# 1. Introduction

In this paper we introduce a new transformation of continuous-time random walks (CTRW) that we will call "random coarse graining" (RCG for short) since it resembles the coarse-graining methods in statistical physics. We show that this transformation leads to a new coupled CTRW, and we investigate the asymptotic behavior of this process in detail (Theorems 5.1–5.12). Our motivation comes from the theory of relaxation in disordered systems.

Over the last decade advanced methods of probability theory have been successfully applied in modeling dielectric relaxation in solids. The phenomenon of approach to equilibrium of a dipolar system driven out of equilibrium by a step or alternating external electric field is one of the most intensively researched topics in modern physics. It has been widely examined experimentally [8, 22, 29, 30] but the problem of understanding its nature is as yet largely open. It has been observed for hundreds of different materials that they exhibit very similar time and frequency dependencies of dynamic dielectric characteristics (like the time decay of the depolarization current, or the complex susceptibility function) [22, 28, 30]. All dielectric data can be represented well enough by a few empirical functions with the power-law asymptotic behavior [8, 29]. However, these various model functions characterize the relaxation processes without in any way indicating the physical mechanisms involved.

A considerable effort devoted to finding a theoretical explanation of the empirically observed results points to the two most widespread and at the same time least understood properties of the relaxation responses; namely, the characteristic power-law asymptotic behavior and the fact that this property is common to a wide range of materials with very different physical and chemical interactions. As a consequence, in theoretical attempts to model relaxation it has been commonly assumed that the empirical relaxation laws reflect a kind of general behavior which is independent of the details of systems under study [7, 10, 11, 38, 40, 62, 64, 66]. In recent attempts to find the origins of the nonexponential relaxation patterns the idea of complex systems as "structures with variations" [18] that are characterized through a large diversity of elementary units and strong interactions between them is of special importance. The time evolution of physical properties of a complex system is unpredictable or anomalous, and the main feature of all dynamical processes in such a system is their stochastic background. In the framework of statistical models the fact that the large scale behavior of complex systems exhibits universality, i.e., that it is to some extent independent of the precise local nature of the system, should come as no surprise [24, 36]. Intuitively, one expects "averaging principles" like the law of large numbers to be in force. However, it turns out to be very hard to make

this intuition precise in concrete examples of stochastic systems with a large number of locally interacting components [11, 21, 32, 33, 40, 50, 62, 63, 67].

The empirical facts stress the need for a completely novel approach to the modeling of dielectric relaxation and, to a certain extent, also mechanical relaxation, photoconduction, photoluminescence and chemical reaction kinetics (sharing some common features, see [30]). The need to understand the connections between the macroscopic response of the relaxing complex system and the statistical properties of individual molecular or dipolar species requires the introduction into relaxation theory of advanced stochastic methods, going beyond the classical techniques of statistical physics. As shown by our work [10, 33, 34, 35, 64] a general formalism of limit theorems of probability theory plays an important role in constructing tools to relate local random characteristics of a complex system to empirical, deterministic relaxation laws, regardless of the specific nature of the system. The significance of this approach lies in the fact that no other one has provided in a simple and plausible way a rigorous explanation for the most widely observed forms of time and frequency dependencies of dielectric characteristics. Thereby, it opens up a new and powerful way of interpreting relaxation phenomena not only in the dielectric context.

The scenario of relaxation proposed and explored in [31, 32, 35, 65] implies a special construction of random sums involving an operation that resembles the deterministic coarse-graining and renormalization-group methods of statistical physics, although, unlike them, has itself a stochastic nature. Limit theorems for such random sums provide the class of limiting distributions that yield straightforwardly the empirically established formulas representing the dielectric data, and hence indicate the stochastic reasons for the applicability of the formulas as fitting functions. In the framework of this model the dielectric responses related to the formulas considered appear as a result of the statistical rules the large complex system follows in its spatio-temporal evolution. In this paper we extend the mathematical construction behind the stochastic scenario of dielectric relaxation described in [32] to adapt it to the concept of CTRW.

The notion of CTRW was introduced by Montroll and Weiss [53] as a walk with random time intervals between subsequent jumps. Since then, it has been applied in physics to model a wide variety of phenomena connected with anomalous diffusion; for instance, fully developed turbulence, transport in disordered or fractal media, intermittent chaotic systems, and relaxation processes [9, 15, 19–21, 38, 52, 56, 57, 60].

In applications of the CTRW theory, the asymptotic distribution of the total distance reached up to a large time $t$ by a walking particle initially at the origin is of great importance, and hence, the large-time behavior of CTRW has been studied intensively. In most approaches the analysis is performed by means of the Fourier–Laplace transform for the total distance [47, 57, 61, 69] and often leads to the fractional-differential-equations description [2, 3, 15, 23, 51, 52, 54, 59]. The inconvenience of this method is that useful, explicit inversion formulas can be provided only under some restrictive assumptions. Another method is based directly on the definition of CTRW as a random walk subordinated to a renewal counting process, which allows using the technique of randomly indexed sequences and limit theorems for stochastic processes (see [4, 5, 19, 20, 25, 41, 42, 48, 49, 66]). In the latter approach, in contrast to the very popu-

lar Tauberian analysis of the Fourier–Laplace transform of the total distance, the limiting distribution can be identified precisely and given in an easy-to-follow form, convenient for further applications. The aim of this paper is to give a contribution to the CTRW theory by introducing a new RCG transformation of the process. Since, in general, the proposed operation establishes a stochastic dependence between time and space steps, we thus provide an interesting class of coupled CTRWs that is general but easy to investigate. Studying the asymptotic behavior of the transformed CTRW we obtain a new class of possible limiting distributions for the large-time total distance reached by the walking particle. The mathematical construction proposed in this paper enables us to solve some open problems in the CTRW theory. Moreover, it allows a natural interpretation in stochastic models, and hence is promising from the point of view of further applications.

The paper is structured as follows: In Section 2 we gather several results on stable distributions, renewal processes, and randomly indexed sequences that are needed in what follows. In Section 3 we recall the definition of CTRW and provide some well known examples of processes that have a CTRW form. Moreover, in Table 1 we briefly present what is known about the asymptotic behavior of the walk according to [41]. The form of presentation allows us to indicate briefly the classes of CTRW already investigated. Also, it clearly points out the open questions.

The next two sections contain the main results of the paper. Namely, in Section 4 we introduce a new notion of RCG transformation of CTRWs. We show that, in general, the transformation leads to a new, coupled CTRW having the form of a random walk subordinated to a compound counting process different from the renewal counting process subordinating the walk before. Then we study the limiting properties of the compound counting process. In Section 5, devoted to investigation of the asymptotic behavior of randomly coarse grained CTRWs, we prove several limit theorems. For the reader's convenience, we endow the section with a guide (Tables 2 and 3). In the theorems both the large-time limits in distribution of the normalized total distance reached by the particle and the normalizing functions are precisely determined. The limiting probability laws obtained are collected in Tables 4–6 and briefly analyzed.

In the next section we interpret the results of Section 5 in the context of CTRW theory, and we partly solve the problems pointed out in Table 1. Moreover, we answer some questions raised in Section 5, concerning the essentiality of some assumptions of the limit theorems. Finally, in Section 7 we propose a slight modification of the randomly coarse grained CTRW concept that enlarges the class of the limiting distributions by the one connected with the model of dielectric relaxation.

## 2. Preliminaries

**2.1. Stable distributions.** For any sequence $X = \{X_i,\, i = 1, 2, \ldots\}$ of i.i.d. random variables define the partial-sum sequence $\{S_X(n),\, n = 0, 1, 2, \ldots\}$, where

(2.1) $$S_X(0) = 0, \quad S_X(n) = \sum_{i=1}^{n} X_i \quad \text{for } n \geq 1.$$

Let

(2.2) $$K_X(t) = \min\{n : S_X(n) > t\}$$

so that $\{K_X(t), t \geq 0\}$ is the first-passage process.

   The asymptotics of $S_X(n)$ for large $n$ has been investigated since the beginnings of probability theory. The first results, namely, the Bernoulli law of large numbers that was proved by Jakob Bernoulli (1655–1705) and appeared in his "Ars Conjectandi" published posthumously in 1713, and the De Moivre–Laplace theorem discovered by Abraham de Moivre about 1730, are acknowledged the most important early contributions to probability theory. Their generalizations like Kolmogorov's strong law of large numbers or the Lindeberg–Lévy central limit theorem have found important applications in many areas.

   Full information about the possible asymptotic behavior of $S_X(n)$ (as $n \to \infty$) is provided by the theory of stable distributions and their domains of attraction [16, 17, 26, 55, 60, 70]. The distribution of $X_i$ is said to belong to the *domain of attraction* of the distribution of the random variable $\mathcal{Z}$ if for some constants $a_n > 0, b_n$ we have

(2.3) $$\frac{S_X(n) - b_n}{a_n} \xrightarrow[n\to\infty]{d} \mathcal{Z}$$

(where "$\xrightarrow{d}$" denotes convergence in distribution). It is well known [16, 17, 70] that among nondegenerate distributions only the stable ones (with the normal distribution as a special case) have nonempty domains of attraction. Moreover, Theorem 1 of [16, XVII 5] yields

PROPOSITION 2.1. (a) *The distribution of $X_i$ belongs to the domain of attraction of a stable law with index of stability $\alpha$, $0 < \alpha < 2$, if and only if*

$$\lim_{x\to\infty} \frac{P(|X_i| > xy)}{P(|X_i| > x)} = y^{-\alpha} \quad \text{for each } y > 0$$

   *and*

   *the limits* $\quad \lim_{x\to\infty} \dfrac{P(X_i > x)}{P(|X_i| > x)} \quad$ *and* $\quad \lim_{x\to\infty} \dfrac{P(X_i < -x)}{P(|X_i| > xy)} \quad$ *exist.*

   (b) *The distribution of $X_i$ belongs to the domain of attraction of a normal law if and only if*

$$\lim_{x\to\infty} \frac{\int_{|X_i|<xy} X_i^2 \, dP}{\int_{|X_i|<x} X_i^2 \, dP} = 1 \quad \text{for any } y > 0.$$

   There are many different ways to define stable laws (see e.g. [55, 70]). In this paper we consider the stable distributions as corresponding to the four-parameter family of characteristic functions

(2.4) $$\varphi_{\alpha,\beta,m,c}(t) = \exp\{imt - |ct|^\alpha(1 - i\beta l(t))\},$$

where $0 < \alpha \leq 2$, $|\beta| \leq 1$, $m \in \mathbb{R}$, $c > 0$, and

$$l(t) = \begin{cases} \operatorname{sgn}(t) \tan(\pi\alpha/2) & \text{for } \alpha \neq 1, \\ -\operatorname{sgn}(t)\dfrac{2}{\pi} \ln|t| & \text{for } \alpha = 1. \end{cases}$$

The parameter $\alpha$ is the *index of stability*; $\beta$ and $m$ are the *skewness* and *shift parameters*, respectively; and $c$ is the *scale parameter* (however, in case $\alpha = 1$ this name is not fully justified [55]). Putting $\alpha = 2$ in formula (2.4) one obtains the characteristic function of the normal distribution, independently of the value of $\beta$. Moreover, $\varphi_{2,\beta,0,c}(t)$ with $c = 1/\sqrt{2}$ and any $\beta$ ($|\beta| \leq 1$) corresponds to the standard normal law $\mathcal{N}(0,1)$.

REMARK 2.1. Throughout this paper $\mathcal{S}_{\alpha,\beta}$ with $0 < \alpha < 2$ and $|\beta| \leq 1$ denotes a stable random variable corresponding to the characteristic function $\varphi_{\alpha,\beta,0,1}(t)$, and $\mathcal{G}$ is a random variable with the standard normal distribution $\mathcal{N}(0,1)$. For any $m \in \mathbb{R}$ and $c > 0$ the characteristic function $\varphi_{\alpha,\beta,m,c}(t)$ corresponds to $c\mathcal{S}_{\alpha,\beta} + m$ if $\alpha \neq 1$, while $\varphi_{1,\beta,m,c}(t)$ corresponds to $c\mathcal{S}_{1,\beta} + m + 2\beta(c\ln c)/\pi$ (see [55, Section 1.2]).

Stable distributions are completely asymmetric in case $0 < \alpha < 1$ and $|\beta| = 1$ only. A random variable $\mathcal{S}_{\alpha,1}$, with $0 < \alpha < 1$, that is positive with probability 1 is often called a *stable subordinator*. It can be used to transform one stable random variable to another. Namely, we have ([55, Section 1.4])

PROPOSITION 2.2. *Let* $0 < \gamma < 1$.

(a) *Let* $0 < \kappa < 2$, $\kappa \neq 1$, *and* $|\beta| \leq 1$. *If* $\mathcal{S}_{\kappa,\beta}$ *and* $\mathcal{S}_{\gamma,1}$ *are independent, then*

$$(\mathcal{S}_{\gamma,1})^{1/\kappa} \mathcal{S}_{\kappa,\beta} \stackrel{d}{=} \begin{cases} c_1 \mathcal{S}_{\kappa\gamma,\beta_1} & \text{if } \kappa\gamma \neq 1, \\ c_1 \mathcal{S}_{1,0} + m_1 & \text{if } \kappa\gamma = 1, \end{cases}$$

*where the constants* $c_1, \beta_1, m_1$ *depend on* $\kappa, \gamma,$ *and* $\beta$; $c_1 > 0$, $|\beta_1| \leq 1$ ($\stackrel{d}{=}$ *denotes equality of distributions*).

(b) *If* $\mathcal{G}$ *and* $\mathcal{S}_{\gamma,1}$ *are independent, then*

$$(\mathcal{S}_{\gamma,1})^{1/2}\mathcal{G} \stackrel{d}{=} c_2 \mathcal{S}_{2\gamma,0},$$

*where* $c_2 > 0$ *depends on* $\gamma$.

The parameters $c_1$, $c_2$, $\beta_1$, $m_1$ can be shown to be of the form

(2.5)
$$\begin{aligned} c_1 &= (\cos(\gamma\Theta)/\cos(\pi\gamma/2))^{1/(\kappa\gamma)}(1 + \beta^2 \tan^2(\pi\kappa/2))^{1/(2\kappa)}, \\ c_2 &= 2^{-1/2}(\cos(\pi\gamma/2))^{-1/(2\gamma)}, \\ \beta_1 &= \tan(\gamma\Theta)/\tan(\gamma\pi\kappa/2), \\ m_1 &= (\sin(\gamma\Theta)/\cos(\pi\gamma/2))(1 + \beta^2 \tan^2(\pi/(2\gamma)))^{\gamma/2}, \end{aligned}$$

where

$$\Theta = \arctan(\beta \tan(\pi\kappa/2)).$$

Necessary and sufficient conditions for the distribution of $X_i$ to belong to the domain of attraction of a stable or normal law (given by Proposition 2.1) are often replaced by simpler sufficient conditions. Namely, in the case of the stable distribution with $0 < \alpha < 2$

it is enough that for some $\sigma_0 > 0$ and $|\beta| \leq 1$,

(A) $$\lim_{x \to \infty} \frac{P(|X_i| > x)}{(x/\sigma_0)^{-\alpha}} = 1 \quad \text{and} \quad \lim_{x \to \infty} \frac{P(X_i > x)}{P(|X_i| > x)} = \frac{1 + \beta}{2};$$

while in the case of the normal law it suffices that the variance of $X_i$ is finite and positive (in fact, this is the well known Lindeberg–Lévy central limit theorem). Moreover, the following can be shown ([16, XVII 5, Th. 2]; [17, Ch. 7, Par. 35, Ths. 4 and 5]):

PROPOSITION 2.3. (a) $X_i$ satisfies condition (A) with some $\alpha$, $\sigma_0$, and $\beta$ if and only if (2.3) holds with

$$a_n = \begin{cases} (q(\alpha))^{1/\alpha} \sigma_0 n^{1/\alpha} & \text{if } \alpha \neq 1, \\ (\pi/2)\sigma_0 n & \text{if } \alpha = 1, \end{cases}$$

(2.6) $$b_n = \begin{cases} 0 & \text{if } \alpha < 1, \\ n^2(\pi/2)\sigma_0 \mathrm{E}\sin(2X_i/(\pi\sigma_0 n)) & \text{if } \alpha = 1, \\ n\mathrm{E}X_i & \text{if } \alpha > 1, \end{cases}$$

and the limit $\mathcal{Z}$ equals $\mathcal{S}_{\alpha,\beta}$. Here and throughout the paper

$$q(\alpha) = \frac{\Gamma(2 - \alpha)\cos(\pi\alpha/2)}{1 - \alpha}.$$

(b) $\mathrm{D}^2 X_i = \sigma^2$ for some $0 < \sigma < \infty$ if and only if (2.3) holds with $a_n = \sigma n^{1/2}$, $b_n = n\,\mathrm{E}X_i$, and the limit $\mathcal{Z}$ is $\mathcal{G}$.

REMARK 2.2. It is easy to check that if the distribution of $X_i$ satisfies condition (A) with $0 < \alpha < 1$, then $\mathrm{E}X_i$ does not exist or is infinite. On the other hand [17], if $1 < \alpha < 2$, then $\mathrm{E}X_i$ exists (although $\mathrm{D}^2 X_i$ is infinite) and hence the constant $b_n$ in (2.6) is well defined.

