To Agnieszka Podkówka

Summary

Our work is divided into six chapters. In Chapter I we introduce necessary notions and present most important facts. We also present our main results. Chapter I covers the following topics:

• Extremal plurisubharmonic functions: the relative extremal function and the pluricomplex Green function;

• The analytic discs method of E. Poletsky: disc functionals, envelope of a disc functional, examples of disc functionals;

• The Poisson functional: We present properties of the most important functional, including the main result of the paper, plurisubharmonicity of the envelope of the Poisson functional on a class of complex manifolds. We also prove the product property of the relative extremal function;

• The Riesz functional: We state some properties of the Riesz functional which follow from the properties of the Poisson functional and the Poisson–Jensen formula. Since these results are contained in other papers, we do not give the proofs.

• The Green and Lelong functionals: We concentrate mainly on the product property of the Green functional.

Chapter II is devoted to the general properties of disc functionals (Section 2.1, Propositions 2.1–2.5) and properties of analytic discs in complex manifolds (Section 2.3). In Section 2.2 we study a class of complex manifolds which is important in Poletsky's theory.

In Chapter III we give the main results of the paper. We show that the envelope of the Poisson functional on any complex manifold is upper semicontinuous (Theorem 3.5). Section 3.2 contains the most important (and most difficult) result of the paper. In Theorem 3.10 we show the plurisubharmonicity of the Poisson functional on a class of complex manifolds. Section 3.3 contains properties of the Poisson functional on Liouville manifolds. Using Poletsky's theory, we give a characterization of Liouville manifolds in terms of analytic discs (Theorem 3.21).

Product properties of the Poisson and Green functionals are presented in Chapter IV (Theorems 4.1 and 4.9).

In Chapter V we give applications of the results obtained. In Section 5.1 we state some properties of the relative extremal function. In Section 5.2, using the product property of the relative extremal function for open sets (Theorem 5.3.) we show the product property of the plurisubharmonic measure in bounded domains in $\mathbb{C}^n$ (Theorem 5.6). Section 5.3 is devoted to the pluricomplex Green function. We obtain the product property of the pluricomplex Green function as a corollary of the product property of the

relative extremal function (Theorem 5.8). In Section 5.4 we give simple results related to the polynomial hulls of compact sets in $\mathbb{C}^n$ (Theorem 5.10).

Chapter VI contains remarks related to Poletsky's theory. We concentrate mainly on holomorphically invariant pseudodistances (Section 6.4).

Most of the prerequisites that we use may be found in the following books: [15], [17], [20], [24].

Some of the results contained in this work may be found in the following papers: [7], [8], [9], [10], [11].

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1. Introduction

1.1. Extremal plurisubharmonic functions. Let X be a complex manifold (1). We denote by $\mathrm{PSH}(X)$ the set of all plurisubharmonic functions on X (2) (3).

Let $\mathfrak{U} \subset \mathrm{PSH}(X)$. We put

$$P_{\mathfrak{U}}(x) = \sup\{u(x) : u \in \mathfrak{U}\}, \qquad x \in X. \tag{1.1}$$

It is well known that if the family $\mathfrak{U}$ is locally bounded from above, then $P^*_{\mathfrak{U}}$ is a plurisubharmonic function on X (cf. [20]), where $v^*(x) = \limsup_{y\to x} v(y)$ denotes the upper semicontinuous regularization of a function v.

The construction (1.1) plays an important role in pluripotential theory (see e.g. [20]). Let us consider some examples.

Let X be a complex manifold and let E be any subset of X. We put

$$\omega(x, E, X) = \sup\{u(x) : u \in \mathrm{PSH}(X),\ u \le -1 \text{ on } E,\ u \le 0 \text{ on } X\}, \qquad x \in X.$$

The function $\omega(\cdot, E, X)$, introduced in 1969 by J. Siciak [42], is called the *relative extremal function*. As mentioned above, the function $\omega^*(\cdot, E, X)$ is plurisubharmonic in X (4). Since in the one-dimensional case the function $\omega^*(\cdot, E, X)$ is closely related to the notion of harmonic measure (see e.g. [37]), in higher dimensions it is sometimes called the plurisubharmonic measure of E relative to X (cf. [20], [40]).

In 1991 Nguyen Thanh Van and J. Siciak [32] proved the following product property of the relative extremal function.

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(1) All complex manifolds considered in the paper are assumed to be connected.

(2) We assume that the constant function $-\infty$ is plurisubharmonic.

(3) For convenience of the reader we list some standard notation in the section "List of symbols".

(4) Note that if U is an open set in X, then $\omega(\cdot, U, X) = \omega^*(\cdot, U, X)$ and, therefore, $\omega(\cdot, U, X) \in \mathrm{PSH}(X)$.

THEOREM 1.1. *Let $D \subset \mathbb{C}^n$ and $G \subset \mathbb{C}^m$ be pseudoconvex domains* (5) *and let $U \subset D$, $V \subset G$ be open subsets. Then*

$$\omega((x,y), U \times V, D \times G) = \max\{\omega(x,U,D), \omega(y,V,G)\}, \qquad (x,y) \in D \times G.$$

The proof of Theorem 1.1 given in [32] extensively uses the pseudoconvexity of D and G. The question whether Theorem 1.1 is true for any (non-pseudoconvex) domains in $\mathbb{C}^n$ (or, more generally, complex manifolds) remained open until 1997.

In 1985 M. Klimek [20] introduced another extremal function, which is also important in pluripotential theory, as follows. Let Ω be a domain in $\mathbb{C}^n$. Define

$$g_\Omega(x,p) := \sup\{u(x) : u \in \mathrm{PSH}^-(\Omega),\ u(y) - \log\|y-p\| \le O(1) \text{ as } y \to p\}, \qquad x, p \in \Omega,$$

where $\mathrm{PSH}^-(\Omega)$ denotes the set of all negative plurisubharmonic functions on the domain Ω. The function $g_\Omega(\cdot,p)$ is called the *pluricomplex Green function* with pole at p. It may be viewed as a natural analogue of the Green function from the classical potential theory (cf. [37]).

In 1989 P. Lelong [27] defined on a domain $\Omega \subset \mathbb{C}^n$ the *pluricomplex Green function* with poles at $p_1, \ldots, p_N \in \Omega$ and weights $\nu_1, \ldots, \nu_N \in (0,\infty)$, where $p_i \ne p_j$, $i \ne j$, as follows:

$$\begin{aligned} g_\Omega(x;(p_1,\nu_1),\ldots,(p_N,\nu_N)) := \sup\{&u(x) : u \in \mathrm{PSH}(\Omega),\ u<0,\\ &u(y) - \nu_j \log\|y-p_j\| \le O(1) \text{ as } y \to p_j,\ j=1,\ldots,N\}, \qquad x \in \Omega. \end{aligned}$$

We see that $g_\Omega(\,\cdot\,;p) = g_\Omega(\,\cdot\,;(p,1))$. P. Lelong [27] proved that in any hyperconvex domain Ω (6) the pluricomplex Green function with poles at $p_1,\ldots,p_N$ and weights $\nu_1,\ldots,\nu_N$ is the unique solution of the Dirichlet problem (7). One can easily extend Lelong's definition to complex manifolds.

It seems that the most general pluricomplex Green function was introduced by A. Zeriahi [47] (see also [25]) as follows. Let X be a complex manifold and let $\alpha : X \to [0,\infty)$ be a function. Define the *pluricomplex Green function* with *pole function* α by the formula

$$g_X(x,\alpha) := \sup\{u(x) : u \in \mathrm{PSH}(X),\ u \le 0,\ \nu(\cdot,u) \ge \alpha\},$$

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(5) Recall that a domain $\Omega \subset \mathbb{C}^n$ is called *pseudoconvex* if there exists a plurisubharmonic exhaustion function u for Ω, i.e. $\{x \in \Omega : u(x) < \beta\}$ is relatively compact in Ω for any $\beta \in \mathbb{R}$ (see e.g. [24]).

(6) Recall that a bounded domain $\Omega \subset \mathbb{C}^n$ is called *hyperconvex* if there exists a negative plurisubharmonic exhaustion function u for Ω, i.e. $\{x \in \Omega : u(x) < \beta\}$ is relatively compact in Ω for any $\beta < 0$ (see e.g. [20]).

(7) More precisely, $g_\Omega(x;(p_1,\nu_1),\ldots,(p_N,\nu_N))$ is the unique solution of the following Dirichlet problem:

$$\begin{cases} u \in C(\Omega \setminus \{p_1,\ldots,p_N\}) \cap \mathrm{PSH}(\Omega),\\ (dd^c u)^n = 0 \text{ in } \Omega \setminus \{p_1,\ldots,p_N\},\\ u(x) - \nu_j \log\|x-p_j\| = O(1) \text{ as } x \to p_j,\ j=1,\ldots,N,\\ u(x) \to 0 \text{ as } x \to \partial\Omega, \end{cases}$$

where $(dd^c u)^n$ is the Monge–Ampère operator (see e.g. [20], [23]).

where $\nu(\cdot,u)$ denotes the Lelong number of u ([8]). Note that for a plurisubharmonic function u in a neighborhood of $x_0 \in \mathbb{C}^n$ we have $\nu(x_0,u) \geq \nu_0 > 0$ if and only if $u(x) - \nu_0 \log\|x - x_0\| \leq O(1)$ when $x \to x_0$. Therefore, in the case $\operatorname{supp}\alpha = \{p_1,\dots,p_N\}$, $\alpha(p_j) = \nu_j$, $j = 1,\dots,N$, we have the equality

$$g_X(\,\cdot\,;(p_1,\nu_1),\dots,(p_N,\nu_N)) = g_X(\,\cdot\,,\alpha).$$

We have the following equivalent definition of the pluricomplex Green function with pole function α.

PROPOSITION 1.2 (see Proposition 5.10 below). *Let X be a complex manifold and let α be a non-negative function on X. Then*

$$g_X(x,\alpha) = \sup\{u(x) : u \in \mathrm{PSH}(X),\ u \leq \inf_{p\in X} \alpha(p) g_X(\cdot,p)\},$$

and, therefore, $g_X(\cdot,\alpha)$ is a plurisubharmonic function on X.

There is an interesting relation between the relative extremal function and the pluricomplex Green function. We need some more definitions. We set $\|\!|x|\!\| := \max\{|z_1|,\dots,|z_n|\}$, $x = (z_1,\dots,z_n) \in \mathbb{C}^n$, and $P(x,r) := \{y \in \mathbb{C}^n : \|\!|y - x|\!\| < r\}$, $r > 0$. We put $P(x,0) = \emptyset$, $x \in \mathbb{C}^n$. Set $P(r) = P(0,r)$.

For a complex manifold X and a family of local coordinates $\{(U_x,\zeta_x)\}_{x\in X}$ such that $\zeta_x(x) = 0$ and $\zeta_x(U_x) = P(1)$ we put

$$\mathfrak{P}(r,\alpha) = \bigcup_{y\in X} \zeta_y^{-1}[P(r^{1/\alpha(y)})], \qquad r \in (0,1),$$

where $r^{1/0} = 0$ for $r \in (0,1)$. We have the following result (cf. [10]).

THEOREM 1.3 (Theorem 5.11). *Let X be a complex manifold and let $\alpha : X \to [0,\infty)$ be any function. Assume that $\{(U_x,\zeta_x)\}_{x\in X}$ is a family of local coordinates such that $\zeta_x(x) = 0$ and $\zeta_x(U_x) = P(1)$. Then*

$$(-\log r)\omega(x,\mathfrak{P}(r,\alpha),X) \underset{r\to 0}{\searrow} g_X(x,\alpha), \qquad x \in X.$$

Using the method from [10], as a corollary of Theorem 1.3 we get the product property of the pluricomplex Green function.

THEOREM 1.4 (Theorem 5.12). *Let X_1 and X_2 be complex manifolds. Assume that for any open subsets $E_1 \subset X_1$ and $E_2 \subset X_2$ we have the following product property:*

$$\omega((x_1,x_2),E_1\times E_2,X_1\times X_2) = \max\{\omega(x_1,E_1,X_1),\omega(x_2,E_2,X_2)\}, \qquad (x_1,x_2) \in X_1\times X_2.$$

Then for arbitrary functions $\alpha_1 : X_1 \to \{0,1\}$ and $\alpha_2 : X_2 \to \{0,1\}$ we have

$$g_{X_1\times X_2}((x_1,x_2),\alpha_1\otimes\alpha_2) = \max\{g_{X_1}(x_1,\alpha_1), g_{X_2}(x_2,\alpha_2)\}, \qquad (x_1,x_2) \in X_1\times X_2,$$

where $(\alpha_1\otimes\alpha_2)(x_1,x_2) := \alpha_1(x_1)\alpha_2(x_2)$.

([8]) Recall that the *Lelong number* of u at the point $x \in \Omega$ is defined by

$$\nu(x,u) := \lim_{r\to 0+} \frac{M_u(x,r)}{\log r},$$

where $M_u(x_0,r) := \sup_{x\in\mathbb{B}_n(x_0,r)} u(x)$ and $\mathbb{B}_n(x,r) := \{y \in \mathbb{C}^n : \|y - x\| < r\}$. We put $\nu(\cdot,-\infty) \equiv \infty$. The Lelong number is a biholomorphic invariant (see e.g. [4], [5]).

As a corollary of Theorems 1.4 and 1.1 we have the following.

THEOREM 1.5. *Let $D \subset \mathbb{C}^n$ and $G \subset \mathbb{C}^m$ be pseudoconvex domains. Then for arbitrary functions $\alpha : D \to \{0,1\}$ and $\beta : G \to \{0,1\}$ we have*

$$g_{D\times G}((x,y),\alpha\otimes\beta) = \max\{g_D(x,\alpha), g_G(y,\beta)\}, \qquad (x,y)\in D\times G.$$

In particular,

$$g_{D\times G}((x,y),(p,q)) = \max\{g_D(x,p), g_G(y,q)\}, \qquad (x,y),(p,q)\in D\times G. \tag{1.2}$$

The product property of the pluricomplex Green function for one pole in pseudoconvex domains (i.e. (1.2)) was proved by M. Jarnicki and P. Pflug [17]. Later, the same authors proved that it suffices to assume the pseudoconvexity of one of the domains D or G [18]. In the meantime, they conjectured that it is true for arbitrary domains, but until 1997 it was an open problem.

In the classical pluripotential theory ([9]) pseudoconvexity of a domain (or more precisely, hyperconvexity) is very important. It seems to be interesting to find methods which work equally well on any domain. The purpose of the paper is to present such a method, *the analytic discs method of E. Poletsky*. This technique allows us to study problems not only in any domain in $\mathbb{C}^n$ but also on a large class of complex manifolds.

1.2. Analytic discs method of E. Poletsky. In the previous section we described extremal functions which are defined with the help of plurisubharmonic functions. In 1991 E. Poletsky [35] proposed a method of characterization of some of them by the family $\mathcal{O}(\overline{\mathbb{D}}, X)$ ([10]). This approach turns out to be very successful in solving many problems in complex analysis which were unaccessible before.

The main idea is to study disc functionals and their envelopes. More precisely, we proceed as follows. Let X be a complex manifold. A *disc functional* on X is a function $H : \mathcal{O}(\overline{\mathbb{D}}, X) \to \overline{\mathbb{R}}$ ([11]). The *envelope* of H is a function $E_H : X \to \overline{\mathbb{R}}$ defined by the formula

$$E_H(x) := \inf\{H(f) : f\in\mathcal{O}(\overline{\mathbb{D}}, X),\ f(0)=x\}, \qquad x\in X.$$

Now, we give examples of disc functionals, some of which are related to the functions considered above. The presented functionals will be studied more carefully later. The functionals $\mathfrak{F}_1, \mathfrak{F}_2, \mathfrak{F}_4$ were introduced by E. Poletsky in [35], Examples 3.1–3.3. Following [25], we call them the *Poisson functional*, and *Riesz functional*, and *Lelong functional* respectively. The first two are motivated by the Poisson–Jensen formula. The functional $\mathfrak{F}_3$ was suggested by E. Poletsky in [34] (see also [7], [9]).

Poisson functional. Let X be a complex manifold and let $\varphi : X \to \overline{\mathbb{R}}$ be a measurable function. Assume that φ is locally bounded from above or below (i.e. φ is locally bounded

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([9]) Here, by classical we mean methods and techniques which are gathered in [20].

([10]) $\mathcal{O}(\overline{\mathbb{D}}, X)$ denotes the family of all holomorphic mappings $f : \mathbb{D} \to X$ which are holomorphic in a neighborhood of the closure $\overline{\mathbb{D}}$.

([11]) We put $\overline{\mathbb{R}} = [-\infty, \infty]$.

from above everywhere on X or φ is locally bounded from below everywhere on X) ([12]). Define the functional $\mathfrak{F}_1 = \mathfrak{F}_1^{\varphi}$ by the formula

$$\mathfrak{F}_1(f) = \frac{1}{2\pi}\int_0^{2\pi} \varphi(f(e^{i\theta}))\,d\theta, \qquad f \in \mathcal{O}(\overline{\mathbb{D}}, X).$$

Riesz functional. Let X be a complex manifold and let v be a plurisubharmonic function on X. We define the functional $\mathfrak{F}_2 = \mathfrak{F}_2^{v}$ as follows. If $f \in \mathcal{O}(\overline{\mathbb{D}}, X)$ and $v \circ f$ is not identically $-\infty$, then

$$\mathfrak{F}_2(f) = \frac{1}{\pi}\int_{\mathbb{D}} (\log|\cdot|)\triangle(v \circ f),$$

where $\triangle u$ denotes the generalized Laplacian of a subharmonic function u (see e.g. [37], Chapter 3). If $f \in \mathcal{O}(\overline{\mathbb{D}}, X)$ and $v \circ f \equiv -\infty$, then we put $\mathfrak{F}_2(f) := 0$.

Green functional. Let X be a complex manifold and let $\alpha : X \to [0,\infty)$ be an arbitrary function. We define the functional $\mathfrak{F}_3 = \mathfrak{F}_3^{\alpha}$ by the formula

$$\mathfrak{F}_3(f) = \sum_{z\in\mathbb{D}_*} \alpha(f(z))\log|z|, \qquad f \in \mathcal{O}(\overline{\mathbb{D}}, X),$$

where $\mathbb{D}_* := \mathbb{D} \setminus \{0\}$. The sum, which may be uncountable, is defined as the infimum of finite partial sums.

Lelong functional. Let X be a complex manifold and let $\alpha : X \to [0,\infty)$ be an arbitrary function. We define the functional $\mathfrak{F}_4 = \mathfrak{F}_4^{\alpha}$ by the formula

$$\mathfrak{F}_4(f) = \sum_{z\in\mathbb{D}_*} \alpha(f(z))\,\mathrm{ord}_z(f)\log|z|, \qquad f \in \mathcal{O}(\overline{\mathbb{D}}, X),$$

where $\mathrm{ord}_z(f)$ denotes the multiplicity of f at z.

Lempert functional. Let X be a complex manifold and let $\alpha : X \to [0,\infty)$ be an arbitrary function. We define the functional $\mathfrak{F}_5 = \mathfrak{F}_5^{\alpha}$ by the formula

$$\mathfrak{F}_5(f) = \inf\{\alpha(f(z))\log|z| : z \in \mathbb{D}_*\}, \qquad f \in \mathcal{O}(\overline{\mathbb{D}}, X).$$

Motivated by the Lempert function (see e.g. [17]), the functional $\mathfrak{F}_5$, in case $\operatorname{supp}\alpha = \{x\}$ and $\alpha(x) = 1$, was introduced in [7]. For any α it was introduced in [25].

The main point of E. Poletsky's theory is the plurisubharmonicity of the envelopes of disc functionals. In the paper we present systematically the plurisubharmonicity of the envelope of the Poisson functional on a class of complex manifolds, containing all domains in $\mathbb{C}^n$. We get as a corollary the proof of the product property of the relative extremal function and, consequently, of the pluricomplex Green function on any domains in $\mathbb{C}^n$.

It is still an open problem whether the envelope of the Poisson functional is plurisubharmonic on *any* complex manifold. So, it seems to be interesting to give a class of complex manifolds as large as possible on which the property holds.

([12]) Note that in this case for any $f \in \mathcal{O}(\overline{\mathbb{D}}, X)$ the integral $\frac{1}{2\pi}\int_0^{2\pi}\varphi(f(e^{i\theta}))\,d\theta$ is well defined. Note also that it may attain the value $-\infty$ or ∞.

1.3. A class of complex manifolds. The disc functional method of E. Poletsky was first extended to a class $\mathcal{P}$ of complex manifolds by F. Lárusson and R. Sigurdsson [25] in 1997.

Recall that an n-dimensional complex manifold X is said to be a *Stein manifold* if:

(a) X has a countable basis;

(b) X is holomorphically convex, i.e. for any compact set $K \subset X$ the set
$$\widehat{K}^{\mathcal{O}(X)} := \{x \in X : |f(x)| \le \|f\|_K \text{ for any } f \in \mathcal{O}(X)\}$$
is compact in X, where $\|f\|_K := \sup\{|f(x)| : x \in K\}$;

(c) $\mathcal{O}(X)$ separates points in X, i.e. for any points $x, y \in X$, $x \neq y$, there exists an $f \in \mathcal{O}(X)$ such that $f(x) \neq f(y)$;

(d) for any point $x \in X$ there exists a holomorphic mapping $F : X \to \mathbb{C}^n$ such that F is injective in a neighborhood of x.

Recall that for domains in $\mathbb{C}^n$, Steinness coincides with pseudoconvexity (see e.g. [15], Chapter VII).

Define $\mathcal{P}$ as the class of complex manifolds X for which there exists a finite sequence of complex manifolds and holomorphic maps
$$X_0 \xrightarrow{h_1} X_1 \xrightarrow{h_2} \ldots \xrightarrow{h_m} X_m = X, \qquad m \ge 0,$$
where X_0 is a domain in a Stein manifold and each h_i, $i = 1, \ldots, m$, is either a holomorphic covering or a finite branched covering ([13]).

Apart from domains in Stein manifolds, $\mathcal{P}$ contains for instance all Riemann surfaces and all covering spaces of projective manifolds ([14]) (see [25]).

The definition of the class $\mathcal{P}$, which may look unnatural, follows from the properties of disc functionals. Studying more carefully these properties we propose an even larger class of complex manifolds.

We say that a complex manifold X belongs to the class $\widetilde{\mathcal{P}}$ if there exists a sequence of domains $X_1 \subset X_2 \subset X_3 \subset \ldots$ in X, each in the class $\mathcal{P}$, such that $X = \bigcup_{k=1}^{\infty} X_k$.

As was shown by J. E. Fornæss [13], there exists a sequence $X_1 \subset X_2 \subset X_3 \subset \ldots$ of Stein manifolds such that $X = \bigcup_{k=1}^{\infty} X_k$ is not Stein. So, it seems ([15]) that the class $\mathcal{P}$ is a proper subclass of the class $\widetilde{\mathcal{P}}$.

We have the following properties of the class $\widetilde{\mathcal{P}}$.

Proposition 1.6 (Proposition 2.8). *Let Y be a domain in a complex manifold X. If X is of class $\widetilde{\mathcal{P}}$, then Y is also of class $\widetilde{\mathcal{P}}$.*

Proposition 1.7 (Proposition 2.9). *Let X, Y be complex manifolds of class $\widetilde{\mathcal{P}}$. Then the product $X \times Y$ is also of class $\widetilde{\mathcal{P}}$.*

([13]) We say that a holomorphic mapping $F : X \to Y$ is a *finite branched covering* if $\dim X = \dim Y$ and F is a proper holomorphic mapping, i.e. for any compact set $K \subset Y$ the set $F^{-1}(K)$ is compact. Note that F is surjective and there exists $k \in \mathbb{N}$ such that $\#F^{-1}(y) = k$ for any $y \in Y \setminus F(Z)$, where $Z = \{x \in X : \operatorname{rank} dF_x < \dim X\}$ (cf. [39]). We call Z the *branched locus* of F.

([14]) By a *projective manifold* we mean a complex submanifold of complex projective space $\mathbb{P}_n$.

([15]) We do not yet have an example showing that $\mathcal{P} \subsetneq \widetilde{\mathcal{P}}$.

1.4. The Poisson functional. The most important functional in applications is the Poisson functional. Let us start with the following basic result.

THEOREM 1.8 (Theorem 3.2). *Let X be a complex manifold and let $\varphi : X \to \overline{\mathbb{R}}$ be a measurable function which is locally bounded from above or below. Then*

$$\sup\{v \in \mathrm{PSH}(X) : v \le \varphi\} \le E_{\mathfrak{F}_1^\varphi} \le \varphi \quad \text{on } X. \tag{1.3}$$

Therefore, if $E_{\mathfrak{F}_1^\varphi}$ is a plurisubharmonic function on X, then

$$E_{\mathfrak{F}_1^\varphi} = \sup\{v \in \mathrm{PSH}(X) : v \le \varphi\} \quad \text{on } X. \tag{1.4}$$

In the course of his study of disc functionals, E. Poletsky [35] introduced the class of *approximately upper semicontinuous functions* ([16]) ([17]). We think that the class of functions given below, which is related to the class of approximately upper semicontinuous functions, is more natural and more handy.

