1. Introduction

The aim of this paper is to present the main results concerning free boundary problems for nonstationary Navier–Stokes equations. We review free boundary problems for equations of motion of both incompressible and compressible viscous fluids. The equations which we consider here are derived and described for example by L. Landau and E. Lifschitz [LanLif] or by J. Serrin [Ser].

A free boundary problem for equations describing the motion of a viscous fluid can be formulated as follows: find a domain $\Omega_t \subset \mathbb{R}^n$ $(n = 2, 3)$ with boundary $S_t = S_1 \cup S_{2t}$ (S_1 is a fixed part of S_t independent of time t; S_{2t} is a free part of S_t) as well as a velocity vector field $v = v(x,t)$ and pressure $p = p(x,t)$ in the case of an incompressible fluid (or a velocity $v = v(x,t)$, density $\varrho = \varrho(x,t)$ and temperature $\theta = \theta(x,t)$ in the case of a compressible fluid), satisfying for $x \in \Omega_t$, $t \in (0,T)$, $T > 0$, the Navier–Stokes system (or the compressible Navier–Stokes system) with the initial conditions $\Omega_t|_{t=0} = \Omega$, $v|_{t=0} = v_0$ in Ω (additionally $\varrho|_{t=0} = \varrho_0$, $\theta|_{t=0} = \theta_0$ for the compressible fluid), the Dirichlet boundary condition for v (and for θ in the compressible heat-conducting case) on $S_1 \times (0,T)$ and the Neumann type condition for the stress tensor (and for θ in the compressible heat-conducting case) on $\bigcup_{t\in(0,T)} S_{2t} \times \{t\}$.

Thus, in the most general case of a compressible viscous heat-conducting fluid the equations under consideration are as follows:

$$\varrho[v_t + (v\cdot\nabla)v] - \operatorname{div}\mathbb{T}(v,p) = \varrho\widetilde{f} \quad \text{in } \widetilde{\Omega}^T, \tag{1.1}$$

$$\varrho_t + \operatorname{div}(\varrho v) = 0 \quad \text{in } \widetilde{\Omega}^T, \tag{1.2}$$

$$\begin{aligned} &\varrho c_v(\theta_t + v\cdot\nabla\theta) - \operatorname{div}(\varkappa\theta) \\ &\quad + \theta p_\theta \operatorname{div} v - \frac{\mu}{2}\sum_{i,j=1}^{3}(v_{ix_j} + v_{jx_i})^2 \\ &\quad - (\nu-\mu)(\operatorname{div} v)^2 = \varrho r, \quad \text{in } \widetilde{\Omega}^T, \end{aligned} \tag{1.3}$$

where $T > 0$, $\widetilde{\Omega}^T \equiv \bigcup_{t\in(0,T)} \Omega_t \times \{t\}$, $\Omega_t \subset \mathbb{R}^n$ is an unknown domain at time t with boundary $S_t = S_1 \cup S_{2t}$; $\mathbb{T} = \mathbb{T}(v,p)$ is the stress tensor given by

$$\mathbb{T}(v,p) = \{-p\delta_{ij} + \mu(v_{ix_j} + v_{jx_i}) + (\nu-\mu)\operatorname{div} v\,\delta_{ij}\}_{i,j=1,\dots,n}.$$

Moreover, $\widetilde{f} = \widetilde{f}(x,t)$ is the force per unit mass, acting on the fluid; $r = r(x,t)$ denotes the heat sources per unit mass; $p = p(\varrho,\theta)$ is the pressure, $\varkappa = \varkappa(\varrho,\theta)$ the coefficient of heat conductivity, $c_v = c_v(\varrho,\theta)$ the specific heat at constant volume; $\nu = \nu(\varrho,\theta)$ and $\mu = \mu(\varrho,\theta)$ are the viscosity coefficients.

The functions $\varkappa$, c_v, ν, μ are positive and $\nu > (1/3)\mu$.

Equations (1.1)–(1.3) correspond to the conservation laws of: momentum, mass and energy, respectively.

We complete equations (1.1)–(1.3) with the following initial conditions:

$$\Omega_t|_{t=0} = \Omega, \quad S_t|_{t=0} = S, \tag{1.4}$$

$$v|_{t=0} = v_0 \quad \text{in } \Omega, \tag{1.5}$$

$$\varrho|_{t=0} = \varrho_0 \quad \text{in } \Omega, \tag{1.6}$$

$$\theta|_{t=0} = \theta_0 \quad \text{in } \Omega. \tag{1.7}$$

We also complete system (1.1)–(1.3) with boundary conditions which differ in dependence on the geometry of the domain Ω_t and its boundary S_t. We consider in this paper two kinds of free boundary problems with respect to the geometry of the domain Ω_t.

Problem I

This is the problem of describing the motion of an isolated mass of a viscous fluid bounded by a free boundary. In this case $\Omega_t \subset \mathbb{R}^n$ $(n = 2, 3)$ is a bounded domain with boundary $S_t = S_{2t}$ $(S_1 = \emptyset)$.

We can imagine that such an isolated mass of a fluid can be for example a drop of a liquid or a gas star. Therefore in what follows such problems will be called *drop problems* for simplicity.

For a drop problem the following boundary conditions are assumed:

$$\mathbb{T}\overline{n} - \sigma H\overline{n} = -p_0\overline{n} \quad \text{on } \widetilde{S}^T \equiv \bigcup_{t\in(0,T)} S_t \times \{t\}, \tag{1.8}$$

$$v \cdot \overline{n} = -\frac{\phi_t}{|\nabla\phi|} \quad \text{on } \widetilde{S}^T, \tag{1.9}$$

$$\varkappa\frac{\partial\theta}{\partial n} = \overline{\theta} \quad \text{on } \widetilde{S}^T \tag{1.10}$$

or

$$\varkappa\frac{\partial\theta}{\partial n} = \varkappa_a(\theta_a - \theta) \quad \text{on } \widetilde{S}^T, \tag{1.10$'$}$$

where $\overline{n}$ is the unit outward vector normal to S_t; σ is the constant coefficient of surface tension; $p_0 = p_0(x, t)$ is the external pressure; $\overline{\theta} = \overline{\theta}(x, t)$ the heat flow per unit surface; $\varkappa_a$ the coefficient of outer heat conductivity; θ_a the atmospheric temperature; $\phi(x, t) = 0$ describes the boundary S_t. In the two-dimensional case H denotes the curvature of S_t, and if $n = 3$, H is the double mean curvature of S_t expressed by

$$H\overline{n} = \Delta_{S_t}(t)x,$$

where $\Delta_{S_t}(t)$ is the Laplace–Beltrami operator on S_t.

Two cases of boundary conditions (1.8) can be taken into account: with $\sigma > 0$ and with $\sigma = 0$. If $\sigma > 0$ we say that the free boundary is *governed by surface tension*. In the absence of surface tension, that is, if $\sigma = 0$, condition (1.8) takes the form

$$\mathbb{T}\overline{n} = -p_0\overline{n} \quad \text{on } \widetilde{S}^T. \tag{1.11}$$

Condition (1.9) is called the *kinematic boundary condition*. It means that the fluid particles do not cross the free boundary.

In drop problems it is usually assumed that

$$\widetilde{f} = f + k\nabla U, \tag{1.12}$$

where $f = f(x,t)$ denotes the external force field per unit mass; k is the constant coefficient of gravitation, and $U(x,t) = \int_{\Omega_t} \frac{\varrho(y,t)}{|x-y|}\,dy$ is the self-gravitational potential.

The second term on the right-hand side of (1.12) is called the *self-gravitational force.* In the case when it is taken into account, that is, in the case of $k > 0$, equations (1.1)–(1.10) describe the motion of a viscous compressible heat-conducting, self-gravitating fluid.

The existence and stability results for Problem I are discussed in Sections 4 and 5.

Problem II

This is the *surface waves problem*, i.e. the problem of describing the motion of a fluid occupying a semifinite domain in $\mathbb{R}^n$ $(n = 2,3)$ bounded above by a free surface S_{2t} and below by the fixed part of the boundary S_t, that is, by S_1. In this case the domain Ω_t is defined as follows:

$$\Omega_t \equiv \{x = (x', x_n) \in \mathbb{R}^n : x' \in \mathbb{R}^{n-1}, -b(x') < x_n < F(x',t)\},$$

where b is a given function, and F is an unknown function.

Its free boundary part is given by

$$S_{2t} \equiv \{x = (x', x_n) \in \mathbb{R}^n : x' \in \mathbb{R}^{n-1},\ x_n = F(x',t)\},$$

and the fixed part of the boundary is defined by

$$S_1 \equiv \{x \in \mathbb{R}^n : x' \in \mathbb{R}^{n-1},\ x_n = -b(x')\}.$$

Therefore, initial condition (1.4) takes the form

$$F|_{t=0} = F_0(x'), \qquad x' \in \mathbb{R}^2. \tag{1.13}$$

For a surface waves problem the following boundary conditions are assumed:

$$\mathbb{T}\overline{n} - \sigma H\overline{n} = -p_0\overline{n} \qquad \text{on } \widetilde{S}_2^T \equiv \textstyle\bigcup_{t\in(0,T)} S_{2t} \times \{t\}, \tag{1.14}$$

$$v \cdot \overline{n} = -\frac{F_t}{\sqrt{1 + |\nabla'_x F|^2}} \qquad \text{on } \widetilde{S}_2^T, \tag{1.15}$$

$$\varkappa\frac{\partial\theta}{\partial n} = \overline{\theta} \qquad \text{on } \widetilde{S}_2^T, \tag{1.16}$$

or

$$\varkappa\frac{\partial\theta}{\partial n} = \varkappa_a(\theta_a - \theta) \qquad \text{on } \widetilde{S}_2^T, \tag{1.16'}$$

$$v = 0 \qquad \text{on } S_1 \times (0,T), \tag{1.17}$$

$$\theta = \theta_b \qquad \text{on } S_1 \times (0,T), \tag{1.18}$$

where $\overline{n}$ is the unit outward vector normal to S_t; in (1.14), $\sigma > 0$ or $\sigma = 0$; θ_b is the temperature at S_1, $\nabla'_x = \nabla_{x'}$; (1.15) is the kinematic condition in this case.

Moreover, in surface waves problems it is usually assumed that

$$\widetilde{f} = f - ge_3, \tag{1.19}$$

where $f = f(x,t)$ is an external force field per unit mass, g denotes the acceleration of gravity and $e_3 = {}^t(0,0,1)$.

Thus, the second term on the right-hand side of (1.19) is the gravity.

The results concerning Problem II are presented in Section 6.

Apart from Problems I and II which are formulated in two- or three-dimensional cases, one-dimensional free boundary problems can be studied for equations (1.1)–(1.3). In the one-dimensional case the unknown domain Ω_t has one of the following forms:

$$\Omega_t = \{x \in \mathbb{R} : 0 < x < y(t)\} \quad \text{or} \quad \Omega_t = \{x \in \mathbb{R} : y_1(t) < x < y_2(t)\},$$

where $y(t)$, $y_1(t)$ and $y_2(t)$ are unknown functions. System (1.1)–(1.3) is then considered together with initial conditions (1.4)–(1.7) and with boundary conditions (1.11), (1.9), (1.10) or (1.17)–(1.18) and (1.9)–(1.11).

System (1.1)–(1.3) with initial and boundary conditions (1.4)–(1.10) in the case of the drop problem or (1.5)–(1.7), (1.13)–(1.18) in the case of the surface waves problem describes the motion of a general viscous compressible heat-conducting fluid. In the paper we will consider some special cases of system (1.1)–(1.3).

1. *Barotropic compressible fluid.* This is a fluid with the state equation $p = p(\varrho)$. The free boundary problem for such a fluid is described by equations (1.1)–(1.2) (where the viscosity coefficients ν and μ depend only on ϱ) with conditions (1.4)–(1.6), (1.8)–(1.9) or (1.5)–(1.6), (1.13)–(1.15), (1.17).

The free boundary one-dimensional problem for such a fluid can also be examined.

2. *Incompressible fluid.* Assuming that $\varrho = \text{const}$ (let for simplicity $\varrho = 1$) equations (1.1)–(1.2) take the form of the classical Navier–Stokes equations

$$v_t + (v \cdot \nabla)v - \nu\Delta v + \nabla p = \widetilde{f} \quad \text{in } \widetilde{\Omega}^T, \tag{1.20}$$

$$\operatorname{div} v = 0 \quad \text{in } \widetilde{\Omega}^T, \tag{1.21}$$

where $p = p(x,t)$.

The incompressibility of the fluid is expressed by equation (1.21) which yields the conservation of the measure of the domain Ω_t, i.e.

$$|\Omega_t| = |\Omega| \quad \text{for } t \in (0,T). \tag{1.22}$$

From (1.22) it follows that incompressible free boundary problems can only be studied if $\Omega_t \subset \mathbb{R}^n$ with $n \geq 2$.

Problem I for an incompressible fluid takes the form of system (1.20)–(1.21) with initial conditions (1.4)–(1.5) and boundary conditions (1.8)–(1.9), where

$$\mathbb{T}(v,p) = \{T_{ij}\}_{i,j=1,\dots,n} = \{-p\delta_{ij} + \nu(v_{ix_j} + v_{jx_i})\}_{i,j=1,\dots,n}.$$

Existence, stability and asymptotic behaviour results for incompressible Problem I are presented in Section 4.

One can also consider boundary condition (1.8) with the surface tension σ depending on the temperature. Such a problem is described in Subsection 4.1.3.

Incompressible Problem II consists of equations (1.20)–(1.21) together with initial conditions (1.5), (1.13) and with boundary conditions (1.14)–(1.15), (1.17). The main results concerning an incompressible surface waves problem are discussed in Subsection 6.1.

First results for free boundary problems presented above were local existence theorems. The first local existence theorem was published in 1977 by V. A. Solonnikov [Sol4].

This result concerned Problem I. A local existence theorem for Problem II was first proved in 1980 by J. T. Beale [B1].

The following years brought other local existence theorems for equations of motion of incompressible fluids. These results can be found in [Sol5, Sol7, Sol8, Sol12, LagSol, Sol13, MogSol, Al1, Al2, T2, Ter1, Ter2, MZaj1, Scw, Wag]. Local solutions obtained in all the above papers belong either to anisotropic Sobolev spaces or to Hölder spaces. Most of the papers give also the uniqueness of local solutions.

The next step in investigating free boundary problems for incompressible Navier–Stokes equations was to obtain global existence theorems initiated in 1984 by the paper of J. T. Beale [B2]. This paper was devoted to the motion of a fluid contained in a three-dimensional infinite ocean, i.e. to Problem II. The first global existence result concerning the motion of a fixed mass of a fluid bounded by a free surface S_t appeared in the paper of V. A. Solonnikov [Sol6]. The methods used to prove the global existence results in the two papers mentioned above are completely different. Beale examined the surface waves problem after transforming it to the equilibrium domain $\Omega_1 = \{x : x' \in \mathbb{R}^2, -b(x') < x_3 < 0\}$, while Solonnikov applied Lagrangian coordinates and this way transformed the considered drop problem to the initial domain Ω.

Other global existence theorems can be found in [Sol8, Sol9, Sol10, Syl, TTan]. All global existence results for incompressible fluids are obtained for initial data sufficiently close to an equilibrium solution. Moreover, global existence is usually proved together with the stability of the equilibrium solution.

For the case of an incompressible fluid some asymptotic results as $t \to \infty$ were proved in [Sol9, Sol10, BNis].

The first free boundary problems for compressible viscous fluids were one-dimensional problems; A. V. Kazhikov was the first mathematician who concentrated on those problems. In [Kaz1] he proved a global existence and uniqueness theorem in a special case, for the one-dimensional free boundary problem for equations of motion of a viscous compressible barotropic fluid. He examined this problem in $\Omega_t = \{x \in \mathbb{R} : 0 < x < y(t)\}$, $t \in (0, T)$, where y is an unknown function. His global existence theorem was obtained for large initial data ν_0 and ϱ_0. He also proved a regularity result for the solution obtained.

In [Kaz2] A. V. Kazhikov proved a similar result in the case of viscous compressible heat-conducting fluid and under the assumption that $\Omega_t = \{x \in \mathbb{R} : y_1(t) < x < y_2(t)\}$, $t \in (0, T)$, where y_1 and y_2 are unknown functions. Other global existence and uniqueness results for the one-dimensional case are presented in [Ok, M, Nag, FPadNov, D2-D4].

A further direction in the study of one-dimensional problems is the asymptotic behaviour of a global solution. This subject has been taken up in [Nag, Ok, D2–D4, M]. In particular, it follows from the above mentioned papers that the assumption of the positivity of the external pressure is crucial to proving the asymptotic convergence of solutions to corresponding stationary solutions.

The methods developed to study one-dimensional problems have also been applied in proving global existence results in the spherically symmetric case. A spherically symmetric model is convenient for example for astrophysicists in examining stellar structures. A trouble with it is that the equations of motion written in spherical coordinates have a singular point at $r = 0$, the centre of the domain Ω_t, which is assumed to be a ball of a radius $r(t)$. For this reason the authors of papers [D1, D4, FBen, OkMak], concerning the spherically symmetric case, simplify the model by assuming that at time t the fluid occupies the domain $\Omega_t = \{x : R < |x| < r(t)\}$, where $R > 0$ is a constant. Similarly to the one-dimensional case all global existence results are obtained for large initial data.

As in the case of incompressible fluids, the first results concerning the general compressible three-dimensional problem were local existence theorems initiated in 1981 by the paper of A. Tani [T1]. This paper was followed in 1983 by the paper of P. Secchi and A. Valli [SVal]. Other papers devoted to local existence and uniqueness theorems are [S1, S2, S3, SolT1, SolT2, StZaj, Zaj2, Zaj3, Zaj5, ZZaj1, ZZaj9, ZZaj11].

The first global existence theorems for equations describing the motion of compressible fluids were proved by V. A. Solonnikov and A. Tani [SolT3] and independently by W. Zajączkowski [Zaj3, Zaj4]. Both [SolT3] and [Zaj3, Zaj4] are concerned with the barotropic case, but in [Zaj3, Zaj4] it is assumed that the pressure of the fluid has the form $p = a\varrho^\gamma$, where $a > 0$ and $\gamma > 1$ are constants. A global existence result for the more general form of pressure, i.e. $p = p(\varrho)$, has been obtained in [SolT3] and [ZZaj10]. Moreover, global existence theorems for viscous compressible heat-conducting fluids can be found in [Z1–Z2, ZZaj6, ZZaj10, ZZaj16].

All the global existence results mentioned above are concerned with a fixed mass of fluid bounded by a free surface, i.e. with Problem I . The only papers devoted to Problem II are [TanT] and [JinPad], but [TanT] merely signals the main results without giving proofs.

It should be underlined that similarly to incompressible flows, all global existence theorems for compressible fluids have been proved under the assumption that the initial data are sufficiently close to an equilibrium state. Moreover, in the papers mentioned above, the stability of the equilibrium state is also usually proved. [SolT3] also brings an asymptotic result in the barotropic case.

Some characteristic features of free boundary problems for Navier–Stokes equations are worth pointing out. First, notice that most of the existence results are obtained after transforming the free boundary problem to a problem in a fixed domain. The most frequently used transformation connects Eulerian coordinates x with Lagrangian coordinates ξ, which are defined as the initial data for the following Cauchy problem:

$$\frac{dx}{dt} = v(x,t), \quad x(0) = \xi, \quad \xi = (\xi_1, \dots, \xi_n). \tag{1.23}$$

Hence, the transformation connecting x and ξ coordinates has the form

$$x = \xi + \int_0^t u(\xi, t')\,dt' \equiv X_u(\xi, t), \tag{1.24}$$

where $u(\xi, t) = v(X_u(\xi, t), t)$.

In ξ coordinates Problems I and II have the unknown functions u, $\eta(\xi, t) = \varrho(X_u(\xi, t), t)$, $\vartheta(\xi, t) = \theta(X_u(\xi, t), t)$ (u and $q(\xi, t) = p(X_u(\xi, t), t)$ in the incompressible case) in a fixed domain $\Omega^T \equiv \Omega \times (0, T)$. For example equations (1.20)–(1.21) in

Lagrangian coordinates take the form

$$u_t - \nu\nabla_u^2 u + \nabla_u q = \widetilde{g} \quad \text{in } \Omega^T, \tag{1.25}$$

$$\nabla_u \cdot u = 0 \quad \text{in } \Omega^T, \tag{1.26}$$

where $\nabla_u = \xi_{ix}\partial_{\xi_i} = (\xi_{ix_j}\partial_{\xi_i})_{j=1,\dots,n}$, ξ_{ix_j} are the elements of the matrix ξ_x which is inverse to $x_\xi = I + \int_0^t u_\xi(\xi,t')\,dt'$ and the summation convention over repeated indices is assumed.

Most of the local existence and uniqueness theorems are obtained for free boundary problems written in Lagrangian coordinates. These solvability results are obtained in various spaces of more or less regular functions. However, considering equations (1.25)–(1.26) it is apparent that the transformation (1.24) involves nonlinear terms. For this reason we have to require the solutions to be so regular that

$$\Big\|\int_0^T u_\xi\,dt'\Big\|_{L_\infty(\Omega)} \le T^{1/2}\Big(\int_0^T \|u_\xi\|^2_{L_\infty(\Omega)}\,dt'\Big)^{1/2} < \infty. \tag{1.27}$$

Therefore, for free boundary problems for Navier–Stokes equations we cannot expect the existence of solutions as weak as for initial-boundary value problems for Navier–Stokes system in fixed domains. That is why for the considered problems we can obtain only local existence theorems or global existence theorems for initial data sufficiently close to equilibrium states. The exceptions are the one-dimensional and spherically symmetric problems for which global existence theorems with arbitrarily large initial data are proved.

Thus, there is always the question about the space of functions with the lowest possible regularity, in which we can obtain the solvability of the above free boundary problems. For example, the lowest possible regularity of a local solution of the three-dimensional incompressible Problem I with $\sigma = 0$ is such that $u \in W_r^{2,1}(\Omega^T)$, $q \in W_r^{1,0}(\Omega^T)$ for $r > 3$ (see [Sol8] or Theorem 4.2 of this paper). Then obviously (1.27) is satisfied. However, in the case of $\sigma > 0$, the above function spaces for u and q are insufficient to prove existence, since the trace of $\Delta_{S_t}(t)u$ on S does not exist for $u \in W_r^{2,1}(\Omega^T)$, $r > 3$. For this reason, the L_2-approach is applied. Thus, for both the incompressible Problem I and Problem II with $\sigma > 0$, the sharp regularity of local solutions is such that $u \in W_2^{2+\alpha,1+\alpha/2}(\Omega^T)$, $q \in W_2^{1+\alpha,1/2+\alpha/2}(\Omega^T)$, $\alpha \in (1/2,1)$ (see [Sol13], [T2] or Theorems 4.5 and 6.2 of this paper).

For comparison, for the general three-dimensional compressible Problem I (i.e. problem (1.1)–(1.3), (1.4)–(1.7), (1.8)–(1.10)) the lowest regularity of a local solution is such that u, $\vartheta \in W_2^{2+\alpha,1+\alpha/2}(\Omega^T)$, $\eta \in W_2^{1+\alpha,1/2+\alpha/2}(\Omega^T) \cap C([0,T]; W_2^{1+\alpha}(\Omega))$, $\alpha \in [3/4,1)$ (see [Z2], [ZZaj11] or Theorem 5.9 in Section 5), while in the case of a barotropic compressible fluid it suffices to examine the solvability in the above spaces with $\alpha \in (1/2,1)$ (see [SolT2] or Theorem 5.1 in Section 5). The higher regularity of a local solution in the general heat-conducting case is connected with the strong nonlinearities of the terms $\varrho c_v(\varrho,\theta)\theta_t$, $\operatorname{div}\mathbb{T}(v,p)$ and $\operatorname{div}(\varkappa(\varrho,\theta)\theta)$. For constant c_v, ν, μ, $\varkappa$ it is possible to look for a solution of problem (1.1)–(1.3), (1.4)–(1.7), (1.8)–(1.10) in spaces with $\alpha \in (1/2,1)$.

However, it should be mentioned that the methods used to prove global existence sometimes force us to look for a solution in spaces of functions of a greater regularity than in the proofs of local existence theorems.

Furthermore, it is worth mentioning that in the case of $\sigma > 0$ the Hilbert spaces $W_2^{2+\alpha,1+\alpha/2}$ were used, because existence theorems for linearized problems (both for incompressible and compressible fluid) have been proved in such spaces. Then by using the method of successive approximations, local solvability results in these spaces were obtained for nonlinear problems. One could also examine the solvability of the linearized problems and next of the nonlinear ones in the spaces $W_r^{2+\alpha,1+\alpha/2}$, $r > 2$, but this has not been done so far.

As already stated above, global existence theorems for the two- or three-dimensional Problems I and II are proved for initial data close to equilibrium states. For incompressible motions this means that the initial velocity v_0 is assumed to be small. Moreover, for problems with free boundary governed by surface tension it is assumed that the boundary of the initial domain is close to a sphere of radius $R_0 = \left(\frac{3}{4\pi}|\Omega|\right)^{1/3}$ in the case of Problem I or to a plane in the case of Problem II. Under the above assumptions, together with the global existence of solutions, the stability of the equilibrium solution is also proved, i.e. it is proved that the velocity v of the fluid remains small, the pressure is close to a certain constant and the free boundary S_t remains close to the same sphere or to the same plane as the initial boundary S for all $t > 0$.

Similarly to the incompressible case, global existence theorems for three-dimensional compressible problems are also proved for initial data close to equilibrium states. For example, under the assumptions that v_0 is small, the initial density ϱ_0 and the initial temperature θ_0 are close to certain constants, and the boundary S of the initial domain Ω is close to a certain sphere it is proved for the compressible heat-conducting drop problem with surface tension that the velocity remains small, the density and the temperature remain close to the same constants, and the free boundary remains close to the same sphere for all $t > 0$.

One of the greatest difficulties of the free boundary problems considered lies in controlling the free boundary. The differences in the ways of controlling free boundaries for cases $\sigma > 0$ and $\sigma = 0$, respectively, are thoroughly described in Section 7 (see Subsections 7.3 and 7.4). Summarizing the considerations from Subsection 7.3, we notice that if the free boundary S_t is governed by surface tension, then we assume boundary condition (1.8) which has the form of an elliptic equation. Then to prove that S_t has the same regularity as S for all $t > 0$, and that it remains close to an equilibrium sphere for all $t > 0$, we use the regularity properties of elliptic equations on S_t.

In contrast, to control a free boundary without surface tension we cannot use boundary condition (1.11). In this case proving global solvability relies on deriving a certain differential inequality which implies that the norm of a local solution is majorized by a decreasing exponential function. This allows one to show that the shape of the free boundary does not change much in time and to extend the solution step by step for all $t > 0$.

The method of controlling the free boundary via a differential inequality is described in details in Subsection 7.4. This method is applied in [Sol8] in the incompressible case and in [ZZaj10, ZZaj16, Zaj3] in the compressible case. In all these papers it is assumed that the external force vanishes.

Finally, some open problems are worth mentioning. Namely, there are no global existence results for free boundary problems for equations of motion of self-gravitating fluids bounded by a free surface without surface tension, both in the incompressible and compressible cases. The only global existence theorem with the self-gravitational force taken into account appears in the paper of V. A. Solonnikov [Sol10] and it refers to an incompressible fluid with a free boundary governed by surface tension. However, since surface tension helps to control the free boundary it seems essential to Solonnikov's proof.

The difficulties connected with the self-gravitational force in drop problems are described in Subsection 7.5.

This paper is divided into seven sections. In Section 2 we present notation, especially concerning the function spaces used. In Section 3 we describe the results relating to the one-dimensional and spherically symmetric cases. Section 4 reviews the existence results for the motion of an incompressible viscous fluid drop. Some asymptotic results are also presented. A special attention is given to the proofs of Theorems 4.7 and 4.8 since they are the first global existence and stability theorems for the drop problems. Both of them were proved by Solonnikov. Theorem 4.7 comes from [Sol8] and is concerned with the incompressible problem without surface tension, i.e. problem (1.20), (1.21), (1.11), (1.9), (1.4), (1.5). Theorem 4.8 coming from [Sol6] is a global existence and stability result for the free boundary incompressible drop problem with $\sigma > 0$.

Since some ideas from the proofs of the above theorems are used to obtain global existence and stability results for free boundary compressible problems, the main steps of these proofs are presented in Section 4 in a fairly detailed way.

Section 5 is devoted to the case of a compressible viscous fluid drop. In Subsection 5.1 existence theorems for the equations of motion of a barotropic compressible viscous fluid are described. However, the main stress has been laid in Section 5 on the presentation of the proofs of Theorems 5.11 and 5.14 which are global existence and stability theorems for the general compressible problem with $\sigma > 0$ (i.e. problem (1.1)–(1.3), (1.4)–(1.7), (1.8)–(1.10)). Theorem 5.11 was proved in [Z1]. Since the proof in [Z1] is very sketchy, it is presented in Subsection 5.2 in detail. This proof is compared in Subsection 7.2 with the proof of Theorem 4.8 in order to show differences and similarities in the approaches to the compressible and incompressible problems.

In Section 6 the main results concerning surface waves problems can be found. In particular, we describe the idea of the proof of global existence and stability for the incompressible Problem II with surface tension (see Theorem 6.5). The theorem comes from [B2] and it was the first global existence theorem for free boundary problems for Navier–Stokes equations. We also present a sketch of the proof of Theorem 6.6, which was proved in [TTan], and which yields global solvability for problem (1.20)–(1.21), (1.5), (1.13)–(1.15), (1.17) with the lowest possible regularity of solutions in the L_2-approach.

Section 7 brings an overview of the problems presented in the previous sections. In Subsection 7.1 we consider the influence of the geometry of the domain Ω_t on the approach to the corresponding free boundary problem, that is, we compare the approaches to drop and surface waves problems.

In Subsection 7.2 we compare the methods applied to obtain existence results for incompressible and compressible problems. To this end we use the proofs of Theorems 4.8 and 5.11 presented extensively in Sections 4 and 5, respectively. We conclude that most of the differences are due to the different natures of the continuity equations in both cases, i.e. equations (1.2) and (1.21), respectively.

To obtain global in time existence of solutions we derive, as usual, some estimates for the local solution of the problem considered. First, we use the conservation laws in order to estimate the L_2-norms of the solution by norms of initial data. Next, we have to derive an estimate of the solution in spaces in which we would like to have global solvability (these spaces are determnied by local existence theorems) by the L_2-norms of this solution.

The most striking (though not only) difference between the incompressible and compressible cases occurs in the way of obtaining the latter estimate. The method applied in the incompressible case is relatively simple and bases on an estimate derived earlier for the solution of an auxiliary linear problem in the proof of the local existence theorem. This method, due to Solonnikov [Sol6], cannot be applied in the compressible case. The main drawback here is the hyperbolic continuity equation (1.2). In the compressible case we have to derive the lacking estimate independently of the estimates obtained earlier for solutions of auxiliary linear parabolic problems. This missing estimate, which usually has the form of a differential inequality, is connected with very long and arduous calculations.

On the other hand, we point out that the method used to prove a global existence and stability theorem in the general compressible case is universal enough to be applied also for incompressible motions.

Both Subsections 7.3 and 7.4 are devoted to the methods of controlling the free boundary. In Subsection 7.3 we discuss the significance of surface tension in this respect, and likewise for obtaining global solvability and stability results. Then in Subsection 7.4 we describe in detail the way of controlling a free boundary which is not governed by surface tension. We underline the role played by an appropriate differential inequality.

As already mentioned, there is no global existence result for Problem I with $\sigma = 0$ and $k > 0$ (i.e. with the self-gravitational force taken into account). The aim of Subsection 7.5 is to describe the case of $\sigma > 0$ and $k > 0$ which was investigated in [Sol10] for the incompressible motion, and at the same time to present the difficulties connected with the case of $\sigma = 0$ and $k > 0$.

The results obtained for free boundary problems for Navier–Stokes equations are mainly existence and stability theorems. However, there are also some asymptotic results which are presented throughout this paper. We summarize those results in the final Subsection 7.6.

2. Notation

Let $f = f(x_1, \dots, x_n)$ be a scalar-valued function defined on a domain $\Omega \subset \mathbb{R}^n$. We denote the gradient of f by ∇f or f_x, sometimes also $\partial_x f$. By f_{xx} we denote the matrix $\{f_{x_i x_j}\}_{i,j=1,\dots,n}$.

Let now $f = f(x_1, \ldots, x_n)$ be a vector-valued function defined on $\Omega \subset \mathbb{R}^n$, i.e., $f : \Omega \to \mathbb{R}^m$, $m > 1$. Then ∇f or f_x denotes the matrix $\{f_{ix_j}\}$, where $i = 1, \ldots, m$, $j = 1, \ldots, n$. Moreover, let X be any function space. We write $f \in X$ if $f_i \in X$ for $i = 1, \ldots, n$.

Let $\Omega \subset \mathbb{R}^n$ be a domain with boundary S and let $T > 0$. Let X be a space of functions defined on Ω or S.

We let $C^k([0,T];X)$, $k \in \mathbb{N} \cup \{0\}$, denote the space of functions $u : [0,T] \to X$ with the norm given by

$$\|u\|_{C^k([0,T];X)} = \sum_{i=0}^{k} \sup_{0 \le t \le T} \left\| \frac{d^k}{dt^k} u(t) \right\|_X .$$

$C^\alpha([0,T];X)$, $0 < \alpha \le 1$, denotes the space of X-valued Hölder continuous functions with the norm

$$\|u\|_{C^\alpha([0,T];X)} = \|u\|_{C^0([0,T];X)} + \sup_{t,t' \in [0,T], t \ne t'} \frac{\|u(t) - u(t')\|_X}{|t - t'|^\alpha}.$$

By $C^k_B(Q)$ ($Q \subset \mathbb{R}^n$ is a domain) we denote the space of functions $u \in C^k(Q)$ such that $D^\sigma u$ ($0 \le |\sigma| \le k$) is bounded on Q with the norm

$$\|u\|_{C^k_B(Q)} = \max_{0 \le |\sigma| \le k} \sup_{x \in Q} |D^\sigma u(x)|,$$

where $\sigma = (\sigma_1, \ldots, \sigma_n)$ is a multiindex and $D^\sigma = \partial^{\sigma_1}_{x_1} \ldots \partial^{\sigma_n}_{x_n}$, $\partial^{\sigma_k}_{x_k} = \partial^{\sigma_k}/\partial^{\sigma_k}_{x_k}$.

Analogously, $L_r(0,T;X)$, $1 \le r < \infty$, is the space of functions u which are measurable and such that the Lebesgue integrals $\int_0^T \|u(t)\|^r_X \, dt$ are finite. The norm in this space is defined by

$$\|u\|_{L^r(0,T;X)} = \Big(\int_0^T \|u(t)\|^r_X \, dt \Big)^{1/r}.$$

$C^l(\Omega)$ (where $l > 0$ and l is noninteger) denotes the Hölder space of functions u defined on Ω with the norm given by

$$\|u\|_{C^l(\Omega)} = \sum_{|\gamma| \le [l]} \sup_{x \in \Omega} |D^\gamma u(x)| + \sum_{|\gamma| = [l]} \sup_{x,x' \in \Omega,\, x \ne x'} \frac{|D^\gamma u(x) - D^\gamma u(x')|}{|x - x'|^{l - [l]}}, \tag{2.1}$$

where $\gamma = (\gamma_1, \ldots, \gamma_n)$ is a multiindex.

Analogously, we let $C^l(\overline{\Omega})$ denote the Hölder space of functions u defined on $\overline{\Omega}$ with the norm (2.1).

If $u \in C^l(\Omega)$ then u can be defined on S in such a way that the resulting function belongs to $C^l(\overline{\Omega})$.

The space $C^l(S)$ can be defined similarly to $C^l(\Omega)$ and $C^l(\overline{\Omega})$ by using local coordinates and partitions of unity.

In what follows we shall use the notation: $\Omega^T = \Omega \times (0,T)$, $S^T = S \times (0,T)$, $T > 0$.

We denote by $C^{l_1,l_2}(\Omega^T)$ (where $l_1, l_2 > 0$ and l_1, l_2 are noninteger) the anisotropic Hölder space of functions u defined on Ω^T. The norm in this space is given as follows:

$$\|u\|_{C^{l_1,l_2}(\Omega^T)} = \sum_{|\gamma|\le[l_1]} \sup_{(x,t)\in\Omega^T} |D_x^\gamma u(x,t)| + \sum_{j=0}^{[l_2]} \sup_{(x,t)\in\Omega^T} |D_t^j u(x,t)| + \sum_{|\gamma|=[l_1]} \sup_{x,x',t} \frac{|D_x^\gamma u(x,t) - D_{x'}^\gamma u(x',t)|}{|x-x'|^{l_1-[l_1]}} + \sup_{x,t,t'} \frac{|D_t^{[l_2]} u(x,t) - D_{t'}^{[l_2]} u(x,t')|}{|t-t'|^{l_2-[l_2]}}, \tag{2.2}$$

where $D_t^j = \partial_t^j$, $D_x^\gamma = \partial_{x_1}^{\gamma_1}\dots\partial_{x_n}^{\gamma_n}$ and $\gamma = (\gamma_1,\dots,\gamma_n)$.

In the case of integer l_1, $C^{l_1,l_2}(\Omega^T)$ is the space with the norm (2.2), where the third term on the right-hand side of (2.2) is omitted. Analogously, in the case of integer l_2 the fourth term on the right-hand side of (2.2) is omitted.

$C^{l_1,l_2}(\overline{\Omega}\times[0,T])$ denotes the Hölder space of functions u defined on $\overline{\Omega}\times[0,T]$ with the norm (2.2).

The space $C^{l_1,l_2}(S^T)$ is defined similarly by using local coordinates and partitions of unity.

In Section 4 the space $\widetilde{C}_\beta^{1+\alpha}(\Omega^T)$, where $\alpha,\beta\in(0,1)$, $\Omega\subset\mathbb{R}^3$, occurs. This is the space of functions u with the finite norm

$$\|u\|_{\widetilde{C}_\beta^{1+\alpha}(\Omega^T)} = \sup_{(x,t)\in\Omega^T} |u(x,t)| + \sum_{i=1}^3 \left\| \frac{\partial u}{\partial x_i} \right\|_{C^{\alpha,\alpha/2}(\Omega^T)} + \sup_{x,x',t,t'} \frac{|u(x,t)-u(x',t)-u(x,t')+u(x',t')|}{|x-x'|^\beta |t-t'|^{(1+\alpha-\beta)/2}}.$$

Next, $W_r^l(\Omega)$, $l\in\mathbb{R}_+\cup\{0\}$, $1\le r<\infty$, is the Sobolev–Slobodetskiĭ space with the norm $\|u\|_{W_r^l(\Omega)}$ defined by

$$\|u\|_{W_r^l(\Omega)}^r = \sum_{0\le|\gamma|\le[l]} \int_\Omega |u(x)|^r\,dx + \sum_{|\gamma|=[l]} \int_\Omega\int_\Omega \frac{|D^\gamma u(x)-D^\gamma u(x')|^r}{|x-x'|^{n+r(l-[l])}}\,dx\,dx', \tag{2.3}$$

where in the case of integer l the second term on the right-hand side of (2.3) is omitted.

$W_{r,\mathrm{loc}}^l(\Omega)$ is the space of functions $u\in W_r^l(\Omega')$ for any $\Omega'\subset\subset\Omega$.

The space $W_r^l(S)$, where $S=\partial\Omega$, is defined in a standard way by means of local coordinates and partitions of unity.

$\Gamma_r^l(\Omega)$, $l\in\mathbb{R}_+$, denotes the space of functions u with the norm

$$\|u\|_{\Gamma_r^l(\Omega)} = \sum_{i\le[l/2]} \|\partial_t^i u\|_{W_2^{l-2i}(\Omega)}.$$

Now, let us introduce the differences

$$\Delta_i(h)u(x) = u(x+he_i) - u(x),$$

where $h\in\mathbb{R}$, $x\in\mathbb{R}^n$ and e_i, $i=1,\dots,n$, are the standard unit vectors. Then we define inductively the m-difference

$$\Delta_i^m(h)u(x) = \Delta_i(h)(\Delta_i^{m-1}(h)u(x)) = \sum_{j=0}^m (-1)^{m-j} c_{jm} u(x+jhe_i),$$

where $c_{jm} = m!/(j!(m-j)!)$.

Similarly, we introduce the differences

$$\Delta(y)u(x) = u(x+y) - u(x), \qquad x, y \in \mathbb{R}^n,$$
$$\Delta^m(y)u(x) = \Delta(y)(\Delta^{m-1}(y)u(x)).$$

Since $\Delta(x-y)u(y) = u(x) - u(y)$ we have

$$\Delta^m(x-y)u(y) = \sum_{i=1}^n \Delta^m((x-y)\cdot e_i)u(y) = \sum_{i=1}^n \Delta_i^m(h)u(y),$$

where $h = (x-y)\cdot e_i$.

Now, we define the Besov space $B_r^l(\mathbb{R}^n)$ by introducing the norm

$$\|u\|_{B_r^l(\mathbb{R}^n)} = \|u\|_{L_r(\mathbb{R}^n)} + \sum_{i=1}^n \bigg(\int_0^{h_0} dh \int_{\mathbb{R}^n} dx\, \frac{|\Delta_i^m(h)\partial_{x_i}^k u|^r}{h^{1+(l-k)r}}\bigg)^{1/r},$$

where $m > l-k$; m, $k \in \mathbb{N}\cup\{0\}$, $l \in \mathbb{R}_+$, $l \notin \mathbb{Z}$.

All norms $\|u\|_{B_r^l(\mathbb{R}^n)}$ are equivalent for all m, k satisfying $m > l-k$ (see [Gol]). Moreover, the norms of $B_r^l(\mathbb{R}^n)$ and $W_r^l(\mathbb{R}^n)$ are equivalent for $l \notin \mathbb{Z}$.

We also define the following norms:

$$\|u\|_{\widetilde{B}_r^l(\mathbb{R}^n)} = \|u\|_{L_r(\mathbb{R}^n)} + \bigg(\int_{\mathbb{R}^n} dx \int_{\mathbb{R}^n} dy\, \frac{|\Delta^m(x-y)\partial_y^k u(y)|^r}{|x-y|^{n+r(l-k)}}\bigg)^{1/r},$$

where $m > l-k$, $\partial_y^k u = \sum_{|\alpha|=k} D_y^\alpha u$, and

$$\|u\|_{\widetilde{W}_r^l(\mathbb{R}^n)} = \|u\|_{L_r(\mathbb{R}^n)} + \bigg(\int_{\mathbb{R}^n} dx \int_{\mathbb{R}^n} dy\, \frac{|\Delta(x-y)\partial_y^{[l]} u(y)|^r}{|x-y|^{n+r(l-[l])}}\bigg)^{1/r}.$$

The last norm coincides with the norm of the Sobolev–Slobodetskiĭ space $W_r^l(\mathbb{R}^n)$ given by (2.3). Moreover, it can be shown that the spaces $W_r^l(\mathbb{R}^n)$, $\widetilde{W}_r^l(\mathbb{R}^n)$, $\widetilde{B}_r^l(\mathbb{R}^n)$ and $B_r^l(\mathbb{R}^n)$ (with $h_0 = \infty$) all coincide for $l \notin \mathbb{Z}$ and have equivalent norms.

In many papers referred to in what follows, various imbedding theorems and interpolation inequalities in Sobolev and Besov spaces are used. One of the most applicable lemmas is as follows.

LEMMA 2.1 (see [BesIlN]). *Let $l \in \mathbb{R}_+$, $\varrho \in \mathbb{R}_+\cup\{0\}$ and $1 \le r \le q \le \infty$. The following imbedding holds*:

$$W_r^l(\mathbb{R}^n) \subset W_q^\varrho(\mathbb{R}^n) \qquad \text{for } n/r - n/q + \varrho < l.$$

Moreover, *the following interpolation inequality holds*:

$$\|u\|_{W_q^\varrho(\mathbb{R}^n)} \le \varepsilon^{1-\kappa}\|u\|_{W_r^l(\mathbb{R}^n)} + c\varepsilon^{-\kappa}\|u\|_{L_r(\mathbb{R}^n)},$$

where $\kappa = (1/l)(n/r - n/q + \varrho)$.

The above lemma also holds for spaces of functions defined in domains $\Omega \subset \mathbb{R}^n$ with sufficiently regular boundary S.

$W_r^l(0,T;X)$, where l is noninteger, $1 \le r < \infty$, is the space of X-valued functions u in W_r^l with the norm defined by

$$\|u\|^r_{W^l_r(0,T;X)} = \|u\|^r_{W^{[l]}_r(0,T;X)} + \int_0^T\int_0^T \frac{\|u(t)-u(t')\|^r_X}{|t-t'|^{1+r(l-[l])}}\,dt\,dt',$$

where $\|u\|^r_{W^{[l]}_r(0,T;X)} = \sum_{i=0}^{[l]}\int_0^T \|\frac{d^i}{dt^i}u(t)\|^r_X\,dt$.

We denote by $W^{l,m}_r(\Omega^T)$, where $l,m\in\mathbb{R}_+\cup\{0\}, 1\le r<\infty$, the anisotropic Sobolev–Slobodetskiĭ space with the norm

$$\begin{aligned}(2.4)\qquad \|u\|^r_{W^{l,m}_r(\Omega^T)} = {} & \int_{\Omega^T}|u(x,t)|^r\,dx\,dt + \sum_{0<|\gamma|\le[l]}\int_{\Omega^T}|D^\gamma_x u(x,t)|^r\,dx\,dt \\ & + \sum_{0<i\le[m]}\int_{\Omega^T}|D^i_t u(x,t)|^r\,dx\,dt \\ & + \sum_{|\gamma|=[l]}\int_0^T dt\int_\Omega\int_\Omega \frac{|D^\gamma_x u(x,t)-D^\gamma_{x'}u(x',t)|^r}{|x-x'|^{n+r(l-[l])}}\,dx\,dx' \\ & + \int_\Omega dx\int_0^T\int_0^T \frac{|D^{[m]}_t u(x,t)-D^{[m]}_{t'}u(x,t')|^r}{|t-t'|^{1+r(m-[m])}}\,dt\,dt'.\end{aligned}$$

In (2.4), $D^i_t=\partial^i_t$, $D^\gamma_x=\partial^{\gamma_1}_{x_1}\dots\partial^{\gamma_n}_{x_n}$. In the case of integer l the fourth term on the right-hand side of (2.4) is omitted, and in the case of integer m the fifth term is omitted.

The space $W^{l,m}_r(S^T)$, where $S=\partial\Omega$, is defined in a standard way by using local coordinates and partitions of unity.

$W^{l,l/2}_{2,\varkappa}(\Omega^T)$, $l\in\mathbb{R}_+$, denotes the space of functions u with the norm

$$\|u\|_{W^{l,l/2}_{2,\varkappa}(\Omega^T)} = \|u\|_{W^{l,l/2}_2(\Omega^T)} + \bigg(\sum_{|\gamma|=[l]}\int_0^T \frac{\|D^\gamma_{x,t}u\|^2_{L_2(\Omega)}}{t^{2\varkappa}}\,dt\bigg)^{1/2},$$

$\varkappa\in(0,1)$, where $D^\gamma_{x,t}=\partial^{\gamma_0}_t\partial^{\gamma_1}_{x_1}\dots\partial^{\gamma_n}_{x_n}$, $|\gamma|=2\gamma_0+\gamma_1+\dots+\gamma_n$. The above space occurs in [Zaj2].

In Section 5 the following notation is used:

$$\begin{aligned}(\|u\|^{(2+\alpha,1+\alpha/2)}_{\Omega^T})^2 &= \|u\|^2_{W^{2+\alpha,1+\alpha/2}_2(\Omega^T)} + T^{-\alpha}\Big(\|u_t\|^2_{L_2(\Omega^T)} \\ &\quad + \sum_{|\gamma|=2}\|D^\gamma_x u\|^2_{L_2(\Omega^T)}\Big) + \sup_{t\le T}\|u(\cdot,t)\|^2_{W^{1+\alpha}_2(\Omega)}, \\ (\|u\|^{(\alpha,\alpha/2)}_{Q^T})^2 &= \|u\|^2_{W^{\alpha,\alpha/2}_2(Q^T)} + T^{-\alpha}\|u\|^2_{L_2(Q^T)},\end{aligned}$$

where $0<\alpha<1$, $Q\in\{\Omega,S\}$.

In Section 6, we denote by $K^l(\Omega\times(0,T))$, $l\ge0$, the space $W^{l,l/2}_2(\Omega\times(0,T))$. In particular, if $T=\infty$, the interval $(0,\infty)$ is written as $\mathbb{R}_+$.

$K^l_{(0)}(\Omega\times\mathbb{R}_+)$ denotes the subspace of $K^l(\Omega\times\mathbb{R}_+)$ consisting of functions u so that $\partial^k_t u(\cdot,0)=0$ for $2k<l-1$. This space has the property that if $u\in K^l_{(0)}(\Omega\times\mathbb{R}_+)$ then the extension of u by zero for $t<0$ belongs to $K^l(\Omega\times\mathbb{R})$.

Let χ be the characteristic function of the set $\{(\xi,\tau)\in\mathbb{R}^2\times\mathbb{R} : |\xi|\le 1,\ |\tau|\le\tau_0\}$. Let $S_{1F}=\{x\in\mathbb{R}^3 : x_3=0\}$. Then

$$\begin{aligned}\widetilde{K}^l_{(0)}(S_{1F}\times\mathbb{R}_+)=\{u : e^{-t}u\in K^l_{(0)}(S_{1F}\times\mathbb{R}_+),\\ ((1-\chi)u^{\wedge\wedge})^{\vee\vee}\in K^l(S_{1F}\times\mathbb{R}),\\ (|\xi|^2+|\tau|)^{1/2}\chi u^{\wedge\wedge}\in L_2(\mathbb{R}^2\times\mathbb{R})\},\end{aligned}$$

where $u^{\wedge\wedge}$ denotes the space-time Fourier transform of u.

Let now ω be a smooth increasing function such that $\omega(t)=0$ for $t\le 1$ and $\omega(t)=1$ for $t>2$. Then

$$\widetilde{K}^l(S_{1F}\times\mathbb{R}_+)=\{u : (1-\omega)u\in K^l(S_{1F}\times\mathbb{R}_+),\ \omega u\in\widetilde{K}^l_{(0)}(S_{1F}\times\mathbb{R}_+)\}.$$

The above definition is independent of the choice of ω.

Also, the following interpolation lemma is used.

Lemma 2.2 (see [BesIlN]). *Let $u\in W^{l,m}_r(\mathbb{R}^n\times(0,T))$, $l,m\in\mathbb{R}_+$. If $q\ge r$ and*

$$\kappa=\sum_{i=1}^n\left(\gamma_i+\frac{1}{r}-\frac{1}{q}\right)\frac{1}{l}+\left(\delta+\frac{1}{r}-\frac{1}{q}\right)\frac{1}{m}<1,$$

then for all $\varepsilon\in(0,1)$,

$$\|D^\delta_t D^\gamma_x u\|_{L_q(\mathbb{R}^n\times(0,T))}\le\varepsilon^{1-\kappa}\|u\|_{W^{l,m}_r(\mathbb{R}^n\times(0,T))}+c\varepsilon^{-\kappa}\|u\|_{L_r(\mathbb{R}^n\times(0,T))},$$

where $\gamma=(\gamma_1,\dots,\gamma_n)$.

The above lemma also holds for spaces of functions defined in domains $\Omega\subset\mathbb{R}^n$ with sufficiently regular boundary S.

3. One-dimensional and spherically symmetric free boundary problems

3.1. One-dimensional case. In this section we will describe the results concerning the one-dimensional free boundary problem for the compressible Navier–Stokes system. It has been discussed in [Kaz1, Kaz2, Ok, M, Nag, D2-D4, FPadNov]. Below we present separately the cases of a viscous barotropic fluid and of a general viscous fluid.

3.1.1. *The motion of a barotropic viscous fluid.* The relevant free boundary problem is as follows:

$$\varrho(v_t+vv_x)-\mu v_{xx}+p_x=\varrho f \qquad \text{for } 0<x<y(t),\ t\in(0,T), \tag{3.1}$$

$$\varrho_t+v\varrho_x+\varrho v_x=0 \qquad \text{for } 0<x<y(t),\ t\in(0,T), \tag{3.2}$$

$$v|_{x=0}=0 \qquad \text{for } t\in(0,T), \tag{3.3}$$

$$\frac{dy(t)}{dt}=v(y(t),t),\ (\mu v_x-p)|_{x=y(t)}=-P \qquad \text{for } t\in(0,T), \tag{3.4}$$

$$v|_{t=0}=v_0(x),\quad \varrho|_{t=0}=\varrho_0(x) \qquad \text{for } 0<x<y(0)=1, \tag{3.5}$$

where $v=v(x,t)$ is the velocity, $\varrho=\varrho(x,t)$ the density of the fluid, and $y=y(t)$ the unknown function; $T>0$, $p=\varrho^\gamma$ with $\gamma\ge 1$ is the pressure; $f=f(x,t)$ is the external force per unit mass acting on the fluid; P is the external constant pressure; μ is the viscosity coefficient and $0 < \varrho_1 \le \varrho_0(x) \le \varrho_2 < \infty$.