Let us add that condition (A) is necessary and sufficient for the distribution of $X_i$ to belong to the so-called domain of normal attraction of a stable law if $\alpha \neq 1$ [16, 17]. There are many different continuous and discrete distributions satisfying (A). Classical examples of continuous ones are stable laws themselves, and also the Pareto and Burr distributions with an appropriate choice of parameters ([1]) [16, 27]. One can get discrete distributions with property (A) by applying a quantizer transformation to some of the above continuous examples. Namely, for a fixed real number $\delta > 0$ one can transform any random variable $X$ into a discrete random variable $X_\delta$ by putting

$$X_\delta = \delta k \iff \delta(k - 1) < X \leq \delta k \text{ for } k = \ldots, -2, -1, 0, 1, 2, \ldots .$$

We have $P(X_\delta \geq \delta k) = P(X > \delta(k - 1))$ and $P(X_\delta \leq -\delta k) = P(X \leq -\delta k)$. Therefore for $x > 0$,

$$P(|X_\delta| > x) = P\left(|X_\delta| > \delta\left[\frac{x}{\delta}\right]\right) = P\left(X \leq -\delta\left(\left[\frac{x}{\delta}\right] + 1\right)\right) + P\left(X > \delta\left[\frac{x}{\delta}\right]\right)$$

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([1]) The Burr distribution with parameters $x_0, a, b > 0$, given by the distribution function

$$F(x) = \begin{cases} 1 - (1 + (x/x_0)^a)^{-b} & \text{if } x \geq 0, \\ 0 & \text{if } x < 0, \end{cases}$$

satisfies (A) with $\sigma_0 = x_0$, $\alpha = ab$, and $\beta = 1$. We have $0 < \alpha < 2$ if and only if $0 < ab < 2$. The Pareto distribution is a special case with $a = 1$.

and hence $P(|X_\delta| > x + 2\delta) \le P(|X| > x + \delta) \le P(|X_\delta| > x) \le P(|X| > x - \delta)$. Similarly $P(X_\delta > x + 2\delta) \le P(X > x + \delta) \le P(X_\delta > x) \le P(X > x - \delta)$. As a consequence, $X$ satisfies condition (A) with some $\sigma_0 > 0$, $0 < \alpha < 2$, $\beta$ such that $|\beta| \le 1$ if and only if $X_\delta$ does with the same parameters. Moreover, we have $0 \le X^+ \le X_\delta^+ \le X^+ + \delta \le X_\delta^+ + \delta$ and $0 \le X_\delta^- \le X^- \le X_\delta^- + \delta \le X^- + \delta$, where $X^+ = \max(0, X)$, $X_\delta^+ = \max(0, X_\delta)$, $X^- = \max(0, -X)$, $X_\delta^- = \max(0, -X_\delta)$. Hence $\mathrm{E}X^+ < \infty \Leftrightarrow \mathrm{E}X_\delta^+ < \infty$, and $\mathrm{E}X^- < \infty \Leftrightarrow \mathrm{E}X_\delta^- < \infty$, so that $\mathrm{E}|X| < \infty \Leftrightarrow \mathrm{E}|X_\delta| < \infty$. Therefore $\mathrm{E}X$ exists if and only if $\mathrm{E}X_\delta$ exists. Moreover, $\mathrm{E}X \le \mathrm{E}X_\delta \le \mathrm{E}X + 2\delta$ and $\mathrm{E}|X| - \delta \le \mathrm{E}|X_\delta| \le \mathrm{E}|X| + \delta$.

**2.2. Renewal theory.** Renewal theory concerns the case when $P(X_i > 0) = 1$ so that $X$ can be interpreted as a sequence of lifetimes. In this case the partial-sum sequence $\{S_X(n), n = 1, 2, \ldots\}$ is called a *renewal process*. Its asymptotic behavior for large $n$ can obviously be determined, for example by Proposition 2.3. Note that for $X_i$ positive with probability 1 the expected value is always determined, finite or infinite, and $0 < \mathrm{E}X_i \le \infty$. Moreover, the only possible value of $\beta$ in (A) for such a random variable is $\beta = 1$, and the condition simplifies to

(B) $$\lim_{x \to \infty} \frac{P(X_i > x)}{(x/\sigma_0)^{-\alpha}} = 1$$

for some $\sigma_0 > 0$ and $0 < \alpha < 2$.

The properties of the Laplace transform and the Tauberian theorems (see [16, XIII]) yield:

PROPOSITION 2.4. *A random variable $X_i$ (positive with probability 1):*

(a) *has a finite expected value $\mathrm{E}X_i = \mu$ if and only if*

$$\lim_{t \to 0+} \frac{1 - \psi_X(t)}{\mu t} = 1;$$

(b) *satisfies condition* (B) *with some $\sigma_0 > 0$ and $0 < \alpha < 1$ if and only if*

$$\lim_{t \to 0+} \frac{1 - \psi_X(t)}{(\sigma_0 t)^\alpha} = \Gamma(1 - \alpha).$$

*Here $\psi_X(t)$ denotes the Laplace transform of $X_i$ (i.e. $\psi_X(t) = \mathrm{E}e^{-tX_i}$).*

In renewal theory the *counting renewal process* $\{N_X(t), t \ge 0\}$, where

(2.7) $$N_X(t) = \max\{n : S_X(n) \le t\},$$

is usually studied instead of the first-passage process $\{K_X(t), t \ge 0\}$, defined by (2.2). In fact, the two processes are closely related in this case ([20, III, Th. 3.1]): $K_X(t) = N_X(t) + 1$. Moreover, it is quite easy to show ([20, p. 49])

PROPOSITION 2.5. *For any $t \ge 0$ we have $P(N_X(t) < \infty) = 1$; moreover,*

(2.8) $$\{N_X(t) \ge n\} = \{S_X(n) \le t\} \quad \text{for } n = 1, 2, \ldots$$

*and hence $N_X(t) \xrightarrow{a.s.} \infty$ as $t \to \infty$.*

More information on the asymptotics of $N_X(t)$ is provided by the following limit theorems:

PROPOSITION 2.6 ([20, II, Th. 5.1]). $N_X(t)/t \xrightarrow{a.s.} 1/\mu$ as $t \to \infty$, where $\mu = \mathrm{E}X_i$. (For $\mu = \infty$ the assertion holds with $1/\mu = 0$.)

PROPOSITION 2.7. (a) Let $X_i$ satisfy condition (B) with some $\sigma_0 > 0$ and $0 < \alpha < 2$.

(a1) If $0 < \alpha < 1$, then

$$\frac{N_X(t)}{t^\alpha} \xrightarrow[t\to\infty]{d} C\frac{1}{\mathcal{S}_{\alpha,1}^\alpha}, \quad \text{where} \quad C = \sigma_0^{-\alpha}(q(\alpha))^{-1}.$$

(a2) If $1 < \alpha < 2$, then

$$\frac{N_X(t) - t/\mu}{t^{1/\alpha}} \xrightarrow[t\to\infty]{d} C\mathcal{S}_{\alpha,-1}, \quad \text{where} \quad C = \frac{\sigma_0}{\mu}\left(\frac{q(\alpha)}{\mu}\right)^{1/\alpha}.$$

(b) If $\mathrm{D}^2 X_i = \sigma^2$ for some $0 < \sigma < \infty$, then

$$\frac{N_X(t) - t/\mu}{t^{1/2}} \xrightarrow[t\to\infty]{d} C\mathcal{G}, \quad \text{where} \quad C = \frac{\sigma}{\mu^{3/2}}.$$

*Proof.* We follow the idea presented in [16, XI 5]. From (2.8), for any fixed $x \geq 0$ we have

$$\left\{\frac{N_X(t)}{t^\alpha} > x\right\} = \{S_X([xt^\alpha] + 1) \leq t\} = \left\{\left(\frac{[xt^\alpha] + 1}{xt^\alpha}\right)^{1/\alpha} \frac{S_X([xt^\alpha] + 1)}{([xt^\alpha] + 1)^{1/\alpha}} \leq \frac{1}{x^{1/\alpha}}\right\}$$

(where $[\cdot]$ denotes the integer part). Applying Proposition 2.3 and Lemma 2 of [16, VIII 2], in case $0 < \alpha < 1$ one obtains

$$P\left(\frac{N_X(t)}{t^\alpha} > x\right) \xrightarrow[t\to\infty]{} P(\sigma_0(q(\alpha))^{1/\alpha}\mathcal{S}_{\alpha,1} < x^{-1/\alpha}),$$

which yields (a1).

Similarly,

$$\left\{\frac{N_X(t) - t/\mu}{t^{1/\alpha}} > x\right\} = \{S_X([xt^{1/\alpha} + t/\mu] + 1) \leq t\}$$
$$= \left\{\frac{S_X([xt^{1/\alpha} + t/\mu] + 1) - \mu([xt^{1/\alpha} + t/\mu] + 1)}{([xt^{1/\alpha} + t/\mu] + 1)^{1/\alpha}} + \frac{\mu([xt^{1/\alpha} + t/\mu] + 1) - t}{([xt^{1/\alpha} + t/\mu] + 1)^{1/\alpha}} \leq 0\right\}.$$

By Proposition 2.3 and Lemma 2 of [16, VIII 2], for $1 < \alpha < 2$ one gets

$$P\left(\frac{N_X(t) - t/\mu}{t^{1/\alpha}} > x\right) \xrightarrow[t\to\infty]{} P(\sigma_0(q(\alpha))^{1/\alpha}\mathcal{S}_{\alpha,1} + \mu^{1+1/\alpha}x \leq 0),$$

proving (a2) since $\mathcal{S}_{\alpha,-1}$ has the same distribution as $-\mathcal{S}_{\alpha,1}$. Part (b) can be proved the same way (or see [20, II, Th. 5.2]). ∎

REMARK 2.3. The conclusions of Propositions 2.5–2.7 also hold for the first-passage process $\{K_X(t), t \geq 0\}$ since $K_X(t) = N_X(t) + 1$.

The notions of the *residual lifetime* $S_X(K_X(t)) - t$ and of the age of the object that is alive at time $t$, i.e. $t - S_X(N_X(t))$, are also of interest in renewal theory. The asymptotics of these two random variables for $t \to \infty$ are presented below.

PROPOSITION 2.8 ([16, XIV 3]; [20, I, Th. 2.3; II, Ths. 5.1 and 6.2]). (a) *Let* $\mathrm{E}X_i = \mu < \infty$. *Then*

$$\frac{t - S_X(N_X(t))}{t} \xrightarrow[t\to\infty]{a.s.} 0, \qquad \frac{S_X(K_X(t)) - t}{t} \xrightarrow[t\to\infty]{a.s.} 0.$$

*Moreover,*

(a1) *if* $X_i$ *has a* $\delta$-*arithmetic distribution, i.e.* $\sum_{n=1}^{\infty} P(X_i = n\delta) = 1$, *and* $\delta > 0$ *is the largest constant with this property, then*

$$n\delta - S_X(N_X(n\delta)) \xrightarrow[n\to\infty]{d} \mathcal{X}_X, \qquad S_X(K_X(n\delta)) - n\delta \xrightarrow[n\to\infty]{d} \mathcal{X}_X + \delta,$$

*where*

$$P(\mathcal{X}_X = k\delta) = \frac{\delta}{\mu} P(X_i > k\delta), \qquad k = 0, 1, 2, \ldots;$$

(a2) *otherwise,*

$$t - S_X(N_X(t)) \xrightarrow[t\to\infty]{d} \mathcal{X}_X, \qquad S_X(K_X(t)) - t \xrightarrow[t\to\infty]{d} \mathcal{X}_X,$$

*where*

$$P(\mathcal{X}_X \le x) = \frac{1}{\mu} \int_0^x P(X_i > s)\, ds, \qquad x > 0.$$

(b) *If* $X_i$ *satisfies condition* (B) *with some* $\sigma_0 > 0$ *and* $0 < \alpha < 1$ *then*

$$\frac{t - S_X(N_X(t))}{t} \xrightarrow[t\to\infty]{d} 1 - \mathcal{B}_\alpha, \qquad \frac{S_X(K_X(t)) - t}{t} \xrightarrow[t\to\infty]{d} \frac{1}{\mathcal{B}_\alpha} - 1,$$

*where* $\mathcal{B}_\alpha$ *has generalized arcsine distribution with parameter* $\alpha$ *(for details see Remark 2.4 below).*

REMARK 2.4. Throughout this paper $\mathcal{B}_\alpha$ with $0 < \alpha < 1$ denotes a random variable with generalized arcsine distribution with parameter $\alpha$, i.e. the beta distribution with parameters $p = \alpha$ and $q = 1 - \alpha$. Its density function is

$$f_\alpha(x) = \begin{cases} \dfrac{\sin(\pi\alpha)}{\pi} x^{\alpha-1}(1-x)^{-\alpha} & \text{for } 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

The density function of $1/\mathcal{B}_\alpha$ takes the form

$$g_\alpha(x) = \begin{cases} \dfrac{\sin(\pi\alpha)}{\pi} x^{-1}(x-1)^{-\alpha} & \text{for } x > 1, \\ 0 & \text{otherwise.} \end{cases}$$

**2.3. Randomly indexed sequences.** The random variable $S_X(N_X(t))$ is the partial sum of the sequence $X$, randomly indexed by $N_X(t)$. Asymptotic properties of sequences of random variables (partial-sum sequences, in particular) with random indices have been widely investigated (see e.g. [1, 12, 13, 20, 43–46, 63]). Classical results for the case when the random indices are independent of the sequence are as follows:

PROPOSITION 2.9 ([20, I, Th. 1.1]). *Let* $\{Y(n), n = 1, 2, \ldots\}$ *be a sequence of random variables such that* $Y(n) \xrightarrow{d} \mathcal{Y}$ *as* $n \to \infty$. *Suppose further that* $\{J(t), t \ge 0\}$ *is a family of positive, integer-valued random variables, independent of* $\{Y(n), n = 1, 2, \ldots\}$ *and such that* $J(t) \xrightarrow{a.s.} \infty$ *as* $t \to \infty$. *Then* $Y(J(t)) \xrightarrow{d} \mathcal{Y}$ *as* $t \to \infty$.

PROPOSITION 2.10 ([13]). *Let $\{Y(n),\, n = 1, 2, \ldots\}$ be a sequence of random variables such that $Y(n)/(c_1 n^{\alpha_1}) \xrightarrow{d} \mathcal{Y}$ as $t \to \infty$ for some positive constants $c_1$, $\alpha_1$. Suppose further that $\{J(t), t \geq 0\}$ is a family of positive, integer-valued random variables, independent of $\{Y(n),\, n = 1, 2, \ldots\}$ and such that $J(t)/(c_2 t^{\alpha_2}) \xrightarrow{d} \mathcal{J}$ as $n \to \infty$ for some positive constants $c_2, \alpha_2$. Let the limit random variable $\mathcal{J}$ be independent of $\mathcal{Y}$. Then*

$$\frac{Y(J(t))}{c_1 c_2^{\alpha_1} t^{\alpha_1 \alpha_2}} \xrightarrow[t \to \infty]{d} \mathcal{J}^{\alpha_1} \mathcal{Y}.$$

In the general case, we have:

PROPOSITION 2.11 ([20, I, Th. 2.1]). *Let $\{Y(n),\, n = 1, 2, \ldots\}$ be a sequence of random variables such that $Y(n) \xrightarrow{a.s.} \mathcal{Y}$ as $n \to \infty$. Suppose further that $\{J(t),\, t \geq 0\}$ is a family of positive, integer-valued random variables such that $J(t) \xrightarrow{a.s.} \infty$ as $t \to \infty$. Then $Y(J(t)) \xrightarrow{a.s.} \mathcal{Y}$ as $t \to \infty$.*

Moreover, Theorem 5.2 of [68] implies

PROPOSITION 2.12. *Let $\{Y(n),\, n = 1, 2, \ldots\}$ be the partial-sum sequence for some sequence of independent and identically distributed random variables such that $Y(n)/b(n) \xrightarrow{d} \mathcal{Y}$ as $n \to \infty$ for some sequence $\{b(n),\, n = 1, 2, \ldots\}$, where $\mathcal{Y}$ has a stable distribution. Suppose further that $\{J(t),\, t \geq 0\}$ is a family of positive, integer-valued random variables such that $J(t) \xrightarrow{a.s.} \infty$ and $J(t)/a(t) \xrightarrow{a.s.} 1$ as $t \to \infty$ for a normalizing function $a(t)$. Then $Y(J(t))/b(a(t)) \xrightarrow{d} \mathcal{Y}$ as $t \to \infty$.*

Note that a.s. convergence in Propositions 2.9 and 2.12 can be replaced by convergence in probability. In the next section we present an important example of a randomly indexed sequence.

## 3. Continuous-time random walks (CTRW)

The notion of CTRW generalizes a simple random walk by allowing a random waiting time between jumps. The definition of this stochastic process can be formulated in the following manner:

DEFINITION 3.1. Let $(T, R) = \{(T_i, R_i),\, i = 1, 2, \ldots\}$ be a sequence of i.i.d. random vectors such that $P(T_i > 0) = 1$. The *continuous-time random walk* generated by $(T, R)$ is the stochastic process $\{\widetilde{R}(t),\, t \geq 0\}$ such that

$$\widetilde{R}(t) = S_R(N_T(t)),$$

where $S_R(n)$ is the partial sum (2.1) of the i.i.d. sequence $R = \{R_i,\, i = 1, 2, \ldots\}$, while $N_T(t)$ is obtained via formula (2.7) from the i.i.d. sequence $T = \{T_i,\, i = 1, 2, \ldots\}$.

The random variable $T_i$ is usually interpreted as the waiting time of a moving particle for the $i$th jump, and $R_i$ as the $i$th jump parameter (indicating both the length and direction of the jump). Hence the renewal process $\{S_T(n),\, n = 1, 2, \ldots\}$ represents the instants of time when the sequential jumps occur, while the process $\{N_T(t),\, t \geq 0\}$ counts the jumps. The random variable $\widetilde{R}(t)$ represents the total distance covered by the particle
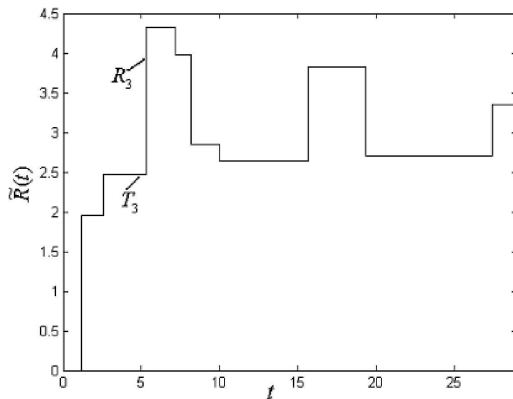
Fig. 1. An example of the CTRW trajectory

until time $t$ (see Figure 1). The CTRW process is called *decoupled* if the random variables $T_i$ and $R_i$ are independent, and *coupled* otherwise.

As we shall see below, several well known processes can be constructed according to Definition 3.1. Note that, in view of Example 3.1, CTRW is a discrete-time random walk subordinated to a renewal counting process.

