuss the weak convergence of the averaged measures $\overline{\nu}_{k,\varepsilon}$ as defined in (2.5.2). For this purpose we first derive some estimate for the integrals $\overline{\nu}_{k,\varepsilon}(f_\delta^w)$, which is the analogue to Lemma 6.2.2 for the averaged measures $\overline{\nu}_{k,\varepsilon}$.

LEMMA 6.2.3. *Let $w \in \Sigma^*$. For all ε and δ such that $0 < \varepsilon \leq \delta \leq \rho r_w$ we have*

$$|\overline{\nu}_{k,\varepsilon}(f_\delta^w) - r_w^s \overline{\nu}_{k,\varepsilon r_w^{-1}}(\mathbb{R}^d)| \leq 2c\delta^\gamma + \frac{\ln \delta}{\ln \varepsilon} 2(c\delta^\gamma + M),$$

where $c = c(w)$ is the constant of Lemma 6.2.1 and M is the one of Lemma 2.3.2.

Proof. Fix $w \in \Sigma^*$. Since $f_w^\delta \leq \mathbf{1}_{(S_w O)_\delta}$,

$$\overline{\nu}_{k,\varepsilon}^\pm(f_\delta^w) \leq \overline{\nu}_{k,\varepsilon}^\pm((S_w O)_\delta) = \frac{1}{|\ln \varepsilon|} \int_\varepsilon^1 \tilde{\varepsilon}^{s-k} C_k^\pm(F_{\tilde{\varepsilon}}, (S_w O)_\delta) \frac{d\tilde{\varepsilon}}{\tilde{\varepsilon}}.$$

For $\varepsilon \leq \delta$, we split the integral into two parts, one over the interval $[\varepsilon, \delta]$ and one over $(\delta, 1)$, and apply the first inequality of Lemma 6.2.1 to the first part and Lemma 2.3.2 to the second part. Then for all $\varepsilon \leq \delta \leq \rho r_w$ we obtain

$$\overline{\nu}_{k,\varepsilon}^\pm(f_\delta^w) \leq \frac{1}{|\ln \varepsilon|} \int_\varepsilon^\delta \left(r_w^s \nu_{k,\tilde{\varepsilon} r_w^{-1}}^\pm(\mathbb{R}^d) + c\delta^\gamma \right) \frac{d\tilde{\varepsilon}}{\tilde{\varepsilon}} + \frac{1}{|\ln \varepsilon|} \int_\delta^1 M \frac{d\tilde{\varepsilon}}{\tilde{\varepsilon}}$$

$$\leq r_w^s \overline{\nu}_{k,\varepsilon r_w^{-1}}^\pm(\mathbb{R}^d) + \frac{c\delta^\gamma}{|\ln \varepsilon|} \int_\varepsilon^\delta \frac{d\tilde{\varepsilon}}{\tilde{\varepsilon}} + \frac{M}{|\ln \varepsilon|} \int_\delta^1 \frac{d\tilde{\varepsilon}}{\tilde{\varepsilon}}$$

$$\leq r_w^s \overline{\nu}_{k,\varepsilon r_w^{-1}}^\pm(\mathbb{R}^d) + c\delta^\gamma + \frac{\ln \delta}{\ln \varepsilon} (c\delta^\gamma + M). \tag{6.2.12}$$

In a similar way we derive the corresponding lower bounds. Since $\mathbf{1}_{S_w O} \leq f_w^\delta$, we have

$$\overline{\nu}_{k,\varepsilon}^\pm(f_\delta^w) \geq \overline{\nu}_{k,\varepsilon}^\pm(S_w O) \geq \frac{1}{|\ln \varepsilon|} \int_\varepsilon^\delta \tilde{\varepsilon}^{s-k} C_k^\pm(F_{\tilde{\varepsilon}}, S_w O) \frac{d\tilde{\varepsilon}}{\tilde{\varepsilon}}.$$