Problem (3.1)–(3.5) has been examined by Kazhikov [Kaz1] in the case of $f \equiv 0$ and $P \equiv 0$. The main results are a global existence theorem for large initial data ν_0 and ϱ_0 of class W_2^1 and a regularity result. Kazhikov uses Lagrangian coordinate ξ defined as the initial data for the Cauchy problem

$$\frac{dx}{dt} = v(x,t), \qquad x(0) = \xi. \tag{3.6}$$

Integrating (3.6) we get

$$x = \xi + \int_0^t u(\xi,t')\,dt' \equiv X_u(\xi,t),$$

where $u(\xi,t) = v(X_u(\xi,t),t)$. In coordinates (ξ,t) problem (3.1)–(3.5) with $f \equiv 0$ and $P \equiv 0$ takes the form

$$\eta u_t - \mu J(Ju_\xi)_\xi + Jp_\xi = 0 \qquad \text{for } \xi \in \Omega,\ t \in (0,T), \tag{3.7}$$

$$\eta_t + \eta J u_\xi = 0 \qquad \text{for } \xi \in \Omega,\ t \in (0,T), \tag{3.8}$$

$$u|_{\xi=0} = 0, \qquad \text{for } t \in (0,T), \tag{3.9}$$

$$\mu J(1,t) u_\xi|_{\xi=1} - p(1,t) = 0 \qquad \text{for } t \in (0,T), \tag{3.10}$$

$$u|_{t=0} = v_0(\xi), \quad \eta|_{t=0} = \varrho_0(\xi) \qquad \text{for } \xi \in \Omega, \tag{3.11}$$

where $\eta(\xi,t) = \varrho(X_u(\xi,t),t)$, $J(\xi,t) = \xi_x(\xi,t) = (1 + \int_0^t u_\xi(\xi,t')\,dt')^{-1}$, $p = \eta^\gamma$, $\gamma \ge 1$, $\Omega = (0,1)$. Equation (3.8) implies

$$\begin{aligned} \eta(\xi,t) &= \varrho_0(\xi)\exp\Big[-\int_0^t Ju_\xi(\xi,t')\,dt'\Big] \\ &= \varrho_0(\xi)\exp\Big[-\int_0^t \Big(1+\int_0^{t'} u_\xi(\xi,t'')\,dt''\Big)^{-1} u_\xi(\xi,t')\,dt'\Big] \\ &= \varrho_0(\xi)\exp\Big\{-\int_0^t \Big[\ln\Big(1+\int_0^{t'} u_\xi(\xi,t'')\,dt''\Big)\Big]_{,t'}\,dt'\Big\} = \varrho_0(\xi)J(\xi,t). \end{aligned} \tag{3.12}$$

Hence problem (3.7)–(3.11) can be rewritten as

$$\varrho_0 u_t - \mu(Ju_\xi)_\xi + p_\xi = 0, \qquad \xi \in \Omega,\ t \in (0,T), \tag{3.13}$$

$$J(\xi,t) = \Big(1 + \int_0^t u_\xi(\xi,t')\,dt'\Big)^{-1}, \qquad \xi \in \Omega,\ t \in (0,T), \tag{3.14}$$

$$u|_{\xi=0} = 0, \qquad t \in (0,T), \tag{3.15}$$

$$\mu J(1,t) u_\xi|_{\xi=1} - p(1,t) = 0, \qquad t \in (0,T), \tag{3.16}$$

$$u|_{t=0} = v_0(\xi), \quad J|_{t=0} = 1, \qquad \xi \in \Omega, \tag{3.17}$$

where $p(\xi,t) = \varrho_0^\gamma(\xi) J^\gamma(\xi,t), \gamma \ge 1$.

The existence of a local in time solution of the above problem can be proved by the method of successive approximations. This method will be presented in Sections 4–6 for two-and three-dimensional problems. Now, we formulate a global existence theorem for problem (3.13)–(3.17) and we present a sketch of the proof. Both can be found in [Kaz1].

THEOREM 3.1. *If* $v_0 \in W_2^1(\Omega)$, $\varrho_0 \in W_2^1(\Omega)$ *and* $0 < \varrho_1 \le \varrho_0(\xi) \le \varrho_2 < \infty$, *then for every* $0 < T < \infty$ *there exists a unique solution of problem* (3.13)–(3.17) *with the properties*: $\sup_{0\le t\le T} \|u(t)\|_{W_2^1(\Omega)} \le K_1$, $\|u_{\xi\xi}\|_{L_2(\Omega^T)} + \|u_t\|_{L_2(\Omega^T)} \le K_2$, $\sup_{0\le t\le T}[\|J_t\|_{L_2(\Omega)} + \|J_\xi\|_{L_2(\Omega)}] \le K_3$, $0 < m_0 \le J(\xi,t) \le M_0 < \infty$, *where* $\Omega^T = \Omega \times (0,T)$, K_i $(i = 1,\dots,3)$, m_0 *and* M_0 *are constants.*

Sketch of proof. The proof relies on deriving the above estimates with constants K_1, K_2, K_3, m_0 and M_0 depending on the initial data, T, μ and γ, but independent of the time of local existence. Therefore the local solution can be extended to the interval $[0,T]$.

STEP 1. Let $\gamma > 1$. Multiplying (3.13) by u and integrating with respect to ξ yields

$$\frac{1}{2}\frac{d}{dt}\int_0^1 \varrho_0 u^2\,d\xi + \mu\int_0^1 J u_\xi^2\,d\xi = \int_0^1 p u_\xi\,d\xi. \tag{3.18}$$

Hence

$$\sup_{0\le t\le T}\int_0^1 \varrho_0(\xi)u^2(\xi,t)\,d\xi \le \int_0^t \varrho_0(\xi)v_0^2(\xi)\,d\xi + \frac{2}{\gamma-1}\int_0^1 \varrho_0^\gamma(\xi)\,d\xi \equiv C_1 < \infty. \tag{3.19}$$

In the case $\gamma = 1$,

$$\frac{1}{2}\frac{d}{dt}\int_0^1 \varrho_0(\xi)w^2(\xi,t)\,d\xi + \mu\int_0^1 J(\xi,t)w_\xi^2(\xi,t)\,d\xi = 0, \tag{3.20}$$

where $w(\xi,t) \equiv u(\xi,t) - \frac{1}{\mu}\int_0^\xi \varrho_0(\xi')\,d\xi'$. This implies

$$\sup_{0\le t\le T}\int_0^1 \varrho_0(\xi)u^2(\xi,t)\,d\xi \le C'\Big(\int_0^1 \varrho_0(\xi)v_0^2(\xi)\,d\xi + \|\varrho_0\|^2_{L_1(\Omega)}\Big) \equiv C_2 < \infty. \tag{3.21}$$

STEP 2

LEMMA 1. *There exist positive constants* m_0 *and* M_0 $(m_0 < \infty, M_0 < \infty)$ *such that*

$$m_0 \le J(\xi,t) \le M_0, \tag{3.22}$$

where m_0 *and* M_0 *are constants depending on* ϱ_1, ϱ_2, γ, μ, T *and the right-hand sides of* (3.19) *if* $\gamma > 1$ *or* (3.21) *if* $\gamma = 1$.

In order to get (3.22) one has to use the following equality for the function $I(\xi,t) = J^\gamma(\xi,t)$, which results from (3.13)–(3.14):

$$\frac{\partial}{\partial\xi}\Big[\ln I(\xi,t) + \frac{\gamma}{\mu}\varrho_0^\gamma(\xi)\int_0^t I(\xi,t')\,dt'\Big] = \frac{\gamma}{\mu}\,\varrho_0(\xi)[v_0(\xi) - u(\xi,t)]. \tag{3.23}$$

After some calculations the following form of $I(\xi,t)$ can be obtained:

$$I(\xi,t) = \frac{\exp\{(\gamma/\mu)\int_\xi^1 \varrho_0(\xi')[u(\xi',t) - v_0(\xi')]\,d\xi'\}}{1 + (\gamma/\mu)\varrho_0^\gamma(\xi)\int_0^t \exp\{(\gamma/\mu)\int_\xi^1 \varrho_0(\xi')[u(\xi',t) - v_0(\xi')]\,d\xi'\}\,dt'}. \tag{3.24}$$

By using the inequality

$$\Big|\int_\xi^1 \varrho_0(\xi')u(\xi',t)\,d\xi'\Big| \le \sup_{0\le t\le T}\Big(\int_0^1 \varrho_0(\xi')u^2(\xi',t)\,d\xi'\Big)^{1/2}\Big(\int_0^1 \varrho_0(\xi')\,d\xi'\Big)^{1/2},$$

estimate (3.22) follows from (3.21) and (3.24).

Step 3. Equality (3.18) and estimates (3.19), (3.22) in the case $\gamma > 1$, and equality (3.20) and estimates (3.21)–(3.22) if $\gamma = 1$ give

$$\|u_\xi\|_{L_2(\Omega^T)} \le C_3. \tag{3.25}$$

Step 4. Since $J_t = -J^2 u_\xi$, estimates (3.22) and (3.25) imply

$$\|J_t\|_{L_2(\Omega^T)} \le C_4 < \infty.$$

Moreover, by (3.23),

$$\sup_{0\le t\le T}\|J_\xi\|_{L_2(\Omega)} \le C_5 < \infty. \tag{3.26}$$

Next, the following inequality is derived:

$$\sup_{0\le t\le T}\int_0^t J\Big(u_\xi - \frac{1}{\mu}\varrho_0^\gamma J^{\gamma-1}\Big)^2 d\xi + \int_0^T\int_0^1 \Big[\Big(Ju_\xi - \frac{1}{\mu}p\Big)_{,\xi}\Big]^2 d\xi\,dt \le C_6 < \infty, \tag{3.27}$$

which together with the boundedness of J and ϱ_0 yields

$$\sup_{0\le t\le T}\|u_\xi\|_{L_2(\Omega)} \le C_7 < \infty. \tag{3.28}$$

Hence

$$\sup_{0\le t\le T}\|J_t\|_{L_2(\Omega)} \le C_8. \tag{3.29}$$

Furthermore, since $(Ju_\xi - \frac{1}{\mu}p)_{,\xi} = \frac{1}{\mu}u_t$, inequality (3.27) implies

$$\|u_t\|_{L_2(\Omega^T)} \le C_9 < \infty. \tag{3.30}$$

Step 5. Estimate (3.27) also yields

$$\int_0^T\int_0^1 (Ju_{\xi\xi} + J_\xi u_\xi)^2\,d\xi\,dt \le C_{10} < \infty. \tag{3.31}$$

In view of (3.22), (3.25) and (3.26), from (3.31) it follows that

$$\|u_{\xi\xi}\|_{L_2(\Omega^T)} \le C_{11} < \infty. \tag{3.32}$$

All C_i in the above estimates depend on the same quantities as m_0 and M_0. Estimates (3.19), (3.21), (3.22), (3.26), (3.28)–(3.30), (3.32) give the assertion of the theorem. ■

For more regular initial data v_0 and ϱ_0 the following regularity result is also proved in [Kaz1].

Theorem 3.2. *Let $v_0 \in C^{2+\alpha}(\overline{\Omega})$, $\varrho_0 \in C^{1+\alpha}(\overline{\Omega})$, $0 < \alpha < 1$ and let the following compatibility conditions be satisfied*: $v_0(0) = 0$, $(\mu v_0'' - \gamma\varrho_0^{\gamma-1}\varrho_0')|_{\xi=0} = 0$, $(\mu v_0' - \varrho_0^\gamma)|_{\xi=1} = 0$. *Then* $u \in C^{2+\alpha,1+\alpha/2}(\overline{\Omega}\times[0,T])$, $J \in C^{1+\alpha,1+\alpha/2}(\overline{\Omega}\times[0,T])$, $J_t \in C^{1+\alpha,\alpha/2}(\overline{\Omega}\times[0,T])$.

Sometimes it is convenient to write problem (3.1)–(3.5) in Lagrangian mass coordinate. Assuming that $\int_0^1 \varrho_0(x)\,dx = 1$, which means that the total mass of the fluid is equal to 1, Lagrangian mass coordinate is given by

$$\xi = \int_0^x \varrho(x', t)\,dx' \tag{3.33}$$

and its inverse transformation is defined by

$$x = \int_0^\xi \overline{v}(\xi', t)\,d\xi',$$

where $\overline{v}(\xi, t) = 1/\eta(\xi, t)$ is the specific volume and η denotes the density ϱ written in coordinate ξ. In coordinate (3.33) problem (3.1)–(3.5) takes the form

$$u_t - (\mu u_\xi/\overline{v} - p)_\xi = g \quad \text{for } \xi \in \Omega,\ t \in (0, T), \tag{3.34}$$

$$\overline{v}_t - u_\xi = 0 \quad \text{for } \xi \in \Omega,\ t \in (0, T), \tag{3.35}$$

$$u|_{\xi=0} = 0, \quad (\mu u_\xi/\overline{v} - q)|_{\xi=1} = -P \quad \text{for } t \in (0, T), \tag{3.36}$$

$$u|_{t=0} = v_0(\xi), \quad \overline{v}|_{t=0} = 1/\varrho_0(\xi) \quad \text{for } \xi \in \Omega, \tag{3.37}$$

where u denotes v and g denotes f written in coordinate ξ, $\Omega = (0, 1)$.

In [M] Mucha proves a global existence and uniqueness theorem for problem (3.34)–(3.37) under the assumptions that $0 < \varrho_1 \le \varrho_0(x) \le \varrho_2 < \infty$, $\int_0^1 \varrho_0(x)\,dx = 1$, $p = a\varrho^\gamma$, $a > 0$, $\gamma > 1$; $P > 0$, $f \le 0$ and $f(\xi) = \varphi'(\xi)$, where $\varphi \in C^2(\mathbb{R})$.

The solution obtained is such that $\overline{v}_\xi$, $u_\xi \in L_\infty((0, \infty); L_2(\Omega))$. Moreover, in [M] the following asymptotic theorem is proved.

THEOREM 3.3. *Let $1/\varrho_0 \in W_2^2(\Omega)$, $v_0 \in W_2^3(\Omega)$, let $\|g'\|_{L_\infty}$ be sufficiently small and let the above conditions on ϱ_0, p, P and f be satisfied. Then $u_t \in W_2^{2,1}(\Omega \times (0, \infty))$ and*

$$\|\overline{v} - w\|_{W_2^2(\Omega)} + \|u\|_{W_2^3(\Omega)} \le ce^{-\alpha t}, \tag{3.38}$$

where $\alpha > 0$, $w = \varrho_e^{-1}(x)$ and $\varrho_e = \varrho_e(\xi)$ is the stationary solution of problem (3.1)–(3.5), *i.e. ϱ_e satisfies*

$$(a\varrho_e^\gamma)_x = \varrho_e f,$$
$$a\varrho_e^\gamma|_{y(\infty)} = P,$$
$$\int_0^{y(\infty)} \varrho_e(x)\,dx = 1.$$

The assumption that $P > 0$ is crucial to proving estimate (3.38).

The free boundary problem for system (3.1)–(3.2) with initial conditions (3.5) is also considered in [Ok], where at $x = 0$ boundary condition (3.3) is assumed, and ϱ is assumed to vanish on the free boundary. Okada proves the existence of a global weak solution. A similar problem in the spherically symmetric case has been examined in [OkMak]; it will be described more thoroughly in Section 3.2.

3.1.2. *The motion of a general viscous fluid.* First, we will concentrate on the paper of Kazhikov [Kaz2] which is concerned with the one-dimensional motion of a viscous polytropic heat-conducting ideal fluid. Such a motion is described by the system

$$\varrho(v_t + vv_x) - \mu v_{xx} + p_x = 0 \quad \text{for } y_0(t) < x < y_1(t),\ t \in (0,T), \tag{3.39}$$

$$\varrho_t + v\varrho_x + v_x\varrho = 0 \quad \text{for } y_0(t) < x < y_1(t),\ t \in (0,T), \tag{3.40}$$

$$c_v\varrho(\theta_t + v\theta_x) - \varkappa\theta_{xx} - \mu v_x^2 + pv_x = 0, \quad \text{for } y_0(t) < x < y_1(t),\ t \in (0,T), \tag{3.41}$$

$$(\mu v_x - p)|_{x=y_i(t)} = -P(t) \quad i = 0,1,\ t \in (0,T), \tag{3.42}$$

$$\theta_x|_{x=y_i(t)} = 0 \quad i = 0,1,\ t \in (0,T), \tag{3.43}$$

$$\frac{dy_i(t)}{dt} = v(y_i(t),t) \quad i = 0,1,\ t \in (0,T), \tag{3.44}$$

$$v|_{t=0} = v_0, \quad \theta|_{t=0} = \theta_0, \quad \varrho|_{t=0} = \varrho_0 \quad \text{for } 0 < x < 1, \tag{3.45}$$

where $p = R\varrho\theta$, $\theta = \theta(x,t)$ is the temperature of the fluid; $R > 0$ is the gas constant; $c_v > 0$ is the constant specific heat at constant volume; $\varkappa > 0$ is the constant coefficient of heat conductivity, and $y_i = y_i(t)$ $(i = 1,2)$ are unknown functions.

Similarly to the barotropic case, problem (3.39)–(3.45) can be written in Lagrangian coordinates as follows:

$$\varrho_0 u_t - \mu(Ju_\xi)_\xi + p_\xi = 0 \quad \text{for } \xi \in \Omega,\ t \in (0,T), \tag{3.46}$$

$$J_t + J^2u_\xi = 0 \quad \text{for } \xi \in \Omega,\ t \in (0,T), \tag{3.47}$$

$$\varrho_0 c_v\vartheta_t - \varkappa(J\vartheta_\xi)_\xi - \mu Ju_\xi^2 + pu_\xi = 0 \quad \text{for } \xi \in \Omega,\ t \in (0,T), \tag{3.48}$$

$$\mu Ju_\xi - p = -P(t) \quad \text{for } \xi = 0,\ \xi = 1,\ t \in (0,T), \tag{3.49}$$

$$\vartheta_\xi = 0 \quad \text{for } \xi = 0,\ \xi = 1,\ t \in (0,T), \tag{3.50}$$

$$u|_{t=0} = v_0, \quad J|_{t=0} = 1, \quad \vartheta|_{t=0} = \theta_0 \quad \text{for } 0 < \xi < 1, \tag{3.51}$$

where $\Omega = (0,1), \vartheta(\xi,t) = \theta(X_u(\xi,t),t)$.

The main result of [Kaz2] is the following theorem analogous to Theorem 3.1 which holds for the barotropic case with $P \equiv 0$.

THEOREM 3.4. *Let* $P \equiv 0$, $v_0 \in W_2^1(\Omega)$, $\theta_0 \in W_2^1(\Omega)$, $\varrho_0 \in W_2^1(\Omega)$ *and* $m_* = \min(\inf_\Omega \varrho_0, \inf_\Omega \theta_0) > 0$, $m^* = \max(\sup_\Omega \varrho_0, \sup_\Omega \theta_0) < \infty$. *Then for every* $0 < T < \infty$ *there exists a unique solution of problem* (3.46)–(3.51) *with the properties*:

$$\sup_{0\le t\le T}\{\|u(t)\|_{W_2^1(\Omega)} + \|\vartheta(t)\|_{W_2^1(\Omega)} + \|\eta_t(t)\|_{L_2(\Omega)} + \|\eta_\xi(t)\|_{L_2(\Omega)}\} \le K_4, \tag{3.52}$$

$$\|u_t\|_{L_2(\Omega^T)} + \|u_{\xi\xi}\|_{L_2(\Omega^T)} + \|\vartheta_t\|_{L_2(\Omega^T)} + \|\vartheta_{\xi\xi}\|_{L_2(\Omega^T)} \le K_5, \tag{3.53}$$

where K_4, K_5 *are positive constants depending on the data and* T. *Moreover*, $\eta(\xi,t) > 0$, $\vartheta(\xi,t) > 0$ *for* $(\xi,t) \in \Omega^T$.

Sketch of proof. As in the barotropic case, the local existence of a solution for a small interval $[0,t_0)$ can be proved by using the method of successive approximations. The proof of Theorem 3.4 relies on obtaining estimates (3.52) and (3.53) which are derived for the local solution with constants K_4 and K_5 independent of t_0. Therefore, these estimates are true for an interval $[0,T]$, where T is arbitrary.

To obtain (3.52)–(3.53) it is assumed for simplicity that $\varrho_0 \equiv 1$, $\mu = c_v = R = 1$. Then from (3.12) it follows that $\eta(\xi,t) = J(\xi,t)$.

STEP 1. From (3.46), (3.48), (3.49) the following conservation energy law is derived:

$$\frac{d}{dt}\int_0^1\left[\vartheta(\xi,t)+\frac{1}{2}u^2(\xi,t)\right]d\xi=0 \quad \text{for all } t. \tag{3.54}$$

This yields

$$\|\theta(t)\|_{L_1(\Omega)}+\frac{1}{2}\|u(t)\|^2_{L_2(\Omega)}=\|\theta_0\|_{L_1(\Omega)}+\frac{1}{2}\|v_0\|^2_{L_2(\Omega)}\equiv N_0<\infty. \tag{3.55}$$

STEP 2. For $\eta(\xi,t)$ the following formula analogous to (3.24) is obtained:

$$\eta(\xi,t)=\frac{\exp\{\int_0^\xi[v_0(\xi')-u(\xi',t)]\,d\xi'\}}{1+\int_0^t\theta(\xi,t')\exp\{\int_0^\xi[v_0(\xi')-u(\xi',t')]\,d\xi\}\,dt'}. \tag{3.56}$$

Relations (3.56) and (3.55) yield

$$M_\eta(t)\le N\Big(1+N^{-1}\int_0^t m_\vartheta(t')\,dt'\Big)^{-1}, \tag{3.57}$$

$$m_\eta(t)\ge N^{-1}\Big(1+N\int_0^t M_\vartheta(t')\,dt'\Big)^{-1}, \tag{3.58}$$

where $M_\eta(t)=\sup_{0\le\xi\le1}\eta(\xi,t)$, $m_\eta(t)=\inf_{0\le\xi\le1}\eta(\xi,t)$, $M_\vartheta(t)=\sup_{0\le\xi\le1}\vartheta(\xi,t)$, $m_\vartheta(t)=\inf_{0\le\xi\le1}\vartheta(\xi,t)$, $N=\exp\{\|v_0\|_{L_1(\Omega)}+\|v_0\|_{L_2(\Omega)}+\sqrt{2}\|\theta_0\|^{1/2}_{L_1(\Omega)}\}$.

STEP 3. Next, it is necessary to show that $m_\vartheta(t)$ is positive for all $t\in[0,T]$. To do this we use (3.57) and the properties of the parabolic equation with the unknown function $1/\vartheta$ which arises from (3.46) by dividing it by $-\vartheta^2$. We prove that

$$m_\vartheta(t)\ge m_*(1+N_1t)^{-\lambda}\ge m_0>0 \quad \text{for all } t\in[0,T], \tag{3.59}$$

where $N_1=m_*(N^2+4)/(4N)$, $\lambda=N^2/(N^2+4)<1$, $m_0=m_*[1+N_1T]^{-\lambda}$. Integrating (3.59) gives

$$\int_0^t m_\vartheta(t')\,dt'\ge N[(1+N_1t)^{1-\lambda}-1].$$

Hence from (3.57) it follows that

$$M_\eta(t)\le N(1+N_1t)^{\lambda-1}. \tag{3.60}$$

STEP 4. Let $\phi(\xi,t)=\vartheta(\xi,t)-\int_0^1\vartheta(\xi',t)\,d\xi'$. Obviously, $\int_0^1\phi(\xi,t)\,d\xi=0$. Therefore, for any t there exists $\xi_1=\xi_1(t)\in[0,1]$ such that $\phi(\xi_1(t),t)=0$. By using

$$|\phi(\xi,t)|^{3/2}=\frac{3}{2}\int_{\xi_1(t)}^{\xi}|\phi(\xi',t)|^{1/2}\operatorname{sign}\phi(\xi',t)\phi_{\xi'}(\xi',t)\,d\xi'$$

and inequality (3.58), the following estimates are proved:

$$\int_0^T\int_0^1\eta(\xi,t)\vartheta^2_\xi(\xi,t)\,d\xi\,dt\le C_1<\infty, \tag{3.61}$$

$$\sup_{0\le t\le T}[\|u(t)\|_{L_4(\Omega)}+\|\vartheta(t)\|_{L_2(\Omega)}]\le C_2<\infty, \tag{3.62}$$

$$\int_0^T M_\vartheta(t)\,dt \le C_3 < \infty. \tag{3.63}$$

Estimates (3.58) and (3.63) imply

$$m_\eta(t) \ge N^{-1}(1+NC_3)^{-1} \equiv n > 0 \quad \text{for all } t \in [0,T]. \tag{3.64}$$

Next, inequalities (3.61), (3.63) and (3.64) give

$$\int_0^T \|\vartheta_\xi(t)\|^2_{L_2(\Omega)} \le n^{-1}C_1, \qquad \int_0^T M^3_\vartheta(t)\,dt \le C_4 < \infty. \tag{3.65}$$

STEP 5. Now, the estimates for derivatives of u, ϑ and η are derived. First, multiplying (3.46) by u one obtains

$$\sup_{0\le t\le T} \|u(t)\|^2_{L_2(\Omega)} + \int_0^T \|u_\xi(t)\|^2_{L_2(\Omega)}\,dt \le C_5 < \infty. \tag{3.66}$$

Next, multiplying (3.46) by $[\eta(u_\xi - \vartheta)]_\xi$ and using (3.64)–(3.66) yields

$$\sup_{0\le t\le T} \|u_\xi(t) - \vartheta(t)\|^2_{L_2(\Omega)} + \int_0^T \|[\eta(u_\xi - \vartheta)]_\xi(t)\|^2_{L_2(\Omega)}\,dt \le C_6. \tag{3.67}$$

Hence, by (3.46),

$$\int_0^T \|u_t(t)\|^2_{L_2(\Omega)}\,dt \le C_6 < \infty. \tag{3.68}$$

Moreover, by (3.67) and (3.62),

$$\sup_{0\le t\le T} \|u_\xi(t)\|_{L_2(\Omega)} \le C_7 < \infty. \tag{3.69}$$

In view of (3.69), equation (3.47) gives the estimate

$$\sup_{0\le t\le T} \|\eta_t(t)\|_{L_2(\Omega)} \le C_8 < \infty.$$

Next, by using formula (3.56) one can calculate that

$$\sup_{0\le t\le T} \|\eta_\xi(t)\|_{L_2(\Omega)} \le C_9 < \infty. \tag{3.70}$$

Finally, estimates (3.67) and (3.70) give

$$\int_0^T \|u_{\xi\xi}(t)\|^2_{L_2(\Omega)} \le C_{10} < \infty.$$

The constants C_i $(i = 1,\ldots,10)$ in the above estimates depend on the initial data and T.

STEP 6. In the same way the necessary estimates for the temperature are derived. The uniqueness is proved in the standard way, by considering problem (3.46)–(3.51) for the differences of two possible solutions. Thus, the assertion of the theorem follows. ■

By means of differentiating equation (3.47) with respect to ξ, and equations (3.46), (3.48) with respect to t, higher-order estimates for the functions u, ϑ, η are obtained. These estimates yield the theorem below.

THEOREM 3.5 (see [Kaz2]). *Let the assumptions of Theorem* 3.4 *be satisfied. Moreover, let* $v_0 \in C^{2+\alpha}(\overline{\Omega})$, $\theta_0 \in C^{2+\alpha}(\overline{\Omega})$, $\varrho_0 \in C^{1+\alpha}(\overline{\Omega})$, $0 < \alpha < 1$ *and assume that the following compatibility conditions are satisfied*:

$$\theta_0' = \mu v_0' - R\varrho_0\theta_0 = 0 \quad \textit{for } \xi = 0,\ \xi = 1.$$

Then $u \in C^{2+\alpha,1+\alpha/2}(\overline{\Omega}\times[0,T])$, $\vartheta \in C^{2+\alpha,1+\alpha/2}(\overline{\Omega}\times[0,T])$, $\eta \in C^{1+\alpha,1+\alpha/2}(\overline{\Omega}\times[0,T])$.

Nagasawa [Nag] examined problem (3.39)–(3.45) in the case $P(t) > 0$ for $t \in [0,\infty)$, after writing it in Lagrangian mass coordinates (3.33). Assuming $P \in C^1([0,\infty))$, $v_0 \in C^{2+\alpha}(\overline{\Omega})$, $\theta_0 \in C^{2+\alpha}(\overline{\Omega})$, $1/\varrho_0 \in C^{1+\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$ he proved a global existence and uniqueness theorem with $u \in C^{2+\alpha,1+\alpha/2}(\overline{\Omega}\times[0,T])$, $\vartheta \in C^{2+\alpha,1+\alpha/2}(\overline{\Omega}\times[0,T])$, $\overline{v} \in C^{1+\alpha,1+\alpha/2}(\overline{\Omega}\times[0,T])$; $u_{\xi t}, \vartheta_{\xi t}, \overline{v}_{tt} \in L_2(\Omega^T)$, $\overline{v} > 0$ and $\vartheta > 0$. The second important result of [Nag] is an asymptotic theorem. Under the additional assumption that $\int_0^\infty |P'(t)|\,dt < \infty$ he proved that the limits $\overline{P} \equiv \lim_{t\to\infty} P(t)$ and $\lim_{t\to\infty}\int_0^t P'(t')\int_0^1 \overline{v}(\xi,t')\,d\xi\,dt'$ exist and the solution $(u,\vartheta,\overline{v})$ converges to the stationary state $(0,\theta_e,\overline{v}_e)$ in $W_2^1(\Omega)\cap C(\overline{\Omega})$ as $t\to\infty$, where θ_e and $\overline{v}_e$ are positive constants given by

$$\theta_e = \frac{1}{c_v + R}\bigg\{\int_0^1 \bigg(\frac{1}{2}v_0^2 + c_v\theta_0 + P(0)/\varrho_0\bigg)\,d\xi + \int_0^\infty P'(t')\int_0^1 \overline{v}(\xi,t')\,d\xi\,dt'\bigg\},$$
$$\overline{v}_e = \frac{R\theta_e}{\overline{P}}.$$

The assumption that $P(t) > 0$ for $t \in [0,\infty)$ is crucial to proving the asymptotic result.

Paper [FPadNov] is also concerned with the solvability of problem (3.39)–(3.45) but in contrast to [Kaz2], Fujita-Yashima, Padula and Novotny prove the existence of a global weak solution in the case when $\inf_{0\le\xi\le1}\varrho_0(\xi) \ge 0$.

Several papers ([D1–D4]) are devoted to the evolution of stellar objects. In these papers various models of self-gravitating viscous heat-conducting fluids occurring in classical astrophysics to describe the motion of gaseous stars are considered. The boundary of a stellar structure is not known, so the problems considered by Ducomet are free boundary problems. In [D2] he studies the simplified one-dimensional case. The problem written in Lagrangian mass coordinates has the form

$$u_t - \bigg(\mu\frac{u_\xi}{\overline{v}} - p\bigg)_\xi = -G\bigg(\xi - \frac{1}{2}\bigg) \quad \text{for } \xi \in \Omega,\ t \in (0,T), \tag{3.71}$$

$$\overline{v}_t - u_\xi = 0, \quad \text{for } \xi \in \Omega,\ t \in (0,T), \tag{3.72}$$

$$c_v\vartheta_t - \varkappa\bigg(\frac{\vartheta_\xi}{\overline{v}}\bigg)_\xi + pu_\xi - \mu\frac{u_\xi^2}{\overline{v}} = 0 \quad \text{for } \xi \in \Omega,\ t \in (0,T), \tag{3.73}$$

$$\mu\frac{u_\xi}{\overline{v}} - p = -P \quad \text{for } \xi = 0, \xi = 1,\ t \in (0,T), \tag{3.74}$$

$$\frac{\varkappa}{\overline{v}}\vartheta_\xi - \lambda\vartheta = 0 \quad \text{for } \xi = 0,\ t \in (0,T), \tag{3.75}$$

$$\frac{\varkappa}{\overline{v}}\vartheta_\xi + \lambda\vartheta = 0 \quad \text{for } \xi = 1,\ t \in (0,T), \tag{3.76}$$

$$u|_{t=0} = v_0(\xi), \quad \vartheta|_{t=0} = \theta_0(\xi), \quad \overline{v}|_{t=0} = 1/\varrho_0(\xi) \quad \text{for } \xi \in \Omega, \tag{3.77}$$

$$(3.78)\qquad (\vartheta,\theta_0,\overline{v},1/\varrho_0,e)(1/2+\xi,t) = (\vartheta,\theta_0,\overline{v},1/\varrho_0,e)(1/2-\xi,t) \quad \text{for } 0\le\xi\le 1/2,\ t\in[0,T],$$

$$(3.79)\qquad (u,v_0)(1/2+\xi,t) = -(u,v_0)(1/2-\xi,t) \quad \text{for } 0\le\xi\le 1/2,\ t\in[0,T],$$

where as before $\Omega=(0,1)$, $p=R\vartheta/\overline{v}$, $-G(\xi-1/2)$ is the gravitational term which is chosen in such a way that $\xi=1/2$ is the symmetry centre for the slab; $P\ge 0$ is the constant external pressure; $\lambda\ge 0$ is a flux parameter; e is the internal energy.

In [D2] under the same assumptions on v_0, θ_0 and ϱ_0 as in [Nag] Ducomet proves a global existence and uniqueness theorem analogous to the result of [Nag]. His methods are similar to those used by Nagasawa and they take its origin in the papers of Kazhikov.

Next, using again the methods of Nagasawa, Ducomet proves that in the case of $\lambda=0$ and $P>0$ the solution of problem (3.71)–(3.79) converges to the stationary state as $t\to\infty$. The rate of convergence is exponential. If $\lambda=0$ and $P=0$ the corresponding stationary solution is unstable, and if $\lambda>0$ the solution tends to the singular limit $(0,0,0)$ which corresponds to the gravitational collapse of the slab into a plane with an infinite specific volume. The case of a more general state equation is also examined in [D2].

In [D3] Ducomet considers a more general problem, describing the motion of a viscous compressible heat-conducting reacting self-gravitating gas. The corresponding system consists of (3.71)–(3.72) and the following equations (see [LedWal]):

$$(3.80)\qquad e_t+Q_\xi+\vartheta p_\vartheta u_\xi-\mu\frac{u_\xi^2}{\overline{v}}-\lambda\phi(\vartheta,Z)=0 \quad \text{for } \xi\in\Omega,\ t\in(0,T),$$

$$(3.81)\qquad Z_t-\left(\frac{d}{\overline{v}^2}Z_\xi\right)_\xi+\phi(\vartheta,Z)=0 \quad \text{for } \xi\in\Omega,\ t\in(0,T),$$

where $e(\overline{v},\vartheta)=c_v\vartheta+a\overline{v}\vartheta^4$ is the internal energy; $p(\overline{v},\vartheta)=R\vartheta/\overline{v}+a\vartheta^4/3$ is the pressure; a is the Stefan–Boltzmann constant; $Z=Z(\xi,t)$ is the (unknown) fraction of reactant; $\lambda\ge 0$ and $d\ge 0$ are two "chemical" constants; the function $\phi(\theta)$ is given by the Arrhenius law: $\phi(\vartheta)=AZ\vartheta^\beta e^{-E/B\vartheta}$; A, β, B, E are positive constants; $Q(\overline{v},\vartheta)=-\varkappa(\overline{v},\vartheta)\vartheta_\xi/\overline{v}$ is the flux with the conductivity given by $\varkappa=\kappa_1+\kappa_2\overline{v}\vartheta^q$; κ_1, κ_2 and q are positive constants.

Together with boundary condition (3.74) the following boundary conditions are assumed:

$$(3.82)\qquad Q=0 \quad \text{for } \xi=0,\ \xi=1,\ t\in(0,T),$$

$$(3.83)\qquad Z_\xi=0 \quad \text{for } \xi=0,\ \xi=1,\ t\in(0,T).$$

To the initial condition (3.77) the initial condition on Z is added:

$$(3.84)\qquad Z|_{t=0}=Z_0(\xi),\qquad \xi\in\Omega.$$

Moreover, together with condition (3.79) the following symmetry condition is assumed:

$$(3.85)\qquad (\vartheta,\theta_0,\overline{v},1/\varrho_0,Z,Z_0)(1/2+\xi,t)=(\vartheta,\theta_0,\overline{v},1/\varrho_0,Z,Z_0)(1/2-\xi,t) \quad \text{for } 0\le\xi\le 1/2,\ t\in(0,T).$$

The following theorem is proved in [D4].

THEOREM 3.6. *Let* $v_0 \in C^{2+\alpha}(\overline{\Omega}), \theta_0 \in C^{2+\alpha}(\overline{\Omega})$, $Z_0 \in C^{2+\alpha}(\overline{\Omega})$, $1/\varrho_0 \in C^{1+\alpha}(\overline{\Omega})$, $0 < \alpha < 1$. *Assume that* θ_0, ϱ_0, Z_0 *are positive on* $[0,1]$ *and that the compatibility conditions between the boundary conditions and the initial data hold. Then for* $q \geq 4$ *and for every* $0 < T < \infty$ *there exists a unique solution of problem* (3.71), (3.72), (3.80)–(3.83), (3.77), (3.84), (3.85) *with the properties*: $u \in C^{2+\alpha,1+\alpha/2}(\overline{\Omega}\times[0,T])$, $\vartheta \in C^{2+\alpha,1+\alpha/2}(\overline{\Omega}\times[0,T])$, $Z \in C^{2+\alpha,1+\alpha/2}(\overline{\Omega}\times[0,T])$, $\overline{v} \in C^{1+\alpha,1+\alpha/2}(\overline{\Omega}\times[0,T])$, $\overline{v}_t \in C^{1+\alpha,\alpha/2}(\overline{\Omega}\times[0,T])$, $u_{\xi t} \in L_2(\Omega^T)$, $\vartheta_{\xi t} \in L_2(\Omega^T)$, $Z_{\xi t} \in L_2(\Omega^T)$, $\overline{v}_{tt} \in L_2(\Omega^T)$; $\vartheta > 0$, $\overline{v} > 0$, $Z > 0$ *on* $[0,1]\times[0,\infty)$.

The proof is as usual based on conservation laws and a priori estimates which are derived by using arguments from [Kaw] and [J].

Some partial results concerning the asymptotic behaviour of the solution of problem (3.71), (3.72), (3.80)–(3.83), (3.77), (3.84), (3.85) are also given in [D4].

3.2. Spherically symmetric case

3.2.1. *Spherically symmetric motion of a viscous barotropic fluid.* Such a motion is studied by Okada and Makino [OkMak]. They consider an atmosphere surrounding a solid star of radius 1 and mass M. The motion of the atmosphere is described by the system

$$\varrho(v_t + vv_r) + p_r - \mu\left(v_{rr} + \frac{2}{r}v_r - \frac{2}{r^2}v\right) = -\frac{\varrho M}{r^2} \quad \text{for } 1 < r < r_1(t),\ t \in (0,T), \tag{3.86}$$

$$\varrho_t + v\varrho_r + \varrho v_r + \frac{2}{r}\varrho v = 0 \quad \text{for } 1 < r < r_1(t),\ t \in (0,T), \tag{3.87}$$

$$v|_{r=1} = 0, \quad \varrho|_{r=r_1(t)} = 0 \quad \text{for } t \in (0,T), \tag{3.88}$$

$$\varrho|_{t=0} = \varrho_0(r), \quad v|_{t=0} = v_0(r) \quad \text{for } 1 < r < r_1(0), \tag{3.89}$$

where $p = a\varrho^\gamma$, $a > 0$, $1 < \gamma \leq 2$. Problem (3.86)–(3.89) arises from (5.1)–(5.5) by introducing spherical coordinates and assuming that the motion is spherically symmetric.

Now, by introducing Lagrangian mass coordinates given by

$$\xi = 4\pi\int_1^r \varrho(s,t)s^2\,ds$$

and by assuming as before that the total mass M of the fluid is equal to 1, problem (3.86)–(3.89) takes the form

$$u_t + 4\pi r^2 p_\xi - 16\pi^2\mu(r^4\eta u_\xi)_\xi + 2\mu\frac{u}{r^2\eta} + \frac{1}{r^2} = 0 \quad \text{for } \xi \in \Omega,\ t \in (0,T), \tag{3.90}$$

$$\eta_t + 4\pi\eta^2(r^2u)_\xi = 0 \quad \text{for } \xi \in \Omega,\ t \in (0,T), \tag{3.91}$$

$$u|_{\xi=0} = 0, \quad \eta|_{\xi=1} = 0 \quad \text{for } t \in (0,T), \tag{3.92}$$

$$\eta|_{t=0} = \eta_0(\xi), \quad u|_{t=0} = u_0(\xi), \quad \text{for } \xi \in \Omega, \tag{3.93}$$

where u and η are the velocity v and the density ϱ written in ξ coordinates; $r = \left[1 + \frac{3}{4\pi}\int_0^\xi \frac{d\xi}{\eta(t,\xi)}\right]^{1/3}$, $p = a\eta^\gamma$, $\Omega = (0,1)$.

In order to prove global existence for (3.90)–(3.93) Okada and Makino discretize this problem with respect to ξ. Thus, they obtain a sequence of approximate Cauchy problems for systems of ordinary equations with respect to t which are locally solvable by the elementary theory of ordinary differential equations. Then they prove the existence of a global solution of an approximate problem by deriving appropriate estimates for the solution with constants on the right-hand sides independent of the time horizon of the local solution. The derived estimates give the convergence of the approximate solutions to functions $u, \eta \in L_\infty([0,T] \times [0,1]) \cap C^1([0,T]; L^2(0,1))$ such that $\eta u_\xi \in L_\infty([0,T] \times [0,1]) \cap C^{1/2}([0,T]; L_2(\Omega))$ for any T and such that (u,η) is a global weak solution of problem (3.90)–(3.93). This global solution is obtained under the assumptions that $u_0 \in C([0,1])$, $\eta_0 \in C([0,1])$, $\eta_0(\xi) > 0$ for $0 \le \xi < 1$, $\eta_0(1) = 0$ and some other assumptions concerning the initial data.

3.2.2. *Spherically symmetric motion of a viscous heat-conducting fluid.* Fujita-Yashima and Benabidallah [FBen] consider the motion of a viscous heat-conducting ideal gas which is symmetric with respect to the origin. They assume that the gas occupies the domain between two surfaces: the rigid surface $\{|x| = r_\Gamma\}$ $(r_\Gamma > 0)$ and the free surface $\{|r| = r_1(t)\}$ $(r_\Gamma < r_1 < \infty)$. If $n = 1$ the motion is one-dimensional; if $n = 2$ the motion is axially symmetric; if $n \ge 3$ the motion is spherically symmetric. Therefore the following system of equations is considered:

$$(3.94)\quad \varrho(v_t + vv_r) - \mu\left(v_{rr} + (n-1)\frac{1}{r}v_r - (n-1)\frac{1}{r^2}v\right) + R(\varrho\theta)_r = \varrho f, \qquad \text{for } r_\Gamma < r < r_1(t),\ t \in (0,T),$$

$$(3.95)\quad \varrho_t + (\varrho v)_r + (n-1)\frac{1}{r}\varrho v = 0, \qquad \text{for } r_\Gamma < r < r_1(t),\ t \in (0,T),$$

$$(3.96)\quad \begin{aligned} &\varrho c_v(\theta_t + v\theta_r) - \varkappa\theta_{rr} - (n-1)\frac{1}{r}\theta_r \\ &\quad + R\varrho\theta\Big(v_r + (n-1)\frac{1}{r}v\Big) \\ &= \mu\left(v_r + (n-1)\frac{1}{r}v\right)^2 - 4(n-1)\mu'\frac{1}{r}vv_r \\ &\quad - 2(n-1)(n-2)\mu'\frac{1}{r^2}v^2, \qquad \text{for } r_\Gamma < r < r_1(t),\ t \in (0,T), \end{aligned}$$

with the boundary conditions

$$(3.97)\quad v|_{r=r_\Gamma} = 0 \qquad \text{for } t \in (0,T),$$

$$(3.98)\quad \theta_r|_{r=r_\Gamma} = 0 \qquad \text{for } t \in (0,T),$$

$$(3.99)\quad \left\{\mu\left(v_r + (n-1)\frac{1}{r}v\right) - 2(n-1)\mu'\frac{1}{r}v - R\varrho\theta\right\}\bigg|_{r=r_1(t)} = 0 \qquad \text{for } t \in (0,T),$$

$$(3.100)\quad \theta_r|_{r=r_1(t)} = 0 \qquad \text{for } t \in (0,T)$$

and with the initial conditions

$$(3.101)\quad v|_{t=0} = v_0(r), \quad \varrho|_{t=0} = \varrho_0(r), \quad \theta|_{t=0} = \theta_0(r) \quad \text{for } r_\Gamma < r < r_1(0),$$

where $\mu = \frac{2(n-1)}{n}\mu' + \zeta$; $\mu' \ge 0$ and $\zeta > 0$ are viscosity coefficients; c_v is the constant specific heat at constant volume, $r_\Gamma > 0$.

As before, to prove global existence and uniqueness for problem (3.94)–(3.101) it is useful to write it in Lagrangian mass coordinates. First, using the methods of Kazhikov [Kaz1, Kaz2], under the assumptions that the total mass of the fluid is 1 and that $u_0, \vartheta_0 \in C^{2+\alpha}(\overline{\Omega})$, $\eta_0 \in C^{1+\alpha}(\overline{\Omega})$, $\alpha \in (0,1)$, $f \in C^2([r_\Gamma, \infty))$, $\inf_{0\le\xi\le1}\eta_0(\xi) > 0$,

$\inf_{0\le\xi\le 1}\vartheta_0(\xi)>0$, $u_0|_{\xi=0}=0$, $\vartheta_{0\xi}|_{\xi=0,1}=0$, $[\mu\eta_0(r_0^{n-1}u_0)_\xi-2(n-1)\mu'\frac{1}{r_0}u_0-R\eta_0\vartheta_0]|_{\xi=1}=0$, Fujita-Yashima and Benabidallah prove the global existence and uniqueness of a solution such that $u,\vartheta\in C^{2+\alpha,1+\alpha/2}(\overline{\Omega}\times[0,T])$, $\eta\in C^{1+\alpha,1+\alpha/2}(\overline{\Omega}\times[0,T])$. Here $\Omega=(0,1)$; u_0, ϑ_0, η_0 are the initial data v_0, θ_0 and ϱ_0, respectively, written in Lagrangian mass coordinates; u, ϑ, η are the velocity, temperature and density of the fluid written in Lagrangian mass coordinates, and $T>0$ is arbitrary.

Using the above global existence result and the argument of [FPadNov] Fujita-Yashima and Benabidallah prove then the existence of a unique global weak solution. More precisely, they prove the following theorem.

THEOREM 3.7. *Let* $\eta_0(\xi)\ge 0$ *for* $\xi\in\overline{\Omega}$, $\operatorname{ess\,sup}_{0\le\xi\le1}\eta_0(\xi)<\infty$, $\eta_0^{-1}\in L_1(\Omega)$, $u_0\in L_4(\Omega)$, $\eta_0^{1/2}u_{0\xi}\in L_2(\Omega)$, $\vartheta_0\in L_2(\Omega)$, $\inf_{0\le\xi\le1}\vartheta_0(\xi)\ge 0$, $f\in C([r_\Gamma,\infty))\cap L_\infty([r_\Gamma,\infty))$, $F(s)=\int_{r_\Gamma}^s f(s')\,ds'\in L_\infty([r_\Gamma,\infty))$, $n\mu>2(n-1)\mu'$. *Moreover, assume that there exists* $\delta>0$ *and* $K_A>0$ *such that for almost every* $\xi\in\overline{\Omega}$ *there exists an interval* $I(\xi)$ *with the properties*: $\xi\in I(\xi)$, $|I(\xi)|=\delta$, $\eta_0(\xi)\le K_A\eta_0(\xi')$ *for almost every* $\xi'\in I(\xi)$. *Then for every* $0<T<\infty$ *there exists a weak solution of problem* (3.94)–(3.101) *such that* $u\in L_\infty(0,T;L_4(\Omega))$, $\vartheta\in L_\infty(0,T;L_2(\Omega))$, $\eta\in L_\infty(\Omega^T)$; u_t, $(\eta(\mu(r^{n-1}u)_\xi-R\vartheta))_\xi\in L_2(0,T;L_2(\Omega))$; $\eta_t,\eta_0^{1/2}(r^{n-1}u)_\xi\in L_\infty(0,T;L_2(\Omega))$; $\eta_0^{1/2}\vartheta_\xi\in L_2(0,T;L_2(\Omega))$; $r\in C(\overline{\Omega}\times[0,T])$ *and for every* $t\in[0,T]$ *the function* $r(\xi,t)$ *is strictly increasing with respect to* ξ, *where* $r(\xi,t)=r_0(\xi)+\int_0^t u(\xi,t')\,dt'$, $r_0(\xi)=[r_\Gamma^n+n\int_0^\xi\frac{1}{\eta_0(\xi')}\,d\xi']^{1/n}$, $\xi\in\overline{\Omega}$, $t\in[0,\infty)$. *Moreover, there exist constants* J_1,J_2 *such that* $0<J_1\le J_2<\infty$ *and* $J_1\le\eta(\xi,t)/\eta_0(\xi)\le J_2$ *for almost every* $(\xi,t)\in\overline{\Omega}\times[0,T]$.

In [D1, D4] the three-dimensional spherically symmetric version of problem (3.71), (3.72), (3.80), (3.81), (3.74)–(3.76), (3.83), (3.77), (3.84), (3.79) and (3.85) with a hard core at $r=r_\Gamma$ is considered. By the methods of [Kaz1, Kaz2] and [FBen] the global existence and uniqueness of a classical solution is proved. This result is analogous to that from [FBen] (for the case of the initial density and initial temperature greater than zero). Moreover, the asymptotic behaviour of a solution is examined in [D4].

4. Two- and three-dimensional free boundary problems for a drop of an incompressible fluid

This section is concerned with a free boundary problem for a drop of an incompressible fluid. The problem is to find a bounded domain $\Omega_t\subset\mathbb{R}^n$, $n=2,3$, a velocity vector field $v=v(x,t)$ ($v=(v_1,\dots,v_n)$) and a pressure $p=p(x,t)$, satisfying the following system with boundary and initial conditions:

$$v_t+(v\cdot\nabla)v-\nu\Delta v+\nabla p=f+k\nabla U,\qquad x\in\Omega_t,\ t\in(0,T),\tag{4.1}$$

$$\operatorname{div}v=0,\qquad x\in\Omega_t,t\in(0,T),\tag{4.2}$$

$$\mathbb{T}\overline{n}-\sigma H\overline{n}=-p_0(x,t)\overline{n},\qquad x\in S_t,t\in(0,T),\tag{4.3}$$

$$v\cdot\overline{n}=-\phi_t/|\nabla\phi|,\qquad x\in S_t,\ t\in(0,T),\tag{4.4}$$

$$v(x,0)=v_0(x),\qquad x\in\Omega_0\equiv\Omega,\tag{4.5}$$

where $T > 0$, $\overline{n}$ is the unit outward vector normal to the boundary $S_t = \partial\Omega_t$; $\phi(x,t) = 0$ describes S_t at least locally; $f = f(x,t)$ is the external force field per unit mass; $U(x,t) = \int_{\Omega_t} |x-y|^{-1}\,dy$ is the self-gravitational potential; $p_0 = p_0(x,t)$ is the external pressure; v_0 is the given initial velocity; Ω is the given initial domain; ν, k and σ are the constant coefficients of viscosity, of gravitation and of surface tension, respectively.

By $\mathbb{T} = \mathbb{T}(v,p)$ we denote the stress tensor of the form

$$\mathbb{T}(v,p) = \{T_{ij}\}_{i,j=1,\dots,n} = \{-p\delta_{ij} + 2\nu S_{ij}(v)\}_{i,j=1,\dots,n}, \tag{4.6}$$

where $\mathbb{S}(v) = \{\frac{1}{2}(v_{ix_j} + v_{jx_i})\}_{i,j=1,\dots,n}$ is the velocity deformation tensor, and $I = \{\delta_{ij}\}_{i,j=1,2,3}$ is the unit matrix.

Moreover, in the three-dimensional case $H = H(x,t)$ is the double mean curvature of S_t at the point x, which is negative for convex domains and which can be expressed as

$$H\overline{n} = \Delta_{S_t}(t)x, \tag{4.7}$$

where $\Delta_{S_t}(t)$ is the Laplace–Beltrami operator on S_t.

Let S_t be determined locally by $x = x(s_1,s_2,t)$, $(s_1,s_2) \in V \subset \mathbb{R}^2$, where V is an open set. Then

$$\Delta_{S_t}(t) = g^{-1/2}\frac{\partial}{\partial s_\alpha}\left(g^{1/2}g^{\alpha\beta}\frac{\partial}{\partial s_\beta}\right), \quad (\alpha,\beta = 1,2), \tag{4.8}$$

where $g = \det\{g_{\alpha\beta}\}_{\alpha,\beta=1,2}$, $g_{\alpha\beta} = \frac{\partial x}{\partial s_\alpha}\cdot\frac{\partial x}{\partial s_\beta}$, $\{g^{\alpha\beta}\}$ is the inverse matrix to $\{g_{\alpha\beta}\}$.

In (4.8) and in what follows we assume the summation convention over repeated indices.

In the two-dimensional case H denotes the curvature of S_t.

In view of equation (4.2) and the kinematic condition (4.4) the measure of Ω_t is conserved, i.e.

$$|\Omega_t| = \int_{\Omega_t} dx = \int_{\Omega} dx = |\Omega|.$$

4.1. Local existence. The papers concerned with local solutions of various special cases of three-dimensional problem (4.1)–(4.5) are [MZaj1, MogSol, Sol4, Sol5, Sol7, Sol8, Sol13].