EXAMPLE 3.1 (Classical random walk, see e.g. [16]). If $T_i = \Delta t$ for some positive constant $\Delta t$ then $\widetilde{R}(t) = S_R([t/\Delta t])$ so that $\{\widetilde{R}(t), t \geq 0\}$ is an ordinary, *discrete-time random walk*. If additionally $P(R_i = 1) = p$ and $P(R_i = -1) = 1 - p$ for some $0 < p < 1$, then for $\Delta t = 1$ we obtain a *classical random walk*.

EXAMPLE 3.2 (Poisson process [6]). If $R_i = \Delta r$ for some nonzero constant $\Delta r$ then $\widetilde{R}(t) = \Delta r N_T(t)$ and the CTRW $\{\widetilde{R}(t), t \geq 0\}$ is just the scaled *counting process*. If additionally $T_i$ is exponentially distributed with parameter $\lambda$, for $\Delta r = 1$ we find that $\{\widetilde{R}(t), t \geq 0\}$ is a *Poisson process* with intensity $\lambda$.

EXAMPLE 3.3 (Compound Poisson process [6]). If $T_i$ and $R_i$ are independent and $T_i$ is exponentially distributed with parameter $\lambda$, then $\{\widetilde{R}(t), t \geq 0\}$ is a *compound Poisson process*, known to have independent and stationary increments. If additionally $R_i$ has a zero-one distribution, i.e. $P(R_i = 1) = p$ and $P(R_i = 0) = 1 - p$ for some $0 < p < 1$, then $\{\widetilde{R}(t), t \geq 0\}$ is just a new Poisson process with intensity $\lambda p$.

EXAMPLE 3.4 (Lévy flight [9, 52]). If $T_i$ and $R_i$ are independent, $\mathrm{E}T_i$ is finite and $R_i$ is a symmetric stable random variable (i.e. with $\beta = m = 0$) with index of stability $\alpha$ such that $1 < \alpha < 2$ (so that $\mathrm{E}R_i = 0$ and $\mathrm{D}^2 R_i = \infty$), then the process $\{\widetilde{R}(t), t \geq 0\}$ is referred to as the *Lévy flight*.

EXAMPLE 3.5 (Age of object that is alive at time $t$ [16, 20]). Taking $R_i = T_i$ we obtain a simple example of a coupled CTRW. In this case $t - \widetilde{R}(t)$ coincides with *the age of the object that is alive at time $t$*, considered in renewal theory.

EXAMPLE 3.6 (Lévy walk [38, 42]). Another example of a coupled CTRW is the *Lévy walk*, obtained when $R_i = Y_i T_i^\nu + m$, where $\nu \geq 0$ and $m$ are constants; $\{Y_i\}$ is a se-

**Table 1.** Classes of CTRWs considered in the literature. The cases for which nondegenerate large-time limiting distributions have been determined are marked by "+"; and open questions, by " ?" and "(?)" (the latter indicates the questions that are partly answered in this paper). "$R_i \perp T_i$" means that the random variables $R_i$ and $T_i$ are independent. $C_0$ is a nonzero constant. Note that in case $\mathrm{E}T_i = \tau < \infty$ and $\mathrm{E}R_i = \varrho \neq 0$ a degenerate, nonzero constant limit has been derived.

| | $\mathbf{E}\boldsymbol{T_i} = \boldsymbol{\tau} \leq \boldsymbol{\infty}$ | | |
|---|---|---|---|
| | $T_i$ satisfies (B) with $1 < \alpha_T < 2$ | $\mathrm{D}^2 T_i < \infty$ | otherwise |
| $\mathbf{E}\boldsymbol{R_i} = \boldsymbol{\varrho} \neq \mathbf{0}$ | | | |
| $R_i = (\varrho/\tau)T_i$ | + | | |
| otherwise | | | |
| $R_i$ satisfies (A) with $1 < \alpha_R < 2$ — $\alpha_R \neq \alpha_T$ | + | + | |
| $R_i$ satisfies (A) with $1 < \alpha_R < 2$ — $\alpha_R = \alpha_T$ — $R_i - (\varrho/\tau)T_i$ satisfies (A) | + | + | |
| $R_i$ satisfies (A) with $1 < \alpha_R < 2$ — $\alpha_R = \alpha_T$ — otherwise | $(?)_1$ | + | |
| $\mathrm{D}^2 R_i < \infty$ | + | + | |
| otherwise | | | ? |
| $\mathbf{E}\boldsymbol{R_i} = \mathbf{0}$ | | | |
| $R_i$ satisfies (A) with $1 < \alpha_R < 2$ | + | | |
| $\mathrm{D}^2 R_i < \infty$ | + | | |
| otherwise | | | ? |
| $\mathbf{E}\boldsymbol{R_i}$ **does not exist** | | | |
| $R_i$ satisfies (A) with $0 < \alpha_R < 1$ | + | | |
| otherwise | | | ? |

| | $\mathbf{E}\boldsymbol{T_i} = \boldsymbol{\infty}$ | | |
|---|---|---|---|
| | $T_i$ satisfies (B) with $0 < \alpha_T < 1$ | | otherwise |
| $\mathbf{E}\boldsymbol{R_i} \neq \mathbf{0}$ | + | | ? |
| $\mathbf{E}\boldsymbol{R_i} = \mathbf{0}$ | | | |
| $R_i$ satisfies (A) with $1 < \alpha_R < 2$ — $R_i \perp T_i$ | | + | |
| $R_i$ satisfies (A) with $1 < \alpha_R < 2$ — otherwise | | $(?)_2$ | |
| $\mathrm{D}^2 R_i < \infty$ — $R_i \perp T_i$ | | + | |
| $\mathrm{D}^2 R_i < \infty$ — otherwise | | $(?)_3$ | |
| otherwise | | | ? |
| $\mathbf{E}\boldsymbol{R_i}$ **does not exist** | | | |
| $R_i$ satisfies (A) with $0 < \alpha_R < 1$ — $R_i = C_0 T_i$ | | + | |
| $R_i$ satisfies (A) with $0 < \alpha_R < 1$ — $R_i \perp T_i$ | | + | |
| $R_i$ satisfies (A) with $0 < \alpha_R < 1$ — otherwise | | $(?)_4$ | |
| otherwise | | | $(?)_5$ |

quence of i.i.d. random variables, independent of $\{T_i\}$, such that $P(Y_i = 1) = p$ and $P(Y_i = -1) = 1 - p$ for some $0 < p < 1$; and $T_i$ satisfies condition (B) with $\alpha > 0$, $\alpha \neq 2$.

It is very difficult to determine the properties of $\{\widetilde{R}(t),\, t \geq 0\}$ in general. What is examined quite well is the behavior of CTRW for $t$ tending to $\infty$. Recently, a general representation of the scaling limit of both decoupled and coupled CTRWs has been found, also in the case of $d$-dimensional jumps [4, 5, 48, 49]; however, the limiting probability laws have not been clearly identified there. A systematic study of the possible total-distance limiting distributions for one-dimensional CTRWs has been done in [41]. In Table 1 we briefly summarize the considerations presented there. Namely, we collect the assumptions under which the long-time asymptotic behavior of CTRW has been determined ([2]), and we indicate the cases that need to be investigated. As the table shows, some results for the case $\mathrm{E}T_i = \infty$ concern the class of decoupled CTRWs only. In the next section we propose a new transformation of CTRWs that in most cases leads to a coupled process. The analysis of the behavior of the transformed process for $t \to \infty$ will enable us to answer some of the open questions pointed out in Table 1.

## 4. Random coarse graining (RCG) of CTRW

**4.1. Definition.** We now introduce a new transformation of a CTRW process that we call "random coarse graining" (RCG) since it resembles the coarse-graining methods for rescaling that smooth out relatively small length-scale structures while preserving larger length-scale ones. The proposed operation is analogous to one of the crucial tools used by physicists to understand complex phenomena in condensed matter problems; namely, the renormalization group approach in statistical physics in the form proposed by Sinai [14, 24, 58]. The RCG transformation has the same feature of averaging the details of the system, although, in contrast to the deterministic classical methods, has a stochastic nature. Another stochastic analog of renormalization-group transformation has been proposed in [61].

DEFINITION 4.1. Let $(T, R) = \{(T_i, R_i),\, i = 1, 2, \ldots\}$ be a sequence of i.i.d. random vectors such that $P(T_i > 0) = 1$, and let $M = \{M_i,\, i = 1, 2, \ldots\}$ be a sequence of i.i.d. random variables such that

$$\sum_{n=1}^{\infty} P(M_i = n) = 1$$

(i.e. $M_i$ is a positive, integer-valued random variable). Assume that $M$ is independent of $(T, R)$. The *CTRW generated by* $(T, R)$ *and randomly coarse grained by means of* $M$ is the process $\{\widetilde{R}_M(t),\, t \geq 0\}$ obtained according to Definition 3.1 from the sequence $(\overline{T}, \overline{R}) = \{(\overline{T}_j, \overline{R}_j),\, j = 1, 2, \ldots\}$, where, with $S_M(j)$ being the $j$th partial sum of $M$,

$$(4.1) \qquad (\overline{T}_j, \overline{R}_j) = \sum_{i=S_M(j-1)+1}^{S_M(j)} (T_i, R_i).$$

---

([2]) The results themselves are not presented since they follow directly from the theorems proved in Section 5; see the comment at the beginning of Section 6.
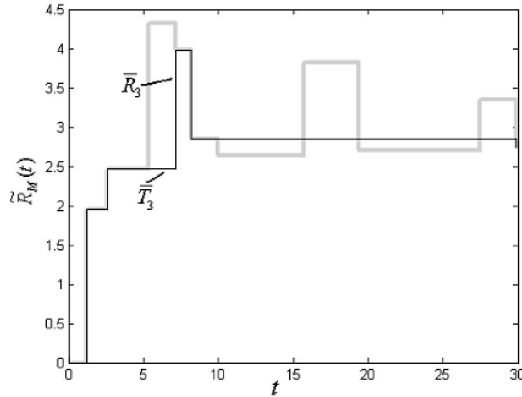
Fig. 2. An example of the randomly-coarse-grained-CTRW trajectory. The grey line is the trajectory of the CTRW process before the RCG transformation

Observe that $P(\overline{T}_j > 0) = 1$ and $(\overline{T}, \overline{R})$ is a sequence of i.i.d. random vectors so that $\{\widetilde{R}_M(t), t \geq 0\}$ is well defined. In most cases, for fixed $j$ the random variables $\overline{R}_j$ and $\overline{T}_j$ are not independent (even if $R_i$ and $T_i$ are) ($^3$). Hence the RCG transformation given by Definition 4.1 converts the CTRW $\{\widetilde{R}(t), t \geq 0\}$, generated by $(T, R)$, into a new CTRW $\{\widetilde{R}_M(t), t \geq 0\}$, generated by $(\overline{T}, \overline{R})$, which, in general, is coupled; see Figure 2. On the other hand, we obtain

THEOREM 4.1. *For any $t > 0$,*

(4.2) $$\widetilde{R}_M(t) = S_R(L(t)),$$

*where*

(4.3) $$L(t) = S_M(N_M(N_T(t)));$$

*so that $\widetilde{R}_M(t)$ is the partial sum of the sequence $\{R_i\}$, randomly indexed by $L(t)$.*

*Proof.* Observe that $\overline{T}_j$ and $\overline{R}_j$, given by (4.1), are equal to

$$\overline{T}_j = S_T(S_M(j)) - S_T(S_M(j-1)), \quad \overline{R}_j = S_R(S_M(j)) - S_R(S_M(j-1)),$$

and therefore

$$S_{\overline{T}}(n) = S_T(S_M(n)), \quad S_{\overline{R}}(n) = S_R(S_M(n)).$$

As a consequence,

(4.4) $$\widetilde{R}_M(t) = S_R(S_M(N_{\overline{T}}(t))).$$

Moreover, for $k = 0, 1, \ldots,$

$$N_{\overline{T}}(t) = k \;\Leftrightarrow\; S_T(S_M(k)) \leq t < S_T(S_M(k+1))$$
$$\Leftrightarrow\; S_M(k) \leq N_T(t) < S_M(k+1) \;\Leftrightarrow\; N_M(N_T(t)) = k.$$

Hence, $N_{\overline{T}}(t) = N_M(N_T(t))$, which together with (4.4) yields (4.2) and (4.3). ■

($^3$) A necessary condition for $\overline{R}_j$ and $\overline{T}_j$ to be independent (if $R_i$ and $T_i$ are independent) is that $g_M(\phi_R(t)\phi_T(t)) = g_M(\phi_R(t))g_M(\phi_T(t))$ for any $t$, where $g_M(s) = \mathrm{E}s^{M_i}$ is the generating function of $M_i$, and $\phi_R(t)$, $\phi_T(t)$ are the characteristic functions of $R_i$, $T_i$, respectively. This condition cannot hold if, for example, $M_i$ has a Poisson distribution, except for the degenerate case of $R_i = 0$ with probability 1.

Theorem 4.1 shows that a randomly coarse-grained CTRW is a discrete-time random walk $\{S_R([t]),\, t \geq 0\}$ subordinated to the compound counting process $\{L(t),\, t \geq 0\}$ defined by (4.3). The process $\{L(t),\, t \geq 0\}$ is different from the renewal counting process $\{N_T(t),\, t \geq 0\}$ subordinating the walk before the RCG transformation. From this point of view, RCG is an operation changing the number of jump steps till time $t$. Taking $M_i = 1$ one obtains $L(t) = N_T(t)$ and hence $\widetilde{R}_M(t) = \widetilde{R}(t)$. Therefore the construction of Definition 4.1 can be considered as a generalization of the concept of CTRW.

**4.2. Some properties of the compound counting process.** Since $M_i$ is integer-valued, $L(t) = S_M(N_M(N_T(t))) \geq N_M(N_T(t))$. From Propositions 2.5 and 2.11, $N_M(N_T(t)) \xrightarrow{\text{a.s.}} \infty$ as $t \to \infty$, and as a consequence we have

THEOREM 4.2. $L(t) \xrightarrow{\text{a.s.}} \infty$ as $t \to \infty$.

Detailed analysis of asymptotic properties of the compound counting process $\{L(t),\, t \geq 0\}$ leads to the following results:

THEOREM 4.3. *Let* $\mathrm{E}M_i < \infty$.

   (a) *If* $\mathrm{E}T_i = \tau < \infty$ *then*

$$\frac{L(t)}{t} \xrightarrow[t\to\infty]{\text{a.s.}} \frac{1}{\tau}.$$

   (b) *If* $T_i$ *satisfies condition* (B) *with* $\alpha = \lambda$ *for some* $0 < \lambda < 1$, *and with* $\sigma_0 = \tau_0 > 0$, *then*

$$(4.5) \qquad \frac{L(t)}{t^\lambda} \xrightarrow[t\to\infty]{d} C\,\frac{1}{\mathcal{S}_{\lambda,1}}, \quad \text{where} \quad C = \tau_0^{-\lambda}(q(\lambda))^{-1}.$$

*Proof.* From Proposition 2.8(a), $S_M(N_M(t))/t \xrightarrow{\text{a.s.}} 1$ as $t \to \infty$. Moreover, $N_T(t) \xrightarrow{\text{a.s.}} \infty$ (Proposition 2.5). Hence, according to Proposition 2.11,

$$\frac{L(t)}{N_T(t)} = \frac{S_M(N_M(N_T(t)))}{N_T(t)} \xrightarrow[t\to\infty]{\text{a.s.}} 1.$$

Thus, in case $\mathrm{E}T_i = \tau < \infty$, we deduce from Proposition 2.6 that

$$\frac{L(t)}{t} = \frac{L(t)}{N_T(t)} \cdot \frac{N_T(t)}{t} \xrightarrow[t\to\infty]{\text{a.s.}} 1 \cdot \frac{1}{\tau}.$$

If instead $\lim_{t\to\infty} \Pr(T_i > t)/(t/\tau_0)^{-\lambda} = 1$ for some $\tau_0 > 0$ and $0 < \lambda < 1$, then it follows from Proposition 2.7(a) and from Lemma 2 of [16, VIII 2] that

$$\frac{L(t)}{t^\lambda} = \frac{L(t)}{N_T(t)} \cdot \frac{N_T(t)}{t^\lambda} \xrightarrow[t\to\infty]{d} 1 \cdot C\,\frac{1}{\mathcal{S}_{\lambda,1}},$$

where $C$ is given by (4.5). $\blacksquare$

Observe that the conclusions of Theorem 4.3 remain true even if the sequences $\{M_i\}$ and $\{T_i\}$ are not independent.

THEOREM 4.4. *Let the random variable* $M_i$ *satisfy condition* (B) *with* $\alpha = \gamma$ *for some* $0 < \gamma < 1$, *and with some* $\sigma_0 > 0$.

(a) *If* $\mathrm{E}T_i = \tau < \infty$ *then*

$$\frac{L(t)}{t} \xrightarrow[t\to\infty]{d} C\mathcal{B}_\gamma, \quad where \quad C = \frac{1}{\tau}$$

and $\mathcal{B}_\gamma$ *has a generalized arcsine distribution with parameter* $\gamma$ *(see Remark 2.4).*

(b) *If* $T_i$ *satisfies condition* (B) *with* $\alpha = \lambda$ *for some* $0 < \lambda < 1$, *and with* $\sigma_0 = \tau_0 > 0$, *then*

$$\frac{L(t)}{t^\lambda} \xrightarrow[t\to\infty]{d} C\mathcal{B}_\gamma \frac{1}{\mathcal{S}_{\lambda,1}^\lambda},$$

where $C$ *is as in* (4.5), *and the random variables* $\mathcal{B}_\gamma$ *and* $\mathcal{S}_{\lambda,1}$ *are independent.*

*Proof.* From Proposition 2.5, $N_T(t) \xrightarrow{\text{a.s.}} \infty$ as $t \to \infty$. Moreover, $\{N_T(t)\}$ is independent of $\{S_M(N_M(t))\}$. From Propositions 2.8(b), 2.9, 2.6 and Lemma 2 of [16, VIII 2], we obtain

$$\frac{L(t)}{t} = \frac{S_M(N_M(N_T(t)))}{N_T(t)} \cdot \frac{N_T(t)}{t} \xrightarrow[t\to\infty]{d} \mathcal{B}_\gamma \cdot \frac{1}{\tau}$$

in case $\mathrm{E}T_i = \tau < \infty$. If instead $\lim_{t\to\infty} \Pr(T_i > t)/(t/\tau_0)^{-\lambda} = 1$ for some $\tau_0 > 0$ and $0 < \lambda < 1$, then according to Propositions 2.7(a) and 2.8(b),

$$\frac{S_M(N_M(t))}{t} \xrightarrow[t\to\infty]{d} \mathcal{B}_\gamma \quad and \quad \frac{N_T(t)}{t^\lambda} \xrightarrow[t\to\infty]{d} C\frac{1}{\mathcal{S}_{\lambda,1}^\lambda},$$

where $C$ is given by (4.5). From Proposition 2.10 we obtain (b). ∎

As shown in Theorem 4.3(a), $L(t)/t$ tends with probability 1 to the constant $1/\tau$ as $t \to \infty$ if both $\mathrm{E}M_i$, $\mathrm{E}T_i$ are finite ($\tau = \mathrm{E}T_i$). The next theorem concerns the asymptotic behavior of the difference $L(t)/t - 1/\tau$.