Applying the second estimate of Lemma 6.2.1, we infer that for all $\varepsilon \leq \delta \leq \rho r_w$,

$$
\overline{\nu}_{k,\varepsilon}^{\pm}(f_\delta^w) \geq \frac{1}{|\ln \varepsilon|} \int_\varepsilon^\delta (r_w^s \nu_{k,\tilde{\varepsilon} r_w^{-1}}^{\pm}(\mathbb{R}^d) - c\delta^\gamma) \, \frac{d\tilde{\varepsilon}}{\tilde{\varepsilon}}
$$

$$
\geq r_w^s \overline{\nu}_{k,\tilde{\varepsilon} r_w^{-1}}^{\pm}(\mathbb{R}^d) - \frac{1}{|\ln \varepsilon|} \int_\delta^1 r_w^s \nu_{k,\tilde{\varepsilon} r_w^{-1}}^{\pm}(\mathbb{R}^d) \, \frac{d\tilde{\varepsilon}}{\tilde{\varepsilon}} - \frac{c\delta^\gamma}{|\ln \varepsilon|} \int_\varepsilon^\delta \frac{d\tilde{\varepsilon}}{\tilde{\varepsilon}}.
$$

Since, by Lemma 2.3.2, $\nu_{k,\tilde{\varepsilon} r_w^{-1}}^{\pm}(\mathbb{R}^d) \leq M$ and $r_w^s \leq 1$, the second term is bounded from below by $-\frac{\ln \delta}{\ln \varepsilon} M$. Therefore we obtain

$$
\overline{\nu}_{k,\varepsilon}^{\pm}(f_\delta^w) \geq r_w^s \overline{\nu}_{k,\tilde{\varepsilon} r_w^{-1}}^{\pm}(\mathbb{R}^d) - c\delta^\gamma - \frac{\ln \delta}{\ln \varepsilon}(c\delta^\gamma + M). \tag{6.2.13}
$$

Applying inequalities (6.2.12) and (6.2.13) to $\overline{\nu}_{k,\varepsilon}(f_\delta^w) = \overline{\nu}_{k,\varepsilon}^+(f_\delta^w) - \overline{\nu}_{k,\varepsilon}^-(f_\delta^w)$ yields the asserted inequality in a similar way to the one for $\nu_{k,\varepsilon}(f_\delta^w)$ in the proof of Lemma 6.2.2. ∎

Proof of the convergence $\overline{\nu}_{k,\varepsilon} \xrightarrow{\mathrm{w}} \overline{C}_k^f(F)\mu_F$. The proof for the averaged measures $\overline{\nu}_{k,\varepsilon}$ is almost the same as the one for $\nu_{k,\varepsilon}$ in the non-arithmetic case. It is easily seen that also the families $\{\overline{\nu}_{k,\varepsilon}^+\}_{\varepsilon \in (0,1]}$ and $\{\overline{\nu}_{k,\varepsilon}^-\}_{\varepsilon \in (0,1]}$ are tight and thus, by Prokhorov's Theorem, relatively compact. Let $\{\varepsilon_n\}$ be a null sequence such that

$$
\overline{\nu}_{k,\varepsilon_n}^+ \xrightarrow{\mathrm{w}} \overline{\nu}_k^+ \quad \text{and} \quad \overline{\nu}_{k,\varepsilon_n}^- \xrightarrow{\mathrm{w}} \overline{\nu}_k^- \quad \text{as } \varepsilon \to 0,
$$

for some limit measures $\overline{\nu}_k^+$ and $\overline{\nu}_k^-$. Then $\overline{\nu}_{k,\varepsilon_n} \xrightarrow{\mathrm{w}} \overline{\nu}_k := \overline{\nu}_k^+ - \overline{\nu}_k^-$. Now we have to show that the limit measure $\overline{\nu}_k$ coincides with $\overline{\mu}_k := \overline{C}_k^f(F)\mu_F$, from which we can conclude the asserted convergence as $\varepsilon \to 0$.

Again we work with the family $\mathcal{A}_F$. By Theorem 6.1.2, the measures $\overline{\nu}_k$ and $\overline{\mu}_k$ coincide if they coincide for all $A \in \mathcal{A}_F$. Since, by (6.1.2), $\overline{\mu}_k(C) = \overline{C}_k^f(F)\mu_F(C) = 0$ for $C \in \mathcal{C}_F$ and, by (6.1.1), $\mu_k(S_w O) = \overline{C}_k^f(F) r_w^s$ for $w \in \Sigma^*$, we have to show that for all $w \in \Sigma^*$,

$$
\overline{\nu}_k(S_w O) = \overline{C}_k^f(F) r_w^s, \tag{6.2.14}
$$

and for all $C \in \mathcal{C}_F$,

$$
\overline{\nu}_k(C) = 0, \tag{6.2.15}
$$

in order to complete the proof that $\overline{\nu}_{k,\varepsilon} \xrightarrow{\mathrm{w}} \overline{C}_k^f(F)\mu_F$.