The most often applied method to prove local existence is to write problem (4.1)–(4.5) in Lagrangian coordinates ξ which are the initial data for the following Cauchy problem:

$$\frac{dx}{dt} = v(x,t), x(0) = \xi, \qquad \xi = (\xi_1,\xi_2,\xi_3). \tag{4.9}$$

As in the one-dimensional case, integrating (4.9) we obtain a transformation which connects Eulerian x and Lagrangian ξ coordinates:

$$x = x(\xi,t) \equiv \xi + \int_0^t u(\xi,t')\,dt' \equiv X_u(\xi,t), \tag{4.10}$$

where $u(\xi,t) = v(X_u(\xi,t),t)$. From (4.4) we have $\Omega_t = \{x \in \mathbb{R}^3 : x = X_u(\xi,t), \xi \in \Omega\}$ and $S_t = \{x \in \mathbb{R}^3 : x = X_u(\xi,t),\ \xi \in S\}$. Problem (4.1)–(4.5) in Lagrangian coordinates has the form

$$u_t - \nu\nabla_u^2 u + \nabla_u q = g + k\nabla_u U_u \quad \text{in } \Omega^T \equiv \Omega \times (0,T), \tag{4.11}$$

$$(4.12)\qquad \nabla_u \cdot u = 0 \qquad \text{in } \Omega^T,$$

$$(4.13)\qquad \mathbb{T}_u \overline{n}_u - \sigma \Delta_u(t) X_u = -q_0 \overline{n}_u \qquad \text{on } S^T \equiv S \times (0,T),$$

$$(4.14)\qquad u|_{t=0} = v_0 \qquad \text{in } \Omega,$$

where $q(\xi,t) = p(X_u(\xi,t),t)$, $g(\xi,t) = f(X_u(\xi,t),t)$, $q_0(\xi,t) = p_0(X_u(\xi,t),t)$, $\overline{n}_u(\xi,t) = \overline{n}(X_u(\xi,t),t)$, $\nabla_u = \xi_{ix}\partial_{\xi_i} = (\xi_{ix_j}\partial_{\xi_i})_{j=1,2,3}$, $\mathbb{T}_u(u,q) = -qI + \mathbb{D}_u(u)$, $\mathbb{D}_u(u) = \nu\{\partial_{x_i}\xi_k\partial_{\xi_k}u_j + \partial_{x_j}\xi_k\partial_{\xi_k}u_i\}_{i,j=1,2,3}$, ($\partial_{x_i}\xi_k$ are the elements of the matrix ξ_x inverse to $x_\xi = I + \int_0^t u_\xi(\xi,t')\,dt'$), $U_u(\xi,t) = \int_\Omega \frac{J_{X_u(\xi',t)}\,d\xi'}{|X_u(\xi,t)-X_u(\xi',t)|}$, J_{X_u} is the Jacobian of transformation (4.10); Δ_u is given by (4.8).

4.1.1. *The case of $\sigma = 0$.* Papers [MZaj1], [Sol4] and [Sol8] are devoted to the case $\sigma = 0$. The results of all these papers are based on the solvability of the following Cauchy–Neumann problem for the Stokes system:

$$(4.15)\qquad u_t - \nu\Delta u + \nabla p = F \qquad \text{in } \Omega^T,$$

$$(4.16)\qquad \nabla \cdot u = G \qquad \text{in } \Omega^T,$$

$$(4.17)\qquad \mathbb{T}(u,p)\overline{n}_0 = D \qquad \text{on } S^T,$$

$$(4.18)\qquad u|_{t=0} = v_0 \qquad \text{in } \Omega,$$

where $\overline{n}_0$ is the unit outward normal vector to S.

Existence theorems for problem (4.15)–(4.18) can be found in [MZaj2], [Sol3] and [Sol8]. In particular, in [Sol8] the following theorem is formulated.

THEOREM 4.1. *Let $r > 3$, $S \in W_r^{2-1/r}$, $F \in L_r(\Omega^T)$, $G \in W_r^{1,0}(\Omega^T)$, $G = \nabla \cdot R$, $R \in L_r(\Omega^T)$, $R_t \in L_r(\Omega^T)$, $v_0 \in W_r^{2-2/r}(\Omega)$, $D \in W_r^{1-1/r,1/2-1/(2r)}(S^T)$. Assume also that $D|_{t=0} = 0$, $G|_{t=0} = 0$ and that the following compatibility conditions are satisfied*:

$$(4.19)\qquad \nabla \cdot v_0 = 0 \qquad \textit{in } \Omega,$$

$$\mathbb{S}(v_0)\overline{n}_0 - \overline{n}_0(\overline{n}_0 \cdot \mathbb{S}(v_0)\overline{n}_0) = 0 \qquad \textit{on } S.$$

Then there exists a unique solution of problem (4.15)–(4.18) *such that $u \in W_r^{2,1}(\Omega^T)$, $p \in W_r^{1,0}(\Omega^T)$, $p \in W_r^{1-1/r,1/2-1/(2r)}(S^T)$. For this solution the following estimate is satisfied*:

$$(4.20)\qquad \|u\|_{W_r^{2,1}(\Omega^T)} + \sup_{t\le T}\|u\|_{W_r^{2-2/r}(\Omega)} + \|p\|_{W_r^{1,0}(\Omega^T)} + \|p\|_{W_r^{1-1/r,1/2-1/(2r)}(S^T)}$$
$$\le c(T)(\|F\|_{L_r(\Omega^T)} + \|G\|_{W_r^{1,0}(\Omega^T)} + \|R_t\|_{L_r(\Omega^T)} + \|v_0\|_{W_r^{2-2/r}(\Omega)} + \|D\|_{W_r^{1-1/r,1/2-1/(2r)}(S^T)}),$$

where $c(T)$ is an increasing positive function of T.

The proof of Theorem 4.1 is done in several steps. First, problem (4.15)–(4.18) with $F = 0$, $G = 0$, $v_0 = 0$ is considered in the halfspace $x_3 > 0$. By applying the Fourier transform with respect to x_1, x_2 and t, this problem becomes a system of ordinary differential equations which can be easily solved. There are two different ways to obtain estimates of a solution of the latter system. One of them, applied by Solonnikov [Sol2], relies on calculating explicitly the inverse Fourier transform of the solution and expressing it in the form of potentials. Then it can be estimated in suitable norms. The second method, probably simpler, is used by Mucha and Zajączkowski [MZaj2] and relies on the direct estimation of this solution by means of the Marcinkiewicz multiplier theorem [Mar, Mik].

Next, in the same way (by applying the Fourier transform) the existence of a unique solution of the Cauchy problem with $G = 0$, $v_0 = 0$ is proved. The results obtained for these two problems give the existence theorem and the appropriate estimate for problem (4.15)–(4.18) with $F \neq 0$, $G \neq 0$, $D \neq 0$, $v_0 = 0$ in the halfspace.

The existence of a unique solution in a bounded domain and estimate (4.20) are proved by using the technique of regularizers. Therefore problem (4.15)–(4.18) is considered locally in a neighbourhood of either an interior point or a boundary point. The boundary neighbourhood problem (4.15)–(4.18) is transformed to a problem in the halfspace, for which the estimates obtained earlier can be used.

A theorem analogous to Theorem 4.1 concerning the solvability of problem (4.15)–(4.18) in Hölder spaces is proved in [Sol3].

The solvability results for problem (4.15)–(4.18) and the method of successive approximations are used to prove the existence of a unique solution to problem (4.11)–(4.14) with $\sigma = 0$. The following problems are considered:

$$
\begin{aligned}
&\partial_t u_{m+1} - \nu\nabla^2 u_{m+1} + \nabla q_{m+1} = f(X_m(\xi,t),t) \\
&\qquad + \nu(\nabla^2_{u_m} - \nabla^2)u_m + (\nabla - \nabla_{u_m})q_m + k\nabla_{u_m}U_{u_m} && \text{in } \Omega^T, \\
&\nabla\cdot u_{m+1} = (\nabla - \nabla_{u_m})\cdot u_m && \text{in } \Omega^T, \\
&\mathbb{T}(u_{m+1},q_{m+1})\overline{n}_0 = \mathbb{T}(u_m,q_m)(\overline{n}_0 - \overline{n}_{u_m}) \\
&\qquad + [\mathbb{T}(u_m,q_m) - \mathbb{T}_{u_m}(u_m,q_m)]\overline{n}_{u_m} - p_0(X_{u_m}(\xi,t),t)\overline{n}_{u_m} && \text{in } \Omega^T, \\
&u_{m+1}|_{t=0} = v_0 && \text{in } \Omega,
\end{aligned}
\tag{4.21}
$$

where $m = 0,1,\ldots$ In (4.21), u_m and q_m are treated as given functions, and $u_0 = 0, q_0 = 0$.

By applying Theorem 4.1 to problems (4.21) Solonnikov [Sol8] proved the following local existence theorem for the case without the self-gravitational force (i.e. for $k = 0$).

THEOREM 4.2. *Assume that $p_0 = 0$ and the function $f(x,t)$ defined for $x \in \mathbb{R}^3$, $0 \le t \le T_0$ is bounded and satisfies the Lipschitz condition with respect to x. Let $r > 3$, $v_0 \in W_r^{2-2/r}(\Omega)$, $S \in W_r^{2-1/r}$ and let compatibility conditions* (4.18) *be satisfied. Then there exists a unique solution $u \in W_r^{2,1}(\Omega^{T_1})$, $q \in W_r^{1,0}(\Omega^{T_1})$ of problem* (4.11)–(4.14), *where $T_1 \le T_0$ depends on $\|v_0\|_{W_r^{2-2/r}(\Omega)}$ and* $\sup|f(x,t)|$ *(T_1 is a decreasing function of these norms). In the case $T_0 = \infty$ we have $T_1 \to \infty$ if the above norms tend to zero.*

In the case of $k > 0$ and $p_0 = \text{const}$, the local existence theorem with the same regularity of a solution as in Theorem 4.2 was proved by Mucha and Zajączkowski [MZaj1].

It should be underlined that the regularity of the local solution obtained in [Sol8] and [MZaj1] is the sharp regularity for this problem in the L_r-approach with $r > 3$.

In the two-dimensional case Theorem 4.2 holds with $r > 2$.

The local solvability of problem (4.11)–(4.14) in Hölder spaces has been examined by Solonnikov [Sol4], where by using the results of [Sol2, Sol3] the following theorem has been proved.

THEOREM 4.3. *Let $k = 0$, $S \in C^{2+\alpha}$, $v_0 \in C^{2+\alpha}(\Omega)$, $f \in C^{\alpha,\alpha/2}(\mathbb{R}^3 \times (0,T))$, $f_x \in C^{\alpha,\alpha/2}(\mathbb{R}^3 \times (0,T))$, $p_0 \in \widetilde{C}^{1+\alpha}_\beta(\mathbb{R}^3 \times (0,T))$, $p_{0x} \in \widetilde{C}^{1+\alpha}_\beta(\mathbb{R}^3 \times (0,T))$; $\alpha,\beta \in (0,1)$. Then* *there exists a unique solution of problem* (4.11)–(4.14) *such that* $u \in C^{2+\alpha,1+\alpha/2}(\Omega^{T'})$, $q \in \widetilde{C}_{\beta}^{1+\alpha}(\Omega^{T'})$, *where*

$$T' = \{t \le T : c(1+R(t))^{(2+\alpha)/(1+\alpha)} R(t) e^{c(1+R(t))^{(2+\alpha)/(1+\alpha)}}(t+t^{\gamma}) \le \delta\},$$
$$\gamma = \min((1-\alpha)/2, \beta/2), \qquad \delta \le 1/8,$$
$$R(t) = \|v_0\|_{C^{2+\alpha}(\Omega)} + c_1\|f\|_{C^{\alpha,\alpha/2}(\mathbb{R}^3\times(0,T))} + c_2\|p_0\|_{\widetilde{C}_{\beta}^{1+\alpha}(\mathbb{R}^3\times(0,T))};$$

$c_1 > 0$ *and* $c_2 > 0$ *are constants independent of* t.

4.1.2. *The case of* $\sigma > 0$. Local existence theorems in the case of $\sigma > 0$ can be found in [MogSol, Sol5, Sol7, Sol13, Scw]. Paper [Sol7] is a review of results concerning both local and global existence for problem (4.11)–(4.14). Similarly to the case without the surface tension the basis of proof of local existence is an existence result for the linear problem (4.15), (4.16), (4.18) with the boundary condition

$$\mathbb{T}(u,p)\overline{n}_0 - \sigma\overline{n}_0\Big(\overline{n}_0 \cdot \Delta_S \int_0^t u\,dt'\Big) = D(x,t) \quad \text{on } S^T. \tag{4.22}$$

To obtain estimates necessary to prove existence it is convenient to project condition (4.22) onto the tangent plane and onto the normal direction to S. Then boundary condition (4.22) can be written in the form

$$\nu\Pi_0\mathbb{S}(u)\overline{n}_0 = \Pi_0 D \qquad \text{on } S^T, \tag{4.23}$$

$$\overline{n}_0 \cdot \mathbb{T}\overline{n}_0 - \sigma\overline{n}_0 \cdot \Delta_S \int_0^t u\,dt' = \overline{n}_0 \cdot D \quad \text{on } S^T, \tag{4.24}$$

where $\Pi_0 f = f - \overline{n}_0(\overline{n}_0 \cdot f)$.

The solvability result for problem (4.15)–(4.16), (4.18), (4.23)–(4.24), obtained in [Sol11] (see Theorem 1.1 of [Sol11]), yields the existence of a unique solution of this problem such that $u \in W_2^{2+\alpha,1+\alpha/2}(\Omega^T)$, $p \in W_2^{\alpha,\alpha/2}(\Omega^T)$, $\nabla p \in W_2^{\alpha,\alpha/2}(\Omega^T)$, $p|_{S^T} \in W_2^{\alpha+1/2,\alpha/2+1/4}(S^T)$ (with $\alpha \in (1/2,1)$), satisfying the estimate

$$\begin{aligned} &\|u_t\|_{\Omega^T}^{(\alpha,\alpha/2)} + \sum_{|\alpha|=2}\|D_{\xi}^{\alpha}u\|_{\Omega^T}^{(\alpha,\alpha/2)} + \sum_i \|u_{\xi_i}\|_{L_2(\Omega^T)} \\ &\quad + \|u\|_{L_2(\Omega^T)} + \|\nabla p\|_{\Omega^T}^{(\alpha,\alpha/2)} + \|p\|_{\Omega^T}^{(\alpha,\alpha/2)} + \|p\|_{W_2^{\alpha+1/2,\alpha/2+1/4}(S^T)} \\ &\le c(T)(\|F\|_{\Omega^T}^{(\alpha,\alpha/2)} + \|G\|_{W_2^{\alpha+1,0}(\Omega^T)} + \|R\|_{W_2^{0,\alpha/2+1}(\Omega^T)} + T^{-\alpha/2}\|R_t\|_{L_2(\Omega^T)} \\ &\quad + \|\Pi_0 D\|_{W_2^{\alpha+1/2,\alpha/2+1/4}(S^T)} + \|D'\|_{W_2^{\alpha+1/2,\alpha/2+1/4}(S^T)} \\ &\quad + T^{-\alpha/2}\|D'\|_{W_2^{\alpha/2,0}(S^T)} + \sigma\|D''\|_{S^T}^{(\alpha-1/2,\alpha/2-1/4)} + \|v_0\|_{W_2^{\alpha+1}(\Omega)}), \end{aligned} \tag{4.25}$$

where it was assumed that $\overline{n}_0 \cdot D = D' + \sigma\int_0^t D''\,dt'$, and $c(T)$ is a positive continuous nondecreasing function of T.

The general method of proof of this result is similar to the method applied in the proof of Theorem 4.1. The differences in these proofs arise from the fact that necessary estimates are derived in different Sobolev–Slobodetskiĭ spaces.

The existence theorem for problem (4.15)–(4.16), (4.18), (4.23)–(4.24) is applied to prove the solvability of the following auxiliary linear problem:

$$u_t - \nu\nabla_w^2 u + \nabla_w q = F \quad \text{in } \Omega^T, \tag{4.26}$$

$$\nabla_w \cdot u = G \quad \text{in } \Omega^T, \tag{4.27}$$

$$\nu \Pi_0 \Pi_w \mathbb{S}_w(u)\overline{n}_w = \Pi_0 D \quad \text{on } S^T, \tag{4.28}$$

$$\overline{n}_0 \cdot \mathbb{T}_w(u,q)\overline{n}_w - \sigma\overline{n}_0 \cdot \Delta_w(t)\int_0^t u\,dt' = D' + \sigma\int_0^t D''\,dt' \quad \text{on } S^T, \tag{4.29}$$

$$u|_{t=0} = v_0 \quad \text{in } \Omega, \tag{4.30}$$

where $\mathbb{S}_w(u) = \frac{1}{2}\{\partial_{x_i}\xi_k\partial_{\xi_k}u_j + \partial_{x_j}\xi_k\partial_{\xi_k}u_i\}_{i,j=1,2,3}$, $\Pi_w f = f - \overline{n}_w(\overline{n}_w \cdot f)$, $\overline{n}_w(\xi,t) = \overline{n}(X_w(\xi,t),t)$, $X_w(\xi,t) = \xi + \int_0^t w(\xi,t')\,dt'$; $\Delta_w(t)$ is given by (4.8) with $x = X_w(\xi,t)$.

Using the method of successive approximations together with estimate (4.25) yields the following theorem.

Theorem 4.4 (see Theorem 1.2 of [Sol13]). *Let* $\alpha \in (1/2,1)$, $S \in W_2^{5/2+\alpha}$, $F \in W_2^{\alpha,\alpha/2}(\Omega^T)$, $\nabla G \in W_2^{\alpha,\alpha/2}(\Omega^T)$, $D \in W_2^{\alpha+1/2,\alpha/2+1/4}(S^T)$, $D' \in W_2^{\alpha+1/2,\alpha/2+1/4}(S^T)$, $D'' \in W_2^{\alpha-1/2,\alpha/2-1/4}(S^T)$, $v_0 \in W_2^{1+\alpha}(\Omega)$. *Let* $G = \nabla\cdot R$, $R \in L_2(\Omega^T)$, $R_t \in W_2^{0,\alpha/2}(\Omega^T)$ *and assume that the following compatibility conditions are satisfied*:

$$\nabla\cdot v_0 = G(\xi,0), \qquad \nu\Pi_0\mathbb{S}(v_0)\overline{n}_0|_{\xi\in S} = \Pi_0 D|_{t=0}.$$

Moreover, assume that $w \in W_2^{2+\alpha,1+\alpha/2}(\Omega^T)$ *and* $T^{1/2}\|w\|_{W_2^{2+\alpha,1+\alpha/2}(\Omega^T)} \le \delta$ *with sufficiently small* δ. *Then there exists a unique solution of problem* (4.26)–(4.30) *such that* $u \in W_2^{2+\alpha,1+\alpha/2}(\Omega^T)$, $q \in W_2^{\alpha,\alpha/2}(\Omega^T)$, $\nabla q \in W_2^{\alpha,\alpha/2}(\Omega^T)$, $q|_{S^T} \in W_2^{\alpha+1/2,\alpha/2+1/4}(S^T)$ *and estimate* (4.25) *holds with* $c = c' + c''T^{1/2-\alpha/2}\|w(\cdot,0)\|_{W_2^\alpha(\Omega)}$; c' *and* c'' *are positive continuous nondecreasing functions of* T.

Now, Theorem 4.4 and the method of successive approximations imply the following theorem, proved in [Sol13].

Theorem 4.5. *Let* $\alpha \in (1/2,1)$, $S \in W_2^{5/2+\alpha}$, $p_0 = 0$, $v_0 \in W_2^{1+\alpha}(\Omega)$ *and the following compatibility conditions are satisfied*:

$$\nabla\cdot v_0 = 0 \quad \text{in } \Omega, \qquad \nu\Pi_0\mathbb{S}(v_0)\overline{n}_0 = 0 \quad \text{on } S.$$

Assume that the vector field $f(x,t)$ *is continuously differentiable with respect to* x *in* $\mathbb{R}^3\times(0,T_0)$ *and satisfies the Lipschitz condition with respect to* x *and the Hölder condition with exponent* $\beta \ge 1/2$ *with respect to* t. *Then there exists a unique solution of problem* (4.11)–(4.14) *such that* $u \in W_2^{2+\alpha,1+\alpha/2}(\Omega^{T'})$, $q \in W_2^{\alpha,\alpha/2}(\Omega^{T'})$, $\nabla q \in W_2^{\alpha,\alpha/2}(\Omega^{T'})$, $q|_{S^{T'}} \in W_2^{\alpha+1/2,\alpha/2+1/4}(S^{T'})$, *where* $T' \le T_0$. T' *depends on the mean curvature of* S *and the norms of* f *and* v_0.

In the earlier paper of Solonnikov [Sol5] the local existence theorem (analogous to Theorem 4.5) is formulated for the case without the self-gravitational force.

In [MogSol] Mogilevskiĭ and Solonnikov prove local solvability for problem (4.11)–(4.14) with $p_0 = 0$ in Hölder spaces.

Schweizer [Scw] obtains local existence and uniqueness for small initial data by applying the semigroup approach.

4.1.3. *The case of σ dependent on the temperature.* Papers [LagSol], [Sol12] and [Wag] are concerned with the local motion of a fluid bounded by a free boundary which is under surface tension depending on the temperature. In this case the problem to solve is as follows:

$$v_t + (v\cdot\nabla)v - \nu\Delta v + \nabla p = f, \qquad x\in\Omega_t,\ t\in(0,T), \tag{4.31}$$

$$\operatorname{div} v = 0, \qquad x\in\Omega_t,\ t\in(0,T), \tag{4.32}$$

$$\theta_t + (v\cdot\nabla)\theta - \varkappa\Delta\theta = \lambda|\mathbb{S}(v)|^2 + r, \qquad x\in\Omega_t,\ t\in(0,T), \tag{4.33}$$

$$\mathbb{T}\overline{n} - \sigma(\theta)H\overline{n} = \nabla_\tau\sigma(\theta), \qquad x\in S_t,\ t\in(0,T), \tag{4.34}$$

$$\partial\theta/\partial n + \beta\theta = h, \qquad x\in S_t,\ t\in(0,T), \tag{4.35}$$

$$v\cdot\overline{n} = -\phi_t/|\nabla\phi|, \qquad x\in S_t,\ t\in(0,T), \tag{4.36}$$

$$v|_{t=0} = v_0,\ \ \theta|_{t=0} = \theta_0, \qquad x\in\Omega, \tag{4.37}$$

where $\theta = \theta(x,t)$ is the temperature of the fluid; $\varkappa$ is the constant coefficient of heat conductivity; λ and β are positive constants; $\sigma(\theta)\geq\sigma_0>0$ is the coefficient of surface tension which is a smooth function of θ; $\nabla_\tau\sigma = \nabla\sigma - \overline{n}\frac{\partial\theta}{\partial n}$ is the gradient of σ at the surface S_t; f, r and h are given forces. Similarly to the case of σ independent of the temperature we have to rewrite problem (4.31)–(4.37) using Lagrangian coordinates. Then the above problem takes the form

$$u_t - \nu\nabla_u^2 u + \nabla_u q = g \qquad \text{in } \Omega^T, \tag{4.38}$$

$$\nabla_u\cdot u = 0 \qquad \text{in } \Omega^T, \tag{4.39}$$

$$\vartheta_t - \varkappa\nabla_u^2\vartheta - \lambda|\mathbb{S}_u(u)|^2 = r_1 \qquad \text{in } \Omega^T, \tag{4.40}$$

$$\mathbb{T}_u\overline{n}_u - \sigma(\vartheta)\Delta_u(t)X_u = (\nabla_u - \overline{n}_u(\overline{n}_u\cdot\nabla_u))\sigma(\vartheta) \qquad \text{on } S^T, \tag{4.41}$$

$$\overline{n}_u\cdot\nabla_u\vartheta + \beta\vartheta = h_1 \qquad \text{on } S^T, \tag{4.42}$$

$$u|_{t=0} = v_0, \quad \vartheta|_{t=0} = \theta_0 \qquad \text{in } \Omega, \tag{4.43}$$

where $r_1(\xi,t) = r(X_u(\xi,t),t)$, $h_1(\xi,t) = h(X_u(\xi,t),t)$.

By using the estimates for solutions of linear boundary-value problems derived in [MogSol] and the method of successive approximations the following theorem is proved in [Sol12].

THEOREM 4.6. *Let Ω be a bounded domain with boundary $S\in C^{3+\alpha}$, $\alpha\in(0,1)$, and let $r=h=0$, $f\in C^{\alpha,(\alpha+\varepsilon)/2}(\mathbb{R}^3\times(0,T))$, $f_x\in C^{\alpha,(\alpha+\varepsilon)/2}(\mathbb{R}^3\times(0,T))$, where $\varepsilon\in(0,1-\alpha)$. For arbitrary $\theta_0\in C^{2+\alpha}(\Omega)$, $v_0\in C^{2+\alpha}(\Omega)$ satisfying the compatibility conditions*

$$\nabla\cdot v_0 = 0 \quad \text{in } \Omega, \qquad \nu(\mathbb{S}(v_0)\overline{n}_0 - \overline{n}_0(\overline{n}_0\cdot\mathbb{S}(v_0)\overline{n}_0) = \Big(\nabla - \overline{n}_0\frac{\partial}{\partial\overline{n}_0}\Big)\sigma(\theta_0) \quad \text{on } S,$$
$$\partial\theta_0/\partial\overline{n}_0 + \beta\theta_0 = 0 \qquad \text{on } S,$$

problem (4.38)–(4.43) *has a unique solution in the interval* $(0,T')$, $T'\leq T$, *with the following differentiability properties*: $u\in C^{2+\alpha,1+\alpha/2}(\Omega^{T'})$, $\theta\in C^{2+\alpha,1+\alpha/2}(\Omega^{T'})$, $q\in C^{1+\alpha,(1+\alpha)/2}(S^{T'})$, $\nabla q\in C^{\alpha,\alpha/2}(\Omega^{T'})$. *The magnitude of T' depends on the norms of the data v_0,θ_0 and S.*

Local solvability of the above problem in Sobolev spaces is examined by Wagner [Wag]. He proves the local existence and uniqueness of a solution assuming that $\lambda = 0$, $\beta = 0$, and that the forces f, r, h satisfy the conditions

$$\int_{\Omega_t} f\,dx = 0, \qquad \int_{\Omega_t} r\,dx = \int_{S_t} h\,dS.$$

Moreover, boundary condition (4.34) is replaced in his paper by

$$\mathbb{T}\overline{n} - \mathrm{Ma}\,\mathrm{Pr}\,\nabla\theta = 2\,\mathrm{Cr}^{-1}\,\mathrm{Pr}\,H\overline{n},$$

where Ma denotes the Marangoni number which gives the ratio of surface tension tractions generated by temperature inhomogeneities at the surface to the dissipation and heat conduction; Pr is the Prandtl number and Cr is the Crispation number.

The proof of the local existence in [Wag] is as follows. First, it is assumed that for given data, a sufficiently smooth stationary solution has been found and by linearizing around this solution, some estimates for the corresponding stationary problem are derived. Next, by using the method of Rothe these estimates are carried over to estimates for the nonstationary linear problem. Finally, by using the considerations for the nonstationary linear problem and the Banach fixed point theorem, the nonlinear problem is solved.

4.2. Global existence and stability. Global existence and stability theorems for the three-dimensional problem (4.1)–(4.5) can be found in [Sol6, Sol8, Sol9, Sol10]. Moreover, [Sol14] is concerned with global existence of solutions to problem (4.31)–(4.37).

4.2.1. *The case of $\sigma = 0$.* Let $\sigma = 0$, $f = 0$, $p_0 = 0$, $k = 0$. The following theorem is proved in [Sol8].

THEOREM 4.7. *Let $r > 3$, $v_0 \in W_r^{2-2/r}(\Omega)$; let compatibility conditions* (4.19) *be satisfied and assume that*

$$\int_{\Omega} v_0 \cdot \eta\, d\xi = 0,$$

where $\eta = a + b \times \xi$, and a and b are arbitrary constant vectors. If $\|v_0\|_{W_r^{2-2/r}(\Omega)} \le \varepsilon$, where $\varepsilon > 0$ is a sufficiently small constant, then a solution of problem (4.11)–(4.14) *exists for all $t > 0$ and*

$$\|u\|_{W_r^{2,1}(Q_{t,t+1})} + \|q\|_{W_r^{1,0}(Q_{t,t+1})} + [q]_{G_{t,t+1}} \le c\|v_0\|_{W_r^{2-2/r}(\Omega)} e^{-\mu t}, \tag{4.44}$$

where $Q_{t,t+1} = \Omega \times (t, t+1), G_{t,t+1} = S \times (t, t+1), \mu > 0$ and

$$[q]_{G_{t_1,t_2}} = \left(\int_{t_1}^{t_2} dt' \int_0^{t'} \|q(\xi,t') - q(\xi,t'-h)\|^r_{L_r(S)} \frac{dh}{h^{(1+r)/2}}\right)^{1/r}.$$

Now, we present the main steps of the proof from [Sol8].

Proof of Theorem 4.7

STEP 1. The first step is to derive the following conservation laws satisfied by a solution of problem (4.1)–(4.5):

$$\int_{\Omega_t} v\cdot(a+b\times x)\,dx=\int_{\Omega} v_0\cdot\eta\,d\xi \quad \text{(momentum conservation law)}, \tag{4.45}$$

$$\frac{d}{dt}\int_{\Omega_t}|v|^2\,dx+\nu E(v)=0 \quad \text{(energy conservation law)}, \tag{4.46}$$

where $E(v)=\sum_{i,j=1}^{3}\int_{\Omega_t}(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i})^2\,dx$. Equality (4.46) and the Korn inequality

$$\|v\|_{W_2^1(\Omega_t)}\le cE(v),$$

which holds in view of (4.45), yield

$$\int_{\Omega_t}|v|^2\,dx=\int_{\Omega}|u|^2\,d\xi\le e^{-2\mu t}\int_{\Omega}|v_0|^2\,d\xi, \tag{4.47}$$

where $\mu>0$ is a constant depending on ν and c.

Step 2. The following lemma holds.

Lemma 1 (see Theorem 4 of [Sol8]). *Let $u\in W_r^{2,1}(\Omega^T)$, $q\in W_r^{1,0}(\Omega^T)$, $q|_{S^T}\in W_r^{1-1/r,1/2-1/(2r)}(S^T)$ be the solution of (4.11)–(4.14) satisfying*

$$\int_0^T\|u\|_{W_r^2(\Omega)}\,dt+\sup_{t\le T}\|u\|_{L_r(\Omega)}+\sup_{t\le T}\int_0^T\|u\|_{W_r^2(\Omega)}\frac{dt'}{(t-t')^{1/2-1/(2r)}}\le\delta, \tag{4.48}$$

where $\delta>0$ is sufficiently small, and let $0\le t_0\le t_0+1\le T$. Then

$$\|u\|_{W_r^{2,1}(Q(\lambda))}+\|q\|_{W_r^{1,0}(Q(\lambda))}+[q]_{G(\lambda)}\le c\lambda^{-(7/4-3/(2r))}\|u\|_{L_{2,r}(Q(0))}, \tag{4.49}$$

where $Q(\lambda)=\Omega\times(t_0+\lambda,t_0+1)$, $\lambda\in(0,1/2)$, $G(\lambda)=S\times(t_0+\lambda,t_0+1)$, $Q_0=Q(0)$, $\|u\|_{L_{2,r}(Q(0))}=(\int_{t_0}^{t_0+1}\|u\|_{L_2(\Omega)}^r\,dt')^{1/r}$.

Proof. We multiply (4.11)–(4.14) by a function $\zeta_\lambda=\zeta_\lambda(t)$ of class C^∞ such that $\zeta_\lambda(t)=1$ for $t\ge t_0+\lambda$, $\zeta_\lambda(t)=0$ for $t\le t_0+\lambda/2$ and $0\le\zeta_\lambda(t)\le1$, $|\zeta_\lambda'(t)|\le c\lambda^{-1}$, $\lambda\in(0,1/2)$. Then the functions $u_\lambda=\zeta_\lambda u$ and $q_\lambda=\zeta_\lambda q$ satisfy a linear problem of the form (4.15)–(4.18):

$$u_{\lambda t}-\nu\nabla^2u_\lambda+\nabla q_\lambda=\nu(\nabla_u^2-\nabla^2)u_\lambda+(\nabla-\nabla_u)q_\lambda+u\zeta_\lambda'(t)\equiv F, \tag{4.50}$$

$$\nabla\cdot u_\lambda=(\nabla-\nabla_u)\cdot u_\lambda\equiv G, \tag{4.51}$$

$$\mathbb{T}(u_\lambda,q_\lambda)\overline{n}_0|_{\xi\in S}=\mathbb{T}(u_\lambda,q_\lambda)(\overline{n}_0-\overline{n}_u)+[\mathbb{T}(u_\lambda,q_\lambda)-\mathbb{T}_u(u_\lambda,q_\lambda)]\overline{n}_u|_{\xi\in S}\equiv D, \tag{4.52}$$

$$u_\lambda|_{t=t_0}=0 \tag{4.53}$$

Moreover, $G=\nabla\cdot R$, $R=(I-\xi_x)u_\lambda$.

In view of Theorem 4.1 the following estimate holds:

$$\begin{aligned}\|u_\lambda\|_{W_r^{2,1}(Q_0)}&+\|q_\lambda\|_{W_r^{1,0}(Q_0)}+[q_\lambda]_{G_0}\\&\le c_1(\|F\|_{L_r(Q_0)}+\|G\|_{W_r^{1,0}(Q_0)}+\|R_t\|_{L_r(Q_0)}+\|D\|_{W_r^{1-1/r,1/2-1/(2r)}(G_0)}),\end{aligned} \tag{4.54}$$

where $G_0=G(0)=S\times(t_0,t_0+1)$.

Direct calculations by using (4.48) show that

$$\begin{aligned}\|F\|_{L_r(Q_0)}+\|G\|_{W_r^{1,0}(Q_0)}&\le\delta c_2(\|u_\lambda\|_{W_r^{2,0}(Q_0)}+\|\nabla q_\lambda\|_{L_r(Q_0)})\\&\quad+c_3\lambda^{-1}\|u\|_{L_r(Q(\lambda/2))},\end{aligned} \tag{4.55}$$

$$\|D\|_{W_r^{1-1/r,1/2-1/(2r)}(G_0)} \le c_4\delta(\|u_\lambda\|_{W_r^{2,1}(Q_0)} + \|q_\lambda\|_{W_r^{1,0}(Q_0)} + [q_\lambda]_{G_0}), \tag{4.56}$$

$$\|R_t\|_{L_r(Q_0)} \le c_5\delta\|u_{\lambda t}\|_{L_r(Q_0)} + \delta[\varepsilon^{2r/(r+3)}\|u\|_{W_r^{2,0}(Q(\lambda/2))} + (c_6\varepsilon)^{-2r/(r-3)}\|u\|_{L_r(Q(\lambda/2))}], \tag{4.57}$$

where interpolation inequalities have been used.

Inequalities (4.54)–(4.57) yield, for sufficiently small δ,

$$\|u_\lambda\|_{W_r^{2,1}(Q_0)} + \|q_\lambda\|_{W_r^{1,0}(Q_0)} + [q_\lambda]_{G_0} \le 2c_1c_3\lambda^{-1}\|u\|_{L_r(Q(\lambda/2))} + 2c_1\delta[\varepsilon^{2r/(r+3)}\|u\|_{W_r^{2,0}(Q(\lambda/2))} + (c_6\varepsilon)^{-2r/(r+3)}\|u\|_{L_r(Q(\lambda/2))}]. \tag{4.58}$$

To estimate the first and third terms on the right-hand side of (4.58), the following interpolation inequality is used:

$$\|u\|_{L_r(\Omega)} \le \kappa\|u\|_{W_r^2(\Omega)} + c_7\kappa^{-3/2(1/2-1/r)}\|u\|_{L_2(\Omega)}$$

with $\kappa = \varepsilon_1\lambda$ and $\kappa = \varepsilon_2$, respectively. Consequently,

$$\begin{aligned}\|u_\lambda\|_{W_r^{2,1}(Q_0)} + \|q_\lambda\|_{W_r^{1,0}(Q_0)} + [q_\lambda]_{G_0} &\le [2c_1c_3\varepsilon_1 + 2c_1\delta\varepsilon^{2r/(r+3)} + 2c_1\delta\varepsilon_2(c_6\varepsilon)^{-2r/(r+3)}]\|u\|_{W_r^{2,1}(Q(\lambda/2))} \\ &\quad + c_8\lambda^{-7/4+3/2r}\|u\|_{L_{2,r}(Q(\lambda/2))} \\ &\le \varepsilon_0\|u\|_{W_r^{2,1}(Q(\lambda/2))} + c_8\lambda^{-7/4+3/2r}\|u\|_{L_{2,r}(Q(\lambda/2))},\end{aligned} \tag{4.59}$$

if we assume that $2c_1c_3\varepsilon_1 + 2c_1\delta\varepsilon^{2r/(r+3)} + 2c_1\delta\varepsilon_2(c_6\varepsilon)^{-2r/(r+3)} \le \varepsilon_0$. Hence

$$U(\lambda) \le \varepsilon_0 U(\lambda/2) + c_8\lambda^{-7/4+3/2r}\|u\|_{L_{2,r}(Q_0)},$$

where $U(\lambda) = \|u\|_{W_r^{2,1}(Q(\lambda))}$. Therefore

$$U(\lambda) \le \varepsilon_0^j U(\lambda/2^j) + c_8\lambda^{-7/4+3/2r}\|u\|_{L_{2,r}(Q_0)}[1 + \varepsilon_0 2^{7/4-3/2r} + \ldots + (\varepsilon_0 2^{7/4-3/2r})^{j-1}]. \tag{4.60}$$

Assuming that $\varepsilon_0 < 2^{3/2r-7/4}$ estimate (4.60) yields, as $j \to \infty$,

$$U(\lambda) \le \frac{c_8\lambda^{-7/4+3/2r}}{1 - \varepsilon_0 2^{7/4-3/2r}}\|u\|_{L_{2,r}(Q_0)}. \tag{4.61}$$

Now (4.59) and (4.61) imply the assertion of the lemma.

STEP 3. From Theorem 4.2 it follows that for ε sufficiently small there exists a solution in the interval $[0,2]$ and moreover, the following inequality holds:

$$\|u\|_{W_r^{2,1}(\Omega\times(0,2))} + \|q\|_{W_r^{1,0}(\Omega\times(0,2))} + [q]_{G_{0,2}} \le c_9\|v_0\|_{W_r^{2-2/r}(\Omega)}. \tag{4.62}$$

Hence

$$\int_0^2 \|u\|_{W_r^2(\Omega)}\,dt + \sup_{t\le 2}\int_0^2 \|u\|_{W_r^2(\Omega)}\frac{dt'}{(t-t')^{1/2-1/(2r)}} + \sup_{t\le 2}\|u\|_{L_r(\Omega)} \le c_{10}\|v_0\|_{W_r^{2-2/r}(\Omega)} \le \delta/2, \tag{4.63}$$

if $c_{10}\varepsilon \le \delta/2$.

STEP 4. Now, it remains to show that the solution can be extended step by step, first to the interval $(2,3]$, next to $(3,4]$, etc. The proof is by induction.

Assume that the solution exists for $0 \le t \le l$ $(l \ge 2)$ and that condition (4.48) with the constant $\delta/2$ is satisfied for $0 \le t \le l$. As a consequence, the shape of Ω_t changes in $[0, l]$, $l > 2$, no more than it does in $[0, 2]$. Hence, the Korn inequality holds in $[0, l]$ with the same constant as in $[0, 2]$ and estimate (4.47) is satisfied for $t \in [0, l]$. Thus, Lemma 1 yields

$$\|u\|_{W_r^{2,1}(Q^{(j)})} + \|q\|_{W_r^{1,0}(Q^{(j)})} + [q]_{G^{(j)}} \le c_{11}\|v_0\|_{L_2(\Omega)} e^{-\mu j} \quad \text{for } 1 \le j \le l-1, \tag{4.64}$$

where $Q^{(j)} = \Omega \times (j, j+1)$, $G^{(j)} = S \times (j, j+1)$.

Hence, by using the imbedding $W_r^{2,1}(Q^{(j)}) \subset C([j, j+1]; W_r^{2-2/r}(\Omega))$ and by passing to Eulerian coordinates we have

$$\|v(x, l)\|_{W_r^{2-2/r}(\Omega_l)} \le c_{12}\|v_0\|_{L_2(\Omega)} e^{-\mu l}. \tag{4.65}$$

Under the assumption that ε is sufficiently small, estimate (4.65) and Theorem 4.2 imply the existence of a unique solution to problem (4.1)–(4.5) for $x \in \Omega_t$, $l \le t \le l+1$, with the initial condition $v(x, l)$. This solution is such that $u^{(l)} \in W_r^{2,1}(\Omega_l \times (l, l+1))$, $q^{(l)} \in W_r^{1,0}(\Omega_l \times (l, l+1))$, where $u^{(l)}$ and $q^{(l)}$ denote v and p written in Lagrangian coordinates $\xi \in \Omega_l$. Moreover, by (4.65) we get

$$\|u^{(l)}\|_{W_r^{2,1}(\Omega_l)} \le c_{13}\|v\|_{W_r^{2-2/r}(\Omega_l)} \le c_{13}c_{12}\|v_0\|_{L_2(\Omega)} e^{-\mu l}.$$

This way we have extended the solution to $(l, l+1]$ and by the above estimate we have

$$\|u\|_{W_r^{2,1}(Q^{(l)})} \le c_{14}\|v_0\|_{L_2(\Omega)} e^{-\mu l}.$$

Hence, assuming that ε is sufficiently small we obtain

$$\int_l^{l+1} \|u\|_{W_r^2(\Omega)}\, dt + \sup_{l \le t \le l+1} \|u\|_{L_r(\Omega)} + \sup_{l \le t \le l+1} \int_l^t \|u\|_{W_r^2(\Omega)} \frac{dt'}{(t-t')^{1/2r'}} \le \frac{\delta}{2} \quad \text{for } t \in (l, l+1).$$

Thus, condition (4.48) with the constant δ is satisfied in $[0, l+1]$. This implies that for δ sufficiently small, the shape of Ω_t changes for $t \in [0, l+1]$ no more than it does for $t \in [0, l]$. Hence, inequality (4.47) for $t \le l+1$ and estimate (4.64) are satisfied.

It remains to show that inequality (4.48) with the constant $\delta/2$ is satisfied for $t \in [0, l+1]$. By (4.64) we get

$$\begin{aligned}
\int_0^{l+1} \|u\|_{W_r^2(\Omega)}\, dt &\le \sum_{j=0}^{l} \Big(\int_j^{j+1} \|u\|_{W_r^2(\Omega)}^r\, dt \Big)^{1/r} \\
&\le c_9\|v_0\|_{W_r^{2-2/r}(\Omega)} + c_{14}\|v_0\|_{L_2(\Omega)} \sum_{j=1}^{l} e^{-\mu j} \le c_{15}\|v_0\|_{W_r^{2-2/r}(\Omega)},
\end{aligned}$$

$$\|u(\xi, t)\|_{L_r(\Omega)} \le \|v_0\|_{L_r(\Omega)} + \int_0^t \|u_\tau\|_{L_r(\Omega)} d\tau \le (c_{15}+1)\|v_0\|_{W_r^{2-2/r}(\Omega)}$$

$$\int_0^t \|u\|_{W_r^2(\Omega)} \frac{dt'}{(t-t')^{1/2r'}} \le 2^{1/r'} c_{15}\|v_0\|_{W_r^{2-2/r}(\Omega)},$$

where $r' = r/(r-1)$. Assume that ε is so small that $[(2^{1/r'}+2)c_{16}+1]\varepsilon \le \delta/2$. Then the above estimates imply (4.48) with $\delta/2$ in $[0, l+1]$. The proof of Theorem 4.6 is complete. ∎

4.2.2. *The case of* $\sigma > 0$*.* Now, we discuss the results of [Sol6, Sol9, Sol10, Sol14]. The global existence of solutions to problem (4.1)–(4.5) in the case: $\sigma > 0$, $k = 0$, $p_0 = 0$ is proved in [Sol6]. The main result of that paper is the following theorem.

THEOREM 4.8. *Let the assumptions of Theorem* 4.5 *be satisfied and assume that* $f = 0$*. Assume that* Ω *is diffeomorphic to a ball and* S *is described by the equation*

$$|\xi| = \widetilde{R}(\omega), \qquad \omega \in S^1,$$

where S^1 *is the unit sphere. Moreover, assume that*

$$\|v_0\|_{W_2^{1+\alpha}(\Omega)} + \|\widetilde{R} - R_0\|_{W_2^{5/2+\alpha}(S^1)} \le \varepsilon,$$

where $R_0 = (3|\Omega|)^{1/3}(4\pi)^{-1/3}$ *is the radius of a ball of volume* $|\Omega|$*. If* $\varepsilon > 0$ *is sufficiently small then problem* (4.1)–(4.5) *has a solution for all* $t > 0$ *such that* Ω_t *is diffeomorphic to a ball and* S_t *is described by the equation* $|x| = R(\omega, t)$, $\omega \in S^1$, $t > 0$*. Moreover,*

$$\begin{aligned}(4.66)\qquad \sup_{t\ge t_1}\|R(\cdot,t)-R_0\|_{W_2^{5/2+\alpha}(S^1)} + \sup_{t\ge t_1}\|v\|_{W_2^{2+\alpha}(\Omega_t)} &\\ + \sup_{t\ge t_1}\|v_t\|_{W_2^{\alpha}(\Omega_t)} + \sup_{t\ge t_1}\|p-q_0\|_{W_2^{1+\alpha}(\Omega_t)} &\\ \le c(t_1)(\|v_0\|_{L_2(\Omega)} + |S| - 4\pi R_0^2), &\end{aligned}$$

where $t_1 > 0$, $q_0 = 2\sigma/R_0, |S|$ *is the area of* S*.*

The general procedure used to prove the above thoerem is the same as in the case of Theorem 4.7. However, there are major differences in details between the cases with and without surface tension.

Proof of Theorem 4.8 (see [Sol6]).

STEP 1. First, by using estimate (4.25) the following inequality is derived:

$$\begin{aligned}(4.67)\qquad \|u\|_{W_2^{2+\alpha,1+\alpha/2}(\Omega^T)} + \|q-q_0\|_{W_2^{\alpha,\alpha/2}(\Omega^T)} &\\ + \|q-q_0\|_{W_2^{1/2+\alpha,1/4+\alpha/2}(S^T)} + \|\nabla q\|_{W_2^{\alpha,\alpha/2}(\Omega^T)} &\\ \le C(\|v_0\|_{W_2^{1+\alpha}(\Omega)} + \|H(\cdot,0)+2/R_0\|_{W_2^{1/2+\alpha}(S)}), &\end{aligned}$$

where $T > 0$ is the time of local existence. Moreover, it follows from the proof of Theorem 4.5 (see [Sol13]) that if $\varepsilon \to 0$ then $T \to \infty$.

STEP 2. The following lemma is proved.

LEMMA 1. *Let* S_t $(t \le T)$ *be described by the equation* $|x| = R(\omega, t)$, $\omega \in S^1$ *and let the origin of coordinates coincide with the barycentre of* Ω_t*. There exists a constant* $\widehat{\delta} \in (0, 1/2)$ *such that if*

$$(4.68)\qquad \sup_{S^1}|R(\omega,t)-R_0| + \sup_{S^1}|\nabla R(\omega,t)| \le \widehat{\delta} R_0 \quad \textit{for } t \le T,$$

where ∇ *is the gradient on* S^1*, then*

$$(4.69)\qquad \int_{S^1}((R(\omega,t)-R_0)^2 + |\nabla R(\omega,t)|^2)d\omega \le c_1(|S_t| - 4\pi R_0^2),$$

where $c_1 > 0$ *is a constant independent of* δ *and* R_0*.*

STEP 3. The following equation is considered:

$$H[R] + 2/R_0 = h(\omega), \tag{4.70}$$

where $H[R]$ is the double mean curvature of S_t expressed in spherical coordinates.

LEMMA 2. *Let $\alpha \in (1/2, 1)$ and let $R \in W_2^{3/2+\alpha}(S^1)$ be a solution of* (4.70) *satisfying* (4.68) *with sufficiently small $\widehat{\delta}$. If $h \in W_2^{\mu}(S^1)$, $\mu \in (0,1)$, then*

$$\|R - R_0\|_{W_2^{2+\mu}(S^1)} \le c_2 \|h\|_{W_2^{\mu}(S^1)} + c_3 \|R - R_0\|_{L_2(S^1)}.$$

If $h \in W_2^{1+\mu_1}(S^1)$, $\mu_1 \in (0,1)$, then

$$\|R - R_0\|_{W_2^{3+\mu_1}(S^1)} \le c_4 \|h\|_{W_2^{1+\mu_1}(S^1)} + c_5 \|R - R_0\|_{L_2(S^1)}. \tag{4.71}$$

The constants c_3 and c_5 may depend on $\|R\|_{W_2^{\alpha+3/2}(S^1)}$.

STEP 4. As in the case $\sigma = 0$ the conservation laws for momentum and energy are crucial to the proof. In this case the conservation laws have the forms:

$$\frac{d}{dt} \int_{\Omega_t} v \cdot (a + b \times x)\, dx = 0, \tag{4.72}$$

$$\frac{d}{dt}\Big(\int_{\Omega_t} |v|^2\, dx + 2\sigma |S_t| \Big) + \nu E(v) = 0. \tag{4.73}$$

Integrating (4.72) yields

$$\int_{\Omega_t} v \cdot (a + b \times x)\, dx = \int_{\Omega} v_0 \cdot (a + b \times \xi)\, d\xi,$$

and integrating (4.73) and then using (4.69) (under the assumption (4.68)) gives

$$\int_{\Omega_t} |v|^2\, dx + \frac{2\sigma}{c_1} \|R - R_0\|^2_{W_2^1(S^1)} + \nu \int_0^t E(v)\, dt' \le \int_{\Omega} |v_0|^2\, d\xi + 2\sigma(|S| - 4\pi R_0^2). \tag{4.74}$$

STEP 5. Now, the following estimate is derived for the local solution of (4.11)–(4.14) (see Theorem 6 of [Sol6]):

$$\begin{aligned} \|u\|_{W_2^{2+\alpha,1+\alpha/2}(Q(\lambda))} &+ \|q - q_0\|_{W_2^{\alpha,\alpha/2}(Q(\lambda))} \\ &+ \|\nabla q\|_{W_2^{\alpha,\alpha/2}(Q(\lambda))} + \|q - q_0\|_{W_2^{1/2+\alpha,1/4+\alpha/2}(G(\lambda))} \\ &\le c_6 \lambda^{-s} \big(\|u\|_{L_2(Q(0))} + \|R - R_0\|_{L_2(S^1 \times (t_0,T))} \big), \end{aligned} \tag{4.75}$$

under the assumption that $T^{1/2} \|u\|_{W_2^{2+\alpha,1+\alpha/2}(\Omega^{T'})} \le \delta$ with δ sufficiently small. In (4.75), $\lambda \in (0,1)$, $t_0 + \lambda < T$, $Q(\lambda) = \Omega \times (t_0 + \lambda, T)$, $G(\lambda) = S \times (t_0 + \lambda, T)$, $t_0 > 0$, $s > 0$.

Proof of estimate (*4.75*). The above estimate is analogous to estimate (4.49) which holds for the case $\sigma = 0$. To prove (4.75) we use the same function ζ_λ as before. Then the functions $u_\lambda = u\zeta_\lambda$ and $q_\lambda = (q - \frac{2\sigma}{R_0})\zeta_\lambda$ are the solution of the problem

$$u_{\lambda t} - \nu \nabla_u^2 u_\lambda + \nabla_u q_\lambda = u \zeta'_\lambda \quad \text{in } \Omega^T, \tag{4.76}$$

$$\nabla_u \cdot u_\lambda = 0 \quad \text{in } \Omega^T, \tag{4.77}$$

$$\Pi_0 \Pi_u \mathbb{S}_u(u_\lambda) \overline{n}_u = 0 \quad \text{on } S^T, \tag{4.78}$$

$$
\begin{aligned}
(4.79)\qquad & \overline{n}_0\cdot \mathbb{T}_u(u_\lambda,q_\lambda)\overline{n}_u-\sigma\overline{n}_0\cdot\int_0^t \Delta_u(t')u_\lambda\,dt'\\
&=\sigma\int_0^t\bigg[\frac{1}{\sigma}\zeta_\lambda'(t')\overline{n}_0\cdot\mathbb{T}_u\Big(u,q-\frac{2\sigma}{R_0}\Big)\overline{n}_u\\
&\quad+\zeta_\lambda(t')\overline{n}_0\cdot\Delta_u'(t')\xi+\zeta_\lambda(t')\overline{n}_0\cdot\Delta_u'(t')\int_0^{t'}u\,dt''\\
&\quad+\frac{2}{R_0}\zeta_\lambda(t')\frac{\partial}{\partial t'}(\overline{n}_0\cdot\overline{n}_u)\bigg]\,dt'\\
&\equiv\sigma\sum_{i=1}^4\int_0^t B_i\,dt' \qquad\text{on } S^T,\\
(4.80)\qquad & u_\lambda|_{t=0}=0 \qquad\text{in } \Omega,
\end{aligned}
$$

where Δ_u' denotes the operator obtained from Δ_u by differentiation of the coefficients with respect to t.