THEOREM 4.5. *Let* $\mathrm{E}M_i < \infty$ *and* $\mathrm{E}T_i = \tau < \infty$.

(a) *If* $T_i$ *satisfies condition* (B) *with* $\alpha = \lambda$ *for some* $1 < \lambda < 2$, *and with* $\sigma_0 = \tau_0 > 0$, *then*

$$\frac{L(t) - t/\tau}{t^{1/\lambda}} \xrightarrow[t\to\infty]{d} C\mathcal{S}_{\lambda,-1}, \quad where \quad C = \frac{\tau_0}{\tau}\left(\frac{q(\lambda)}{\tau}\right)^{1/\lambda}.$$

(b) *If* $\mathrm{D}^2 T_i = \sigma^2$ *for some* $0 < \sigma < \infty$ *then*

$$\frac{L(t) - t/\tau}{t^{1/2}} \xrightarrow[t\to\infty]{d} C\mathcal{G}, \quad where \quad C = \frac{\sigma}{\tau^{3/2}}.$$

*Proof.* The random variable $M_i$ is integer valued. Let $\delta > 0$ be the largest integer constant such that $\sum_{n=1}^\infty P(M_i = n\delta) = 1$. Then $N_M(t) = N_M([t/\delta]\delta)$ and we have

$$L(t) - t/\tau = S_M\left(N_M\left(\left[\frac{N_T(t)}{\delta}\right]\delta\right)\right) - \left[\frac{N_T(t)}{\delta}\right]\delta$$

$$+ \left(\left[\frac{N_T(t)}{\delta}\right] - \frac{N_T(t)}{\delta}\right)\delta$$

$$+ (N_T(t) - t/\tau).$$

(Here $[\cdot]$ denotes the integer part.) From Proposition 2.8(a), $n\delta - S_M(N_M(n\delta)) \xrightarrow{d} \mathcal{X}_M$ as $n \to \infty$, where $\mathcal{X}_M$ is a well defined random variable. Since $N_T(t) \xrightarrow{\text{a.s.}} \infty$ as $t \to \infty$

(Proposition 2.5) and $\{N_T(t)\}$ is independent of $\{S_M(N_M(t))\}$, the first term on the right-hand side above divided by $t^{1/\lambda}$ (or $t^{1/2}$) tends almost surely to 0 as $t \to \infty$ from Proposition 2.9 and Lemma 2 of [16, VIII 2]. The absolute value of the second term is less than $\delta$ so that, divided by $t^{1/\lambda}$ (or $t^{1/2}$), it also tends to 0. As a consequence, the asymptotics of the difference $L(t) - t/\tau$ is determined by the behavior of the third term and the conclusions follow from Proposition 2.7. ∎

## 5. Asymptotic behavior of randomly coarse grained CTRWs

In this section we prove several limit theorems providing information on the asymptotic distribution of $\widetilde{R}_M(t)$ for $t$ tending to infinity. We consider different classes of randomly coarse grained CTRWs. For the reader's convenience, in Tables 2 and 3 we provide a guide to the results of this part of the paper.

In the first two theorems we study the case $R_i \propto T_i$.

THEOREM 5.1. *Assume that* $EM_i = \mu < \infty$ *and that* $R_i = CT_i$ *for some constant* $C \neq 0$.

(a) *If* $ET_i = \tau < \infty$ *then*

$$\frac{\widetilde{R}_M(t)}{t} \xrightarrow[t\to\infty]{a.s.} C.$$

*Moreover*:

(a1) *if the distribution of* $T_i$ *is arithmetic then*

$$\widetilde{R}_M(n\delta) - Cn\delta \xrightarrow[n\to\infty]{d} -C\mathcal{X}_{T,M},$$

*where* $\delta > 0$ *is the largest constant such that* $\sum_{n=1}^\infty P(\overline{T}_j = n\delta) = 1$, *and*

$$P(\mathcal{X}_{T,M} = k\delta) = \frac{\delta}{\tau\mu} P(\overline{T}_j > k\delta), \quad k = 0, 1, 2, \ldots;$$

(a2) *otherwise*,

$$\widetilde{R}_M(t) - Ct \xrightarrow[t\to\infty]{d} -C\mathcal{X}_{T,M},$$

*where*

$$P(\mathcal{X}_{T,M} \leq x) = \frac{1}{\tau\mu} \int_0^x P(\overline{T}_j > s)\, ds, \quad x > 0.$$

(b) *If* $T_i$ *satisfies condition* (B) *with* $\alpha = \lambda$ *for some* $0 < \lambda < 1$, *and with* $\sigma_0 = \tau_0 > 0$, *then*

$$\frac{\widetilde{R}_M(t)}{t} \xrightarrow[t\to\infty]{d} C\mathcal{B}_\lambda.$$

THEOREM 5.2. *Assume that* $M_i$ *satisfies condition* (B) *with* $\alpha = \gamma$ *for some* $0 < \gamma < 1$, *and with some* $\sigma_0 > 0$, *and that* $R_i = CT_i$ *for some constant* $C \neq 0$.

(a) *If* $ET_i = \tau < \infty$ *then*

$$\frac{\widetilde{R}_M(t)}{t} \xrightarrow[t\to\infty]{d} C\mathcal{B}_\gamma.$$

**Table 2.** Guide to the limit theorems for randomly coarsed grained CTRWs; case $\mathrm{E}M_i < \infty$. $C_0$ is a nonzero constant.

| | | $\mathbf{E}\boldsymbol{T}_i = \boldsymbol{\tau} < \infty$ | | |
|---|---|---|---|---|
| | | $T_i$ satisfies (B) with $1 < \alpha_T < 2$ | $\mathrm{D}^2T_i < \infty$ | otherwise |
| $\mathbf{E}\boldsymbol{R}_i = \boldsymbol{\varrho} \neq \mathbf{0}$ | | Th. 5.9(a) and | | |
| $\quad R_i = (\varrho/\tau)T_i$ | | Th. 5.1(a) | | |
| $\quad$ otherwise | | | | |
| $\quad\quad R_i$ satisfies (A) with $1 < \alpha_R < 2$ | $\alpha_R \neq \alpha_T$ | Th. 5.11(a1,a3) | | |
| | $\alpha_R = \alpha_T$: $R_i - (\varrho/\tau)T_i$ satisfies (A) | Th. 5.11(a2), Rem. 5.1 | Th. 5.11(b) | |
| | otherwise | ? | | |
| $\quad\quad \mathrm{D}^2R_i < \infty$ | | Th. 5.12(a) | Th. 5.12(b) | |
| $\quad\quad$ otherwise | | | | ? |
| $\mathbf{E}\boldsymbol{R}_i = \mathbf{0}$ | | | | |
| $\quad R_i$ satisfies (A) with $1 < \alpha_R < 2$ | | Th. 5.5(a) | | |
| $\quad \mathrm{D}^2R_i < \infty$ | | Th. 5.7(a) | | |
| $\quad$ otherwise | | | | ? |
| $\mathbf{E}\boldsymbol{R}_i \textbf{ does not exist}$ | | | | |
| $\quad R_i$ satisfies (A) with $0 < \alpha_R < 1$ | | Th. 5.3(a) | | |
| $\quad$ otherwise | | | | ? |

| | | $\mathbf{E}\boldsymbol{T}_i = \infty$ | |
|---|---|---|---|
| | | $T_i$ satisfies (B) with $0 < \alpha_T < 1$ | otherwise |
| $\mathbf{E}\boldsymbol{R}_i \neq \mathbf{0}$ | | Th. 5.9(b) | ? |
| $\mathbf{E}\boldsymbol{R}_i = \mathbf{0}$ | | | |
| $\quad R_i$ satisfies (A) with $1 < \alpha_R < 2$ | $R_i \perp\!\!\!\perp T_i$ | Th. 5.5(b) | |
| | otherwise | ? | |
| $\quad \mathrm{D}^2R_i < \infty$ | $R_i \perp\!\!\!\perp T_i$ | Th. 5.7(b) | |
| | otherwise | ? | |
| $\quad$ otherwise | | | ? |
| $\mathbf{E}\boldsymbol{R}_i \textbf{ does not exist}$ | | | |
| $\quad R_i$ satisfies (A) with $0 < \alpha_R < 1$ | $R_i = C_0 T_i$ | Th. 5.1(b) | |
| | $R_i \perp\!\!\!\perp T_i$ | Th. 5.3(b) | |
| | otherwise | ? | |
| $\quad$ otherwise | | | ? |

**Table 3.** Guide to the limit theorems for randomly coarsed grained CTRWs; case when $M_i$ satisfies (B) with $0 < \alpha_M < 1$. $C_0$ is a nonzero constant.

| | | | $\mathbf{E}T_i < \infty$ | |
|---|---|---|---|---|
| $\mathbf{E}R_i \neq 0$ | | | Th. 5.10(a) | |
| $\mathbf{E}R_i = 0$ | | | | |
| | $R_i$ satisfies (A) with $1 < \alpha_R < 2$ | $R_i \perp T_i$ | Th. 5.6(a) | |
| | | otherwise | | ? |
| | $D^2 R_i < \infty$ | $R_i \underline{\perp} T_i$ | Th. 5.8(a) | |
| | | otherwise | | ? |
| | otherwise | | | ? |
| $\mathbf{E}R_i$ **does not exist** | | | | |
| | $R_i$ satisfies (A) with $0 < \alpha_R < 1$ | $R_i \perp T_i$ | Th. 5.4(a) | |
| | | otherwise | | ? |
| | otherwise | | | ? |

| | | | $\mathbf{E}T_i = \infty$ | |
|---|---|---|---|---|
| | | | $T_i$ satisfies (B) with $0 < \alpha_T < 1$ | otherwise |
| $\mathbf{E}R_i \neq 0$ | | | Th. 5.10(b) | ? |
| $\mathbf{E}R_i = 0$ | | | | |
| | $R_i$ satisfies (A) with $1 < \alpha_R < 2$ | $R_i \underline{\perp} T_i$ | Th. 5.6(b) | |
| | | otherwise | ? | |
| | $D^2 R_i < \infty$ | $R_i \perp T_i$ | Th. 5.8(b) | |
| | | otherwise | ? | |
| | otherwise | | | ? |
| $\mathbf{E}R_i$ **does not exist** | | | | |
| | $R_i$ satisfies (A) with $0 < \alpha_R < 1$ | $R_i = C_0 T_i$ | Th. 5.2(b) | |
| | | $R_i \perp T_i$ | Th. 5.4(b) | |
| | | otherwise | ? | |
| | otherwise | | | ? |

(b) If $T_i$ satisfies condition (B) with $\alpha = \lambda$ for some $0 < \lambda < 1$, and with $\sigma_0 = \tau_0 > 0$, then

$$\frac{\widetilde{R}_M(t)}{t} \xrightarrow[t\to\infty]{d} C\mathcal{B}_{\lambda\gamma}.$$

*Proof of Theorems 5.1 and 5.2.* When $R_i = CT_i$, we have $\overline{R}_j = C\overline{T}_j$ and hence $\widetilde{R}_M(t) = CS_{\overline{T}}(N_{\overline{T}}(t))$. Therefore the conclusions follow directly from Proposition 2.8 and the following lemma:

LEMMA 5.1. (a) *Assume that* $\mathbf{E}M_i = \mu < \infty$.

(a1) *If* $\mathbf{E}T_i = \tau < \infty$ *then* $\mathbf{E}\overline{T}_j = \tau\mu < \infty$.

(a2) *If* $T_i$ *satisfies condition* (B) *with* $\alpha = \lambda$ *for some* $0 < \lambda < 1$, *and with* $\sigma_0 = \tau_0 > 0$, *then so does* $\overline{T}_j$ *with* $\alpha = \lambda$ *and* $\sigma_0 = \tau_0\mu^{1/\lambda}$.

(b) *Assume that $M_i$ satisfies condition* (B) *with* $\alpha = \gamma$ *for some* $0 < \gamma < 1$, *and with* $\sigma_0 = \mu_0 > 0$.

   (b1) *If* $\mathrm{E}T_i = \tau < \infty$ *then* $\overline{T}_j$ *satisfies condition* (B) *with* $\alpha = \gamma$ *and* $\sigma_0 = \tau\mu_0$.

   (b2) *If* $T_i$ *satisfies condition* (B) *with* $\alpha = \lambda$ *for some* $0 < \lambda < 1$, *and with* $\sigma_0 = \tau_0 > 0$, *then so does* $\overline{T}_j$ *with* $\alpha = \lambda\gamma$ *and*

$$\sigma_0 = \tau_0 \mu_0^{1/\lambda} \left( \frac{\Gamma(1-\gamma)(\Gamma(1-\lambda))^\gamma}{\Gamma(1-\lambda\gamma)} \right)^{1/\lambda\gamma}.$$

(c) *The distribution of* $\overline{T}_j$ *is* $\overline{\delta}$-*arithmetic (for some* $\overline{\delta} > 0$) *if and only if* $T_i$ *has a* $\delta$-*arithmetic distribution (with some* $0 < \delta \le \overline{\delta}$).

*Proof of Lemma 5.1.* Part (a1) is a standard result for random sums (see e.g. [16, V 9-11]). Since the Laplace transform $\psi_{\overline{T}}$ of $\overline{T}_j$ is

$$\psi_{\overline{T}}(t) = \psi_M(-\ln \psi_T(t)),$$

where $\psi_M$ and $\psi_T$ are the Laplace transforms of $M_i$ and $T_i$, respectively, parts (a2), (b1), and (b2) can be obtained by the technique of Proposition 2.4. Namely, for $x = -\ln \psi_T(t) \to 0+$ as $t \to 0+$ we have

- in case (a2),

$$\frac{1 - \psi_{\overline{T}}(t)}{(\mu^{1/\lambda}\tau_0 t)^\lambda} = \frac{1 - \psi_M(x)}{\mu x} \cdot \frac{x}{e^x - 1} \cdot \frac{1 - \psi_T(t)}{(\tau_0 t)^\lambda},$$

   and from Proposition 2.4,

$$\lim_{t\to 0+} \frac{1 - \psi_{\overline{T}}(t)}{(\sigma_0 t)^\lambda} = \Gamma(1 - \lambda)$$

   with $\sigma_0$ of the given form;

- in case (b1),

$$\frac{1 - \psi_{\overline{T}}(t)}{(\mu_0 \tau t)^\gamma} = \frac{1 - \psi_M(x)}{(\mu_0 x)^\gamma} \cdot \left( \frac{x}{e^x - 1} \right)^\gamma \cdot \left( \frac{1 - \psi_T(t)}{\tau t} \right)^\gamma,$$

   and from Proposition 2.4,

$$\lim_{t\to 0+} \frac{1 - \psi_{\overline{T}}(t)}{(\sigma_0 t)^\gamma} = \Gamma(1 - \gamma)$$

   with $\sigma_0$ of the given form;

- in case (b2),

$$\frac{1 - \psi_{\overline{T}}(t)}{(\mu_0^{1/\lambda}\tau_0 t)^{\lambda\gamma}} = \frac{1 - \psi_M(x)}{(\mu_0 x)^\gamma} \cdot \left( \frac{x}{e^x - 1} \right)^\gamma \cdot \left( \frac{1 - \psi_T(t)}{(\tau_0 t)^\lambda} \right)^\gamma,$$

   and from Proposition 2.4,

$$\lim_{t\to 0+} \frac{1 - \psi_{\overline{T}}(t)}{(\sigma_0 t)^{\lambda\gamma}} = \Gamma(1 - \lambda\gamma)$$

   with $\sigma_0$ of the given form.

Applying Proposition 2.4 again we obtain the respective conclusions.

In order to prove (c) let us mention the well known fact ([16, XV 1, Lemma 3]) that for a positive random variable $X$, we have $\sum_{n=1}^{\infty} P(X = n\delta) = 1$ if and only if $\phi_X(2\pi/\delta) = 1$,

where $\phi_X$ is the characteristic function of $X$. Hence, if $T_i$ has a $\delta$-arithmetic distribution then its characteristic function $\phi_T$ satisfies $\phi_T(2\pi/\delta) = 1$ and as a consequence, for $\phi_{\overline{T}}$, the characteristic function of $\overline{T}_j$, one gets $\phi_{\overline{T}}(2\pi/\delta) = \sum_{m=1}^{\infty}(\phi_T(2\pi/\delta))^m P(M_i = m)$ = 1. Therefore $\sum_{n=1}^{\infty} P(\overline{T}_j = n\delta) = 1$ so that the distribution of $\overline{T}_j$ is $\overline{\delta}$-arithmetic for some $\overline{\delta} \geq \delta$.