Let Ω be a domain in $\mathbb{C}^n$ and let $\varphi : \Omega \to \overline{\mathbb{R}}$ be a measurable function locally bounded from above or below. We say that φ is a *weakly integrally upper semicontinuous function* if for any $x_0 \in \Omega$ we have

$$\limsup_{r\to 0+} \frac{1}{b_n r^{2n}} \Big[\sup_{x \in \mathbb{B}_n(x_0,r)} \int_{\mathbb{B}_n(x,r)} \varphi(y)\, d\mathcal{L}^{2n}(y) \Big] \le \varphi(x_0),$$

where $b_n := \mathcal{L}^{2n}(\mathbb{B}_n)$ ([18]).

The following result gives non-trivial examples of weakly integrally upper semicontinuous functions.

PROPOSITION 1.9 (Proposition 3.3). *Let Ω be a domain in $\mathbb{C}^n$ and let φ be a superharmonic on Ω. Then φ is weakly integrally upper semicontinuous on Ω.*

Let X be a complex manifold and let $\varphi : X \to \overline{\mathbb{R}}$ be a measurable function locally bounded from above or below. We say that φ is an *integrally upper semicontinuous function* (written $\varphi \in \mathcal{IC}^\uparrow(X)$) if for any domain $\Omega \subset \mathbb{C}^m$, $m \ge 1$, and any holomorphic mapping $F : \Omega \to X$ the function $\varphi \circ F$ is weakly integrally upper semicontinuous on Ω ([19]).

([16]) Let $\varphi : \Omega \to \overline{\mathbb{R}}$ be a function, where $\Omega \subset \mathbb{C}^n$ is a domain. We say that φ is an *approximately upper semicontinuous function* if for any $x \in \Omega$ and any $\varepsilon > 0$ there exists a measurable set $F \subset \mathbb{C}^n$ such that $\{y \in \Omega : \varphi(y) > \varphi(x) + \varepsilon\} \subset F$ and

$$\limsup_{r\to 0} \mathcal{L}^{2n}(F \cap \mathbb{B}_n(x,r))/\mathcal{L}^{2n}(\mathbb{B}_n(x,r)) = 0,$$

where $\mathcal{L}^{2n}$ denotes the Lebesgue measure in $\mathbb{C}^n$.

([17]) As shown by E. Poletsky, there exists a lower semicontinuous function φ on the unit ball $\mathbb{B}_n := \mathbb{B}_n(0,1)$ in $\mathbb{C}^n$ such that $E_{\mathfrak{F}_1^\varphi}$ is not plurisubharmonic. So, it seems interesting to give a class of functions φ for which $E_{\mathfrak{F}_1^\varphi}$ is plurisubharmonic and which contains upper semicontinuous and plurisuperharmonic functions. The latter is important in the study of the Riesz functional (see Section 1.5).

([18]) Let $\varphi(z) = 1$ if $z = 1/n$, $n \in \mathbb{N}$, and 0 otherwise. Note that φ is not an upper semicontinuous function on $\mathbb{C}$, but it is weakly integrally upper semicontinuous.

([19]) Note that the relation between weakly integrally upper semicontinuous and integrally upper semicontinuous functions is similar to the relation between superharmonic and plurisuperharmonic functions.

We see from the definition that any upper semicontinuous function is integrally upper semicontinuous. The following proposition shows that there exist integrally upper semicontinuous functions which are not upper semicontinuous.

Proposition 1.10 (Corollary 3.4). *Let X be a complex manifold and let $\varphi \in \mathrm{PSH}(X)$. Then $-\varphi \in \mathcal{IC}^{\uparrow}(X)$.*

One of the main results connected with the Poisson functional is the following.

Theorem 1.11 (Theorem 3.5). *Let X be a complex manifold. Assume that*

(a) *$\varphi \in \mathcal{IC}^{\uparrow}(X)$ is locally bounded from above or*
(b) *φ is a plurisuperharmonic function on X, $\varphi \not\equiv \infty$.*

Then $E_{\mathfrak{F}_1^\varphi}$ is upper semicontinuous.

The next result gives us a class of integrally upper semicontinuous functions for which we have $E_{\mathfrak{F}_1^\varphi} < \infty$.

Proposition 1.12 (Proposition 3.9). *Let X be a complex manifold. Assume that*

(a) *$\varphi \in \mathcal{IC}^{\uparrow}(X)$ is locally bounded from above or*
(b) *φ is a plurisuperharmonic function on X, $\varphi \not\equiv \infty$.*

Then $E_{\mathfrak{F}_1^\varphi} < \infty$.

We are able to prove the plurisubharmonicity of the envelope for complex manifolds of class $\widetilde{\mathcal{P}}$. As mentioned above, it is the main point of Poletsky's theory.

Theorem 1.13 (Theorem 3.10). *Let X be a complex manifold of class $\widetilde{\mathcal{P}}$. Assume that*

(a) *$\varphi \in \mathcal{IC}^{\uparrow}(X)$ is locally bounded from above or*
(b) *φ is a plurisuperharmonic function on X, $\varphi \not\equiv \infty$.*

Then $E_{\mathfrak{F}_1^\varphi}$ is a plurisubharmonic function and, therefore,

$$E_{\mathfrak{F}_1^\varphi} = \sup\{v \in \mathrm{PSH}(X) : v \le \varphi\} \quad \text{on } X.$$

Theorem 1.13 for upper semicontinuous functions and for domains in $\mathbb{C}^n$ was proved by E. Poletsky [34], [35]. For upper semicontinuous functions on complex manifolds of class $\mathcal{P}$ it was proved by F. Lárusson and R. Sigurdsson [25]. For plurisuperharmonic functions and the same class of complex manifolds the proof was given by the author [11].

Now, let us consider the following special case of the Poisson functional. Let X be a complex manifold and let U be an open subset of X. We put

$$\begin{aligned}\widetilde{\omega}(x,U,X) &:= \inf\left\{\frac{1}{2\pi}\int_0^{2\pi} -\chi_U(f(e^{i\theta}))\,d\theta : f \in \mathcal{O}(\overline{\mathbb{D}},X),\ f(0)=x\right\}\\ &= -\sup\left\{\frac{1}{2\pi}\sigma(\{\tau\in\mathbb{T} : f(\tau)\in U\}) : f\in\mathcal{O}(\overline{\mathbb{D}},X),\ f(0)=x\right\}, \qquad x\in X,\end{aligned}$$

where χ_U denotes the characteristic function of U and σ denotes the arc length measure on the unit circle $\mathbb{T}$. Note that $-\chi_U$ is upper semicontinuous.

As a corollary of Theorem 1.13 we obtain the following result.

Corollary 1.14. *Let X be a complex manifold of class $\widetilde{\mathcal{P}}$ and let U be an open subset of X. Then*

$$\omega(x,U,X)=\widetilde{\omega}(x,U,X). \tag{1.5}$$

The main result relating to this special function is the following product property, first proven in [12].

Theorem 1.15 (Theorem 4.1). *Let X_1 and X_2 be complex manifolds and let $U_1\subset X_1$, $U_2\subset X_2$ be open sets. Then*

$$\widetilde{\omega}((x_1,x_2),U_1\times U_2,X_1\times X_2)=\max\{\widetilde{\omega}(x_1,U_1,X_1),\widetilde{\omega}(x_2,U_2,X_2)\},\qquad (x_1,x_2)\in X_1\times X_2.$$

Using Corollary 1.14 and Theorem 1.15 we obtain the product property for the relative extremal function (see [12]).

Theorem 1.16 (Theorem 5.5). *Let X_1 and X_2 be complex manifolds of class $\widetilde{\mathcal{P}}$ and let $E_1\subset X_1$, $E_2\subset X_2$ be open or compact subsets. Then*

$$\omega((x_1,x_2),E_1\times E_2,X_1\times X_2)=\max\{\omega(x_1,E_1,X_1),\omega(x_2,E_2,X_2)\},\qquad (x_1,x_2)\in X_1\times X_2.$$

1.5. The Riesz functional. By the Riesz representation, for a plurisubharmonic function v on a complex manifold X and a holomorphic mapping $f\in\mathcal{O}(\overline{\mathbb{D}},X)$ such that $v\circ f\not\equiv-\infty$ we have

$$\mathfrak{F}_2^v(f)=v(f(0))-\frac{1}{2\pi}\int_0^{2\pi}v(f(e^{i\theta}))\,d\theta.$$

So,

$$\mathfrak{F}_2^v(f)=v(f(0))+\mathfrak{F}_1^{-v}(f)\qquad\text{and}\qquad E_{\mathfrak{F}_2^v}=v+E_{\mathfrak{F}_1^{-v}}. \tag{1.6}$$

As a simple corollary of Theorem 1.11 and (1.6) we have the following.

Corollary 1.17. *Let X be a complex manifold and let v be a plurisubharmonic function on X. Then $E_{\mathfrak{F}_2^v}$ is an upper semicontinuous function on X.*

Recall the following result.

Theorem 1.18 (see Theorem 4.4 in [25]). *Let X be a complex manifold and let v be a plurisubharmonic function on X. Then*

$$\sup\{u\in\mathrm{PSH}(X):u\le 0,\ \mathcal{L}(u)\ge\mathcal{L}(v)\}\le E_{\mathfrak{F}_2^v}\quad \text{on } X,$$

where $\mathcal{L}(v)$ denotes the Levi form $i\partial\overline{\partial}v$ of v, which is a closed positive $(1,1)$-current on X (see e.g. [20]) ([20]). Moreover, if $E_{\mathfrak{F}_2^v}$ is plurisubharmonic then

$$\sup\{u\in\mathrm{PSH}(X):u\le 0,\ \mathcal{L}(u)\ge\mathcal{L}(v)\}= E_{\mathfrak{F}_2^v}\quad \text{on } X.$$

As a corollary of Theorem 1.13, Theorem 1.18, and (1.6) we get we following

Theorem 1.19. *Let X be a complex manifold of class $\widetilde{\mathcal{P}}$ and let v be a plurisubharmonic function on X. Then $E_{\mathfrak{F}_2^v}$ is a plurisubharmonic function on X and, therefore,*

$$\sup\{u\in\mathrm{PSH}(X):u\le 0,\ \mathcal{L}(u)\ge\mathcal{L}(v)\}= E_{\mathfrak{F}_2^v}\quad \text{on } X.$$

([20]) We put $\mathcal{L}(-\infty)=0$.

Theorem 1.19 for plurisubharmonic functions on domains in $\mathbb{C}^n$ was stated by E. Poletsky (see [35]). For continuous plurisubharmonic functions on complex manifolds of class $\mathcal{P}$ it was proved by F. Lárusson and R. Sigurdsson (see [25]). It seems that the first complete proof for any plurisubharmonic function was given by the author (see [11]).

1.6. The Green and Lelong functionals. Since both functionals are related to the pluricomplex Green function and are very similar, we decided to present their properties together.

Let X be a complex manifold and let $\alpha : X \to [0,\infty)$ be any function. Note that $\mathfrak{F}_4^\alpha \le \mathfrak{F}_3^\alpha$ and, therefore, $E_{\mathfrak{F}_4^\alpha} \le E_{\mathfrak{F}_3^\alpha}$.

For both functionals we have the following duality property (see Proposition 5.1 in [25], see also [35], [7]).

THEOREM 1.20. *Let X be a complex manifold and let $\alpha : X \to [0,\infty)$ be any function. Put $u = E_{H^\alpha}$, where $H = \mathfrak{F}_3$ or $\mathfrak{F}_4$. Then*

$$\sup\{v \in \mathrm{PSH}(X) : v \le 0,\ \nu(\cdot, v) \ge \alpha\} \le u \le 0 \quad \text{on } X.$$

Moreover, if u is a plurisubharmonic function on X, then $\nu(\cdot, u) \ge \alpha$ on X. Therefore, in this case we have

$$\sup\{v \in \mathrm{PSH}(X) : v \le 0,\ \nu(\cdot, v) \ge \alpha\} = u \quad \text{on } X.$$

We have the following result, related to the plurisubharmonicity of the functionals $\mathfrak{F}_3^\alpha$ and $\mathfrak{F}_4^\alpha$.

THEOREM 1.21 (see Theorem 1 in [7], Theorem 5.3 in [25]). *Let X be a domain in a Stein manifold and let α be a non-negative function on X. Then*

$$E_{\mathfrak{F}_3^\alpha} = E_{\mathfrak{F}_4^\alpha} = \sup\{v \in \mathrm{PSH}(X) : v \le 0,\ \nu(\cdot, v) \ge \alpha\}.$$

In the case when X is a domain in $\mathbb{C}^n$, $\operatorname{supp}\alpha = \{x\}$, and $\alpha(x) = 1$, Theorem 1.21 was proved by the author. The general case is proved by a similar method. One has to use the Remmert–Bishop–Narasimhan embedding theorem and proceed in $\mathbb{C}^n$ as in [7] (for more details see [25]).

We have the following product property.

THEOREM 1.22 (Theorem 4.10 for $\mathfrak{F}_3$). *Let X_1 and X_2 be complex manifolds and let $\alpha_1 : X_1 \to \{0,1\}$ and $\alpha_2 : X_2 \to \{0,1\}$ be arbitrary functions. Assume that $H = \mathfrak{F}_3$ or $\mathfrak{F}_4$. Then*

$$E_{H^{\alpha_1 \otimes \alpha_2}}(x_1, x_2) = \max\{E_{H^{\alpha_1}}(x_1), E_{H^{\alpha_2}}(x_2)\}, \qquad (x_1, x_2) \in X_1 \times X_2.$$

The study of the product property of disc functionals was initiated by the author (see [8] where Theorem 1.22 is proved for the case $\operatorname{supp}\alpha = \{x\}$ and $\alpha(x) = 1$). Later extensions to the general case and to the Poisson functional are modifications of the original proof. Theorem 1.22 for the Green functional was proved by the author [9] and for the Lelong functional by F. Lárusson and R. Sigurdsson [26].

2. Preliminary results

2.1. General properties of disc functionals. In this section we present simple but important properties of disc functionals.

PROPOSITION 2.1. *Let X be a complex manifold and let $\{X_j\}_{j=1}^\infty$, $X_j \subset X_{j+1}$, be a sequence of domains in X such that $X = \bigcup_{j=1}^\infty X_j$. Assume that $H : \mathcal{O}(\overline{\mathbb{D}}, X) \to \overline{\mathbb{R}}$ is a disc functional. Then*

$$E_{H|_{X_j}} \searrow E_H \qquad \textit{as } j \to \infty.$$

Moreover, if $E_{H|_{X_j}} \in \mathrm{PSH}(X_j)$ for any $j \ge 1$, then $E_H \in \mathrm{PSH}(X)$.

Proof. Note that

$$E_{H|_{X_j}} \ge E_{H|_{X_{j+1}}} \ge E_H \quad \text{on } X_j,\ j = 1, 2, \ldots$$

Fix an $x_0 \in X$. We may assume that $E_H(x_0) < \infty$. Fix $\beta \in \mathbb{R}$ such that $E_H(x_0) < \beta$. Then there exists an $f \in \mathcal{O}(\overline{\mathbb{D}}, X)$ such that $f(0) = x_0$ and $H(f) < \beta$. There exists $j_0 \ge 1$ such that $f(\overline{\mathbb{D}}) \subset X_j$ for $j \ge j_0$. Then $E_{H|_{X_j}}(x_0) \le H(f) < \beta$ for $j \ge j_0$. Hence, $E_{H|_{X_j}}(x_0) \searrow E_H(x_0)$ as $j \to \infty$.

Recall that if $\{u_j\}_{j\in\mathbb{N}} \subset \mathrm{PSH}(Y)$ is a decreasing sequence of plurisubharmonic functions on a complex manifold Y, then $u = \lim_{j\to\infty} u_j \in \mathrm{PSH}(Y)$ (see e.g. Theorem 2.9.14 in [20]). So, if $E_{H|_{X_j}} \in \mathrm{PSH}(X_j)$ for any $j \ge 1$, then $E_H \in \mathrm{PSH}(X_j)$ for any $j \ge 1$ and, therefore, $E_H \in \mathrm{PSH}(X)$. ∎

PROPOSITION 2.2. *Let X be a complex manifold and let H be a disc functional. Assume that H_j, $j = 1, 2, \ldots$, is a sequence of disc functionals such that $H_j(f) \searrow H(f)$ for any $f \in \mathcal{O}(\overline{\mathbb{D}}, X)$. Then*

$$E_{H_j} \searrow E_H \qquad \textit{as } j \to \infty.$$

Proof. Fix an $x_0 \in X$. We may assume that $E_H(x_0) < \infty$. Fix $\beta \in \mathbb{R}$ such that $E_H(x_0) < \beta$. Then there exists an $f \in \mathcal{O}(\overline{\mathbb{D}}, X)$ such that $f(0) = x_0$ and $H(f) < \beta$. There exists $j_0 \ge 1$ such that $H_j(f) < \beta$ and, therefore, $E_{H_j}(x_0) < \beta$ for $j \ge j_0$. ∎

PROPOSITION 2.3. *Let X be a complex manifold. Assume that $H : \mathcal{O}(\overline{\mathbb{D}}, X) \to \overline{\mathbb{R}}$ is a disc functional. Then*

$$E_H(x) = \inf\{E_{H|_Y}(x) : Y \textit{ is a relatively compact domain in } X,\ x \in Y\}, \qquad x \in X.$$

Proof. Note that the inequality "$\le$" follows from the definition of the envelope of a disc functional.

Fix $x_0 \in X$. We may assume that $E_H(x_0) < \infty$. Fix $\beta \in \mathbb{R}$ such that $E_H(x_0) < \beta$. Then there exists an $f \in \mathcal{O}(\overline{\mathbb{D}}, X)$ such that $f(0) = x_0$ and $H(f) < \beta$. There exists a relatively compact domain Y such that $f(\overline{\mathbb{D}}) \subset Y$. Then $E_{H|_Y}(x_0) \le H(f) < \beta$ and, therefore,

$$\inf\{E_{H|_Y}(x_0) : Y \text{ is a relatively compact domain in } X,\ x_0 \in Y\} < \beta. \ ∎$$

Let X and Y be complex manifolds and let $F : X \to Y$ be a holomorphic mapping. If H is a disc functional on Y, then the pullback disc functional F^*H on X is defined by the formula

$$F^*H(f) = H(F \circ f), \qquad f \in \mathcal{O}(\overline{\mathbb{D}}, X).$$

Proposition 2.4. *Let X and Y be complex manifolds and let $F : X \to Y$ be a holomorphic mapping. Assume that H is a disc functional on Y. Then*

$$E_H \circ F \le E_{F^*H}.$$

Moreover, if F is a holomorphic covering, then $E_{F^*H} = E_H \circ F$.

Proof. Fix $x_0 \in X$. We may assume that $E_{F^*H}(x_0) < \infty$. Fix $\beta \in \mathbb{R}$ such that $E_{F^*H}(x_0) < \beta$. There exists an $f \in \mathcal{O}(\overline{\mathbb{D}}, X)$ such that $f(0) = x_0$ and $F^*H(f) < \beta$. Hence, $H(F \circ f) < \beta$. Put $\widetilde{f} = F \circ f$. Note that $\widetilde{f} \in \mathcal{O}(\overline{\mathbb{D}}, Y)$ and $\widetilde{f}(0) = F(x_0)$. Therefore, $E_H(F(x_0)) < \beta$.

Assume that F is a holomorphic covering. Fix $x_0 \in X$ and fix $\beta \in \mathbb{R}$ such that $E_H(F(x_0)) < \beta$. There exists an $f \in \mathcal{O}(\overline{\mathbb{D}}, Y)$ such that $f(0) = F(x_0)$ and $H(f) < \beta$. Since F is a holomorphic covering, there exists an $\widetilde{f} \in \mathcal{O}(\overline{\mathbb{D}}, X)$ such that $\widetilde{f}(0) = x_0$ and $f = F \circ \widetilde{f}$. Then $H(f) = H(F \circ \widetilde{f})$ and, therefore, $E_{F^*H}(x_0) < \beta$. ∎

Proposition 2.5. *Let X and Y be complex manifolds and let $F : X \to Y$ be a holomorphic finite branched covering. Let Z be the branched locus of F. Assume that H is a disc functional on Y such that $E_{H|_{\widetilde{Y}}} = E_H$, where $\widetilde{Y} := Y \setminus F(Z)$. Then*

$$E_{\widetilde{H}} \circ F = E_{F^*\widetilde{H}} \quad \text{on } \widetilde{X},$$

where $\widetilde{X} := X \setminus Z$ and $\widetilde{H} := H|_{\widetilde{Y}}$.

Moreover, if $E_H = E_{\widetilde{H}}$ on $\widetilde{Y}$, then $E_{F^*H} = E_{F^*\widetilde{H}}$ on $\widetilde{X}$.

Proof. Note that $F|_{\widetilde{X}} : \widetilde{X} \to \widetilde{Y}$ is a holomorphic covering. Hence,

$$E_{\widetilde{H}} \circ F = E_{F^*\widetilde{H}} \quad \text{on } \widetilde{X}.$$

We have $E_H \circ F \le E_{F^*H}$, $E_H \circ F \le E_{\widetilde{H}} \circ F$, and $E_{F^*H} \le E_{F^*\widetilde{H}}$. So, if $E_H = E_{\widetilde{H}}$ on $\widetilde{Y}$, then $E_{F^*H} = E_{F^*\widetilde{H}}$ on $\widetilde{X}$. ∎

2.2. A class of complex manifolds. Let us start with the following two results, which are crucial for our considerations. The first one is well known (see e.g. [15], Chapter VII).

Theorem 2.6 (Remmert–Bishop–Narasimhan embedding theorem). *Let X be a Stein manifold. Then there exists a holomorphic embedding of X into $\mathbb{C}^N$ for some $N \in \mathbb{N}$* ([21]).

Theorem 2.7. *Let X be a domain in a Stein manifold. Then there exist a domain Y in $\mathbb{C}^n$ and holomorphic mappings $\Phi : X \to Y$, $\Psi : Y \to X$ such that $\Psi \circ \Phi = \mathrm{id}_X$.*

Proof. Let X be a domain in a Stein manifold $\widetilde{X}$. There exists a biholomorphic mapping $\widetilde{\Phi} : \widetilde{X} \to Z$, where Z is a submanifold of $\mathbb{C}^N$.

By a theorem of Docquier and Grauert (see e.g. [15], VIII.C.8) there exists a connected neighborhood $\widetilde{Y}$ of Z in $\mathbb{C}^N$ and a holomorphic retraction $\xi : \widetilde{Y} \to Z$. We put $\Phi := \widetilde{\Phi}|_X$ and $\Psi := \widetilde{\Phi}^{-1} \circ \xi|_Y$, where Y is a connected component of $\xi^{-1}(\Phi(X))$ which contains $\Phi(X)$.

It is easy to check that $\Psi \circ \Phi = \mathrm{id}_X$. ∎

Note that if X, Y are complex manifolds, X is of class $\mathcal{P}$ and $h : X \to Y$ is either a holomorphic covering or a finite branched covering, then Y is also of class $\mathcal{P}$.

([21]) Recall that a holomorphic mapping $F : X \to \mathbb{C}^N$ is called an *embedding* if F is injective, proper, and for any $x \in X$ the tangent mapping $dF_x : T_x X \to T_{F(x)}\mathbb{C}^N$ is injective.

Proposition 2.8 (cf. Proposition 3.5 in [25]). *Let Y be a domain in a complex manifold X. If X is of class $\widetilde{\mathcal{P}}$, then Y is also of class $\widetilde{\mathcal{P}}$.*

Proof. First note that it follows immediately from the definition that any complex manifold of class $\widetilde{\mathcal{P}}$ has countable base.

Let $Y = \bigcup_{j=1}^{\infty} Y_j$, where $Y_j \subset Y_{j+1}$, $j = 1, 2, \dots$, are relatively compact domains in Y. It suffices to prove that Y_j are of class $\mathcal{P}$, $j = 1, 2, \dots$

So, we may assume that Y is relatively compact in X. It suffices to prove that if X is of class $\mathcal{P}$, then Y is also of class $\mathcal{P}$. Note that if $h : \widetilde{X} \to X$ is a holomorphic covering (resp. a finite branched covering), Y is a domain in X and $\widetilde{Y}$ is a connected component of $h^{-1}(Y)$, then $h|_{\widetilde{Y}} : \widetilde{Y} \to Y$ is a holomorphic covering (resp. a finite branched covering).

Let

$$X_0 \xrightarrow{h_1} X_1 \xrightarrow{h_2} \dots \xrightarrow{h_m} X_m = X, \qquad m \geq 0,$$

be a sequence as in the definition of the class $\mathcal{P}$.