For the solution (u_λ,q_λ) of problem (4.76)–(4.80) inequality (4.25) is satisfied. To obtain (4.75) it suffices to estimate the norms $\|B_i\|_{W_2^{\alpha-1/2,\alpha/2-1/4}(S^T)}$. By direct calculations one can get

$$
\begin{aligned}
(4.81)\qquad & \|u_\lambda\|_{W_2^{2+\alpha,1+\alpha/2}(\Omega^T)}+\|\nabla q_\lambda\|_{W_2^{\alpha,\alpha/2}(\Omega^T)}+\|q_\lambda\|_{W_2^{\alpha,\alpha/2}(\Omega^T)}+\|q_\lambda\|_{W_2^{\alpha+1/2,\alpha/2+1/4}(S^T)}\\
&\le c(\lambda^{-1}\|u\|_{W_2^{\alpha,\alpha/2}(Q(\lambda/2))}+\lambda^{-3/4-\alpha/2}\|u\|_{L_2(Q(\lambda/2))}+\lambda^{-1}\|D_\xi u\|_{W_2^{\alpha-1/2,\alpha/2-1/4}(G(\lambda/2))}\\
&\quad+\lambda^{-3/4-\alpha/2}\|D_\xi u\|_{L_2(G(\lambda/2))}+\lambda^{-\alpha/2+1/4}\|D_\xi^2 u\|_{L_2(G(\lambda/2))}\\
&\quad+\lambda^{-1}\|q-2\sigma/R_0\|_{W_2^{\alpha-1/2,\alpha/2-1/4}(G(\lambda/2))}+\lambda^{-3/4-\alpha/2}\|q-2\sigma/R_0\|_{L_2(G(\lambda/2))}).
\end{aligned}
$$

To estimate $\|q-2\sigma/R_0\|_{L_2(G(\lambda/2))}$ the boundary condition $\overline{n}_u\cdot\mathbb{T}_u(u,q-2\sigma/R_0)\overline{n}_u=\sigma(H+2/R_0)$ is used. Thus, applying an interpolation inequality we have

$$
\begin{aligned}
(4.82)\qquad & \|q-2\sigma/R_0\|_{L_2(G(\lambda/2)}\le c\|D_\xi u\|_{L_2(G(\lambda/2))}+\sigma\|H+2/R_0\|_{L_2(G(\lambda/2)}\\
&\le c\Big[\|D_\xi u\|_{L_2(G(\lambda/2))}+\Big(\int_{t_0}^T\|R-R_0\|^2_{W_2^2(S^1)}\,dt\Big)^{1/2}\Big]\\
&\le c\|D_\xi u\|_{L_2(G(\lambda/2))}+\kappa^{\alpha-1/2}\Big(\int_{t_0}^T\|R-R_0\|^2_{W_2^{3/2+\alpha}(S^1)}\Big)^{1/2}+c\kappa^{-2}\Big(\int_{t_0}^T\|R-R_0\|^2_{L_2(S^1)}\Big)^{1/2},
\end{aligned}
$$

where κ is a sufficiently small constant. Next, using Lemma 2 from Step 3 and the boundary condition yields

$$
\begin{aligned}
(4.83)\qquad & \|R-R_0\|_{W_2^{3/2+\alpha}(S^1)}\le c(\|H[R]+2/R_0\|_{W_2^{\alpha-1/2}(S^1)}+\|R-R_0\|_{L_2(S^1)})\\
&\le c(\|q-2\sigma/R_0\|_{W_2^{\alpha-1/2}(S)}+\|D_\xi u\|_{W_2^{\alpha-1/2}(S)}+\|R-R_0\|_{L_2(S^1)}).
\end{aligned}
$$

Combining (4.82) and (4.83) we obtain

$$
\begin{aligned}
(4.84)\qquad & \|q-2\sigma/R_0\|_{L_2(G(\lambda/2))}\le c[\kappa^{\alpha-1/2}(\|D_\xi u\|_{W_2^{\alpha-1/2,\alpha/2-1/4}(G(\lambda/2))}\\
&\quad+\|q-2\sigma/R_0\|_{W_2^{\alpha-1/2}(G(\lambda/2))})+\|D_\xi u\|_{L_2(G(\lambda/2))}+\kappa^{-2}\|R-R_0\|_{L_2(S^1\times(t_0,T))}].
\end{aligned}
$$

Taking into account (4.81), (4.84) and using appropriate intepolation inequalities to the norms $\|D_\xi u\|_{L_2(G(\lambda/2))}$, $\|D_\xi u\|_{W_2^{\alpha-1/2,\alpha/2-1/4}(G(\lambda/2))}$, $\|u\|_{W_2^{\alpha,\alpha/2}(Q(\lambda/2))}$ and $\|q-2\sigma/R_0\|_{W_2^{\alpha-1/2,\alpha/2-1/4}(G(\lambda/2))}$ we get

$$U(\lambda)\le \varepsilon U(\lambda/2)+C(\varepsilon)\lambda^{-s}(\|u\|_{L_2(Q(0))}+\|R-R_0\|_{L_2(S^1\times(t_0,T))}),$$

where $s=(\frac{3}{2}+\alpha)\frac{2+\alpha}{1+\alpha}$, $\varepsilon<2^{-s}$, $U(\lambda)=\|u\|_{W_2^{2+\alpha,1+\alpha/2}(Q(\lambda))}+\|q-2\sigma/R_0\|_{W_2^{\alpha+1/2,\alpha/2+1/4}(G(\lambda))}$ $+\|\nabla q\|_{W_2^{\alpha,\alpha/2}(Q(\lambda))}+\|q-2\sigma/R_0\|_{W_2^{\alpha,\alpha/2}(Q(\lambda))}$. Hence

$$U(\lambda)\le \frac{C(\varepsilon)\lambda^{-s}}{1-2^s\varepsilon}(\|u\|_{L_2(Q(0))}+\|R-R_0\|_{L_2(S^1\times(t_0,T))}).$$

The last inequality coincides with (4.75).

Step 6. Now define the differences $u^{(s)}(\xi,t)=u_\lambda(\xi,t)-u_\lambda(\xi,t-s)$, $q^{(s)}(\xi,t)=q_\lambda(\xi,t)-q_\lambda(\xi,t-s)$ (where $s<t_0$, $\lambda=(T+t_0)/2$). The functions $u^{(s)}$ and $q^{(s)}$ satisfy a linear problem which is implied by problem (4.76)–(4.80). Therefore, estimate (4.25) applied to $u^{(s)}$ and $q^{(s)}$ and (4.75) yield the inequality

$$\begin{aligned}\|u^{(s)}\|_{W_2^{2+\alpha,1+\alpha/2}(Q(\lambda))}+\|\nabla q^{(s)}\|_{W_2^{\alpha,\alpha/2}(Q(\lambda))}+\|q^{(s)}\|_{W_2^{\alpha,\alpha/2}(Q(\lambda))}&\\ +\|q^{(s)}\|_{W_2^{1/2+\alpha,1/4+\alpha/2}(G(\lambda))}&\\ \le c_7(\|u\|_{L_2(Q(0))}+\|R-R_0\|_{L_2(S^1\times(t_0,T))})s^\beta,&\end{aligned}\tag{4.85}$$

where $\beta>1/2$. From (4.85) and the imbedding properties in Besov spaces we get the estimate

$$\begin{aligned}\sup_{t_1<t<T'}\|u\|_{W_2^{2+\alpha}(\Omega)}+\sup_{t_1<t<T'}\|q-q_0\|_{W_2^{1+\alpha}(\Omega)}&\\ \le c_8(\|u\|_{L_2(Q(0))}+\|R-R_0\|_{L_2(S^1\times(t_0,T))})\quad \text{for } t_1>t_0.&\end{aligned}\tag{4.86}$$

Step 7. Assume that ε is so small that problem (4.11)–(4.14) is locally solvable in the interval $(0,1)$ and estimate (4.67) holds. Then estimates (4.74), (4.75) and (4.86) imply

$$\|v(x,t)\|_{W_2^{1+\alpha}(\Omega_t)}\le c_9\varepsilon,\quad t\in(t_0,1].\tag{4.87}$$

Moreover, estimates (4.71), (4.69) and (4.86) give

$$\|R(\omega,t)-R_0\|_{W_2^{5/2+\alpha}(S^1)}\le c_{10}\varepsilon,\quad t\in(t_0,1].\tag{4.88}$$

Estimates (4.87)–(4.88) enable extending the solution to the interval $(1,2]$ if $\varepsilon>0$ is sufficiently small. Then it is shown by induction that in view of the estimates and relations occurring in Steps 1–6 the solution can be extended for $t>0$. ■

In [Sol9] Solonnikov considers the same problem as in [Sol6], but he also studies the asymptotic behaviour of the solution as $t\to\infty$. He proves that under the assumptions of Theorem 4.8 and assuming that $\int_\Omega v_0\,d\xi=0$, the solution of problem (4.1)–(4.5) tends as $t\to\infty$ to a quasi-stationary solution of this problem which corresponds to a rotation of the fluid as a rigid body around the axis parallel to the vector $m=\int_\Omega(v_0\times\xi)\,d\xi$. The free boundary then tends to an equilibrium figure.

The two-dimensional case is analyzed in detail and it is shown that in this case the equilibrium figure is the circle $S_\infty = \{x : |x| = R_0\}$. Moreover, it is proved that

$$v \to v_\infty = a(x_1, -x_2), \qquad p \to p_\infty = \frac{a^2}{2}|x|^2 + \frac{\sigma}{R_0} - \frac{a^2}{2}R_0^2 \quad \text{as } t \to \infty,$$

where $a = 2b(\pi R_0^4)^{-1}$, $b = \int_\Omega v_0 \cdot (\xi_2, -\xi_1)\, d\xi$. The above convergence is uniform in x. Moreover, the first derivatives of v with respect to x tend uniformly to the first derivatives of v_∞.

The case when the self-gravitational force exists, i.e. when $k > 0$, is considered in [Sol10]. The method used to prove global existence in this case is the same as for the case $k = 0$. Under the assumption that $f = 0$ the momentum conservation law (4.72) holds, and the energy conservation law takes the form

$$\frac{d}{dt}\bigg(\int_{\Omega_t} |v|^2\, dx + 2\sigma|S_t| - k\int_{\Omega_t}\int_{\Omega_t} \frac{dx\,dy}{|x-y|}\bigg) + \nu E(v) = 0. \tag{4.89}$$

Under the assumptions of Lemma 1 from the proof of Theorem 4.8, equality (4.89) and the above-mentioned Lemma 1 yield the following estimate analogous to (4.74):

$$\begin{aligned}\int_{\Omega_t} |v|^2\, dx + (C_1\sigma + C_2k)\int_{S^1} (R(\omega,t) - R_0)^2 d\omega + C_3\sigma\int_{S^1} |\nabla R(\omega,t)|^2 d\omega + \nu\int_0^t E(v)\, dt' \\ \le \int_\Omega |v_0|^2\, d\xi + (C_4\sigma + C_5k)\int_\Omega (R(\omega,0) - R_0)^2 d\omega + C_6\int_{S^1} |\nabla R(\omega,0)|^2 d\omega,\end{aligned} \tag{4.90}$$

where C_i $(i = 1,\dots,6)$ are positive constants; ∇ is the gradient on S^1.

It is proved that in this case estimates (4.75) and (4.86) also hold with q replaced by $q' = q - kU_u - 2\sigma/R_0 + (4/3)\pi kR_0^2$ and with the right-hand sides replaced by $C_7\lambda^{-s}(\|u\|_{L_2(Q(0))} + (\sigma+k)\|R - R_0\|_{L_2(S^1\times(t_0,T))} + \sigma\|\nabla R\|_{L_2(S^1\times(t_0,T))})$ and $C_8(\|u\|_{L_2(Q(0))} + \|R - R_0\|_{L_2(S^1\times(t_0,T))} + \|\nabla R\|_{L_2(S^1\times(t_0,T))})$, respectively. Thus, estimate (4.86) has the following form in this case:

$$\begin{aligned}\sup_{t_1<t<T} \|u\|_{W_2^{2+\alpha}(\Omega)} + \sup_{t_1<t<T} \|u_t\|_{L_2(\Omega)} + \sup_{t_1<t<T} \|q'\|_{W_2^{1+\alpha}(\Omega)} \\ \le C_8(\|u\|_{L_2(Q(0))} + \|R - R_0\|_{L_2(S^1\times(t_0,T)} + \|\nabla R\|_{L_2(S^1\times(t_0,T)}) \quad \text{for } t_1 > t_0.\end{aligned} \tag{4.91}$$

Therefore, the proof of global existence in this case can be done in the same way as in the proof of Theorem 4.8. Thus, under the additional assumption that $\int_\Omega v_0\, d\xi = 0$ the assertion of Theorem 4.8 holds with p replaced by $p' = p - kU - 2\sigma/R_0 + (4/3)\pi kR_0^3$.

Moreover, it is proved in [Sol10] that

$$v(x,t) - a\eta_3(x) \to 0, \quad R(\omega,t) \to r(\omega), \quad p(x,t) \to p_\infty(x) = \frac{a^2}{2}(x_1^2 + x_2^2) + \frac{2\sigma}{R_0} - \frac{4}{3}\pi kR_0^2 + k_1$$

as $t \to \infty$ and

$$\Omega_\infty = \{x \in \mathbb{R}^3 : |x| < r(x/|x|)\},$$

where $\eta_3(x) = (x_2, -x_1, 0)$; $a = b(\int_{\Omega_\infty}(x_1^2 + x_2^2)\, dx)^{-1}$; b is such that $\int_\Omega (v_0 \times \xi)\, d\xi =$

$(0,0,b)$; $r=\varrho(\omega)+R_0$ and ϱ is the solution of the equation

$$\frac{\sigma}{R_0}(\Delta\varrho+2\varrho)-\frac{4}{3}\pi kR_0^2\varrho+kR_0^2\int_{S^1}\varrho(\omega')\frac{d\omega'}{|\omega-\omega'|}+R_0k_1+\varrho k_1=F. \tag{4.92}$$

In (4.92),

$$F=\frac{\sigma}{R_0}(f\Delta\varrho+\nabla f\cdot\nabla\varrho)-\frac{2\sigma|\nabla\varrho|^2}{\sqrt{g}(r+\sqrt{g})}-\frac{a^2}{2}r^3(\omega_1^2+\omega_2^2)$$
$$-k\varrho(U_\infty(r\omega)-\frac{4}{3}\pi R_0^2)-kr\int_0^1(1-s)\frac{d^2}{ds^2}U(r\omega;s)\,ds$$

and k_1 is defined by the equation

$$k_1(4\pi R_0+Q[\varrho])=\int_{S^1}F d\omega-\frac{2\sigma}{R_0}Q[\varrho]-\frac{8}{3}\pi kR_0^2Q[\varrho],$$

where $Q[\varrho]=-\frac{1}{R_0}\int_{S^1}\varrho^2d\omega-\frac{1}{3R_0^2}\int_{S^1}\varrho^3d\omega$.

Moreover, in the above formulas ∇ is the gradient and Δ is the Laplacian on S^1; $g=r^2+|\nabla r|^2$, $f=(2R_0\varrho+\varrho^2+|\nabla\varrho|^2)/(\sqrt{g}(R_0+\sqrt{g}))$, $U_\infty(x)=\int_{\Omega_\infty}|x-y|^{-1}\,dy$, $U(x;s)=\int_{\Omega(s)}|x-y|^{-1}\,dy$, $\Omega(s)=\{|x|<R_s(x/|x|)\}$, $R_s=R_0+s\varrho$.

As before, the above convergence is uniform in x or ω. Moreover, the first derivatives of v with respect to x tend uniformly to the first derivatives of v_∞, and the first and second derivatives of R tend uniformly to the first and second derivatives of r.

In [Sol10] Solonnikov examines the solvability of equation (4.92). It is shown that if b is sufficiently small then there exists a solution $\varrho\in C^\infty(S^1)$ of (4.92). Moreover, this solution is unique in the class of functions $\varrho\in C^2(S^1)$ satisfying the inequality $\sup_{S^1}|\varrho|+\sup_{S^1}|\nabla\varrho|\le\delta R_0$ with sufficiently small δ and some other conditions which are consequences of the conservation laws.

Paper [Sol14] is devoted to the case of σ depending on the temperature. The global solvability of problem (4.31)–(4.37) in Hölder spaces is proved. More precisely, Solonnikov proves the following theorem.

THEOREM 4.9. *Let Ω be a domain defined by the inequality $|x|<R(\omega,0)$, $\omega=x/|x|$ with $R\in C^{3+\alpha}(S^1)$, $\alpha\in(0,1)$, let the assumptions of Theorem 4.6 be satisfied and assume that $f=0$. Moreover, assume that*

$$\|v_0\|_{C^{2+\alpha}(\Omega)}+\|\theta_0\|_{C^{2+\alpha}(\Omega)}+\|R(\cdot,0)-R_0\|_{C^{3+\alpha}(S^1)}\le\varepsilon.$$

Then for sufficiently small ε there exists a solution of problem (4.31)–(4.37) with the properties: $R\in C^{3+\alpha,(3+\alpha)/2}(S^1\times(0,\infty))$, $R_t\in C^{2+\alpha,(2+\alpha)/2}(S^1\times(0,\infty))$, $v\in C^{2+\alpha,(2+\alpha)/2}(Q^\infty)$, $\nabla p\in C^{\alpha,\alpha/2}(Q^\infty)$, $p\in C^{1+\alpha,(1+\alpha)/2}(\Sigma^\infty)$, where $Q^\infty=\{x\in\Omega_t: t>0\}$, $\Sigma^\infty=\{x\in S_t: t>0\}$.

The general scheme of proof is the same as in the case of $\sigma=\text{const}$. The crucial point as usual are the conservation laws and their consequences. Moreover, assuming that the solution of problem (4.31)–(4.37) is defined for $t\in(0,t_0+1)$ (where $t_0\ge 0$), estimates for higher order derivatives of w,s,θ (where $w=v-v_\infty$, $s=p-p_\infty$, v_∞ and p_∞ are the same as above) are derived in the interval $(t_0+1/2,t_0+1)$ in terms of lower order norms of these functions in the interval (t_0, t_0+1). To obtain these estimates Solonnikov uses the theory of parabolic initial-boundary value problems developed in [Sol1] and [LadSolUr], estimates from [Mog] and [MogSol] and interpolation inequalities.

5. Three-dimensional free boundary problem for a drop of a compressible fluid

5.1. The motion of a compressible viscous barotropic fluid. This section is devoted to the motion of a fixed mass of a viscous compressible barotropic fluid bounded by a free surface. Such a motion is described by the following system of equations with boundary and initial conditions:

$$\varrho[v_t + (v\cdot\nabla)v] - \operatorname{div}\mathbb{T}(v,p) = \varrho(f + k\nabla U), \quad x\in\Omega_t,\ t\in(0,T), \tag{5.1}$$

$$\varrho_t + \operatorname{div}(\varrho v) = 0, \quad x\in\Omega_t,\ t\in(0,T), \tag{5.2}$$

$$\mathbb{T}\overline{n} - \sigma H\overline{n} = -p_0(x,t)\overline{n}, \quad x\in S_t,\ t\in(0,T), \tag{5.3}$$

$$v\cdot\overline{n} = -\phi_t/|\nabla\phi|, \quad x\in S_t,\ t\in(0,T), \tag{5.4}$$

$$v(x,0) = v_0(x), \quad \varrho(x,0) = \varrho_0(x), \quad x\in\Omega_0\equiv\Omega, \tag{5.5}$$

where $\Omega_t\subset\mathbb{R}^3$ is a bounded domain at time t which is unknown together with the velocity $v = v(x,t)$ and the density $\varrho = \varrho(x,t)$ of the fluid, $T > 0$. Moreover, $p = p(\varrho)$ is the pressure, U is the self-gravitational potential which has the following form in this case:

$$U(x,t) = \int_{\Omega_t}\frac{\varrho(y,t)}{|x-y|}\,dy, \tag{5.6}$$

and H is the double mean curvature of $S_t = \partial\Omega_t$ given by (4.7).

As before, by $\mathbb{T} = \mathbb{T}(v,p)$ we denote the stress tensor which in the case of compressible fluid has the form

$$\mathbb{T}(v,p) = \{T_{ij}\}_{i,j=1,2,3} = \{-p\delta_{ij} + 2\mu S_{ij}(v) + (\nu-\mu)\delta_{ij}\operatorname{div} v\}_{i,j=1,2,3}, \tag{5.7}$$

where as in Section 4, $\mathbb{S}(v) = \{S_{ij}\}_{i,j=1,2,3}$ is the velocity deformation tensor, $\mathbf{I} = \{\delta_{ij}\}_{i,j=1,2,3}$ is the unit matrix, and μ and ν are the constant viscosity coefficients such that $\nu > (1/3)\mu > 0$.

The remaining quantities in problem (5.1)–(5.5) have the same meaning as in problem (4.1)–(4.5) of Section 4.

Since the fluid is compressible, from the continuity equation (5.2) and the kinematic condition (5.4) it follows that the total mass of the fluid is conserved, i.e.

$$\int_{\Omega_t}\varrho(x,t)\,dx = \int_{\Omega}\varrho_0(\xi)\,d\xi = M, \tag{5.8}$$

where M is a constant.

5.1.1. *Local existence.* Local existence theorems for problem (5.1)–(5.5) can be found in [SolT1, SolT2, StZaj, Zaj2, Zaj3, Zaj5]. In all these papers in order to prove the local existence of solutions, problem (5.1)–(5.5) is rewritten in Lagrangian coordinates ξ which are the initial data of problem (4.9).

Let $u(\xi,t)=v(X_u(\xi,t),t)$, $\eta(\xi,t)=\varrho(X_u(\xi,t),t)$, $\overline{n}_u(\xi,t)=\overline{n}(X_u(\xi,t),t)$, $q_0(\xi,t)=p_0(X_u(\xi,t),t)$, $g(\xi,t)=f(X_u(\xi,t),t)$, where $X_u(\xi,t)$ is given by (4.10). Then problem (5.1)–(5.5) takes the following form in Lagrangian coordinates:

(5.9) $$\eta u_t-\mu\nabla_u^2 u-\nu\nabla_u\nabla_u\cdot u+\nabla_u p(\eta)=\eta(g+k\nabla_u U_u)\quad \text{in } \Omega^T,$$

(5.10) $$\eta_t+\eta\nabla_u\cdot u=0\quad \text{in } \Omega^T,$$

(5.11) $$\mathbb{T}_u(u,p)\overline{n}_u-\sigma\Delta_u(t)X_u=-q_0(\xi,t)\overline{n}_u\quad \text{on } S^T,$$

(5.12) $$u|_{t=0}=v_0,\quad \eta|_{t=0}=\varrho_0\quad \text{in } \Omega,$$

where

$$U_u(\xi,t)=\int_\Omega \frac{\eta(\xi',t)}{|X_u(\xi,t)-X_u(\xi',t)|}J_{X_u(\xi',t)}\,d\xi',$$

$$\begin{aligned}\mathbb{T}_u(u,p)&=-pI+\mathbb{D}_u(u)\\&=\{-p(\eta,\vartheta)\delta_{ij}+\mu(\partial_{x_i}\xi_k\partial_{\xi_k}u_j+\partial_{x_j}\xi_k\partial_{\xi_k}u_i)+(\nu-\mu)\delta_{ij}\nabla_u\cdot u\}_{i,j=1,2,3};\end{aligned}$$

∇_u, Δ_u and ξ_x are defined in Section 4.

A local solution of the problem with the lowest possible regularity in the L_2-approach is obtained by Solonnikov and Tani [SolT1, SolT2]. In [SolT2] they prove the following theorem.

THEOREM 5.1. *Let* $\alpha\in(1/2,1)$, $S\in W_2^{5/2+\alpha}$, $v_0\in W_2^{1+\alpha}(\Omega)$, $\varrho_0\in W_2^{1+\alpha}(\Omega)$, $\inf_{\xi\in\Omega}\varrho_0(\xi)\ge\varrho_*>0$, $p\in C^3(\mathbb{R}_+)$ *and assume that* f *has continuous derivatives of order one and two*, p_0 *is three times continuously differentiable with respect to* x *and that* f, f_x *satisfy the Hölder condition with exponent* $\beta\ge 1/2$, *and* p_0, p_{0x} *satisfy the Lipschitz condition with respect to* t. *Moreover, assume that the following compatibility condition is satisfied*:

$$-p(\varrho_0)\overline{n}_0+(\nu-\mu)(\nabla\cdot v_0)\overline{n}_0+2\mu\mathbb{S}(v_0)\overline{n}_0=(\sigma H\overline{n}_0-p_0\overline{n}_0)|_{t=0}\quad \textit{on } S,$$

where $\overline{n}_0$ *is the unit outward vector normal to* S. *Then problem* (5.9)–(5.12) *has a unique solution* $(u,\eta)\in W_2^{2+\alpha,1+\alpha/2}(\Omega^{T^*})\times C([0,T];W_2^{1+\alpha}(\Omega))\cap W_2^{1+\alpha,1/2+\alpha/2}(\Omega^{T^*})$ *on a finite time interval* $(0,T^*)$, *the length of which depends on the data*, *i.e. on norms of* f, p_0, v_0, ϱ_0 *and on the mean curvature of* S.

Sketch of proof. First, the following linear problem is considered:

(5.13) $$u_t-\varrho_0^{-1}(\mu\Delta u+\nu\nabla\operatorname{div}u)=F\quad \text{in } \Omega^T,$$

(5.14) $$\mu\Pi_0\mathbb{S}(u)\overline{n}_0=\Pi_0 b_1\quad \text{on } S^T,$$

(5.15) $$\overline{n}_0\cdot\mathbb{D}(u)\overline{n}_0-\sigma\overline{n}_0\cdot\Delta_S\int_0^t u\,dt'=b_2\quad \text{on } S^T,$$

(5.16) $$u|_{t=0}=u_0\quad \text{in } \Omega,$$

where $\mathbb{D}(u)=2\mu\mathbb{S}(u)+(\nu-\mu)\operatorname{div}u\mathbf{I}$, Π_0 is defined in Section 4. A solvability result for the above problem is obtained. The method used to obtain the existence of a unique solution for problem (5.13)–(5.16) is similar to the methods from [Sol5, Sol11, Sol13].

Next, the following linear problem is studied:

$$u_t - \varrho_0^{-1}(\mu\nabla_w^2 u + \nu\nabla_w\nabla_w\cdot u) = F \quad \text{in } \Omega^T, \tag{5.17}$$

$$\mu\Pi_0\Pi_w\mathbb{S}_w(u)\overline{n}_w = \Pi_0 b_1 \quad \text{on } S^T, \tag{5.18}$$

$$\overline{n}_0\cdot\mathbb{D}_w(u)\overline{n}_w - \sigma\overline{n}_0\cdot\Delta_w(t)\int_0^t u\,dt' = b_2 \quad \text{on } S^T, \tag{5.19}$$

$$u|_{t=0} = u_0 \quad \text{in } \Omega, \tag{5.20}$$

where $\Pi_w D = D - \overline{n}_w(\overline{n}_w\cdot D)$, $\mathbb{D}_w(u) = 2\mu\mathbb{S}_w(u) + (\nu-\mu)\nabla_w\cdot u\mathbf{I}$, $\mathbb{S}_w(u) = \frac{1}{2}\{\partial_{x_i}\xi_k\partial_{\xi_k}u_j + \partial_{x_j}\xi_k\partial_{\xi_k}u_i\}_{i,j=1,2,3}$, $\overline{n}_w(\xi,t) = \overline{n}(X_w(\xi,t),t)$ and $X_w(\xi,t) = \xi + \int_0^t w(\xi,t')\,dt'$.

By using the solvability result for problem (5.13)–(5.16) and the Banach fixed point theorem the following lemma is proved.

LEMMA. *Let* $\alpha\in(1/2,1)$, $S\in W_2^{2/3+\alpha}$, $\varrho_0\in W_2^{1+\alpha}(\Omega)$, $\varrho_0\ge C_0>0$ *and suppose that*

$$T^{1/2}\|w\|_{\Omega^T}^{(2+\alpha,1+\alpha/2)} \le \delta,$$

where $\delta>0$ *is sufficiently small. Then for arbitrary* $F\in W_2^{\alpha,\alpha/2}(\Omega^T)$, $u_0\in W_2^{1+\alpha}(\Omega)$, $b_1\in W_2^{\alpha+1/2,\alpha/2+1/4}(S^T)$ *and* $b_2 = b' + \sigma\int_0^t B\,dt'$ *with* $b'\in W_2^{\alpha+1/2,\alpha/2+1/4}(S^T)$, $B\in W_2^{\alpha-1/2,\alpha/2-1/4}(S^T)$ *satisfying the compatibility conditions*

$$\mu\Pi_0\mathbb{S}(u_0)\overline{n}_0 = \Pi_0 b_1|_{t=0} \quad \text{on } S,$$
$$\overline{n}_0\cdot\mathbb{D}(u_0)\overline{n}_0 = b'|_{t=0} \quad \text{on } S,$$

problem (5.17)–(5.20) *is uniquely solvable in* $W_2^{2+\alpha}(\Omega^T)$ *and*

$$\|u\|_{\Omega^T}^{(2+\alpha,1+\alpha/2)} \le c(T)(\|F\|_{\Omega^T}^{(\alpha,\alpha/2)} + \|u_0\|_{W_2^{1+\alpha}(\Omega)} + \|b_1\|_{W_2^{\alpha+1/2,\alpha/2+1/4}(S^T)} + \|b'\|_{W_2^{\alpha+1/2,\alpha/2+1/4}(S^T)} + \sigma\|B\|_{S^T}^{(\alpha-1/2,\alpha/2-1/4)}), \tag{5.21}$$

where $c(T)$ *is a nondecreasing function of* T.

By using the fact that $\eta(\xi,t) = \varrho_0(\xi)\exp(-\int_0^t\nabla_u\cdot u\,dt') = \varrho_0(\xi)J_{X_u(\xi,t)}^{-1}$ and by applying the above lemma together with the method of successive approximations, the assertion of the theorem follows. ■

Local solvability of problem (5.9)–(5.12) in the case of $\sigma=0$ and $k>0$ is studied by Ströhmer and Zajączkowski [StZaj]. By a method diffrent from that of Solonnikov and Tani [SolT2] they obtain a local solution which is more regular than the solution ensured by Theorem 5.1. Ströhmer and Zajączkowski consider first an auxiliary linear problem in a bounded domain and prove by the Galerkin method the existence of a weak solution of this problem. Next, they increase the regularity of the solution by applying some regularization techniques.

Local solvability of problem (5.9)–(5.12) is also examined in papers of Zajączkowski [Zaj2, Zaj3, Zaj5], in various function spaces. All these papers are devoted to the case with the absence of the self-gravitating force, i.e. $k=0$.

In [Zaj2] under the assumptions that $\varrho_0\in W_2^{l+1}(\Omega)$, $v_0\in W_2^{l+1}(\Omega)$, $f\in C^{l+1}(\mathbb{R}^3\times(0,T))$, $S\in W_2^{l+5/2}$, $l>3/2$, $l\notin\mathbb{Z}$ and $l/2 = n+3/4+\kappa$, $n\in\mathbb{N}\cup\{0\}$, $\kappa\in(0,1/4)$ it is proved that there exists a unique local solution (u,η) of problem (5.9)–(5.12) in the case of $\sigma > 0$. This solution is such that $u \in W_{2,\kappa}^{l+2,l/2+1}(\Omega^T)$, $\eta \in W_{2,\kappa}^{l+1,l/2+1/2}(\Omega^T) \cap C([0,T];\Gamma_2^{l+1}(\Omega))$.

In [Zaj5] using the results of [Zaj1] Zajączkowski proves local existence and uniqueness of a solution (u,η) of problem (5.9)–(5.12) such that $u \in W_2^{4,2}(\Omega^T)$, $\eta \in C([0,T];\Gamma_0^3(\Omega))$, $\eta_t \in L_2(0,T;W_2^3(\Omega))$, $\eta_{tt} \in L_2(0,T;W_2^1(\Omega))$. This result is obtained for $\sigma > 0$, v_0, $\varrho_0 \in W_2^3(\Omega)$, $1/\varrho_0 \in L_\infty(\Omega)$, $S \in W_2^{7/2}$ and $g \in W_2^{2,1}(\Omega^T)$.

In [Zaj3] local solvability of problem (5.9)–(5.12) with $\sigma = 0$ in similar function spaces to those in [Zaj2] is examined.

5.1.2. *Global existence and stability*

The case of $\sigma = 0$, $k = 0$ *and* $p_0 = \text{const}$. There are only two papers concerning the case of $\sigma = 0$, $p_0 = \text{const}$ and $k = 0$, i.e. [ZZaj10] and [Zaj3]. In both, the existence of a global solution sufficiently close to an equilibrium state is proved. Moreover, the stability of this equilibrium state is shown.

To define an equilibrium state consider the equation

$$p(\varrho) = p_0, \tag{5.22}$$

where $p \in C^3(\mathbb{R}_+)$ and $p' > 0$ for $\varrho > 0$.

Definition 5.1. Let $f = 0$. By an *equilibrium state* we mean a solution (v,ϱ) of problem (5.1)–(5.5) such that $v = 0$, $\varrho = \varrho_e$ and $\Omega_t = \Omega_e$ for $t \ge 0$, where ϱ_e is a solution of (5.22) and Ω_e is a domain of volume $|\Omega_e| = M/\varrho_e$.

Before formulating the main result of [ZZaj10] some notation should be introduced. We set

$$p_\sigma = p - p_0, \quad \varrho_\sigma = \varrho - \varrho_e, \quad \varrho_{\sigma 0} = \varrho_0 - \varrho_e,$$
$$\varphi(t) = |v(t)|^2_{2,0,\Omega_t} + |\varrho_\sigma(t)|^2_{2,0,\Omega_t},$$
$$\Phi(t) = |v(t)|^2_{3,1,\Omega_t} + \|\varrho_\sigma(t)\|^2_{W_2^2(\Omega_t)} + \|\varrho_{\sigma t}(t)\|^2_{W_2^2(\Omega_t)} + \|\varrho_{\sigma tt}(t)\|^2_{W_2^1(\Omega_t)},$$
$$\mathfrak{N}(t) = \{(v,\varrho_\sigma) : \varphi(t) < \infty\}, \quad \mathfrak{M}(t) = \Big\{(v,\varrho_\sigma) : \sup_{0\le t'\le t} \varphi(t') + \int_0^t \Phi(t')\,dt' < \infty\Big\},$$

where

$$|f(t)|_{l,k,Q} \equiv \sum_{i\le l-k} \|\partial_t^i f(t)\|_{W_2^{l-i}(Q)}, \quad l \in \mathbb{N}\cup\{0\},\ Q \subset \mathbb{R}^3. \tag{5.23}$$

Then the global existence theorem proved in [ZZaj10] is as follows.

Theorem 5.2. *Let* $f = 0$, $p \in C^3(\mathbb{R}_+)$ *with* $p' > 0$ *for* $\varrho > 0$, $(v,\varrho_\sigma) \in \mathfrak{N}(0)$, $S \in W_2^{5/2}$ *and let the following compatibility condition be satisfied*:

$$\partial_t^i\{[\mathbb{D}(v) - (p(\varrho) - p_0)]\overline{n}\}|_{t=0} = 0, \quad i = 0,1, \quad \textit{on } S.$$

Moreover, let the following assumptions be satisfied:

$$\varphi(0) \le \varepsilon; \tag{5.24}$$

$$\|v_0\|^2_{L_2(\Omega)} + \|\varrho_{\sigma 0}\|^2_{L_2(\Omega)} \le \delta; \tag{5.25}$$

$l > 0$ is a constant such that $\varrho_e - l > 0$ and $\varrho_1 < \varrho_0 < \varrho_2$, where $\varrho_1 = \varrho_e - l$, $\varrho_2 = \varrho_e + l$;

$$\int_\Omega \varrho_0 v_0 \cdot (a + b \times \xi)\, d\xi = 0,$$

where a and b are arbitrary constant vectors. Then for sufficiently small constants ε and δ there exists a global solution of problem (5.1)–(5.5) *such that $(v, \varrho_\sigma) \in \mathfrak{M}(t)$ for $t > 0$, $S_t \in W_2^{5/2}$ for $t > 0$ and*

$$\varphi(t) \le c\varepsilon \quad \text{for } t > 0,$$

where $c > 0$ is a constant depending on ϱ_1, ϱ_2 and the form of p.

Sketch of proof

Step 1. The first step is to derive the following inequality for the local solution to the problem considered:

$$\begin{aligned}(5.26)\qquad &\|u\|^2_{\mathcal{A}_{T,\Omega}} + \|\eta_\sigma\|^2_{\mathcal{B}_{T,\Omega}}\\ &\le \psi_1(T)(\|\varrho_{\sigma 0}\|^2_{W_2^2(\Omega)} + \|v_0\|^2_{W_2^2(\Omega)} + \|u_t(0)\|^2_{W_2^1(\Omega)} + \|u_{tt}(0)\|^2_{L_2(\Omega)}),\end{aligned}$$

where the spaces $\mathcal{A}_{T,\Omega}$ and $\mathcal{B}_{T,\Omega}$ are given by (5.79) and (5.80) below; T is the time of local existence; ψ_1 is a positive continuous increasing function of T. Inequality (5.26) is derived by using Lemmas 3.5, 2.3 and Theorem 4.2 of [ZZaj9]. From (5.26) it follows that

$$(5.27)\qquad \sup_{0\le t'\le t} \varphi(t') + \int_0^t \Phi(t')\, dt' \le c_1\varphi(0) \quad \text{for } t \le T,$$

where $c_1 > 0$ is a constant depending on T and $\int_0^T \|v\|^2_{W_2^3(\Omega_t)}\, dt$.

Step 2. For a sufficiently smooth local solution the following differential inequality can be proved:

$$\begin{aligned}(5.28)\qquad \frac{d\overline{\varphi}}{dt} + c_2\Phi &\le c_3\Big(\varphi + \int_0^t \|v\|^2_{W_2^3(\Omega_{t'})}\, dt'\Big)\\ &\quad\cdot \Big[1 + \Big(\varphi + \int_0^t \|v\|^2_{W_2^3(\Omega_{t'})}\, dt'\Big)^2\Big]\Phi + c_4\|p_\sigma\|^2_{L_2(\Omega_t)} \quad \text{for } t \le T.\end{aligned}$$

where c_i $(i = 2, 3, 4)$ are positive constants depending on ϱ_1, ϱ_2, ν, μ, $\int_0^t \|v\|^2_{W_2^3(\Omega_{t'})}\, dt'$, $\|S\|_{W_2^{5/2}}$, T and the constants of imbedding theorems and Korn inequalities. Moreover, $\overline{\varphi}$ is a function satisfying

$$(5.29)\qquad c_5\varphi(t) \le \overline{\varphi}(t) \le c_6\varphi(t) \quad \text{for } t \le T,$$

where c_5, $c_6 > 0$ are constants depending on ϱ_1, ϱ_2 and the form of p.

Step 3. Inequalities (5.27)–(5.28), assumption (5.24) with ε sufficiently small and the argument of Lemma 6.2 of [Zaj3] used to estimate the norm $\|p_\sigma\|^2_{L_2(\Omega_t)}$ imply

$$(5.30)\qquad \overline{\varphi}(t) \le c_7\overline{\varphi}(0)e^{-c_8 t} \quad \text{for } t \le T,$$

where the positive constants $c_7 > 1$ and $c_8 > 0$ depend on the same quantities as c_2, c_3, c_4.

STEP 4. The following lemma is proved.

LEMMA. *If estimates* (5.24) *and* (5.25) *hold then*

$$\|v\|^2_{L_2(\Omega_t)} + \|\varrho_\sigma\|^2_{L_2(\Omega_t)} \le c_9\varepsilon^2 + c_{10}c_{11}\delta \quad \text{for } t \le T,$$

where c_{10} *and* c_{11} *are positive constants depending on* ϱ_1, ϱ_2 *such that*

$$\frac{1}{c_{11}}(\|v\|^2_{L_2(\Omega_t)} + \|\varrho_\sigma\|^2_{L_2(\Omega_t)}) \le \frac{1}{2}\int\limits_{\Omega_t}\left(\varrho v^2 + \frac{p_1}{\varrho}\varrho_\sigma^2\right)dx$$
$$\le c_{10}(\|v\|^2_{L_2(\Omega_t)} + \|\varrho_\sigma\|^2_{L_2(\Omega_t)}) \quad \text{for } t \le T;$$

p_1 *is a positive function such that* $p_\sigma = (\varrho - \varrho_\sigma)\int_0^1 p'(\varrho_e + s(\varrho - \varrho_e))\,ds \equiv p_1\varrho_\sigma$; $c_9 > 0$ *is a constant depending on the same quantities as* c_2, c_3, c_4. *Moreover*,

$$\|p_\sigma\|^2_{L_2(\Omega_t)} \le c_{12}(c_9\varepsilon^2 + c_{10}c_{11}\delta), \tag{5.31}$$

where $c_{12} > 0$ *is a constant depending on* ϱ_1, ϱ_2 *and* p.

STEP 5. Assumptions (5.24)–(5.25) and inequalities (5.27), (5.28), (5.31) yield, for sufficiently small α and δ,

$$\overline{\varphi}(t) \le c_6\varepsilon \quad \text{for } t \le T. \tag{5.32}$$

Moreover, by (5.29)

$$\varphi(t) \le \frac{c_6}{c_5}\varepsilon \quad \text{for } t \le T.$$

STEP 6. The solution is extended step by step, first from $[0, T]$ to $[T, 2T]$, next from $[T, 2T]$ to $[2T, 3T]$ and so on. Estimate (5.32) allows us to extend the solution to $[T, 2T]$. In order to extend the solution onto $\mathbb{R}_+$ we have to verify step by step that both the volume and shape of the domain Ω_t do not change much in time. To do this inequality (5.30) is used. ∎

In an earlier paper [Zaj3] Zajączkowski proves global existence of a solution close to the equilibrium state determined by Definition 5.1 in the special case of $p = a\varrho^\gamma$, where $a > 0$, $\gamma > 1$. Global existence and stability of the equilibrium state is proved in [Zaj3] in spaces of functions of a greater regularity than in [ZZaj10]. More precisely, under assumptions appropriately stronger than those of Theorem 5.2 it is proved that there exists a global solution (v, ϱ) of problem (5.1)–(5.5) such that $(v, p_\sigma) \in \mathfrak{M}(t)$ for $t > 0$, $S_t \in W_2^{7/2}$ for $t > 0$, where $\mathfrak{M}(t) = \{(v, p_\sigma) : \sup_{0 \le t' \le t}\varphi(t') + \int_0^t \Phi(t')\,dt' < \infty\}$, and $\varphi(t) = |v(t)|^2_{3,0,\Omega_t} + |p_\sigma(t)|^2_{3,0,\Omega_t}$, $\Phi(t) = |v(t)|^2_{4,1,\Omega_t} + |p_\sigma(t)|^2_{3,0,\Omega_t}$.

Although the proof in this case is also based on an appropriate differential inequality which is similar to inequality (5.28), it is much more complicated than the proof of Theorem 5.2.

The case of $\sigma > 0$, $k = 0$ *and* $p_0 =$ const. There are two papers concerning global existence in this case, namely [SolT3] and [Zaj4]. Although the authors of the two papers were working on them at the same time, the paper of Solonnikov and Tani was published two years earlier, in 1992. Since the methods used in these papers are different, both of them will be described here.

First, we present the paper of Zajączkowski. As in the case $\sigma = 0$, assuming that the initial data are sufficiently close to an equilibrium state it is proved in [Zaj4] that there exists a global solution which remains sufficiently close to the equilibrium state at all times. This result is obtained under the assumption that p_0 is a constant.

The definition of the equilibrium state introduced in [Zaj4] is as follows.

DEFINITION 5.2. Let $f = 0$ and let a functional dependence $p = p(\varrho)$ be given. By an *equilibrium state* we mean a solution (v, ϱ, Ω_t) of problem (5.1)–(5.5) such that $v = 0$, $\varrho = \varrho_e$, $\Omega_t = \Omega_e$ for $t \ge 0$, where $\varrho_e = M/|\Omega_e|$ and Ω_e is a ball of radius R_e, which is a solution of the equation

$$p\biggl(\frac{M}{\frac{4}{3}\pi R_e^3}\biggr) = \frac{2\sigma}{R_e} + p_0.$$

Below, the following conditions will be assumed:

$$f = 0, \quad p_0 > 0, \tag{5.33}$$

$$p = a\varrho^\gamma, \quad a > 0, \quad \gamma > 1. \tag{5.34}$$

An essential role in the proof of global existence is played by the following lemma which collects the conservation laws for problem (5.1)–(5.5).

LEMMA 5.1. *Sufficiently smooth solutions to problem* (5.1)–(5.5) *satisfy*

$$\frac{d}{dt}\biggl[\int_{\Omega_t}\Bigl(\frac{1}{2}\varrho v^2 + \varrho h(\varrho)\Bigr)\,dx + p_0|\Omega_t| + \sigma|S_t|\biggr] + \frac{\mu}{2}E(v) + (\nu-\mu)\|\operatorname{div} v\|^2_{L_2(\Omega_t)} = 0 \quad (\textit{the energy conservation law}), \tag{5.35}$$

where $E(v) = \sum_{i,j=1}^3 (v_{ix_j} + v_{jx_i})^2\,dx$, $|S_t|$ *is the surface area of* S_t *and* $h(\varrho) = \int \frac{p(\varrho)}{\varrho^2}\,d\varrho$. *Moreover*,

$$\frac{d}{dt}\int_{\Omega_t} \varrho v\cdot(a + b\times x)\,dx = 0 \quad (\textit{the momentum and angular momentum conservation laws}), \tag{5.36}$$

where a, b *are arbitrary constant vectors*, *and*

$$\frac{d}{dt}\int_{\Omega_t}\varrho x\,dx = \int_{\Omega_t}\varrho v\,dx. \tag{5.37}$$

The energy conservation law (5.35) yields

$$\frac{1}{2}\int_{\Omega_t}\varrho v^2\,dx + \int_{\Omega_t}\overline{\psi}(\varrho)\,dx + p_0|\Omega_t| + \sigma|S_t| \le \frac{1}{2}\int_{\Omega}\varrho_0 v_0^2\,d\xi + \int_{\Omega}\overline{\psi}(\varrho_0)\,d\xi + p_0|\Omega| + \sigma|S| \equiv d, \tag{5.38}$$

where $\overline{\psi}(\varrho) = \frac{a}{\gamma-1}\varrho^\gamma$.

Multiplying (5.38) by $|\Omega_t|^{\gamma-1}$ and using the Hölder inequality together with (5.8) gives

$$y(|\Omega_t|) + \frac{1}{2}|\Omega_t|^{\gamma-1}\int_{\Omega_t}\varrho v^2\,dx + \sigma(|S_t| - 4\pi R_t^2)|\Omega_t|^{\gamma-1} + \frac{a}{\gamma-1}\biggl(|\Omega_t|^{\gamma-1}\int_{\Omega_t}\varrho^\gamma\,dx - \Bigl(\int_{\Omega_t}\varrho\,dx\Bigr)^\gamma\biggr) \le 0,$$

where

$$y(x) = p_0 x^\gamma + \widetilde{c}\sigma x^{\gamma-1/3} - dx^{\gamma-1} + \frac{aM^\gamma}{\gamma-1}, \tag{5.39}$$

$x = |\Omega_t|$, $\widetilde{c} = (36\pi)^{1/3}$, $R_t = \left(\frac{3}{4\pi}|\Omega_t|\right)^{1/3}$. Using the properties of the function (5.39) it is shown that under some assumptions on the data p_0, σ, d, M, γ, the volume of Ω_t does not change much in time. In fact, the minimum points of (5.39) are determined by the equation

$$y'(x) = [p_0\gamma x + \widetilde{c}\sigma(\gamma - 1/3)x^{2/3} - d(\gamma-1)]x^{\gamma-2} = 0. \tag{5.40}$$

Viète's formulas imply that there exists a unique positive root x_0 of (5.40).

To calculate $y(x_0)$ equation (5.40) is rewritten in the form

$$\omega^3 + 3q\omega + 2r = 0, \tag{5.41}$$

where $\omega = x^{1/3} + \mu_0$, $q = -\mu_0^2$, $r = \mu_0^3 - \nu_0$, $\mu_0 = \frac{\widetilde{c}\sigma(\gamma-1/3)}{3p_0\gamma}$, $\nu_0 = \frac{d(\gamma-1)}{2p_0\gamma}$. Let

$$D = r^2 + q^3 = \nu_0(\nu_0 - 2\mu_0^3).$$

One of the three possibilities holds:

$$\text{if } \nu_0 \in (2\mu_0^3, \infty) \equiv I_1, \quad \text{then } D > 0, \tag{5.42$_1$}$$

$$\text{if } \nu_0 \in (\mu_0^3, 2\mu_0^3] \equiv I_2, \quad \text{then } D \le 0, \tag{5.42$_2$}$$

$$\text{if } \nu_0 \in (0, \mu_0^3] \equiv I_3, \quad \text{then } D < 0. \tag{5.42$_3$}$$

For $\nu_0 \in I_i$ $(i = 1,2,3)$ the following functions ψ_i are defined:

$$\cosh\psi_1 \equiv \nu_0/\mu_0^3 - 1, \quad \text{where } \nu_0 \in I_1, \tag{5.43$_1$}$$

$$\cos\psi_2 \equiv \nu_0/\mu_0^3 - 1, \quad \text{where } \nu_0 \in I_2, \tag{5.43$_2$}$$

$$\cos\psi_3 \equiv 1 - \nu_0/\mu_0^3, \quad \text{where } \nu_0 \in I_3. \tag{5.43$_3$}$$

Now, let $(5.42)_1$ be satisfied. Then

$$\begin{aligned} y(x_0) = & -(\gamma-1)^{-1}p_0\mu_0^{3\gamma}\left(2\cosh\frac{\psi_1}{3} - 1\right)^{3(\gamma-1)} \\ & \cdot\left[2(\cosh\psi_1 + 1) - \frac{\gamma-1}{\gamma-1/3}\left(2\cosh\frac{\psi_1}{3} - 1\right)^2\right] \\ & + aM^\gamma/(\gamma-1) \equiv -\overline{\Phi}_1(\mu_0,\psi_1,p_0,\gamma) + aM^\gamma/(\gamma-1) \\ \equiv & -\Phi_1(\mu_0,\psi_1,p_0,\gamma,a,M). \end{aligned} \tag{5.44$_1$}$$

Let $(5.42)_2$ be satisfied. Then

$$\begin{aligned} y(x_0) = & -(\gamma-1)^{-1}p_0\mu_0^{3\gamma}\left(2\cos\frac{\psi_2}{3} - 1\right)^{3(\gamma-1)} \\ & \cdot\left[2\bigg(\cos\psi_2 + 1) - \frac{\gamma-1}{\gamma-(1/3)}\left(2\cos\frac{\psi_2}{3} - 1\right)^2\right] \\ & + aM^\gamma/(\gamma-1) \equiv -\overline{\Phi}_2(\mu_0,\psi_2,p_0,\gamma) + aM^\gamma/(\gamma-1) \\ \equiv & -\Phi_2(\mu_0,\psi_2,p_0,\gamma,a,M). \end{aligned} \tag{5.44$_2$}$$

In the case when $(5.42)_3$ is satisfied, $y(x_0)$ is given by

$$(5.44)_3\qquad \begin{aligned} y(x_0) = &-(\gamma-1)^{-1}p_0\mu_0^{3\gamma}\Big[2\cos\Big(\frac{\pi}{3}-\frac{\psi_3}{3}\Big)-1\Big]^{3(\gamma-1)}\\ &\cdot\Big\{2(1-\cos\psi_3)-\frac{\gamma-1}{\gamma-(1/3)}\Big[2\cos\Big(\frac{\pi}{3}-\frac{\psi_3}{3}\Big)-1\Big]^2\Big\}\\ &+aM^\gamma/(\gamma-1)\equiv-\overline{\Phi}_3(\mu_0,\psi_3,p_0,\gamma)+aM^\gamma/(\gamma-1)\\ \equiv&-\Phi_3(\mu_0,\psi_3,p_0,\gamma,a,M). \end{aligned}$$

The following lemma holds [Zaj4, Lemma 2.2].

LEMMA 5.2. *Assume that the parameters μ_0, ν_0, p_0, γ, a, M satisfy one of the relations*:

$$(5.45)_i\qquad \nu_0\in I_i,\qquad 0<\Phi_i(\mu_0,\psi_i,p_0,\gamma,a,M)\le\delta_0,$$

where $i=1,2,3$, I_i is defined by $(5.42)_i$ and Φ_i is determined by $(5.44)_i$. Then there exist constants c_1 and c_2 independent of δ_0 such that

$$\sup_{t\le T}\operatorname{var}|\Omega_t|\le c_1\delta,\qquad \sup_{t\le T}\operatorname{var}\int_{\Omega_t}\varrho^\gamma\,dx\le c_2\delta,$$

where T is the time of existence of the solution of problem (5.1)–(5.5), $\delta^2=c\delta_0$. *Moreover, if $(5.45)_i$ is satisfied then*

$$\big||\Omega_t|-Q_i\big|\le c_3\delta\qquad \textit{for } t\le T,$$

where

$$Q_1=\mu_0^3\Big(2\cosh\frac{\psi_1}{3}-1\Big)^3,\qquad Q_2=\mu_0^3\Big(2\cos\frac{\psi_2}{3}-1\Big)^3,\qquad Q_3=\mu_0^3\Big[2\cos\Big(\frac{\pi}{3}-\frac{\psi_3}{3}\Big)-1\Big]^3.$$

Let $|\Omega^*|=\max_{t\le T}|\Omega_t|$, $|\Omega_*|=\min_{t\le T}|\Omega_t|$, $\overline{\psi}^*=\max_{t\le T}\int_{\Omega_t}\varrho^\gamma\,dx$, and $\overline{\psi}_*=\min_{t\le T}\int_{\Omega_t}\varrho^\gamma\,dx$. Lemma 5.2 implies that $|\Omega^*|-|\Omega_*|\le c_1\delta$ and $\overline{\psi}^*-\overline{\psi}_*\le c_2\delta$. Let $|S_*|=4\pi R_*^2$, where R_* is determined by $(4\pi/3)R_*^3=|\Omega_*|$. Then $|S_t|-|S_*|\ge0$. Therefore, estimate (5.38) yields

$$(5.46)\qquad \begin{aligned} &\frac12\int_{\Omega_t}\varrho v^2\,dx+\frac{a}{\gamma-1}\Big(\int_{\Omega_t}\varrho^\gamma\,dx-\overline{\psi}_*\Big)+p_0(|\Omega_t|-|\Omega_*|)+\sigma(|S_t|-|S_*|)\\ &\le\frac12\int_\Omega\varrho_0v_0^2\,d\xi+\frac{a}{\gamma-1}\Big(\int_\Omega\varrho_0^\gamma\,d\xi-\overline{\psi}_*\Big)+p_0(|\Omega|-|\Omega_*|)+\sigma(|S|-|S_*|). \end{aligned}$$

In order to formulate the main result of [Zaj4] we set

$$\varphi(t)=|v(t)|^2_{3,0,\Omega_t}+|p_\sigma(t)|^2_{3,0,\Omega_t}+|v(t)|^2_{3,1,S_t}\equiv\varphi_0(t)+|v(t)|^2_{3,1,S_t},$$

where notation (5.23) is used. To prove global existence Zajączkowski assumes that

$$(5.47)\qquad \overline{\varphi}(0)+|v(0)|^2_{4,0,\Omega}\le\varepsilon,$$

where $\varepsilon>0$ is a sufficiently small constant; $\overline{\varphi}(t)$ is a function satisfying the estimate

$$c_4\varphi(t)\le\overline{\varphi}(t)\le c_5\varphi(t)\qquad \text{for } t\le T.$$

In the above inequality c_4, c_5 are constants depending on $\varrho_*=\min_{0\le t\le T,x\in\overline{\Omega}_t}\varrho(x,t)$, $\varrho^*=\max_{0\le t\le T,x\in\overline{\Omega}_t}\varrho(x,t)$ and T is the time of local existence of the solution.