On the other hand, if $\overline{T}_j$ has a $\overline{\delta}$-arithmetic distribution then $\phi_{\overline{T}}(2\pi/\overline{\delta}) = 1$ and hence $\sum_{m=1}^{\infty}(1 - |\phi_T(2\pi/\overline{\delta})|^m)P(M_i = m) = 0$. Therefore $|\phi_T(2\pi/\overline{\delta})| = 1$ so that, according to [16, XV, 1, Lemma 4], $\sum_{n=1}^{\infty} P(T_i = b + n\overline{\delta}) = 1$ and $\phi_T(2\pi/\overline{\delta}) = e^{i2\pi b/\overline{\delta}}$ for some $b$. Moreover, the constant $b$ has to satisfy $\sum_{m=1}^{\infty}(1 - \cos(2\pi bm/\overline{\delta}))P(M_i = m) = 0$, and hence, $b = \overline{\delta}k/m_0$ for some positive integer $k$ and for $m_0$ such that $P(M_j = m_0) > 0$. As a consequence, we obtain $\sum_{n=1}^{\infty} P(T_i = n\delta_0) = 1$ for $\delta_0 = \overline{\delta}/m_0$. ∎

Let us now consider more general classes of CTRWs:

THEOREM 5.3. *Assume that* $\mathrm{E}M_i < \infty$ *and* $R_i$ *satisfies condition* (A) *with* $\alpha = \kappa$ *for some* $0 < \kappa < 1$, *with* $\sigma_0 = \varrho_0 > 0$, *and some* $\beta$, $|\beta| \leq 1$.

(a) *If* $\mathrm{E}T_i = \tau < \infty$ *then*

$$(5.1) \qquad \frac{\widetilde{R}_M(t)}{t^{1/\kappa}} \xrightarrow[t\to\infty]{d} C\mathcal{S}_{\kappa,\beta}, \quad \text{where} \quad C = \varrho_0\left(\frac{q(\kappa)}{\tau}\right)^{1/\kappa}.$$

(b) *If* $T_i$ *satisfies condition* (B) *with* $\alpha = \lambda$ *for some* $0 < \lambda < 1$, *and with* $\sigma_0 = \tau_0 > 0$, *and if* $T_i$ *and* $R_i$ *are independent, then*

$$(5.2) \qquad \frac{\widetilde{R}_M(t)}{t^{\lambda/\kappa}} \xrightarrow[t\to\infty]{d} C\frac{\mathcal{S}_{\kappa,\beta}}{\mathcal{S}_{\lambda,1}^{\lambda/\kappa}}, \quad \text{where} \quad C = \frac{\varrho_0}{\tau_0^{\lambda/\kappa}}\left(\frac{q(\kappa)}{q(\lambda)}\right)^{1/\kappa},$$

*and the stable random variables* $\mathcal{S}_{\kappa,\beta}$ *and* $\mathcal{S}_{\lambda,1}$ *are independent.*

*Proof.* From Proposition 2.3,

$$(5.3) \qquad \frac{S_R(n)}{n^{1/\kappa}} \xrightarrow[n\to\infty]{d} C_1\mathcal{S}_{\kappa,\beta}, \quad \text{where} \quad C_1 = \varrho_0(q(\kappa))^{1/\kappa}.$$

If $\mathrm{E}T_i = \tau < \infty$, then $L(t)/t \xrightarrow{\text{a.s.}} 1/\tau$ as $t \to \infty$ by Theorem 4.3(a); and from Proposition 2.12 and Theorem 4.2 we obtain (a).

If instead $\lim_{t\to\infty} \Pr(T_i > t)/(t/\tau_0)^{-\lambda} = 1$ for some $\tau_0 > 0$ and $0 < \lambda < 1$, then it follows from Theorem 4.3(b) that $L(t)/t^{\lambda} \xrightarrow{d} C_1(1/\mathcal{S}_{\lambda,1}^{\lambda})$ as $t \to \infty$, where $C_1$ is as in (4.5). If $R_i$ and $T_i$ are independent, the family $\{L(t), t \geq 0\}$ is independent of $\{R_i, i = 1, 2, \ldots\}$, and by Proposition 2.10 one obtains (b). ∎

THEOREM 5.4. *Assume that* $M_i$ *satisfies condition* (B) *with* $\alpha = \gamma$ *for some* $0 < \gamma < 1$, *and with some* $\sigma_0 > 0$. *Suppose that* $R_i$ *and* $T_i$ *are independent, and* $R_i$ *satisfies condition* (A) *with* $\alpha = \kappa$ *for some* $0 < \kappa < 1$, *with* $\sigma_0 = \varrho_0 > 0$, *and some* $\beta$, $|\beta| \leq 1$.

(a) *If* $\mathrm{E}T_i = \tau < \infty$ *then*

$$\frac{\widetilde{R}_M(t)}{t^{1/\kappa}} \xrightarrow[t\to\infty]{d} C(\mathcal{B}_\gamma)^{1/\kappa}\mathcal{S}_{\kappa,\beta},$$

*where* $C$ *is as in* (5.1), *and the random variables* $\mathcal{B}_\gamma$ *and* $\mathcal{S}_{\kappa,\beta}$ *are independent.*

(b) *If $T_i$ satisfies condition* (B) *with $\alpha = \lambda$ for some $0 < \lambda < 1$, and with $\sigma_0 = \tau_0 > 0$, then*

$$\frac{\widetilde{R}_M(t)}{t^{\lambda/\kappa}} \xrightarrow[t\to\infty]{d} C(\mathcal{B}_\gamma)^{1/\kappa} \frac{\mathcal{S}_{\kappa,\beta}}{\mathcal{S}_{\lambda,1}^{\lambda/\kappa}},$$

*where $C$ is as in* (5.2), *and $\mathcal{B}_\gamma$, $\mathcal{S}_{\kappa,\beta}$, and $\mathcal{S}_{\lambda,1}$ are independent.*

*Proof.* From Proposition 2.3, $S_R(n)/n^{1/\kappa} \xrightarrow{d} C_1 \mathcal{S}_{\kappa,\beta}$ as $n \to \infty$, where $C_1$ is as in (5.3). If $\mathrm{E}T_i = \tau < \infty$, then $L(t)/t \xrightarrow{d} (1/\tau)\mathcal{B}_\gamma$ as $t \to \infty$ from Theorem 4.4(a). If instead $\lim_{t\to\infty} \Pr(T_i > t)/(t/\tau_0)^{-\lambda} = 1$ for some $\tau_0 > 0$ and $0 < \lambda < 1$, then it follows from Theorem 4.4(b) that $L(t)/t^\lambda \xrightarrow{d} C_1 \mathcal{B}_\gamma(1/\mathcal{S}_{\lambda,1}^\lambda)$ as $t \to \infty$, where $C_1$ is as in (4.5). Since $R_i$ and $T_i$ are independent, the family $\{L(t),\, t \geq 0\}$ is independent of $\{R_i,\, i = 1, 2, \ldots\}$, and by Proposition 2.10 one obtains both (a) and (b). ∎

THEOREM 5.5. *Let* $\mathrm{E}R_i = 0$. *Moreover, suppose $R_i$ satisfies condition* (A) *with $\alpha = \kappa$ for some $1 < \kappa < 2$, with $\sigma_0 = \varrho_0 > 0$, and some $\beta$, $|\beta| \leq 1$. Assume that $\mathrm{E}M_i < \infty$.*

(a) *If* $\mathrm{E}T_i = \tau < \infty$ *then*

$$\frac{\widetilde{R}_M(t)}{t^{1/\kappa}} \xrightarrow[t\to\infty]{d} C\mathcal{S}_{\kappa,\beta},$$

*where $C$ is as in* (5.1).

(b) *If $T_i$ satisfies condition* (B) *with $\alpha = \lambda$ for some $0 < \lambda < 1$, and with $\sigma_0 = \tau_0 > 0$, and if $T_i$ and $R_i$ are independent, then*

$$\frac{\widetilde{R}_M(t)}{t^{\lambda/\kappa}} \xrightarrow[t\to\infty]{d} C \frac{\mathcal{S}_{\kappa,\beta}}{\mathcal{S}_{\lambda,1}^{\lambda/\kappa}},$$

*where $C$ is as in* (5.2), *and the stable random variables $\mathcal{S}_{\kappa,\beta}$ and $\mathcal{S}_{\lambda,1}$ are independent.*

*Proof.* Since from Proposition 2.3, $S_R(n)/n^{1/\kappa} \xrightarrow{d} C_1 \mathcal{S}_{\kappa,\beta}$ as $n \to \infty$, where $C_1$ is as in (5.3), the proof is parallel to that of Theorem 5.3. ∎

THEOREM 5.6. *Assume that $R_i$ and $T_i$ are independent, $\mathrm{E}R_i = 0$, and $R_i$ satisfies condition* (A) *with $\alpha = \kappa$ for some $1 < \kappa < 2$, with $\sigma_0 = \varrho_0 > 0$, and some $\beta$, $|\beta| \leq 1$. Moreover, assume that $M_i$ satisfies condition* (B) *with $\alpha = \gamma$ for some $0 < \gamma < 1$, and with some $\sigma_0 > 0$.*

(a) *If* $\mathrm{E}T_i = \tau < \infty$ *then*

$$\frac{\widetilde{R}_M(t)}{t^{1/\kappa}} \xrightarrow[t\to\infty]{d} C(\mathcal{B}_\gamma)^{1/\kappa} \mathcal{S}_{\kappa,\beta},$$

*where $C$ is as in* (5.1), *and the random variables $\mathcal{B}_\gamma$ and $\mathcal{S}_{\kappa,\beta}$ are independent.*

(b) *If $T_i$ satisfies condition* (B) *with $\alpha = \lambda$ for some $0 < \lambda < 1$, and with $\sigma_0 = \tau_0 > 0$, then*

$$\frac{\widetilde{R}_M(t)}{t^{\lambda/\kappa}} \xrightarrow[t\to\infty]{d} C(\mathcal{B}_\gamma)^{1/\kappa} \frac{\mathcal{S}_{\kappa,\beta}}{\mathcal{S}_{\lambda,1}^{\lambda/\kappa}},$$

*where $C$ is as in* (5.2), *and $\mathcal{B}_\gamma$, $\mathcal{S}_{\kappa,\beta}$ and $\mathcal{S}_{\lambda,1}$ are independent.*

*Proof.* From Proposition 2.3, $S_R(n)/n^{1/\kappa} \overset{d}{\to} C_1 \mathcal{S}_{\kappa,\beta}$ as $n \to \infty$, where $C_1$ is as in (5.3). Hence, the proof is parallel to that of Theorem 5.4. ∎

THEOREM 5.7. *Let* $\mathrm{E}R_i = 0$, $\mathrm{D}^2 R_i = \sigma^2$ *for some* $0 < \sigma < \infty$, *and* $\mathrm{E}M_i < \infty$.

(a) *If* $\mathrm{E}T_i = \tau < \infty$ *then*

$$(5.4) \qquad\qquad \frac{\widetilde{R}_M(t)}{t^{1/2}} \xrightarrow[t\to\infty]{d} C\mathcal{G}, \quad where \quad C = \frac{\sigma}{\tau^{1/2}}.$$

(b) *If* $T_i$ *satisfies condition* (B) *with* $\alpha = \lambda$ *for some* $0 < \lambda < 1$, *and with* $\sigma_0 = \tau_0 > 0$, *and if* $T_i$ *and* $R_i$ *are independent, then*

$$(5.5) \qquad\qquad \frac{\widetilde{R}_M(t)}{t^{\lambda/2}} \xrightarrow[t\to\infty]{d} C\frac{\mathcal{G}}{\mathcal{S}_{\lambda,1}^{\lambda/2}}, \quad where \quad C = \frac{\sigma}{\tau_0^{\lambda/2}}(q(\lambda))^{-1/2},$$

*and the random variables* $\mathcal{G}$ *and* $\mathcal{S}_{\lambda,1}$ *are independent.*

*Proof.* Since from Proposition 2.3, $S_R(n)/n^{1/2} \overset{d}{\to} \sigma\mathcal{G}$ as $n \to \infty$, the proof is parallel to that of Theorem 5.3. ∎

THEOREM 5.8. *Assume that* $R_i$ *and* $T_i$ *are independent,* $\mathrm{E}R_i = 0$, *and* $\mathrm{D}^2 R_i = \sigma^2$ *for some* $0 < \sigma < \infty$. *Moreover, assume that* $M_i$ *satisfies condition* (B) *with* $\alpha = \gamma$ *for some* $0 < \gamma < 1$, *and with some* $\sigma_0 > 0$.

(a) *If* $\mathrm{E}T_i = \tau < \infty$ *then*

$$\frac{\widetilde{R}_M(t)}{t^{1/2}} \xrightarrow[t\to\infty]{d} C(\mathcal{B}_\gamma)^{1/2}\mathcal{G},$$

*where* $C$ *is as in* (5.4), *and the random variables* $\mathcal{B}_\gamma$ *and* $\mathcal{G}$ *are independent.*

(b) *If* $T_i$ *satisfies condition* (B) *with* $\alpha = \lambda$ *for some* $0 < \lambda < 1$, *and with* $\sigma_0 = \tau_0 > 0$, *then*

$$\frac{\widetilde{R}_M(t)}{t^{\lambda/2}} \xrightarrow[t\to\infty]{d} C(\mathcal{B}_\gamma)^{1/2}\frac{\mathcal{G}}{\mathcal{S}_{\lambda,1}^{\lambda/2}},$$

*where* $C$ *is as in* (5.5), *and* $\mathcal{B}_\gamma$, $\mathcal{G}$ *and* $\mathcal{S}_{\lambda,1}$ *are independent.*

*Proof.* From Proposition 2.3, $S_R(n)/n^{1/2} \overset{d}{\to} \sigma\mathcal{G}$ as $n \to \infty$. Hence, the proof is parallel to that of Theorem 5.4. ∎

Let us note that Theorems 5.3(b), 5.5(b), 5.6, 5.7(b) and 5.8 have been proved only for $T_i$ and $R_i$ independent. It follows from Theorem 5.1(b) that this assumption cannot be just omitted in Theorem 5.3(b). However, it is an open question if it is necessary. For more comments, see Section 6.

THEOREM 5.9. *Let* $\mathrm{E}R_i = \varrho \neq 0$ *and let* $\mathrm{E}M_i < \infty$.

(a) *If* $\mathrm{E}T_i = \tau < \infty$ *then*

$$\frac{\widetilde{R}_M(t)}{t} \xrightarrow[t\to\infty]{a.s.} \frac{\varrho}{\tau}.$$

(b) *If* $T_i$ *satisfies condition* (B) *with* $\alpha = \lambda$ *for some* $0 < \lambda < 1$, *and with* $\sigma_0 = \tau_0 > 0$, *then*

$$(5.6) \qquad \frac{\widetilde{R}_M(t)}{t^\lambda} \xrightarrow[t\to\infty]{d} C \frac{1}{\mathcal{S}_{\lambda,1}^\lambda}, \quad \text{where} \quad C = \frac{\varrho}{\tau_0^\lambda}\, (q(\lambda))^{-1}.$$

*Proof.* From the strong law of large numbers $S_R(n)/n \xrightarrow{\text{a.s.}} \varrho$ as $n \to \infty$. Since $L(t) \xrightarrow{\text{a.s.}} \infty$ as $t \to \infty$ (Theorem 4.2),

$$\frac{\widetilde{R}_M(t)}{L(t)} = \frac{S_R(L(t))}{L(t)} \xrightarrow[t\to\infty]{\text{a.s.}} \varrho$$

according to Proposition 2.11. Then, in case $\mathrm{E}T_i = \tau < \infty$, we deduce from Theorem 4.3(a) that

$$\frac{\widetilde{R}_M(t)}{t} = \frac{\widetilde{R}_M(t)}{L(t)} \cdot \frac{L(t)}{t} \xrightarrow[t\to\infty]{\text{a.s.}} \varrho \cdot \frac{1}{\tau}.$$

If instead $\lim_{t\to\infty} \Pr(T_i > t)/(t/\tau_0)^{-\lambda} = 1$ for some $\tau_0 > 0$ and $0 < \lambda < 1$, then it follows from Theorem 4.3(b) and from Lemma 2 of [16, VIII 2] that

$$\frac{\widetilde{R}_M(t)}{t^\lambda} = \frac{\widetilde{R}_M(t)}{L(t)} \cdot \frac{L(t)}{t^\lambda} \xrightarrow[t\to\infty]{d} \varrho \cdot C_1 \frac{1}{\mathcal{S}_{\lambda,1}^\lambda},$$

where $C_1$ is given by (4.5). ∎

THEOREM 5.10. *Let* $\mathrm{E}R_i = \varrho \neq 0$. *Assume that* $M_i$ *satisfies condition* (B) *with* $\alpha = \gamma$ *for some* $0 < \gamma < 1$, *and with some* $\sigma_0 > 0$.

(a) *If* $\mathrm{E}T_i = \tau < \infty$ *then*

$$\frac{\widetilde{R}_M(t)}{t} \xrightarrow[t\to\infty]{d} \frac{\varrho}{\tau} \mathcal{B}_\gamma.$$

(b) *If* $T_i$ *satisfies condition* (B) *with* $\alpha = \lambda$ *for some* $0 < \lambda < 1$, *and with* $\sigma_0 = \tau_0 > 0$, *then*

$$\frac{\widetilde{R}_M(t)}{t^\lambda} \xrightarrow[t\to\infty]{d} C\mathcal{B}_\gamma \frac{1}{\mathcal{S}_{\lambda,1}^\lambda},$$

*where* $C$ *is as in* (5.6), *and* $\mathcal{B}_\gamma$ *and* $\mathcal{S}_{\lambda,1}$ *are independent.*

*Proof.* The proof is parallel to that of Theorem 5.9 (with Theorem 4.4 used instead of Theorem 4.3). ∎

Observe that Theorems 5.9(a) and 5.10(a) cover the special case $R_i = CT_i$ for which the conclusions coincide indeed with those of Theorems 5.1(a) and 5.2(a). More detailed investigations of the case considered in Theorem 5.9(a) lead to the following results on the asymptotic behavior of the difference between $\widetilde{R}_M(t)/t$ and its constant limit $\varrho/\tau$:

THEOREM 5.11. *Let* $\mathrm{E}M_i < \infty$, $\mathrm{E}R_i = \varrho \neq 0$ *and* $\mathrm{E}T_i = \tau < \infty$. *Moreover, suppose* $R_i$ *satisfies condition* (A) *with* $\alpha = \kappa$ *for some* $1 < \kappa < 2$, *with* $\sigma_0 = \varrho_0 > 0$, *and* $\beta = \beta_0$ *for some* $|\beta_0| \leq 1$.