If $m = 0$, then X is a domain in a Stein manifold and, therefore, Y is also a domain in a Stein manifold. Hence, Y is a manifold of class $\mathcal{P}$. If $m \geq 1$, then we define a sequence

$$Y_0 \xrightarrow{\ell_1} Y_1 \xrightarrow{\ell_2} \dots \xrightarrow{\ell_m} Y_m = Y$$

by induction as follows. For $i = m, \dots, 1$, let $Y_{i-1} \subset X_{i-1}$ be a connected component of $h_i^{-1}(Y_i)$ and let $\ell_i = h_i|_{Y_{i-1}}$. Then Y_0 is a domain in a Stein manifold, and we see that Y is a complex manifold of class $\mathcal{P}$. ∎

Proposition 2.9 (cf. Proposition 3.6 in [25]). *Let X, Y be complex manifolds of class $\widetilde{\mathcal{P}}$. Then the product $X \times Y$ is also of class $\widetilde{\mathcal{P}}$.*

Proof. It suffices to prove that if X, Y are of class $\mathcal{P}$, then $X \times Y$ is also of class $\mathcal{P}$. Note that if $h : \widetilde{X} \to X$ is a holomorphic covering (resp. a finite branched covering) and Z is a complex manifold, then $h \times \mathrm{id} : \widetilde{X} \times Z \to X \times Z$ is a holomorphic covering (resp. a finite branched covering). Note also that the product of Stein manifolds is a Stein manifold.

Let

$$X_0 \xrightarrow{h_1} X_1 \xrightarrow{h_2} \dots \xrightarrow{h_m} X_m = X, \qquad Y_0 \xrightarrow{\ell_1} Y_1 \xrightarrow{\ell_2} \dots \xrightarrow{\ell_m} Y_m = Y,$$

be sequences as in the definition of class $\mathcal{P}$. We may assume that they are of the same length, because such sequences can always be extended by identity mappings. Now we replace each mapping $X_i \xrightarrow{h_{i+1}} X_{i+1}$ by the composition $X_i \xrightarrow{h_{i+1}} X_{i+1} \xrightarrow{\mathrm{id}} X_{i+1}$ and each mapping $Y_i \xrightarrow{\ell_{i+1}} Y_{i+1}$ by the composition $Y_i \xrightarrow{\mathrm{id}} Y_i \xrightarrow{\ell_{i+1}} Y_{i+1}$. Then the sequence

$$X_0 \times Y_0 \xrightarrow{h_1 \times \mathrm{id}} X_1 \times Y_0 \xrightarrow{\mathrm{id} \times \ell_1} X_1 \times Y_1 \xrightarrow{h_2 \times \mathrm{id}} \dots \xrightarrow{h_m \times \mathrm{id}} X_m \times Y_{m-1} \xrightarrow{\mathrm{id} \times \ell_m} X_m \times Y_m = X \times Y$$

shows that $X \times Y$ is a complex manifold of class $\mathcal{P}$. ∎

2.3. Variation of analytic discs. This part of the paper is based mainly on the results from [25]. For the sake of completeness we give proofs.

Theorem 2.10. *Let X be a complex manifold. Let $f : \mathbb{D} \to X$ be a holomorphic mapping. Then there exist open sets $W \subset \mathbb{D} \times X$, $\widetilde{W} \subset \mathbb{D} \times \mathbb{C}^n$ $(n = \dim X)$, and a biholomorphic* *mapping* $\Psi : W \to \widetilde{W}$ *such that*

$$\Psi(z, f(z)) = (z, 0), \qquad z \in \mathbb{D}.$$

Proof. Consider the graph

$$\Gamma = \{(z, f(z)) : z \in \mathbb{D}\} \subset \mathbb{D} \times X.$$

Then Γ is a Stein submanifold of $\mathbb{D} \times X$. By Siu's theorem ([22]) there exist a Stein neighborhood $W \subset \mathbb{D} \times X$ of Γ and a biholomorphic map $\widetilde{\Psi}$ of W onto a neighborhood of the zero section of the normal bundle of Γ, which identifies Γ with the zero section. It is well known that the normal bundle of Γ is holomorphically trivial ([23]) and, therefore, it is biholomorphic to $\Gamma \times \mathbb{C}^n$. From this we conclude that there exists a biholomorphic map $\Psi : W \to \widetilde{W}$ such that $\Psi(z, f(z)) = (z, 0)$ for all $z \in \mathbb{D}$, where $\widetilde{W}$ is a neighborhood of $\mathbb{D} \times \{0\}$. ∎

Corollary 2.11 (cf. Lemma 1.1 in [46]). *Let X be a complex manifold and let $f_0 \in \mathcal{O}(\mathbb{D}_{r_0}, X)$, $r_0 > 1$. Then for any $r \in (1, r_0)$ there exist an open set $U \subset \mathbb{D}_{r_0} \times X$ and a biholomorphic mapping $\Phi : U \to \mathbb{D}_r \times \mathbb{D}^n$ such that*

(i) $\{(z, f_0(z)) : z \in \mathbb{D}_r\} \subset U$,
(ii) $\Phi(z, f_0(z)) = (z, 0)$, $z \in \mathbb{D}_r$.

Proof. By Theorem 2.10 there exist open sets $W \subset \mathbb{D}_{r_0} \times X$, $\widetilde{W} \subset \mathbb{D}_{r_0} \times \mathbb{C}^n$, and a biholomorphic mapping $\Psi : W \to \widetilde{W}$ such that

$$\Psi(z, f_0(z)) = (z, 0), \qquad z \in \mathbb{D}_{r_0}.$$

Fix $r \in (1, r_0)$. Note that $\{(z, 0) : z \in \mathbb{D}_r\}$ is relatively compact in $\widetilde{W}$. Therefore, there exists $R > 0$ such that

$$\widetilde{U} := \{(z, z_1, \ldots, z_n) : z \in \mathbb{D}_r,\ |z_j| < R,\ j = 1, \ldots, n\} \subset \widetilde{W}.$$

Put $U = \Psi^{-1}(\widetilde{U})$ and $\Phi(z, x) = (\Psi_1(z, x), (1/R)\Psi_2(z, x))$, where $\Psi = (\Psi_1, \Psi_2)$. ∎

Corollary 2.12 (see [25], Lemma 2.3; cf. [6], Theorem 1.1). *Let X be a complex manifold and let $f_0 \in \mathcal{O}(\mathbb{D}_{r_0}, X)$, $r_0 > 1$. Then for any $r \in (1, r_0)$ there exist an open neighborhood V of $x_0 = f_0(0)$ and $f \in \mathcal{O}(\mathbb{D}_r \times V, X)$ such that*

(i) *$f(z, x_0) = f_0(z)$ for all $z \in \mathbb{D}_r$,*
(ii) *$f(0, x) = x$ for all $x \in V$.*

Moreover, if f_0 is non-constant, then for every finite set $M \subset \mathbb{D}_*$ we can find an f such that

(iii) $f(w, x) = f_0(w)$,

([22]) Theorem (see [43], Corollary 1). *Suppose that V is a complex submanifold of a complex manifold M. If V is Stein, then there exists a biholomorphic map from an open neighborhood W of V in M onto an open neighborhood of the zero cross section of the normal bundle of V in M such that its restriction to V agrees with the canonical map from V onto the zero cross section. As a consequence, there is a holomorphic retraction from W onto V.*

([23]) Theorem (see e.g. [14], Theorem 30.4). *Let X be a non-compact Riemann surface, i.e. a 1-dimensional connected complex manifold. Then every holomorphic vector bundle on X is holomorphically trivial.*

(iv) $\operatorname{ord}_w f(\,\cdot\,,x) = \operatorname{ord}_w f_0(\cdot)$ *for all* $w \in M$ *and all* $x \in V$.

If f_0 *is constant, then for every finite set* $M \subset \mathbb{D}_*$ *and every* $N \in \mathbb{N}$ *we can find an* f *such that*

(iii′) $f(w,x) = f_0(w)$,
(iv′) $\operatorname{ord}_w f(\,\cdot\,,x) \geq N$ *for all* $w \in M$ *and all* $x \in V$.

Proof. Fix $r \in (1,r_0)$ and $\widetilde{r} \in (r,r_0)$. According to Corollary 2.11 there exist a neighborhood $U \subset \mathbb{D}_{r_0} \times X$ of $\{(z,f_0(z)) : z \in \mathbb{D}_{\widetilde{r}}\}$ and a biholomorphic mapping $\Phi : U \to \mathbb{D}_{\widetilde{r}} \times \mathbb{D}^n$ such that $\Phi(z,f_0(z)) = (z,0)$, $z \in \mathbb{D}_{\widetilde{r}}$.

If $M = \emptyset$, then set $P \equiv 1$. If $M \neq \emptyset$, then we take $m \in \mathbb{N}$ such that $m \geq \operatorname{ord}_w(f_0)+1$, $w \in M$, if f_0 is non-constant, and $m \geq N+1$ if f_0 is constant. Define

$$P(z) = \Big[\prod_{w\in M}(1 - z/w)\Big]^m.$$

Note that $\Phi(0,x_0) = (0,0)$. Hence, there exists a neighborhood V of x_0 such that $(z,0) + P(z)\Phi(0,x) \in \mathbb{D}_{\widetilde{r}} \times \mathbb{D}^n$ for all $z \in \mathbb{D}_r$ and $x \in V$.

Let $\operatorname{pr} : \mathbb{C} \times X \to X$ be the natural projection. If we define the holomorphic mapping by

$$f(z,x) = \operatorname{pr}(\Phi^{-1}((z,0) + P(z)\Phi(0,x))), \qquad z \in \mathbb{D}_r,\ x \in V,$$

then all the conditions are satisfied. ∎

COROLLARY 2.13. *Let* X *be a complex manifold and let* $f_0 \in \mathcal{O}(\mathbb{D}_{r_0},X)$, $r_0 > 1$. *Suppose that* $\{w_1,\ldots,w_\ell\} \subset \mathbb{D}_*$ *are different points. Then for any* $r \in (1,r_0)$ *there exist disjoint neighborhoods* $U_j \subset \mathbb{D}_*$ *of* w_j, $j = 1,\ldots,\ell$, *and* $f \in \mathcal{O}(\mathbb{D}_r \times U_1 \times \ldots \times U_\ell, X)$ *such that*

(i) $f(0,z_1,\ldots,z_\ell) = f_0(0)$,
(ii) $f(z_j,z_1,\ldots,z_\ell) = f_0(w_j)$ *for* $j = 1,\ldots,\ell$ *and for all* $z_1 \in U_1, \ldots, z_\ell \in U_\ell$.

Proof. Fix $r \in (1,r_0)$ and $\widetilde{r} \in (r,r_0)$. According to Corollary 2.11 there exist a neighborhood $U \subset \mathbb{D}_{r_0} \times X$ of $\{(z,f_0(z)) : z \in \mathbb{D}_{\widetilde{r}}\}$ and a biholomorphic mapping $\Phi : U \to \mathbb{D}_{\widetilde{r}} \times \mathbb{D}^n$ such that $\Phi(z,f_0(z)) = (z,0)$, $z \in \mathbb{D}_{\widetilde{r}}$.

Since $w_1,\ldots,w_\ell$ are different points, there exist disjoint neighborhoods $U_1,\ldots,U_\ell \subset \mathbb{D}_*$ of $w_1,\ldots,w_\ell$. Consider the polynomial $P(z,z_1,\ldots,z_\ell) := (z-z_1)\ldots(z-z_\ell)$ and the holomorphic mapping

$$\widetilde{f}(z,z_1,\ldots,z_\ell) = \Big[\sum_{j=1}^{\ell} \frac{P(z,z_1,\ldots,z_\ell)}{P'_z(z_j,z_1,\ldots,z_\ell)(z-z_j)} \cdot \frac{w_j}{z_j}\Big](z,0), \tag{2.1}$$

$$z \in \mathbb{D}_{r_0},\ z_1 \in U_1,\ldots,z_\ell \in U_\ell.$$

Note that

$$\sum_{j=1}^{\ell} \frac{P(z,z_1,\ldots,z_\ell)}{P'_z(z_j,z_1,\ldots,z_\ell)(z-z_j)} = 1, \qquad z \in \mathbb{D}_{r_0},\ z_1 \in U_1,\ldots,z_\ell \in U_\ell,$$

and, therefore,

$$\widetilde{f}(z, z_1, \dots, z_\ell) - (z, 0) = \Bigg[\sum_{j=1}^{\ell} \frac{P(z, z_1, \dots, z_\ell)}{P'_z(z_j, z_1, \dots, z_\ell)(z - z_j)} \cdot \Big(\frac{w_j}{z_j} - 1\Big)\Bigg](z, 0),$$
$$z \in \mathbb{D}_{r_0},\ z_1 \in U_1, \dots, z_\ell \in U_\ell.$$

Hence, taking even smaller $U_1, \dots, U_\ell$ we may assume that $\widetilde{f}(z, z_1, \dots, z_\ell) \in \mathbb{D}_{\widetilde{r}} \times \mathbb{D}^n$ for all $z \in \mathbb{D}_r$, $z_j \in U_j$, $j = 1, \dots, \ell$.

Note that $\widetilde{f}(z_j, z_1, \dots, z_\ell) = (w_j, 0)$, $j = 1, \dots, \ell$, and $\widetilde{f}(0, z_1, \dots, z_\ell) = 0$, $z_1 \in U_1$, $\dots, z_\ell \in U_\ell$. Put $f(z, z_1, \dots, z_\ell) = \mathrm{pr} \circ \Phi^{-1}(\widetilde{f}(z, z_1, \dots, z_\ell))$. It is easy to see that all the conditions are satisfied. ■

3. The Poisson functional

3.1. Upper semicontinuity of the Poisson functional. In this section we study the upper semicontinuity of the envelope of the Poisson functional.

As a corollary of Proposition 2.4 we have the following result.

THEOREM 3.1. *Let X and Y be complex manifolds and let $F : X \to Y$ be a holomorphic mapping. Let $\varphi : Y \to \overline{\mathbb{R}}$ be a measurable function which is locally bounded from above or below. Then*

$$E_{\mathfrak{F}_1^\varphi} \circ F \le E_{\mathfrak{F}_1^{\varphi \circ F}}.$$

Moreover, if F is a holomorphic covering, then

$$E_{\mathfrak{F}_1^\varphi} \circ F = E_{\mathfrak{F}_1^{\varphi \circ F}}.$$

Proof. Note that $F^*\mathfrak{F}_1^\varphi = \mathfrak{F}_1^{\varphi \circ F}$ and use Proposition 2.4. ■

We have the following duality for the Poisson functional (see [34], [35], [25]).

THEOREM 3.2. *Let X be a complex manifold and let $\varphi : X \to \overline{\mathbb{R}}$ be a measurable function which is locally bounded from above or below. Then*

$$\sup\{v \in \mathrm{PSH}(X) : v \le \varphi\} \le E_{\mathfrak{F}_1^\varphi} \le \varphi \quad \text{on } X. \tag{3.1}$$

Therefore, if $E_{\mathfrak{F}_1^\varphi}$ is a plurisubharmonic function on X, then

$$\sup\{v \in \mathrm{PSH}(X) : v \le \varphi\} = E_{\mathfrak{F}_1^\varphi} \quad \text{on } X.$$

For the sake of completeness we give a proof.

Proof. Take a function $f \in \mathcal{O}(\overline{\mathbb{D}}, X)$ and a plurisubharmonic function $v \le \varphi$. Then

$$v(f(0)) \le \frac{1}{2\pi} \int_0^{2\pi} v(f(e^{i\theta}))\, d\theta \le \frac{1}{2\pi} \int_0^{2\pi} \varphi(f(e^{i\theta}))\, d\theta.$$

So, we have the left inequality of (3.1). For the right inequality it suffices to take constant functions. ■

The following proposition served as a motivation for introducing (weakly) integrally upper semicontinuous functions. It gives examples of integrally upper semicontinuous functions which are not upper semicontinuous.

PROPOSITION 3.3. *Let Ω be a domain in $\mathbb{C}^n$ and let $\varphi : \Omega \to (-\infty, \infty]$ be a superharmonic function. Then φ is weakly integrally upper semicontinuous on Ω* ([24]).

Proof. Fix $x_0 \in \Omega$ and $\varepsilon > 0$. We may assume that $\varphi(x_0) \neq \infty$. Put $\varepsilon_1 := \varepsilon/(2^{2n} - 1)$. Since φ is lower semicontinuous, there exists $r_0 > 0$ such that

$$\varphi(x) + \varepsilon_1 \geq \varphi(x_0), \qquad x \in \mathbb{B}_n(x_0, 2r_0) \Subset \Omega.$$

Fix $x \in \mathbb{B}_n(x_0, r)$, $r \in (0, r_0)$. We have

$$\begin{aligned}
\varphi(x_0) &\geq \frac{1}{b_n(2r)^{2n}} \int\limits_{\mathbb{B}_n(x_0,2r)} \varphi(y)\, d\mathcal{L}^{2n}(y) \\
&= \frac{1}{b_n(2r)^{2n}} \int\limits_{\mathbb{B}_n(x,r)} \varphi(y)\, d\mathcal{L}^{2n}(y) + \frac{1}{b_n(2r)^{2n}} \int\limits_{\mathbb{B}_n(x_0,2r)\setminus\mathbb{B}_n(x,r)} \varphi(y)\, d\mathcal{L}^{2n}(y) \\
&\geq \frac{1}{b_n(2r)^{2n}} \int\limits_{\mathbb{B}_n(x,r)} \varphi(y)\, d\mathcal{L}^{2n}(y) + \frac{1}{b_n(2r)^{2n}} (\varphi(x_0) - \varepsilon_1)(b_n(2r)^{2n} - b_n r^{2n}) \\
&= \frac{1}{b_n(2r)^{2n}} \int\limits_{\mathbb{B}_n(x,r)} \varphi(y)\, d\mathcal{L}^{2n}(y) + (\varphi(x_0) - \varepsilon_1)\left(1 - \frac{1}{2^{2n}}\right) \\
&= \frac{1}{b_n(2r)^{2n}} \int\limits_{\mathbb{B}_n(x,r)} \varphi(y)\, d\mathcal{L}^{2n}(y) + \varphi(x_0) - \varphi(x_0)\frac{1}{2^{2n}} - \varepsilon_1\left(1 - \frac{1}{2^{2n}}\right).
\end{aligned}$$

So,

$$\varphi(x_0) + \varepsilon \geq \frac{1}{b_n r^{2n}} \int\limits_{\mathbb{B}_n(x,r)} \varphi(y)\, d\mathcal{L}^{2n}(y) = \frac{1}{b_n} \int\limits_{\mathbb{B}_n} \varphi(x + ry)\, d\mathcal{L}^{2n}(y). \quad \blacksquare$$

As a corollary of Proposition 3.3 we get

COROLLARY 3.4. *Let X be a complex manifold and let $\varphi \in \mathrm{PSH}(X)$. Then $-\varphi \in \mathcal{IC}^{\uparrow}(X)$.*

The main result of this section is the following

THEOREM 3.5. *Let X be a complex manifold. Assume that*

(a) *$\varphi \in \mathcal{IC}^{\uparrow}(X)$ is locally bounded from above or*
(b) *φ is a plurisuperharmonic function on X, $\varphi \not\equiv \infty$.*

Then $E_{\mathfrak{F}_1^{\varphi}}$ is upper semicontinuous.

Before we go into the proof we need the following results.

LEMMA 3.6. *Let $\varphi : \mathbb{T} \times \mathbb{B}_n \to \overline{\mathbb{R}}$ be an integrable function. Then*

$$\frac{1}{2\pi b_n} \int\limits_0^{2\pi} \int\limits_{\mathbb{B}_n} \varphi(e^{i\theta}, y)\, d\mathcal{L}^{2n}(y)\, d\theta = \frac{1}{2\pi b_n} \int\limits_0^{2\pi} \int\limits_{\mathbb{B}_n} \varphi(e^{i\theta}, e^{i\theta} y)\, d\mathcal{L}^{2n}(y)\, d\theta. \tag{3.2}$$

([24]) Note that we may define weakly integrally upper semicontinuity at a point $x_0 \in \Omega$. Then Proposition 3.3 may be reformulated as follows.

PROPOSITION. *Let Ω be a domain in $\mathbb{C}^n$ and let $\varphi : \Omega \to (-\infty, \infty]$ be a lower semicontinuous function. Assume that $\limsup_{r\to 0}(b_n r^{2n})^{-1} \int_{\mathbb{B}_n(x_0,r)} \varphi(y)\mathcal{L}^{2n}(y) \leq \varphi(x_0)$. Then φ is weakly integrally upper semicontinuous at x_0.*

Therefore, there exists $y_0 \in \mathbb{B}_n$ *such that*

$$\frac{1}{2\pi b_n}\int_0^{2\pi}\int_{\mathbb{B}_n}\varphi(e^{i\theta},y)\,d\mathcal{L}^{2n}(y)\,d\theta \ge \frac{1}{2\pi}\int_0^{2\pi}\varphi(e^{i\theta},e^{i\theta}y_0)\,d\theta.$$

Proof. This follows immediately from measure theory (use change of variables). ∎

LEMMA 3.7. *Let* X *be a complex manifold. Assume that*

(a) $\varphi \in \mathcal{IC}^{\uparrow}(X)$ *is locally bounded from above or*
(b) φ *is a plurisuperharmonic function on* X, $\varphi \not\equiv \infty$.

Suppose that Ω *is a domain in* $\mathbb{C}^m$. *Let* $f : \mathbb{D}_r \times \Omega \to X$, $r > 1$, *be a holomorphic mapping such that* $\varphi \circ f(e^{i\theta},\cdot) \not\equiv \infty$, $\theta \in [0,2\pi)$ ([25]). *Then*

$$F(y) := \frac{1}{2\pi}\int_0^{2\pi}\varphi(f(e^{i\theta},y))\,d\theta, \qquad y \in \Omega,$$

is an integrally upper semicontinuous function on Ω.

Proof. Note that if φ is a plurisuperharmonic function, then F is also plurisuperharmonic. So, in case (b) the result follows from Corollary 3.4.

Hence, we may assume that we have case (a). Note that it suffices to prove that F is weakly integrally upper semicontinuous. Fix $y_0 \in \Omega$. We may assume that $F(y_0) < \infty$. Fix $\beta > F(y_0)$. Suppose that there exist $r_m \searrow 0$ and $y_m \in \mathbb{B}_n(y_0, r_m)$ such that

$$\frac{1}{b_n}\int_{\mathbb{B}_n}F(y_m + r_m y)\,d\mathcal{L}^{2n}(y) \ge \beta.$$

Note that

$$\frac{1}{b_n}\int_{\mathbb{B}_n}F(y_m + r_m y)\,d\mathcal{L}^{2n}(y) = \frac{1}{2\pi}\int_0^{2\pi}\left[\frac{1}{b_n}\int_{\mathbb{B}_n}\varphi(f(e^{i\theta},y_m + r_m y))\,d\mathcal{L}^{2n}(y)\right]d\theta.$$

By Fatou's theorem

$$\begin{aligned}\limsup_{m\to\infty}\frac{1}{2\pi}\int_0^{2\pi}&\left[\frac{1}{b_n}\int_{\mathbb{B}_n}\varphi(f(e^{i\theta},y_m + r_m y))\,d\mathcal{L}^{2n}(y)\right]d\theta\\ &\le \frac{1}{2\pi}\int_0^{2\pi}\limsup_{m\to\infty}\left[\frac{1}{b_n}\int_{\mathbb{B}_n}\varphi(f(e^{i\theta},y_m + r_m y))\,d\mathcal{L}^{2n}(y)\right]d\theta.\end{aligned}$$

But for any fixed $\theta \in [0,2\pi)$, $\varphi(f(e^{i\theta},\cdot))$ is an integrally upper semicontinuous function and, therefore,

$$\limsup_{m\to\infty}\left[\frac{1}{b_n}\int_{\mathbb{B}_n}\varphi(f(e^{i\theta},y_m + r_m y))\,d\mathcal{L}^{2n}(y)\right] \le \varphi(f(e^{i\theta},y_0)).$$

So,

$$\beta \le \limsup_{m\to\infty}\frac{1}{2\pi}\int_0^{2\pi}\left[\frac{1}{b_n}\int_{\mathbb{B}_n}\varphi(f(e^{i\theta},y_m + r_m y))\,d\mathcal{L}^{2n}(y)\right]d\theta \le F(y_0).$$

The contradiction finishes the proof. ∎

LEMMA 3.8. *Let* X *be a complex manifold. Suppose that*

([25]) Note that $\varphi \circ f(e^{i\theta},\cdot) \equiv \infty$, $\theta \in [0,2\pi)$, is possible only in case (b).

(a) *$\varphi \in \mathcal{IC}^{\uparrow}(X)$ is locally bounded from above or*
(b) *φ is a plurisuperharmonic function on X, $\varphi \not\equiv \infty$.*

Assume that $x_0 \in X$, $\beta \in \mathbb{R}$, and $f_0 \in \mathcal{O}(\overline{\mathbb{D}}, X)$ is such that $f_0(0) = x_0$ and $\mathfrak{F}_1(f_0) < \beta$, where $\mathfrak{F}_1 := \mathfrak{F}_1^{\varphi}$. Then there exist a neighborhood V of x_0 in X, $r > 1$, and $f \in \mathcal{O}(\mathbb{D}_r \times \mathbb{B}_n(r) \times V, X)$ such that:

(a) *$f(0,0,x) = f(0,y,x) = x$ for any $x \in V$ and any $y \in \mathbb{B}_n$;*
(b) *$f(z,0,x_0) = f_0(z)$ for any $z \in \overline{\mathbb{D}}$;*
(c)
$$\frac{1}{b_n}\int\limits_{\mathbb{B}_n} \mathfrak{F}_1(f(\,\cdot\,,y,x))\,d\mathcal{L}^{2n}(y) < \beta \quad \textit{for all } x \in V.$$

Proof. According to Corollary 2.12, there exist an $\widetilde{r} > 1$, an open neighborhood $\widetilde{V}$ of x_0, and an $\widetilde{f} \in \mathcal{O}(\mathbb{D}_{\widetilde{r}} \times \widetilde{V}, X)$ such that $\widetilde{f}(z,x_0) = f_0(z)$ for all $z \in \mathbb{D}_{\widetilde{r}}$ and $\widetilde{f}(0,x) = x$ for all $x \in \widetilde{V}$.