Proof of (6.2.14). The arguments are very similar to those in the proof of (6.2.7). Fix $w \in \Sigma^*$ and let $r = r_w$. Lemma 6.2.3 ensures that for all n and δ such that $\varepsilon_n \leq \delta \leq \rho r$,

$$
|\overline{\nu}_{k,\varepsilon_n}(f_\delta^w) - r_w^s \overline{\nu}_{k,\varepsilon_n r_w^{-1}}(\mathbb{R}^d)| \leq 2c\delta^\gamma + \frac{\ln \delta}{\ln \varepsilon_n} 2(c\delta^\gamma + M).
$$

If we keep δ fixed and let $n \to \infty$, the weak convergence implies $\overline{\nu}_{k,\varepsilon_n}(f_\delta^w) \to \overline{\nu}_k(f_\delta^w)$, while $\overline{\nu}_{k,\varepsilon_n r_w^{-1}}(\mathbb{R}^d) \to \overline{C}_k^f(F)$, by Theorem 2.3.6. On the right hand side the second term vanishes. Hence the above inequality yields

$$
|\overline{\nu}_k(f_\delta^w) - r_w^s \overline{C}_k^f(F)| \leq 2c\delta^\gamma
$$

for each $\delta \leq \rho r$. If we now let $\delta \to 0$, the integrals $\overline{\nu}_k(f_\delta^w)$ converge to $\overline{\nu}_k(S_w O)$, while the right hand side of the inequality vanishes, proving assertion (6.2.14). ∎

Proof of (6.2.15). Fix $r > 0$. For the functions g_δ defined in (6.2.11) there exists $c > 0$ such that for all $\varepsilon \leq \delta \leq \rho r$,

$$\overline{\nu}_{k,\varepsilon}^\pm(g_\delta) \leq c\delta^\gamma + \frac{\ln \delta}{\ln \varepsilon}(c\delta^\gamma + M),$$

which is derived in a similar way to (6.2.12).

Having established this estimate, the arguments are the same as those in the proof of (6.2.8). Set $\varepsilon = \varepsilon_n$ in the above inequality and let first $n \to \infty$ and afterwards $\delta \to 0$. Then the right hand side converges to 0, while on the left hand side we end up with $\overline{\nu}_k^\pm(O(r))$, which therefore equals zero. Now the assertion (6.2.15) is an immediate consequence. ∎

This completes the proof of Theorem 2.5.1.

Proof of Theorem 2.5.2. Suppose now that F is an arbitrary self-similar set satisfying OSC, i.e. the parallel sets F_ε are not necessarily polyconvex. We will briefly outline how the arguments in the above proof have to be modified to obtain a proof of Theorem 2.5.2.

By Gatzouras's theorem, the average Minkowski content $\overline{M}(F)$ always exists and is strictly positive, implying that always $s_d = s - d$. Going again through the proof of Theorem 2.5.1, it is easily seen that for $k = d$ most of the arguments remain valid in the general case. Some parts even simplify due to the positivity of the measure. In Lemmas 6.2.1 we simply replace $C_d^{\mathrm{var}}(F_\varepsilon, \cdot)$ by $\lambda_d(F_\varepsilon \cap \cdot)$ and use Lemma 5.6.2 instead of Lemma 5.2.1 to obtain the corresponding general estimate. Lemmas 6.2.2 and 6.2.3 remain valid as stated, when the generalized definitions (2.5.4) and (2.5.5) of $\nu_{d,\varepsilon}$ and $\overline{\nu}_{d,\varepsilon}$ are used. The proofs of both lemmas even simplify, since these measures are not signed.

To prove the convergence $\nu_{d,\varepsilon} \overset{\mathrm{w}}{\to} M(F)\mu_F$ for non-arithmetic sets F, Prokhorov's Theorem can now be applied directly to $\nu_{d,\varepsilon}$, without switching to the signed measures first. Choose any sequence $\{\varepsilon_n\}$ such that $\nu_{d,\varepsilon_n} \overset{\mathrm{w}}{\to} \nu_d$ and show that the limit measure ν_d of this sequence equals $\mu_d := M(F)\mu_F$ and is thus independent of the chosen sequence. The equivalence of the measures is obtained with the same arguments as in in the proof of Theorem 2.5.1 taking into account Lemma 5.6.2 and the generalized Lemma 6.2.2. The proof of the convergence $\overline{\nu}_{d,\varepsilon} \overset{\mathrm{w}}{\to} \overline{M}(F)\mu_F$ is adapted in a similar way.