The main result of paper [Zaj4] is the following theorem.

THEOREM 5.3. *Let the assumptions* (5.33), (5.34), (5.47) *and the assumptions of Lemma* 5.2 *be satisfied. Let the constant parameters* ν, μ, σ, M, a, γ, $p_0, |\Omega|, |S|$ *of problem* (5.1)–(5.5) *be such that* $(5.45)_i$ *implies the smallness of* $|S_t - S_{t'}|$ *for every* $t, t' > 0$ *and let*

$$\frac{1}{2}\int\limits_{\Omega} \varrho_0 v_0^2 \, d\xi \le \delta_1. \tag{5.48}$$

Assume that $\varrho_0 \in W_2^3(\Omega)$, $v_0 \in W_2^4(\Omega)$ *are such that*

$$\int\limits_{\Omega} \varrho_0 \xi \, d\xi = 0, \qquad \int\limits_{\Omega} \varrho_0 v_0 \cdot (a + b \times \xi) \, d\xi = 0,$$

where a, b *are arbitrary constant vectors. Assume that* $S \in W_2^{4+1/2}$, S *is described by the equation*

$$|\xi| = \widetilde{R}(\omega), \qquad \omega \in S^1,$$

(S^1 *is the unit sphere*) *and* Ω *is diffeomorphic to a ball. Moreover, let*

$$\|H(\cdot, 0) + 2/R_e\|^2_{W_2^2(S^1)} \le \varepsilon_1.$$

Assume also the following compatibility conditions:

$$D_\xi^\alpha \partial_t^i (\mathbb{T}\overline{n} - \sigma H \overline{n} + p_0 \overline{n})|_{t=0,S} = 0 \qquad \text{for } |\alpha| + i \le 2.$$

Then for sufficiently small δ_0, δ_1, ε, ε_1 *there exists a global solution of problem* (5.1)–(5.5) *such that* $(v, p_\sigma) \in \mathfrak{M}(t)$ *for* $t > 0$, $S_t \in W_2^{4+1/2}$ *for* $t > 0$ *and*

$$\varphi(t) \le \varepsilon, \qquad \|H(\cdot, t) + 2/R_e\|^2_{W_2^2(S^1)} \le \varepsilon_1 \qquad \text{for } t > 0,$$

where $\mathfrak{M}(t) = \{(v, p_\sigma) : \sup_{0 \le t' \le t} \varphi_0(t) + \int_0^t \Phi(t') \, dt' < \infty\}$, $\Phi(t) = |v(t)|^2_{4,1,\Omega_t} + |p_\sigma(t)|^2_{3,0,\Omega_t}$, $p_\sigma = p - p_0$.

Sketch of proof

STEP 1. Assuming that there exists a sufficiently regular local solution of problem (5.1)–(5.2) and using Lemmas 5.1–5.2, assumption (5.48) and inequality (5.46), the following estimate is proved:

$$\|v\|^2_{L_2(\Omega_t)} \le \delta_3, \tag{5.49}$$

where $\delta_3 = \delta_3(\delta_0, \delta_1)$ and $\delta_3 \to 0$ if $\delta_0 \to 0$ and $\delta_1 \to 0$. Moreover, using some other consequences of the conservation laws it is proved that S_t is described by the equation $|x| = R(\omega, t)$ for $t \le T$, $\omega \in S^1$, where $R(\omega, 0) = \widetilde{R}(\omega)$ and

$$\sup_{0 \le t \le T} \|R(\cdot, t) - R_e\|^2_{L_2(S^1)} \le \delta_4, \tag{5.50}$$

where δ_4 is a sufficiently small constant.

STEP 2. For a sufficiently regular solution of problem (5.1)–(5.5) a differential inequality similar to (5.28) (with functions φ, $\overline{\varphi}$ and Φ defined above) is derived.

Step 3. Local existence of a solution to problem (5.1)–(5.5) is proved. This solution is such that $(v, p_\sigma) \in \mathfrak{M}(t)$ for $t \le T$ and satisfies

$$\varphi(t) + \int_0^t \Phi(t')\,dt' \le c_6(\varepsilon + \varepsilon_1 + \delta_4), \tag{5.51}$$

where ε, ε_1 and δ_4 are the constants occurring above.

Step 4. By using (5.49)–(5.51) the following estimate is derived:

$$\|p_\sigma\|^2_{L_2(\Omega_t)} \le \delta_5, \tag{5.52}$$

where $\delta_5 = \delta_5(\varepsilon, \varepsilon_1, \delta_3, \delta_4)$ is an increasing function of its arguments.

Step 5. The next step of the proof is to increase the regularity of the local solution. Namely, it is shown that

$$\sup_{t_1 \le t \le T} \|v\|^2_{W_2^4(\Omega_t)} \le c(t_1)\|v\|^2_{W_2^{4,2}(\Omega_t^T)}, \tag{5.53}$$

where T is the time of local existence and $t_1 > 0$.

Step 6. By using the differential inequality obtained in Step 2, estimates (5.49)–(5.51) and assumption (5.47), it is proved that

$$\varphi(t) \le \varepsilon \quad \text{for } t \le T. \tag{5.54}$$

Next, using estimates (5.49) and (5.52)–(5.54) yields

$$\|H(\cdot, t) + 2/R_e\|^2_{W_2^2(S^1)} \le \varepsilon_1 \quad \text{for } t \le T.$$

Step 7. The solution is extended step by step to all $t > 0$ by using the estimates derived in Steps 1–6. ■

Notice that Theorem 5.3 is concerned with the case $p_0 > 0$. The case $p_0 = 0$ is also considered in [Zaj4] and a global existence and stability theorem analogous to Theorem 5.3 is proved.

The paper of Solonnikov and Tani [SolT3] is concerned with the same problem as [Zaj4] but they examine global solvability in the anisotropic Sobolev–Slobodetskiĭ spaces used in [SolT2] to prove local existence. Thus, they obtain a global solution of problem (5.1)–(5.5) which has the lowest regularity in the L_2-approach. Paper [SolT3] is written in a sketchy way. The main result of this paper is the following theorem.

Theorem 5.4. *Let $\sigma > 0$, $k = 0$, $f = 0$, $p_0 = 0$. Suppose that the assumptions of Theorem* 5.1 *are satisfied and that S is defined by the equation*

$$|\xi| = \widetilde{R}(\omega), \quad \omega = \xi/|\xi|,$$

where $\widetilde{R} \in W_2^{5/2+\alpha}(S^1)$. Assume also that

$$\|v_0\|^2_{W_2^{\alpha+1}(\Omega)} + \|\varrho_0 - \varrho_e\|^2_{W_2^{\alpha+1}(\Omega)} + \|\widetilde{R} - R_e\|^2_{W_2^{\alpha+5/2}(S^1)} \le \varepsilon < 1,$$

where ϱ_e and R_e are two positive constants satisfying the relation

$$p(\varrho_e) = 2\sigma/R_e.$$

Moreover, let $p'(\varrho_e) - p(\varrho_e)/(3\varrho_e) > 0$ and let assumption (5.58) *below be satisfied with sufficiently small ε_1. Then the time of local existence is an increasing function of $1/\varepsilon$*

which tends to infinity as $\varepsilon \to 0$. *Moreover, if* ε *is sufficiently small, then the solution may be extended to the infinite time interval* $t > 0$. *The free surface* S_t *is determined by the equation*

$$|x - x_0| = R(\omega, t),$$

where $x_0 = Vt$, $V = (\int_\Omega \varrho_0 \, d\xi)^{-1} \int_\Omega \varrho_0 v_0 \, d\xi$, $R(\cdot, t) \in W_2^{\alpha+5/2}(S^1)$ *for any* $t > 0$ *and* $\widetilde{R}(\omega) = R(\omega, 0)$. *As* $t \to \infty$, *the solution tends to a quasi-stationary solution of problem* (5.1)–(5.5) *corresponding to a rotation of liquid as a rigid body about an axis which is parallel to the vector* $\int_\Omega \varrho_0 [v_0 \times \xi] \, d\xi = m$ *and which is moving uniformly with a constant speed* V.

The existence of a quasi-stationary solution of problem (5.1)–(5.5) corresponding to the rotation of the liquid as a rigid body about an axis which is parallel to the vector m (it is supposed that it is the x_3-axis) is proved in [SolT4]. The authors look for a solution $(v_\infty, \varrho_\infty, \Omega_\infty)$ of equations (5.1)–(5.2) independent of time which has the form

$$v_\infty(x) = \beta(x_2, -x_1, 0) = \beta[e_3 \times x], \qquad e_{3i} = \delta_{3i},$$

where β is a constant;

$$\varrho_\infty(x) = P^{-1}\Big(\frac{\beta^2}{2}|x'|^2 + N\Big),$$

where $P(\varrho) = \int_{\varrho_1}^{\varrho} \frac{p'(s)}{s} ds$, $\varrho_1 \geq 0$; N is a constant, $x' = (x_1, x_2, 0)$ and Ω_∞ is a domain with boundary S_∞ given by the equation

$$|x| = R_\infty(x/|x|).$$

The unknown quantities β, N and R_∞ are determined by the boundary condition on S_∞, i.e.

$$\sigma H_\infty + p\bigg(P^{-1}\Big(\frac{\beta^2}{2}|x'|^2 + N\Big)\bigg) = 0, \tag{5.55}$$

where

$$H_\infty = \bigg(\frac{1}{R_\infty}\frac{1}{\sqrt{R_\infty^2 + |\nabla R_\infty|^2}}\Delta R_\infty + \nabla\frac{1}{\sqrt{R_\infty^2 + |\nabla R_\infty|^2}} \cdot \nabla R_\infty - \frac{2}{\sqrt{R_\infty^2 + |\nabla R_\infty|^2}} R_\infty\bigg)$$

is the double mean curvature of S_∞; ∇ is the gradient and Δ is the laplacian on S^1. Moreover, the above quantities are determined by the equation for the angular momentum, i.e. $\int_{\Omega_\infty} \varrho_\infty (v_\infty \times x) \, dx = m \equiv \int_\Omega \varrho_0 (v_0 \times \xi) \, d\xi$, which implies

$$\beta \int\limits_{\Omega_\infty} |x'|^2 \varrho_\infty(x) \, dx = m_3 \equiv \gamma, \tag{5.56}$$

and by the equation for the total mass of the liquid

$$\int\limits_{\Omega_\infty} \varrho_\infty(x) \, dx = M, \tag{5.57}$$

where $M = \int_\Omega \varrho_0(\xi) \, d\xi$.

The main result of paper [SolT4] is the following theorem.

Theorem 5.5. *Let $p(\varrho)$ be a positive increasing function of class $C^{1+\delta}(\mathbb{R}_+)$, $\delta\in(0,1)$ and let $p'(\varrho_e)-\frac{1}{3\varrho_e}p(\varrho_e)\neq 0$. For arbitrary γ and M satisfying*

$$|\gamma|+|M-M_e|<\varepsilon_1 \tag{5.58}$$

with sufficiently small ε_1, there exists a unique solution $(R_\infty,\beta,N)\in\widetilde{C}^{2+\delta}(S^1)\times\mathbb{R}\times\mathbb{R}$ of (5.55)–(5.57) *satisfying the estimate*

$$\|R_\infty-R_e\|_{C^{2+\delta}(S^1)}+|\beta|+|P^{-1}(N)-\varrho_e|\le C(|\gamma|+|M-M_e|),$$

where $M_e=\frac{4}{3}\pi R_e^3\varrho_e$, $C>0$ is a constant.

In the above theorem $\widetilde{C}^{2+\delta}(S^1)$ denotes the subspace of $C^{2+\delta}(S^1)$ consisting of rotationally symmetric functions (i.e. functions depending on $|\omega|=\sqrt{\omega_1^2+\omega_2^2}$ and ω_3) which are even with respect to ω_3.

Now, we present a sketch of proof of Theorem 5.4.

Sketch of proof

Step 1. Inequality (5.21) yields the following estimate for the local solution of (5.1)–(5.5):

$$\|u\|^2_{W_2^{2+\alpha,1+\alpha/2}(\Omega^T)}\le c_1(T)(\|v_0\|^2_{W_2^{1+\alpha}(\Omega)}+\|\varrho_0-\varrho_e\|^2_{W_2^{1+\alpha}(\Omega)}+\|R(\cdot,0)-R_e\|^2_{W_2^{5/2+\alpha}(S^1)}),$$

where $C_1(T)$ is an increasing function of T.

Step 2. A consequence of the conservation laws (5.35)–(5.37) and (5.8) is the estimate

$$\begin{aligned}\frac12\int_{\Omega_t}\varrho v^2\,dx+c_2\sigma\int_{S^1}(|R-R_e|^2+|\nabla R|^2)\,d\omega+c_3\int_{\Omega_t}(\varrho-\varrho_e)^2\,dx\\+\int_0^t\Big[\frac{\mu}{2}E(v)+(\nu-\mu)\|\operatorname{div}v\|^2_{L_2(\Omega_t)}\Big]\,dt'\\\le c_4\int_\Omega\varrho_0v_0^2\,d\xi+c_5\sigma\int_{S^1}(|R(\omega,0)-R_e|^2+|\nabla R(\omega,0)|^2)\,d\omega+c_6\int_\Omega(\varrho_0-\varrho_e)^2\,d\xi,\end{aligned}\tag{5.59}$$

where the constants c_i $(i=2,\dots,6)$ are independent of t.

Step 3. By using the Korn inequality, the following estimate is derived:

$$\|w\|_{L_2(\Omega_t)}\le c_7\|\mathbb{S}(v)\|_{L_2(\Omega_t)}+c_8|\gamma|\Big(\int_{S^1}|R-R_\infty|\,d\omega+\int_{\Omega_t}|\tau|\,dx\Big),\tag{5.60}$$

where $w=v-v_\infty$, $\tau=\varrho-\varrho_\infty$. Moreover, by applying the methods from [Sol10], one can prove

$$\|\tau\|_{W_2^1(\Omega_t)}+\|R-R_\infty\|_{W_2^1(S^1)}\le c_9(\|w_t\|_{L_2(\Omega_t)}+\|w\|_{W_2^2(\Omega_t)}+\|(w\cdot\nabla)w\|_{L_2(\Omega_t)}).\tag{5.61}$$

The constants c_7, c_8, c_9 in (5.60)–(5.61) are independent of γ and t.

Step 4. In order to obtain the estimate which plays a crucial role in the proof, new coordinates y are introduced. They are connected with the coordinates x by the formula

$$x(y,t)=y(1+\Phi(y,t)),\qquad y\in B_{R_e}\equiv\{y:|y|\le R_e\},\tag{5.62}$$

where $\Phi(y,t) = (1/R_e)(R((y/|y|),t) - R_e)$ on the sphere $S_{R_e} : |y| = R_e$ and for $y \in B_{R_e}$, $\Phi(y,t)$ is an extension of the above function such that

$$\|\Phi(\cdot,t)\|_{W_2^{3+\alpha}(B_{R_e})} \le c_{10}\|\Phi(\cdot,t)\|_{W_2^{5/2+\alpha}(S_{R_e})} \le c_{11}\|R(\cdot,t) - R_e\|_{W_2^{5/2+\alpha}(S^1)}.$$

The transformation defined by (5.62) maps B_{R_e} onto Ω_t.

Let $\widetilde{w}$ and $\widetilde{\tau}$ be w and τ respectively, written in y coordinates. Then the following lemma holds.

LEMMA. *Let $(w,\tau,R-R_\infty)$ be defined for $0<t<T$ and let*

$$\begin{aligned} N^2_{\alpha,T}[\widetilde{w},\widetilde{\tau},R-R_\infty] \equiv{}& \|\widetilde{w}\|^2_{W_2^{2+\alpha,1+\alpha/2}(B^T_{R_e})} + \int_0^T \|\widetilde{\tau}\|^2_{W_2^{1+\alpha}(B^T_{R_e})}\,dt + \sup_{t\in(0,T)} \|\widetilde{w}\|^2_{W_2^{1+\alpha}(B_{R_e})} \\ &+ \sup_{t\in(0,T)} \|\widetilde{\tau}\|^2_{W_2^{1+\alpha}(B_{R_e})} + \sup_{t\in(0,T)} \|\widetilde{\tau}_t\|^2_{W_2^{\alpha}(B_{R_e})} + \sup_{t\in(0,T)} \|R-R_\infty\|^2_{W_2^{2+\alpha}(S^1)} \\ &+ \|R-R_\infty\|^2_{L_2(0,T;W_2^{5/2+\alpha}(S^1))} \le \delta, \end{aligned}$$

where δ is a small positive constant. Then

$$N^2_{\alpha,T}[\widetilde{w},\widetilde{\tau},R-R_\infty] \le c_{12}(\|v_0\|^2_{W_2^{1+\alpha}(\Omega)} + \|\varrho_0-\varrho_e\|^2_{W_2^{1+\alpha}(\Omega)} + \|R(\cdot,0)-R_e\|^2_{W_2^{5/2+\alpha}(S^1)}),$$

where c_{12} is a constant independent of T.

The above lemma is proved by using estimates (5.59)–(5.61).

STEP 5. In this step we increase the regularity of v by means of the estimate

$$\sup_{t\in(t_0,T)} \|u(\cdot,t)\|^2_{W_2^{2+\alpha}(\Omega)} \le c(t_0)\|u\|^2_{W_2^{2+\alpha,1+\alpha/2}(\Omega^T)},$$

where $t_0 > 0$.

STEP 6. Applying the above inequalities step by step infinitely many times yields the boundedness of $N_{\alpha,\infty}[\widetilde{w},\widetilde{\tau},R-R_\infty]$. Moreover, it follows that $\widetilde{w}$, $\widetilde{\tau}$ and $R-R_\infty$ tend to zero as $t\to\infty$. ■

5.2. The motion of a compressible viscous heat-conducting fluid. In this section we consider a free boundary problem for equations describing the motion of a general compressible viscous heat-conducting fluid. The problem considered is given by the following system of equations and boundary and initial conditions:

$$\varrho[v_t + (v\cdot\nabla)v] - \operatorname{div}\mathbb{T}(v,p) = \varrho(f + k\nabla U),\ x\in\Omega_t,\quad t\in(0,T), \tag{5.63}$$

$$\varrho_t + \operatorname{div}(\varrho v) = 0, \qquad x\in\Omega_t,\ t\in(0,T), \tag{5.64}$$

$$\begin{aligned} &\varrho c_v(\theta_t + v\cdot\nabla\theta) - \operatorname{div}(\varkappa\nabla\theta) + \theta p_\theta \operatorname{div} v \\ &\quad - \frac{\mu}{2}\sum_{i,j=1}^3 (v_{ix_j} + v_{jx_i})^2 - (\nu-\mu)(\operatorname{div})^2 = \varrho r, \qquad x\in\Omega_t,\ t\in(0,T), \end{aligned} \tag{5.65}$$

$$\mathbb{T}\overline{n} - \sigma H\overline{n} = -p_0\overline{n}, \qquad x\in S_t,\ t\in(0,T), \tag{5.66}$$

$$v\cdot\overline{n} = -\phi_t/|\nabla\phi|, \qquad x\in S_t,\ t\in(0,T), \tag{5.67}$$

$$\varkappa\partial\theta/\partial n = \overline{\theta}, \qquad x\in S_t,\ t\in(0,T), \tag{5.68}$$

$$v|_{t=0} = v_0,\quad \varrho|_{t=0} = \varrho_0,\quad \theta|_{t=0} = \theta_0 \qquad \text{in } \Omega, \tag{5.69}$$

where as before $\Omega_t \subset \mathbb{R}^3$ is a bounded unknown domain at time t; $T > 0$; $\varrho = \varrho(x,t)$, $v = v(x,t)$ and $\theta = \theta(x,t)$ are the density, velocity and temperature of the fluid, respectively; $\varkappa = \varkappa(\varrho, \theta)$ is the positive coefficient of heat conductivity; $r = r(x,t)$ denotes the heat sources per unit mass; $\overline{\theta} = \overline{\theta}(x,t)$ is the heat flow per unit surface; $p_0 = p_0(x,t)$ is the external pressure. Moreover, in this case the pressure of the fluid p, the specific heat at constant volume c_v and the viscosity coefficients ν and μ are functions of the density and the temperature, i.e. $p = p(\varrho, \theta)$, $c_v = c_v(\varrho, \theta)$, $\nu = \nu(\varrho, \theta)$ and $\mu = \mu(\varrho, \theta)$. The functions c_v, ν, μ are positive and $\nu > \frac{1}{3}\mu$. The self-gravitational potential U, the stress tensor $\mathbb{T}$ and the double mean curvature of $S_t = \partial\Omega_t$ are given by (5.6), (5.7) and (4.7), respectively.

In this case the mass conservation law (5.8) also holds.

5.2.1. *Local existence.* Local solvability of problem (5.63)–(5.69) has been examined in [SVal, S1-S3, T1, ZZaj1, ZZaj9, ZZaj11]. Just as in the case of an incompressible fluid and a compressible barotropic fluid, we write problem (5.63)–(5.69) in Lagrangian coordinates. Then it takes the form

$$\eta u_t - \operatorname{div}_u \mathbb{T}_u(u,p) = \eta(g + k\nabla_u U_u) \quad \text{in } \Omega^T, \tag{5.70}$$

$$\eta_t + \eta \nabla_u \cdot u = 0 \quad \text{in } \Omega^T, \tag{5.71}$$

$$\begin{aligned} \eta c_v(\eta,\vartheta)\vartheta_t - \nabla_u \cdot (\varkappa \nabla_u \vartheta) &= -\vartheta p_\vartheta(\eta,\vartheta)\nabla_u \cdot u \\ &+ \frac{\mu}{2}\sum_{i,j=1}^{3}(\xi_{x_i}\cdot\partial_\xi u_j + \xi_{x_j}\cdot\partial_\xi u_i)^2 - (\nu-\mu)(\nabla_u\cdot u)^2 = \eta h \quad \text{in } \Omega^T, \end{aligned} \tag{5.72}$$

$$\mathbb{T}_u(u,p)\overline{n}_u - \sigma\Delta_u(t)X_u = -q_0\overline{n}_u \quad \text{on } S^T, \tag{5.73}$$

$$\varkappa(\eta,\vartheta)\overline{n}_u \cdot \nabla_u\vartheta = \overline{\vartheta} \quad \text{on } S^T, \tag{5.74}$$

$$u|_{t=0} = v_0, \quad \eta|_{t=0} = \varrho_0, \quad \vartheta|_{t=0} = \theta_0, \quad \text{in } \Omega, \tag{5.75}$$

where $h(\xi,t) = r(X_u(\xi,t),t)$, $\overline{\vartheta}(\xi,t) = \overline{\theta}(X_u(\xi,t),t)$, $\mathbb{T}_u(u,p) = \{-p(\eta,\vartheta)\delta_{ij} + \mu(\eta,\vartheta)(\partial_{x_i}\xi_k\partial_{\xi_k}u_j + \partial_{x_j}\xi_k\partial_{\xi_k}u_i) + (\nu(\eta,\vartheta) - \mu(\eta,\vartheta))\delta_{ij}\nabla_u\cdot u\}_{i,j=1,2,3}$, $\mathbf{I} = \{\delta_{ij}\}_{i,j=1,2,3}$, $\operatorname{div}_u \mathbb{T}_u(u,p) = \{\partial_{x_j}\xi_k\partial_{\xi_k}T_{uij}(u,p)\}_{i=1,2,3}$.

The first paper devoted to the local solvability of the above problem in the case of $\sigma = 0$, $k = 0$ and with boundary condition (5.74) replaced by

$$\varkappa\overline{n}_u \cdot \nabla_u\vartheta + \varkappa_e\vartheta = \varkappa_e\vartheta_e \quad \text{on } S^T, \tag{5.76}$$

where $\varkappa_e = \varkappa_e(\xi,t)$ is the external heat conductivity and $\vartheta_e = \vartheta_e(\xi,t)$ the external temperature, was the paper of Tani [T1]. In this paper Tani, using the method of successive approximations, proves local existence and uniqueness of a solution of problem (5.70)–(5.73), (5.75), (5.76) in Hölder spaces.

In [SVal] Secchi and Valli consider problem (5.70)–(5.75) in the case of constant coefficients ν, μ, $\varkappa$ and $\sigma = k = \overline{\vartheta} = 0$. They prove local existence of solutions by using the Schauder fixed point theorem. Therefore, the uniqueness result is given separately. The existence theorem of [SVal] is as follows.

Theorem 5.6. *Let S be of class C^4. Suppose that $f \in W_2^{2,1}(B_R^{T_0})$, $r \in W_2^{2,1}(B_R^{T_0})$ for each $R > 0$, $p_0 \in W_2^{3,3/2}(B_R^{T_0})$ with $p_0 \in W_2^{1/4}(0,T_0;W_2^1(B_R)) \cap L_2(0,T_0;C^\alpha(\overline{B}_R))$ for* *each* $R>0$ (*where* $B_R=\{x\in\mathbb{R}^3:|x|<R\}$), $1/4<\alpha\le 1$, $p\in C^3(\mathbb{R}^2)$, $c_v\in C^3(\mathbb{R}^2)$, $v_0\in W_2^3(\Omega)$, $\theta_0\in W_2^3(\Omega)$, $\varrho_0\in W_2^3(\Omega)$ *with* $\min_{\xi\in\overline{\Omega}}\varrho_0(\xi)\equiv\varrho_*>0$. *Assume that the compatibility conditions*

$$\partial_\xi^\beta[\mathbb{D}(v_0)\overline{n}_0-p(\varrho_0,\theta_0)\overline{n}_0+q_0(\xi,0)\overline{n}_0]|_S=0,\quad |\beta|\le 1,$$
$$\partial_\xi^\beta(\partial\theta_0/\partial n)|_S=0,\quad |\beta|\le 1,$$

are satisfied. Then there exist $T^*\in(0,T_0)$, $u\in W_2^{4,2}(\Omega^{T^*})$, $\vartheta\in W_2^{4,2}(\Omega^{T^*})$, $\eta\in W_2^1(0,T^*;W_2^3(\Omega))\cap W_2^2(0,T^*;W_2^1(\Omega))$ *such that* $\eta>0$ *in* $\overline{\Omega}\times[0,T^*]$ *and a diffeomorphism* $X_u\in W_2^1(0,T^*;W_2^4(\Omega))\cap W_2^3(0,T^*;L_2(\Omega))$ (*where* X_u *is given by* (4.10)) *such that* (u,ϑ,η,X_u) *is a solution of the problem in* Ω^{T^*}.

Moreover, Secchi and Valli [SVal] prove the following uniqueness theorem.

THEOREM 5.7. *Let* S *be of class* C^1. *Suppose that* $f\in L_1(0,T;\mathrm{Lip}(\overline{B}_R))$, $r\in L_1(0,T;\mathrm{Lip}(\overline{B}_R))$, $p\in C^1(\mathbb{R}^2)$ *with* $p_\vartheta\in C^1(\mathbb{R}^2)$, $c_v\in C^1(\mathbb{R}^2)$, $p_0\in L_2(0,T;\mathrm{Lip}(\overline{B}_R))$ *with* $\nabla p_0\in L_1(0,T;\mathrm{Lip}(\overline{B}_R))$ *for each* $R>0$ (*where* $\mathrm{Lip}(\overline{B}_R)$ *denotes the space of Lipschitz continuous functions on* $\overline{B}_R$). *Moreover, assume that* $\eta\ge\overline{\varrho}_*>0$ *in* Ω^T, $c_v\ge\overline{c}_v>0$, $\det[X_{u_\xi}]\ge a_0>0$ *in* Ω^T, $X_u(\cdot,t)$ *is injective in* $\overline{\Omega}$ *for each* $t\in[0,T]$. *Then the solution of the problem considered is unique in the class of functions* $\eta\in L_\infty(\Omega^T)$ *with* $\eta_\xi\in L_2(0,T;L_\infty(\Omega))$, $u\in L_\infty(\Omega^T)$ *with* $u_\xi\in L_2(0,T;L_\infty(\Omega))$ *and* $u_{\xi\xi}\in L_1(0,T;L_\infty(\Omega))$; $\vartheta\in L_\infty(\Omega^T)$ *with* $\vartheta_\xi\in L_2(0,T;L_\infty(\Omega))$ *and* $\vartheta_{\xi\xi}\in L_1(0,T;L_\infty(\Omega))$; $X_u\in L_\infty(\Omega^T)$ *with* $X_{u\xi}\in L_\infty(\Omega^T)$ *and* $X_{u\xi\xi}\in L_2(0,T;L_\infty(\Omega))$.

The methods of [SVal] are also applied in [S1–S3]. In contrast to [SVal] in [S3] problem (5.70)–(5.75) with the self-gravitational force taken into account is considered. Moreover, boundary condition (5.74) is replaced by (5.76) with $\varkappa_e$ being a positive constant. As in [SVal], Secchi assumes that $\sigma=0$ and that the coefficients $\varkappa$, ν, μ are positive constants. The local solution obtained in [S3] is less regular than the solution from [SVal]. More precisely, assuming that S is of class C^3; f, $r\in L_2(0,T_0;W_2^1(B_R))\cap L_2(0,T_0;C(\overline{B}_R))$ for each $R>0$; p,$c_v\in C^2(\mathbb{R}^3)$, $c_v>0$; p_0, $\vartheta_e\in W_2^{3/4}(0,T_0;W_2^1(B_R))\cap L_\infty(0,T_0;W_2^2(B_R))$ for each $R>0$; v_0, $\varrho_0,\theta_0\in W_2^2(\Omega)$; $\min_{\xi\in\overline{\Omega}}\varrho_0(\xi)>0$ and assuming appropriate compatibility conditions Secchi [S3] proves the existence of a local solution of problem (5.70)–(5.73), (5.75), (5.76) such that $u,\vartheta\in L_2(0,T^*;W_2^3(\Omega))\cap W_2^1(0,T^*;W_2^1(\Omega))$, $\eta\in W_2^1(0,T^*;W_2^2(\Omega))$, $\eta_t\in L_\infty(0,T^*;W_2^1(\Omega))$, $\eta>0$ in $\overline{\Omega}\times[0,T]$, $X_u\in W_2^1(0,T^*;W_2^3(\Omega))\cap W_2^2(0,T^*;W_2^1(\Omega))$ for some $T^*\in(0,T_0)$.

Papers [S1, S2] are devoted to a similar problem but additionally the effect of radiation is taken into account. Therefore equation (5.72) is replaced by

$$\eta c_v(\eta,\vartheta)\vartheta_t=\nabla_u\cdot(\varkappa\nabla_u\vartheta)+\nabla_u\cdot\left(\frac{3}{4}\frac{ac}{\kappa(\eta,\vartheta)\eta}\vartheta^3\nabla_u\vartheta\right)$$
$$-\vartheta p_\vartheta\nabla_u\cdot u+\eta\varepsilon(\eta,\vartheta)+\frac{\mu}{2}\sum_{i,j=1}^{3}(\xi_{x_i}\cdot\partial_\xi u_j+\xi_{x_j}\cdot\partial_\xi u_i)^2-(\nu-\mu)(\nabla_u\cdot u)^2=0\quad\text{in }\Omega^T,\tag{5.77}$$

where $\varkappa$, ν and μ are assumed to be positive constants; a is the Stefan–Boltzmann constant; c is the light velocity; $\kappa=\kappa(\eta,\vartheta)$ is the Rosseland mean absorption coefficient; $\varepsilon=\varepsilon(\eta,\vartheta)$ is the rate of liberation of nuclear energy. Moreover, boundary condition (5.74) is replaced by

$$\left(\varkappa+\frac{4ac}{3\kappa(\eta,\vartheta)\eta}\vartheta^3\right)\frac{\partial\vartheta}{\partial n}+\varkappa_e\vartheta=0 \quad \text{on } S^T, \tag{5.78}$$

where $\varkappa_e$ is a positive constant.

In [S2] Secchi proves local existence of a solution of problem (5.70)–(5.73), (5.77), (5.78), (5.75) with the same regularity as in [S3]. In [S1] uniqueness of this solution is proved.

Papers [ZZaj1, ZZaj9, ZZaj11] are also concerned with local solvability of problem (5.70)–(5.75). In [ZZaj1] a similar result to Theorem 5.6 is obtained for two cases: $\sigma=0$ and $\sigma>0$. As in [SVal] it is assumed in [ZZaj1] that $\varkappa$, ν, μ are positive constants and $k=0$. The proof of the local existence is different from that in [SVal] because it is based on the method of successive approximations.

In [ZZaj9], by using the methods of [StZaj2], local existence in the case of constant positive coefficients $\varkappa$, ν, μ, constant p_0, $\sigma=0$ and $k=0$, is proved. This local solvability is obtained in function spaces similar to [StZaj2]. The main result of [ZZaj9] is the following theorem.

THEOREM 5.8. *Assume that* $S\in W_2^{5/2}$, $v_0\in W_2^2(\Omega)$, $\theta_0\in W_2^2(\Omega)$, $\varrho_0\in W_2^2(\Omega)$, $u_t(0)\in W_2^1(\Omega)$, $\vartheta_t(0)\in W_2^1(\Omega)$, $u_{tt}(0)\in L_2(\Omega)$, $\vartheta_{tt}(0)\in L_2(\Omega)$ (*where* $u_t(0)$, $u_{tt}(0)$, $\vartheta_t(0)$, $\vartheta_{tt}(0)$ *are calculated from equations* (5.70) *and* (5.72)), $u_{0t}(0)\in W_2^1(\Omega)$, $\vartheta_{0t}(0)\in W_2^1(\Omega)$, $u_{0tt}(0)\in L_2(\Omega)$, $\vartheta_{0tt}(0)\in L_2(\Omega)$ (*where* u_0 *and* ϑ_0 *satisfy problems* (5.81) *and* (5.82)). *Let* $f\in L_2(0,T;W^2_{2,\mathrm{loc}}(\mathbb{R}^3))$, $r\in L_2(0,T;W^2_{2,\mathrm{loc}}(\mathbb{R}^3))$; $f_t\in L_2(0,T;W^1_{2,\mathrm{loc}}(\mathbb{R}^3))$, $r_t\in L_2(0,T;W^1_{2,\mathrm{loc}}(\mathbb{R}^3))$; $f_{tt}\in L_2(0,T;L_{2,\mathrm{loc}}(\mathbb{R}^3))$, $r_{tt}\in L_2(0,T;L_{2,\mathrm{loc}}(\mathbb{R}^3))$, $\overline{\theta}\in L_2(0,T;W^3_{2,\mathrm{loc}}(\mathbb{R}^3))\cap C([0,T];W^2_{2,\mathrm{loc}}(\mathbb{R}^3))$, $\overline{\theta}_t\in L_2(0,T;W^2_{2,\mathrm{loc}}(\mathbb{R}^3))$, $\overline{\theta}_{tt}\in L_2(0,T;W^1_{2,\mathrm{loc}}(\mathbb{R}^3))$, $p\in C^3(\mathbb{R}^2)$, $c_v\in C^2(\mathbb{R}^2)$ *and assume that the following compatibility conditions are satisfied*:

$$\begin{aligned}&\partial_t^i\{[\mathbb{D}_u(u)-(p(\eta,\vartheta)-p_0)]\overline{n}_u\}|_{t=0}=0, \quad i=0,1, && \textit{on } S,\\ &\partial_t^i[\overline{n}_u\cdot\nabla_u\vartheta]|_{t=0}=\partial_t^i\overline{\vartheta}|_{t=0}, \quad i=0,1, && \textit{on } S.\end{aligned}$$

Then there exists $T^*\in(0,T)$ *such that there exists a unique solution* $(u,\vartheta,\eta)\in\mathcal{A}_{T^*,\Omega}\times\mathcal{A}_{T^*,\Omega}\times\mathcal{B}_{T^*,\Omega}$ *of problem* (5.70)–(5.75), *where*

$$\mathcal{A}_{T^*,\Omega}\equiv\mathcal{B}_{T^*,\Omega}\cap L_2(0,T^*;W_2^3(\Omega)), \tag{5.79}$$

$$\begin{aligned}\mathcal{B}_{T^*,\Omega}\equiv\{w\in C([0,T^*];W_2^2(\Omega)):\ & w_t\in C([0,T^*];W_2^1(\Omega))\cap L_2(0,T^*;W_2^2(\Omega)),\\ & w_{tt}\in C([0,T^*];L_2(\Omega))\cap L_2(0,T^*;W_2^1(\Omega))\}.\end{aligned} \tag{5.80}$$

To prove the above theorem the method of successive approximations is applied. The zero step functions, u_0,ϑ_0,η_0, are chosen to satisfy the following problems:

$$\begin{aligned}&u_{0t}-\operatorname{div}\mathbb{D}(u_0)=0 && \text{in } \Omega^T,\\ &\mathbb{D}(u_0)\overline{n}_0=(p(\varrho_0,\theta_0)-p_0)\overline{n}_0 && \text{on } S^T,\\ &u_0|_{t=0}=v_0 && \text{in } \Omega,\end{aligned} \tag{5.81}$$

where $\mathbb{D}(u_0)=\{\mu(u_{0i\xi_j}+u_{0j\xi_i})+(\nu-\mu)\delta_{ij}\operatorname{div}u_0\}_{i,j=1,2,3}$;

$$\vartheta_{0t}-\widetilde{\varkappa}(\varrho_0,\theta_0)\nabla_\xi^2\vartheta_0=F(u_0,\varrho_0,\theta_0) \quad \text{in } \Omega^T, \tag{5.82}$$

$$\begin{aligned} &\overline{n}_0 \cdot \nabla_\xi \vartheta_0 = \overline{\vartheta}_0 && \text{on } S^T, \\ &\vartheta_0|_{t=0} = \theta_0 && \text{in } \Omega, \end{aligned}$$

where $\overline{\vartheta}_0(\xi,t) = \overline{\theta}(X_{u_0}(\xi,t),t)$, the functions $\widetilde{\varkappa}(\varrho_0,\theta_0)$ and $F(u_0,\varrho_0,\theta_0)$ are such that $\partial_t\vartheta_0|_{t=0} = \partial_t\vartheta|_{t=0}$ and

$$\begin{aligned} \eta_{0t} + \eta_0 \operatorname{div} u_0 &= 0 && \text{in } \Omega^T, \\ \eta_0|_{t=0} &= \varrho_0 && \text{in } \Omega. \end{aligned}$$

Paper [Z2] is concerned with the case of $\sigma > 0$. The aim of [Z2] was to examine solvability of problem (5.70)–(5.75) in the class of functions having the lowest possible regularity in the L_2-approach. Thus, the general method of treating this problem is the same as that used by Solonnikov and Tani [SolT2] for the barotropic case. However, these two papers differ in details. As in [SolT2], in order to prove local existence, the method of successive approximations is applied in [Z3]. Convergence of these approximations can be proved under the assumption that $\alpha \in [3/4, 1)$. This assumption is stronger than in the barotropic case, where it is sufficient to assume that $\alpha \in (1/2, 1)$ (see [SolT2]). The main result of [Z2] can be formulated as follows.

THEOREM 5.9. *Let* $\alpha \in [3/4,1)$, $S \in W_2^{5/2+\alpha}$, $v_0 \in W_2^{1+\alpha}(\Omega)$, $\varrho_0 \in W_2^{1+\alpha}(\Omega)$, $\theta_0 \in W_2^{1+\alpha}(\Omega)$, $\inf_{\xi\in\Omega} \varrho_0(\xi) > 0$, $p \in C^3(\mathbb{R}^2)$, $c_v \in C^2(\mathbb{R}^2)$, $\nu \in C^3(\mathbb{R}^2)$, $\mu \in C^3(\mathbb{R}^2)$, $\varkappa \in C^3(\mathbb{R}^2)$; $f \in C_B^2(\mathbb{R}^3 \times \mathbb{R}_+)$, $r \in C_B^2(\mathbb{R}^3 \times \mathbb{R}_+)$, $\overline{\theta} \in C_B^3(\mathbb{R}^3 \times \mathbb{R}_+)$ *and let the following compatibility conditions be satisfied*:

$$\begin{aligned} &\Pi_0 \mathbb{D}(v_0)\overline{n}_0 = 0 && \textit{on } S, \\ &\overline{n}_0 \cdot \mathbb{D}(v_0)\overline{n}_0 = \overline{n}_0 \cdot (p(\varrho_0,\theta_0) - p_0)\overline{n}_0 + \sigma \overline{n}_0 \cdot \Delta_S(0)\xi && \textit{on } S, \\ &\overline{n}_0 \cdot \nabla\theta_0 = \overline{\theta}|_{t=0} && \textit{on } S. \end{aligned}$$

Then there exists $T > 0$ (*depending on the norms of* v_0, θ_0, ϱ_0, S) *such that there exists a unique solution* $(u,\vartheta,\eta) \in W_2^{2+\alpha,1+\alpha/2}(\Omega^T) \times W_2^{2+\alpha,1+\alpha/2}(\Omega^T) \times C([0,T]; W_2^{1+\alpha}(\Omega)) \cap W_2^{1+\alpha,1/2+\alpha/2}(\Omega^T)$ *of problem* (5.70)–(5.75).

REMARK 5.1. In fact, it suffices to assume in Theorem 5.9 that $f \in C^2(\mathbb{R}^3 \times \mathbb{R}_+)$, $r \in C^2(\mathbb{R}^3 \times \mathbb{R}_+)$, $\overline{\theta} \in C^3(\mathbb{R}^3 \times \mathbb{R}_+)$.

REMARK 5.2. An analogous theorem in the case of constant ν, μ and $\varkappa$ has been proved earlier in [ZZaj11].

REMARK 5.3. The assumption that $\alpha \in [3/4, 1)$ is connected with the strong nonlinearities of the terms $\eta c_v(\eta,\vartheta)\vartheta_t$, $\operatorname{div}_u \mathbb{T}_u(u,p)$ and $\nabla_u \cdot (\varkappa \nabla_u \vartheta)$. If we assume that c_v, ν and μ are constants, then Theorem 5.9 holds for $\alpha \in (1/2, 1)$.

5.2.2. *Global existence and stability*

The case of $\sigma = 0$, $k = 0$ *and* $p_0 = \text{const}$. This case of problem (5.63)–(5.69) is examined in [ZZaj2–ZZaj4, ZZaj6, ZZaj14 –ZZaj16]. In [ZZaj2–ZZaj4, ZZaj6] the above problem is considered under the restrictive assumption on the form of the internal energy $e = e(\varrho,\theta)$. Namely, it is assumed that $e = e(\varrho,\theta)$ has the form

$$e(\varrho,\theta) = a_0\varrho^\alpha + \overline{h}(\varrho,\theta), \tag{5.83}$$

where $a_0 > 0$, $\alpha > 0$, $\overline{h}(\varrho,\theta) \ge h_* > 0$ and a_0, α, h_* are constants.

Moreover, in all the papers mentioned above the following condition is assumed:

$$\begin{gathered}\nu,\mu,\varkappa,p_0 \text{ are constants and}\\ \nu>\frac{1}{3}\mu>0,\quad \varkappa>0,\quad c_v>0;\quad p_\varrho>0,\quad p_\theta>0\quad \text{for } \varrho,\theta>0.\end{gathered}\tag{5.84}$$

Gobal existence for problem (5.63)–(5.69) satisfying (5.83)–(5.84) is proved in [ZZaj6]; in the proof some results of [ZZaj2, ZZaj4] are used. The solution $(v,\theta_\sigma,\varrho_\sigma)$ obtained (where $\theta_\sigma=\theta-\theta_e$, $\varrho_\sigma=\varrho-\varrho_e$; θ_e and ϱ_e are defined by (5.85) in Definition 5.3 below) is such that $\sup_{0<t'<t}(|v(t')|^2_{3,0,\Omega_{t'}}+|\theta_\sigma(t')|^2_{3,0,\Omega_{t'}}+|\varrho_\sigma(t')|^2_{3,0,\Omega_{t'}})+\int_0^t(|v(t')|^2_{4,1,\Omega_{t'}}+|\theta_\sigma(t')|^2_{4,1,\Omega_{t'}}+|\varrho_\sigma(t')|^2_{3,0,\Omega_{t'}})\,dt'<\infty$ for all $t>0$ (where the norms $|f(t)|_{l,k,Q}$ are given by (5.23)) and $S_t\in W_2^{7/2}$ for $t>0$.

In [ZZaj16] the authors also prove global in time existence of solutions of problem (5.63)–(5.69) which are sufficiently close to an equilibrium state. However, in contrast to paper [ZZaj6] no restrictions on the form of the internal energy e are assumed. Moreover, the regularity of solutions obtained in [ZZaj16] is lower than the regularity of solutions from [ZZaj6].

The definition of an equilibrium state in this heat-conducting case with $\sigma=0$ and $k=0$ is as follows.

Definition 5.3. Let $f=0$, $r=\overline{\theta}=0$. An *equilibrium state* is a solution $(v,\theta,\varrho,\Omega_t)$ of (5.63)–(5.69) such that $v=0$, $\theta=\theta_e$, $\varrho=\varrho_e$, $\Omega_t=\Omega_e$ for $t\ge 0$, where θ_e,ϱ_e are positive constants satisfying the state equation

$$p(\varrho_e,\theta_e)=p_0\tag{5.85}$$

and Ω_e is a domain of volume $|\Omega_e|=M/\varrho_e$.

Now, introduce the notation:

$$\begin{aligned}
&p_\sigma=p-p_0,\qquad \theta_\sigma=\theta-\theta_e,\qquad \varrho_\sigma=\varrho-\varrho_e,\\
&\varphi(t)=|v(t)|^2_{2,0,\Omega_t}+|\theta_\sigma(t)|^2_{2,0,\Omega_t}+|\varrho_\sigma(t)|^2_{2,0,\Omega_t},\\
&\Phi(t)=|v(t)|^2_{3,1,\Omega_t}+|\theta_\sigma(t)|^2_{3,1,\Omega_t}+\|\varrho_\sigma(t)\|^2_{W_2^2(\Omega_t)}\\
&\qquad\quad+\|\varrho_{\sigma t}(t)\|^2_{W_2^2(\Omega_t)}+\|\varrho_{\sigma tt}(t)\|^2_{W_2^1(\Omega_t)},\\
&\mathfrak{N}(t)=\{(v,\theta_\sigma,\varrho_\sigma):\varphi(t)<\infty\},\\
&\mathfrak{M}(t)=\Big\{(v,\theta_\sigma,\varrho_\sigma):\sup_{0\le t'\le t}\varphi(t')+\int_0^t\Phi(t')\,dt'<\infty\Big\}.
\end{aligned}$$

The norms $|f(t)|_{k,l,Q}$ are given by formula (5.23).

The following theorem is proved in [ZZaj16].

Theorem 5.10. *Let* (5.84) *and the assumptions of Theorem* 5.8 *be satisfied. Let* $f=0$, $r=\overline{\theta}=0$, $(v,\theta_\sigma,\varrho_\sigma)\in\mathfrak{N}(0)$, $S\in W_2^{5/2}$. *Moreover*, *let the following assumptions be satisfied*:

$$\varphi(0)\le\varepsilon;\tag{5.86}$$

$l>0$ *is a constant such that* $\varrho_e-l>0$, $\theta_0-l>0$ *and*

$$\varrho_1<\varrho_0<\varrho_2,\qquad \theta_1<\theta_0<\theta_2,$$

where $\varrho_1 = \varrho_e - l,\ \varrho_2 = \varrho + l,\ \theta_1 = \theta_e - l,\ \theta_2 = \theta_e + l$;

$$\int_\Omega \varrho_0 v_0 \cdot (a + b \times \xi)\, d\xi = 0,$$

where a *and* b *are arbitrary constant vectors*;

$$\int_\Omega \varrho_0\, d\xi = M.$$

Then for sufficiently small ε *there exists a unique global solution of* (5.63)–(5.69) *such that* $(v, \theta_\sigma, \varrho_\sigma) \in \mathfrak{M}(t)$ *for* $t \in \mathbb{R}_+$, $S_t \in W_2^{5/2}$ *for* $t \in \mathbb{R}_+$ *and*

$$\varphi(t) \le c\varepsilon \quad \textit{for } t \in \mathbb{R}_+,$$

where $c > 0$ *is a constant depending on* $\Omega, \varrho_1, \varrho_2, \theta_1, \theta_2, p, c_v, \nu, \mu, \varkappa$.

Two inequalities are basic in the proof of Theorem 5.10. The first follows from the proof of Theorem 5.8 and from Lemmas 3.5–3.6 and 2.3 of [ZZaj9]. Namely, for sufficiently small time T of local existence, the local solution of (5.70)–(5.75) satisfies

$$\|u\|^2_{\mathcal{A}_{T,\Omega}} + \|\vartheta_\sigma\|^2_{\mathcal{A}_{T,\Omega}} + \|\eta_\sigma\|^2_{\mathcal{B}_{T,\Omega}} \le C_1(T)\varphi(0), \tag{5.87}$$

where $\mathcal{A}_{T,\Omega}$, $\mathcal{B}_{T,\Omega}$ are given by (5.79) and (5.80), and C_1 is an increasing function of T. Inequality (5.87) rewritten in Eulerian coordinates yields, for $t \le T$,

$$\sup_{0\le t'\le t} \varphi(t') + \int_0^t \Phi(t')\, dt' \le C_2(T)\varphi(0), \tag{5.88}$$

where C_2 is an increasing function of T.

In the process of extending the solution step by step to all $t > 0$, estimate (5.87) implies step by step that ϱ and θ remain in the intervals (ϱ_1, ϱ_2) and (θ_1, θ_2), respectively, for all t. Moreover, this estimate implies that for ε sufficiently small, the shape of Ω_t does not change much for $t \le T$. In order to extend the solution step by step and to control the shape of the fluid, the following differential inequality is also used:

$$\frac{d\overline{\varphi}}{dt} + c_1\Phi \le c_2\Big[\varphi(1+\varphi^2) + \int_0^t \|v\|^2_{W_2^3(\Omega_{t'})}\, dt'\Big]\Phi \quad \text{for } t \le T, \tag{5.89}$$

where c_1, c_2 are positive constants depending on ϱ_1, ϱ_2, θ_1, θ_2, ν, μ, $\varkappa$, c_v, p, $\|S\|_{W_2^{5/2}}$, T and the constants from imbedding theorems and Korn inequalities (c_1 and c_2 are also nondecreasing continuous functions of $\int_0^T \|v\|^2_{W_2^3(\Omega_{t'})}\, dt'$). Moreover, $\overline{\varphi}$ in (5.89) is a function satisfying the estimate

$$c_3\varphi(t) \le \overline{\varphi}(t) \le c_4\varphi(t) \quad \text{for } t \le T,$$

where c_3, c_4 are positive constants depending on ϱ_1, ϱ_2, θ_1, θ_2, μ, $\varkappa$, c_v, p, $\|S\|_{W_2^{5/2}}$, T and the constants from imbedding theorems.

Now, since ε is sufficiently small, assumption (5.86) and inequalities (5.88)–(5.89) yield

$$\frac{d\overline{\varphi}}{dt} + c_5\Phi \le 0. \tag{5.90}$$

It follows from (5.90) that

$$\overline{\varphi}(t) \le \overline{\varphi}(0)e^{-c_6 t} \quad \text{for } t \le T, \tag{5.91}$$

where c_6 is a constant depending on the same quantities as c_1 and c_2.

As in the barotropic case, inequalities (5.91) and (5.90) allow one to extend the solution and to control the shape of Ω_t. Thus, if we assume that we have proved the existence of a solution in an interval $[0,(k-1)T]$ for $k \ge 2$ and the estimate

$$|x-\xi| = \Big|\int_0^t v(x,t')\,dt'\Big| \le c_7\varepsilon \quad \text{for } x \in \overline{\Omega}_t,\ 0 \le t \le (k-1)T, \tag{5.92}$$

with sufficiently small ε, then thanks to (5.90), we can prove that estimate (5.92) also holds for $x \in \overline{\Omega}_t$, $0 \le t \le kT$.

Moreover, if $\|\int_0^t u(\xi,t')\,dt'\|_{W_2^3(\Omega)} \le c_8\varepsilon$ for $0 \le t \le (k-1)T$ then assuming that ε is sufficiently small and using inequality (5.90) we can prove the same estimate for $0 \le t \le kT$ (see Subsection 7.4).

The differential inequality (5.89) is derived in [ZZaj14].

Notice that in contrast to inequality (5.28) from the proof of Theorem 5.2 which has been obtained for the barotropic case, the left-hand side of (5.89) consists only of the "nonlinear term". No L_2-norms occur on the left-hand side of (5.89), so there is no need to estimate additionally the sum of the norms $\|p_\sigma\|^2_{L_2(\Omega_t)} + \|\theta_\sigma\|^2_{L_2(\Omega_t)}$.

Obviously, (5.28) could be replaced in the barotropic case by an inequality of the form (5.89), which simplified the proof of Theorem 5.2.

The case of $\sigma > 0$, $k = 0$ *and* $p_0 = \text{const}$. Global existence of solutions to problem (5.63)–(5.69) which are sufficiently close to an equilibrium state, in the case of capillary fluids, is studied in [Z1–Z3, ZZaj3, ZZaj5, ZZaj7, ZZaj8, ZZaj12, ZZaj13] together with the stability of the equilibrium state.

The definition of an equilibrium state in this case is as follows.