(a) *If* $T_i$ *satisfies condition* (B) *with* $\alpha = \lambda$ *for some* $1 < \lambda < 2$, *and with* $\sigma_0 = \tau_0 > 0$, *then*

(a1) *in case* $\lambda > \kappa$,

$$\frac{\widetilde{R}_M(t) - (\varrho/\tau)t}{t^{1/\kappa}} \xrightarrow[t\to\infty]{d} C\mathcal{S}_{\kappa,\beta_0},$$

*where* $C$ *is as in* (5.1);

(a2) *in case $\lambda = \kappa$, if $R_i$ and $T_i$ are independent,*

$$\frac{\widetilde{R}_M(t) - (\varrho/\tau)t}{t^{1/\kappa}} \xrightarrow[t\to\infty]{d} C\mathcal{S}_{\kappa,\beta_1}, \quad where \quad C = \varrho_1 \left( \frac{q(\kappa)}{\tau} \right)^{1/\kappa}$$

*with*

$$\varrho_1 = [\varrho_0^\kappa + (|\varrho|\tau_0/\tau)^\kappa]^{1/\kappa},$$

$$\beta_1 = \frac{\beta_0 \varrho_0^\kappa - \operatorname{sgn}(\varrho)(|\varrho|\tau_0/\tau)^\kappa}{\varrho_0^\kappa + (|\varrho|\tau_0/\tau)^\kappa};$$

(a3) *in case $\lambda < \kappa$,*

$$(5.7) \qquad \frac{\widetilde{R}_M(t) - (\varrho/\tau)t}{t^{1/\lambda}} \xrightarrow[t\to\infty]{d} C\mathcal{S}_{\lambda,-1}, \quad where \quad C = \frac{\varrho\tau_0}{\tau}\left( \frac{q(\lambda)}{\tau} \right)^{1/\lambda}.$$

(b) *If $\mathrm{D}^2 T_i < \infty$ then*

$$\frac{\widetilde{R}_M(t) - (\varrho/\tau)t}{t^{1/\kappa}} \xrightarrow[t\to\infty]{d} C\mathcal{S}_{\kappa,\beta_0},$$

*where $C$ is as in* (5.1).

*Proof.* For any $a$ we have

$$(5.8) \qquad \frac{\widetilde{R}_M(t) - (\varrho/\tau)t}{t^a} = \frac{S_R(L(t)) - \varrho L(t)}{t^a} + \frac{\varrho(L(t) - t/\tau)}{t^a}.$$

It follows from Propositions 2.3, 2.12, and Theorems 4.2, 4.3 that

$$\frac{S_R(L(t)) - \varrho L(t)}{t^{1/\kappa}} \xrightarrow[t\to\infty]{d} \frac{C_1}{\tau^{1/\kappa}} \mathcal{S}_{\kappa,\beta},$$

where $C_1$ is as in (5.3); while the nondegenerate asymptotic distribution of $L(t) - t/\tau$ divided by $t^{1/\lambda}$ (or $t^{1/2}$) has been determined by Theorem 4.5. By Lemma 2 of [16, VIII 2], one finds that for $a = 1/\kappa$ the second term of the right-hand side of (5.8) tends to 0 with probability 1 if $\lambda > \kappa$ or if $\mathrm{D}^2 T_i < \infty$, and hence (a1) and (b) hold. Similarly, if $\lambda < \kappa$ then for $a = 1/\lambda$ the first summand in (5.8) tends to 0 with probability 1, proving (a3).

In order to study the case $\lambda = \kappa$ consider the random variable $R_i^* = R_i - (\varrho/\tau)T_i$. Observe that $\mathrm{E}R_i^* = 0$ and, since $R_i$ and $T_i$ are independent, $R_i^*$ satisfies (A) with $\alpha = \kappa$, $\sigma_0 = \varrho_1$, and $\beta = \beta_1$ (see [16, VIII 8, Example (c)]). Since $(T, R^*) = \{(T_i, R_i^*)\}$ is an i.i.d. sequence independent of $M$ and $\overline{R_j^*} = \overline{R}_j - (\varrho/\tau)\overline{T}_j$, we have

$$(5.9) \qquad \frac{\widetilde{R}_M(t) - (\varrho/\tau)t}{t^{1/\kappa}} = \frac{\widetilde{R}_M^*(t)}{t^{1/\kappa}} + \frac{\varrho}{\tau}\left( S_{\overline{T}}(N_{\overline{T}}(t)) - t \right) \cdot \frac{1}{t^{1/\kappa}},$$

where $\{\widetilde{R}_M^*(t)\}$ is the CTRW generated by $(T, R^*)$ and randomly coarse grained by $M$. It follows from Theorem 5.1, Proposition 2.8, and Lemma 2 of [16, V 9-11] ($^4$) that the

---

($^4$) Moreover, in case $T_i$ has an arithmetic distribution, considerations similar to those used in the proof of Theorem 4.5 are needed.

second term on the right-hand side of (5.9) tends to 0 with probability 1. Then one can apply Theorem 5.5 to the process $\{\widetilde{R}_M^*(t)\}$ to infer that the first term tends in distribution to $C\mathcal{S}_{\kappa,\beta_1}$, which shows (a2). ∎

REMARK 5.1. It can be seen from Theorem 5.1 that the assumption of Theorem 5.11(a2) that $T_i$ and $R_i$ are independent cannot be just omitted. However, as the proof of this part suggests, instead of the independence one can assume that $R_i - (\varrho/\tau)T_i$ satisfies condition (A) with $\alpha = \kappa_1$, $\sigma_0 = \varrho_1$, and $\beta = \beta_1$ for some $1 < \kappa_1 < 2$, $\varrho_1 > 0$, and $|\beta_1| \leq 1$, to obtain

$$\frac{\widetilde{R}_M(t) - (\varrho/\tau)t}{t^{1/\kappa_1}} \xrightarrow[t\to\infty]{d} C\mathcal{S}_{\kappa_1,\beta_1}, \quad \text{where} \quad C = \varrho_1\left(\frac{q(\kappa_1)}{\tau}\right)^{1/\kappa_1}.$$

THEOREM 5.12. Let $\mathrm{E}M_i < \infty$, $\mathrm{E}R_i = \varrho \neq 0$, $\mathrm{E}T_i = \tau < \infty$, and $\mathrm{D}^2 R_i = \sigma^2 < \infty$.

    (a) If $T_i$ satisfies condition (B) with $\alpha = \lambda$ for some $1 < \lambda < 2$, and with $\sigma_0 = \tau_0 > 0$, then

$$\frac{\widetilde{R}_M(t) - (\varrho/\tau)t}{t^{1/\lambda}} \xrightarrow[t\to\infty]{d} C\mathcal{S}_{\lambda,-1},$$

    where $C$ is as in (5.7).

    (b) If $\mathrm{D}^2 T_i < \infty$ and $P(R_i = (\varrho/\tau)T_i) < 1$, then

$$\frac{\widetilde{R}_M(t) - (\varrho/\tau)t}{t^{1/2}} \xrightarrow[t\to\infty]{d} C\mathcal{G},$$

    where

$$C = \left(\frac{\sigma^2 + (\varrho/\tau)^2\mathrm{D}^2 T_i - 2(\varrho/\tau)\mathrm{Cov}(R_i,T_i)}{\tau}\right)^{1/2}.$$

*Proof.* Part (a) can be shown similarly to Theorem 5.11(a3).

    For (b) consider $R_i^* = R_i - (\varrho/\tau)T_i$. Observe that $\mathrm{E}R_i^* = 0$ and

$$\mathrm{D}^2 R_i^* = \sigma^2 + \left(\frac{\varrho}{\tau}\right)^2 \mathrm{D}^2 T_i - 2\left(\frac{\varrho}{\tau}\right)\mathrm{Cov}(R_i,T_i) < \infty;$$

also $\mathrm{D}^2 R_i^* > 0$ since $P(R_i = (\varrho/\tau)T_i) < 1$. Hence, we can apply Theorem 5.7 to the randomly coarse grained CTRW $\{\widetilde{R}_M^*(t)\}$ generated by $(T_i, R_i^*)$ and $M$, and similarly to Theorem 5.11(a2), we obtain the conclusion of (b). ∎

    Note that, as Theorem 5.1 shows, the assumption that $P(R_i = (\varrho/\tau)T_i) < 1$ is essential in Theorem 5.12(b).

    The probability laws that have been obtained as weak limits of $\widetilde{R}_M(t)/(Ct^a)$ for $t \to \infty$ with parameters $C > 0$ and $a$ appropriately chosen are collected in Tables 4 and 5. Except for the stable, normal, and generalized arcsine distributions one can recognize there the transstable probability law of $1/\mathcal{S}_{\gamma,1}^\gamma$, $0 < \gamma < 1$ [16, 70], and the fractional stable distributions of $\mathcal{G}/\mathcal{S}_{\gamma,1}^{\gamma/2}$ and $\mathcal{S}_{\alpha,\beta}/\mathcal{S}_{\gamma,1}^{\gamma/\alpha}$, $0 < \alpha < 2$, $0 < \gamma < 1$, $|\beta| \leq 1$ [39, 60]. Other limits in distribution are mixtures of those.
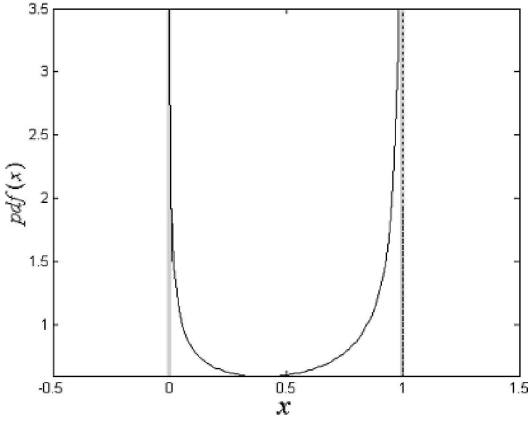
Fig. 3. $ER_i \neq 0$, $ET_i < \infty$, and $EM_i < \infty$ (dashed line) or $M_i$ satisfies (B) with $\alpha_M = 0.6$ (solid line). The grey lines indicate the asymptotes.
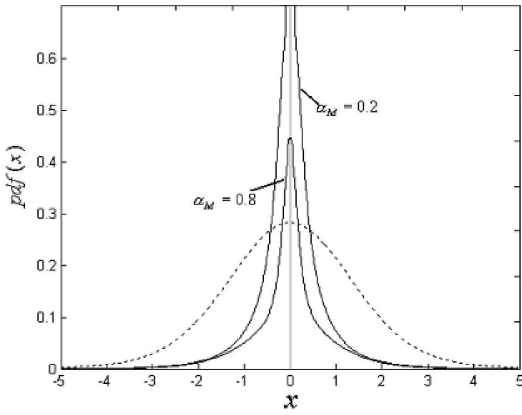


Fig. 4. $ER_i = 0$, $R_i$ satisfies (A) with $\alpha_R = 1.8 \in (1, 2)$ and $\beta_R = 0$; $ET_i < \infty$; and $EM_i < \infty$ (dashed line) or $M_i$ satisfies (B) with $\alpha_M = 0.8$ and $0.2$ (solid lines); $R_i$ is independent of $T_i$ if $EM_i = \infty$. The grey line is the asymptote of the density function corresponding to $\alpha_M = 0.2 \leq 0.5$.
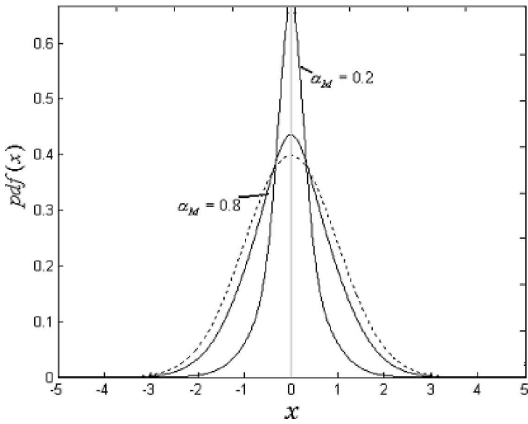


Fig. 5. $ER_i = 0$, $D^2 R_i < \infty$; $ET_i < \infty$; and $EM_i < \infty$ (dashed line) or $M_i$ satisfies (B) with $\alpha_M = 0.8$ and $0.2$ (solid lines); $R_i$ is independent of $T_i$ if $EM_i = \infty$. The grey line is the asymptote of the density function corresponding to $\alpha_M = 0.2 \leq 0.5$.
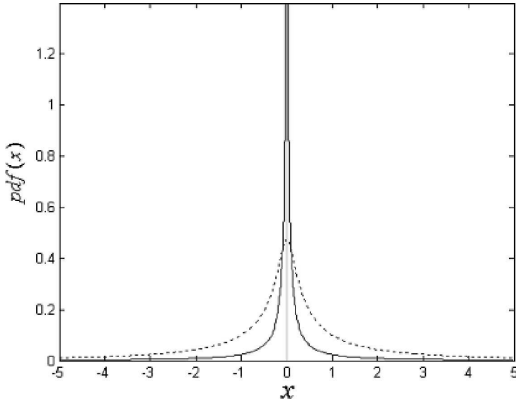
Fig. 6. $R_i$ satisfies (A) with $\alpha_R = 0.6 < 1$ and $\beta_R = 0$; $\mathrm{E}T_i < \infty$; and $\mathrm{E}M_i < \infty$ (dashed line) or $M_i$ satisfies (B) with $\alpha_M = 0.2$ (solid line); $R_i$ is independent of $T_i$ if $\mathrm{E}M_i = \infty$. The grey line indicates the asymptote of the density function corresponding to $M_i$ satisfying (B).
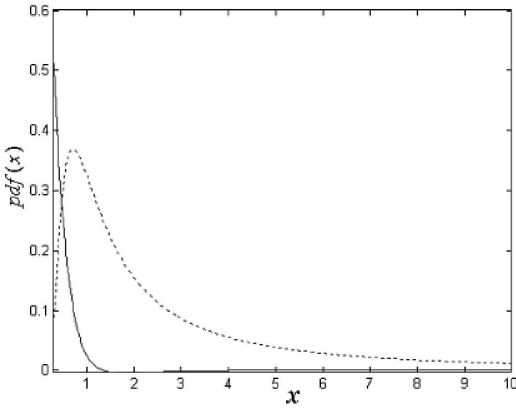


Fig. 7. $\mathrm{E}R_i \neq 0$; $T_i$ satisfies (B) with $\alpha_T = 0.7 < 1$; and $\mathrm{E}M_i < \infty$ (dashed line) or $M_i$ satisfies (B) with $\alpha_M = 0.2$ (solid line).
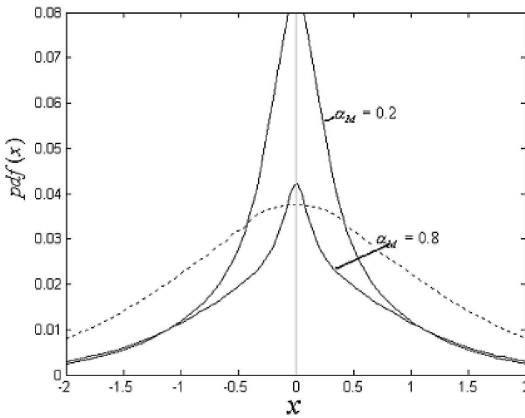


Fig. 8. $\mathrm{E}R_i = 0$, $R_i$ satisfies (A) with $\alpha_R = 1.8 \in (1,2)$ and $\beta_R = 0$; $T_i$ satisfies (B) with $\alpha_T = 0.6 < 1$; $R_i$ is independent of $T_i$; and $\mathrm{E}M_i < \infty$ (dashed line) or $M_i$ satisfies (B) with $\alpha_M = 0.8$ and $0.2$ (solid lines). The grey line is the asymptote of the density function corresponding to $\alpha_M = 0.2 \leq 0.5$.
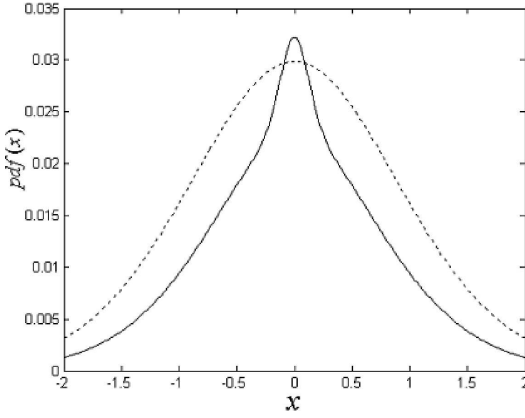
Fig. 9. $ER_i = 0$, $D^2R_i < \infty$; $T_i$ satisfies (B) with $\alpha_T = 0.8 < 1$; $R_i$ independent of $T_i$; and $EM_i < \infty$ (dashed line) or $M_i$ satisfies (B) with $\alpha_M = 0.8$ (solid line).



Fig. 10. $R_i \propto T_i$, $T_i$ satisfies (B) with $\alpha_T = 0.8 < 1$; and $EM_i < \infty$ (dashed line) or $M_i$ satisfies (B) with $\alpha_M = 0.8$ (solid line). The grey lines indicate the asymptotes.