6.3. Proof of Theorems 2.5.3 and 2.5.4. For $k = d-1$ and $k = d$, we show the weak convergence of the normalized curvature measures $\nu_{k,\varepsilon}^1 \overset{\mathrm{w}}{\to} \mu_F$. First we discuss the case $k = d - 1$. The arguments for $k = d$ are very similar as we will briefly outline afterwards.

Proof of Theorem 2.5.3. Let F be a self-similar set as assumed in Theorem 2.5.3. In particular it was assumed that there exists a constant $b > 0$ such that

$$\liminf_{\varepsilon \to 0} \varepsilon^{s-d+1} C_{d-1}(F_\varepsilon) = b, \tag{6.3.1}$$

which immediately implies $s_{d-1} = s - d + 1$. By definition, for all $\varepsilon > 0$,

$$\nu_{d-1,\varepsilon}^1 = (\varepsilon^{s-d+1} C_{d-1}(F_\varepsilon))^{-1} \nu_{d-1,\varepsilon}.$$

If F is non-arithmetic, then the weak convergence $\nu_{d-1,\varepsilon}^1 \overset{\mathrm{w}}{\to} \mu_F$ follows immediately, since, by Theorems 2.3.6 and 2.5.1, $\varepsilon^{s-d+1} C_{d-1}(F_\varepsilon)$ converges to $C_{d-1}^f(F)$, while $\nu_{d-1,\varepsilon} \overset{\mathrm{w}}{\to} C_{d-1}^f(F)\mu_F$. By the assumptions, $C_{d-1}^f(F) > 0$.

So assume now that F is arithmetic. In this case we have to work a bit more. Similarly to the above proofs, let $\{\varepsilon_n\}$ a null sequence such that $\nu^1_{d-1,\varepsilon_n} \xrightarrow{\mathrm{w}} \nu^1_{d-1}$ as $n \to \infty$ for some limit measure ν^1_{d-1}. We prove that ν^1_{d-1} is independent of the choice of the sequence $\{\varepsilon_n\}$ by showing that $\nu^1_{d-1} = \mu_F$. By Theorem 6.1.2 and the considerations in Section 6.1, it suffices to show that for all $w \in \Sigma^*$,

$$\nu^1_{d-1}(S_w O) = r^s_w, \tag{6.3.2}$$

and for all $C \in \mathcal{C}_F$,

$$\nu^1_{d-1}(C) = 0. \tag{6.3.3}$$

Proof of (6.3.2). Fix $w \in \Sigma^*$. Dividing the inequality in Lemma 6.2.2 by $\nu_{d-1,\varepsilon}(\mathbb{R}^d)$, we infer that for all $\varepsilon_n \leq \delta \leq \rho r_w$,

$$|\nu^1_{d-1,\varepsilon_n}(f^w_\delta) - r^s_w q_w(\varepsilon_n)| \leq 2c\delta^\gamma (\nu_{d-1,\varepsilon_n}(\mathbb{R}^d))^{-1},$$

where

$$q_w(\varepsilon) := \frac{\nu_{d-1,\varepsilon r_w^{-1}}(\mathbb{R}^d)}{\nu_{d-1,\varepsilon}(\mathbb{R}^d)} = \frac{(\varepsilon r_w^{-1})^{s-d+1} C_{d-1}(F_{\varepsilon r_w^{-1}})}{\varepsilon^{s-d+1} C_{d-1}(F_\varepsilon)}.$$

We show that this quotient converges to 1 as $\varepsilon \to 0$. Since F was assumed to be arithmetic, there is some $h > 0$ such that, for each $i = 1, \ldots, N$, there exists $n_i \in \mathbb{N}$ such that $-\ln r_i = n_i h$. We infer from Remark 4.1.5 that the expression $g(\varepsilon) = \varepsilon^{s-d+1} C_k(F_\varepsilon)$ is asymptotic to some periodic function $G(\varepsilon)$ of (multiplicative) period $\zeta = e^{-h}$. Noting that $r_w = \zeta^n$ is some integer power n of the period, we conclude that numerator and denominator of $q_w(\varepsilon)$ are asymptotic to the same function $G(\varepsilon r_w^{-1}) = G(\varepsilon)$. This implies convergence of the quotient $q_w(\varepsilon)$ to 1.