DEFINITION 5.4. Let $f = 0$, $r = \overline{\theta} = 0$. By an *equilibrium state* we mean a solution $(v,\theta,\varrho,\Omega_t)$ of (5.63)–(5.69) such that $v = 0$, $\theta = \theta_e$, $\varrho = \varrho_e$, $\Omega_t = \Omega_e$ for $t \ge 0$, where $\varrho_e = (M/(4/3)\pi R_e^3)$; Ω_e is a ball of radius R_e; $R_e > 0$ and $\theta_e > 0$ satisfy the equation

$$p\bigg(\frac{M}{(4/3)\pi R_e^3}, \theta_e\bigg) = p_0 + \frac{2\sigma}{R_e}.$$

In [ZZaj3, ZZaj5, ZZaj7, ZZaj8] problem (5.63)–(5.69) under the restrictive assumption (5.83) is examined.

Existence of a global solution and stability of the equilibrium state are proved in [ZZaj8]. The approach to the global solvability in the above mentioned paper is similar to that applied in the barotropic case in [Zaj4].

Papers [ZZaj5, ZZaj7] contain some auxiliary results, used in [ZZaj8]. In [ZZaj7] a differential inequality, crucial to the proof of the global existence, is derived, while [ZZaj5] is devoted to some consequences of the conservation laws used in [ZZaj8].

The regularity of the global solution $(v, \theta_\sigma, \varrho_\sigma)$ obtained in [ZZaj8] is such that $\sup_{0<t'<t} \varphi_0(t') + \int_0^t \Phi(t')\,dt' < \infty$ and $S_t \in W_2^{9/2}$ for all $t > 0$, where

$$\varphi_0(t) = |v(t)|^2_{3,0,\Omega_t} + |\varrho_\sigma(t)|^2_{3,0,\Omega_t} + |\theta_\sigma(t)|^2_{3,0,\Omega_t}; \tag{5.93}$$

$$\Phi(t) = |v(t)|^2_{4,1,\Omega_t} + |\theta_\sigma(t)|^2_{4,1,\Omega_t} + |\varrho_\sigma(t)|^2_{3,0,\Omega_t}; \tag{5.94}$$

$\varrho_\sigma = \varrho - \varrho_e$, $\theta_\sigma = \theta - \theta_e$ and the norms $|f(t)|_{k,l,Q}$ are given by (5.23).

The paper [Z1] generalizes the result of [ZZaj8] in such a way that assumption (5.83) is removed. The proof of global existence in [Z1] is very sketchy. Therefore, we present below the global existence theorem of [Z1] together with its proof, which will be discussed thoroughly. We assume the following conditions:

(5.95) $f = 0,\ p_0 > 0,\ \overline{\theta} \geq 0;$

(5.96) $\|r\|^2_{C^3_B(\mathbb{R}^3\times(0,\infty))} + \|\overline{\theta}\|^2_{C^4_B(\mathbb{R}^3\times(0,\infty))} \leq \overline{\delta}$
and $\overline{\theta} \in L_1(\mathbb{R}^3 \times (0,\infty))$, where $\overline{\delta} > 0$ is a sufficiently small constant;

(5.97) $\varrho_1 < \varrho_0 < \varrho_2,\ \theta_1 < \theta_0 < \theta_2$ for all $\xi \in \Omega$,
where ϱ_1, ϱ_2, θ_1, θ_2 are positive constants;

(5.98) $e_1 < e(\varrho, \theta) < e_2$ for all $\varrho \in (\varrho_1, \varrho_2),\ \theta \in (\theta_1, \theta_2)$,
where $0 < e_1 < e_2$ are constants;

(5.99) $c_v \in C^3(\mathbb{R}^2),\ p \in C^4(\mathbb{R}^2),\ e \in C^1(\mathbb{R}_+ \times \mathbb{R}_+)$.

Just as in the barotropic case, the conservation laws are very important in the proof of global existence. The energy conservation law has the following form in this case:

$$\frac{d}{dt}\bigg[\int_{\Omega_t} \varrho\bigg(\frac{v^2}{2} + e\bigg)\,dx + p_0|\Omega_t| + \sigma|S_t|\bigg] - \int_{S_t} \overline{\theta}\,ds = 0. \tag{5.100}$$

The conservation laws (5.36)–(5.37) and the mass conservation law also hold in this case. The energy conservation law and assumptions (5.98), (5.113) imply

$$\begin{aligned}\frac{e_1}{\varrho_2^\beta}\int_{\Omega_t} \varrho^\gamma\,dx &+ \int_{\Omega_t} \frac{\varrho v^2}{2}\,dx + p_0|\Omega_t| + \sigma|S_t| \\ &\leq \int_{\Omega} \varrho_0\bigg(\frac{v_0^2}{2} + e_0\bigg)\,d\xi + p_0|\Omega| + \sigma|S| + \int_0^\infty dt \int_{\mathbb{R}^3} \overline{\theta}(x,t)\,dx \equiv d,\end{aligned} \tag{5.101}$$

where $\gamma = \beta + 1$, $\beta > 0$ is a constant, $e_0 = e(\varrho_0, \theta_0)$, $t \leq T$, T is the time of local existence.

As in the barotropic case, multiplying (5.101) by $|\Omega_t|^\beta$ we get

$$\begin{aligned}y(|\Omega_t|) + \frac{e_1}{\varrho_2^\beta}\bigg[|\Omega_t|^\beta \int_{\Omega_t} \varrho^\gamma\,dx - \bigg(\int_{\Omega_t} \varrho\,dx\bigg)^\gamma\bigg] \\ + |\Omega_t|^\beta \int_{\Omega_t} \varrho\frac{v^2}{2}\,dx + \sigma|\Omega_t|^\beta(|S_t| - 4\pi R_t^2) \leq 0,\end{aligned}$$

where

$$y(x) = p_0 x^\gamma + \widetilde{c}\sigma x^{\gamma-1/3} - d x^{\gamma-1} + \frac{e_1}{\varrho_2^\beta} M^\gamma, \quad \widetilde{c} = (36\pi)^{1/3}, \quad R_t = \bigg(\frac{3}{4\pi}|\Omega_t|\bigg)^{1/3}.$$

We use the properties of $y(x)$ to prove that under some assumptions on the data p_0, σ, d, M, e_1, ϱ_2, γ, the volume of Ω_t does not change much in time. We proceed as in the barotropic case. Namely, the minimum points of $y(x)$ are determined by equation (5.41) with d defined now by the right-hand side of (5.101). Since we consider $y(x)$ for $x > 0$ we look for positive minimum points of $y(x)$. Viète's formulas imply that there exists a unique positive root x_0 of (5.41). We want to calculate $y(x_0)$. In order to do this we have to consider three cases $(5.42)_1$–$(5.42)_3$, one of which holds. For each of these cases the function Φ_i $(i = 1, 2, 3)$ has the form

$$\Phi_i(\mu_0, \psi_i, p_0, \gamma, e_1, \varrho_2, M) = \overline{\Phi}_i(\mu_0, \psi_i, p_0, \gamma) - \frac{e_1}{\varrho_2^\beta} M^\gamma,$$

where ψ_i are given by of $(5.42)_i$ and $(5.43)_i$, and $\overline{\Phi}_i$ $(i = 1, 2, 3)$ are defined in $(5.44)_i$.

In the case $(5.43)_i$ we obtain

$$y(x_0) = -\Phi_i(\mu_0, \psi_i, p_0, \gamma, e_1, \varrho_2, M).$$

It can be proved that if $0 < -y(x_0) \le \delta_0$ with sufficiently small δ_0 then the volume of Ω_t does not change much in time (see [ZZaj5]).

Therefore, we assume that the parameters μ_0, ν_0, p_0, γ, e_1, ϱ_2, M satisfy one of the relations

$$(5.102)_i \qquad \nu_0 \in I_i, \ 0 < \Phi_i(\mu_0, \psi_i, p_0, \gamma, e_1, \varrho_2, M) \le \delta_0,$$

where $1 \le i \le 3$.

Moreover, assume that

$$(5.103) \qquad ||\Omega| - |\Omega_e|| \le \delta_1$$

$$(5.104) \qquad \int_\Omega \varrho_0 \frac{v_0^2}{2}\, d\xi + \int_\Omega \varrho_0(e_0 - e_1)\, d\xi + \sigma(|S| - 4\pi R_0^2) + \int_0^\infty dt \int_{\mathbb{R}^3} \overline{\theta}(x,t)\, dx \le \delta_2,$$

where R_0 is the radius of a ball of volume $|\Omega|$, $\delta_2 \in (0,1)$.

We introduce the spaces:

$$\mathfrak{N}(\mathrm{t}) = \{(v, \theta_\sigma, \varrho_\sigma) : \varphi(t) < \infty\},$$

$$\mathfrak{M}(\mathrm{t}) = \Big\{(v, \theta_\sigma, \varrho_\sigma) : \sup_{0 \le t' \le t} \varphi_0(t') + \int_0^t \Phi(t')\, dt' < \infty\Big\},$$

where $\varphi(t) = \varphi_0(t) + |v(t)|^2_{3,1,S_t}$; φ_0 and Φ are the functions given by (5.93) and (5.94), respectively.

THEOREM 5.11. *Let the assumptions* (5.84), (5.95)–(5.96), (5.99) *with* $e_\varrho > 0$ *for* $\varrho, \theta > 0$, (5.103)–(5.104) *and one of the conditions* $(5.102)_i$ *be fulfilled. Let* $(v, \theta_\sigma, \varrho_\sigma) \in \mathfrak{N}(0)$ *and*

$$(5.105) \qquad \varphi(0) \le \alpha_1,$$

where $\alpha_1 \in (0,1)$ *and* ϱ_e, θ_e *and* R_e *satisfy conditions of Definition* 5.4. *Let the following compatibility conditions be satisfied*:

$$D_\xi^\alpha \partial_t^i (\mathbb{T}\overline{n} - \sigma H \overline{n} + p_0 \overline{n})|_{t=0} = 0 \quad \textit{on } S,\ |\alpha| + i \le 2,$$
$$D_\xi^\alpha \partial_t^i (\overline{n} \cdot \nabla\theta - \overline{\theta})|_{t=0} = 0 \quad \textit{on } S,\ |\alpha| + i \le 2.$$

Assume that $l > 0$ *is a constant such that* $\varrho_e - l > 0$, $\theta_e - l > 0$ *and* (5.97)–(5.98) *hold with* $\varrho_1 = \varrho_e - l$, $\varrho_2 = \varrho_e + l$, $\theta_1 = \theta_e - l$, $\theta_2 = \theta_e + l$. *Let*

$$\int_\Omega \varrho_0 \, d\xi = M, \quad \int_\Omega \varrho_0 \xi \, d\xi = 0, \quad \int_\Omega \varrho_0 v_0 \, d\xi = 0. \tag{5.106}$$

Moreover, assume that Ω *is diffeomorphic to a ball and let* S *be described by* $|\xi| = \widetilde{R}(\omega)$, $\omega \in S^1$ (S^1 *is a unit sphere*), *where* $\widetilde{R}$ *satisfies*

$$\|\widetilde{R} - R_e\|^2_{W_2^1(S^1)} \le \alpha_2. \tag{5.107}$$

Finally, assume that $S \in W_2^{4+1/2}$ *and*

$$\|H(\cdot, 0) + 2/R_e\|^2_{W_2^{2+1/2}(S)} \le \alpha_1. \tag{5.108}$$

Then for sufficiently small constants $\alpha_1, \alpha_2, \overline{\delta}$ *and* δ_i ($i = 0, \ldots, 3$) *there exists a unique global solution to problem* (5.63)–(5.69) *such that* $(v, \theta_\sigma, \varrho_\sigma) \in \mathfrak{M}(t)$ *for* $t \in \mathbb{R}_+$ *and*

$$\sup_{jT \le t \le (j+1)T} \varphi_0(t) + \int_{jT}^{(j+1)T} \Phi(t') \, dt' \le \widehat{c}_1 \alpha_1 \quad \textit{for } j \in \mathbb{N} \cup \{0\}, \tag{5.109}$$

where T *is the time of local existence defined by Theorem* 4.2 *of* [ZZaj1], $\widehat{c}_1 > 0$ *is a constant. Moreover,* $S_t \in W_2^{4+1/2}$ *for* $t \in \mathbb{R}_+$, Ω_t *satisfies condition* (5.126) *for* $t \in \mathbb{R}_+$ *and*

$$\sup_{t_1 > 0} \|R(\cdot, t) - R_e\|^2_{W_2^{4+1/2}(S^1)} \le \widehat{c}_2(\alpha_1 + \overline{\delta}) + \widehat{c}_3 \alpha_2, \tag{5.110}$$

where the constant $\widehat{c}_2$ *depends on* $0 < t_1 < T$.

Auxiliary results for the proof of Theorem 5.11

PART 1. We start with an estimate for the local solution (u, ϑ, η) of problem (5.70)–(5.75) such that $(u, \vartheta, \eta) \in W_2^{4,2}(\Omega^T) \times W_2^{4,2}(\Omega^T) \times C([0,T]; W_2^3(\Omega))$ and $\eta_t \in C([0,T]; W_2^2(\Omega)) \cap L_2(0,T; W_2^3(\Omega)$, $\eta_{tt} \in L_2(0,T; W_2^1(\Omega))$. Local existence of such a solution is proved in [ZZaj1]. To derive the estimate we write the system (5.70)–(5.75) with $g = 0$, $k = 0$, the constant coefficients ν, μ, $\varkappa$ and constant p_0 as follows:

$$\begin{aligned}
&\eta u_t - \operatorname{div}_u \mathbb{T}_u(u, p_\sigma) = 0 && \text{in } \Omega^T,\\
&\eta_{\sigma t} + \eta \nabla_u \cdot u = 0 && \text{in } \Omega^T,\\
&\eta c_v(\eta, \vartheta)\vartheta_{\sigma t} - \nabla_u \cdot (\varkappa \nabla_u \vartheta_\sigma) = -\vartheta p_\vartheta(\eta, \vartheta) \nabla_u \cdot u &&\\
&\qquad + \frac{\mu}{2} \sum_{i,j=1}^3 (\xi_{x_i} \cdot \partial_\xi u_j + \xi_{x_j} \cdot \partial_\xi u_i)^2 &&\\
&\qquad - (\nu - \mu)(\nabla_u \cdot u)^2 = \eta h && \text{in } \Omega^T,\\
&\mathbb{T}_u(u, p_\sigma)\overline{n}_u - \sigma(H + 2/R_e)\overline{n}_u = 0 && \text{on } S^T,\\
&\overline{n}_u \cdot \nabla_u \vartheta_\sigma = \overline{\vartheta} && \text{on } S^T,\\
&u|_{t=0} = v_0, \quad \eta_\sigma|_{t=0} = \varrho_{\sigma 0}, \quad \vartheta_\sigma|_{t=0} = \theta_{\sigma 0}, && \text{in } \Omega,
\end{aligned} \tag{5.111}$$

where $p_\sigma = p - p_0 - 2\sigma/R_e$; $\varrho_{\sigma 0} = \varrho_0 - \varrho_e$, $\theta_{\sigma 0} = \theta_0 - \theta_e$; u, ϑ_σ, η_σ denote v, θ_σ, ϱ_σ written in Lagrangian coordinates $\xi \in \Omega$.

We can treat problem (5.111) as a linear problem with respect to u, ϑ_σ and η_σ. Then from theorems concerning such linear problems (Theorems 3.6, 4.1 and Lemma 3.3 of [ZZaj1]) we get the estimate

$$\begin{aligned}(5.112)\qquad &\|u\|^2_{W_2^{4,2}(\Omega^t)}+\|\vartheta_\sigma\|^2_{W_2^{4,2}(\Omega^t)}+\|\eta_\sigma\|^2_{C([0,t];W_2^3(\Omega))}+\|\eta_{\sigma t}\|^2_{C([0,t];W_2^2(\Omega))}\\&\qquad+\|\eta_{\sigma t}\|^2_{L_2(0,t;W_2^3(\Omega))}+\|\eta_{\sigma tt}\|^2_{L_2(0,t;W_2^1(\Omega))}\\&\le\varphi^{(1)}(T)\bigg[\|v_0\|^2_{W_2^3(\Omega)}+\|\varrho_{\sigma 0}\|^2_{W_2^3(\Omega)}+\|\theta_{\sigma 0}\|^2_{W_2^3(\Omega)}+\|u_t(0)\|^2_{W_2^1(\Omega)}\\&\quad+\|\vartheta_{\sigma t}(0)\|_{W_2^1(\Omega)}+\|H(\cdot,0)+2/R_e\|^2_{W_2^{2+1/2}(S)}+\|h\|^2_{W_2^{2,1}(\Omega^T)}+\|\overline{\vartheta}\|^2_{W_2^{3-1/2,3/2-1/4}(S^T)}\\&\quad+\bigg(\int_0^T\frac{\|D^2_{\xi,t}\overline{\vartheta}\|^2_{L_2(S)}}{t^{1/2}}\,dt\bigg)^{1/2}\bigg],\end{aligned}$$

where $t\le T$, T is the time of local existence (depending on α_1); $\varphi^{(1)}$ is a positive nondecreasing continuous function of T; $u_t(0)$ and $\vartheta_{\sigma t}(0)$ are calculated from system (5.111).

Part 2. We also need some lemmas which yield estimates for the L_2-norms of v,θ_σ and p_σ. These lemmas are consequences of the conservation laws of energy and mass.

Let (v,θ,ϱ) be the local solution of problem (5.63)–(5.69) which is guaranteed by Theorem 4.1 of [ZZaj1]. Assume

$$(5.113)\qquad \varrho_1<\varrho(x,t)<\varrho_2,\ \theta_1<\theta(x,t)<\theta_2\quad\text{for all } x\in\Omega_t,\ t\in[0,T],$$

where T is the time of local existence.

Lemma 5.3 (see Theorem 2.3 of [Z1]). *Let conditions* (5.95), (5.96), (5.98), (5.113) *be satisfied. Let $\delta_0\in(0,1)$ be given. Assume that the parameters μ_0, ν_0, β, e_1, ϱ_2, M satisfy one of the relations* $(5.102)_i$. *Then there exists a constant $c_1>0$ independent of δ_0* (*it can depend on the parameters*) *such that*

$$\operatorname*{var}_{0\le t\le T}|\Omega_t|\le c_1\delta,$$

where $\operatorname{var}_{0\le t\le T}|\Omega_t|=\sup_{0\le t\le T}|\Omega_t|-\inf_{0\le t\le T}|\Omega|$, $\delta^2=c_2\delta_0$, $c_2>0$ *is a constant. Moreover*, *in the case* $(5.102)_i$ *we have*

$$\big||\Omega_t|-Q_i\big|\le c_3\delta\quad \textit{for } t\in[0,T],$$

where Q_i $(i=1,2,3)$ are defined in Lemma 5.2, *$c_3>0$ is a constant.*

The above lemma is analogous to Lemma 5.2 of this paper which holds for the barotropic fluid. The proof of Lemma 5.3 is the same as the proof of Theorem 1 of [ZZaj5] (see also Lemma 2.2 of [Zaj4]).

Remark 5.4. For each $1\le i\le 3$ and for each δ_0 there exist parameters γ, d, p_0, σ, M, e_1, ϱ_2 such that condition $(5.102)_i$ is satisfied (see [ZZaj5].

For example, let $\nu_0=2\mu_0^3$. Then

$$\frac{d(\gamma-1)}{2p_0}=\frac{2l^3(\gamma-1/3)^3}{27\gamma^2},\qquad l=\frac{\widetilde{c}\sigma}{p_0}$$

and

$$\Phi_2 = \frac{p_0(9\gamma-1)}{(3\gamma-1)(\gamma-1)}\left[\frac{l(3\gamma-1)}{9\gamma}\right]^{3\gamma} - \frac{e_1}{\varrho_2^\beta}M^\gamma$$
$$= \frac{27\gamma^2 d(9\gamma-1)}{4(\gamma-1/3)^3(3\gamma-1)l^3}\left(\frac{l}{3}\right)^{3\gamma}\left(1-\frac{1}{3\gamma}\right)^{3\gamma} - e_1\varrho_2\left(\frac{M}{\varrho_2}\right)^\gamma.$$

We see that $\lim_{\gamma\to 1^+}\Phi_2 = d - e_1 M > 0$. On the other hand, assuming that $l/3 \le 1$ we get

$$\lim_{\gamma\to+\infty}\Phi_2 \begin{cases} =0 & \text{if } M/\varrho_2 < 1,\\ -\infty & \text{if } M/\varrho_2 > 1,\\ -e_1\varrho_2 & \text{if } M/\varrho_2 = 1.\end{cases}$$

Moreover, if we assume $l/3 > 1$ and $M/\varrho_2 \ge (l/3)^3$, we obtain $\lim_{\gamma\to+\infty}\Phi_2 = -\infty$.

Therefore in this case for each δ_0 we can find γ such that $0 \le \Phi_2 \le \delta_0$.

LEMMA 5.4. *Let the assumptions of Lemma* 5.3 *be satisfied. Moreover*, *assume* (5.104). *Then*

$$\|v\|^2_{L_2(\Omega_t)} \le \delta_3 \quad \text{for } t \le T,$$

where $\delta_3 = c_4(\delta+\delta_2)$; $c_4 > 0$ *is a constant depending on* ϱ_1, ϱ_2, θ_1,θ_2, M, β, d, p_0, σ *and the form of internal energy* e.

Proof. Let R_t and R_* be the radii of balls of volumes $|\Omega_t|$ and $\inf_{0\le t\le T}|\Omega_t|$, respectively. Then by Lemma 5.3,

$$|\Omega_t| - \inf_t|\Omega_t| \le c_1\delta, \qquad R_t - R_* \le c\delta \qquad \text{for } 0 \le t \le T,$$

where $c > 0$ is a constant. Since $|S_t| - 4\pi R_t^2 \ge 0$ we also have $|S_t| - 4\pi R_*^2 \ge 0$. Therefore, using the conservation laws of energy (5.100) and of mass we get

$$\begin{aligned}(5.114)\qquad &\int_{\Omega_t}\varrho\frac{v^2}{2}\,dx + \int_{\Omega_t}\varrho(e(\varrho,\theta)-e_1)\,dx + p_0(|\Omega_t| - \inf_t|\Omega_t|) + \sigma(|S_t| - 4\pi R_*^2)\\ &\le \int_\Omega \varrho_0\frac{v_0^2}{2}\,d\xi + \int_\Omega \varrho_0(e(\varrho_0,\theta_0)-e_1)\,d\xi + p_0(|\Omega| - \inf_t|\Omega_t|) + \sigma(|S| - 4\pi R_*^2)\\ &\quad + \int_0^\infty dt\int_{R^3}\overline{\theta}(s,t')\,dt' \le c(\delta+\delta_2).\end{aligned}$$

This completes the proof. ■

LEMMA 5.5. *Let assumption* (5.113) *be satisfied. Then*

$$(5.115)\qquad \|p_\sigma\|^2_{L_2(\Omega_t)} \le c_5[\varepsilon(\|u\|^2_{W_2^{4,2}(\Omega^T)} + \|u\|^4_{W_2^{4,2}(\Omega^T)}) + c(\varepsilon,T)(\|u\|^2_{L_2(\Omega^T)} + \|u\|^4_{L_2(\Omega^T)})]$$

and

$$\begin{aligned}(5.116)\qquad &\|\varrho_\sigma\|^2_{L_2(\Omega_t)} + \|\theta_\sigma\|^2_{L_2(\Omega_t)} \le c_6[\varepsilon\|u\|^2_{W_2^{4,2}(\Omega^T)} + c(\varepsilon,T)\|u\|^2_{L_2(\Omega^T)}\\ &+\|v_x\|^2_{L_2(\Omega_t)} + \|\theta_{\sigma x}\|^2_{L_2(\Omega_t)} + \|v\|^4_{W_2^1(\Omega_t)} + \|\varrho_\sigma\|^4_{W_2^1(\Omega_t)} + \|\theta_\sigma\|^4_{W_2^1(\Omega_t)}] \quad \text{for } t \le T,\end{aligned}$$

where c_5, $c_6 > 0$ *are constants depending on* ϱ_1, ϱ_2, θ_1, θ_2; $\varepsilon \in (0,1)$ *is a constant*; $c(\varepsilon,T)$ *is a positive constant depending on* ε *and* T.

Proof. Let w_1 be a solution of the problem

$$\begin{aligned} &\operatorname{div} w_1 = p_\sigma && \text{in } \Omega_t,\\ &w_1 = 0 && \text{on } S_t. \end{aligned} \tag{5.117}$$

In view of Lemma 2.2 of [LadSol] there exists a solution of problem (5.117) such that $w_1 \in W_2^1(\Omega_t)$ and

$$\|w_1\|_{W_2^1(\Omega_t)} \le c\|p_\sigma\|_{L_2(\Omega_t)}. \tag{5.118}$$

Now, multiplying equation (5.63) by w_1 and integrating over Ω_t we obtain

$$\|p_\sigma\|^2_{L_2(\Omega_t)} \le \varepsilon\|w_1\|^2_{W_2^1(\Omega_t)} + c(\varepsilon)\|v_t\|^2_{L_2(\Omega_t)} + c(\|v_x\|^2_{L_2(\Omega_t)} + \|v\|^4_{W_2^1(\Omega_t)}),$$

where $\varepsilon > 0$ is a sufficiently small constant. Hence using (5.118) yields

$$\|p_\sigma\|^2_{L_2(\Omega_t)} \le c(\|v_t\|^2_{L_2(\Omega_t)} + \|v_x\|^2_{L_2(\Omega_t)} + \|v\|^4_{W_2^1(\Omega_t)}). \tag{5.119}$$

Next, let us rewrite equation (5.63) in the form

$$\varrho[v_t + (v\cdot\nabla)v] - \operatorname{div}\mathbb{D}(v) + p_\varrho\nabla\varrho_\sigma + p_\theta\nabla\theta_\sigma = 0 \tag{5.120}$$

and let w_2 be a solution of the problem

$$\begin{aligned} &\operatorname{div} w_2 = \varrho_\sigma && \text{in } \Omega_t,\\ &w_2 = 0 && \text{on } S_t. \end{aligned} \tag{5.121}$$

There exists $w_2 \in W_2^1(\Omega_t)$ satisfying (5.121) and

$$\|w_2\|_{W_2^1(\Omega_t)} \le c\|\varrho_\sigma\|_{L_2(\Omega_t)}. \tag{5.122}$$

Now, we multiply (5.120) by w_2 and integrate over Ω_t. In view of (5.113), the positivity and continuity of p_ϱ we get

$$\begin{aligned} \|\varrho_\sigma\|^2_{L_2(\Omega_t)} &\le \varepsilon\|w_2\|^2_{W_2^1(\Omega_t)} + c(\varepsilon)\|v_t\|^2_{L_2(\Omega_t)}\\ &\quad + c(\|v_x\|^2_{L_2(\Omega_t)} + \|\theta_{\sigma x}\|^2_{L_2(\Omega_t)} + \|v\|^4_{W_2^1(\Omega_t)} + \|\varrho_\sigma\|^4_{W_2^1(\Omega_t)} + \|\theta_\sigma\|^4_{W_2^1(\Omega_t)}), \end{aligned} \tag{5.123}$$

where we used the integration by parts in $\int_{\Omega_t} p_\varrho\nabla\varrho_\sigma w_2\,dx$.

The relation

$$p_\sigma = p_\varrho\varrho_\sigma + p_\theta\theta_\sigma$$

(where the values of p_ϱ and p_θ are taken at a point $(\varrho_e + s(\varrho-\varrho_e), \theta_e + s(\theta-\theta_e))$, $s\in(0,1)$) implies

$$\|\theta_\sigma\|^2_{L_2(\Omega_t)} \le c(\|p_\sigma\|^2_{L_2(\Omega_t)} + \|\varrho_\sigma\|^2_{L_2(\Omega_t)}), \tag{5.124}$$

where c is a constant depending on ϱ_1, ϱ_2, θ_1, θ_2.

Therefore, taking into account (5.118), (5.119), (5.122)–(5.124) we have

$$\begin{aligned} \|\varrho_\sigma\|^2_{L_2(\Omega_t)} + \|\theta_\sigma\|^2_{L_2(\Omega_t)} &\le c(\|u_t\|^2_{L_2(\Omega)} + \|v_x\|^2_{L_2(\Omega_t)}\\ &\quad + \|\theta_{\sigma x}\|^2_{L_2(\Omega_t)} + \|v\|^4_{W_2^1(\Omega_t)} + \|\varrho_\sigma\|^4_{W_2^1(\Omega_t)} + \|\theta_\sigma\|^4_{W_2^1(\Omega_t)}), \end{aligned} \tag{5.125}$$

where we used the fact that the local existence is proved for T so small that $\overline{c}_1 \le |\xi_x| \le \overline{c}_2$ for some positive constants $\overline{c}_1$ and $\overline{c}_2$.

Now, using in (5.125) the interpolation inequality

$$\sup_{0\le t\le T}\|u_t\|^2_{L_2(\Omega)} \le \varepsilon\|u\|^2_{W_2^{4,2}(\Omega^T)} + c(\varepsilon,T)\|u\|^2_{L_2(\Omega^T)}$$

we obtain (5.116).

Estimate (5.115) follows by using in (5.119) the above interpolation inequality and the inequality

$$\sup_{0\le t\le T}\|u\|^2_{W^1_2(\Omega)} \le \varepsilon\|u\|^2_{W^{4,2}_2(\Omega^T)} + c(\varepsilon,T)\|u\|^2_{L_2(\Omega^T)}. \ \blacksquare$$

PART 3. In this part we use the momentum conservation law (5.36) and (5.37) to derive an estimate for a function describing the free boundary S_t.

We assume the following condition:

(5.126) Ω_t is diffeomorphic to a ball and S_t is described by the equation

$$|x| = R(\omega,t), \qquad \omega\in S^1.$$

LEMMA 5.6. *Let condition* (5.126) *and the assumptions of Theorem* 7.1 *be satisfied for* $t\le T$. *Then for* $0\le t_*\le t\le T$,

$$\begin{aligned}(5.127)\qquad &\int_{t_*}^t \|R(\cdot,t')-R(\cdot,0)\|^2_{W^4_2(S^1)}\,dt' \le c_7\Big[\varepsilon\Big(\int_{t_*}^t\|u\|^2_{W^4_2(\Omega)}\,dt'\\ &+\int_{t_*}^t\|\eta_\sigma\|^2_{W^3_2(\Omega)}\,dt' + \int_{t_*}^t\|\vartheta_\sigma\|^2_{W^3_2(\Omega)}\,dt'\Big) + c(\varepsilon)(\|u\|^2_{L_2(\Omega\times(t_*,t))} + \|q_\sigma\|^2_{L_2(\Omega\times(t_*,t))})\\ &+(t-t_*)(\|H(\cdot,0)+2/R_e\|^2_{W^2_2(S)} + \sup_{0\le t'\le t}\|R(\cdot,t')-R_e\|^2_{L_2(S^1)})\Big],\end{aligned}$$

where $\varepsilon\in(0,1)$ *is a sufficiently small constant and the constant* c_7 *can depend on* $\|R(\cdot,t)\|_{W^{3+1/2}_2(S^1)}$. *Moreover, for* $0\le t_*\le t\le T$ *we have*

$$\begin{aligned}(5.128)\qquad &\int_{t_*}^t \|R(\cdot,t')-R(\cdot,0)\|^2_{W^{4+1/2}_2(S^1)}\,dt'\\ &\le c_8\Big[\int_{t_*}^t\|u\|^2_{W^4_2(\Omega)}\,dt' + \int_{t_*}^t\|\eta_\sigma\|^2_{W^3_2(\Omega)}\,dt' + \int_{t_*}^t\|\vartheta_\sigma\|^2_{W^3_2(\Omega)}\,dt'\\ &\quad+(t-t_*)(\|H(\cdot,0)+2/R_e\|^2_{W^{2+1/2}_2(S)} + \sup_{0\le t'\le t}\|R(\cdot,t')-R_e\|^2_{L_2(S^1)})\Big],\end{aligned}$$

where the constant c_8 *can depend on* $\|R(\cdot,t)\|_{W^{3+1/2}_2(S^1)}$.

Proof. Applying Theorem 7.1 we obtain

$$\begin{aligned}&\int_{t_*}^t \|R(\cdot,t')-R(\cdot,0)\|^2_{W^{4+1/2}_2(S^1)}\,dt' \le 2\Big[\int_{t_*}^t\|R(\cdot,t')-R_e\|^2_{W^{4+1/2}_2(S^1)}\,dt'\\ &\qquad+(t-t_*)\|R(\cdot,0)-R_e\|^2_{W^{4+1/2}_2(S^1)}\Big]\\ &\le c\Big[\int_{t_*}^t\|H(\cdot,t')+2/R_e\|^2_{W^{2+1/2}_2(S_{t'})}\,dt' + (t-t_*)(\|H(\cdot,0)+2/R_e\|^2_{W^{2+1/2}_2(S)}\\ &\qquad+\sup_{0\le t'\le t}\|R(\cdot,t')-R_e\|^2_{L_2(S^1)}\Big] \quad \text{for } 0\le t_*\le t\le T.\end{aligned}$$

Hence, using boundary condition (5.66) we get (5.128). Estimate (5.127) is proved in the same way. ∎

REMARK 5.5. The constants c_7 and c_8 depend only on R_e and $\|\widetilde{R}\|_{W_2^{3+1/2}(S^1)}$ if the constants α_i $(i=1,2)$ are sufficiently small. In fact, for $t\le T$ we can estimate

$$\|R(\cdot,t)\|_{W_2^{3+1/2}(S^1)}\le \varepsilon\|\widetilde{R}-R_e\|_{W_2^{4+1/2}(S^1)}+c(\varepsilon)\|\widetilde{R}-R_e\|_{L_2(S^1)}+c'(R_e+T^{1/2}\|u\|_{W_2^4(\Omega^T)}).$$

Hence, by using Theorem 7.1, assumptions (5.96), (5.107), (5.108) and estimate (5.112), for $t\le T$ and for sufficiently small ε, α_1, α_2 and $\overline{\delta}$ we have

$$\|R(\cdot,t)\|_{W_2^{3+1/2}(S^1)}\le \varepsilon\alpha_1^{1/2}+c(\varepsilon)\alpha_2^{1/2}+c'\{R_e+[Tc''(\alpha_1+\overline{\delta})^{1/2}]\}<2c'R_e.$$

Similarly for $t\le T$ and $\omega\in S^1$ we obtain

$$\begin{aligned}|R(\omega,t)|^2&\ge \widehat{c}'|\widetilde{R}(\omega)|^2-\widehat{c}''T(\alpha_1+\overline{\delta})\ge \widehat{c}'(R_e-|\widetilde{R}(\omega)-R_e|_{L_\infty(S^1)})^2-\widehat{c}''T(\alpha_1+\overline{\delta})\\&\ge \widehat{c}'(R_e-\varepsilon\alpha_1^{1/2}-\widehat{c}^{(3)}\alpha_2^{1/2})^2-\widehat{c}''T(\alpha_1+\overline{\delta})>\frac{1}{2}\widehat{c}'R_e^2,\end{aligned}$$

if ε, α_1, α_2 and $\overline{\delta}$ are sufficiently small. Therefore

$$\|R(\cdot,t)\|_{W_2^{3+1/2}(S^1)}>\widehat{c}^{(4)}R_e\quad \text{for } t\le T.$$

In view of the above estimates we see that in fact the constants c_8 and c_9 depend only on R_e.

To estimate $\|R(\cdot,t)-R_e\|^2_{L_2(S^1)}$ we need the following lemma analogous to Theorem 3 of [Sol6] (formulated in this paper as Theorem 4.7) which holds in the incompressible case.

LEMMA 5.7. *Assume that Ω_t satisfies condition* (5.126) *and suppose the origin coincides with the barycentre of Ω_t. Let $\varrho(x,t)$ be the density defined for $x\in\Omega_t$ and set $\varrho_*=\inf_{t\in[0,T]}\inf_{x\in\overline{\Omega}}\varrho(x,t)$. Then if there exists a constant $\widehat{\delta}\in(0,1/2)$ such that*

$$\sup_{S^1}|R(\omega,t)-R_t|+\sup_{S^1}|\nabla R(\omega,t)|\le\widehat{\delta}R_t\quad \textit{for } t\in[0,T], \tag{5.129}$$

where R_t is the radius of a ball of volume $|\Omega_t|$, i.e. $R_t=\big(\frac{3}{4\pi}|\Omega_t|\big)^{1/3}$, then

$$\begin{aligned}&\int_{S^1}(|R(\omega,t)-R_t|^2+|\nabla R(\omega,t)|^2)\,d\omega\\&\quad\le c_9(|S_t|-4\pi R_t^2)+(c_{10}/(|\Omega_t|^{4/3}\varrho_*))\Big(\int_{\Omega_t}(\varrho-\varrho_*)\,dx\Big)^2\quad \textit{for } t\in[0,T],\end{aligned}\tag{5.130}$$

where c_9, c_{10} are constants which do not depend on $\widehat{\delta}$ and R_t.

Lemma 5.7 is analogous to Lemma 1 of Theorem 4.8 which holds for an incompressible motion. However, in the incompressible case the only term on the right-hand side of the estimate is $c(|S_t|-4\pi R_0^2)$, where $R_0=(\frac{3}{4\pi}|\Omega|)^{1/3}$. Thus, the second term on the right-hand side of (5.130) is associated with the compressibility of the fluid.

The assumption that the origin coincides with the barycentre of Ω_t means that $\int_{\Omega_t}\varrho x\,dx=0$ for $t\le T$. This last condition is implied by the assumptions: $\int_\Omega\varrho_0\xi\,d\xi=0$, $\int_\Omega\varrho_0v_0\,d\xi=0$ and by the conservation laws (5.36)–(5.37).

The proof of the above lemma depends on the formula

$$|S_t| - 4\pi R_t^2 = \int\limits_{S^1} (R\sqrt{R^2 + |\nabla R|^2} - R_t^2)\, d\omega.$$

By the assumption that the origin coincides with the barycentre of Ω_t we have

$$\int\limits_{\Omega_t} (\varrho - \varrho_*)x\, dx + \varrho_*\Big(\int\limits_{\Omega_t} x\, dx - \int\limits_{K(0,R_t)} x\, dx\Big) = 0,$$

where $K(0, R_t)$ is the ball with center 0 and radius R_t. Hence

$$\frac{1}{\varrho_*}\int\limits_{\Omega_t} (\varrho - \varrho_*)x\, dx + \frac{1}{4}\int\limits_{S^1} (R^4(\omega, t) - R_t^4)\nu(\omega)\, d\omega = 0,$$

where $\nu(\omega) = (\cos\varphi_1 \cos\varphi_2, \sin\varphi_1 \cos\varphi_2, \sin\varphi_2)$; φ_1, φ_2 are spherical coordinates.

The integral $\int_{S^1}(R\sqrt{R^2 + |\nabla R|^2} - R_t^2)\, d\omega$ is estimated by using the above equality and the equality

$$\int\limits_{S^1} (R^3(\omega, t) - R_t^3)\, d\omega = 0,$$

which holds, since $|\Omega_t| = \frac{4}{3}\pi R_t^3$.

In the incompressible case, the formula

$$|S_t| - 4\pi R_0^2 = \int\limits_{S^1} (R\sqrt{R^2 + |\nabla R|^2} - R_0^2)\, d\omega$$

is used together with the conditions

$$\int\limits_{S^1} (R^3(\omega, t) - R_0^3)\, d\omega = 0, \qquad \int\limits_{S^1} (R^4(\omega, t) - R_0^4)\nu(\omega)\, d\omega = 0.$$

The second condition above means that the origin coincides with the barycentre of Ω_t in this case.

Lemma 5.7 yields the estimate of $\|R(\omega, t) - R_t\|^2_{W_2^1(S^1)}$ and therefore it is useful in the next lemma which shows that if the data of the problem are sufficiently small then $\sup_{0\le t\le T} \|R(\omega, t) - R_e\|^2_{W_2^1(S^1)}$ is also small.

Lemma 5.8. *Let the assumptions of Lemma* 5.4 *be satisfied. Let* $e \in C^1(\mathbb{R}_+ \times \mathbb{R}_+)$, $e_\varrho > 0$, $e_\theta = c_v > 0$, $v_0 \in W_2^3(\Omega)$, $\theta_0 \in W_2^3(\Omega)$, $\varrho_0 \in W_2^3(\Omega)$ *and let*

$$\|v_0\|^2_{W_2^3(\Omega)} + \|\theta_{\sigma 0}\|^2_{W_2^3(\Omega)} + \|\varrho_{\sigma 0}\|^2_{W_2^3(\Omega)} \le \overline{\varepsilon}, \tag{5.131}$$

where $\overline{\varepsilon} \in (0, 1)$. *Let assumptions* (5.96), (5.103) *and* (5.106) *hold. Moreover*, *assume that* Ω *is diffeomorphic to a ball and* S *is described by*

$$|\xi| = \widetilde{R}(\omega), \qquad \omega \in S^1, \tag{5.132}$$

where $\widetilde{R}$ *satisfies* (5.107). *Finally*, *assume* (5.108) *with* α_1 *replaced by a constant* $\widehat{\varepsilon} \in (0, 1)$. *Then for* Ω_t $(t \le T)$ *condition* (5.126) *is satisfied and for sufficiently small constants* δ_0, δ_1, $\delta_4, \overline{\delta}$, α_2, $\overline{\varepsilon}$, $\widehat{\varepsilon}$ *we have*

$$\sup_{0\le t\le T} \|R(\cdot, t) - R_e\|^2_{W_2^1(S^1)} \le \alpha_2. \tag{5.133}$$

First, assuming smallness of $\widehat{\varepsilon}$, $\overline{\varepsilon}$, $\overline{\delta}$, δ_3 it is proved that condition (5.129) is satisfied. Then (5.130) and inequality (5.114) yield (5.133) provided constants δ_0, δ_1 and δ_3 are sufficiently small. Therefore α_2 does not depend on $\overline{\varepsilon}$ and $\widehat{\varepsilon}$.

Proof of Lemma 5.8. By assumption (5.132) and by (4.10), condition (5.126) with $R(\omega,t)$ such that $R(\omega,0)=\widetilde{R}(\omega)$ is satisfied for Ω_t $(t\le T)$.

Using assumptions (5.131), (5.96), (5.106), (5.108), estimate (5.112) and the interpolation inequality, for sufficiently small $\overline{\varepsilon}$, $\overline{\delta}$ and $\widehat{\varepsilon}$ we get

$$|R(\omega,t)-\widetilde{R}(\omega)| = \frac{\||x|^2-|\xi|^2|}{|x|+|\xi|} \le c_{11}\Big\|\int_0^t u\,dt'\Big\|_{L_\infty(\Omega)} \le \varepsilon_1\int_0^t \|u\|_{W_2^2(\Omega)}\,dt' + c(\varepsilon_1)\|u\|_{L_2(\Omega^t)}.$$

Similarly

$$|\nabla R(\omega,t)-\nabla\widetilde{R}(\omega)| \le \varepsilon_2\int_0^t \|u\|_{W_2^2(\Omega)}\,dt' + c(\varepsilon_2)\|u\|_{L_2(\Omega^t)}.$$

Hence applying once again the interpolation inequality, for $t\le T$ we obtain

$$\begin{aligned}(5.134)\quad & |R(\omega,t)-R_t|^2+|\nabla R(\omega,t)|^2 \\ &\le c_{12}\bigg[\varepsilon\|\widetilde{R}-R_e\|^2_{W_2^{4+1/2}(S^1)} + c(\varepsilon)\|\widetilde{R}-R_e\|^2_{L_2(S^1)} + \bigg|\Big(\frac{3}{4\pi}|\Omega_t|\Big)^{1/3}-\Big(\frac{3}{4\pi}|\Omega_e|\Big)^{1/3}\bigg|^2 \\ &\quad + \varepsilon_3\int_0^t \|u\|^2_{W_2^2(\Omega)}\,dt' + c(\varepsilon_3)\|u\|^2_{L_2(\Omega^t)}\bigg],\end{aligned}$$

where ε, $\varepsilon_3\in(0,1)$ are constants.

In view of Theorem 7.1, assumptions: (5.131), (5.96), (5.103), (5.106)–(5.108), estimates (5.112), (5.134) and Lemmas 5.3–5.4, for $t\le T$ we get

$$|R(\omega,t)-R_t|^2+|\nabla R(\omega,t)|^2 \le c_{13}(\varepsilon+\varepsilon_3)\widehat{\varepsilon}+c_{14}[\alpha_2+\varepsilon_3(\overline{\delta}+\overline{\varepsilon})]+c_{15}(\delta+\delta_1+\delta_3),$$

where the constants c_i $(i=13,14,15)$ depend on T. Since by (5.101) and (5.113),

$$\Big(\frac{3}{4\pi}\,\frac{M}{\varrho_2}\Big)^{1/3} \le R_t \le \Big(\frac{3}{4\pi}\,\frac{d}{p_0}\Big)^{1/3} \quad \text{for } t\le T,$$

for sufficiently small constants ε, ε_3, α_2, δ, δ_1 and δ_3 we obtain

$$\sup_{S^1}|R(\omega,t)-R_t| + \sup_{S^1}|\nabla R(\omega,t)| \le \widehat{\delta}R_t \quad \text{for } t\le T,$$

where $\widehat{\delta}\in(0,1/2)$ is a constant. Then by Lemma 5.7 estimate (5.130) holds.

Now, we will estimate the terms on the right-hand side of (5.130). By (5.114) we get

$$(5.135)\qquad |S_t|-4\pi R_t^2 \le |S_t|-4\pi R_*^2 \le c_{16}(\delta+\delta_2).$$

Now, consider

$$e(\varrho,\theta)-e(\varrho_*,\theta_*) = e_\varrho(\overline{\varrho},\overline{\theta})(\varrho-\varrho_*)+e_\theta(\overline{\varrho},\overline{\theta})(\theta-\theta_*),$$

where $\overline{\varrho} = \varrho_* + s(\varrho - \varrho_*)$, $\overline{\theta} = \theta_* + s(\theta - \theta_*)$, $\varrho_* = \inf_{t\in[0,T]} \inf_{x\in\overline{\Omega}_t} \varrho(x,t)$, $\theta_* = \inf_{t\in[0,T]} \inf_{x\in\overline{\Omega}_t} \theta(x,t)$, $s \in (0,1)$. Since $e_\varrho > 0$, $e_\theta > 0$ assumption (5.113), the above relation and estimate (5.114) yield

$$\int_{\Omega_t} (\varrho - \varrho_*)\, dx \le c_{17}(\delta + \delta_2), \tag{5.136}$$

where $c_{17} > 0$ is a constant depending on ϱ_1, ϱ_2, θ_1, θ_2 and the form of e.

In view of (5.130), (5.135), (5.136), assumption (5.103) and Lemma 5.3, we get

$$\begin{aligned}\sup_{0\le t\le T} \|R(\cdot,t) - R_e\|^2_{W^1_2(S^1)} &\le 2\Big(\sup_{0\le t\le T} \|R(\cdot,t) - R_t\|^2_{W^1_2(S^1)} + |S^1|\|R_t - R_e|^2\Big)\\ &\le c_{18}(\delta + \delta_1 + \delta_2),\end{aligned}$$

where $c_{18} > 0$ is a constant depending on ϱ_1, ϱ_2, θ_1, θ_2, e, p_0, σ, d, M. Assuming that δ, δ_1 and δ_2 are so small that

$$c_{18}(\delta + \delta_1 + \delta_2) \le \alpha_2$$

we obtain (5.133).

This completes the proof of the lemma. ■

Let us notice that in the case of ideal gas, i.e., when $e = c_v\theta$, $p = R\varrho\theta$ (where $c_v > 0$ is a constant and $R > 0$ is a constant) we have $e_\varrho = 0$, so we cannot apply Lemma 5.8. However, replacing assumption (5.104) by a stronger one we obtain the following lemma.

LEMMA 5.9. *Let $p = a\varrho e$ (where $a > 0$ is a constant). Let the assumptions of Lemma* 5.4 *be satisfied apart from* (5.104). *Let $e \in C^1(\mathbb{R}_+ \times \mathbb{R}_+)$, $e_\theta = c_v > 0$, $p_\varrho > 0$ and let assumptions* (5.96), (5.106), (5.103), (5.131) *be fulfilled. Moreover, assume that Ω and S satisfy the same conditions as in Lemma* 5.8 (*with* (5.107) *and* (5.108)). *Finally, assume*

$$\int_{\Omega} \varrho_0 \frac{v_0^2}{2}\, d\xi + \int_{\Omega} (\varrho_0 e(\varrho_0,\theta_0) - \varrho_1 e_1)\, d\xi + +\sigma[|S| - 4\pi R_0^2] + \varkappa \int_0^\infty dt \int_{\mathbb{R}^3} \overline{\theta}(s,t)\, dt \le \delta_2. \tag{5.137}$$

Then the assertions of Lemmas 5.4 *and* 5.8 *hold.*

Proof. Set $p_1 = a\varrho_1 e_1$. By using Lemma 5.3 we get in the same way as in Lemma 5.4 the following estimate:

$$\begin{aligned}&\int_{\Omega_t} \varrho \frac{v^2}{2}\, dx + \frac{1}{a}\int_{\Omega_t} (p(\varrho,\theta) - p_1)\, dx + p_0(|\Omega_t| - \inf_t |\Omega_t|) + \sigma(|S_t| - 4\pi R_*^2)\\ &\quad\le \int_{\Omega} \varrho_0 \frac{v_0^2}{2}\, d\xi + \frac{1}{a}\int_{\Omega} (p(\varrho_0,\theta_0) - p_1)\, d\xi + \frac{p_1}{a}\|\Omega| - |\Omega_t\|\\ &\qquad + p_0(|\Omega| - \inf_t |\Omega_t|) + \sigma(|S| - 4\pi R_0^2) + \int_0^\infty dt \int_{\mathbb{R}^3} \overline{\theta}(s,t')\, dt' \le c(\delta + \delta_2).\end{aligned} \tag{5.138}$$

Since

$$0 \le p(\varrho,\theta) - p(\varrho_*,\theta_*) = p_\varrho(\overline{\varrho},\overline{\theta})(\varrho - \varrho_*) + p_\theta(\overline{\varrho},\overline{\theta})(\theta - \theta_*) < p(\varrho,\theta) - p_1,$$

estimates (5.138) and (5.130) yield the assertion of the lemma. ■

Lemma 5.9 is used instead of Lemma 5.10 in the proof of Theorem 5.12 (see below).

PART 4. The next estimate shows an increase of regularity of the local solution. Such an estimate is used to control the regularity of the free boundary S_t.

LEMMA 5.10. *Let* $(u,\vartheta,\eta)\in W_2^{4,2}(\Omega^T)\times W_2^{4,2}(\Omega^T)\times C([0,T];W_2^3(\Omega))$ *with* $\eta_t\in C([0,T];$ $W_2^2(\Omega))\cap L_2(0,T;W_2^3(\Omega))$ *be the local solution of problem* (5.70)–(5.75). *Then for any* $0<t_1<T$,

$$\sup_{t_1\le t\le T}\|u\|^2_{W_2^4(\Omega)}+\sup_{t_1\le t\le T}\|\vartheta_\sigma\|^2_{W_2^4(\Omega)} \le c(K)(K_1+\|r\|^2_{C_B^3(\mathbb{R}^3\times\mathbb{R}_+)}+\|\overline{\theta}\|^2_{C_B^4(\mathbb{R}^3\times\mathbb{R}_+)}), \tag{5.139}$$

where $K_1=\|u\|^2_{W_2^{4,2}(\Omega^T)}+\|\vartheta_\sigma\|^2_{W_2^{4,2}(\Omega^T)}$,

$$K=K_1+\sup_{0\le t\le T}\|u\|^2_{W_2^3(\Omega)}+\sup_{0\le t\le T}\|\vartheta_\sigma\|^2_{W_2^3(\Omega)} +\sup_{0\le t\le T}\|\eta_\sigma\|^2_{W_2^3(\Omega)}+\|\eta_{\sigma t}\|^2_{L_2(0,T;W_2^1(\Omega))}+\sup_{0\le t\le T}\|\eta_{\sigma t}\|^2_{W_2^1(\Omega)},$$

$c(K)$ *is a positive nondecreasing continuous function of* K *depending also on* t_1.

Inequality (5.139) is similar to the inequality from [ZZaj12] which was proved for the local solution such that $(u,\vartheta,\eta)\in W_2^{2+\alpha,1+\alpha/2}(\Omega^T)\times W_2^{2+\alpha,1+\alpha/2}(\Omega^T)\times W_2^{1+\alpha,1/2+\alpha/2}(\Omega^T)$ $\cap\, C([0,T];W_2^{1+\alpha}(\Omega))$ $(\alpha\in(3/4,1))$ in the case of $r=\overline{\theta}=0$. The general idea of the proof of inequality (5.139) comes from the paper of Solonnikov [Sol6], where the analogous estimate is obtained for the velocity and pressure of an incompressible fluid.

REMARK 5.6. Choose the constant C such that $c(0)<C$. Then by estimate (5.112) and assumption (5.105) for sufficiently small α_1 and $\overline{\delta}$ we have $c(K)<C$.

PART 5. In the case of an incompressible fluid to obtain global existence it is sufficient to use estimates derived for the linear problem and estimates similar to those of Parts 1–4 (see Sections 4 and 7). However, the nature of the equations describing the motion of compressible fluids is such that we need another two estimates for the local solution of problem (5.63)–(5.69).