Fig. 11. $R_i$ satisfies (A) with $\alpha_R = 0.6 < 1$ and $\beta_R = 0$; $T_i$ satisfies (B) with $\alpha_T = 0.4 < 1$; $R_i$ is independent of $T_i$; and $EM_i < \infty$ (dashed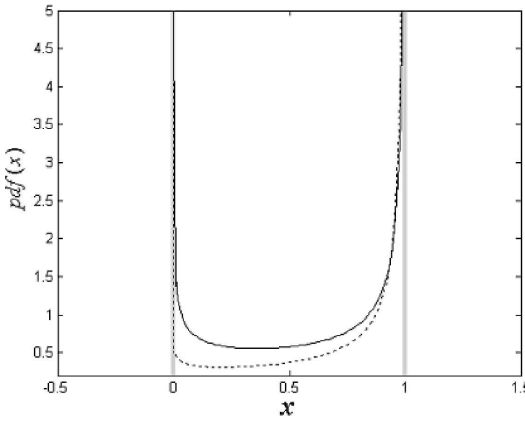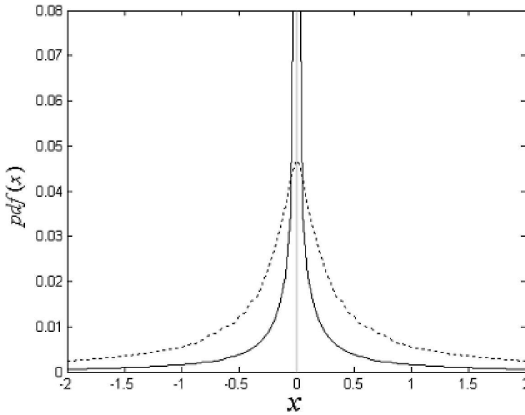 line) or $M_i$ satisfies (B) with $\alpha_M = 0.8$ (solid line). The grey line is the asymptote of the density function corresponding to $M_i$ satisfying (B)

**Table 4.** Limits in distribution for $\widetilde{R}_M(t)/(Ct^a)$ as $t \to \infty$ with $C > 0$ and $a$ appropriately chosen; case $\mathrm{E}M_i < \infty$. $C_0$ is a nonzero constant.

| | $\mathbf{E}T_i < \infty$ | |
|---|:---:|:---:|
| $\mathbf{E}R_i \neq 0$ | 1 | |
| $\mathbf{E}R_i = 0$ | | |
| $\quad R_i$ satisfies (A) with $1 < \alpha_R < 2$ | $\mathcal{S}_{\alpha_R,\beta_R}$ | |
| $\quad\quad \mathrm{D}^2 R_i < \infty$ | $\mathcal{G}$ | |
| $\quad\quad\quad$ otherwise | | ? |
| $\mathbf{E}R_i$ **does not exist** | | |
| $\quad R_i$ satisfies (A) with $0 < \alpha_R < 1$ | $\mathcal{S}_{\alpha_R,\beta_R}$ | |
| $\quad\quad$ otherwise | | ? |

| | | $\mathbf{E}T_i = \infty$ | | |
|---|---|:---:|:---:|:---:|
| | | $T_i$ satisfies (B) with $0 < \alpha_T < 1$ | | otherwise |
| $\mathbf{E}R_i \neq 0$ | | $\dfrac{1}{\mathcal{S}_{\alpha_T,1}^{\alpha_T}}$ | | ? |
| $\mathbf{E}R_i = 0$ | | | | |
| $\quad R_i$ satisfies (A) with $1 < \alpha_R < 2$ | $R_i \perp\!\!\!\perp T_i$ | $\dfrac{\mathcal{S}_{\alpha_R,\beta_R}}{\mathcal{S}_{\alpha_T,1}^{\alpha_T/\alpha_R}}$ | | |
| | otherwise | | ? | |
| $\quad\quad \mathrm{D}^2 R_i < \infty$ | $R_i \perp\!\!\!\perp T_i$ | $\dfrac{\mathcal{G}}{\mathcal{S}_{\alpha_T,1}^{\alpha_T/2}}$ | | |
| | otherwise | | ? | |
| $\quad\quad$ otherwise | | | | ? |
| $\mathbf{E}R_i$ **does not exist** | | | | |
| $\quad R_i$ satisfies (A) with $0 < \alpha_R < 1$ | $R_i = C_0 T_i$ | $\mathcal{B}_{\alpha_T}$ | | |
| | $R_i \perp\!\!\!\perp T_i$ | $\dfrac{\mathcal{S}_{\alpha_R,\beta_R}}{\mathcal{S}_{\alpha_T,1}^{\alpha_T/\alpha_R}}$ | | |
| | otherwise | | ? | |
| $\quad\quad$ otherwise | | | | ? |

All nondegenerate distributions quoted in Tables 4 and 5 are continuous. Figures 3–11 present their densities, numerically evaluated for some values of the parameters. In each figure we compare the limiting density functions resulting from the RCG transformation of CTRWs from the same class in the cases of $\mathrm{E}M_i$ finite and of $M_i$ satisfying condition (B) with $\alpha = \gamma$ for some $0 < \gamma < 1$.

**Table 5.** Limits in distribution for $\widetilde{R}_M(t)/(Ct^a)$ as $t \to \infty$; case when $M_i$ satisfies (B) with $0 < \alpha_M < 1$. $C_0$ is a nonzero constant.

| | | $\mathbf{E}T_i < \infty$ | |
|---|---|---|---|
| **$\mathbf{E}R_i \neq 0$** | | $\mathcal{B}_{\alpha_M}$ | |
| **$\mathbf{E}R_i = 0$** | | | |
| $R_i$ satisfies (A) with $1 < \alpha_R < 2$ | $R_i \perp T_i$ | $\mathcal{B}_{\alpha_M}^{1/\alpha_R} \mathcal{S}_{\alpha_R,\beta_R}$ | |
| | otherwise | | ? |
| $\mathrm{D}^2 R_i < \infty$ | $R_i \perp T_i$ | $\mathcal{B}_{\alpha_M}^{1/2} \mathcal{G}$ | |
| | otherwise | | ? |
| otherwise | | | ? |
| **$\mathbf{E}R_i$ does not exist** | | | |
| $R_i$ satisfies (A) with $0 < \alpha_R < 1$ | $R_i \perp T_i$ | $\mathcal{B}_{\alpha_M}^{1/\alpha_R} \mathcal{S}_{\alpha_R,\beta_R}$ | |
| | otherwise | | ? |
| otherwise | | | ? |

| | | $\mathbf{E}T_i = \infty$ | |
|---|---|---|---|
| | | $T_i$ satisfies (B) with $0 < \alpha_T < 1$ | otherwise |
| **$\mathbf{E}R_i \neq 0$** | | $\mathcal{B}_{\alpha_M} \dfrac{1}{\mathcal{S}_{\alpha_T,1}^{\alpha_T}}$ | ? |
| **$\mathbf{E}R_i = 0$** | | | |
| $R_i$ satisfies (A) with $1 < \alpha_R < 2$ | $R_i \perp T_i$ | $\mathcal{B}_{\alpha_M}^{1/\alpha_R} \dfrac{\mathcal{S}_{\alpha_R,\beta_R}}{\mathcal{S}_{\alpha_T,1}^{\alpha_T/\alpha_R}}$ | |
| | otherwise | ? | |
| $\mathrm{D}^2 R_i < \infty$ | $R_i \perp T_i$ | $\mathcal{B}_{\alpha_M}^{1/2} \dfrac{\mathcal{G}}{\mathcal{S}_{\alpha_T,1}^{\alpha_T/2}}$ | |
| | otherwise | ? | |
| otherwise | | | ? |
| **$\mathbf{E}R_i$ does not exist** | | | |
| $R_i$ satisfies (A) with $0 < \alpha_R < 1$ | $R_i = C_0 T_i$ | $\mathcal{B}_{\alpha_T \alpha_M}$ | |
| | $R_i \perp T_i$ | $\mathcal{B}_{\alpha_M}^{1/\alpha_R} \dfrac{\mathcal{S}_{\alpha_R,\beta_R}}{\mathcal{S}_{\alpha_T,1}^{\alpha_T/\alpha_R}}$ | |
| | otherwise | ? | |
| otherwise | | | ? |

Table 6 concerns the asymptotics of the difference between $\widetilde{R}_M(t)/t$ and its constant limit $\varrho/\tau$ (assuming that $\varrho = \mathbf{E}R_i \neq 0$, $\tau = \mathbf{E}T_i$; and $\mathbf{E}M_i < \infty$). The limiting distributions derived in this case are mainly the stable ones (including the normal law); however, for $R_i \propto T_i$ other limits are possible.

**Table 6.** Limits in distribution for $(\widetilde{R}_M(t) - t\varrho/\tau)/(Ct^a)$ as $t \to \infty$ with $\varrho = \mathrm{E}R_i \neq 0$, $\tau = \mathrm{E}T_i$; case $\mathrm{E}M_i < \infty$.

| | | | $\mathbf{ET_i = \tau < \infty}$ | | |
|---|---|---|---|---|---|
| | | | $T_i$ satisfies (B) with $1<\alpha_T<2$ | $\mathrm{D}^2T_i < \infty$ | otherwise |
| $\mathbf{ER_i = \rho \neq 0}$ | | | | | |
| $R_i = (\rho/\tau)T_i$ | | | $\mathcal{X}_{T,M}$ | | |
| otherwise | | | | | |
| $R_i$ satisfies (A) with $1<\alpha_R<2$ | $\alpha_R \neq \alpha_T$ | | $\mathcal{S}_{\alpha_R,\beta_R}$ or $\mathcal{S}_{\alpha_T,-1}$ | $\mathcal{S}_{\alpha_R,\beta_R}$ | |
| | $\alpha_R = \alpha_T$ | $R_i - (\rho/\tau)T_i$ satisfies (A) with some $\alpha_{RT}<2$ | $\mathcal{S}_{\alpha_{RT},\beta_{RT}}$ | | |
| | | otherwise | ? | | |
| $\mathrm{D}^2 R_i < \infty$ | | | $\mathcal{S}_{\alpha_T,-1}$ | $\mathcal{G}$ | |
| otherwise | | | | | ? |

# 6. Consequences of the RCG transformation for CTRW theory

The considerations of Section 5 on the asymptotics of CTRWs randomly coarse grained by means of $M$ such that $\mathrm{E}M_i < \infty$ are summarized in Table 2. All limit theorems enumerated in this table concern $\{\widetilde{R}(t), t \geq 0\}$, the CTRW generated by $(T, R)$, as a special case since $\widetilde{R}(t) = \widetilde{R}_M(t)$ for $M_i = 1$ for which obviously $\mathrm{E}M_i < \infty$. Moreover, neither the limiting distributions (quoted in Table 4) nor the normalizing functions derived in the theorems from Table 2 depend on $M$. Hence, in case $\mathrm{E}M_i < \infty$, the RCG does not change the asymptotic properties of CTRW. On the contrary, if $M_i$ satisfying condition (B) with $\alpha = \gamma$ for some $0 < \gamma < 1$ is applied, the transformation yields a different limiting distribution as follows from the comparison of Tables 4 and 5. (Note that the normalizing functions keep the same form.)

As a consequence of the theorems proved in Section 5, some of the open questions pointed out in Table 1 can be answered. Let us first examine the influence of RCG on sequences $(T, R)$ generating CTRWs. The transformation converts $(T, R)$ into $(\overline{T}, \overline{R})$ given by (4.1). It follows from Lemma 5.1 that RCG keeps the properties of waiting times in case $\mathrm{E}M_i < \infty$; and that it leads to time steps satisfying (B) with $\alpha$ less than 1 if $M_i$ satisfies (B) with $\alpha = \gamma$ for some $0 < \gamma < 1$. This remains generally true if jumps instead of time steps are considered; however, there are some exceptions as shown in the following generalizations of Lemma 5.1, concerning the properties of the random sum $\overline{R}_j$:

THEOREM 6.1. *Assume that* $\mathrm{E}M_i = \mu < \infty$.

(a) *If* $\mathrm{E}R_i = \varrho$ *exists then* $\mathrm{E}\overline{R}_j = \varrho\mu$. *Moreover*:

(a1) *In case* $\varrho \neq 0$:

(a1.1) *if* $R_i$ *satisfies condition* (A) *with* $\alpha = \kappa$ *for some* $1 < \kappa < 2$, *with* $\sigma_0 = \varrho_0 > 0$ *and* $\beta = \beta_0$ *for some* $\beta_0$, $|\beta_0| \leq 1$, *and if* $M_i$ *satisfies*

condition (B) with $\alpha = \gamma$ for some $1 < \gamma < 2$, $\gamma \neq \kappa$, and with $\sigma_0 = \mu_0 > 0$, then $\overline{R}_j$ satisfies (A) with $\alpha = \min(\kappa, \gamma)$,

$$\sigma_0 = \begin{cases} \varrho\mu_0 & \text{if } \kappa > \gamma, \\ \varrho_0\mu^{1/\kappa} & \text{if } \kappa < \gamma, \end{cases} \qquad \beta = \begin{cases} 1 & \text{if } \kappa > \gamma, \\ \beta_0 & \text{if } \kappa < \gamma; \end{cases}$$

(a1.2) if $R_i$ satisfies condition (A) with $\alpha = \kappa$ for some $1 < \kappa < 2$, with $\sigma_0 = \varrho_0 > 0$ and $\beta = \beta_0$ for some $\beta_0$, $|\beta_0| \leq 1$; and if $\mathrm{D}^2 M_i < \infty$, then $\overline{R}_j$ satisfies (A) with $\alpha = \kappa$, $\sigma_0 = \varrho_0\mu^{1/\kappa}$ and $\beta = \beta_0$;

(a1.3) if $\mathrm{D}^2 R_i < \infty$, and if $M_i$ satisfies condition (B) with $\alpha = \gamma$ for some $1 < \gamma < 2$, and with $\sigma_0 = \mu_0 > 0$, then $\overline{R}_j$ satisfies (A) with $\alpha = \gamma$, $\sigma_0 = \varrho\mu_0$ and $\beta = 1$;

(a1.4) if $\mathrm{D}^2 R_i < \infty$ and $\mathrm{D}^2 M_i < \infty$, then $\mathrm{D}^2 \overline{R}_j = \mu\mathrm{D}^2 R_i + \varrho^2\mathrm{D}^2 M_i < \infty$.

(a2) *In case $\varrho = 0$:*

(a2.1) if $R_i$ satisfies condition (A) with $\alpha = \kappa$ for some $1 < \kappa < 2$, with $\sigma_0 = \varrho_0 > 0$ and $\beta = \beta_0$ for some $\beta_0$, $|\beta_0| \leq 1$, then $\overline{R}_j$ satisfies (A) with $\alpha = \kappa$, $\sigma_0 = \varrho_0\mu^{1/\kappa}$ and $\beta = \beta_0$;

(a2.2) if $\mathrm{D}^2 R_i < \infty$, then $\mathrm{D}^2\overline{R}_j = \mu\mathrm{D}^2 R_i < \infty$.

(b) *If $R_i$ satisfies condition (A) with $\alpha = \kappa$ for some $0 < \kappa < 1$, with $\sigma_0 = \varrho_0 > 0$ and $\beta = \beta_0$ for some $\beta_0$, $|\beta_0| \leq 1$, then $\overline{R}_j$ satisfies (A) with $\alpha = \kappa$, $\sigma_0 = \varrho_0\mu^{1/\kappa}$ and $\beta = \beta_0$.*

*Proof.* Observe that $S_{\overline{R}}(n) = S_R(S_M(n))$, $n = 0, 1, \ldots$, where the random indices $\{S_M(n)\}$ are independent of the sequence $\{S_R(n)\}$. Moreover, from the strong law of large numbers

$$(6.1) \qquad \frac{S_M(n)}{n} \xrightarrow[n\to\infty]{d} \mu.$$

A classical result for the random sums $\mathrm{E}\overline{R}_j = \varrho\mu$ can be obtained under the assumptions of (a) by means of the conditional-expected-value technique (see e.g. [16, V 9-11]). Also, (a1.4) and (a2.2) can be shown this way.

Under the assumptions of (a1.1), from Proposition 2.3(a) we have

$$\frac{S_M(n) - n\mu}{(q(\gamma))^{1/\gamma}\mu_0 n^{1/\gamma}} \xrightarrow[n\to\infty]{d} \mathcal{S}_{\gamma,1}.$$

Moreover, it follows from Propositions 2.3(a) and 2.12, and from (6.1), that

$$\frac{S_R(S_M(n)) - S_M(n)\varrho}{(\mu q(\kappa))^{1/\kappa}\varrho_0 n^{1/\kappa}} \xrightarrow[n\to\infty]{d} \mathcal{S}_{\kappa,\beta_0}.$$

Since for any $a$

$$\frac{S_{\overline{R}}(n) - n\mu\varrho}{n^a} = \frac{S_R(S_M(n)) - S_M(n)\varrho}{n^{1/\kappa}}\frac{1}{n^{a-1/\kappa}} + \varrho\frac{S_M(n) - n\mu}{n^{1/\gamma}}\frac{1}{n^{a-1/\gamma}},$$

by Lemma 2 of [16, VIII 2] one obtains

- $\dfrac{S_{\overline{R}}(n) - n\mu\varrho}{(q(\gamma))^{1/\gamma}\mu_0 n^{1/\gamma}} \xrightarrow[n\to\infty]{d} \mathcal{S}_{\gamma,1}$ if $\kappa > \gamma$,

- $\dfrac{S_{\overline{R}}(n) - n\mu\varrho}{(\mu q(\kappa))^{1/\kappa}\varrho_0 n^{1/\kappa}} \xrightarrow[n\to\infty]{d} \mathcal{S}_{\kappa,\beta_0}$ if $\kappa < \gamma$;

that yields the conclusion of (a1.1) according to Proposition 2.3(a). Parts (a1.2) and (a1.3) can be shown in a parallel way.