If we now let $n \to \infty$ in the above inequality, the quotient $q_w(\varepsilon_n)$ converges to 1 while the right hand side is bounded from above by b^{-1}, by the assumption (6.3.1). Hence

$$|\nu^1_{d-1}(f^w_\delta) - r^s_w| \leq 2cb^{-1}\delta^\gamma,$$

and, by taking the limit $\delta \to 0$, we derive that $\nu^1_{d-1}(S_w O) = r^s_w$, completing the proof of (6.3.2). ∎

Proof of (6.3.3). It suffices to show that $\nu^1_{d-1}(O(r)^c) = 0$ for each $r > 0$. So fix some r. Analogously to the proof of (6.2.8), we conclude from Lemma 5.2.1 that the functions g_δ (defined in (6.2.11)) satisfy the inequality

$$\nu^1_{d-1,\varepsilon_n}(g_\delta) = \frac{\nu_{d-1,\varepsilon_n}(g_\delta)}{\nu_{d-1,\varepsilon_n}(\mathbb{R}^d)} \leq c\delta^\gamma (\nu_{d-1,\varepsilon_n}(\mathbb{R}^d))^{-1}$$

for all $\varepsilon_n \leq \delta \leq \rho r$. Here note that $C^{\mathrm{var}}_{d-1}(F_\varepsilon, \cdot) = C_{d-1}(F_\varepsilon, \cdot)$. Now the assertion easily follows by letting first $n \to \infty$ and then $\delta \to 0$ and taking into account (6.3.1). ∎

We have shown that for each null sequence $\{\varepsilon_n\}$, for which as $n \to \infty$ the measures $\nu^1_{d-1,\varepsilon_n}$ converge at all, they converge to μ_F. Hence $\nu^1_{d-1,\varepsilon} \xrightarrow{\mathrm{w}} \mu_F$ as $\varepsilon \to 0$, as we stated in Theorem 2.5.3.

Proof of Theorem 2.5.4. Note that, by Gatzouras's theorem, a condition similar to (6.3.1) is always satisfied: There exists some $b > 0$ such that $\liminf_{\varepsilon \to 0} \varepsilon^{s-d} \lambda_d(F_\varepsilon) = b$. Hence

there is no extra assumption required in this case. Now the arguments of the above proof carry over to the case $k = d$ and to arbitrary self-similar sets satisfying OSC, when Theorem 2.3.10 and Lemma 5.6.2 are taken into account.

Appendix: Signed measures and weak convergence

We summarize a few facts and definitions concerning signed measures, which we always regard as totally finite signed measures here. In particular, we clarify the notion of weak convergence of a sequence of signed measures. For more details on signed measures we refer to the text books on measure theory, e.g. the ones by Elstrodt [5], Doob [4] or Jacobs [16], and for the weak convergence of measures to Billingsley [3].

Signed measures. Let X a metric space and $\mathcal{X}$ the σ-algebra of Borel sets of X. A function $\mu : \mathcal{X} \to \mathbb{R}$ is called a *signed measure* if it is σ-additive, i.e. for any sequence of pairwise disjoint sets $A_1, A_2, \ldots \in \mathcal{X}$

$$\mu\left(\bigcup_i A_i\right) = \sum_i \mu(A_i).$$

In particular, this definition implies $\mu(\emptyset) = 0$ and $|\mu(X)| < \infty$. μ is called a *measure* or *positive measure* if $\mu(A) \geq 0$ for all $A \in \mathcal{X}$.

We define the set functions μ^+, μ^- and μ^{var} by setting, for each $A \in \mathcal{X}$,

$$\mu^+(A) := \sup_{B \subseteq A} \mu(B), \quad \mu^-(A) := -\inf_{B \subseteq A} \mu(B), \quad \mu^{\mathrm{var}}(A) := \mu^+(A) + \mu^-(A).$$

It can be shown that μ^+, μ^- and μ^{var} are finite positive measures on $\mathcal{X}$. They are called respectively the *positive, negative* and *total variation* measures of μ.