First, we need the following energy type inequality which is derived in [ZZaj12]:

$$\begin{aligned}&\frac12\frac{d}{dt}\int_{\Omega_t}\left(\varrho v^2+\frac{p_1}{\varrho}\varrho_\sigma^2+\frac{\varrho c_v p_2}{\theta p_\theta}\theta_\sigma^2\right)dx\\&\quad+\frac{\sigma}{2}\frac{d}{dt}\int_{S_t}g^{\gamma\delta}\int_0^t v_{s_\gamma}\,dt'\cdot\overline{n}\int_0^t v_{s_\delta}\,dt'ds\cdot\overline{n}+c_{19}(\|v\|^2_{W_2^1(\Omega_t)}+\|\theta_{\sigma x}\|^2_{L_2(\Omega_t)})\\&\le\varepsilon(\|\varrho_\sigma\|^2_{L_2(\Omega_t)}+\|\theta_\sigma\|^2_{L_2(\Omega_t)})+\overline{\varepsilon}_1\left(\|v\|^2_{W_2^2(\Omega_t)}+\left\|\int_0^t v\,dt'\right\|^2_{L_2(\Omega_t)}+\|H(\cdot,0)+2/R_e\|^2_{L_2(S)}\right)\\&\quad+c_{20}(\|v\|^2_{L_2(\Omega_t)}+\|r\|^2_{L_2(\Omega_t)}+\|\overline{\theta}\|^2_{L_2(S_t)})\\&\quad+c_{21}[(\|\theta_\sigma\|^2_{W_2^{1+\alpha}(\Omega_t)}+\|\varrho_\sigma\|^2_{W_2^{1+\alpha}(\Omega_t)})(\|\theta_{\sigma t}\|^2_{L_2(\Omega_t)}+\|\varrho_{\sigma t}\|^2_{L_2(\Omega_t)})\\&\quad+(\|\varrho_\sigma\|^2_{W_2^1(\Omega_t)}+\|\theta_\sigma\|^2_{W_2^1(\Omega_t)})\|\theta_\sigma\|^2_{W_2^{1+\alpha}(\Omega_t)}+\|v\|^2_{W_2^1(\Omega_t)}(\|\varrho_\sigma\|^2_{W_2^2(\Omega_t)}+\|\theta_\sigma\|^2_{W_2^2(\Omega_t)})],\end{aligned}\tag{5.140}$$

where $\alpha \in (1/2,1)$, ε and $\overline{\varepsilon}_1$ are sufficiently small constants; c_{19}, c_{20} and c_{21} are positive constants depending on ϱ_1, ϱ_2, θ_1, θ_2; $p_1(\varrho,\theta) = \int_0^1 p_\varrho(\varrho_e + s(\varrho-\varrho_e),\theta)ds$, $p_2(\theta) = \int_0^1 p_\theta(\varrho_e,\theta_e + s(\theta-\theta_e))\,ds$.

We also need a differential inequality which gives the possibility of estimating the highest norms of the solution by the following terms: the nonlinear terms consisting of products of the highest norms of v, θ_σ, ϱ_σ and the linear terms, i.e. the L_2-norms of these functions and the norms of r, $\overline{\theta}$, $H(\cdot,0)+2/R_e$, $R(\cdot,t)-R(\cdot,0)$.

To obtain this inequality we derive first some auxiliary estimates by using both Eulerian and Lagrangian coordinates. Some auxiliary estimates are derived in Eulerian coordinates. These are energy-type estimates for $(v,\theta_\sigma,\varrho_\sigma)$ (where (v,θ,ϱ) is the solution of problem (5.63)–(5.69)) and its time derivatives up to order three. The remaining auxiliary estimates are derived by using Lagrangian coordinates and considering problem (5.70)–(5.75) locally.

Therefore, we introduce a family $\{\widetilde{\Omega}_i\}$ of open sets such that $\overline{\Omega} \subset \bigcup_{i\in\mathcal{M}\cup\mathcal{N}} \widetilde{\Omega}_i$, where $\widetilde{\Omega}_i$ for $i\in\mathcal{M}$ are interior subdomains and $\widetilde{\Omega}_i \cap \Omega$ for $i\in\mathcal{N}$ are boundary subdomains, i.e. $\overline{\widetilde{\Omega}_i} \subset \Omega$ for $i\in\mathcal{M}$ and $\widetilde{\Omega}_i \cap S \neq \emptyset$ for $i\in\mathcal{N}$. With this covering of Ω we associate a partition of unity $\{\zeta_i\}$ such that $\sum_{i\in\mathcal{M}\cup\mathcal{N}} \zeta_i(\xi) = 1$ for $\xi\in\Omega$, $0\le\zeta_i\le 1$, $\zeta_i\in C_0^\infty(\widetilde{\Omega}_i)$.

Assume that $\widetilde{\Omega}_i\cap\Omega$ is a boundary subdomain and $\zeta_i(\xi)=1$ for $\xi\in\omega_i$, where ω_i is a set such that $\overline{\omega}_i \subset \widetilde{\Omega}_i$. Let $\beta\in\overline{\omega}_i\cap S\subset\widetilde{\Omega}_i\cap S\equiv\widetilde{S}_i$. Introduce local coordinates connected with ξ by

$$y_k = \alpha_{kl}(\xi_l-\beta_l),\qquad \alpha_{3k}=n_k(\beta),\qquad k=1,2,3, \tag{5.141}$$

where $\{\alpha_{kl}\}$ is a constant orthogonal matrix such that $\widetilde{S}_i$ is determined by

$$y_3 = F(y_1,y_2),\qquad |y_j|<d,\ j=1,2$$

with $F(0)=0$, $\nabla F(0)=0$, $F\in W_2^{4+1/2}$. We assume that $\operatorname{diam}(\widetilde{\Omega}_i\cap\Omega)<2d$, where d is sufficiently small. Next, we introduce functions u',ϑ' and η' by

$$u_k'(y)=\alpha_{kl}u_l(\xi)|_{\xi=\xi(y)},\quad k=1,2,3;\quad \vartheta'(y)=\vartheta(\xi)|_{\xi=\xi(y)},\quad \eta'(y)=\eta(\xi)|_{\xi=\xi(y)},$$

where $\xi=\xi(y)$ is the inverse transformation to (5.141).

Further, we want to straighten the boundary, so we define new coordinates by

$$z_j=y_j,\quad j=1,2,\quad z_3=y_3-\widetilde{F}(y),$$

which will be denoted by $z=\Phi(y)$, where $\widetilde{F}$ is an extension of F to $\mathbb{R}^3_+$ such that $\|\widetilde{F}\|_{W_2^5(\mathbb{R}^3_+)}\le c\|F\|_{W_2^{4+1/2}(U)}$ and $U=\{z'=(z_1,z_2)\in\mathbb{R}^2 : |z_j|<d,\ j=1,2\}$. Let

$$\widehat{\Omega}_i\equiv\Phi(\widetilde{\Omega}_i\cap\Omega),\qquad \widehat{S}_i\equiv\Phi(\widetilde{S}_i).$$

Then $|z_j|<d$, $j=1,2$, for $z\in\widehat{\Omega}_i$ and $\operatorname{diam}\widehat{\Omega}_i<cd$. Define

$$\widehat{f}(z)=f'(y)|_{y=\Phi^{-1}(z)},\qquad f\in\{u,\vartheta,\eta\}.$$

Set $\widehat{\nabla}_k=\xi_{lx_k}z_{i\xi_l}\nabla_{z_i}|_{\xi=\chi^{-1}(z)}$, where $\chi(\xi)=\Phi(\psi(y))$ and $y=\psi(\xi)$ is defined by (5.141) and write

$$\widetilde{u}^{(i)}(\xi)=u(\xi)\zeta_i(\xi),\quad \widetilde{\vartheta}^{(i)}_\sigma(\xi)=\vartheta_\sigma(\xi)\zeta_i(\xi),\quad \widetilde{\eta}^{(i)}_\sigma(\xi)=\eta_\sigma(\xi)\zeta_i(\xi),\quad \xi\in\widetilde{\Omega}_i,\ i\in\mathcal{M};$$

$$\widetilde{u}^{(i)}(z)=\widehat{u}(z)\widehat{\zeta}_i(z),\quad \widetilde{\vartheta}_\sigma^{(i)}(z)=\widehat{\vartheta}_\sigma(z)\widehat{\zeta}_i(z),\quad \widetilde{\eta}_\sigma^{(i)}(z)=\widehat{\eta}_\sigma(z)\widehat{\zeta}(z),\quad z\in\widehat{\Omega}_i,\ i\in\mathcal{N},$$

where $\widehat{\zeta}(z)=\zeta(\xi)|_{\xi=\chi^{-1}(z)}$.

The functions $\widetilde{u}^{(i)}(\xi)$, $\widetilde{\vartheta}_\sigma^{(i)}(\xi)$, $\widetilde{\eta}_\sigma^{(i)}(\xi)$ for $i\in\mathcal{M}$ satisfy in $\widetilde{\Omega}_i$ a system of equations implied by $(5.111)_1$–$(5.111)_3$, while $\widetilde{u}^{(i)}(z)$, $\widetilde{\vartheta}_\sigma^{(i)}(z)$, $\widetilde{\eta}_\sigma^{(i)}(z)$ for $i\in\mathcal{N}$ satisfy in $\widehat{\Omega}_i$ a system implied by $(5.111)_1$–$(5.111)_3$ together with boundary conditions $(5.111)_4$–$(5.111)_5$. Let $\widetilde{\Omega}$ be one of the $\widetilde{\Omega}_i$'s and let ζ be one of ζ_i's. Moreover, let $\widetilde{u}=(\widetilde{u}_1,\widetilde{u}_2,\widetilde{u}_3)$, $\widetilde{\vartheta}_\sigma$, $\widetilde{\eta}_\sigma$ be the solution of the problem in $\widetilde{\Omega}$. Then the problem in the interior domain $\widetilde{\Omega}$ has the following form:

$$\begin{aligned}
&\eta\widetilde{u}_{kt}-\nabla_{u_l}T_{ukl}(\widetilde{u},\widetilde{p}_\sigma)=-\nabla_{u_l}B_{ukl}(u,\zeta)-T_{ukl}(u,p_\sigma)\nabla_{u_l}\zeta,\qquad k=1,2,3,\\
&\widetilde{\eta}_{\sigma t}+\varrho_e\nabla_u\cdot\widetilde{u}=\varrho_e u\cdot\nabla_u\zeta-\eta_\sigma\nabla_u\cdot u\zeta,\\
&\eta c_v(\eta,\vartheta)\widetilde{\vartheta}_{\sigma t}-\varkappa\nabla_u^2\widetilde{\vartheta}_\sigma+\theta_e p_\vartheta(\varrho_e,\theta_e)\nabla_u\cdot\widetilde{u}\\
&\quad=\eta\widetilde{h}+\bigg[\frac{\mu}{2}\sum_{k,l=1}^3(\xi_{mx_k}\partial_{\xi_m}u_l+\xi_{mx_l}\partial_{\xi_m}u_k)^2+(\nu-\mu)(\nabla_u\cdot u)^2\bigg]\zeta\\
&\qquad+\theta_e p_\vartheta(\varrho_e,\theta_e)u\cdot\nabla_u\zeta+(\theta_e p_\vartheta(\varrho_e,\theta_e)-\vartheta p_\vartheta(\eta,\vartheta))\nabla_u\cdot u\zeta\\
&\qquad-\varkappa(\nabla_u^2\zeta\vartheta_\sigma+2\nabla_u\zeta\cdot\nabla_u\vartheta_\sigma),
\end{aligned}\tag{5.142}$$

where $\nabla_l=\partial_{\xi_l}$, $\nabla_{u_l}=\xi_{mx_l}\partial_{\xi_m}$, $\widetilde{p}_\sigma=p_\sigma\zeta$, $\widetilde{h}=h\zeta$, $\mathbb{B}_u(u,\zeta)=\{B_{ukl}(u,\zeta)\}_{k,l=1,2,3}=\{\mu(u_k\nabla_{u_l}\zeta+u_l\nabla_{u_k}\zeta)+(\nu-\mu)\delta_{kl}u\cdot\nabla_u\zeta\}_{k,l=1,2,3}$.

In a boundary subdomain the problem is as follows:

$$\begin{aligned}
&\widehat{\eta}\widetilde{u}_{kt}-\widehat{\nabla}_l\widehat{T}_{lk}(\widetilde{u},\widetilde{p}_\sigma)=-\widehat{\nabla}_l\widehat{B}_{kl}(\widehat{u},\widehat{\zeta})-\widehat{T}_{kl}(\widehat{u},\widehat{p}_\sigma)\widehat{\nabla}_l\widehat{\zeta},\qquad k=1,2,3,\\
&\widetilde{\eta}_{\sigma t}+\varrho_e\widehat{\nabla}\cdot\widetilde{u}=\varrho_e\widehat{u}\cdot\widehat{\nabla}\widehat{\zeta}-\widehat{\eta}_\sigma\widehat{\nabla}\cdot\widehat{u}\widehat{\zeta},\\
&\widehat{\eta}c_v(\widehat{\eta},\widehat{\vartheta})\widetilde{\vartheta}_{\sigma t}-\varkappa\widehat{\nabla}^2\widetilde{\vartheta}_\sigma+\theta_e p_{\widehat{\vartheta}}(\varrho_e,\theta_e)\nabla\cdot\widetilde{u}\\
&\quad=\widehat{\eta}\widetilde{h}+\bigg[\frac{\mu}{2}\sum_{k,l=1}^3(\widehat{\nabla}_k\widehat{u}_l+\widehat{\nabla}_l\widehat{u}_k)^2+(\nu-\mu)(\widehat{\nabla}\cdot\widehat{u})^2\bigg]\widehat{\zeta}\\
&\qquad+\theta_e p_{\widehat{\vartheta}}(\varrho_e,\theta_e)\widehat{u}\cdot\widehat{\nabla}\widehat{\zeta}+(\theta_e p_{\widehat{\vartheta}}(\varrho_e,\theta_e)-\widehat{\vartheta}p_{\widehat{\vartheta}}(\widehat{\eta},\widehat{\vartheta}))\widehat{\nabla}\cdot\widehat{u}\widehat{\zeta}\\
&\qquad+\theta_e p_{\widehat{\vartheta}}(\varrho_e,\theta_e)(\nabla\cdot\widetilde{u}-\widehat{\nabla}\cdot\widetilde{u})-\varkappa(\widehat{\nabla}^2\widehat{\zeta}\widehat{\vartheta}_\sigma+2\widehat{\nabla}\widehat{\zeta}\cdot\widehat{\nabla}\widehat{\vartheta}_\sigma),\\
&\widehat{\mathbb{T}}(\widetilde{u},\widetilde{p}_\sigma)\widehat{n}=\sigma\Delta_{\widehat{S}}\widehat{\xi}\widehat{\zeta}\cdot\widehat{n}\widehat{n}+\sigma\Delta_{\widehat{S}}\int_0^t\widetilde{u}\,dt'\cdot\widehat{n}\widehat{n}+\widehat{\mathbb{B}}(\widehat{u},\widehat{\zeta})\widehat{n}\\
&\qquad-\sigma\Big(2\widehat{\nabla}\int_0^t\widehat{u}\,dt'\widehat{\nabla}\widehat{\zeta}+\int_0^t\widehat{u}\,dt'\widehat{\nabla}^2\widehat{\zeta}\Big)\cdot\widehat{n}\widehat{n}+\frac{2\sigma}{R_e}\widehat{n}\widehat{\zeta},\\
&\widehat{n}\cdot\nabla\widetilde{\vartheta}_\sigma=\widetilde{\overline{\vartheta}}+\widehat{n}\cdot\widehat{\nabla}\widehat{\zeta}\widehat{\vartheta}_\sigma,
\end{aligned}\tag{5.143}$$

where $\nabla_l=\partial_{z_l}$, $\widehat{\nabla}=(\widehat{\nabla}_k)_{k=1,2,3}$, $\widehat{\mathbb{B}}(\widehat{u},\widehat{\zeta})=\{\widehat{B}_{kl}(\widehat{u},\widehat{\zeta})\}_{k,l=1,2,3}=\{\mu(\widehat{u}_k\widehat{\nabla}_l\widehat{\zeta}+\widehat{u}_l\widehat{\nabla}_k\widehat{\zeta})+(\nu-\mu)\delta_{kl}\widehat{u}\cdot\widehat{\nabla}\widehat{\zeta}\}_{k,l=1,2,3}$, $\widehat{\mathbb{T}}(\widetilde{u},\widetilde{p}_\sigma)=\{\widehat{T}_{kl}(\widetilde{u},\widetilde{p}_\sigma)\}_{k,l=1,2,3}=\{\mu(\widehat{\nabla}_k\widehat{u}_l+\widehat{\nabla}_l\widehat{u}_k)+(\nu-\mu)\delta_{kl}\widehat{\nabla}\cdot\widehat{u}-\widetilde{p}_\sigma\delta_{kl}\}_{k,l=1,2,3}$, $\widehat{n}=\widehat{n}(z,t)$ is the vector $\overline{n}_u=\overline{n}(X_u(\xi,t),t)$ written in z coordinates. Notice that using the interpolation inequalities we obtain the estimates

$$\|F\|_{W_2^4(\widehat{S}_i)}\le\varepsilon\|F\|_{W_2^{4+1/2}(\widehat{S}_i)}+c(\varepsilon)\|F\|_{L_2(\widehat{S}_i)}\le c(\varepsilon+d)\|F\|_{W_2^{4+1/2}(\widehat{S}_i)}$$

and

$$\|\widehat{n}_{0z'}\|_{W_2^2(\widehat{S}_i)} \le \varepsilon\|F\|_{W_2^{4+1/2}(\widehat{S}_i)} + c(\varepsilon)\|\widehat{n}_{0z'}\|_{L_2(\widehat{S}_i)} \le c(\varepsilon+d)\|F\|_{W_2^{5+1/2}(\widehat{S}_i)},$$

where $\widehat{n}_0 = \widehat{n}(z',0)$, $z'=(z_1,z_2)$ and the right-hand sides of the above estimates are as small as we need if ε and d are sufficiently small.

Moreover

$$\Delta_{\widehat{S}}(t) = \frac{1}{\sqrt{g_{\widehat{u}}}}\frac{\partial}{\partial z_\gamma}\left(\sqrt{g_{\widehat{u}}}\, g_{\widehat{u}}^{\gamma\delta}\frac{\partial}{\partial z_\delta}\right) \qquad (\gamma,\delta=1,2),$$

where $\{g_{\widehat{u}}^{\gamma\delta}\}$ is the inverse matrix to $\{(g_{\widehat{u}})_{\gamma\delta}\}$; $(g_{\widehat{u}})_{\gamma\delta} = \frac{\partial x}{\partial z_\gamma}\cdot\frac{\partial x}{\partial z_\delta}$, $x=\widehat{\xi}+\int_0^T \widehat{u}(z,t')\,dt'$, $g_{\widehat{u}} = \det\{(g_{\widehat{u}})_{\gamma\delta}\}$.

Now, introduce the function

$$\begin{aligned}
(5.144)\quad \overline{\varphi}^{(2)}(t) = {}& \sum_{j=0}^{3}\int_{\Omega_t}(a_{1j}|\partial_t^j v|^2 + a_{2j}|\partial_t^j\varrho_\sigma|^2 + a_{3j}|\partial_t^j\theta_\sigma|^2)\,dx\\
&+\sum_{i\in\mathcal{M}}\Big[\sum_{1\le|\alpha|\le3}\int_{\widetilde{\Omega}_i}(b_{1\alpha}|D_\xi^\alpha\widetilde{u}^{(i)}|^2 + b_{2\alpha}|D_\xi^\alpha\widetilde{\eta}_\sigma^{(i)}|^2 + b_{3\alpha}|D_\xi^\alpha\widetilde{\vartheta}_\sigma^{(i)}|^2)A\,d\xi\\
&+\sum_{1\le|\alpha|\le3}\int_{\widetilde{\Omega}_i}(b_{4\alpha}|D_\xi^\alpha\partial_t\widetilde{u}^{(i)}|^2 + b_{5\alpha}|D_\xi^\alpha\partial_t\widetilde{\eta}_\sigma^{(i)}|^2 + b_{6\alpha}|D_\xi^\alpha\partial_t\widetilde{\vartheta}_\sigma^{(i)}|^2)A\,d\xi\\
&+\sum_{|\alpha|=1}\int_{\widetilde{\Omega}_i}(b_{7\alpha}|D_\xi^\alpha\partial_t^2\widetilde{u}^{(i)}|^2 + b_{8\alpha}|D_\xi^\alpha\partial_t^2\widetilde{\eta}_\sigma^{(i)}|^2 + b_{9\alpha}|D_\xi^\alpha\partial_t^2\widetilde{\vartheta}_\sigma^{(i)}|^2)A\,d\xi\Big]\\
&+\sum_{i\in\mathcal{N}}\Big[\sum_{1\le|\alpha|\le3}\int_{\widehat{\Omega}_i}(c_{1\alpha}|D_z^\alpha\widetilde{u}^{(i)}|^2 + c_{2\alpha}|D_z^\alpha\widetilde{\eta}_\sigma^{(i)}|^2 + c_{3\alpha}|D_z^\alpha\widetilde{\vartheta}_\sigma^{(i)}|^2)J\,dz\\
&+\sum_{1\le|\alpha|\le2}\int_{\widehat{\Omega}_i}(c_{4\alpha}|D_z^\alpha\partial_t\widetilde{u}^{(i)}|^2 + c_{5\alpha}|D_z^\alpha\partial_t\widetilde{\eta}_\sigma^{(i)}|^2 + c_{6\alpha}|D_z^\alpha\partial_t\widetilde{\vartheta}_\sigma^{(i)}|^2)J\,dz\\
&+\sum_{|\alpha|=1}\int_{\widehat{\Omega}_i}(c_{7\alpha}|D_z^\alpha\partial_t^2\widetilde{u}^{(i)}|^2 + c_{8\alpha}|D_z^\alpha\partial_t^2\widetilde{\eta}_\sigma^{(i)}|^2 + c_{9\alpha}|D_z^\alpha\partial_t^2\widetilde{\vartheta}_\sigma^{(i)}|^2)J\,dz\Big]
\end{aligned}$$

and

$$\begin{aligned}
\overline{\varphi}(t) = {}&\overline{\varphi}^{(2)}(t) + \sum_{i\in\mathcal{N}}\Big[d_1\int_{\widehat{S}_i}\widetilde{\delta}^{\beta\gamma}\sum_{0\le|\alpha|\le2}\widehat{n}\cdot\int_0^t D_{z'}^\alpha\widetilde{u}^{(i)}_{z_\delta z_\beta}\,dt'\,\widehat{n}\cdot\int_0^t D_{z'}^\alpha\widetilde{u}^{(i)}_{z_\delta z_\gamma}\,dt'\,J\,ds\\
&+d_2\int_{\widehat{S}_i}\sum_{0\le|\alpha|\le2}\Big|\widehat{n}\cdot\int_0^t D_{z'}^\alpha\widetilde{u}^{(i)}_{z_1z_2}\,dt'\Big|^2 J\,ds\\
&+d_3\int_{\widehat{S}_i}\sum_{|\alpha|\le2}\sum_{\beta=1}^2\Big[\frac12\widehat{n}\cdot\int_0^t D_{z'}^\alpha\widetilde{u}^{(i)}_{z_\beta z_\beta}\,dt' + 2D_{z'}^\alpha(H(\cdot,0)+2/R_e)\Big]^2 J\,ds\\
&+d_4\int_{\widehat{S}_i} g_{\widehat{u}}^{\beta\gamma}\Big(\sum_{1\le|\alpha|\le2}\widehat{n}\cdot D_{z'}^\alpha\widetilde{u}^{(i)}_{z_\beta}\widehat{n}\cdot D_{z'}^\alpha\widetilde{u}^{(i)}_{z_\gamma}\\
&+\sum_{|\alpha|=1}\overline{n}\cdot D_{z'}^\alpha\widetilde{u}^{(i)}_{tz_\beta}\overline{n}\cdot D_{z'}^\alpha\widetilde{u}^{(i)}_{tz_\gamma}\Big)J\,ds\\
&+d_5\int_{S_t} g^{\beta\gamma}\Big(\Big(\overline{n}\cdot\int_0^t v\,dt'\Big)_{s_\beta}\Big(\overline{n}\cdot\int_0^t v\,dt'\Big)_{s_\beta} + \sum_{j=0}^2\overline{n}\cdot\partial_t^j v_{s_\beta}\overline{n}\cdot\partial_t^j v_{s_\gamma}\Big)\,ds\Big].
\end{aligned}$$

In the above formulas a_{ij} $(i,j=1,\dots,6)$ are positive continuous functions of ϱ and θ depending also on ν, μ, $\varkappa$ and the forms of p and c_v; $b_{i\alpha}$ $(i=1,\dots,6; 1\le|\alpha|\le 3)$, $b_{i\alpha}$ $(i=7,\dots,9;|\alpha|=1)$, $c_{i\alpha}$ $(i=1,\dots,3;1\le|\alpha|\le 3)$, $c_{i\alpha}$ $(i=4,\dots,6;1\le|\alpha|\le 2)$, $c_{i\alpha}$ $(i=7,\dots,9;|\alpha|=1)$ are positive continuous functions of η and ϑ depending also on ν, μ, $\varkappa$ and the forms of p and c_v; d_i $(i=1,\dots,5)$ are positive constants depending on σ; A is the Jacobian of the transformation $x=x(\xi)$; J is the Jacobian of the transformation $x=x(z)$; $\{\widetilde{\delta}^{\beta\gamma}\}$ is a positive definite matrix, i.e. $\widetilde{\delta}^{\beta\gamma}\tau_\beta\tau_\gamma\ge c_0|\tau|^2$, where $c_0>0$ is a constant, $\tau=(\tau_1,\tau_2)\in\mathbb{R}^2$; $\{g^{\beta\gamma}\}$ is the inverse matrix to $\{g_{\beta\gamma}\}$ $\big(g_{\beta\gamma}=\frac{\partial x}{\partial s_\beta}\cdot\frac{\partial x}{\partial s_\gamma}\big)$ provided S_t is determined locally by $x=x(s_1,s_2,t)$, $(s_1,s_2)\in V\subset\mathbb{R}^2$, V is an open set.

Moreover, the summation over repeated indices is assumed.

The exact form of $\overline{\varphi}$ is given in [ZZaj7]. However, for the proof of global existence the above form of this function is sufficient.

The following lemma is proved in [ZZaj7].

Lemma 5.11 (see Theorem 3.13 of [ZZaj7]). *For a sufficiently smooth solution of problem* (5.63)–(5.69) *with* $f=0$, $k=0$, *the following inequality holds for* $t\le T$:

$$
\begin{aligned}
\frac{d\overline{\varphi}}{dt}+c_{22}\Phi &\le c_{23}P(X)X(1+X^3)\Big(X+\Phi+\int_0^t\|v\|^2_{W_2^4(\Omega_{t'})}\,dt'\Big)\\
&\quad+c_{24}\Big(\|R(\cdot.t)-R(\cdot,0)\|^2_{W_2^{4+1/2}(S^1)}\Big\|\int_0^t v\,dt'\Big\|^2_{W_2^3(S_t)}\\
&\quad+\|R(\cdot,t)-R(\cdot,0)\|^2_{W_2^3(S^1)}\Big\|\int_0^t v\,dt'\Big\|^2_{W_2^4(S_t)}+\|H(\cdot,0)+2/R_e\|^4_{W_2^2(S)}\Big)\\
&\quad+c_{25}(\|v\|^2_{L_2(\Omega_t)}+\|\varrho_\sigma\|^2_{L_2(\Omega_t)}+\|\theta_\sigma\|^2_{L_2(\Omega_t)}+\|R(\cdot,t)-R(\cdot,0)\|^2_{L_2(S^1)})\\
&\quad+\overline{\varepsilon}_2c_{26}(\|H(\cdot,0)+2/R_e\|^2_{W_2^2(S)}+\|R(\cdot,t)-R(\cdot,0)\|^2_{W_2^4(S^1)})+c_{27}F,
\end{aligned}
\tag{5.145}
$$

where

$$
F(t)=\|r\|^2_{C_B^3(\mathbb{R}^3\times\mathbb{R}_+)}+\|\overline{\theta}\|^2_{C_B^4(\mathbb{R}^3\times\mathbb{R}_+)},\qquad X(t)=\varphi_0(t)+\int_0^t\|v(t')\|^2_{W_2^3(\Omega_{t'})}\,dt',
$$

Φ *is given by* (5.94), T *is the time of local existence*; $c_{22}>0$ *is a constant depending on* ϱ_1, ϱ_2, θ_1, θ_2, μ, ν, $\varkappa$, T; c_i $(i=23,\dots,27)$ *are positive constants depending on the same quantities as* c_{22} *and on* T, $\int_0^T\|v\|^2_{W_2^3(\Omega_t)}\,dt$, $\|S\|_{W_2^{4+1/2}}$, *the constants from imbedding lemmas and Korn inequalities*; $\overline{\varepsilon}_2>0$ *is a small parameter*; P *is a positive continuous increasing function.*

The proof of Lemma 5.11 is very technical, so it is omitted here. Notice that some terms of the function $\overline{\varphi}(t)$ are expressed in Eulerian coordinates, while the others in Lagrangian ones. This follows from the fact that in order to obtain inequality (5.145) we have to derive some auxiliary differential inequalities. The following terms of $\overline{\varphi}$ are connected with these inequalities. The terms expressed in Eulerian coordinates arise when we derive estimates for $(v,\theta_\sigma,\varrho_\sigma)$ (where (v,θ,ϱ) is the solution of problem (5.63)–(5.69)) and the time derivatives of this solution up to order three.

The terms written in Lagrangian coordinates are associated with the estimates of spatial derivatives and mixed derivatives of the solution. Then we have to consider the problem locally and to derive these estimates by using systems (5.142)–(5.143).

Now, after integrating (5.145) with respect to $t \in (0,T)$ we obtain on the right-hand side terms of greater regularity than the regularity of the local solution guaranteed by Theorem 4.2 of [ZZaj1]. For this reason we prove that we can increase the regularity of the local solution. This result is formulated as follows.

LEMMA 5.12. *Let $S \in W_2^{4+1/2}$ and v_0, θ_0, ϱ_0 be such that $(v, \theta_\sigma, \varrho_\sigma) \in \mathfrak{N}(0)$. Let the assumptions of either Lemma* 5.8 *or Lemma* 5.9 *be satisfied. Moreover, let*

$$\overline{\varphi}(0) \le \overline{\varepsilon},$$

where $\overline{\varepsilon}$ is the constant from (5.131). *Then the local solution of problem* (5.63)–(5.69) (*determined by Theorem* 4.2 *of* [ZZaj1]) *is such that $(v, \theta_\sigma, \varrho_\sigma) \in \mathfrak{M}(t)$ for $t \le T$ and*

$$\begin{aligned}\overline{\phi}(t) + \int_0^t \Phi(t')\,dt' &\le c_{28}(\overline{\varphi}(0) + \|H(\cdot,0) + 2/R_e\|^2_{W_2^{2+1/2}(S)} \\ &\quad + \sup_{0\le t'\le t} \|R(\cdot,t') - R_e\|^2_{L_2(S^1)} + \sup_{0\le t'\le t} F(t')) \\ &\le c_{28}(\overline{\varepsilon} + \widehat{\varepsilon} + \alpha_2 + \overline{\delta}).\end{aligned}$$

To obtain the above estimate we use inequality (5.112). First, by (5.112) and the imbeddings

$$\sup_{0\le t'\le t}(\|u\|^2_{W_2^3(\Omega)} + \|u_t\|^2_{W_2^1(\Omega)}) \le c(\|u\|^2_{W_2^{4,2}(\Omega^t)} + \|v_0\|^2_{W_2^3(\Omega)} + \|u_t(0)\|^2_{W_2^1(\Omega)})$$

we get

$$\|v(t)\|^2_{W_2^3(\Omega_t)} + \|v_t(t)\|^2_{W_2^1(\Omega_t)} \le c(\overline{\varepsilon} + \widehat{\varepsilon} + \overline{\delta}) \quad \text{for } t \le T,$$

where c is independent of t.

By using the continuity equation (5.71) and inequality (5.112) we obtain the estimate for $\|\varrho_{\sigma tt}(t)\|^2_{L_2(\Omega_t)} + \|\varrho_{\sigma t}(t)\|^2_{W_2^2(\Omega_t)} + \|\varrho_\sigma(t)\|^2_{W_2^3(\Omega_t)} + \int_0^t(\|\varrho_{\sigma tt}(t')\|^2_{W_2^1(\Omega_{t'})} + \int_0^t \|\varrho_{\sigma t}(t')\|^2_{W_2^3(\Omega_t)})\,dt'$.

The estimates of the remaining terms are obtained in such a way that we derive for them step by step differential inequalities similar to the auxiliary inequalities leading to (5.145). Then we integrate each of these inequalities with respect to t and use (5.112) or the inequality obtained in the previous step. For example, proceeding this way we first get

$$\begin{aligned}&\|v_{xxt}(t)\|^2_{L_2(\Omega_t)} + \|\varrho_{\sigma xxt}(t)\|^2_{L_2(\Omega_t)} + \|\theta_{\sigma xxt}(t)\|^2_{L_2(\Omega_t)} \\ &\qquad + \int_0^t (\|v_{xxt}(t')\|^2_{W_2^1(\Omega_{t'})} + \|\varrho_{\sigma xxt}(t')\|^2_{L_2(\Omega_{t'})} + \|\theta_{\sigma xxt}(t')\|^2_{W_2^1(\Omega_{t'})})\,dt' \\ &\quad \le c\varepsilon \int_0^t (\|v_{xttt}(t')\|^2_{L_2(\Omega_{t'})} + \|v_{xxt}(t')\|^2_{W_2^1(\Omega_{t'})} + \|v_{xxtt}(t')\|^2_{L_2(\Omega_{t'})} \\ &\qquad + \|\theta_{\sigma xttt}(t')\|^2_{L_2(\Omega_{t'})} + \|\theta_{\sigma xxt}(t')\|^2_{W_2^1(\Omega_{t'})} + \|\theta_{\sigma xxtt}(t')\|^2_{L_2(\Omega_{t'})})\,dt' \\ &\qquad + c(\overline{\varepsilon} + \widehat{\varepsilon} + \alpha_2 + \overline{\delta}),\end{aligned}$$

where $\varepsilon > 0$ is a sufficiently small constant. The remaining inequalities have similar forms.

Part 6. Finally, we need

Lemma 5.13. *Let* $\widetilde{\Omega}$ *be a domain such that* $\overline{\widetilde{\Omega}} \subset \Omega_t = \{x = \xi + \int_0^t u(\xi, t')\, dt' : \xi \in \Omega\}$ *and let* $u \in W_2^{2+\alpha,1+\alpha/2}(\Omega^t)$, $\alpha \in (1/2, 1)$, *be a function satisfying*

$$\Big\| \int_0^t u\, dt' \Big\|_{L_\infty(\Omega)} + \Big\| \int_0^t u_\xi\, dt' \Big\|_{L_\infty(\Omega)} \le \delta.$$

Let $f = f(x,t) \ge 0$ *be a function which is integrable in* Ω_t. *Then for sufficiently small* δ *we have*

$$\int_{\widetilde{\Omega}} f(x,t)\, dx \le (1 + c\delta) \int_{\widetilde{\Omega}} g(\xi, t)\, d\xi + \omega(\delta),$$

where $g(\xi,t) = f(x(\xi,t),t)$, $c > 0$ *is a constant and* ω *is a positive function such that* $\omega(\delta) \to 0$ *as* $\delta \to 0$.

Proof. Since $\overline{\widetilde{\Omega}} \subset \Omega_t$, for sufficiently small δ we have $\overline{\widetilde{\Omega}} \subset \Omega$. Set

$$A_t = \Big\{\xi \in \Omega : x = \xi + \int_0^t u\, dt' \in \widetilde{\Omega}\Big\}.$$

Then

$$\int_{\widetilde{\Omega}} f(x,t)\, dx = \int_{A_t} g(\xi,t) J\, d\xi,$$

where $J = \det\{x_\xi\}$. Hence

$$\int_{\widetilde{\Omega}} f(x,t)\, dx \le (1 + c\delta) \int_{A_\delta} g(\xi,t)\, d\xi,$$

where $c > 0$ is a constant, $A_\delta = \widetilde{\Omega} \cup B_\delta$, and $B_\delta = \{\xi \in \Omega \setminus \widetilde{\Omega} : \operatorname{dist}(\xi, \partial\widetilde{\Omega}) < \delta\}$. Therefore

$$\int_{\widetilde{\Omega}} f(x,t)\, dx \le (1 + c\delta) \int_{\widetilde{\Omega}} g(\xi,t)\, d\xi + (1 + c\delta) \int_{B_\delta} g(\xi,t)\, d\xi.$$

This completes the proof. ■

Remark 5.7. Let $\widetilde{\Omega} \subset \mathbb{R}^3$ be such that $\partial\Omega_t \cap \widetilde{\Omega} \ne \emptyset$ and $\Omega_t \cap \partial\widetilde{\Omega} \ne \emptyset$. Then under the assumptions of Lemma 5.12 we can prove that for sufficiently small δ,

$$\int_{\Omega_t \cap \widetilde{\Omega}} f(x,t)\, dx \le (1 + c\delta) \int_{\Omega \cap \widetilde{\Omega}} g(\xi,t)\, d\xi + \omega(\delta),$$

where $\omega(\delta) \to 0$ as $\delta \to 0$ and $c > 0$ is a constant.

Proof of Theorem 5.11. First, notice that inequality (5.112), the imbeddings $W_2^{4,2}(\Omega^T) \subset C(\overline{\Omega} \times [0,T])$, $W_2^3(\Omega) \subset C(\overline{\Omega})$ and assumptions (5.96), (5.105), (5.108) yield

$$\sup_{\Omega^T} |u|^2 + \sup_{\Omega^T} |\vartheta_\sigma|^2 + \sup_{\Omega^T} |\eta_\sigma|^2 \le c_{29}(\alpha_1 + \overline{\delta}), \tag{5.146}$$

where $c_{29} > 0$ is a constant depending on T and the constant $c_{30} = c(\Omega, T)$ from the inequality $\sup_{\Omega^T} |u|^2 + \sup_{\Omega^T} |\vartheta_\sigma|^2 + \sup_{\Omega^T} |\eta_\sigma|^2 \le c_{30}(\|u\|^2_{W_2^{4,2}(\Omega^T)} + \|\vartheta_\sigma\|^2_{W_2^{4,2}(\Omega^T)} + \sup_{0\le t\le T} \|\eta_\sigma\|^2_{W_2^3(\Omega)})$.

Therefore, assuming that α_1 and $\overline{\delta}$ are so small that

$$[c_{29}(\alpha_1+\overline{\delta})]^{1/2}<l,$$

(where l is the constant from the assumptions of the theorem) we get

$$\varrho_1<\varrho(x,t)<\varrho_2,\quad \theta_1<\theta<\theta_2\quad \text{for } x\in\Omega_t,\ t\in[0,T]. \tag{5.147}$$

By the assumptions of the theorem, the assertion of Lemma 5.8 is satisfied. Moreover, by Theorem 7.1 we have

$$\|R(\cdot,t)-R_e\|_{W_2^{4+1/2}(S^1)}\le c_{31}\|\mathcal{H}(\cdot,t)+2/R_e\|_{W_2^{2+1/2}(S^1)}+c_{32}\|R(\cdot,t)-R_e\|_{L_2(S^1)}. \tag{5.148}$$

Here, $\mathcal{H}$ denotes the double mean curvature of S_t written in the coordinates ω, i.e. $\mathcal{H}(\omega,t)$ $=H[R]=H(x,t)$ (where $H[R]$ is given by (7.28)). By Remark 5.5 the constants c_{31} and c_{32} depend only on R_e.

Using boundary condition (5.66) rewritten in the form

$$\mathbb{T}(v,p_\sigma)\overline{n}=\sigma(H+2/R_e)\overline{n}, \tag{5.149}$$

Lemma 5.10, estimate (5.112) and assumptions (5.96), (5.105), for $t_1\le t\le T$ we have

$$\begin{aligned}\|H(\cdot,t)+2/R_e\|^2_{W_2^{2+1/2}(S_t)}&\le c_{33}\Big(\sup_{t_1\le t\le T}\|u(t)\|^2_{W_2^4(\Omega)}\\&\quad+\sup_{t_1\le t\le T}\|\vartheta_\sigma(t)\|^2_{W_2^3(\Omega)}+\sup_{0\le t\le T}\|\eta_\sigma(t)\|^2_{W_2^3(\Omega)}\Big)\\&\le c_{34}(\alpha_1+\overline{\delta}),\end{aligned} \tag{5.150}$$

where $t_1>0$ is arbitrary and c_{34} is a constant depending on t_1 and T. By (5.148), (5.150) and Lemma 5.8 we deduce that $S_t\in W_2^{4+1/2}$ for $t\le T$ and

$$\sup_{t_1\le t\le T}\|R(\cdot,t)-R_e\|^2_{W_2^{4+1/2}(S^1)}\le c_{31}c_{34}(\alpha_1+\overline{\delta})+c_{32}\alpha_2. \tag{5.151}$$

Therefore, Lemma 5.3 and estimate (5.151) imply that the volume and shape of Ω_t do not change much for $t\le T$. For this reason we can derive differential inequality (5.145) with the constants c_i $(i=22,\dots,27)$ independent of Ω_t for $t\le T$.

Thus, Lemmas 5.11, 5.5, 5.6, 5.12 and estimate (5.159) below yield, for sufficiently small α_1, α_2 and $\overline{\delta}$,

$$\begin{aligned}\overline{\varphi}(t)+\frac{3}{4}c_{22}\int_0^t\Phi(t')\,dt'&\le c_{35}\Big(\int_0^t\|v\|^2_{W_2^1(\Omega_{t'})}\,dt'+\int_0^t\|\theta_{\sigma x}\|^2_{L_2(\Omega_{t'})}\,dt'+\int_0^t F(t')\,dt'\Big)\\&\quad+c_{36}(\varepsilon t\|u\|^2_{W_2^{4,2}(\Omega^T)}+c(\varepsilon,T)t\|u\|^2_{L_2(\Omega^T)})+c_{37}(t[\overline{\varepsilon}_2(\|H(\cdot,0)+2/R_e\|^2_{W_2^2(S)}\\&\quad+\|H(\cdot,0)+2/R_e\|^4_{W_2^2(S)})+(\sup_{0\le t'\le t}\|R(\cdot,t')-R_e\|^2_{L_2(S^1)}\\&\quad+(\sup_{0\le t'\le t}\|R(\cdot,t')-R_e\|^2_{L_2(S^1)})^2)]+\overline{\varphi}(0),\end{aligned} \tag{5.152}$$

where the constants c_i $(i=35,36,37)$ depend on the same quantities as the constants c_i $(i=22,\dots,27)$.

Integrating estimate (5.140) and using Lemma 5.5 and inequality (5.112) we obtain, for sufficiently small ε and $\overline{\varepsilon}_1$,

$$
\begin{aligned}
(5.153)\quad & \varphi^{(3)}(t) + c_{38}\Big(\int_0^t \|v\|^2_{W^1_2(\Omega_{t'})}\,dt' + \int_0^t \|\theta_{\sigma x}\|^2_{L_2(\Omega_{t'})}\,dt'\Big) \\
&\le c_{39}\Big[\varepsilon t\|u\|^2_{W^{4,2}_2(\Omega^T)} + c(\varepsilon,T)t\|u\|^2_{L_2(\Omega^T)} + \int_0^t \|v\|^2_{L_2(\Omega_{t'})}\,dt' \\
&\quad + \varepsilon\int_0^t \|v\|^2_{W^2_2(\Omega_{t'})}\,dt'\Big] + c_{40}\overline{\varepsilon}_1 t\|H(\cdot,0) + 2/R_e\|^2_{L_2(S)} + c_{41}\Big[\int_0^t \|r\|^2_{L_2(\Omega_{t'})}\,dt' \\
&\quad + \int_0^t \|\overline{\theta}\|^2_{L_2(\Omega_{t'})}\,dt' + (\alpha_1 + \overline{\delta})\Big(\int_0^t \|\theta_\sigma\|^2_{W^2_2(\Omega_{t'})}\,dt' + \int_0^t \|\varrho_\sigma\|^2_{W^2_2(\Omega_{t'})}\,dt'\Big)\Big] + \varphi^{(3)}(0),
\end{aligned}
$$

where

$$
\varphi^{(3)}(t) = \frac{1}{2}\int_{\Omega_t}\Big(\varrho v^2 + \frac{p_1}{\varrho}\,\varrho_\sigma^2 + \frac{\varrho c_v p_2}{\theta p_\theta}\,\theta_\sigma^2\Big)\,dx + \frac{\sigma}{2}\int_S g^{\gamma\delta}\Big(\int_0^t v\,dt'\cdot\overline{n}\Big)_{s_\gamma}\Big(\int_0^t v\,dt'\cdot\overline{n}\Big)_{s_\delta}\,ds.
$$

By using Theorem 7.1, an interpolation inequality and assumptions (5.107), (5.108) we have

$$
\begin{aligned}
(5.154)\quad \|H(\cdot,0) + 2/R_e\|^2_{W^2_2(S)} &\le c_{42}\|\widetilde{R} - R_e\|^2_{W^4_2(S^1)} \\
&\le \varepsilon'\|\widetilde{R} - R_e\|^2_{W^{4+1/2}_2(S^1)} + c(\varepsilon')\|\widetilde{R} - R_e\|^2_{L_2(S^1)} \\
&\le \varepsilon\|H(\cdot,0) + 2/R_e\|^2_{W^{2+1/2}_2(S)} + c_{43}(\varepsilon)\|\widetilde{R} - R_e\|^2_{L_2(S^1)} \le \varepsilon\alpha_1 + c_{43}(\varepsilon)\alpha_2,
\end{aligned}
$$

where the constants c_{42} and c_{43} depend on R_e and $\|\widetilde{R}\|_{W^{3+1/2}_2(S^1)}$.

Now, multiplying (5.153) by a sufficiently large constant c_{44} (so large that $c_{44}c_{38} > c_{35}$), then adding to (5.152) and using inequality (5.112), (5.154), assumptions (5.96), (5.107), (5.108) and Lemmas 5.4 and 5.8, we get the following estimate for sufficiently small α_1, $\overline{\delta}$ and for $t \le T$:

$$
(5.155)\quad \overline{\psi}(t) + \frac{c_{22}}{2}\int_0^t \Phi(t')\,dt' \le t[c_{45}\varepsilon\alpha_1 + c_{46}(\overline{\delta} + \delta_3 + \alpha_2)] + \overline{\psi}(0) \equiv t\overline{\gamma} + \overline{\psi}(0),
$$

where

$$
(5.156)\quad \overline{\psi}(t) = \overline{\varphi}(t) + c_{44}\varphi^{(3)}(t),
$$

$\varepsilon \in (0,1)$ can be chosen sufficiently small; $c_i > 0$ ($i = 45, 46$) depend on the same quantities as c_{35} and c_{46} depends also on ε.

The form of the function $\overline{\varphi}^{(2)}(0)$ implies

$$
(5.157)\quad c_{47}\varphi_0(0) \le \overline{\varphi}^{(2)}(0) \le c_{48}\varphi_0(0),
$$

where $\overline{\varphi}^{(2)}$ is given by (5.144), c_{47}, $c_{48} > 0$ are the constants depending on ϱ_1, ϱ_2, θ_1, θ_2, p, c_v, μ, ν, σ. Hence, by (5.147) we also have

$$
(5.158)\quad c_{47}\varphi_0(t) \le \overline{\varphi}^{(2)}(t) \le c_{48}\varphi_0(t) \quad \text{for } t \le T.
$$

Moreover, assumption (5.105) and (5.154) imply

$$\overline{\varphi}(0) \le \overline{\psi}(0) \le c_{49}[\alpha_1 + \varepsilon\alpha_1 + c_{43}(\varepsilon)\alpha_2] \le c_{50}\alpha_1 \equiv \alpha_0, \tag{5.159}$$

if we choose $c_{50} > c_{49}$ and ε, α_2 are sufficiently small.

Now, choose $c_{51} > c_{50}$. Under the assumption that constants $\overline{\delta}$, δ_3, α_2 and ε are so small that

$$T\overline{\gamma} + \alpha_0 \le c_{51}\alpha_1, \tag{5.160}$$

estimate (5.155) yields

$$\overline{\psi}(t) + \frac{c_{22}}{2}\int_0^t \Phi(t')\,dt' \le c_{51}\alpha_1 \quad \text{for } t \le T. \tag{5.161}$$

Therefore, by (5.158),

$$\varphi_0(t) \le \frac{c_{51}}{c_{47}}\alpha_1 \quad \text{for } t \le T. \tag{5.162}$$

Now, we estimate once more $\|H(\cdot,t)+2/R_e\|^2_{W_2^{2+1/2}(S_t)}$ for $t_1 \le t \le T$ using now (5.161). We obtain

$$\|H(\cdot,t) + 2/R_e\|^2_{W_2^{\alpha+1/2}(S_t)} \le c_{52}\alpha_1 \quad \text{for } t_1 \le t \le T, \tag{5.163}$$

where $t_1 > 0$ is the same as in (5.150) and c_{52} is a constant depending only on the constants c_{33}, c_{22}, c_{51} and C from Remark 5.6.

In view of estimates (5.162) and (5.163) if we assume that α_1 is sufficiently small, then the solution can be extended to the interval $[T, 2T]$.

Hence, for the local solution in $[T, 2T]$ estimate (5.112) holds. Namely, for sufficiently small α_1 we obtain, for $T \le t \le 2T$,

$$\begin{aligned} &\|u\|^2_{W_2^{4,2}(\Omega_T\times(T,t))} + \|\vartheta_\sigma\|^2_{W_2^{4,2}(\Omega_t\times(T,t)} + \|\eta_\sigma\|^2_{C([T,t];W_2^3(\Omega_T))} \\ &\quad + \|\eta_{\sigma t}\|^2_{L_2(T,t;W_2^3(\Omega_T))} + \|\eta_{\sigma t}\|^2_{C([T,t];W_2^2(\Omega_T))} + \|\eta_{\sigma tt}\|^2_{L_2(T,t;W_2^1(\Omega_T))} \\ &\le \varphi^{(1)}(T)\Big[\|u(T)\|^2_{W_2^3(\Omega_T)} + \|\eta_\sigma(T)\|^2_{W_2^3(\Omega_T)} + \|\vartheta_\sigma(T)\|^2_{W_2^3(\Omega_T)} \\ &\quad + \|u_t(T)\|^2_{W_2^1(\Omega_T)} + \|\vartheta_{\sigma t}(T)\|^2_{W_2^1(\Omega_T)} + \|h\|^2_{W_2^{2,1}(\Omega_T\times(T,2T))} \\ &\quad + \|\overline{\vartheta}\|^2_{W_2^{3-1/2,3/2-1/4}(S_T\times(T,2T))} + \bigg(\int_T^{2T} \frac{\|\overline{\vartheta}\|^2_{L_2(\Omega_T)}}{t^{1/2}}\,dt\bigg)^{1/2} \\ &\quad + \|H(\cdot,T) + 2/R_e\|^2_{W_2^{2+1/2}(S_T)}\Big], \end{aligned} \tag{5.164}$$

where $\varphi^{(1)}$ is the same function as in (5.112) and u, η_σ, ϑ_σ, h, $\overline{\vartheta}$ are now v, ϱ_σ, θ_σ, r, $\overline{\theta}$ written in Lagrangian coordinates $\xi_T \in \Omega_T$, $\xi_T = \xi + \int_0^T u(\xi,t)\,dt$, $\xi \in \Omega$.

Now, inequality (5.164), the imbeddings $W_2^{4,2}(\Omega_T \times (T,2T)) \subset C(\overline{\Omega}_T \times [T,2T])$ and $W_2^3(\Omega_T) \subset C(\overline{\Omega}_T)$ together with Lemma 5.3 and (5.151), assumption (5.96) and estimates (5.162)–(5.163) yield

$$\sup_{\Omega_T\times[T,2T]}|u|^2+\sup_{\Omega_T\times[T,2T]}|\vartheta_\sigma|^2+\sup_{\Omega_T\times[T,2T]}|\eta_\sigma|^2\le c_{29}[(c_{51}/c_{47}+c_{52})\alpha_1+\overline{\delta}],$$

where c_{29}, c_{47}, c_{51}, c_{52} are the constants from (5.146) and (5.162)–(5.163), respectively. Therefore, assuming that α_1 and $\overline{\delta}$ are so small that

$$\{c_{29}[(c_{51}/c_{47}+c_{52})\alpha_1+\overline{\delta}]\}^{1/2}<l$$

we obtain

$$\varrho_1<\varrho(x,t)<\varrho_2,\quad \theta_1<\theta(x,t)<\theta_2\quad \text{for } x\in\Omega_t,\ t\in[T,2T]. \tag{5.165}$$

In view of (5.165), (5.162) and the assumptions of the theorem, Lemmas 5.4 and 5.8 give respectively

$$\|v\|^2_{L_2(\Omega_t)}\le\delta_3\quad \text{for } t\le 2T \tag{5.166}$$

and

$$\sup_{0\le t\le 2T}\|R(\cdot,t)-R_e\|^2_{W^1_2(S^1)}\le\alpha_2. \tag{5.167}$$

Using, as before, Theorem 7.1, the interpolation inequality, (5.163) and (5.167) we get

$$\begin{aligned}\|H(\cdot,T)+2/R_e\|^2_{W^2_2(S_T)}&\le c_{42}\|R(\cdot,T)-R_e\|^2_{W^4_2(S^1)}\\&\le\varepsilon'\|R(\cdot,T)-R_e\|^2_{W_2^{4+1/2}(S^1)}+c(\varepsilon')\|R(\cdot,T)-R_e\|^2_{L_2(S^1)}\\&\le\varepsilon\|H(\cdot,T)+2/R_e\|^2_{W_2^{2+1/2}(S)}+c_{43}(\varepsilon)\|R(\cdot,T)-R_e\|^2_{L_2(S^1)}\\&\le\varepsilon c_{52}\alpha_1+c_{43}(\varepsilon)\alpha_2,\end{aligned} \tag{5.168}$$

where the constants c_{42} and c_{43} are the same as in (5.154) and by Remark 5.5 they depend only on R_e and $\|\widetilde{R}\|_{W_2^{3+1/2}(S^1)}$.

Moreover, by using (5.148), boundary condition (5.149), Lemma 5.10 and estimates (5.164), (5.162), (5.163), (5.167) we get

$$\sup_{t_1+T\le t\le 2T}\|R(\cdot,t)-R_e\|^2_{W_2^{4+1/2}(S^1)}\le c_{31}c_{34}[(c_{51}/c_{47}+c_{52})\alpha_1+\overline{\delta}]+c_{32}\alpha_2,$$

where c_{31}, c_{32}, c_{34} are the same constants as in (5.151).