Let (U,ζ) be a local coordinate such that $\zeta(x_0) = 0$. We may assume that $U \subset \widetilde{V}$, $\zeta : U \to \zeta(U) = \mathbb{B}_n$. Consider the function
$$F(w) = \frac{1}{2\pi}\int\limits_0^{2\pi} \varphi(\widetilde{f}(e^{i\theta}, \zeta^{-1}(w)))\,d\theta, \qquad w \in \mathbb{B}_n.$$
By Lemma 3.7, F is an integrally upper semicontinuous function on $\mathbb{B}_n$. Fix an $\varepsilon > 0$ such that $\mathfrak{F}_1(f_0) < \beta - \varepsilon$. Then there exists an $r_0 \in (0,1/2)$ such that
$$\frac{1}{b_n}\int\limits_{\mathbb{B}_n} F(y_1 + r_1 y)\,d\mathcal{L}^{2n}(y) < \beta,$$
for any $y_1 \in \mathbb{B}_n(r_1)$, $r_1 \in (0,r_0)$. Fix $r_1 \in (0,r_0)$. Put $f(z,y,x) := \widetilde{f}(z, \zeta^{-1}(\zeta(x) + r_1 z y))$ and $V := \zeta^{-1}(\mathbb{B}_n(r_1))$. We have
$$\begin{aligned}\frac{1}{b_n}\int\limits_{\mathbb{B}_n} \mathfrak{F}_1(f(\,\cdot\,,y,x))\,d\mathcal{L}^{2n}(y) &= \frac{1}{b_n}\int\limits_{\mathbb{B}_n}\Big[\frac{1}{2\pi}\int\limits_0^{2\pi}\varphi(f(e^{i\theta},y,x))\,d\theta\Big]\,d\mathcal{L}^{2n}(y)\\ &= \frac{1}{b_n}\int\limits_{\mathbb{B}_n}\Big[\frac{1}{2\pi}\int\limits_0^{2\pi}\varphi(\widetilde{f}(e^{i\theta},\zeta^{-1}(\zeta(x)+r_1e^{i\theta}y)))\,d\theta\Big]\,d\mathcal{L}^{2n}(y)\\ &= \frac{1}{b_n}\int\limits_{\mathbb{B}_n}\Big[\frac{1}{2\pi}\int\limits_0^{2\pi}\varphi(\widetilde{f}(e^{i\theta},\zeta^{-1}(\zeta(x)+r_1y)))\,d\theta\Big]\,d\mathcal{L}^{2n}(y)\\ &= \frac{1}{b_n}\int\limits_{\mathbb{B}_n} F(\zeta(x)+r_1y)\,d\mathcal{L}^{2n}(y) < \beta.\end{aligned}$$
Take $r \in (1,\widetilde{r})$ such that $\zeta(x) + r_1 z y \in \mathbb{B}_n$ for any $x \in V$, $z \in \mathbb{D}_r$, $y \in \mathbb{B}_n(r)$. ∎

Proof of Theorem 3.5. Fix an $x_0 \in X$. We may assume that $E_{\mathfrak{F}_1}(x_0) < \infty$, where $\mathfrak{F}_1 = \mathfrak{F}_1^{\varphi}$. Let $\beta > E_{\mathfrak{F}_1}(x_0)$ be fixed. By definition there exists an $f_0 \in \mathcal{O}(\overline{\mathbb{D}}, X)$ such that $f_0(0) = x_0$ and $\mathfrak{F}_1(f_0) < \beta$. According to Lemma 3.8 there exist a neighborhood V of x_0 in X, $r > 1$, and $f \in \mathcal{O}(\mathbb{D}_r \times \mathbb{B}_n(r) \times V, X)$ such that $f(0,0,x) = x$ and
$$\frac{1}{b_n}\int\limits_{\mathbb{B}_n} \mathfrak{F}_1(f(\,\cdot\,,y,x))\,d\mathcal{L}^{2n}(y) < \beta \quad \text{for all } x \in V.$$

Fix an $x \in V$. By Lemma 3.6 there exists $y_0 \in \mathbb{B}_n$ such that

$$\frac{1}{b_n} \int_{\mathbb{B}_n} \mathfrak{F}_1(f(\,\cdot\,, y, x))\, d\mathcal{L}^{2n}(y) \geq \mathfrak{F}_1(g),$$

where $g(z) = f(z, zy_0, x)$. It suffices to note that $g(0) = x$. ∎

PROPOSITION 3.9. *Let X be a complex manifold. Assume that*

(a) $\varphi \in \mathcal{IC}^{\uparrow}(X)$ *is locally bounded from above or*

(b) φ *is a plurisuperharmonic function on X, $\varphi \not\equiv \infty$.*

Then $E_{\mathfrak{F}_1^{\varphi}} < \infty$.

Proof. Note that case (a) is trivial. Assume that φ is a plurisuperharmonic function.

Let $x_0 \in X$ be fixed. We have to show that $E_{\mathfrak{F}_1^{\varphi}}(x_0) < \infty$. Assume that (U, ζ) is a local coordinate such that $\zeta(x_0) = 0$. We may assume that $\zeta : U \to \zeta(U) = \mathbb{B}_n(2)$. Take an $x_1 \in U$, $x_1 \neq x_0$, such that $\varphi(x_1) < \infty$. Consider the superharmonic function $u := \varphi \circ f$, where $f(z) := \zeta^{-1}(z\zeta(x_1)/\|\zeta(x_1)\|)$, $z \in \overline{\mathbb{D}}$. Note that $f(0) = x_0$ and $u(\|\zeta(x_1)\|) = \varphi(x_1) < \infty$. Since $u \not\equiv \infty$,

$$\mathfrak{F}_1^{\varphi}(f) = \frac{1}{2\pi} \int_0^{2\pi} \varphi \circ f(e^{i\theta})\, d\theta = \frac{1}{2\pi} \int_0^{2\pi} u(e^{i\theta})\, d\theta < \infty.$$

Hence, $E_{\mathfrak{F}_1^{\varphi}} \leq \mathfrak{F}_1^{\varphi}(f) < \infty$. ∎

3.2. Plurisubharmonicity of the Poisson functional. In this section we study the Poisson functional on complex manifolds of class $\widetilde{\mathcal{P}}$. First we show plurisubharmonicity on domains in Stein manifolds. Later, using Propositions 2.1, 2.4, and 2.5 we extend the results obtained to the class $\widetilde{\mathcal{P}}$.

The main result of this part (and, actually, of the whole paper) is the following.

THEOREM 3.10. *Let X be a complex manifold of class $\widetilde{\mathcal{P}}$. Assume that*

(a) $\varphi \in \mathcal{IC}^{\uparrow}(X)$ *is locally bounded from above or*

(b) φ *is a plurisuperharmonic function on X, $\varphi \not\equiv \infty$.*

Then $E_{\mathfrak{F}_1^{\varphi}}$ is a plurisubharmonic function on X.

We assume first that X is a domain in a Stein manifold. The idea of the proof in this case goes back to E. Poletsky ([34], [35]). First note that $E_{\mathfrak{F}_1} < \infty$ (use Proposition 3.9) and $E_{\mathfrak{F}_1}$ is an upper semicontinuous function in both cases (use Theorem 3.5), where $\mathfrak{F}_1 = \mathfrak{F}_1^{\varphi}$. Therefore, we have to prove that

$$E_{\mathfrak{F}_1}(h(0)) \leq \frac{1}{2\pi} \int_0^{2\pi} E_{\mathfrak{F}_1}(h(e^{i\theta}))\, d\theta$$

for any $h \in \mathcal{O}(\overline{\mathbb{D}}, X)$. Fix an $h \in \mathcal{O}(\overline{\mathbb{D}}, X)$. Since $E_{\mathfrak{F}_1}$ is upper semicontinuous, there exists a sequence of continuous functions v_n on X such that $v_n \searrow E_{\mathfrak{F}_1}$ (cf. [30], Chapter III). Hence, it suffices to show that for every $\varepsilon > 0$ and $v \in C(X, \mathbb{R})$ with $v > E_{\mathfrak{F}_1}$ there exists

$g \in \mathcal{O}(\overline{\mathbb{D}}, X)$ such that $g(0) = h(0)$ and

$$\mathfrak{F}_1(g) \le \frac{1}{2\pi} \int_0^{2\pi} v(h(e^{i\theta}))\, d\theta + \varepsilon.$$

For the construction of g, we first show that there exist $r > 1$ and $F \in C^\infty(\mathbb{D}_r \times \mathbb{T}, X)$ such that $F(\cdot, w) \in \mathcal{O}(\overline{\mathbb{D}}, X)$, $F(0, w) = h(w)$ for all $w \in \mathbb{T}$, and

$$\frac{1}{2\pi} \int_0^{2\pi} \mathfrak{F}_1(F(\cdot, e^{i\theta}))\, d\theta \le \frac{1}{2\pi} \int_0^{2\pi} v(h(e^{i\theta}))\, d\theta + \varepsilon.$$

Next we show that there exist $s \in (1, r)$ and $G \in \mathcal{O}(\mathbb{D}_s \times \mathbb{D}_s, X)$ such that $G(0, w) = h(w)$ for all $w \in \mathbb{D}_s$ and

$$\frac{1}{2\pi} \int_0^{2\pi} \mathfrak{F}_1(G(\cdot, e^{i\theta}))\, d\theta \le \frac{1}{2\pi} \int_0^{2\pi} \mathfrak{F}_1(F(\cdot, e^{i\theta}))\, d\theta + \varepsilon.$$

Finally, we show that there exists a $\theta_0 \in [0, 2\pi)$ such that if g is defined by the formula $g(z) = G(e^{i\theta_0} z, z)$, then

$$\mathfrak{F}_1(g) \le \frac{1}{2\pi} \int_0^{2\pi} \mathfrak{F}_1(G(\cdot, e^{i\theta}))\, d\theta.$$

As we see, the main steps of the proof coincide with the proof of the plurisubharmonicity of $E_{\mathfrak{F}_1^\varphi}$ for an upper semicontinuous function φ (see the discussion before Lemma 2.3 in [25]). But the proofs of these steps turn out to be technical and complicated.

Let us start with the following result, which follows from measure theory.

LEMMA 3.11. *Let $\psi : \mathbb{T} \to \overline{\mathbb{R}}$ be a measurable function such that $\int_{\mathbb{T}} |\psi|\, d\sigma < \infty$ (i.e. $\psi \in L^1(\mathbb{T})$). Then for any $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\int_I \psi(w)\, d\sigma(w) < \varepsilon$$

for any measurable set $I \subset \mathbb{T}$ with $\sigma(I) < \delta$.

Proof. Fix $\varepsilon > 0$. We have $\int_{\mathbb{T}} |\psi|\, d\sigma < \infty$. Hence, there exists $C > 0$ such that

$$\int_{\{z \in \mathbb{T} : |\psi(z)| \ge C\}} |\psi|\, d\sigma < \frac{\varepsilon}{2}.$$

Take $\delta := \varepsilon/(2C)$. Then

$$\int_I \psi\, d\sigma \le \int_I |\psi|\, d\sigma = \int_{\{z \in I : |\psi(z)| < C\}} |\psi|\, d\sigma + \int_{\{z \in I : |\psi(z)| \ge C\}} |\psi|\, d\sigma \le C\sigma(I) + \frac{\varepsilon}{2} < \varepsilon. \blacksquare$$

LEMMA 3.12 (cf. Lemma 5.5 in [34], Lemma 2.5 in [25]). *Let $h \in \mathcal{O}(\overline{\mathbb{D}}, X)$, $\varepsilon > 0$, and $v \in C(X, \mathbb{R})$ with $v > E_{\mathfrak{F}_1}$. Assume that:*

(a) *φ is an integrally upper semicontinuous function on X locally bounded from above or*

(b) *φ is a plurisuperharmonic function on X such that $\varphi \circ h \not\equiv \infty$.*

Then there exist $r > 1$ and $F \in C^\infty(\mathbb{D}_r \times \mathbb{T}, X)$ such that $F(\cdot, w) \in \mathcal{O}(\mathbb{D}_r, X)$, $F(0, w) = h(w)$ for all $w \in \mathbb{T}$, and

$$\frac{1}{2\pi}\int_0^{2\pi} \mathfrak{F}_1(F(\cdot, e^{i\theta}))\,d\theta < \frac{1}{2\pi}\int_0^{2\pi} v(h(e^{i\theta}))\,d\theta + \varepsilon. \tag{3.3}$$

Proof. Let $w_0 \in \mathbb{T}$. Put $x_0 = h(w_0)$. From Lemma 3.8 it follows that there exist $r_0 > 1$, $f_0 \in \mathcal{O}(\mathbb{D}_{r_0} \times \mathbb{B}_n(r_0) \times V_0, X)$ such that $f_0(0, 0, x) = f_0(0, y, x) = x$, $x \in V_0$, $y \in \mathbb{B}_n(r_0)$, and

$$\frac{1}{b_n}\int_{\mathbb{B}_n} \mathfrak{F}_1(f_0(\,\cdot\,, y, x))\,d\mathcal{L}^{2n}(y) < v(x_0) \quad \text{ for all } x \in V_0.$$

Replacing V_0 by a smaller neighborhood of x_0 we get

$$\frac{1}{b_n}\int_{\mathbb{B}_n} \mathfrak{F}_1(f_0(\,\cdot\,, y, x))\,d\mathcal{L}^{2n}(y) < v(x) + \frac{\varepsilon}{4}, \qquad x \in V_0.$$

We can take an open arc $I_0 \subset \mathbb{T}$ containing w_0 such that $h(w) \in V_0$ for all $w \in I_0$. Define $F_0 : \mathbb{D}_{r_0} \times \mathbb{B}_n(r_0) \times I_0 \to X$ by $F_0(z, y, w) = f_0(z, y, h(w))$. Replacing r_0 by a smaller number in $(1, \infty)$ and I_0 by a smaller open arc containing w_0, we may assume that $F_0(\mathbb{D}_{r_0} \times \mathbb{B}_n(r_0) \times I_0)$ is relatively compact in X.

Using a compactness argument, we see that there exist a covering of $\mathbb{T}$ by open arcs $\{I_\nu\}_{\nu=1}^N$, $r_\nu > 1$, and $F_\nu \in C^\infty(\mathbb{D}_{r_\nu} \times \mathbb{B}_n(r_\nu) \times I_\nu, X)$ such that

a) $F_\nu(\cdot, \cdot, w) \in \mathcal{O}(\mathbb{D}_{r_\nu} \times \mathbb{B}_n(r_\nu), X)$,
b) $F_\nu(0, 0, w) = F_\nu(0, y, w) = h(w)$,
c) $F_\nu(\mathbb{D}_{r_\nu} \times \mathbb{B}_n(r_\nu) \times I_\nu)$ is relatively compact in X,
d)
$$\frac{1}{b_n}\int_{\mathbb{B}_n} \mathfrak{F}_1(F_\nu(\,\cdot\,, y, w))\,d\mathcal{L}^{2n}(y) < v(h(w)) + \frac{\varepsilon}{4}$$

for $y \in \mathbb{B}_n(r_\nu)$, $w \in I_\nu$, $\nu = 1, \dots, N$.

Put $r := \min_\nu r_\nu$. Choose $M \subset X$ to be a compact set such that

$$h(\overline{\mathbb{D}}) \cup \bigcup_{\nu=1}^N F_\nu(\mathbb{D}_r \times \mathbb{B}_n(r) \times I_\nu) \subset M.$$

Let $C > \max\{\sup_M |v|, \sup_M \varphi\}$ in case (a) and let $C > \sup_M |v|$ in case (b).

In the case of a plurisuperharmonic function φ such that $\varphi \circ h \not\equiv \infty$ it is well known that $\varphi \circ h \in L^1(\mathbb{T})$ (cf. [37]). So, by Lemma 3.11 there exists $\delta > 0$ such that for any measurable set $I \subset \mathbb{T}$ with $\sigma(I) < \delta$ we have

$$\int_I \varphi \circ h\,d\sigma < \frac{\pi\varepsilon}{2}.$$

In case (a) we put $\delta = 1$. There exist a subset $A \subset \{1, \dots, N\}$ and disjoint closed arcs $J_\nu \subset I_\nu$, $\nu \in A$, such that $\sigma(\mathbb{T} \setminus \bigcup J_\nu) < \min\{\delta, \pi\varepsilon/(2C)\}$. By possibly removing some arc I_ν from the covering of $\mathbb{T}$, we may assume that $A = \{1, \dots, N\}$. We choose disjoint open arcs K_ν such that $J_\nu \subset K_\nu \subset I_\nu$ and a function $\varrho \in C^\infty(\mathbb{T})$ such that

- $0 \le \varrho \le 1$,
- $\varrho(w) = 1$ for $w \in \bigcup J_\nu$,
- $\varrho(w) = 0$ for $w \in \mathbb{T} \setminus \bigcup K_\nu$.

Note that

$$\int_{J_\nu} \frac{1}{b_n} \int_{\mathbb{B}_n} \mathfrak{F}_1(F_\nu(\cdot, y, w))\, d\mathcal{L}^{2n}(y)\, d\sigma(w) \le \int_{J_\nu} v(h(w))\, d\sigma(w) + \frac{\varepsilon}{4}\sigma(J_\nu).$$

Hence, there exists $y_\nu \in \mathbb{B}_n$ such that

$$\int_{J_\nu} \mathfrak{F}_1(F_\nu(\cdot, y_\nu, w))\, d\sigma(w) \le \int_{J_\nu} v(h(w))\, d\sigma(w) + \frac{\varepsilon}{4}\sigma(J_\nu). \tag{3.4}$$

We define $F : \mathbb{D}_r \times \mathbb{T} \to X$ by

$$F(z, w) = \begin{cases} F_\nu(\varrho(w)z, y_\nu, w), & z \in \mathbb{D}_r,\ w \in K_\nu, \\ h(w), & z \in \mathbb{D}_r,\ w \in \mathbb{T} \setminus \bigcup K_\nu. \end{cases}$$

The choice of ϱ ensures that $F \in C^\infty(\mathbb{D}_r \times \mathbb{T}, X)$, $F(\cdot, w) \in \mathcal{O}(\mathbb{D}_r, X)$, and $F(0, w) = h(w)$, $w \in \mathbb{T}$. We have

$$\begin{aligned} \int_0^{2\pi} \mathfrak{F}_1(F(\cdot, e^{i\theta}))\, d\theta = \sum_\nu \int_{J_\nu} \mathfrak{F}_1(F_\nu(\cdot, y_\nu, w))\, d\sigma(w) \\ + \sum_\nu \int_{K_\nu \setminus J_\nu} \mathfrak{F}_1(F_\nu(\varrho(w)\cdot, y_\nu, w))\, d\sigma(w) + \int_{\mathbb{T} \setminus \bigcup_\nu K_\nu} \varphi(h(w))\, d\sigma(w). \end{aligned}$$

By (3.4) we get

$$\sum_\nu \int_{J_\nu} \mathfrak{F}_1(F_\nu(\cdot, y_\nu, w))\, d\sigma(w) \le \sum_\nu \int_{J_\nu} v(h(w))\, d\sigma(w) + \frac{\varepsilon}{4}\sigma\Big(\bigcup_\nu J_\nu\Big).$$

Let us estimate $\sum_\nu \int_{K_\nu \setminus J_\nu} \mathfrak{F}_1(F_\nu(\varrho(w)\cdot, y_\nu, w))\, d\sigma(w)$. Note that

$$\sum_\nu \int_{K_\nu \setminus J_\nu} \mathfrak{F}_1(F_\nu(\varrho(w)\cdot, y_\nu, w))\, d\sigma(w) = \sum_\nu \int_{K_\nu \setminus J_\nu} \frac{1}{2\pi} \int_0^{2\pi} \varphi(F_\nu(\varrho(w)\cdot, y_\nu, w))\, d\theta\, d\sigma(w).$$

In case (a) we have

$$\sum_\nu \int_{K_\nu \setminus J_\nu} \frac{1}{2\pi} \int_0^{2\pi} \varphi(F_\nu(\varrho(w)e^{i\theta}, y_\nu, w))\, d\theta\, d\sigma(w) \le C\sigma\Big(\bigcup_\nu K_\nu \setminus \bigcup_\nu J_\nu\Big).$$

And in case (b) we have

$$\begin{aligned} \sum_\nu \int_{K_\nu \setminus J_\nu} \frac{1}{2\pi} \int_0^{2\pi} \varphi(F_\nu(\varrho(w)e^{i\theta}, y_\nu, w))\, d\theta\, d\sigma(w) &\le \sum_\nu \int_{K_\nu \setminus J_\nu} \varphi(F_\nu(0, y_\nu, w))\, d\sigma(w) \\ &= \sum_\nu \int_{K_\nu \setminus J_\nu} \varphi(h(w))\, d\sigma(w). \end{aligned}$$

If we combine the inequalities we already have, then in case (a) we get

$$\begin{aligned}\int_0^{2\pi} \mathfrak{F}_1(F(\cdot, e^{i\theta}))\, d\theta &\le \int_{\bigcup_\nu J_\nu} v(h(w))\, d\sigma(w) + \frac{\varepsilon}{4}\sigma\Big(\bigcup_\nu J_\nu\Big) \\ &\quad + C\sigma\Big(\bigcup_\nu K_\nu \setminus \bigcup_\nu J_\nu\Big) + C\sigma\Big(\mathbb{T} \setminus \bigcup_\nu K_\nu\Big) \\ &\le \int_{\mathbb{T}} v(h(w))\, d\sigma(w) - \int_{\mathbb{T}\setminus\bigcup_\nu J_\nu} v(h(w))\, d\sigma(w) + \pi\varepsilon \\ &< \int_{\mathbb{T}} v(h(w))\, d\sigma(w) + 2\pi\varepsilon.\end{aligned}$$

In case (b) we have

$$\begin{aligned}\int_0^{2\pi} \mathfrak{F}_1(F(\cdot, e^{i\theta}))\, d\theta &\le \int_{\bigcup_\nu J_\nu} v(h(w))\, d\sigma(w) + \frac{\varepsilon}{4}\sigma\Big(\bigcup_\nu J_\nu\Big) \\ &\quad + \int_{\bigcup_\nu K_\nu\setminus\bigcup_\nu J_\nu} \varphi(h(w))\, d\sigma(w) + \int_{\mathbb{T}\setminus\bigcup_\nu K_\nu} \varphi(h(w))\, d\sigma(w) \\ &\le \int_{\mathbb{T}} v(h(w))\, d\sigma(w) - \int_{\mathbb{T}\setminus\bigcup_\nu J_\nu} v(h(w))\, d\sigma(w) + \pi\varepsilon \\ &< \int_{\mathbb{T}} v(h(w))\, d\sigma(w) + 2\pi\varepsilon. \ \blacksquare\end{aligned}$$

Recall the following approximation result (see Lemma 2.6 in [25], cf. Lemma 5.6 in [34], Lemma 6 in [7]).

LEMMA 3.13. *Let X be a domain in a Stein manifold. Let $r > 1$, $h \in \mathcal{O}(\mathbb{D}_r, X)$, and $F \in C^\infty(\mathbb{D}_r \times \mathbb{T}, X)$ be such that $F(\cdot, w) \in \mathcal{O}(\mathbb{D}_r, X)$, and $F(0, w) = h(w)$ for all $w \in \mathbb{T}$. Then for any $s \in (1, r)$ there exists a sequence $F_j \in \mathcal{O}(\mathbb{D}_s \times A_j, X)$, $j \ge 1$, where A_j is an open annulus containing $\mathbb{T}$, such that*

(i) *$F_j \to F$ uniformly on $\mathbb{D}_s \times \mathbb{T}$ as $j \to \infty$,*

(ii) *there is an integer $k_j \ge j$ such that the map $(z, w) \mapsto F_j(zw^{k_j}, w)$ can be extended to a map $G_j \in \mathcal{O}(\mathbb{D}^2_{s_j}, X)$, where $s_j \in (1, s)$ and*

(iii) *$G_j(0, w) = h(w)$ for all $w \in \mathbb{D}_{s_j}$.*

Proof. By Theorem 2.7 there exist a domain Y in $\mathbb{C}^n$ and holomorphic mappings $\Phi : X \to Y$, $\Psi : Y \to X$ such that $\Psi \circ \Phi = \mathrm{id}_X$. We define $\widetilde{F} = \Phi \circ F$, $\widetilde{h} = \Phi \circ h$. For any $j \in \mathbb{N}$ we put

$$\widetilde{F}_j(z, w) = \widetilde{h}(w) + \sum_{k=-j}^{j} \bigg(\frac{1}{2\pi} \int_0^{2\pi} (\widetilde{F}(z, e^{i\theta}) - \widetilde{h}(e^{i\theta}))e^{-ik\theta}\, d\theta\bigg) w^k. \tag{3.5}$$

Since the function $\theta \mapsto \widetilde{F}(z, e^{i\theta}) - \widetilde{h}(e^{i\theta})$ is infinitely differentiable with period 2π, its Fourier series converges uniformly on $\mathbb{R}$. Hence the series in (3.5) converges uniformly on $\mathbb{D}_t \times \mathbb{T}$, $t \in (1, r)$.