THEOREM A.1.1. *Let μ be a signed measure on a σ-algebra $\mathcal{X}$. Then*

(i) *(Jordan decomposition)* $\mu = \mu^+ - \mu^-$.
(ii) *(Hahn decomposition)* X *is the disjoint union of two sets* $X^+, X^- \in \mathcal{X}$ *such that* $\mu^-(X^+) = \mu^+(X^-) = 0$. *The sets* X^+ *and* X^- *are unique up to* μ^{var}-*null sets.*

Integration with respect to a signed measure. Recall that for a measurable function $f : X \to \mathbb{R}$ the *integral* with respect to a positive measure μ is defined as follows. For nonnegative simple functions $g = \sum_{i=1}^n c_i \mathbf{1}_{A_i}$, where $A_i \subseteq X$ and $c_i > 0$, set

$$\int_X g \, d\mu = \sum_{i=1}^n c_i \mu(A_i).$$

Any nonnegative measurable function f is approximated from below by a sequence $g_1, g_2, \ldots$ of simple functions and the integral is then defined as the limit

$$\int_X f \, d\mu = \lim_{j \to \infty} \int_X g_j \, d\mu.$$

It can be shown that the limit does not depend on the choice of the sequence $g_1, g_2, \ldots$. The function f is μ-*integrable* if the limit is finite. Finally, an arbitrary measurable

function f is μ-*integrable* if the integrals $\int_X f_+ \, d\mu$ and $\int_X f_- \, d\mu$ are both finite, and for such f the integral is defined as

$$\int_X f \, d\mu = \int_X f_+ \, d\mu - \int_X f_- \, d\mu.$$

Here $f_+(x) = \max\{f(x), 0\}$ and $f_-(x) = f_+(x) - f(x)$ denote the positive and negative part of f, respectively.

The integral with respect to a signed measure μ is now defined with respect to its Jordan decomposition. A measurable function $f : X \to \mathbb{R}$ is μ-*integrable* if and only if it is μ^{var}-integrable. Then the *integral* with respect to μ is defined as

$$\int_X f \, d\mu = \int_X f \, d\mu^+ - \int_X f \, d\mu^-.$$

Weak convergence of signed measures. Having defined the integral with respect to a signed measure, the generalization of the concept of weak convergence to signed measures is straightforward. Let $\mu, \mu_1, \mu_2, \ldots$ be signed measures on $\mathcal{X}$. The sequence $\{\mu_n\}$ is said to *converge weakly* to the limit measure μ, $\mu_n \xrightarrow{\mathrm{w}} \mu$ as $n \to \infty$ if

$$\lim_{n \to \infty} \int_X f \, d\mu_n = \int_X f \, d\mu$$

for all bounded continuous functions f.

It is obvious from the definition that weak convergence of the variation measures μ_n^+ and μ_n^- of μ_n to the variation measures μ^+ and μ^- of μ is sufficient for weak convergence of a sequence of signed measures:

$$\mu_n^+ \xrightarrow{\mathrm{w}} \mu^+ \text{ and } \mu_n^- \xrightarrow{\mathrm{w}} \mu^- \;\Rightarrow\; \mu_n \xrightarrow{\mathrm{w}} \mu.$$

This implication suggests investigating the variation measures, to which the theory of weak convergence of (positive) measures applies, instead of studying the signed measures themselves. Note that the converse implication is not true. This is illustrated by a simple example.

EXAMPLE A.1.2. Let $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$ be two disjoint sequences in X, i.e. in particular $x_n \neq y_n$ for all $n \in \mathbb{N}$, converging both to the same point $x \in X$. For each n let $\mu_n^+ := \delta_{x_n}$ and $\mu_n^- := \delta_{y_n}$ be the Dirac measures of x_n and y_n, respectively. Set $\mu_n := \mu_n^+ - \mu_n^-$. Obviously, $\mu_n^{\pm} \xrightarrow{\mathrm{w}} \delta_x$ and therefore $\mu_n \xrightarrow{\mathrm{w}} \delta_x - \delta_x = 0$. Thus the limit measure μ is the zero measure and so its positive and negative variations μ^+ and μ^- are zero as well. Hence we have $\mu_n \xrightarrow{\mathrm{w}} \mu$ but neither $\mu_n^+ \xrightarrow{\mathrm{w}} \mu^+$ nor t $\mu_n^- \xrightarrow{\mathrm{w}} \mu^-$.

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