Under the assumptions of (a2.1) and (b), we have

$$\frac{S_R(n)}{(q(\kappa))^{1/\kappa}\varrho_0 n^{1/\kappa}} \xrightarrow[n\to\infty]{d} \mathcal{S}_{\kappa,\beta_0},$$

and from (6.1) and Proposition 2.12 we obtain

$$\frac{S_{\overline{R}}(n)}{(\mu q(\kappa))^{1/\kappa}\varrho_0 n^{1/\kappa}} \xrightarrow[n\to\infty]{d} \mathcal{S}_{\kappa,\beta_0}.$$

Applying Proposition 2.3(a) we get the assertions of both parts. ∎

THEOREM 6.2. *Assume that $M_i$ satisfies condition* (B) *with $\alpha = \gamma$ for some $0 < \gamma < 1$, and with $\sigma_0 = \mu_0 > 0$.*

(a) *Suppose* $\mathrm{E}R_i = \varrho$ *exists.*

    (a1) *In case $\varrho \neq 0$, $\overline{R}_j$ satisfies condition* (A) *with $\alpha = \gamma$, $\sigma_0 = \varrho\mu_0$ and $\beta = 1$.*

    (a2) *In case $\varrho = 0$:*

        (a2.1) *if $R_i$ satisfies condition* (A) *with $\alpha = \kappa$ for some $1 < \kappa < 2$, with $\sigma_0 = \varrho_0 > 0$, and $\beta = \beta_0$ for some $\beta_0$, $|\beta_0| \leq 1$, then $\overline{R}_j$ satisfies* (A) *with $\alpha = \kappa\gamma$,*

$$\sigma_0 = \begin{cases} c_0(q(\kappa\gamma))^{-1/(\kappa\gamma)} & \text{if } \kappa\gamma \neq 1, \\ 2c_0/\pi & \text{if } \kappa\gamma = 1, \end{cases}$$

$$\beta = \begin{cases} \tan(\gamma\Theta)/\tan(\gamma\pi\kappa/2) & \text{if } \kappa\gamma \neq 1, \\ 0 & \text{if } \kappa\gamma = 1, \end{cases}$$

       *where*

(6.2) $\qquad c_0 = \varrho_0(\mu_0 q(\kappa))^{1/\kappa}(\Gamma(1-\gamma)\cos(\gamma\Theta))^{1/(\kappa\gamma)}(1 + \beta_0^2 \tan^2(\pi\kappa/2))^{1/(2\kappa)},$

(6.3) $\qquad\qquad\qquad\qquad \Theta = \arctan(\beta_0 \tan(\pi\kappa/2)).$

       *Moreover,* $\mathrm{E}\overline{R}_j = 0$ *if $\kappa\gamma > 1$;*

        (a2.2) *if $\mathrm{D}^2 R_i = \sigma^2$ for some $0 < \sigma < \infty$, then $\overline{R}_j$ satisfies condition* (A) *with $\alpha = 2\gamma$,*

$$\sigma_0 = \begin{cases} c_0(q(2\gamma))^{-1/(2\gamma)} & \text{if } 2\gamma \neq 1, \\ 2c_0/\pi & \text{if } 2\gamma = 1, \end{cases}$$

       *and $\beta = 0$, where $c_0 = \sigma(\mu_0/2)^{1/2}(\Gamma(1-\gamma))^{1/(2\gamma)}$. Moreover, $\mathrm{E}\overline{R}_j = 0$ if $2\gamma > 1$.*

(b) *If $R_i$ satisfies condition* (A) *with $\alpha = \kappa$ for some $0 < \kappa < 1$, with $\sigma_0 = \varrho_0 > 0$, and $\beta = \beta_0$ for some $\beta_0$, $|\beta_0| \le 1$, then so does $\overline{R}_j$ with $\alpha = \kappa\gamma$, $\sigma_0 = c_0(q(\kappa\gamma))^{-1/(\kappa\gamma)}$ and $\beta = \tan(\gamma\Theta)/\tan(\gamma\pi\kappa/2)$, where $c_0$ and $\Theta$ are of the form* (6.2) *and* (6.3), *respectively.*

*Proof.* Observe that $S_{\overline{R}}(n) = S_R(S_M(n))$, $n = 0, 1, \ldots$, where the random indices $\{S_M(n)\}$ are independent of $\{S_R(n)\}$. Moreover, from Proposition 2.3(a),

$$\frac{S_M(n)}{(q(\gamma))^{1/\gamma}\mu_0 n^{1/\gamma}} \xrightarrow[n\to\infty]{d} \mathcal{S}_{\gamma,1}.$$

In case $\varrho \ne 0$ we have $S_R(n)/n \xrightarrow{\text{a.s.}} \varrho$ as $n \to \infty$ from the strong law of large numbers, and it follows from Proposition 2.12 that

$$\frac{S_{\overline{R}}(n)}{\varrho(q(\gamma))^{1/\gamma}\mu_0 n^{1/\gamma}} \xrightarrow[n\to\infty]{d} \mathcal{S}_{\gamma,1}.$$

From Proposition 2.3(a) we obtain the assertion of (a1).

In case $\varrho = 0$, under the assumption of (a2.1), we have

$$\frac{S_R(n)}{(q(\kappa))^{1/\kappa}\varrho_0 n^{1/\kappa}} \xrightarrow[n\to\infty]{d} \mathcal{S}_{\kappa,\beta_0},$$

and from Proposition 2.10 we obtain

$$\frac{S_{\overline{R}}(n)}{Cn^{1/(\kappa\gamma)}} \xrightarrow[n\to\infty]{d} (\mathcal{S}_{\gamma,1})^{1/\kappa}\mathcal{S}_{\kappa,\beta_0}$$

for $C = \varrho_0(\mu_0 q(\kappa))^{1/\kappa}(q(\gamma))^{1/(\kappa\gamma)}$. Applying Proposition 2.2 one shows that

- in case $\kappa\gamma \ne 1$:

$$\frac{S_{\overline{R}}(n)}{c_1 Cn^{1/(\kappa\gamma)}} \xrightarrow[n\to\infty]{d} \mathcal{S}_{\kappa\gamma,\beta_1},$$

- in case $\kappa\gamma = 1$:

$$\frac{S_{\overline{R}}(n) - m_1 c_1 Cn}{c_1 Cn} \xrightarrow[n\to\infty]{d} \mathcal{S}_{1,0},$$

where $c_1$, $\beta_1$, $m_1$ are given by (2.5). Now (a2.1) follows from Proposition 2.3(a).

Parts (a2.2) and (b) can be proved the same way as for (a2.1). ∎

As a consequence of Theorems 6.1 and 6.2 we obtain the following partial answers to the questions pointed out in Table 1:

⟨?⟩₁ Under the assumptions of Theorem 5.12(b), and if additionally $M_i$ satisfies (B) with $\alpha = \gamma$ for some $1 < \gamma < 2$, $\{\widetilde{R}_M(t), t \ge 0\}$ is a CTRW generated by $(\overline{T}, \overline{R})$, where $\overline{T}_j$ and $\overline{R}_j$ satisfy conditions (B) and (A), respectively, with $\alpha_{\overline{T}} = \alpha_{\overline{R}}$, both equal to $\gamma$. However, the normalized difference between $\widetilde{R}_M(t)/t$ and its constant limit has asymptotically normal distribution in this case.

⟨?⟩₂ The processes $\{\widetilde{R}_M(t), t \ge 0\}$ considered in Theorems 5.5(b), 5.6, and 5.8 (assuming additionally $\kappa\gamma > 1$ in Theorem 5.6 and $2\gamma > 1$ in Theorem 5.8) are examples of CTRWs generated by $(\overline{T}, \overline{R})$ such that $\overline{T}_j$ satisfies condition (B) with $0 < \alpha_{\overline{T}} < 1$, $\overline{R}_j$ satisfies (A) with $1 < \alpha_{\overline{R}} < 2$, and $\mathrm{E}\overline{R}_j = 0$. The large-time limiting distribution of the total distance reached by the walking particle obtained

by means of Theorem 5.5(b) is the same as for decoupled walks from this class although $\overline{T}_j$ and $\overline{R}_j$ can be stochastically dependent. Hence the independence between the time and jump steps is not necessary to get this kind of limiting law. On the other hand, Theorems 5.6 and 5.8 reveal other possibilities for the asymptotic behavior of this type of CTRW.

$(?)_3$ The assumptions of Theorem 5.7(b) lead to the CTRW $\{\widetilde{R}_M(t),\, t \geq 0\}$ generated by $(\overline{T}, \overline{R})$, where $\overline{T}_j$ satisfies condition (B) with $0 < \alpha_{\overline{T}} < 1$, while $0 < \mathrm{D}^2 \overline{R}_j < \infty$ and $\mathrm{E}\overline{R}_j = 0$. Moreover, $\overline{T}_j$ and $\overline{R}_j$ can be stochastically dependent. Since the resulting limiting distribution is the same as for decoupled walks from this class, the independence between the time and jump steps is not necessary for this kind of asymptotic behavior in this case. It remains an open question if other limiting laws are possible here.

$(?)_4$ Taking into account the processes $\{\widetilde{R}_M(t),\, t \geq 0\}$ considered in Theorems 5.3(b), 5.4, 5.6, and 5.8 (assuming additionally $\kappa\gamma < 1$ in Theorem 5.6 and $2\gamma < 1$ in Theorem 5.8) we obtain examples of CTRWs generated by $(\overline{T}, \overline{R})$ such that $\overline{T}_j$ satisfies condition (B) with $0 < \alpha_{\overline{T}} < 1$, and $\overline{R}_j$ satisfies (A) with $0 < \alpha_{\overline{R}} < 1$. The large-time limiting distribution of the normalized $\widetilde{R}_M(t)$ obtained by means of Theorem 5.3(b) is the same as for decoupled walks from this class although $\overline{T}_j$ and $\overline{R}_j$ can be stochastically dependent; and hence the independent time and jump steps are not necessarily required to reach the limiting law of that form. However, different asymptotic behaviors for this type of CTRWs are determined in Theorems 5.4, 5.6 and 5.8.

$(?)_5$ If we take $\kappa\gamma = 1$ in Theorem 5.6 or $2\gamma = 1$ in Theorem 5.8, we obtain examples of CTRWs generated by $(\overline{T}, \overline{R})$ such that $\overline{T}_j$ satisfies condition (B) with $0 < \alpha_{\overline{T}} < 1$, while $\overline{R}_j$ satisfies (A) with $\alpha_{\overline{R}} = 1$. Theorems 5.6 and 5.8 thus provide examples of limiting distribution of the large-time total distance reached by the particle for the CTRWs from this class, not studied before.

In a similar way we can examine how essential is the independence of $T_i$ and $R_i$ assumed in some theorems of Section 5 for technical reasons. Namely, take a new sequence $M^* = \{M_i^*, i = 1, 2, \ldots\}$ of i.i.d. positive integer-valued random variables, independent of $(T, R)$ and $M$. By means of $M^*$, we can convert $\{\widetilde{R}_M(t),\, t \geq 0\}$, which is a CTRW generated by $(\overline{T}, \overline{R})$, into a new randomly coarse grained CTRW, say $\{\widetilde{R}_{MM^*}(t),\, t \geq 0\}$. It is easy to show that the resulting process is equal to the CTRW generated by $(T, R)$ and transformed by means of the sequence $\overline{M} = \{\overline{M}_j,\, j = 1, 2, \ldots\}$, where

$$\overline{M}_j = \sum_{i=S_{M^*}(j-1)+1}^{S_{M^*}(j)} M_i.$$

Hence the asymptotic properties of $\widetilde{R}_{MM^*}(t)$ can be determined by Lemma 5.1 and the theorems of Section 5. On the other hand, our construction provides examples of randomly coarse grained CTRW with dependent time and jump steps.

A detailed analysis, similar to the one for the questions from Table 1, shows that assuming independence of $T_i$ and $R_i$ in Theorems 5.3–5.8 is not necessary to get the

conclusions; however, it cannot be completely omitted since other probability laws can appear as weak limits for randomly coarse grained CTRWs from each class considered.

## 7. CTRW-like processes

CTRWs are generated by sequences $(T, R) = \{(T_i, R_i),\ i = 1, 2, \ldots\}$, where $T_i$ is a waiting time of the walking particle for the $i$th jump determined by $R_i$. Let us slightly modify this interpretation of $(T, R)$. Namely, consider $T_i$ as a survival time of the particle at the level reached in $i$ jump steps determined by $R_1, \ldots, R_i$. The modification manifests itself as the change of the process counting the jumps since the total distance, say $\widetilde{R}_0(t)$, reached in this case by the particle at time $t$ is equal to $S_R(K_T(t))$, where $K_T(t)$ is of the form (2.2). The resulting CTRW-like process $\{\widetilde{R}_0(t),\ t \geq 0\}$ is hence a discrete-time random walk $\{S_R([t]),\ t \geq 0\}$ subordinated to the first-passage process $\{K_T(t),\ t \geq 0\}$ (instead of the renewal counting process $\{N_T(t),\ t \geq 0\}$ subordinating the walk in the case of the original CTRW).

The idea of substituting the corresponding first-passage process for the renewal counting process can be applied to the compound counting process $\{L(t),\ t \geq 0\}$, shown to subordinate the random walk $\{S_R([t]),\ t \geq 0\}$ after the RCG transformation of CTRW (Theorem 4.1). We obtain random indices of forms similar to $L(t) = S_M(N_M(N_T(t)))$; namely,

$$L_1(t) = S_M(N_M(K_T(t))),$$
$$L_2(t) = S_M(K_M(N_T(t))),$$
$$L_3(t) = S_M(K_M(K_T(t))),$$

where $\{K_T(t)\}$, $\{K_M(t)\}$ are the first-passage processes generated by $T$, $M$, respectively. For each $i = 1, 2, 3$ the counting process $\{L_i(t),\ t \geq 0\}$ leads to a CTRW-like process $\{\widetilde{R}_{M,i}(t)\}$ of the form

$$\widetilde{R}_{M,i}(t) = S_R(L_i(t)).$$

Moreover, for $M_i = 1$ we get $L_1(t) = L_2(t) = K_T(t)$ and hence $\widetilde{R}_{M,1}(t) = \widetilde{R}_{M,2}(t) = \widetilde{R}_0(t)$.

Observe that for any $t \geq 0$ we have

$$L(t) \leq L_1(t) \leq L_3(t) \quad \text{and} \quad L(t) \leq L_2(t) \leq L_3(t)$$

so that $L_i(t) \xrightarrow{\text{a.s.}} \infty$ as $t \to \infty$, $i = 1, 2, 3$. Moreover, since $K_T(t)$ satisfies the conclusions of Propositions 2.5–2.7 just as $N_T(t)$ does, one can easily show that $\{L_1(t)\}$ and $\{\widetilde{R}_{M,1}(t)\}$ satisfy the conclusions of all theorems of Sections 4 and 5 concerning $\{L(t)\}$ and $\{\widetilde{R}_M(t)\}$ except Theorems 5.1 and 5.2; i.e., in general, the asymptotic behavior of $L_1(t)$ and $\widetilde{R}_{M,1}(t)$ for $t \to \infty$ is the same as the one of $L(t)$ and $\widetilde{R}_M(t)$ except (probably) for the case when $R_i = CT_i$.

For $\{L_i(t)\}$ and $\{\widetilde{R}_{M,i}(t)\}$ with $i = 2, 3$ it follows from Proposition 2.8 that the processes satisfy the conclusions of Theorems 4.3, 4.5, 5.3, 5.5, 5.7, 5.9, 5.11, 5.12, and of theorems similar to 4.4, 5.4, 5.6, 5.8, 5.10 with $1/\mathcal{B}_\gamma$ instead of $\mathcal{B}_\gamma$. Moreover, we

have $\widetilde{R}_{M,2}(t) = S_{\overline{R}}(K_{\overline{T}}(t))$ and hence, for the process $\{\widetilde{R}_{M,2}(t)\}$ one can prove theorems similar to Theorems 5.1 and 5.2:

THEOREM 7.1. *Assume that* $\mathrm{E}M_i = \mu < \infty$ *and* $R_i = CT_i$ *for some constant* $C \neq 0$.

(a) *If* $\mathrm{E}T_i = \tau < \infty$ *then*

$$\frac{\widetilde{R}_{M,2}(t)}{t} \xrightarrow[t \to \infty]{a.s.} C.$$

*Moreover*,

(a1) *if the distribution of* $T_i$ *is arithmetic then*

$$\widetilde{R}_{M,2}(n\delta) - Cn\delta \xrightarrow[n \to \infty]{d} C(\mathcal{X}_{T,M} + \delta),$$

*where* $\delta > 0$ *is the largest constant such that* $\sum_{n=1}^{\infty} P(\overline{T}_j = n\delta) = 1$, *and*

$$P(\mathcal{X}_{T,M} = k\delta) = \frac{\delta}{\tau\mu} P(\overline{T}_j > k\delta), \quad k = 0, 1, 2, \dots;$$

(a2) *otherwise*,

$$\widetilde{R}_{M,2}(t) - Ct \xrightarrow[t \to \infty]{d} C\mathcal{X}_{T,M},$$

*where*

$$P(\mathcal{X}_{T,M} \leq x) = \frac{1}{\tau\mu} \int_0^x P(\overline{T}_j > s)\,ds, \quad x > 0.$$

(b) *If* $T_i$ *satisfies condition* (B) *with* $\alpha = \lambda$ *for some* $0 < \lambda < 1$, *and with* $\sigma_0 = \tau_0 > 0$, *then*

$$\frac{\widetilde{R}_{M,2}(t)}{t} \xrightarrow[t \to \infty]{d} C\frac{1}{\mathcal{B}_\lambda},$$

*where* $1/\mathcal{B}_\lambda$ *is defined in Remark* 2.4.

THEOREM 7.2. *Assume that* $M_i$ *satisfies condition* (B) *with* $\alpha = \gamma$ *for some* $0 < \gamma < 1$, *and with some* $\sigma_0 > 0$. *Let* $R_i = CT_i$ *for some constant* $C \neq 0$.

(a) *If* $\mathrm{E}T_i = \tau < \infty$ *then*

$$\frac{\widetilde{R}_{M,2}(t)}{t} \xrightarrow[t \to \infty]{d} C\frac{1}{\mathcal{B}_\gamma},$$

*where* $1/\mathcal{B}_\gamma$ *is defined in Remark* 2.4.

(b) *If* $T_i$ *satisfies condition* (B) *with* $\alpha = \lambda$ *for some* $0 < \lambda < 1$, *and with* $\sigma_0 = \tau_0 > 0$, *then*

$$\frac{\widetilde{R}_{M,2}(t)}{t} \xrightarrow[t \to \infty]{d} C\frac{1}{\mathcal{B}_{\lambda\gamma}},$$

*where* $1/\mathcal{B}_{\lambda\gamma}$ *is defined in Remark* 2.4.

Let us add that the presented idea of modification of the CTRW concept is justified by the fact that this way we enlarge the class of limiting distributions by the one connected with dielectric responses. The construction and results for $\widetilde{R}_{M,3}(t)$ have already found an application in modeling dielectric relaxation phenomena (see [32]).

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