Let $t \in (s,r)$. Since $\widetilde{F}(z,w) \in Y$ for all $(z,w) \in \mathbb{D}_r \times \mathbb{T}$ and $\widetilde{F}_j \to \widetilde{F}$ uniformly on $\mathbb{D}_t \times \mathbb{T}$, we can choose j_0 so large that $\widetilde{F}_j(z,w) \in Y$ for all $(z,w) \in \mathbb{D}_t \times \mathbb{T}$ and $j \geq j_0$. Since $s \in (1,t)$, by continuity we can choose an open annulus A_j containing $\mathbb{T}$ such that $\widetilde{F}_j(z,w) \in Y$ for all $(z,w) \in \mathbb{D}_s \times A_j$. We define $F_j \in \mathcal{O}(\mathbb{D}_s \times A_j, X)$ by $F_j = \Psi \circ \widetilde{F}_j$. Then (i) holds.

For every $z \in \mathbb{D}_r$ the mapping $w \mapsto \widetilde{F}_j(z,w) - \widetilde{h}(w)$ has a pole of order at most j at the origin, and for every $w \in \mathbb{D}_r \setminus \{0\}$ the mapping $z \mapsto \widetilde{F}_j(z,w) - \widetilde{h}(w)$ has a zero at the origin. Hence $(z,w) \mapsto \widetilde{F}_j(zw^k,w)$ can be extended to a holomorphic mapping $\mathbb{D} \times \mathbb{D} \to \mathbb{C}^n$ for every $k \geq j$.

Since $\widetilde{F}_j(0,w) = \widetilde{h}(w) \in Y$ for all $w \in \mathbb{D}_r \setminus \{0\}$, there exists $\delta > 0$ such that $\widetilde{F}_j(zw^k,w) \in Y$ for all integers $k \geq j$ and $(z,w) \in \mathbb{D}_\delta \times \overline{\mathbb{D}}$. Since $\widetilde{F}_j(z,w) \in Y$ for all $(z,w) \in \bar{D} \times \mathbb{T}$, we can choose $\varrho_j \in (0,1)$ such that $\widetilde{F}_j(z,w) \in Y$ for all $(z,w) \in \overline{\mathbb{D}} \times (\overline{\mathbb{D}} \setminus \mathbb{D}_{\varrho_j})$, so we conclude that $\widetilde{F}_j(zw^k,w) \in Y$ for all $(z,w) \in \overline{\mathbb{D}} \times (\overline{\mathbb{D}} \setminus \mathbb{D}_{\varrho_j})$ and all integers $k \geq j$. Now we take k_j so large that $|zw^{k_j}| < \delta$ for all $(z,w) \in \overline{\mathbb{D}} \times \mathbb{D}_{\varrho_j}$. Then $\widetilde{F}_j(zw^{k_j},w) \in Y$ for all $(z,w) \in \overline{\mathbb{D}} \times \overline{\mathbb{D}}$. We finally choose $s_j \in (1,s)$ such that $\widetilde{F}_j(zw^{k_j},w) \in Y$ for all $(z,w) \in \mathbb{D}_{s_j} \times \mathbb{D}_{s_j}$ and define $G_j(z,w) = \Psi \circ \widetilde{F}_j(zw^{k_j},w)$. Then (ii) and (iii) hold. ∎

Lemma 3.14. *Let X be a domain in a Stein manifold. Let h and F satisfy the conditions of Lemma* 3.12. *Then for every $\varepsilon > 0$ there exist $s \in (1,r)$ and $G \in \mathcal{O}(\mathbb{D}_s \times \mathbb{D}_s, X)$ such that $G(0,w) = h(w)$ for all $w \in \mathbb{D}_s$, and*

$$\frac{1}{2\pi}\int_0^{2\pi} \mathfrak{F}_1(G(\cdot,e^{i\theta}))\,d\theta \leq \frac{1}{2\pi}\int_0^{2\pi} \mathfrak{F}_1(F(\cdot,e^{i\theta}))\,d\theta + \varepsilon.$$

Proof. By Theorem 2.7 there exist a domain Y in $\mathbb{C}^n$ and holomorphic mappings $\Phi : X \to Y$, $\Psi : Y \to X$ such that $\Psi \circ \Phi = \mathrm{id}_X$. We define $\widetilde{F} = \Phi \circ F$, $\widetilde{F}_k = \Phi \circ F_k$, and $\widehat{\varphi} := \varphi \circ \Psi$. Note that $\widehat{\varphi}$ is a weakly approximately upper semicontinuous function on Y.

For any fixed $z,w \in \mathbb{T}$ there exists $r(z,w) > 0$ such that

$$\frac{1}{b_n}\int_{\mathbb{B}_n} \widehat{\varphi}(y_1 + ry)\,d\mathcal{L}^{2n}(y) \leq \widehat{\varphi}(\widetilde{F}(z,w)) + \frac{\varepsilon}{2} = \varphi(F(z,w)) + \frac{\varepsilon}{2}$$

for $y_1 \in \mathbb{B}(F(z,w),r)$, $r \in (0, r(z,w))$. Hence, for any fixed $z,w \in \mathbb{T}$ we have

$$\limsup_{m\to\infty}\limsup_{k\to\infty}\frac{1}{b_n}\int_{\mathbb{B}_n} \widehat{\varphi}\Big(\widetilde{F}_k(z,w) + \frac{1}{m}y\Big)\,d\mathcal{L}^{2n}(y) \leq \varphi(F(z,w)) + \frac{\varepsilon}{2}.$$

By Fatou's theorem, we have

$$\begin{aligned}
&\limsup_{m\to\infty}\limsup_{k\to\infty}\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\Big[\frac{1}{b_n}\int_{\mathbb{B}_n} \widehat{\varphi}\Big(\widetilde{F}_k(e^{i\tau},e^{i\theta}) + \frac{1}{m}y\Big)\,d\mathcal{L}^{2n}(y)\Big]\,d\tau\,d\theta \\
&\quad\leq \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\Big[\limsup_{m\to\infty}\limsup_{k\to\infty}\frac{1}{b_n}\int_{\mathbb{B}_n} \widehat{\varphi}\Big(\widetilde{F}_k(e^{i\tau},e^{i\theta}) + \frac{1}{m}y\Big)\,d\mathcal{L}^{2n}(y)\Big]\,d\tau\,d\theta \\
&\qquad\leq \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} \varphi(F(e^{i\tau},e^{i\theta}))\,d\tau\,d\theta + \frac{\varepsilon}{2}.
\end{aligned}$$

Hence, there exist m_0 and k_0 such that

$$\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\left[\frac{1}{b_n}\int_{\mathbb{B}_n}\widehat{\varphi}\Big(\widetilde{F}_{k_0}(e^{i\tau},e^{i\theta})+\frac{1}{m_0}y\Big)\,d\mathcal{L}^{2n}(y)\right]d\tau\,d\theta \le \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\varphi(F(e^{i\tau},e^{i\theta}))\,d\tau\,d\theta+\varepsilon.$$

So, there exists $y_0\in\mathbb{B}_n$ such that

$$\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\widehat{\varphi}\Big(\widetilde{F}_{k_0}(e^{i\tau},e^{i\theta})+\frac{1}{m_0}e^{i\tau}y_0\Big)\,d\tau\,d\theta\le\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\varphi(F(e^{i\tau},e^{i\theta}))\,d\tau\,d\theta+\varepsilon$$

Put $\widetilde{G}(z,w)=\Phi\circ G_{k_0}(z,w)+(1/m_0)zw^{\ell_{k_0}}y_0$ and $G=\Psi\circ\widetilde{G}$, where G_{k_0} is given by Lemma 3.13. Finally we note that

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}\mathfrak{F}_1(G(\cdot,e^{i\theta}))\,d\theta&=\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\varphi(G(e^{i\tau},e^{i\theta}))\,d\tau\,d\theta\\&=\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\widehat{\varphi}(\widetilde{G}(e^{i\tau},e^{i\theta}))\,d\tau\,d\theta\\&\le\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\varphi(F(e^{i\tau},e^{i\theta}))\,d\tau\,d\theta+\varepsilon=\frac{1}{2\pi}\int_0^{2\pi}\mathfrak{F}_1(F(\cdot,e^{i\theta}))\,d\theta+\varepsilon.\ \blacksquare\end{aligned}$$

LEMMA 3.15. *Let $s>1$ and $G\in\mathcal{O}(\mathbb{D}_s\times\mathbb{D}_s,X)$. Then there exists $g\in\mathcal{O}(\mathbb{D}_s,X)$ such that $g(0)=G(0,0)$ and*

$$\mathfrak{F}_1(g)\le\frac{1}{2\pi}\int_0^{2\pi}\mathfrak{F}_1(G(\cdot,e^{i\theta}))\,d\theta.$$

Proof. Note that

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}\mathfrak{F}_1(G(\cdot,e^{i\theta}))\,d\theta&=\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\varphi(G(e^{i\tau},e^{i\theta}))\,d\tau\,d\theta\\&=\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\varphi(G(e^{i\tau},e^{i\theta+i\tau}))\,d\tau\,d\theta.\end{aligned}$$

So, there exists $\theta_0\in[0,2\pi)$ such that

$$\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\varphi(G(e^{i\tau},e^{i\theta+i\tau}))\,d\tau\,d\theta\ge\frac{1}{2\pi}\int_0^{2\pi}\varphi(G(e^{i\tau},e^{i\theta_0}e^{i\tau}))\,d\tau.$$

Put $g(z)=G(z,e^{i\theta_0}z)$. $\blacksquare$

Now we are going to prove Theorem 3.10 for any manifold of class $\widetilde{\mathcal{P}}$. First recall the following result.

LEMMA 3.16 (see Proposition 2.9.26 in [20], cf. Lemma 3.2 in [25]). *Let X,Y be complex manifolds and let $h:X\to Y$ be a finite branched covering. Let u be a plurisubharmonic* *function on X. Then the function h_*u defined by the formula*

$$h_*u(y) = \max\{u(x) : x \in h^{-1}(y)\}, \qquad y \in Y,$$

is plurisubharmonic on Y (26).

For the sake of completeness we give a proof.

Proof. Let $S \subset Y$ be the branch locus of h. Then $h : X \setminus h^{-1}(S) \to Y \setminus S$ is a finite holomorphic covering, so h_*u is plurisubharmonic on $Y \setminus S$. The restriction $h_*u|_{Y\setminus S}$ extends to a plurisubharmonic function v on Y with

$$v(y) = \limsup_{\widetilde{y}\to y,\, \widetilde{y}\notin S} h_*u(\widetilde{y}), \qquad y \in S.$$

If u is continuous, then h_*u is also continuous and, hence, plurisubharmonic. In the general case, let $p \in S$ and U be an open coordinate ball containing p. We have a finite map $h : h^{-1}(U) \to U$, so $h^{-1}(U)$ is Stein, and the main approximation theorem for plurisubharmonic functions holds on $h^{-1}(U)$. Let V be a relatively compact open ball in U with $p \in V$. Then there are smooth plurisubharmonic functions u_n on $h^{-1}(V)$ such that $u_n \searrow u$. Since h_*u_n are plurisubharmonic and $h_*u_n \searrow h_*u$, we conclude that h_*u is plurisubharmonic on V. ∎

Recall also the following modifications of Propositions 3.1, 3.3 from [25].

PROPOSITION 3.17. *Let X,Y be complex manifolds such that there exists a finite branched covering $h : X \to Y$. If $E_{\mathfrak{F}_1^\varphi} \in \mathrm{PSH}(X)$ for every integrally upper semicontinuous function φ on X locally bounded from above*, *then $E_{\mathfrak{F}_1^\varphi} \in \mathrm{PSH}(Y)$ for every function φ on Y with the same properties.*

Proof. Let $\varphi : Y \to \mathbb{R} \cup \{-\infty\}$ be an integrally upper semicontinuous function locally bounded from above. Let $\psi = E_{\mathfrak{F}_1^{\varphi\circ h}}$. By assumption ψ is plurisubharmonic on X and $E_{\mathfrak{F}_1^\varphi} \circ h \le \psi$. Now $\psi \le \varphi \circ h$, so $h_*\psi \le \varphi$ and $h_*\psi \le E_{\mathfrak{F}_1^\varphi}$. Hence,

$$(h_*\psi) \circ h \le E_{\mathfrak{F}_1^\varphi} \circ h \le \psi.$$

Note that $(h_*\psi) \circ h \le \psi$ implies that $(h_*\psi) \circ h = \psi$. Hence, $E_{\mathfrak{F}_1^\varphi} \circ h = \psi$ and $E_{\mathfrak{F}_1^\varphi} = h_*\psi$ is plurisubharmonic. ∎

PROPOSITION 3.18. *Let X,Y be complex manifolds such that there exists a finite branched covering $h : X \to Y$. If $E_{\mathfrak{F}_1^\varphi} \in \mathrm{PSH}(X)$ for every plurisuperharmonic function φ on X, $\varphi \not\equiv \infty$, then $E_{\mathfrak{F}_1^\varphi} \in \mathrm{PSH}(Y)$ for every such function φ on Y.*

Proof. The same as the proof of Proposition 3.17. ∎

Propositions 2.1, 2.4, 3.17, 3.18 imply Theorem 3.10 for any complex manifold of class $\widetilde{\mathcal{P}}$.

3.3. Liouville manifolds. We say that a complex manifold X is a *Liouville manifold* if any negative plurisubharmonic function on X is constant (27).

(26) Recall the following useful result. Suppose that $x_0 \in X$ is an isolated point of the set $h^{-1}(h(x_0))$. It is well known (see e.g. [41]) that there exist domains $U \subset X$ and $V \subset Y$ such that $h^{-1}(h(x_0)) \cap U = \{x_0\}$ and $h|_U : U \to V$ is a finite branched covering.

(27) Recall that by Liouville's theorem, $\mathbb{C}^n$ is a Liouville manifold (see e.g. [20]).

In this section we study the envelope of the Poisson functional on Liouville manifolds. Let us start with the following result, which is crucial for our considerations.

PROPOSITION 3.19 (cf. [25], Proposition 6.1). *Let X be a Liouville manifold. Let $U \neq \emptyset$ be an open set in X such that $\widetilde{\omega}(\cdot, U, X)$ is a plurisubharmonic function on X. Then for every $x \in X$ and every $\varepsilon > 0$ there exists an $f \in \mathcal{O}(\overline{\mathbb{D}}, X)$ such that $f(0) = x$ and $\sigma(\mathbb{T} \cap f^{-1}(U)) > 2\pi - \varepsilon$.*

Proof. We know that $\widetilde{\omega}(\cdot, U, X) = E_{\mathfrak{F}_1^{-\chi_U}}$. Since $\widetilde{\omega}(\cdot, U, X) \leq 0$, from the assumptions we get $\widetilde{\omega}(\cdot, U, X) \equiv -1$. Hence, for every $x \in X$ and every $\varepsilon > 0$ there exists an $f \in \mathcal{O}(\overline{\mathbb{D}}, X)$ such that $f(0) = x$ and

$$\frac{\varepsilon}{2\pi} - 1 > \frac{1}{2\pi} \int_{\mathbb{T}} (-\chi_U \circ f)\, d\sigma = -\frac{\sigma(\mathbb{T} \cap f^{-1}(U))}{2\pi}.$$

Hence, $\sigma(\mathbb{T} \cap f^{-1}(U)) > 2\pi - \varepsilon$. ∎

PROPOSITION 3.20 (cf. [25], Proposition 6.1). *Let X be a complex manifold. Assume that for any $x \in X$, any open set $U \neq \emptyset$ in X, and any $\varepsilon > 0$ there exists an $f \in \mathcal{O}(\overline{\mathbb{D}}, X)$ such that $f(0) = x$ and $\sigma(\mathbb{T} \cap f^{-1}(U)) > 2\pi - \varepsilon$. Then for every upper semicontinuous function $\varphi : X \to [-\infty, \infty)$ bounded from above, $E_{\mathfrak{F}_1^{\varphi}}$ is constant (and, therefore, plurisubharmonic) on X. In particular, X is a Liouville manifold and $\widetilde{\omega}(\cdot, U, X)$ is a plurisubharmonic function on X for any open subset U of X.*

Proof. Note that $E_{\mathfrak{F}_1^{\varphi}} \geq \inf_X \varphi$ on X.

Suppose that $\varphi < C$ on X. Fix $c \in \mathbb{R}$ such that $c > \inf_X \varphi$. Put $U_c := \{x \in X : \varphi(x) < c\}$. Note that $U_c \neq \emptyset$ is an open subset in X. Fix $\varepsilon > 0$. Let $f \in \mathcal{O}(\overline{\mathbb{D}}, X)$ be such that $f(0) = x$ and $\sigma(\mathbb{T} \cap f^{-1}(U)) > 2\pi - \varepsilon$. Then we have

$$\int_{\mathbb{T}} \varphi \circ f\, d\sigma = \int_{\mathbb{T} \cap f^{-1}(U_c)} \varphi \circ f\, d\sigma + \int_{\mathbb{T} \setminus f^{-1}(U_c)} \varphi \circ f\, d\sigma \leq c\sigma(\mathbb{T} \cap f^{-1}(U_c)) + C\varepsilon \leq 2\pi c + C\varepsilon.$$

So,

$$E_{\mathfrak{F}_1^{\varphi}}(x) \leq \mathfrak{F}_1^{\varphi}(f) \leq c + C\frac{\varepsilon}{2\pi}.$$

Since $\varepsilon > 0$ is arbitrary, we get $E_{\mathfrak{F}_1^{\varphi}}(x) \leq c$. Take $c \searrow \inf_X \varphi$. Hence, $E_{\mathfrak{F}_1^{\varphi}} \equiv \inf_X \varphi$. ∎

Let us give some corollaries of Propositions 3.19 and 3.20. First, we have the following characterization of Liouville manifolds in terms of analytic discs.

THEOREM 3.21. *Let X be a complex manifold from class $\widetilde{\mathcal{P}}$ ([28]). Then X is a Liouville manifold if and only if for any $x \in X$, any $\varepsilon > 0$, and any open set $U \neq \emptyset$ there exists an $f \in \mathcal{O}(\overline{\mathbb{D}}, X)$ such that $f(0) = x$ and $\sigma(\mathbb{T} \cap f^{-1}(U)) > 2\pi - \varepsilon$.*

COROLLARY 3.22. *Let X be a Liouville manifold. Then the following conditions are equivalent:*

(1) *$\widetilde{\omega}(\cdot, U, X)$ is a plurisubharmonic function on X for every open subset U of X;*

(2) *For any $x \in X$, any open subset $U \neq \emptyset$, and any $\varepsilon > 0$ there exists an $f \in \mathcal{O}(\overline{\mathbb{D}}, X)$ such that $f(0) = x$ and $\sigma(\mathbb{T} \cap f^{-1}(U)) > 2\pi - \varepsilon$.*

([28]) Actually, it suffices to assume that $\widetilde{\omega}(\cdot, U, X) \in \mathrm{PSH}(X)$ for any open subset U of X.

Moreover, if X is a compact manifold, then the above conditions are equivalent to

(3) *For any upper semicontinuous function $\varphi : X \to [-\infty, \infty)$ the function $E_{\mathfrak{F}_1^\varphi}$ is plurisubharmonic on X.*

Proof. It suffices to note that on a compact complex manifold any upper semicontinuous function is bounded from above. ∎

We have the following result.

PROPOSITION 3.23 (see Proposition 6.4 in [25]). *Let X, Y be complex manifolds and let $F : X \to Y$ be a surjective holomorphic mapping. Assume that X (and, therefore, Y) is a Liouville manifold and that $\widetilde{\omega}(\cdot, U, X)$ is a plurisubharmonic function on X for every open subset U of X. Then $\widetilde{\omega}(\cdot, V, Y)$ is a plurisubharmonic function on Y for every open subset V of Y. Moreover, if Y is a compact complex manifold, then $E_{\mathfrak{F}_1^\varphi}$ is a plurisubharmonic function on Y for any upper semicontinuous function $\varphi : Y \to [-\infty, \infty)$.*

Proof. Let $V \neq \emptyset$ be an open subset of Y. Then

$$\widetilde{\omega}(F(x), V, Y) \leq \widetilde{\omega}(x, F^{-1}(V), X), \qquad x \in X.$$

By the assumptions $\widetilde{\omega}(\cdot, F^{-1}(V), X) \equiv -1$. Therefore, $\widetilde{\omega}(\cdot, V, Y) \equiv -1$. ∎

PROPOSITION 3.24 (cf. Remark 6.2 in [25]). *Let X be a Liouville manifold. Assume that X is taut* (29). *Then for any $r > 1$ there exists an open subset $U \neq \emptyset$ of X such that*

$$\widetilde{\omega}_r(x, U, X) := \inf \left\{ \frac{1}{2\pi} \int_{\mathbb{T}} (-\chi_U \circ f)\, d\sigma : f \in \mathcal{O}(\mathbb{D}_r, X),\ f(0) = x \right\}$$

is not a plurisubharmonic function on X.

Proof. Fix $r > 1$. Assume that $\widetilde{\omega}_r(\cdot, U, X)$ is a plurisubharmonic function on X for any open subset $U \neq \emptyset$ of X.

Let $x_0 \in X$ and let (U_n) be a decreasing neighborhood basis of a point $y_0 \neq x_0$ in X. Then $\widetilde{\omega}_r(\cdot, U_n, X) \equiv -1$. We get holomorphic mappings $f_n : \mathbb{D}_r \to X$ such that $f_n(0) = x_0$ and $\sigma(\mathbb{T} \cap f_n^{-1}(U_n)) > 2\pi - 1/n$. Since X is taut, there exists a subsequence of (f_n) which converges uniformly on compact sets to a holomorphic mapping $f : \mathbb{D}_r \to X$. Then $f(0) = x_0$ and $f(\mathbb{T}) = \{y_0\}$. Hence, we obtain a contradiction. ∎

REMARK 3.25. Note that $\mathbb{C} \setminus S$ is a Liouville taut domain for any closed polar set $S \subset \mathbb{C}$ with $\#S > 1$ (see e.g. [17]).

3.4. The Poisson functional on domains in $\mathbb{C}^n$. In the definition of the envelope of a disc functional we use *all* analytic discs. In this section we show that for domains in $\mathbb{C}^n$ it suffices to take analytic discs which are restrictions of the polynomial mappings.

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(29) Recall that X is called *taut* if the space $\mathcal{O}(\mathbb{D}, X)$ is normal, i.e. for any sequence $\{f_j\}_{j \in \mathbb{N}} \subset \mathcal{O}(\mathbb{D}, X)$ there exists a subsequence $\{f_{j_\nu}\}$ with $f_{j_\nu} \underset{\nu \to \infty}{\overset{K}{\Longrightarrow}} f \in \mathcal{O}(\mathbb{D}, X)$ or there exists a subsequence $\{f_{j_\nu}\}$ which diverges uniformly on compact sets, i.e. for any two compact sets $K \subset \mathbb{D}$, $L \subset X$ there is an index $\nu_0 \in \mathbb{N}$ such that $f_{j_\nu}(K) \cap L = \emptyset$ if $\nu \geq \nu_0$ (cf. [45]).

THEOREM 3.26. *Let Ω be a domain in $\mathbb{C}^n$ and let $\varphi : \Omega \to [-\infty, \infty)$ be an upper semicontinuous function. Then*

$$E_{\mathfrak{F}_1^\varphi}(x) = \inf\{\mathfrak{F}_1^\varphi(f) : f \in \mathcal{O}(\overline{\mathbb{D}}, \Omega),\ f : \mathbb{C} \to \mathbb{C}^n \text{ is a polynomial},\ f(0) = x\}.$$

Proof. Note that

$$E_{\mathfrak{F}_1^\varphi}(x) \le \inf\{\mathfrak{F}_1^\varphi(f) : f \in \mathcal{O}(\overline{\mathbb{D}}, \Omega),\ f : \mathbb{C} \to \mathbb{C}^n \text{ is a polynomial},\ f(0) = x\}.$$

Fix $x_0 \in X$. We may assume that $E_{\mathfrak{F}_1^\varphi}(x_0) < \infty$. Take $\beta \in \mathbb{R}$ such that $E_{\mathfrak{F}_1^\varphi}(x_0) < \beta$. By the definition, there exists an $f \in \mathcal{O}(\overline{\mathbb{D}}, X)$ such that $f(0) = x_0$ and $\mathfrak{F}_1^\varphi(f) < \beta$. Assume that

$$f(z) = \sum_{k=0}^{\infty} c_k z^k, \qquad z \in \mathbb{D},$$

is the Taylor expansion of f, where $c_k \in \mathbb{C}^n$. Put $f_N := \sum_{k=0}^{N} c_k z^k$. There exists $n_0 \in \mathbb{N}$ such that $f_N(\overline{\mathbb{D}}) \subset \Omega$ for any $N \ge n_0$. We have

$$\limsup_{N\to\infty} \mathfrak{F}_1^\varphi(f_N) \le \frac{1}{2\pi} \int_0^{2\pi} \limsup_{N\to\infty} \varphi(f_N(e^{i\theta}))\, d\theta \le \mathfrak{F}_1^\varphi(f) < \beta.$$

Hence, there exists $N \in \mathbb{N}$ such that $\mathfrak{F}_1^\varphi(f_N) < \beta$. ∎

4. Product property

4.1. Product property of the Poisson functional. The main result of this part is the following product property.

THEOREM 4.1 (cf. [12]). *Let X_1 and X_2 be complex manifolds and let $U_1 \subset X_1$, $U_2 \subset X_2$ be open sets. Then*

$$\widetilde{\omega}((x_1, x_2), U_1 \times U_2, X_1 \times X_2) = \max\{\widetilde{\omega}(x_1, U_1, X_1), \widetilde{\omega}(x_2, U_2, X_2)\}, \qquad (x_1, x_2) \in X_1 \times X_2.$$

For the proof of Theorem 4.1 we need some technical results. First recall the following two of them.

THEOREM 4.2 (see [33], Chapter III). *Let $\pi : \mathbb{D} \to \mathbb{D}$ be an inner function* (30)*. Assume that π is non-constant and not a Blaschke product* (31)*. Then there exists a $\theta \in [0, 2\pi)$ such that $\pi^*(e^{i\theta}) = 0$.*

THEOREM 4.3 (see [33], Chapter II). *Let π be a bounded holomorphic function on the unit disc and let A be a compact polar set in $\mathbb{C}$. Assume that there exists a set $I \subset \mathbb{T}$ of positive measure such that $\pi^*(z) \in A$, $z \in I$. Then π is constant.*

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(30) Recall that a function $\pi : \mathbb{D} \to \mathbb{D}$ is called *inner* if $\pi^*(e^{i\theta}) := \lim_{r\to 1} \pi(re^{i\theta}) \in \mathbb{T}$ for almost all $\theta \in [0, 2\pi)$.

(31) Let $\{a_n\} \subset \mathbb{D}_*$ and let $\sum_n (1 - |a_n|) < \infty$. A function of the form

$$B(z) = e^{i\theta} z^m \prod_n \frac{|a_n|}{a_n} \frac{a_n - z}{1 - \bar{a}_n z}, \qquad \theta \in \mathbb{R},$$

is called a *Blaschke product*. Note that for a Blaschke product we have $B(0) = 0$ for $m \ge 1$ and $|B(0)| = \prod_n |a_n|$ for $m = 0$. It is well known that any Blaschke product is an inner function (see e.g. [38], Theorem 15.24).

Lemma 4.4. *Let A be a compact polar subset of the unit disc $\mathbb{D}$ and let $\pi : \mathbb{D} \to \mathbb{D} \setminus A$ be a universal covering. Then π is an inner function. Moreover, if $0 \notin A$, then π is a Blaschke product.*

Proof. Note that if $\pi^*(e^{i\theta})$ exists for a $\theta \in [0, 2\pi)$, then $\pi^*(e^{i\theta}) \in \mathbb{T} \cup A$. From Theorem 4.3 we deduce that $\pi^*(e^{i\theta}) \in \mathbb{T}$ for almost all $\theta \in [0, 2\pi)$.

If $0 \notin A$, then from Theorem 4.2 we see that π is a Blaschke product. ∎

Lemma 4.5. *Let B be a finite Blaschke product and let $\pi : \mathbb{D} \to \mathbb{D}$ be a holomorphic function. Then π is an inner function if and only if $B \circ \pi$ is inner.*

Proof. It suffices to note that B extends holomorphically to $\overline{\mathbb{D}}$ and $B(\mathbb{T}) \subset \mathbb{T}$. ∎

We need a version of Löwner's theorem (cf. [44], Theorem VIII.30). But first recall the following.

Theorem 4.6 (see [37], Theorem 1.2.4, [44], Theorem IV.1). *Let $\varphi \in L^1(\mathbb{T})$. Put*

$$u_\varphi(z) := \frac{1}{2\pi} \int_0^{2\pi} P(z, \theta)\varphi(e^{i\theta})\, d\theta,$$

where $P(z, \theta) = (1 - |z|^2)/|e^{i\theta} - z|^2$ denotes the Poisson kernel. Then

(a) *u_φ is a harmonic function on $\mathbb{D}$, $\inf_{\mathbb{T}} \varphi \le u_\varphi \le \sup_{\mathbb{T}} \varphi$;*

(b) *$u_\varphi^*(e^{i\theta}) = \lim_{r\to 1} u_\varphi(re^{i\theta})$ exists for almost all $\theta \in [0, 2\pi)$; moreover, $u_\varphi^* = \varphi$ a.e. on $\mathbb{T}$;*

(c) *if φ is continuous at $\zeta_0 \in \mathbb{T}$, then u_φ extends continuously to ζ_0.*

Lemma 4.7 (Löwner's theorem). *Let $\pi : \mathbb{D} \to \mathbb{D}$ be an inner holomorphic function such that $\pi(0) = 0$. Then for any open set $I \subset \mathbb{T}$ we have $\sigma((\pi^*)^{-1}(I)) = \sigma(I)$.*

Proof. Note that any open $I \subset \mathbb{T}$ can be written as $I = \bigcup_{j=1}^\infty I_j$, where I_j are disjoint open arcs. So, we may assume that I is an open arc.

Fix an open arc $I \subset \mathbb{T}$. Note that $J := (\pi^*)^{-1}(I)$ is a measurable set. Put

$$u_I(z) := \frac{1}{2\pi} \int_0^{2\pi} P(z, \theta)\chi_I(e^{i\theta})\, d\theta,$$

$$u_J(z) := \frac{1}{2\pi} \int_0^{2\pi} P(z, \theta)\chi_J(e^{i\theta})\, d\theta,$$

$$u(z) := u_I(\pi(z)) - u_J(z), \qquad z \in \mathbb{D}.$$

Note that u_I is a continuous function on $\mathbb{T} \setminus \partial I$. Put

$$\begin{aligned} A := {} & \{\zeta \in \mathbb{T} : \lim_{r\to 1} u_J(r\zeta) \text{ does not exist}\} \\ & \cup \{\zeta \in \mathbb{T} : u_J^*(\zeta) = \lim_{r\to 1} u_J(r\zeta) \text{ exists and } u_J^*(\zeta) \ne \chi_J\} \\ & \cup \{\zeta \in \mathbb{T} : \lim_{r\to 1} \pi(r\zeta) \text{ does not exist}\} \\ & \cup \{\zeta \in \mathbb{T} : \pi^*(\zeta) = \lim_{r\to 1} \pi(r\zeta) \text{ exists and } \pi^*(\zeta) \in \partial I\}. \end{aligned}$$

Note that $A \subset \mathbb{T}$ is of measure zero (use Theorem 4.3). Moreover, if $e^{i\theta} \in J \setminus A$ (i.e. $u_J^*(e^{i\theta}) = 1$ and $\pi^*(e^{i\theta}) \in I$), then

$$u^*(e^{i\theta}) = \lim_{r\to 1} u(re^{i\theta}) = \lim_{r\to 1}(u_I(\pi(re^{i\theta})) - u_J(re^{i\theta})) = 1 - 1 = 0.$$

If $e^{i\theta} \notin J \cup A$, then $u^*(e^{i\theta}) \le 0$. So, $u^* \le 0$ a.e. on $\mathbb{T}$ (actually, on $\mathbb{T}\setminus A$). Hence, $u \le 0$ ([32]) and, therefore,

$$0 \ge u(0) = u_I(0) - u_J(0) = \frac{\sigma(I)}{2\pi} - \frac{\sigma(J)}{2\pi}.$$

So, $\sigma(I) \le \sigma(J)$.

By the same reason

$$\sigma(\mathbb{T}\setminus I) \le \sigma((\pi^*)^{-1}(\mathbb{T}\setminus I)) \le \sigma(\mathbb{T}\setminus J).$$

Hence, $2\pi - \sigma(I) \le 2\pi - \sigma(J)$ and, therefore, $\sigma(I) \ge \sigma(J)$. ∎

Lemma 4.8. *Let $\{I_j\}_{j=1}^k$ be a family of disjoint open arcs on the unit circle, let $I = \bigcup_{j=1}^k I_j$, and let $\sigma(I) = \alpha > 0$. Then for every $\varepsilon > 0$ there exists a finite Blaschke product $B : \mathbb{D} \to \mathbb{D}$ such that $B(0) = 0$, $B'(z) \ne 0$ for $z \in B^{-1}(0)$, and $B^{-1}(J_\varepsilon) \subset I$, where $J_\varepsilon = \{e^{i\theta} : 0 < \theta < \alpha - \varepsilon\}$.*

Proof. We may assume that $\alpha < 2\pi$. Note that the functions

$$u_j(z) = \frac{1}{2\pi}\int_0^{2\pi} P(z,\theta)\chi_{I_j}\, d\theta, \qquad 1 \le j \le k,$$

are harmonic on $\mathbb{D}$. Let v_j be a conjugate harmonic function to u_j such that $v_j(0) = 0$. So, $h_j = u_j + iv_j : \mathbb{D} \to R = \{z \in \mathbb{C} : 0 < \operatorname{Re} z < 1\}$ is a holomorphic mapping such that $h_j(0) = \sigma(I_j)/(2\pi)$. Actually, it is not difficult to see that

$$h_j(z) = -\frac{i}{\pi}\operatorname{Log}\left(\frac{z - e^{i\theta_{2j}}}{z - e^{i\theta_{1j}}}\right) - \frac{\sigma(I_j)}{2\pi}, \qquad z \in \mathbb{D},$$

where $I_j = \{e^{i\theta} : \theta_{1j} < \theta < \theta_{2j}\}$ (hence, $\sigma(I_j) = \theta_{2j} - \theta_{1j}$) and $\operatorname{Log} : \{z : \operatorname{Im} z > 0\} \to \{z : 0 < \operatorname{Im} z < \pi\}$ is such that $\operatorname{Log} i = \pi i/2$ (cf. [31]). Moreover, h_j extends homeomorphically to $\mathbb{T}\setminus\{e^{i\theta_{1j}}, e^{i\theta_{2j}}\} \to \{z \in \mathbb{C} : \operatorname{Im} z = 0 \text{ or } \operatorname{Im} z = \pi\}$ and $h_j(I_j) = \{z \in \mathbb{C} : \operatorname{Im} z = 0\}$.

The mapping $h = \sum_{j=1}^k h_j$ also maps $\mathbb{D}$ into R and $h(0) = \alpha/(2\pi)$. Moreover, h extends homeomorphically to $\mathbb{T}\setminus\partial I$ and $h|_I : I \to J' := \{z \in \mathbb{C} : \operatorname{Re} z = 2\pi\}$ (and $I = h^{-1}(J')$). Let

$$F(z) = \frac{e^{\pi z i} - e^{\alpha i/2}}{e^{\pi z i} - e^{-\alpha i/2}}, \qquad z \in R.$$

Then $F : R \to \mathbb{D}$ is a conformal mapping such that $F(\alpha) = 0$ and $F(J') = J = \{e^{i\theta} : 0 < \theta < \alpha\}$. Note that F extends homeomorphically to ∂R. Let $B = F \circ h$. Then

([32]) Note that if u is a subharmonic function on $\mathbb{D}$ bounded from above, such that $u^* \le 0$ a.e. on $\mathbb{T}$, then $u \le 0$ on $\mathbb{D}$. Indeed,

$$u(0) \le \limsup_{r\to 1}\frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta})\, d\theta \le \frac{1}{2\pi}\int_0^{2\pi}\limsup_{r\to 1} u(re^{i\theta})\, d\theta \le \frac{1}{2\pi}\int_0^{2\pi} u^*(e^{i\theta})\, d\theta \le 0.$$

Using automorphisms of the unit disc, we get $u \le 0$ on $\mathbb{D}$.

$B(0) = 0$. Note that $B|_{\mathbb{T}\setminus\partial I} : \mathbb{T}\setminus\partial I \to \mathbb{T}$. Moreover, $B(I) = J$ and $B(\mathbb{T}\setminus\bar{I}) = \mathbb{T}\setminus\bar{J}$. A straightforward calculation shows that

$$B(z) = \frac{\prod_{j=1}^k (z - e^{i\theta_{2j}}) - e^{i\alpha}\prod_{j=1}^k (z - e^{i\theta_{1j}})}{\prod_{j=1}^k (z - e^{i\theta_{2j}}) - \prod_{j=1}^k (z - e^{i\theta_{1j}})}.$$

Hence, $B(e^{i\theta_{1j}}) = 1$ and $B(e^{i\theta_{2j}}) = e^{i\alpha}$, $j = 1,\dots,k$. Therefore, $B(\mathbb{T}) \subset \mathbb{T}$ and we deduce that B is a finite Blaschke product (cf. [39], Chapter 7.3.1).

Suppose that

$$B(z) = e^{i\tau}\prod_{j=1}^N \left(\frac{z - a_j}{1 - \bar{a}_j z}\right)^{m_j}.$$

Note that $B(\mathbb{T}\setminus I) \subset \mathbb{T}\setminus J$. Take a closed arc $\widetilde{J} \subset J$ such that $\sigma(\widetilde{J}) \ge \alpha - \varepsilon$. Then

$$B(\mathbb{T}\setminus I) \subset \mathbb{T}\setminus\widetilde{J}.$$

Take different points $a_{j1},\dots,a_{jm_j}$, $j = 1,\dots,N$, sufficiently close to a_j such that $a_j \in \{a_{j1},\dots,a_{jm_j}\}$. It is sufficient to replace B by

$$\widetilde{B}(z) = e^{i\tau}\prod_{j=1}^N\prod_{\ell=1}^{m_j}\left(\frac{z - a_{j\ell}}{1 - \bar{a}_{j\ell}z}\right).$$

If $a_{j1},\dots,a_{jm_j}$ are close enough to a_j, then

$$\widetilde{B}(\mathbb{T}\setminus I) \subset \mathbb{T}\setminus\widetilde{J}.$$

Note that $\widetilde{B}^{-1}(\widetilde{J}) \subset I$. Take $\widehat{B}(z) = \widetilde{B}(e^{i\theta}z)$ with θ such that $\widehat{B}$ satisfies the conclusion of the lemma. ■

The following result is a simple corollary of the definition of $\widetilde{\omega}$ and Proposition 2.4.

Proposition 4.9. *Let X, Y be complex manifolds and let $U \subset X$, $V \subset Y$ be open sets. Suppose that $F : X \to Y$ is a holomorphic mapping such that $F(U) \subset V$. Then*

$$\widetilde{\omega}(x, U, X) \ge \widetilde{\omega}(F(x), V, Y), \qquad x \in X.$$

Moreover, if F is a holomorphic covering and $U = F^{-1}(V)$, then

$$\widetilde{\omega}(x, U, X) = \widetilde{\omega}(F(x), V, Y), \qquad x \in X.$$

Now we are ready to prove Theorem 4.1.

Proof of Theorem 4.1. For the proof of the inequality "$\ge$" it suffices to consider the projections $\mathrm{pr}_1 : X_1 \times X_2 \to X_1$ and $\mathrm{pr}_2 : X_1 \times X_2 \to X_2$ and use Proposition 4.9.

To prove "$\le$", put $u_1 = -\chi_{U_1}$ and $u_2 = -\chi_{U_2}$.

Let $(x_1, x_2) \in X_1 \times X_2$ be fixed and let $\beta \in \mathbb{R}$ be an arbitrary number such that

$$\max\{\widetilde{\omega}(x_1, U_1, X_1), \widetilde{\omega}(x_2, U_2, X_2)\} < \beta.$$

By definition there are holomorphic mappings $f_1 : \overline{\mathbb{D}} \to X_1$ and $f_2 : \overline{\mathbb{D}} \to X_2$ such that $f_1(0) = x_1$, $f_2(0) = x_2$, and

$$\frac{1}{2\pi}\int_0^{2\pi} u_j(f_j(e^{i\theta}))\,d\theta < \beta, \qquad j = 1, 2.$$

Note that ${f_1}^{-1}(U_1)\cap\mathbb{T}$ is an open set in $\mathbb{T}$. So, we may choose a finite set of disjoint open arcs $I_1^1,\dots,I_m^1\subset {f_1}^{-1}(U_1)\cap\mathbb{T}$ on the unit circle $\mathbb{T}$ such that $\sigma(I^1)>-2\pi\beta$, where $I^1=\bigcup_{j=1}^m I_j^1$. Similarly we choose $I_1^2,\dots,I_k^2$ with $I^2=\bigcup_{j=1}^k I_j^2$. By Lemma 4.8 we may find Blaschke products B_1, B_2 and an open arc I on the unit circle with $\sigma(I)>-2\pi\beta$ such that $B_1^{-1}(I)\subset I^1$ and $B_2^{-1}(I)\subset I^2$, $B_j(0)=0$, $B_j'(z)\neq 0$ for $z\in B_j^{-1}(0)$, $j=1,2$.

Let A be the set of critical values of the mappings B_1 and B_2. Note that 0 is not in A. Let π be a holomorphic universal covering of $\mathbb{D}\setminus A$ by $\mathbb{D}$ with $\pi(0)=0$. If $\widetilde{I}=(\pi^*)^{-1}(I)$, then according to Lemma 4.7, $\sigma(\widetilde{I})=\sigma(I)$. There are liftings ψ_1 and ψ_2 of $\mathbb{D}$ into $\mathbb{D}$ such that $\pi=B_1\circ\psi_1=B_2\circ\psi_2$ and $\psi_1(0)=\psi_2(0)=0$. Note that by Lemma 4.5, ψ_1, ψ_2 are inner holomorphic mappings because π is inner. Also the non-tangential boundary values of ψ_1 and ψ_2 on $\widetilde{I}$ belong to I^1 and I^2 respectively. Put $\widetilde{f_1}=f_1\circ\psi_1$ and $\widetilde{f_2}=f_2\circ\psi_2$. Then

$$\frac{1}{2\pi}\int_0^{2\pi}\max\{u_1(\widetilde{f_1}(e^{i\theta}),\widetilde{f_2}(e^{i\theta})),u_2(\widetilde{f_1}(e^{i\theta}),\widetilde{f_2}(e^{i\theta}))\}\,d\theta\le-\frac{\sigma(\widetilde{I})}{2\pi}<\beta.$$

By Fatou's theorem the same inequality holds if we replace $\widetilde{f_j}(z)$, $j=1,2$, with $\widetilde{f_j}(rz)$, where $r<1$ is sufficiently close to 1. Hence, $\widetilde{\omega}((x_1,x_2),U_1\times U_2,X_1\times X_2)<\beta$. Since β was arbitrary, we get the assertion. ∎

4.2. Product property of the Green functional. The main result of this part is the following product property (cf. [8], [9]).

THEOREM 4.10. *Let X_1 and X_2 be complex manifolds. Assume that $\alpha_1:X_1\to\{0,1\}$ and $\alpha_2:X_2\to\{0,1\}$ are arbitrary functions. Then*

$$E_{\mathfrak{F}_3^{\alpha_1\otimes\alpha_2}}(x_1,x_2)=\max\{E_{\mathfrak{F}_3^{\alpha_1}}(x_1),E_{\mathfrak{F}_3^{\alpha_2}}(x_2)\},\qquad (x_1,x_2)\in X_1\times X_2.$$

For the proof of Theorem 4.10 we need some technical results. Let $a_1,\dots,a_\ell\in\mathbb{D}$ and let $m_1,\dots,m_\ell\in\mathbb{N}$. Suppose that

$$B(z)=e^{i\theta}\prod_{j=1}^{\ell}\left(\frac{a_j-z}{1-\overline{a}_jz}\right)^{m_j}.$$

We put $\operatorname{mult}(B)=m_1+\ldots+m_\ell$.

LEMMA 4.11. *For any $c\in\mathbb{D}$,*

$$B_c(z)=\frac{B(z)-c}{1-\overline{c}B(z)}$$

is a Blaschke product and $\operatorname{mult}B_c=\operatorname{mult}B$.

Proof. Since B_c is a proper holomorphic function in $\mathbb{D}$, B_c is a finite Blaschke product (see e.g. [38]). Note that

$$B(z)=\frac{B_c(z)+c}{1+\overline{c}B_c(z)}.$$

Hence, it suffices to show that $\operatorname{mult}(B_c)\ge\operatorname{mult}(B)$. Observe that $B'(z)=0$ if and only if $B_c'(z)=0$. The equation $B_c(z)=-c$ has $\operatorname{mult}(B)$ solutions in $\mathbb{D}$. Let

$$B_c(z)=e^{i\theta_c}\prod_{j=1}^{\ell}\left(\frac{b_j-z}{1-\overline{b}_jz}\right).$$

The equation

$$e^{i\theta_c}\prod_{j=1}^{\ell}(b_j-z)+c\prod_{j=1}^{\ell}(1-\overline{b}_jz)=0$$

has at least $\mathrm{mult}(B)$ solutions. Therefore, $\ell\geq \mathrm{mult}(B)$. ∎

LEMMA 4.12. *Let*

$$B(z)=e^{i\theta}\prod_{j=1}^{\ell}\left(\frac{a_j-z}{1-\overline{a}_jz}\right),$$

where the numbers $a_1,\dots,a_\ell\in\mathbb{D}$ are different. In the above notation

$$B(z)=\frac{B_c(z)+c}{1+\overline{c}B(z)}=\frac{e^{i\theta_c}\prod_{j=1}^{\ell}\left(\frac{b_j-z}{1-\overline{b}_jz}\right)+c}{1+\overline{c}e^{i\theta_c}\prod_{j=1}^{\ell}\left(\frac{b_j-z}{1-\overline{b}_jz}\right)},$$

where $c\in\mathbb{D}$. Let

$$\widetilde{B}(z)=\frac{e^{i\theta_c}\prod_{j=1}^{\ell}\left(\frac{\widetilde{b}_j-z}{1-\overline{\widetilde{b}}_jz}\right)+c}{1+\overline{c}e^{i\theta_c}\prod_{j=1}^{\ell}\left(\frac{\widetilde{b}_j-z}{1-\overline{\widetilde{b}}_jz}\right)}=e^{i\tau}\prod_{j=1}^{\ell}\left(\frac{\widetilde{a}_j-z}{1-\overline{\widetilde{a}}_jz}\right).$$

If the numbers $\widetilde{b}_1,\dots,\widetilde{b}_\ell$ are sufficiently close to $b_1,\dots,b_\ell$, then the numbers $\widetilde{a}_1,\dots,\widetilde{a}_\ell$ are sufficiently close to $a_1,\dots,a_\ell$.

Proof. Note that the numbers $a_1,\dots,a_\ell$ are solutions of the equation $B_c(z)=-c$. Put

$$P(z)=e^{i\theta_c}\prod_{j=1}^{\ell}(b_j-z)+c\prod_{j=1}^{\ell}(1-\overline{b}_jz).$$

Notice that $P(a_1)=\ldots=P(a_\ell)=0$. The polynomial P is of degree ℓ and has ℓ different zeros. Then the polynomial

$$\widetilde{P}(z)=e^{i\theta_c}\prod_{j=1}^{\ell}(\widetilde{b}_j-z)+c\prod_{j=1}^{\ell}(1-\overline{\widetilde{b}}_jz)=0$$

has also ℓ zeros close to $a_1,\dots,a_\ell$ for $\widetilde{b}_1,\dots,\widetilde{b}_\ell$ sufficiently close to $b_1,\dots,b_\ell$. ∎

The following result is a simple corollary of Proposition 2.4.

PROPOSITION 4.13. *Let X and Y be complex manifolds and let $F:X\to Y$ be a holomorphic mapping. Assume that $\alpha:Y\to[0,\infty)$ is any function. Then*

$$E_{\mathfrak{F}_3^{\alpha}}\circ F\leq E_{\mathfrak{F}_3^{\alpha\circ F}}.$$

Moreover, if F is a holomorphic covering, then

$$E_{\mathfrak{F}_3^{\alpha}}\circ F=E_{\mathfrak{F}_3^{\alpha\circ F}}.$$

Proof of Theorem 4.10. Assume first that $\alpha_1 \equiv 0$. Then $\alpha_1 \otimes \alpha_2 \equiv 0$, $E_{\mathfrak{F}_3^{\alpha_1}} \equiv 0$, $E_{\mathfrak{F}_3^{\alpha_1 \otimes \alpha_2}} \equiv 0$ and, therefore,

$$E_{\mathfrak{F}_3^{\alpha_1 \otimes \alpha_2}}(x_1, x_2) = \max\{E_{\mathfrak{F}_3^{\alpha_1}}(x_1), E_{\mathfrak{F}_3^{\alpha_2}}(x_2)\}, \qquad (x_1, x_2) \in X_1 \times X_2.$$

Hence, we may assume that $\alpha_1, \alpha_2 \not\equiv 0$.

For the proof of the inequality "$\geq$", consider the projections $\mathrm{pr}_1 : X_1 \times X_2 \to X_1$ and $\mathrm{pr}_2 : X_1 \times X_2 \to X_2$ and use Proposition 4.13.

To prove "$\leq$", fix $(x_1, x_2) \in X_1 \times X_2$. Suppose that $\beta \in \mathbb{R}$ is such that

$$\max\{E_{\mathfrak{F}_3^{\alpha_1}}(x_1), E_{\mathfrak{F}_3^{\alpha_2}}(x_2)\} < \beta.$$

It is sufficient to prove that

$$E_{\mathfrak{F}_3^{\alpha_1 \otimes \alpha_2}}(x_1, x_2) < \beta.$$

By the definition there are holomorphic mappings $f_1 : \overline{\mathbb{D}} \to X_1$ and $f_2 : \overline{\mathbb{D}} \to X_2$ such that $f_1(0) = x_1$, $f_2(0) = x_2$,

$$\sum_{j=1}^{\nu} \log |z_j| < \beta \quad \text{and} \quad \sum_{j=1}^{\mu} \log |w_j| < \beta,$$

where $\{z_1, \ldots, z_\nu\} \subset f_1^{-1}(\operatorname{supp} \alpha_1)$ and $\{w_1, \ldots, w_\mu\} \subset f_2^{-1}(\operatorname{supp} \alpha_2)$, $z_j, w_j \neq 0$.

We may assume that f_1 and f_2 are such that ν and μ are minimal and that $|z_1| \leq \ldots \leq |z_\nu|$ and $|w_1| \leq \ldots \leq |w_\mu|$.

Then

$$|z_1 \ldots z_\nu| \geq e^\beta |z_\nu|^\nu \quad \text{and} \quad |w_1 \ldots w_\mu| \geq e^\beta |w_\mu|^\mu. \tag{4.1}$$

For, if $|z_1 \ldots z_\nu| < e^\beta |z_\nu|^\nu$ then we may consider the mapping $f_1(z_\nu z)$, and we have a contradiction with the minimality of ν.

If $|z_1 \ldots z_\nu| < |w_1 \ldots w_\mu|$, we replace f_1 with the mapping $\widetilde{f}_1(z) = f_1(tz)$, where $t = (|z_1 \ldots z_\nu| / |w_1 \ldots w_\mu|)^{1/\nu}$. Then $|z_j/t| < 1$, $j = 1, \ldots, \nu$, (use (4.1)) and

$$\left| \frac{z_1}{t} \ldots \frac{z_\nu}{t} \right| = |w_1 \ldots w_\mu|.$$

Hence, we may assume that

$$|z_1 \ldots z_\nu| = |w_1 \ldots w_\mu| =: C < e^\beta.$$

Moreover, replacing $f_1(z)$ with $f_1(e^{-i\theta_1} z)$ and $f_2(z)$ with $f_2(e^{-i\theta_2} z)$, where θ_1, θ_2 are so chosen that $e^{i\theta_1} z_1 \ldots e^{i\theta_1} z_\nu = C$ and $e^{i\theta_2} w_1 \ldots e^{i\theta_2} w_\mu = C$, we may assume that $z_1 \ldots z_\nu = w_1 \ldots w_\mu = C$.

We consider the Blaschke products

$$B_1(z) = \prod_{j=1}^{\nu} \left(\frac{z_j - z}{1 - \overline{z}_j z} \right)$$

and

$$\widetilde{B}_1(z) = \frac{B_1(z) - B_1(0)}{1 - \overline{B}_1(0) B_1(z)} = e^{i\theta} \prod_{j=1}^{\nu} \frac{z - w_j}{1 - \overline{w}_j z}, \qquad z \in \overline{\mathbb{D}}.$$

We choose different w'_j, $1 \le j \le \nu$, as close to w_j as we want, such that $0 \in \{w'_1, \dots, w'_\nu\}$. Define

$$G_1(z) = e^{i\theta} \prod_{j=1}^{\nu} \frac{z - w'_j}{1 - \overline{w}'_j z}.$$

Note that $\widetilde{B}_1^{-1}(-C) = \{z_1, \dots, z_\nu\}$. We can find $w'_1, \dots, w'_\nu$ such that $G_1^{-1}(-C)$ consists of ν different points z'_j, $1 \le j \le \nu$, as close to z_j as we want. Using Corollary 2.13, let us replace the mapping f_1 with $\widetilde{f}_1 : \overline{\mathbb{D}} \to X$ in such a way that $\widetilde{f}_1(0) = f_1(0)$ and $\widetilde{f}_1(z'_j) = f_1(z_j)$, $j = 1, \dots, \nu$.

Repeating this process for f_2 we may assume that for the Blaschke products B_1 and B_2 the derivatives are equal to 0 neither on preimages of C nor at the points z_j or w_j respectively.

Let A be the union of the images of the singular points under the mappings B_1 and B_2. Note that neither 0 nor C are in A. Let π be a holomorphic universal covering of $\mathbb{D} \setminus A$ by $\mathbb{D}$ with $\pi(0) = C$. There are liftings ψ_1 and ψ_2 mapping $\mathbb{D}$ into $\mathbb{D}$ such that $\pi = B_1 \circ \psi_1 = B_2 \circ \psi_2$ and $\psi_1(0) = \psi_2(0) = 0$. If $\pi^{-1}(0) = \{\eta_1, \eta_2, \dots\}$, then $f_1 \circ \psi_1$ and $f_2 \circ \psi_2$ map 0 into z_1 and z_2, and all points η_j into $\operatorname{supp} \alpha_1$ and $\operatorname{supp} \alpha_2$ respectively.

By Theorem 4.2 we see that π is a Blaschke product. Thus

$$\pi(z) = \prod_{j=1}^{\infty} \frac{\overline{\eta}_j}{|\eta_j|} \frac{\eta_j - z}{1 - \overline{\eta}_j z} \quad \text{and} \quad \Big| \prod_{j=1}^{\infty} \eta_j \Big| = \pi(0) = C < e^{\beta}.$$

Since $(f_1 \circ \psi_1, f_2 \circ \psi_2)$ maps $\mathbb{D}$ into $X_1 \times X_2$,

$$E_{\mathfrak{F}_3^{\alpha_1 \otimes \alpha_2}}(x_1, x_2) \le \sum_{j=1}^{\infty} \log |\eta_j| < \beta. \ \blacksquare$$

5. Applications

5.1. The relative extremal function. Let us start with the following simple result.

PROPOSITION 5.1. *Let X be a complex manifold and let $\varphi : X \to [-\infty, \infty)$ be an upper semicontinuous function. Then*

$$v_\varphi(x) := \sup\{u(x) : u \in \mathrm{PSH}(X),\, u \le \varphi\}, \qquad x \in X,$$

is a plurisubharmonic function on X.

Proof. Note that $v_\varphi \le \varphi$ and, therefore, $v_\varphi^* \le \varphi^* = \varphi$. It is well known that v_φ^* is a plurisubharmonic function on X (cf. [20]). Hence, $v_\varphi^* \le v_\varphi$. $\blacksquare$

As an immediate corollary we get

COROLLARY 5.2. *Let X be a complex manifold and let $U \subset X$ be an open set. Then $\omega^*(\cdot, U, X) = \omega(\cdot, U, X) \in \mathrm{PSH}(X)$.*

Now, let us give the formula for the relative extremal function $\omega(\cdot, E, X)$, where E and X are concentric balls with respect to a norm in $\mathbb{C}^n$.

PROPOSITION 5.3 (cf. [20], Lemma 4.5.8). *Let* $q : \mathbb{C}^n \to [0,\infty)$ *be such that* $q(tx) = |t|q(x)$ *for any* $t \in \mathbb{C}$ *and* $x \in \mathbb{C}^n$. *Then for any* R, r *with* $R > r > 0$ *we have*

$$\omega(x, B_q(r), B_q(R)) \le \max\left\{\frac{\log\frac{q(x)}{R}}{\log\frac{R}{r}}, -1\right\} = \frac{\log^+\frac{q(x)}{r}}{\log\frac{R}{r}} - 1, \quad x \in B_q(R), \tag{5.1}$$

where $B_q(\varrho) = \{x \in \mathbb{C}^n : q(x) < \varrho\}$, $\varrho > 0$. *Moreover, if* $\log q \in \mathrm{PSH}(\mathbb{C}^n)$, *then in* (5.1) *we have equality.*

Proof. Fix an $x_0 \in \mathbb{C}^n$. If $q(x_0) < r$, then $x_0 \in B_q(r)$. Therefore,

$$\omega(x_0, B_q(r), B_q(R)) = -1.$$

Assume that $q(x_0) \in (r, R)$. The function

$$u(t) = \omega(tx_0, B_q(r), B_q(R)) - \frac{\log\frac{q(tx_0)}{R}}{\log\frac{R}{r}}$$

is subharmonic in the annulus $A = \mathbb{D}(R/q(x_0)) \setminus \overline{\mathbb{D}}(r/q(x_0))$ ([33]). Moreover,

$$\limsup_{A \ni t \to s} u(t) \le 0, \quad s \in \partial A.$$

By the maximum principle for subharmonic functions $u \le 0$ in A.

If $\log q \in \mathrm{PSH}(\mathbb{C}^n)$, then from the definition of the relative extremal function we have the inequality "$\ge$" in (5.1) and the result follows. ∎

THEOREM 5.4 (cf. [29]). *Let* X *be a complex manifold and let* $E \subset X$ *be any subset. Then*

$$\omega(x, E, X) = \sup\{\omega(x, U, X) : E \subset U \text{ open}\}, \quad x \in X.$$

In particular, if E *is compact, then for any neighborhood basis* $U_1 \supset U_2 \supset \ldots$ *of* E *we have*

$$\omega(x, E, X) = \lim_{j\to\infty} \omega(x, U_j, X), \quad x \in X.$$

Proof. Let $u \in \mathrm{PSH}^-(X)$ be such that $u \le -1$ on E. Fix an $\varepsilon \in (0,1)$. Then $U_\varepsilon = \{x \in X : u < -1 + \varepsilon\}$ is an open subset of X such that $E \subset U_\varepsilon$. Hence,

$$\frac{u}{1-\varepsilon} \le \omega(\cdot, U_\varepsilon, X).$$

Therefore,

$$u(x) \le (1-\varepsilon)\sup\{\omega(x, U, X) : E \subset U \text{ open}\}, \quad x \in X.$$

Taking $\varepsilon \to 0$, we obtain the required result. ∎

Using Corollary 1.14 and Theorem 4.1 we obtain the following product property for the relative extremal function.

THEOREM 5.5. *Let* X_1 *and* X_2 *be complex manifolds of class* $\widetilde{\mathcal{P}}$ *and let* $E_1 \subset X_1$, $E_2 \subset X_2$ *be open or compact subsets. Then*

$$\omega((x_1, x_2), E_1 \times E_2, X_1 \times X_2) = \max\{\omega(x_1, E_1, X_1), \omega(x_2, E_2, X_2)\}, \quad (x_1, x_2) \in X_1 \times X_2. \tag{5.2}$$

Proof. Using Theorem 5.4 we may pass from the open subsets to the subsets E_j, $j = 1, 2$. ∎

([33]) Recall that $\omega(\cdot, B_q(r), B_q(R)) \in \mathrm{PSH}(B_q(R))$.

5.2. Plurisubharmonic measure. Recall that for a complex manifold X and a subset E of X, by the *plurisubharmonic measure* of E relative to X we mean $\omega^*(\cdot, E, X)$. Here, $u^*(x) = \limsup_{y\to x} u(y)$ is the upper semicontinuous regularization of a function $u : X \to [-\infty, \infty)$ locally bounded from above. Recall also that $\omega^*(\cdot, E, X)$ is a plurisubharmonic function on X.

Then we have the following counterpart of Theorem 5.4 for the plurisubharmonic measure.

PROPOSITION 5.6. *Let X be a separable complex manifold and let E be any subset of X. Then there exist open subsets U_j, $j = 1, 2, \dots$, such that $E \subset U_j$, $U_{j+1} \subset U_j$, and*

$$\omega^*(x, E, X) = (\lim_{j\to\infty} \omega(x, U_j, X))^*, \qquad x \in X.$$

Proof. By Choquet's lemma ([34]) there is a family of plurisubharmonic functions $\{v_k\}_{k=1}^{\infty} \subset \mathrm{PSH}(X)$ such that $v_k \le 0$ on X, $v_k \le -1$ on E, and $\omega^*(\cdot, E, X) = (\sup v_k)^*$. Take $u_j = \sup_{k\le j} v_k$. Then $\omega^*(\cdot, E, X) = (\lim_{j\to\infty} u_j)^*$. Put $U_j := \{x \in X : u_j(x) < -1 + 1/j\}$. ∎

For bounded domains in $\mathbb{C}^n$ we also have the following very useful result.

PROPOSITION 5.7 (cf. [1]). *Let Ω be a bounded domain in $\mathbb{C}^n$ and let E be any subset of Ω. Put $E_\varepsilon := \{x \in \Omega : \omega^*(x, E, \Omega) < -1 + \varepsilon\}$ ([35]), where $\varepsilon \in (0, 1)$. Then*

$$\frac{\omega^*(\cdot, E, \Omega)}{1-\varepsilon} \le \omega(\cdot, E_\varepsilon, \Omega) \le \omega^*(\cdot, E, \Omega) \qquad \text{on } \Omega.$$

Therefore, $\omega(\cdot, E_\varepsilon, X) \nearrow \omega^*(\cdot, E, \Omega)$ as $\varepsilon \searrow 0$.

Proof. Put $\mathcal{N} := \{x \in \Omega : \omega(x, E, \Omega) < \omega^*(x, E, \Omega)\}$ and $\widetilde{E} := E \setminus \mathcal{N}$. It is well known (see e.g. Theorem 4.7.6 in [20]) that $\mathcal{N}$ is a pluripolar set, i.e. there exists a negative plurisubharmonic function $v \in \mathrm{PSH}(\Omega)$, $v \not\equiv -\infty$, such that $\mathcal{N} \subset \{x \in \Omega : v(x) = -\infty\}$.

Let us show that $\omega^*(\cdot, E, \Omega) \equiv \omega^*(\cdot, \widetilde{E}, \Omega)$. Since $\widetilde{E} \subset E$, it follows that $\omega^*(\cdot, E, \Omega) \le \omega^*(\cdot, \widetilde{E}, \Omega)$.

Take an $\varepsilon > 0$ and $u \in \mathrm{PSH}(\Omega)$ such that $u \le -1$ on $\widetilde{E}$ and $u \le 0$ on Ω. Then $\widetilde{u} := u + \varepsilon v \in \mathrm{PSH}(\Omega)$ is such that $\widetilde{u} \le -1$ on E and $\widetilde{u} \le 0$ on Ω. Hence, $\widetilde{u} \le \omega(\cdot, E, \Omega) \le \omega^*(\cdot, E, \Omega)$. So, $\omega^*(\cdot, \widetilde{E}, \Omega) + \varepsilon v \le \omega^*(\cdot, E, \Omega)$ on Ω. Take $\varepsilon \to 0$. Then $\omega^*(\cdot, \widetilde{E}, \Omega) \le \omega^*(\cdot, E, \Omega)$ on $\Omega \setminus \{x \in \Omega : v(x) = -\infty\}$ and $\omega^*(\cdot, \widetilde{E}, \Omega) \le \omega^*(\cdot, E, \Omega)$ on Ω ([36]).

Note that

$$\omega^*(x, \widetilde{E}, \Omega) = \omega^*(x, E, \Omega) = \omega(x, E, \Omega) = -1, \qquad x \in \widetilde{E}.$$

Therefore, $\widetilde{E} \subset E_\varepsilon$ and

$$\omega^*(\cdot, E, \Omega) = \omega^*(\cdot, \widetilde{E}, \Omega) \ge \omega(\cdot, E_\varepsilon, \Omega).$$

([34]) CHOQUET'S LEMMA (cf. Lemma 2.3.4 in [20]). *Let X be a separable metric space and let $\{u_\alpha\}_{\alpha\in A}$ be a family of real-valued functions on X. Suppose that this family is locally bounded from above. Then there exists a countable subset B of A such that* $(\sup_{\alpha\in A} u_\alpha)^* = (\sup_{\beta\in B} u_\beta)^*$.

([35]) Note that E_ε is an open set.

([36]) Use Theorem 2.9.2, Corollary 2.9.8, Corollary 2.9.10 of [20].

Put $u(x) := \omega^*(\cdot, E, \Omega)/(1-\varepsilon)$, $x \in \Omega$. Note that u is a plurisubharmonic function such that $u \le 0$ on Ω and $u \le -1$ on E_ε. Hence, $u \le \omega(\cdot, E_\varepsilon, \Omega)$ on Ω. ∎

For the plurisubharmonic measure we have the following product property.

THEOREM 5.8. *Let $\Omega_1 \subset \mathbb{C}^{n_1}$, $\Omega_2 \subset \mathbb{C}^{n_2}$ be bounded domains and let $E_1 \subset \Omega_1$, $E_2 \subset \Omega_2$ be any subsets. Then*

$$\omega^*((x_1, x_2), E_1 \times E_2, \Omega_1 \times \Omega_2) = \max\{\omega^*(x_1, E_1, \Omega_1), \omega^*(x_2, E_2, \Omega_2)\}, \quad (x_1, x_2) \in \Omega_1 \times \Omega_2.$$

For the proof we need the following.

LEMMA 5.9. *Let X_1, X_2 be complex manifolds and let $E_1 \subset X_1$, $E_2 \subset X_2$ be any subsets. Then*

$$\begin{aligned}(5.3)\qquad \max\{\omega(x_1, E_1, X_1), \omega(x_2, E_2, X_2)\} &\le \omega((x_1, x_2), E_1 \times E_2, X_1 \times X_2)\\ &\le -\omega(x_1, E_1, X_1)\omega(x_2, E_2, X_2) \le -\omega^*(x_1, E_1, X_1)\omega^*(x_2, E_2, X_2)\\ &\qquad \textit{for any } (x_1, x_2) \in X_1 \times X_2.\end{aligned}$$

Proof. Note that the first inequality is trivial. Let us show the second inequality. Fix $u \in \mathrm{PSH}(X_1 \times X_2)$ such that $u \le 0$ on $X_1 \times X_2$, $u \le -1$ on $E_1 \times E_2$. Note that

$$\begin{aligned} u(\,\cdot\,, x_2) &\le \omega(\cdot, E_1, X_1) = \omega(\cdot, E_1, X_1) \cdot [-\omega(x_2, E_2, X_2)], \quad x_2 \in E_2,\\ u(x_1, \,\cdot\,) &\le \omega(\cdot, E_2, X_2) = \omega(\cdot, E_2, X_2) \cdot [-\omega(x_1, E_1, X_1)], \quad x_1 \in E_1.\end{aligned}$$

So,

$$u(x_1, x_2) \le -\omega(x_1, E_1, X_1)\omega(x_2, E_2, X_2), \quad (x_1, x_2) \in (E_1 \times X_2) \cup (X_1 \times E_2).$$

Fix $x_1 \in E_1$. If $\omega(x_1, E_1, X_1) = 0$, then

$$u(x_1, x_2) \le 0 = -\omega(x_1, E_1, X_1)\omega(x_2, E_2, X_2), \quad x_2 \in X_2.$$

Hence, we may assume that $\omega(x_1, E_1, X_1) \ne 0$. Put $v(x) := u(x_1, x)/(-\omega(x_1, E_1, X_1))$, $x \in X$. Note that $v \in \mathrm{PSH}(X_2)$, $v \le 0$. Moreover, $v \le -1$ on E_2. So, $v \le \omega(\cdot, E_2, X_2)$ on X_2. ∎

Proof of Theorem 5.8. Fix an $\varepsilon \in (0, 1)$. Then by (5.3),

$$\omega((x_1, x_2), E_1 \times E_2, \Omega_1 \times \Omega_2) \le -(1-\varepsilon)^2 \quad \text{on } (E_1)_\varepsilon \times (E_2)_\varepsilon.$$

So,

$$\omega^*((x_1, x_2), E_1 \times E_2, \Omega_1 \times \Omega_2) \le -(1-\varepsilon)^2 \quad \text{on } (E_1)_\varepsilon \times (E_2)_\varepsilon.$$

It follows that on $\Omega_1 \times \Omega_2$,

$$\frac{\omega^*(\cdot, E_1 \times E_2, \Omega_1 \times \Omega_2)}{(1-\varepsilon)^2} \le \omega(\cdot, (E_1)_\varepsilon \times (E_2)_\varepsilon, \Omega_1 \times \Omega_2) \le \omega^*(\cdot, E_1 \times E_2, \Omega_1 \times \Omega_2).$$

Therefore,

$$\begin{aligned}\omega^*((x_1, x_2), E_1 \times E_2, \Omega_1 \times \Omega_2)&\\ &= \lim_{\varepsilon \to 0} \omega((x_1, x_2), (E_1)_\varepsilon \times (E_2)_\varepsilon, \Omega_1 \times \Omega_2)\\ &= \lim_{\varepsilon \to 0} \max\{\omega(x_1, (E_1)_\varepsilon, \Omega_1), \omega(x_2, (E_2)_\varepsilon, \Omega_2)\}\\ &= \max\{\omega^*(x_1, E_1, \Omega_1), \omega^*(x_2, E_2, \Omega_2)\}, \quad (x_1, x_2) \in \Omega_1 \times \Omega_2. \end{aligned}$$ ∎

Theorem 5.8 for pseudoconvex domains Ω_1, Ω_2 was proved in [32]. The general case is stated in [12]. The proof can be found in Błocki [1].

5.3. The pluricomplex Green function. For the pluricomplex Green function with pole function α we have the following equivalent definition.

PROPOSITION 5.10. *Let X be a complex manifold and let α be a non-negative function on X. Then*

$$g_X(x,\alpha) = \sup\{u(x) : u \in \mathrm{PSH}(X),\ u \le \inf_{p\in X} \alpha(p) g_X(\cdot,p)\},$$

and, therefore, $g_X(\cdot,\alpha)$ is a plurisubharmonic function on X.

Proof. Put $\varphi(x) := \inf_{p\in X} \alpha(p) g_X(x,p)$ and $v(x) := \sup\{u(x) : u \in \mathrm{PSH}(X), u \le \varphi\}$, $p, x \in X$. Note that φ is an upper semicontinuous function on X and, therefore, $v \in \mathrm{PSH}(X)$. It suffices to show that $g_X(\cdot,\alpha) = v$ on X.

Let $u \in \mathrm{PSH}(X)$, $u \le \varphi$ on X. Fix $p \in X$. Then $u \le \alpha(p) g_X(\cdot,p)$ and, therefore, $\nu(p,u) \ge \alpha(p)$. Hence, $\nu(\cdot,u) \ge \alpha$. So, $u \le g_X(\cdot,\alpha)$ and, therefore, $v \le g_X(\cdot,\alpha)$.

Assume that $u \in \mathrm{PSH}(X)$, $u \le 0$ on X, and $\nu(\cdot,u) \ge \alpha$. Fix $p \in X$. Then $\nu(p,u) \ge \alpha(p)$ and, therefore, $u \le \alpha(p) g_X(\cdot,p)$ on X. So, $u \le \varphi$ on X. Hence, $u \le v$ and, therefore, $g_X(\cdot,\alpha) \le v$. ■

In this section we show that the pluricomplex Green function may be considered as an infinitesimal version of the relative extremal function. More precisely, we have the following (cf. [10]).

THEOREM 5.11. *Let X be a complex manifold and let $\alpha : X \to [0,\infty)$ be any function. Assume that $\{(U_x,\zeta_x)\}_{x\in X}$, is a family of local coordinates such that $\zeta_x(U_x) = P(1)$ and $\zeta_x(x) = 0$. Then*

$$(-\log r)\omega(x, \mathfrak{P}(r,\alpha), X) \underset{r\to 0}{\searrow} g_X(x,\alpha),$$

where

$$\mathfrak{P}(r,\alpha) = \bigcup_{y\in X} \zeta_y^{-1}[P(r^{1/\alpha(y)})], \qquad r \in (0,1).$$

Proof. Note that the case $\alpha \equiv 0$ is trivial. Hence, we may assume that $\alpha \not\equiv 0$. The proof will be divided into 3 steps.

STEP 1. We show that for any $r \in (0,1)$ we have

$$(-\log r)\omega(x, \mathfrak{P}(r,\alpha), X) \ge g_X(x,\alpha).$$

Put

$$u(x) := \frac{g_X(x,\alpha)}{-\log r}.$$

Note that u is a negative plurisubharmonic function on X. Take $x \in \mathfrak{P}(r,\alpha)$. Choose $y \in X$ such that $x \in \zeta_y^{-1}(P(r^{1/\alpha(y)})) \subset U_y$. Then

$$u(x) \le \frac{g_{U_y}(x,(y,\alpha(y)))}{-\log r} = \frac{g_{P(1)}(\zeta_y(x),(0,\alpha(y)))}{-\log r} = \alpha(y)\frac{\log \|\zeta_y(x)\|}{-\log r} \le -1.$$

Here, we use the formula (cf. [20])

$$g_{P(1)}(x,(0,\nu)) = \nu \log \|x\|, \qquad x \in P(1).$$

Hence, we have the required inequality.

STEP 2. We show that

$$u_r(x) := (-\log r)\omega(x, \mathfrak{P}(r,\alpha), X), \qquad x \in X,$$

is an increasing sequence of functions with respect to $r \in (0,1)$.

Fix $r < \varrho < 1$. Put

$$u(x) := \frac{u_r(x)}{-\log \varrho}, \qquad x \in X.$$

Note that u is a negative plurisubharmonic function on X. It suffices to show that $u \le -1$ on $\mathfrak{P}(\varrho,\alpha)$. Take $x \in \mathfrak{P}(r,\alpha)$. Choose $y \in X$ such that $x \in \zeta_y^{-1}(P(r^{1/\alpha(y)})) \subset U_y$. Then

$$u(x) \le \frac{(-\log r)\omega(x, \zeta_y^{-1}[P(r^{1/\alpha(y)})], U_y)]}{-\log \varrho} \le -1.$$

Here, we use Proposition 5.3.

STEP 3. Put

$$u(x) := \lim_{r\to 0}(-\log r)\omega(x, \mathfrak{P}(r,\alpha), X), \qquad x \in X.$$

Fix $y \in X$. It suffices to show that

$$u(x) \le \alpha(y) \log \|\zeta_y(x)\| \qquad \text{for any } x \in U_y.$$

Note that u is a negative plurisubharmonic function. Fix $x \in U_y \setminus \{y\}$ and take $r < \|\zeta_y(x)\|^{\alpha(y)}$. Then

$$\begin{aligned} u(x) &\le (-\log r)\omega(x, \mathfrak{P}(r,\alpha), X) \le (-\log r)\omega(x, \zeta_y^{-1}[P(r^{\frac{1}{\alpha(y)}})], U_y) \\ &= (-\log r)\frac{\alpha(y)\log\|\zeta_y(x)\|}{-\log r} = \alpha(y)\log\|\zeta_y(x)\|. \end{aligned}$$

Hence, $u(\cdot) \le g_X(\cdot,\alpha)$. ∎

Using the method from [10], as a corollary of Theorem 5.11 we have the product property of the pluricomplex Green function.

THEOREM 5.12. *Let X_1 and X_2 be complex manifolds. Assume that for any open subsets $E_1 \subset X_1$ and $E_2 \subset X_2$ we have the following product property:*

$$\omega((x_1,x_2), E_1\times E_2, X_1\times X_2) = \max\{\omega(x_1,E_1,X_1), \omega(x_2,E_2,X_2)\}, \qquad (x_1,x_2)\in X_1\times X_2.$$

Then for arbitrary functions $\alpha_1 : X_1 \to \{0,1\}$ and $\alpha_2 : X_2 \to \{0,1\}$ we have

$$g_{X_1\times X_2}((x_1,x_2), \alpha_1\otimes\alpha_2) = \max\{g_{X_1}(x_1,\alpha_1), g_{X_2}(x_2,\alpha_2)\}, \qquad (x_1,x_2)\in X_1\times X_2.$$

Proof. Suppose that $\{(U_{jx},\zeta_{jx})\}_{x\in X_j}$ is a local coordinate centered at x such that $\zeta_{jx}(U_{jx}) = P(1)$ and $\zeta_{jx}(x) = 0$, $j = 1,2$. Then $(U_{1x_1}\times U_{2x_2}, \zeta_{1x_1}\times\zeta_{2x_2})$, $(x_1,x_2)\in X_1\times X_2$, is a local coordinate centered at (x_1,x_2) such that $\zeta_{1x_1}\times\zeta_{2x_2}(U_{1x_1}\times U_{2x_2}) = P(1)$.

Then

$$\begin{aligned}g_{X_1\times X_2}((x_1,x_2),\alpha_1\otimes\alpha_2) &= \lim_{r\to 0}(-\log r)\omega((x_1,x_2),\mathfrak{P}(r,\alpha_1\otimes\alpha_2),X_1\times X_2)\\ &= \lim_{r\to 0}(-\log r)\omega((x_1,x_2),\mathfrak{P}(r,\alpha_1)\times\mathfrak{P}(r,\alpha_2),X_1\times X_2)\\ &= \lim_{r\to 0}(-\log r)\max\{\omega(x_1,\mathfrak{P}(r,\alpha_1),X_1),\omega(x_2,\mathfrak{P}(r,\alpha_2),X_2)\}\\ &= \max\{g_{X_1}(x_1,\alpha_1),g_{X_2}(x_2,\alpha_2)\},\qquad (x_1,x_2)\in X_1\times X_2. \ \blacksquare\end{aligned}$$

As a corollary we get

COROLLARY 5.13. *Let X_1 and X_2 be complex manifolds of class $\widetilde{\mathcal{P}}$. Assume that $\alpha_1 : X_1\to\{0,1\}$ and $\alpha_2 : X_2\to\{0,1\}$ are arbitrary functions. Then*

$$g_{X_1\times X_2}((x_1,x_2),\alpha_1\otimes\alpha_2)=\max\{g_{X_1}(x_1,\alpha_1),g_{X_2}(x_2,\alpha_2)\},\qquad (x_1,x_2)\in X_1\times X_2.$$

5.4. Polynomial hulls. Let K be a compact set in $\mathbb{C}^n$. The *polynomial hull* $\widehat{K}$ of K is defined as follows:

$$\widehat{K}=\{x\in\mathbb{C}^n : |p(x)|\le\|p\|_K \text{ for any polynomial } p:\mathbb{C}^n\to\mathbb{C}\}.$$

We say that $\Omega\subset\mathbb{C}^n$ is a *Runge domain* if for any compact set $K\subset\Omega$ we have $\widehat{K}\subset\Omega$. For a Runge domain $\Omega\subset\mathbb{C}^n$ we have $\widehat{K}=K^{\mathcal{O}(\Omega)}=K^{\mathrm{PSH}(\Omega)}$ (see e.g. [24]), where

$$K^{\mathrm{PSH}(\Omega)}=\{x\in\Omega : u(x)\le\sup_K u \text{ for any } u\in\mathrm{PSH}(\Omega)\}.$$

We have the following characterization of the polynomial hull.

THEOREM 5.14 (cf. Theorem 7.1 in [35], Theorem 7.4 in [25]). *Let K be a compact set in $\mathbb{C}^n$ and let $x_0\in\mathbb{C}^n$. Then the following conditions are equivalent:*

(a) $x_0\in\widehat{K}$;

(b) *there exists a Runge domain Ω in $\mathbb{C}^n$ such that for any neighborhood U of K we have $\omega(x_0,U,\Omega)=-1$;*

(c) *there exists a Runge domain Ω in $\mathbb{C}^n$ such that for any neighborhood U of K and any $\varepsilon>0$ there exists an $f\in\mathcal{O}(\overline{\mathbb{D}},\Omega)$ such that $f(0)=x$ and $\sigma(\mathbb{T}\cap f^{-1}(U))>2\pi-\varepsilon$.*

Proof. (a)⇒(b). Take an open ball B containing K and x_0. Suppose that $x_0\in\widehat{K}$. Then $x_0\in K^{\mathrm{PSH}(B)}$. Let U be a neighborhood of K in B and let $\varphi=-\chi_U$. Then $u=E_{\mathfrak{F}_1^\varphi}$ is a plurisubharmonic function in B. We have $u=-1$ on U. Hence, $u(x_0)=-1$.

(b)⇒(c). Follows immediately from the definition of the envelope of a disc functional.

(c)⇒(a). Let p be a polynomial. Then

$$|p(x)|\le\frac{1}{2\pi}\int_{\mathbb{T}}|p\circ f|\,d\sigma\le\sup_U|p|+\frac{1}{2\pi}\sigma(\mathbb{T}\setminus f^{-1}(U))\sup_\Omega|P|.$$

Take $U\searrow K$ and $\varepsilon\to 0$. Then $|p(x)|\le\|p\|_K$. $\blacksquare$

A more refined characterization of the polynomial hull of a compact set in $\mathbb{C}^n$ can be found in [35].

6. Concluding remarks

6.1. Envelope of disc functionals. For someone who works in the part of complex analysis which is connected with pluripotential theory, the definition of envelope of a disc functional is very natural. But one can give other possible definitions. From the point of view of interpolation theory, probably, it would be natural to consider more general types of envelopes. Namely, let X be a complex manifold and let $H : \mathcal{O}(\overline{\mathbb{D}}, X) \to \overline{\mathbb{R}}$ be a disc functional. Take different points $z_1, \dots, z_\ell \in \mathbb{D}$ and points $x_1, \dots, x_\ell \in X$ and put

$$E_H(x_1, \dots, x_\ell) := \inf\{H(f) : f \in \mathcal{O}(\overline{\mathbb{D}}, X),\ f(z_1) = x_1, \dots, f(z_\ell) = x_\ell\}.$$

6.2. Complex manifolds. As mentioned in the Introduction, it is still an open problem whether the envelope of the Poisson functional is plurisubharmonic on *any* complex manifold.

Peter Pflug noted that any complex manifold of class $\widetilde{\mathcal{P}}$ (and therefore, of class $\mathcal{P}$) has a countable basis. It is known that there exists a simply connected two-dimensional complex manifold M which has no countable basis (cf. [2]). So, the complex manifold M does not belong to the class $\widetilde{\mathcal{P}}$.

Proposition 2.3 shows a possible extension of the class $\widetilde{\mathcal{P}}$, so as to include complex manifolds with non-countable base.

6.3. The Poisson functional on domains in $\mathbb{C}^n$. Dealing with general complex manifolds, we introduced the notion of the integrally upper semicontinuous function. Analyzing the proof of Theorem 3.5 more carefully one can show that for domains in $\mathbb{C}^n$ it suffices to assume only weak integral upper semicontinuity. More precisely, we have the following interesting result.

THEOREM 6.1. *Let Ω be a domain in $\mathbb{C}^n$. Assume that*

(a) $\varphi : \Omega \to [-\infty, \infty)$ *is a weakly integrally upper semicontinuous function locally bounded from above or*

(b) φ *is a superharmonic function on Ω, $\varphi \not\equiv \infty$.*

Then $E_{\mathfrak{F}_1^\varphi}$ is a plurisubharmonic function on Ω.

6.4. Holomorphically invariant pseudodistances. Lempert's theorem. In this section we present some results and definitions from the theory of holomorphically invariant families. As we shall see many properties of analytic disc functions are motivated by the properties of holomorphically invariant families. This is the main point of this section.

Let us start with the following basic result.

THEOREM 6.2 (Schwarz–Pick Lemma). *Let $f \in \mathcal{O}(\mathbb{D}, \mathbb{D})$. Then*

$$p(f(z_1), f(z_2)) \le p(z_1, z_2), \qquad z_1, z_2 \in \mathbb{D},$$

where $p := \tanh^{-1}(m)$ is the Poincaré distance and $m(z, w) := |(z - w)/(1 - \overline{w}z)|$ is the Möbius distance. Moreover, if equality holds for some $z_1 \neq z_2$, then it holds for all $z_1, z_2 \in \mathbb{D}$.

There are many ways of extending the above result to higher dimensions.

In 1927 Carathéodory [3] defined for any domain Ω in $\mathbb{C}^n$ the function

$$c_\Omega(x,y) := \sup\{p(f(x),f(y)) : f \in \mathcal{O}(\Omega,\mathbb{D})\}, \qquad x,y\in\Omega.$$

We call c_Ω the *Carathéodory pseudodistance* of Ω. It is not difficult to see that it is really a pseudodistance ([37]).

Note that $c_{\mathbb{D}} = p$ and $c_{\Omega_2}(F(x),F(y)) \le c_{\Omega_1}(x,y)$ for any domains $\Omega_1 \subset \mathbb{C}^{n_1}$, $\Omega_2 \subset \mathbb{C}^{n_2}$, any holomorphic mapping $F : \Omega_1 \to \Omega_2$, and any points $x,y \in \Omega_1$. More on the Carathéodory pseudodistance can be found in [17].

In 1967 S. Kobayashi [21] (see also [22]) defined the following pseudodistance:

$$k_\Omega(x,y) := \inf\Big\{\sum_{j=1}^N \widetilde{k}_\Omega(x_{j-1},x_j) : N\ge 1,\, x = x_0,x_1,\dots,x_{N-1},\, x_N = y\in\Omega\Big\},$$

where

$$\widetilde{k}_\Omega(x,y) := \inf\{p(z,w) : \text{there is } f\in\mathcal{O}(\mathbb{D},\Omega) \text{ with } f(z)=x,\ f(w)=y\}, \qquad x,y\in\Omega.$$

We call k_Ω the *Kobayashi pseudodistance* and $\widetilde{k}_\Omega$ the *Lempert function* of Ω. One can see that in a general domain Ω the Lempert function $\widetilde{k}_\Omega$ is not a pseudodistance (i.e. it does not satisfy the triangle inequality; see e.g. [17]).

From the Schwarz–Pick Lemma and from the definition it follows immediately that $k_{\mathbb{D}} = \widetilde{k}_{\mathbb{D}} = p$ and

$$k_{\Omega_2}(F(x),F(y)) \le k_{\Omega_1}(x,y), \qquad \widetilde{k}_{\Omega_2}(F(x),F(y)) \le \widetilde{k}_{\Omega_1}(x,y)$$

for any domains $\Omega_1 \subset \mathbb{C}^{n_1}$, $\Omega_2 \subset \mathbb{C}^{n_2}$, any holomorphic mapping $F : \Omega_1 \to \Omega_2$ and any points $x,y\in\Omega_1$.

From these considerations we come to the definition of the (holomorphically) contractible family of functions.

We say that $d := (d_\Omega)_{\Omega \text{ a domain in } \mathbb{C}^n}$, where $d_\Omega : \Omega\times\Omega \to [0,\infty)$, is a (*holomorphically*) *contractible family of functions* if

$$d_{\mathbb{D}} = p,$$
$$d_{\Omega_2}(F(x),F(y)) \le d_{\Omega_1}(x,y) \qquad \text{for any } F\in\mathcal{O}(\Omega_1,\Omega_2),\, x,y\in\Omega_1.$$

It is immediate from the definition that for a biholomorphic mapping $F : \Omega_1 \to \Omega_2$ we have the equality $d_{\Omega_2}(F(x),F(y)) = d_{\Omega_1}(x,y)$, $x,y\in\Omega_1$.

As we have seen, the Carathéodory and Kobayashi pseudodistances, and the Lempert function form holomorphically contractible families of functions.

In view of the Schwarz–Pick Lemma, the Carathéodory pseudodistance is the smallest and the Lempert function is the largest among all holomorphically contractible families of

([37]) Recall that $d : \Omega\times\Omega \to [0,\infty)$ is a *pseudodistance* on Ω if it satisfies the following conditions:

(i) $d(x,x) = 0$, $x\in\Omega$,
(ii) d is symmetric (i.e., $d(x,y) = d(y,x)$, $x,y\in\Omega$),
(iii) d satisfies the triangle inequality (i.e., $d(x,y) \le d(x,z) + d(z,y)$, $x,y,z\in\Omega$).

functions. Therefore, for any holomorphically contractible family of functions $(d_\Omega)_{\Omega\subset\mathbb{C}^n}$ we have

$$c_\Omega \le d_\Omega \le \widetilde{k}_\Omega.$$

Moreover, if d_Ω is a pseudodistance then $d_\Omega \le k_\Omega$.

We have the following properties of the pluricomplex Green function ([20], [17]):

(i) $g_{\mathbb{D}}(z,w) = \log m(x,y)$, $z,w\in\mathbb{D}$;

(ii) for any holomorphic mapping $F:\Omega_1\to\Omega_2$, where Ω_j is a domain in $\mathbb{C}^{n_j}$, $j=1,2$, we have

$$g_{\Omega_2}(F(x),F(y)) \le g_{\Omega_1}(x,y), \qquad x,y\in\Omega_1.$$

In other words, the family $(\widehat{g}_\Omega)_\Omega$ is a contractible family of functions, where $\widehat{g}_\Omega := \tanh^{-1}(e^{g_\Omega})$.

In 1981 L. Lempert [28] proved the following deep result (see also [17]).

Theorem 6.3 (Lempert theorem). *Let Ω be a convex domain in $\mathbb{C}^n$. Then*

$$c_\Omega = \widetilde{k}_\Omega.$$

We see from the Lempert theorem that on convex domains any holomorphically contractible family of functions may be defined in a unique way.

Using automorphisms of the unit disc, it is elementary to show that for the Carathéodory pseudodistance we have

$$c^*_\Omega(x,y) = \sup\{|f(y)| : f\in\mathcal{O}(\Omega,\mathbb{D}),\ f(x)=0\}, \qquad x,y\in\Omega,$$

where $c^*_\Omega := \tanh c_\Omega$. We know that c_Ω is the smallest holomorphically invariant function on Ω. Trying to make it "larger" we have to consider the supremum above over a larger family of functions. It is well known that $\log|f|$ is plurisubharmonic for any holomorphic function f. So, from this point of view we come immediately to the definition of the pluricomplex Green function g_Ω given by M. Klimek.

Using again automorphisms of the unit disc, it is not difficult to see that

$$\widetilde{k}^*_\Omega(x,y) := \inf\{t>0 : \text{there is } f\in\mathcal{O}(\overline{\mathbb{D}},\Omega) \text{ with } f(0)=x,\ t\in f^{-1}(y)\}, \qquad x,y\in\Omega,$$

where $\widetilde{k}^*_\Omega := \tanh\widetilde{k}_\Omega$.

We also know that $\widetilde{k}_\Omega$ is the largest holomorphically invariant function on Ω. Trying to make it "smaller" it seems reasonable to take all the preimages of y in the above definition.

In 1989 ([38]) E. Poletsky [36] defined the following function. Let Ω be a domain in $\mathbb{C}^n$ and let p be a point in Ω. Define

$$\widetilde{g}_\Omega(x,p) = \inf\Big\{\sum_{z\in f^{-1}(p)} \operatorname{ord}_z(f)\log|z| : f\in\mathcal{O}(\overline{\mathbb{D}},\Omega),\ f(0)=x\Big\}.$$

We see that $\widetilde{g}_\Omega = E_{\mathfrak{F}_4^\alpha}$, where $\operatorname{supp}\alpha=\{p\}$ and $\alpha(p)=1$. In a series of papers E. Poletsky ([36], [34], [35]) claimed the equality

$$g_\Omega = \widetilde{g}_\Omega. \tag{6.1}$$

([38]) In 1989 the English translation of the book appeared. The Russian version was published in 1986.

Note that the equality $g_\Omega = \widetilde{g}_\Omega$ is equivalent to the plurisubharmonicity of $\widetilde{g}_\Omega$. It seems that the first complete proof of (6.1) was given in 1997 by the author [7].

The equality (6.1) may be considered as a generalization of the Lempert theorem to all domains in $\mathbb{C}^n$.

We say that a family of holomorphically contractible functions d has the *product property* if for any domains Ω_1, Ω_2 and for any points $(x_1, x_2), (y_1, y_2) \in \Omega_1 \times \Omega_2$ we have

$$d_{\Omega_1\times\Omega_2}((x_1, x_2), (y_1, y_2)) = \max\{d_{\Omega_1}(x_1, y_1), d_{\Omega_2}(x_2, y_2)\}. \tag{6.2}$$

It follows from the contractibility that the inequality "$\geq$" in (6.2) is always fulfilled.

One can show that the Lempert function, the Kobayashi and Carathéodory pseudodistances have the product property (see [17], [19]). The proof of the product property of the pluricomplex Green function for arbitrary domains, given by the author [8], is similar in spirit to the proof of the product property of the Lempert function.

Recall also that all the contractible families of functions discussed above are continuous with respect to increasing sequences of domains (see [17]). More precisely, for any sequence of domains $\{\Omega_j\}_{j=1}^\infty \subset \mathbb{C}^n$, $\Omega_j \subset \Omega_{j+1}$, $\Omega = \bigcup_{j=1}^\infty \Omega_j$ we have (see e.g. [17])

$$d_{\Omega_j} \to d_\Omega \quad \text{as } j \to \infty,$$

where $d = c, g, k$ or $\widetilde{k}$. So, we see that the invariant functions considered behave like disc functionals with respect to increasing sequences of domains (Theorem 2.1):

List of symbols

$\mathbb{C}$:= the field of complex numbers;
$\mathbb{R}$:= the field of real numbers;
$\overline{\mathbb{R}} := [-\infty, \infty]$;
$\mathbb{N}$:= the set of natural numbers ($0 \in \mathbb{N}$);
$\mathbb{D}_r := \{z \in \mathbb{C} : |z| < r\}$, $r > 0$;
$\mathbb{D} := \mathbb{D}_1$ the unit disc in $\mathbb{C}$;
$\mathbb{D}_* := \mathbb{D} \setminus \{0\}$;
$\mathbb{T}$:= the unit circle in $\mathbb{C}$;
σ := the arc length measure on the unit circle $\mathbb{T}$;
χ_U := the characteristic function of a set U;
$\|x\| := \sqrt{|z_1|^2 + \ldots + |z_n|^2}$, $x = (z_1, \ldots, z_n) \in \mathbb{C}^n$;
$\mathbb{B}_n(x, r) := \{y \in \mathbb{C}^n : \|y - x\| < r\}$ = the Euclidean ball with center $x \in \mathbb{C}^n$ and radius $r > 0$;
$\mathbb{B}_n := \mathbb{B}_n(0, 1)$;
$|\!|\!|x|\!|\!| := \max\{|z_1|, \ldots, |z_n|\}$, $x = (z_1, \ldots, z_n) \in \mathbb{C}^n$;
$P(x, r) := \{y \in \mathbb{C}^n : |\!|\!|y - x|\!|\!| < r\}$, $r > 0$;
$P(r) := P(0, r)$;
$\mathcal{L}^{2n}$:= the Lebesgue measure in $\mathbb{C}^n$;
$b_n := \mathcal{L}^{2n}(\mathbb{B}_n)$;
$(\alpha_1 \otimes \alpha_2)(x_1, x_2) := \alpha_1(x_1)\alpha_2(x_2)$.
$\mathcal{O}(X, Y)$:= the set of all holomorphic mappings $F : X \to Y$;
$\mathcal{O}(X) := \mathcal{O}(X, \mathbb{C})$;
$\mathcal{O}(\overline{\mathbb{D}}, X)$:= the set of all holomorphic mappings $f : \overline{\mathbb{D}} \to X$ which extend holomorphically to a neighborhood of the closure $\overline{\mathbb{D}}$ of $\mathbb{D}$;
$\triangle u$:= the generalized Laplacian of a subharmonic function u;
$\mathrm{PSH}(X)$:= the set of all plurisubharmonic functions on X;
$\mathrm{PSH}^-(X)$:= the set of all negative plurisubharmonic functions on X;
$g_X(\cdot, \alpha)$:= the pluricomplex Green function on X with pole function α;
$P_{\mathfrak{U}}(x) := \sup\{u(x) : u \in \mathfrak{U}\}$, $x \in X$, where $\mathfrak{U} \subset \mathrm{PSH}(X)$;
u^* := the upper semicontinuous regularization of u;
$\omega(\cdot, E, X)$:= the relative extremal function of a subset $E \subset X$;
$\omega^*(\cdot, E, X)$:= the plurisubharmonic measure of a subset $E \subset X$;
$\widetilde{\omega}(\cdot, U, X)$:= the special case of the Poisson functional for an open subset U of a complex manifold X;
$g_X(\cdot, p)$:= the pluricomplex Green function with pole at $p \in X$;
$g_X(x; (p_1, \nu_1), \ldots, (p_N, \nu_N))$:= the pluricomplex Green function with poles at $p_1, \ldots, p_N \in X$ and of weights $\nu_1, \ldots, \nu_N \in (0, \infty)$, $p_i \neq p_j$, $i \neq j$;
$\mathfrak{P}(r, \alpha) := \bigcup_{y \in X} \zeta_y^{-1}[P(r^{1/\alpha(y)})]$, where $r \in (0, 1)$, $\{(U_x, \zeta_x)\}_{x \in X}$ is a family of local coordinates on a complex manifold X such that $\zeta_x(x) = 0$ and $\zeta_x(U_x) = P(1)$;
$\|f\|_K := \sup\{|f(x)| : x \in K\}$, where $f : K \to \mathbb{C}$;

$\widehat{K} := \{x \in \mathbb{C}^n : |p(x)| \le \|p\|_K$ for any holomorphic polynomial $p : \mathbb{C}^n \to \mathbb{C}\}$ the polynomial hull of a compact set $K \subset \mathbb{C}^n$;
$\widehat{K}^{\mathcal{O}(X)} := \{x \in X : |f(x)| \le \|f\|_K$ for any $f \in \mathcal{O}(X)\}$ the holomorphic hull of a compact set $K \subset X$;
$E_H(x) :=$ the envelope of a disc functional H;
$\mathfrak{F}_1 :=$ the Poisson functional;
$\mathfrak{F}_2 :=$ the Riesz functional;
$\mathfrak{F}_3 :=$ the Green functional;
$\mathfrak{F}_4 :=$ the Lelong functional;
$\mathfrak{F}_5 :=$ the Lempert functional;
$\mathrm{ord}_z(f) :=$ the multiplicity of a holomorphic mapping $f : \overline{\mathbb{D}} \to X$ at $z \in \mathbb{D}$;
$\mathcal{IC}^{\uparrow}(X) :=$ the class of integrally upper semicontinuous functions on a complex manifold X.

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