To obtain the above estimate for $T\le t\le t_1+T$ we assume that $t_1<T/2$ and we choose $t=T/2$ as the initial point. Then using estimates (5.148), (5.167) for $T/2\le t\le 3T/2$, boundary condition (5.149), Lemma 5.10, inequality (5.164) with T replaced by $T/2$ and Ω_T replaced by $\Omega_{T/2}$ (where u, ϑ_σ, η_σ denote v, θ_σ, ϱ_σ written in Lagrangian coordinates $\xi_{T/2}\in\Omega_{T/2}$) and estimates (5.162), (5.163) we get

$$\sup_{T\le t\le 2T}\|R(\cdot,t)-R_e\|^2_{W_2^{4+1/2}(S^1)}\le c_{31}c_{34}[(c_{51}/c_{47}+c_{52})\alpha_1+\overline{\delta}]+c_{32}\alpha_2. \tag{5.169}$$

Therefore, for α_1, α_2 and $\overline{\delta}$ sufficiently small, the volume and shape of Ω_t do not change more in $[T,2T]$ than they do in $[0,T]$.

Thus, we can use the differential inequality (5.145) which holds for the solution of problem (5.63)–(5.69) for $t\in(T,2T)$ with the same constants c_i ($i=22,\dots,27$) as before. Just as for the interval $(0,T)$ we derive this inequality by using first the problem written in Eulerian coordinates, i.e. problem (5.63)–(5.69). Then in order to obtain some auxiliary estimates we use problem (5.70)–(5.74) in $\Omega_T \times (T, 2T)$ which is now expressed in Lagrangian coordinates $\xi_T \in \Omega_T$, $\xi_T = \xi + \int_0^T u(\xi, t)\,dt$. Now, ∇_u is defined as $\nabla_u = \xi_{Tx}\partial_{\xi_T} = (\xi_{Tix_j}\partial_{\xi_{Ti}})_{j=1,2,3}$, where ξ_{Tx} denotes the inverse matrix to x_{ξ_T}, and $x = \xi_T + \int_T^t u(\xi_T, t')\,dt'$. We use problem (5.70)–(5.74) in order to obtain some auxiliary local estimates. For this reason we have to use systems (5.142)–(5.143) which arise from problem (5.70)–(5.74).

Since the domains Ω and Ω_T do not differ much, in order to obtain now systems (5.142) and (5.143) we can take the covering of $\overline{\Omega}$ together with the family of functions $\{\zeta_i\}_{i\in\mathcal{M}\cup\mathcal{N}}$. Thus, $\overline{\Omega}_T \subset \bigcup_{i\in\mathcal{M}\cup\mathcal{N}} \widetilde{\Omega}_i$ and $0 < n_0 \le \sum_{i\in\mathcal{M}\cup\mathcal{N}} \zeta_i(\xi_T) \le N_0$ for $\xi_T \in \Omega_T$, where n_0 and N_0 are sufficiently close to 1.

We derive inequality (5.145) with the function $\overline{\varphi}_T(t)$ which has the same form as $\overline{\varphi}(t)$. However, the integrals over $(0,t)$ are now replaced by integrals over (T,t), and the integrals over $\widehat{\Omega}_i$ and $\widehat{S}_i$ $(i \in \mathcal{N})$ are replaced by integrals over $\widehat{\Omega}_{T,i} = \Phi_T(\widetilde{\Omega}_i \cap \Omega_T)$ and $\widehat{S}_{T,i} = \Phi_T(\widetilde{\Omega}_i \cap S_T)$, respectively. Φ_T is the transformation which straightens $\widetilde{S}_{T,i} \equiv \widetilde{\Omega}_i \cap S_T$. We introduce local coordinates by

$$\overline{y}_k = \widetilde{\alpha}_{kl}\Big(\xi_{Tl} - \beta_l - \int_0^T u_l(\beta, t')\,dt'\Big), \qquad k = 1,2,3,$$

where $\beta \in \widetilde{\Omega}_i \cap S$ and $\{\widetilde{\alpha}_{kl}\}$ is an orthogonal matrix such that $\widetilde{S}_{T,i}$ is determined by $\overline{y}_3 = \overline{F}(\overline{y}_1, \overline{y}_2)$, where $\overline{F} \in W_2^{4+1/2}$, $\overline{F}(0) = 0$, $\nabla\overline{F}(0) = 0$. We have the relations:

$$|\widetilde{\alpha}_{kl} - \alpha_{kl}| \le c_{53}(\|R(\cdot,T) - \widetilde{R}\|_{L_\infty(S^1)} + \|\nabla R(\cdot,T) - \nabla\widetilde{R}\|_{L_\infty(S^1)}) \quad \text{for } k,l = 1,2,3$$

and

$$\|F - G\|_{W_2^{4+1/2}(u)} \le c_{54}\|R(\cdot,T) - \widetilde{R}\|_{W_2^{4+1/2}(S^1)},$$

where $G(y_1, y_2) = \overline{F}(\overline{y}_1, \overline{y}_2)|_{\overline{y}=\chi_1(y)}$, χ_1 is the transformation connecting coordinates y given by (5.141) and coordinates $\overline{y}$; $\{\alpha_{kl}\}$ is the same orthogonal matrix as in (5.141). Next, we introduce the coordinates $\overline{z}_k = \overline{y}_k$, $k = 1,2$, and $\overline{z}_3 = \overline{y}_3 - \widetilde{\overline{F}}(\overline{y})$, where $\widetilde{\overline{F}}$ is the extension of $\overline{F}$ to $\mathbb{R}^3_+$ such that $\|\widetilde{\overline{F}}\|_{W_2^5(\mathbb{R}^3_+)} \le c\|\overline{F}\|_{W_2^{4+1/2}(U)}$.

We denote by Φ_T the transformation such that $\overline{z} = \Phi_T(\overline{y})$.

Now, consider the function $\overline{\psi}_T(t) = \overline{\varphi}_T(t) + c_{44}\varphi^{(3)}(t)$, where c_{44} is the constant from (5.156). Since

$$\Big\|\int_0^T u\,dt'\Big\|_{L_\infty(\Omega)} + \Big\|\int_0^T u_\xi\,dt'\Big\|_{L_\infty(\Omega)} \le \varepsilon_1\Big(\int_0^T \|u\|^2_{W_2^4(\Omega)}\,dt\Big)^{1/2} + c(\varepsilon_1)\|u\|_{L_2(\Omega^T)} \le \varepsilon_1 c_{51}\alpha_1 + c(\varepsilon_1)\delta_3,$$

the form of $\overline{\psi}_T(t)$, Lemma 5.13, Remark 5.7 and estimate (5.161) with sufficiently small α_1 imply

$$\begin{aligned}\overline{\psi}_T(T) \le [1 + c_{55}[\varepsilon_2(\|u\|_{W_2^{4,2}(\Omega^T)} + \|R(\cdot,T) - \widetilde{R}\|_{W_2^{4+1/2}(S^1)}) \\ + c(\varepsilon_2)(\|u\|_{L_2(\Omega^T)} + \|R(\cdot,T) - \widetilde{R}\|_{L_2(S^1)})]\overline{\psi}(T) \\ + c_{56}\|H(\cdot,T) + 2/R_e\|_{W_2^2(S_T)} + \omega(\varepsilon_1, \delta_3),\end{aligned}$$

where the constants c_i $(i = 55, 56)$ depend on T; ε_1, ε_2 are sufficiently small constants and $\omega(\varepsilon_1, \delta_3)$ is as small as we need if ε_1 and δ_3 are sufficiently small.

Hence using Lemma 5.4 and estimates (5.161), (5.167)–(5.169) we get

$$\overline{\psi}_T(T) \le (1+\kappa_1)\overline{\psi}(T) + \kappa_2, \tag{5.170}$$

where $\kappa_1 = \kappa_1(\varepsilon_2, \delta_3, \overline{\delta}, \alpha_2)$ and $\kappa_2 = \kappa_2(\varepsilon, \alpha_2, \varepsilon_1, \delta_3)$ are constants depending also on α_1 which are as small as we need if ε, ε_1, ε_2, δ_3, $\overline{\delta}$ and α_2 are sufficiently small.

Moreover, from the form of $\overline{\varphi}_T^{(2)}(t)$ and estimate (5.165) it follows that

$$c_{47}\overline{\varphi}_T^{(2)}(t) \le \varphi_0(t) \le c_{48}\overline{\varphi}_T^{(2)}(t) \quad \text{for } T \le t \le 2T, \tag{5.171}$$

where c_{47}, c_{48} are the constants from (5.157).

The function $\overline{\varphi}_T^{(2)}$ has the same form as $\overline{\varphi}^{(2)}$ but some terms of $\overline{\varphi}_T^{(2)}$ are expressed in Lagrangian coordinates $\xi_T \in \Omega_T$ and the integrals over $\widehat{\Omega}_i$ are replaced by the integrals over $\widehat{\Omega}_{T,i}$.

Now, as before, Lemmas 5.11, 5.12, 5.5 and estimates (5.140), (5.165)–(5.167), (5.161), (5.170) yield for sufficiently small α_1, α_2, $\overline{\delta}$, δ_3, ε, ε_1, ε_2,

$$\begin{aligned} \overline{\psi}_T(t) + \frac{c_{22}}{2}\int_0^t \Phi(t')\,dt' &\le (t-T)[\varepsilon c_{45}(c_{51}/c_{47} + c_{52})\alpha_1 \\ &\quad + c_{46}(\overline{\delta} + \delta_3 + \alpha_2)] + \overline{\psi}_T(T) \quad \text{for } T \le t \le 2T, \end{aligned} \tag{5.172}$$

where c_{22} and c_i $(i = 45, 46)$ are the same constants as in (5.155).

Now, we want to extend the local solution to the interval $[2T, 3T]$. To do this we have to prove that provided the constants ε, ε_1, ε_2, δ_3, $\overline{\delta}$, α_2 are sufficiently small, the following estimate holds:

$$\overline{\psi}_T(T) \le \alpha_0, \tag{5.173}$$

where α_0 is the same constant as in (5.159).

First, notice that Lemmas 5.5, 5.11 and estimate (5.140) imply, for $0 \le t_* \le t \le T$,

$$\overline{\psi}(t) + \frac{c_{22}}{2}\int_{t_*}^t \Phi(t')\,dt' \le (t-t_*)\overline{\gamma} + \overline{\psi}(t_*), \tag{5.174}$$

where $\overline{\gamma}$ is the same small constant as in (5.155). From the form of $\overline{\psi}(t)$ it follows that

$$\begin{aligned} \int_{t_*}^t \Phi(t')\,dt' \ge c_{57}\int_{t_*}^t \overline{\psi}(t')\,dt' - c_{58}\Big(\sum_{i\in\mathcal{N}}\int_{t_*}^t \Big\|\int_0^{t'} \widetilde{u}\,dt''\Big\|^2_{W_2^4(\widehat{S}_i)}\,dt' \\ + \int_{t_*}^t \|H(\cdot,0) + 2/R_e\|^2_{W_2^2(S)}\,dt'\Big), \end{aligned} \tag{5.175}$$

where the constants c_{57} and c_{58} depend on the same quantities as c_{48} and c_{49}.

Using the fact that $x = R(\omega, t)\omega$, $\omega \in S^1$ and Lemma 5.6 we get

$$\sum_{i\in\mathcal{N}}\int_{t_*}^t \Big\|\int_0^{t'} \widetilde{u}\,dt''\Big\|^2_{W_2^4(\widehat{S}_i)}\,dt' \le c_{59}\int_{t_*}^t \|R(\cdot,t') - R(\cdot,0)\|^2_{W_2^4(S^1)}\,dt' \tag{5.176}$$

$$\le c_{60}\Big[\varepsilon\int_{t_*}^{t}\Phi(t')\,dt' + c(\varepsilon)\int_{t_*}^{t}(\|v\|^2_{L_2(\Omega_{t'})} + \|p_\sigma\|^2_{L_2(\Omega_{t'})})\,dt'$$

$$+ (t-t_*)(\sup_{0\le t'\le t}\|R(\cdot,t')-R_e\|^2_{L_2(S^1)} + \|H(\cdot,0)+2/R_e\|^2_{W_2^2(S)})\Big].$$

Now, taking into account (5.174)–(5.176), (5.166)–(5.167), (5.115), (5.112), (5.154) and (5.105), (5.96) we obtain

$$\overline{\psi}(t) + c_{61}\int_{t_*}^{t}\overline{\psi}(t')\,dt' \le c_{62}(t-t_*)\overline{\gamma} + \overline{\psi}(t_*),$$

where the constants c_{61}, c_{62} depend on the same quantities as c_i $(i = 22,\dots,27)$. Hence

(5.177) $$\psi(t) + c_{61}\int_{t_*}^{t}\psi(t')\,dt' \le \psi(t_*) \quad \text{for all } t_* \text{ and } t \text{ satisfying } 0\le t_*\le t\le T,$$

where

$$\psi(t) = \overline{\psi}(t) - \frac{c_{62}\overline{\gamma}}{c_{61}}.$$

Assuming that $c_{62}\overline{\gamma}/c_{61} < \alpha_0$, by (5.177) and (5.159) we get

$$\psi(t) \le \left(\alpha_0 - \frac{c_{62}\overline{\gamma}}{c_{61}}\right)e^{-c_{61}t} \quad \text{for } t\le T.$$

Therefore

$$\overline{\psi}(T) \le \frac{c_{62}\overline{\gamma}}{c_{61}} + \left(\alpha_0 - \frac{c_{62}\overline{\gamma}}{c_{61}}\right)e^{-c_{61}T}.$$

Hence, by (5.170)

$$\overline{\psi}_T(T) \le (1+\kappa_1)\left[\frac{c_{62}\overline{\gamma}}{c_{61}} + \left(\alpha_0 - \frac{c_{62}\overline{\gamma}}{c_{61}}\right)e^{-c_{61}T}\right] + \kappa_2.$$

Assuming that the constants ε_1,ε_2, δ_3, α_2, $\overline{\delta}$, ε are so small that

$$(1+\kappa_1)\left[\frac{c_{62}\overline{\gamma}}{c_{61}} + \left(\alpha_0 - \frac{c_{62}\overline{\gamma}}{c_{61}}\right)e^{-c_{61}T}\right] + \kappa_2 \le \alpha_0,$$

we get (5.173).

Now, in view of (5.173) estimate (5.172) yields

(5.178) $$\overline{\psi}_T(t) + \frac{c_{22}}{2}\int_0^t\Phi(t')\,dt'$$

$$\le (t-T)[\varepsilon c_{45}\left(\frac{c_{51}}{c_{47}} + c_{52}\right)\alpha_1 + c_{46}(\overline{\delta}+\delta_3+\alpha_2)] + \alpha_0 \quad \text{for } T\le t\le 2T.$$

Therefore, assuming that the constants ε, $\overline{\delta}$, δ_3, α_2 are so small that

$$T[\varepsilon c_{45}(c_{51}/c_{47} + c_{52})\alpha_1 + c_{46}(\overline{\delta}+\delta_3+\alpha_2)] + \alpha_0 \le c_{51}\alpha_1$$

we get

$$\overline{\psi}_T(t) + \frac{c_{22}}{2}\int_T^t\Phi(t')\,dt' \le c_{51}\alpha_1 \quad \text{for } T\le t\le 2T,$$

where c_{51} is the constant from inequality (5.161). Hence by (5.171)

$$\varphi_0(t) \le \frac{c_{51}}{c_{47}}\,\alpha_1 \quad \text{for } T \le t \le 2T$$

and

$$\sup_{t_1+T\le t\le 2T} \|H(\cdot,t)+2/R_e\|^2_{W_2^{2+1/2}(S_t)} \le c_{52}\alpha_1.$$

Moreover, as before, we have

$$\sup_{0\le t\le 3T} \|R(\cdot,t)-R_e\|^2_{W_2^1(S^1)} \le \alpha_2.$$

Thus, we can extend the solution to the interval $[2T,3T]$. Continuing this process step by step, the solution can be extended for all $t>0$ and inequalities (5.109)–(5.110) hold. This completes the proof. ∎

Remark 5.8. From the proof of Theorem 5.11 it follows that some estimates in the interval $[0,T]$ hold with different constants than in $[T,2T]$. However, thanks to estimate (5.173), the procedure of extending the solution to successive intervals stabilizes and therefore in the intervals $[iT,(i+1)T]$ $(i=2,3,\dots)$ all the estimates used in the proof hold with the same constants as in $[T,2T]$.

Remark 5.9. From the proof of Theorem 5.11 it follows that apart from (5.109)–(5.110), the assumptions of the theorem also yield

$$\overline{\psi}_{jT}(jT) \le \widehat{c}_4\alpha_1 \quad \text{for } j\in\mathbb{N},$$

where $\widehat{c}_4>0$ is the constant.

$\overline{\psi}_{jT}$ are functions having the same forms as $\overline{\psi}$, but the appropriate terms of $\overline{\psi}_{jT}$ are expressed in Lagrangian coordinates $\xi_{jT}\in\Omega_{jT}$ with the integrals over $(0,t)$ replaced by integrals over (jT,t), and the integrals over $\widehat{\Omega}_i$ and $\widehat{S}_i$ replaced by integrals over $\widehat{\Omega}_{jT,i}=\Phi_{jT}(\widetilde{\Omega}_i\cap\Omega_{jT})$ and $\widehat{S}_{jT}=\Phi_{jT}(\widetilde{\Omega}_i\cap S_T)$, respectively. Φ_T is the transformation which straightens $\widetilde{S}_{jT,i}=\widetilde{\Omega}_i\cap S_T$.

Remark 5.10. The assumptions that $\int_\Omega \varrho_0\xi\,d\xi=0$ and $\int_\Omega \varrho_0 v_0\,d\xi=0$ are not restrictive, because we can always choose coordinates in which they hold (see the final part of Subsection 7.6). Therefore, in fact they can be removed from the formulation of Theorems 5.11–5.13.

In the case $e_\varrho=0$ the following theorem analogous to Theorem 5.11 holds.

Theorem 5.12. *Let $e_\varrho=0$, $p=a\varrho e$, where $a>0$ is a constant. Let the other assumptions of Theorem* 5.11 *hold apart from* (5.104). *Moreover, let assumption* (5.137) *be satisfied. Then the assertion of Theorem* 5.11 *holds.*

Now, we shall consider the case of $p_0=0$ and $f=0$. In this case the energy conservation law has the form

$$\frac{d}{dt}\bigg[\int_{\Omega_t}\varrho\Big(\frac{v^2}{2}+e\Big)\,dx+\sigma|S_t|\bigg]-\varkappa\int_{S_t}\overline{\theta}\,ds=0.$$

Hence, assuming (5.98), (5.113) and using the mass conservation law we obtain

$$(5.179)\quad \frac{e_1}{\varrho_2^\beta}\int\limits_{\Omega_t}\varrho^\gamma\,dx+\int\limits_{\Omega_t}\frac{\varrho v^2}{2}\,dx+\sigma|S_t| \le \int\limits_{\Omega}\varrho_0\bigg(\frac{v_0^2}{2}+e_0\bigg)\,d\xi+\sigma|S|+\int\limits_0^\infty dt\int\limits_{R^3}\overline{\theta}(s,t)ds\equiv d_0,$$

where $\beta>0$, $\gamma=\beta+1$. Multiplying inequality (5.179) by $|\Omega_t|^\beta$ yields

$$(5.180)\quad y_0(|\Omega_t|)+\frac{e_1}{\varrho_2^\beta}\bigg[|\Omega_t|^\beta\int\limits_{\Omega_t}\varrho^\gamma\,dx-\Big(\int\limits_{\Omega_t}\varrho\,dx\Big)^\gamma\bigg] +|\Omega_t|^\beta\int\limits_{\Omega_t}\varrho\frac{v^2}{2}\,dx+\sigma|\Omega_t|^\beta(|S_t|-4\pi R_t^2)\le 0,$$

where $y_0(x)=\sigma\widetilde{c}x^{\gamma-1/3}-d_0x^{\gamma-1}+(e_1/\varrho_2^\beta)M^\gamma$, $\widetilde{c}=(36\pi)^{1/3}$.

Since the last three terms of (5.180) are positive it follows that $y_0(|\Omega_t|)\le 0$. The extremum points of $y_0(x)$ are determined from the equation

$$(5.181)\quad y_0'(x)=[\sigma\widetilde{c}(\gamma-1/3)x^{2/3}-d_0(\gamma-1)]x^{\gamma-2}=0.$$

Equation (5.181) has the only positive solution

$$x_0=\bigg[\frac{d_0(\gamma-1)}{\widetilde{c}\sigma(\gamma-1/3)}\bigg]^{3/2}$$

which is the minimum point of $y_0(x)$. Since $\widetilde{c}\sigma(\gamma-1/3)x_0^{2/3}=d_0(\gamma-1)$ we get

$$(5.182)\quad -y_0(x_0)=\frac{2}{3}(\gamma-1)^{3(\gamma-1)/2}\bigg(\gamma-\frac{1}{3}\bigg)^{-(3\gamma-1)/2}(\widetilde{c}\sigma)^{-3(\gamma-1)/2}d_0^{(3\gamma-1)/2}-\frac{e_1}{\varrho_2^\beta}M^\gamma.$$

The following lemma can be proved by using the argument of the proof of Theorem 1 from [ZZaj5].

Lemma 5.13. *Let* $p_0=0$, $f=0$, $\overline{\theta}\ge 0$ *and let assumptions* (5.98), (5.113) *be satisfied. Moreover assume that*

$$(5.183)\quad 0<-y_0(x_0)\le\delta_0,$$

where $\delta_0>0$ *is sufficiently small and* $-y_0(x_0)$ *is given by* (5.182). *Then*

$$\operatorname*{var}_{0\le t\le T}|\Omega_t|\le c_1\delta,$$

where $\delta^2=c_2\delta_0$.

Remark 5.11. From the analysis of the behaviour of the function

$$F(\gamma)=\frac{2}{3}(\gamma-1)^{3(\gamma-1)/2}\bigg(\gamma-\frac{1}{3}\bigg)^{-(3\gamma-1)/2}(\widetilde{c}\sigma)^{-3(\gamma-1)/2}d_0^{(3\gamma-1)/2}-\frac{e_1}{\varrho_2^{\gamma-1}}M^\gamma$$

it follows that for every δ_0 we can find γ and d_0, σ, M, e_1, ϱ_2 such that condition (5.183) is satisfied.

In fact, putting $\beta = \gamma - 1$ we obtain the function

$$G(\beta) = F(\beta+1) = \frac{2}{3(1+\frac{1}{3\beta/2})^{3\beta/2}(\beta+\frac{2}{3})}\left(\frac{d_0}{\widetilde{c}\sigma}\right)^{3\beta/2} d_0 - \frac{e_1}{\varrho_2^\beta} M^{\beta+1}.$$

Hence $\lim_{\beta\to 0^+} G(\beta) = d_0 - e_1 M > 0$. Now, assume that $d_0 \le \widetilde{c}\sigma$. This implies that $|\Omega| < 1$ and $M < \varrho_2$. Then we get $\lim_{\beta\to\infty} G(\beta) = 0$. Therefore, for every $\delta_0 > 0$ we can find γ such that (5.183) holds.

Thus, we can formulate the following theorem.

THEOREM 5.13. *Let $f = p_0 = 0$, $\overline{\theta} \ge 0$ and assume that condition* $(5.102)_i$ *is substituted by* (5.183). *Moreover, let the other assumptions either of Theorem* 5.11 *or Theorem* 5.12 *be satisfied. Then the assertion of Theorem* 5.11 *holds.*

REMARK 5.12. The assertion of Theorem 5.13 also holds for $p_0 \ne 0$, because we can replace in problem (5.63)–(5.69) p by $\overline{p} = p - p_0$ getting boundary condition (5.66) with the external pressure equal to zero.

The local solution determined by Theorem 5.9 can also be extended to a global one if we assume that $\alpha \in (3/4, 1)$ (see [Z2]). More precisely, the following theorem holds.

THEOREM 5.14. *Let $\alpha \in (3/4,1)$, $e \in C^1(\mathbb{R}_+ \times \mathbb{R}_+)$ and let the assumptions of Theorem* 5.9 *hold. Let the assumptions* (5.84), (5.95), (5.106) *and* (5.97)–(5.98) *with the constants ϱ_1, ϱ_2, θ_1, θ_2 defined in Theorem* 5.11 *be satisfied. Let one of the conditions* $(5.102)_i$ *be fulfilled. Assume that either $e_\varrho > 0$ for ϱ, $\theta > 0$ and* (5.104) *holds, or $e_\varrho = 0$, $p = a\varrho e$ $(a > 0)$ and* (5.137) *is satisfied. Moreover, assume that*

$$\|v_0\|^2_{W_2^{1+\alpha}(\Omega)} + \|\theta_{\sigma 0}\|^2_{W_2^{1+\alpha}(\Omega)} + \|\varrho_{\sigma 0}\|^2_{W_2^{1+\alpha}(\Omega)} \le \alpha_1$$

and

$$\|r\|^2_{C_B^2(\mathbb{R}^3\times(0,\infty))} + \|\overline{\theta}\|^2_{C_B^3(\mathbb{R}^3\times(0,\infty))} \le \overline{\delta}.$$

Finally, assume that the function $\widetilde{R}$ describing S satisfies (5.107), $S \in W_2^{5/2+\alpha}$ *and*

$$\|H(\cdot,0) + 2/R_e\|^2_{W_2^{\alpha+1/2}(S)} \le \alpha_1.$$

Then for sufficiently small α_1, α_2, $\overline{\delta}$ and δ_i $(i = 0,\dots,3)$ the solution of problem (5.63)–(5.69) *exists for all $t > 0$ and the following estimates are satisfied*:

$$\|v(t)\|^2_{W_2^{1+\alpha}(\Omega_t)} + \|\theta_\sigma(t)\|^2_{W_2^{1+\alpha}(\Omega_t)} + \|\varrho_\sigma(t)\|^2_{W_2^{1+\alpha}(\Omega_t)} \le \widetilde{c}_1\alpha_1 \quad \text{for } t > 0$$

and

$$\begin{aligned}\|v\|^2_{W_2^{2+\alpha,1+\alpha/2}(\Omega_t\times(jT,(j+1)T)} &+ \|\theta_\sigma\|^2_{W_2^{2+\alpha,1+\alpha/2}(\Omega_t\times(jT,(j+1)T)} \\ &+ \|\varrho_\sigma\|^2_{W_2^{1+\alpha,1/2+\alpha/2}(\Omega_t\times(jT,(j+1)T)} \le \widetilde{c}_2\alpha_1 \quad \text{for } t \in (jT,(j+1)T),\ j \in \mathbb{N}\cup\{0\},\end{aligned}$$

where T is the time of local existence and $\widetilde{c}_1$, $\widetilde{c}_2 > 0$ are constants. Moreover, $S_t \in W_2^{5/2+\alpha}$ for $t \in \mathbb{R}_+$ and

$$\sup_{t_1>0} \|R(\cdot,t) - R_e\|^2_{W_2^{5/2+\alpha}(S^1)} \le \widetilde{c}_3\alpha_1 + \widetilde{c}_4\alpha_2,$$

where $\widetilde{c}_3$, $\widetilde{c}_4 > 0$ are constants and the constant $\widetilde{c}_3$ depends on $0 < t_1 < T$.

The general idea of the proof is the same as in Theorem 5.11. However, some modifications are necessary. First, we replace Lemma 5.5 by

LEMMA 5.14. *Let assumption* (5.113) *be satisfied. Then for* $0 \le t_* < t \le T$ *we have*

$$\|q_\sigma\|^2_{L_2(\Omega\times(t_*,t))} \le \overline{c}_5\Big[\varepsilon\|u\|^2_{W_2^{2+\alpha,1+\alpha/2}(\Omega\times(t_*,T))} + c(\varepsilon,T-t_*)\|u\|^2_{L_2(\Omega\times(t_*,T))} + \sup_{t_*\le t'\le t}\|u\|^2_{W_2^1(\Omega)}\int_{t_*}^t \|u\|^2_{W_2^1(\Omega)}\,dt'\Big]$$

and

$$\begin{aligned}\|\eta_\sigma\|^2_{L_2(\Omega\times(t_*,t))} &+ \|\vartheta_\sigma\|^2_{L_2(\Omega\times(t_*,t))}\\ &\le \overline{c}_5\Big[\varepsilon\|u\|^2_{W_2^{2+\alpha,1+\alpha/2}(\Omega\times(t_*,T))} + c(\varepsilon,T-t_*)\|u\|^2_{L_2(\Omega\times(t_*,T))} + \|u_\xi\|^2_{L_2(\Omega\times(t_*,t))}\\ &\quad + \|\vartheta_{\sigma\xi}\|^2_{L_2(\Omega\times(t_*,t))} + \sup_{t_*\le t'\le t}\|u\|^2_{W_2^1(\Omega)}\int_{t_*}^t \|u\|^2_{W_2^1(\Omega)}\,dt'\\ &\quad + \sup_{t_*\le t'\le t}\|\eta_\sigma\|^2_{W_2^1(\Omega)}\int_{t_*}^t \|\eta_\sigma\|^2_{W_2^1(\Omega)}\,dt' + \sup_{t_*\le t'\le t}\|\vartheta_\sigma\|^2_{W_2^1(\Omega)}\int_{t_*}^t \|\vartheta_\sigma\|^2_{W_2^1(\Omega)}\,dt'\Big],\end{aligned}$$

where $q_\sigma(\xi,t) = p_\sigma(\eta(\xi,t),\vartheta(\xi,t))$; $\overline{c}_5,\overline{c}_6 > 0$ *are constants depending on* ϱ_1, ϱ_2, θ_1, θ_2; $\varepsilon \in (0,1)$ *is a constant*; $c(\varepsilon,T)$ *are positive constants depending on* ε *and* T.

The proof of the above lemma is similar to the proof of Lemma 5.5. The only difference is connected with the fact that we cannot estimate $\sup_{0\le t\le T}\|u_t\|^2_{L_2(\Omega)}$ in this case. Therefore, we use the interpolation inequality

$$\int_{t_*}^T \|u_t\|^2_{L_2(\Omega)}\,dt \le \varepsilon\|u\|^2_{W_2^{2+\alpha,1+\alpha/2}(\Omega\times(t_*,T))} + c(\varepsilon,T-t_*)\|u\|^2_{L_2(\Omega\times(t_*,T))}.$$

Lemmas 5.3, 5.4, 5.7, 5.13 and the inequality (5.140) are applied in the proof of Theorem 5.14 without any changes, while Lemmas 5.6, 5.8 and 5.9 require only slight modifications. Namely, in Lemmas 5.8 and 5.9 we assume that $v_0 \in W_2^{1+\alpha}(\Omega)$, $\theta_0 \in W_2^{1+\alpha}(\Omega)$, $\varrho_0 \in W_2^{1+\alpha}(\Omega)$ and $\|v_0\|^2_{W_2^{1+\alpha}(\Omega)} + \|\theta_{\sigma 0}\|^2_{W_2^{1+\alpha}(\Omega)} + \|\varrho_{\sigma 0}\|^2_{W_2^{1+\alpha}(\Omega)} \le \overline{\varepsilon}$, where $\overline{\varepsilon}$ is sufficiently small. Then the assertions of the lemmas hold. Lemma 5.6 is modified in such a way that we obtain inequalities analogous to (5.127) and (5.128) for $\int_{t_*}^t \|R(\cdot,t') - R(\cdot,0)\|^2_{W_2^{2+\alpha}(S^1)}\,dt'$ and $\int_{t_*}^t \|R(\cdot,t') - R(\cdot,0)\|^2_{W_2^{5/2+\alpha}(S^1)}\,dt'$, respectively.

Next, using Lemmas 3.1–3.2 of [Z2] and Lemma 6.1 of [BurZaj] (see also Lemmas 2.1, 3.1, 3.2 of [ZZaj11]) we see that the following estimate analogous to (5.112) holds:

$$\begin{aligned}(5.184)\qquad &[\|u\|^{(\alpha+2,\alpha/2+1)}_{\Omega^t} + \|\vartheta_\sigma\|^{(\alpha+2,\alpha/2+1)}_{\Omega^t} + \|\eta_\sigma\|_{W_2^{1+\alpha,1/2+\alpha/2}(\Omega^t)} + \sup_{0\le t'\le t}\|\eta_\sigma\|_{W_2^{1+\alpha}(\Omega)}]^2\\ &\le \overline{\varphi}(T)(\|v_0\|^2_{W_2^{1+\alpha}(\Omega)} + \|\theta_{\sigma 0}\|^2_{W_2^{1+\alpha}(\Omega)} + \|\varrho_{\sigma 0}\|^2_{W_2^{1+\alpha}(\Omega)} + \|H(\cdot,0) + 2/R_e\|^2_{W_2^{\alpha+1/2}(S)}\\ &\quad + \|h\|^2_{W_2^{\alpha,\alpha/2}(\Omega^T)} + \|\overline{\vartheta}\|^2_{W_2^{\alpha+1/2,\alpha/2+1/4}(S^T)}),\end{aligned}$$

where $t \le T$, T is the time of local existence depending on α_1; $\overline{\varphi}$ is a positive nondecreasing continuous function.

Moreover, the estimate analogous to (5.139) is proved for the local solution determined by Theorem 5.9. This estimate has the form (see [Z2], [ZZaj11])

$$\sup_{t_1\le t\le T}\|u\|^2_{W_2^{2+\alpha}(\Omega)} + \sup_{t_1\le t\le T}\|\vartheta_\sigma\|^2_{W_2^{2+\alpha}(\Omega)} \le c(K)(K_1 + \|r\|^2_{C_B^2(\mathbb{R}^3\times(0,\infty))} + \|\overline{\theta}\|^2_{C_B^3(\mathbb{R}^3\times(0,\infty))}), \tag{5.185}$$

where $K_1 = \|u\|^2_{W_2^{2+\alpha,1+\alpha/2}(\Omega^T)} + \|\vartheta_\sigma\|^2_{W_2^{2+\alpha,1+\alpha/2}(\Omega^T)}$, $K = K_1 + \sup_{0\le t\le T}\|u\|^2_{W_2^{1+\alpha}(\Omega)} + \sup_{0\le t\le T}\|\vartheta_\sigma\|^2_{W_2^{1+\alpha}(\Omega)} + \sup_{0\le t\le T}\|\eta_\sigma\|^2_{W_2^{1+\alpha}(\Omega)} + \|\eta_\sigma\|^2_{W_2^{1+\alpha,1/2+\alpha/2}(\Omega^T)}$, $c(K)$ is a positive nondecreasing continuous function of K depending also on t_1.

Finally, instead of differential inequality (5.145) we use an inequality which is proved in [ZZaj13]. This inequality written in a more general form is as follows:

$$\begin{aligned}\phi(t,\Omega) &+ \overline{c}_{22}(\|u\|^2_{W_2^{2+\alpha,1+\alpha/2}(\Omega\times(t_*,t))} + \|\vartheta_\sigma\|^2_{W_2^{2+\alpha,1+\alpha/2}(\Omega\times(t_*,t))} \\ &+ \|\eta_\sigma\|^2_{W_2^{1+\alpha,1/2+\alpha/2}(\Omega\times(t_*,t))}) \\ &\le c(t,Z_1,Z_2)[\|u\|^2_{L_2(\Omega\times(t_*,t))} + \|\eta_\sigma\|^2_{L_2(\Omega\times(t_*,t))} + \|\vartheta_\sigma\|^2_{L_2(\Omega\times(t_*,t))} + \|h\|^2_{W_2^{\alpha,\alpha/2}(\Omega^T)} \\ &+ \|\overline{\vartheta}\|^2_{W_2^{\alpha+1/2,\alpha/2+1/4}(S^T)} + \varepsilon_1 Z_4 + c(\varepsilon_1)Z_1Z_3] + \phi(t_*,\Omega),\end{aligned} \tag{5.186}$$

where $0 \le t_* \le t \le T$, $c = c(t, Z_1, Z_2)$ is a positive continuous nondecreasing function with respect to its arguments depending also on ϱ_1, ϱ_2, θ_1, θ_2, $\|S\|_{W_2^{5/2+\alpha}}$ and the constants from imbedding theorems; $\overline{c}_{22} > 0$ is a constant; $\varepsilon_1 \in (0,1)$ is a constant which can be assumed sufficiently small and

$$\begin{aligned} Z_1 &= \|u\|^2_{W_2^{2+\alpha,1+\alpha/2}(\Omega\times(t_*,t))} + \|\vartheta_\sigma\|^2_{W_2^{2+\alpha,1+\alpha/2}(\Omega\times(t_*,t))} + \|\eta_\sigma\|^2_{W_2^{1+\alpha,1/2+\alpha/2}(\Omega\times(t_*,t))}, \\ Z_2 &= \sup_{0\le t\le T}\|u\|^2_{W_2^{1+\alpha}(\Omega)} + \sup_{0\le t\le T}\|\vartheta_\sigma\|^2_{W_2^{1+\alpha}(\Omega)} + \sup_{0\le t\le T}\|\eta_\sigma\|^2_{W_2^{1+\alpha}(\Omega)}, \\ Z_3 &= [\|u\|^{(\alpha+2,\alpha/2+1)}_{\Omega^t} + \|\vartheta_\sigma\|^{(\alpha+2,\alpha/2+1)}_{\Omega^t}]^2 + \|\eta_\sigma\|^2_{W_2^{1+\alpha,1/2+\alpha/2}(\Omega^t)} + \sup_{0\le t'\le t}\|\eta_\sigma\|^2_{W_2^{1+\alpha}(\Omega)}, \\ Z_4 &= (t-t_*)\|H(\cdot,0) + 2/R_e\|^2_{W_2^{\alpha+1/2}(S)} + \int_{t_*}^{t}\|R(\cdot,t') - R(\cdot,0)\|^2_{W_2^{5/2+\alpha}(S^1)}\,dt'. \end{aligned}$$

The function ϕ satisfies the inequality

$$\begin{aligned}\phi(t,\Omega) \le c\Big(&\|u(t)\|^2_{W_2^{1+\alpha}(\Omega)} + \|\vartheta_\sigma(t)\|^2_{W_2^{1+\alpha}(\Omega)} + \|\eta_\sigma(t)\|^2_{W_2^{1+\alpha}(\Omega)} \\ &+ \Big\|\int_0^t u\,dt'\Big\|^2_{W_2^{2+\alpha}(S)} + \|H(\cdot,0) + 2/R_e\|^2_{W_2^{\alpha}(S)}\Big),\end{aligned}$$

where $c > 0$ is a constant depending on ϱ_1, ϱ_2, θ_1, θ_2. The exact form of ϕ is given in [ZZaj13].

Inequalities (5.184), (5.186) and (5.140) together with Lemma 5.14 and the assumptions of the theorem imply the following estimate:

$$\overline{\psi}(t) + \overline{c}_{61}\int_{t_*}^{t}\overline{\psi}(t')\,dt' \le \overline{\gamma}_1 + \overline{c}_{62}(t_*)\overline{\gamma}_2 + \overline{\psi}(t_*), \tag{5.187}$$

where $0 \le t_* < t \le T$, $\overline{c}_{61} > 0$ is a constant; $\overline{c}_{62}(t_*) = c(\varepsilon, T - t_*)$ is the constant from Lemma 5.14; $\overline{\gamma}_1$, $\overline{\gamma}_2$ are sufficiently small constants; $\overline{\psi}(t) = \phi(t,\Omega) + \overline{c}_{44}\varphi^{(3)}(t)$ (see (5.156)) and similarly to c_{44} the constant $\overline{c}_{44}$ is chosen sufficiently large.

The constants $\overline{c}_{61}$, $\overline{\gamma}_1$ and the function $\overline{c}_{62}$ depend on ϱ_1, ϱ_2, θ_1, θ_2, $\|S\|_{W_2^{5/2+\alpha}}$ and the constants from imbedding theorems.

The role of inequality (5.187) in the proof of Theorem 5.14 is the same as inequality (5.177) in the proof of Theorem 5.11. Namely, it is essential in the process of extending the local solution to a global one. In fact, the following lemma holds.

LEMMA 5.15. *Assume that ψ is a nonnegative continuous function defined in $[0,T]$ and satisfying for $0 \le t_* < t \le T$ the inequality*

$$\psi(t) + C_1\int_{t_*}^{t}\psi(t')\,dt' \le \gamma_1 + C_2(t_*)\gamma_2 + \psi(t_*), \tag{5.188}$$

where $C_1 > 0$ and γ_1, $\gamma_2 \ge 0$ are constants; $C_2(t_*) = \omega(T - t_*)$ and $\omega = \omega(t)$ is a positive continuous function defined in $(0,T]$. Let $C_3 = \sup_{t\in[T/2,T]}\omega(t)$. If $\psi(0) \le \alpha_0$ and $\gamma_1 + C_3\gamma_2 \le \min(\frac{\alpha_0}{2}, \frac{C_1T\alpha_0}{4+2C_1T})$ then there exists $T/2 \le T_1 \le T$ such that

$$\psi(T_1) \le (1-\varepsilon_0)\alpha_0, \tag{5.189}$$

where $\varepsilon_0 = \min(\frac{1}{2}, \frac{C_1T}{4+2C_1T})$.

Proof. Let $0 \le T_* \le T$ be such that $\psi(T_*) = \inf_{t\in[0,T]}\psi(t)$. If $0 \le T_* \le T/2$ then by (5.188),

$$\psi(T) + C_1\int_{T_*}^{T}\psi(t')\,dt' \le \gamma_1 + \omega(T - T_*)\gamma_2 + \psi(T_*)$$

and hence

$$\psi(T) \le (1 - C_1\frac{T}{2})\psi(T_*) + \gamma_1 + C_3\gamma_2.$$

If $C_1(T/2) \ge 1$ then $\psi(T) \le \gamma_1 + C_3\gamma_2$. If $C_1(T/2) < 1$ then $\psi(T) \le (1 - C_1T/2)\alpha_0 + \gamma_1 + C_3\gamma_2 \le (1 - C_1T/4)\alpha_0$. Therefore (5.189) holds with $T_1 = T$.

Now, assume that $T/2 < T_* \le T$. Then by (5.188),

$$\psi(T_*) + C_1\int_{0}^{T_*}\psi(t')\,dt' \le \gamma_1 + \omega(T)\gamma_2 + \psi(0),$$

and hence

$$\psi(T_*) \le \frac{\alpha_0}{1 + C_1(T/2)} + \gamma_1 + C_3\gamma_2 = \left(1 - \frac{C_1T}{2 + C_1T}\right)\alpha_0 + \gamma_1 + C_3\gamma_2.$$

Therefore, the assertion holds now with $T_1 = T_*$. This completes the proof. ∎

Now, we present the sketch of the proof of Theorem 5.14.

Sketch of proof. We proceed as in the proof of Theorem 5.11. First, inequalities (5.184), (5.186) and (5.140) together with Lemmas 5.14, 5.4 and the lemmas analogous to Lemmas

5.6 and 5.8 or 5.9 yield, for $t \le T$,

$$\overline{\psi}(t) + \frac{\overline{c}_{22}}{2}(\|u\|^2_{W_2^{2+\alpha,1+\alpha/2}(\Omega^t)} + \|\vartheta_\sigma\|^2_{W_2^{2+\alpha,1+\alpha/2}(\Omega^t)} + \|\eta_\sigma\|^2_{W_2^{1+\alpha,1/2+\alpha/2}(\Omega^t)} \le \overline{c}_{45}\varepsilon(\alpha_1+\overline{\delta}) + c(\varepsilon,T)\delta_3 T + \overline{c}_{46}T[\overline{\delta}+\varepsilon_1(\alpha_1+\alpha_2)] + \overline{\psi}(0) \equiv \overline{\gamma} + \overline{\psi}(0), \tag{5.190}$$

where the constant $c(\varepsilon,T)$ depends also on Ω, and the constants $\overline{c}_{22}$, $\overline{c}_{45}$, $\overline{c}_{46}$ depend on the same quantities as the constants $\overline{c}_{61}$ and $\overline{c}_{62}$ above.

As in the proof of Theorem 5.11 we get (see (5.154))

$$\|H(\cdot,0) + 2/R_e\|^2_{W_2^\alpha(S)} \le \varepsilon_3\alpha_1 + \overline{c}_{43}(\varepsilon_3)\alpha_2.$$

Hence

$$\overline{\psi}(0) \le \overline{c}_{49}[\alpha_1 + \varepsilon_3\alpha_1 + \overline{c}_{43}(\varepsilon_3)\alpha_2] \le \overline{c}_{50}\alpha_1 \equiv \alpha_0, \tag{5.191}$$

if $\overline{c}_{50} > \overline{c}_{49}$ and ε_3, α_2 are sufficiently small. Hence assuming that $\overline{c}_{51} > \overline{c}_{50}$ and ε, ε_1, $\overline{\delta}$, δ_3 are so small that $\overline{\gamma} + \alpha_0 \le \overline{c}_{51}\alpha_1$, we obtain

$$\overline{\psi}(t) + \frac{\overline{c}_{22}}{2}(\|u\|^2_{W_2^{2+\alpha,1+\alpha/2}(\Omega^t)} + \|\vartheta_\sigma\|_{W_2^{2+\alpha,1+\alpha/2}(\Omega^t)} + \|\eta_\sigma\|^2_{W_2^{1+\alpha,1/2+\alpha/2}(\Omega^t)}) \le \overline{c}_{51}\alpha_1 \quad \text{for } t \le T. \tag{5.192}$$

Therefore, by the imbedding $W_2^{2+\alpha,1+\alpha/2}(\Omega^T) \subset C([0,T]; W_2^{1+\alpha}(\Omega))$ we get

$$\|u(t)\|^2_{W_2^{1+\alpha}(\Omega)} + \|\vartheta_\sigma(t)\|^2_{W_2^{1+\alpha}(\Omega)} + \|\eta_\sigma(t)\|^2_{W_2^{1+\alpha}(\Omega)} \le \overline{c}_{51}^{(1)}\alpha_1 \quad \text{for } t \le T.$$

Hence

$$\|u(T_1)\|^2_{W_2^{1+\alpha}(\Omega_{T_1})} + \|\vartheta_\sigma(T_1)\|^2_{W_2^{1+\alpha}(\Omega_{T_1})} + \|\eta_\sigma(T_1)\|^2_{W_2^{1+\alpha}(\Omega_{T_1})} \le \overline{c}_{51}^{(2)}\alpha_1, \tag{5.193}$$

where $0 < T_1 \le T$ and u, ϑ_σ, η_σ denote v, θ_σ, ϱ_σ written in Lagrangian coordinates $\xi_{T_1} \in \Omega_{T_1}$.

Now, the boundary condition (5.149) and inequalities (5.185) and (5.192) imply, for $t_1 \le t \le T$,

$$\|H(\cdot,t) + 2/R_e\|^2_{W_2^{\alpha+1/2}(S)} \le c(\sup_{t_1\le t\le T}\|u\|^2_{W_2^{2+\alpha}(\Omega)} + \sup_{t_1\le t\le T}\|\vartheta_\sigma\|^2_{W_2^{2+\alpha}(\Omega)} + \sup_{t_1\le t\le T}\|\eta_\sigma\|^2_{W_2^{1+\alpha}(\Omega)}) \le \overline{c}_{52}\alpha_1,$$

where $t_1 > 0$; $c > 0$ is a constant depending on t_1; $\overline{c}_{52}$ is a constant such that $C[(\overline{c}_{51} + \overline{c}_{51}^{(1)})\alpha_1 + \overline{\delta}] \le \overline{c}_{52}\alpha_1$ and $C(K) \le C$. Therefore, using Theorem 7.1 and Lemma 5.8 (or 5.9) we get

$$\sup_{t_1\le t\le T}\|R(\cdot,t) - R_e\|^2_{W_2^{5/2+\alpha}(S^1)} \le \widetilde{c}_3\alpha_1 + \widetilde{c}_4\alpha_2.$$

This means $S_t \in W_2^{5/2+\alpha}$ for $t \le T$ and the shape of S_t does not change much in $[0,T]$. To extend the solution outside $[0,T]$ we derive inequality (5.187) with $\overline{\gamma}_2 = \delta_3 T$, $\omega(t) = c(\varepsilon,,t)$ and with $\overline{\gamma}_1 = \overline{\gamma}_1(\overline{\delta},\varepsilon,\varepsilon_1,\varepsilon_3,\alpha_2)$ as small as we need if $\overline{\delta}$, ε, ε_1, ε_3, α_2 are sufficiently small. Then for $\overline{\delta}$, ε, ε_1, ε_3, α_2 and δ_3 so small that $\overline{\gamma}_1 + \sup_{t\in[T/2,T]} c(\varepsilon,t)\overline{\gamma}_2 \le \min(\alpha_0/2, (\overline{c}_{61}T\alpha_0)/(4+2\overline{c}_{61}T))$, estimate (5.191) and Lemma 5.15 imply

(5.194) $$\overline{\psi}(T_1) \le (1-\varepsilon_0)\alpha_0 \quad \text{for } T/2 \le T_1 \le T,$$

where $\varepsilon_0 = \min(1/2, (\overline{c}_{61}T)/(4+2\overline{c}_{61}T))$.

If α_1 is sufficiently small then by (5.193) there exists a local solution in $[T_1, T+T_1]$. It satisfies an inequality analogous to (5.186) with $\overline{\psi}(t)$ replaced by $\overline{\psi}_{T_1}(t)$. The function $\overline{\psi}_{T_1}(t)$ has the same form as $\overline{\psi}(t)$ but it is expressed in Lagrangian coordinates $\xi_{T_1} \in \Omega_{T_1}$.

Using (5.194) and assuming that δ_3, $\overline{\delta}$, α_2 are sufficiently small, we get

$$\overline{\psi}_{T_1}(T_1) \le \alpha_0.$$

The argument for the above estimate is the same as in Theorem 5.11.

Hence repeating the above considerations we extend the solution to $[T_2, T_2+T]$, where $T_1 + T/2 \le T_2 \le T_1 + T$, and then step by step to $\mathbb{R}_+$. ■

REMARK 5.13. Theorem 5.14 is formulated for the case of constant ν, μ and $\varkappa$, but it can also be proved under the assumption that ν, μ and $\varkappa$ are sufficiently regular functions of ϱ and θ (see [Z2]).

REMARK 5.14. Since we prove step by step that the volume and shape of Ω_t do not change much in time, in each step we obtain inequality (5.187) with the same constant $\overline{c}_{61}$ and the same function $\overline{c}_{62}$.

6. Two- and three-dimensional surface waves problems

6.1. The motion of an incompressible fluid. The three-dimensional problem under consideration is formulated as follows: find a domain $\Omega_t \equiv \{x = (x', x_3) \in \mathbb{R}^3 : x' = (x_1, x_2) \in \mathbb{R}^2, -b(x') < x_3 < F(x', t)\}$, a velocity vector field $v = v(x,t)$ $(v = (v_1, v_2, v_3))$ and a pressure $p = p(x,t)$ satisfying the system:

(6.1) $$v_t + (v\cdot\nabla)v - \nu\Delta v + \nabla p = f - ge_3, \qquad x \in \Omega_t,\ t \in (0,T),$$

(6.2) $$\operatorname{div} v = 0, \qquad x \in \Omega_t,\ t \in (0,T),$$

(6.3) $$\mathbb{T}\overline{n} - \sigma H\overline{n} = -p_0\overline{n}, \qquad x \in S_F(t),\ t \in (0,T),$$

(6.4) $$v\cdot\overline{n} = \frac{\partial F/\partial t}{\sqrt{1+|\nabla' F|^2}}, \qquad x \in S_F(t),\ t \in (0,T),$$

(6.5) $$v = 0, \qquad x \in S_B,\ t \in (0,T),$$

(6.6) $$v|_{t=0} = v_0(x), \qquad x \in \Omega_0 \equiv \Omega,$$

(6.7) $$F|_{t=0} = F_0(x'), \qquad x' \in \mathbb{R}^2,$$

where g is the acceleration of gravity, $e_3 = {}^t(0,0,1)$; $S_F(t) \equiv \{x' \in \mathbb{R}^2 : x_3 = F(x',t)\}$ is the free surface with unknown function F, $\nabla' = (\partial/\partial x_1, \partial/\partial x_2)$, $S_B = \{x' \in \mathbb{R}^2 : x_3 = -b(x')\}$ and b is a given function; $\mathbb{T}$ is the stress tensor defined by (4.6); $\sigma \ge 0$ is the surface tension; p_0 is the atmospheric pressure which is a positive constant; $\overline{n}$ is the unit outward vector normal to $S_F(t)$ at x; H is the double mean curvature of $S_F(t)$ at x given by

$$H(x,t) = \nabla' \cdot \left(\frac{\nabla' F}{\sqrt{1+|\nabla' F|^2}}\right).$$

Problem (6.1)–(6.7) describes the motion of a fluid contained in an unbounded domain, such as an ocean of infinite extent and finite depth.

The two-dimensional surface waves problem can be formulated analogously.

6.1.1. *Local existence.* Just as in the case of a fixed mass of a fluid bounded by a free surface (see Sections 4–5) it is convenient to write problem (6.1)–(6.7) in Lagrangian coordinates. Then assuming that $p = \overline{p} - p_0 + gx_3$, problem (6.1)–(6.7) takes the form

$$u_t - \nu\nabla_u^2 u + \nabla_u q = h \quad \text{in } \Omega^T, \tag{6.8}$$

$$\nabla_u \cdot u = 0 \quad \text{in } \Omega^T, \tag{6.9}$$

$$\mathbb{T}_u(u,q)\overline{n}_u - \sigma\Delta_u(t)X_u = -gX_{u,3}\overline{n}_u \quad \text{on } S_F^T, \tag{6.10}$$

$$u = 0 \quad \text{on } S_B^T, \tag{6.11}$$

$$u|_{t=0} = v_0 \quad \text{in } \Omega, \tag{6.12}$$

where $q(\xi,t) = \overline{p}(X_u(\xi,t),t)$, $h(\xi,t) = f(X_u(\xi,t),t)$, $S_F = S_F(0)$.

Local solvability of problem (6.1)–(6.7) is examined in [Al1, Al2, B1, T2]. The first paper concerning problem (6.1)–(6.7) was the paper of Beale [B1], where the case of $\sigma = 0$ was studied. Assuming $f = 0$, Beale examined local solvability of problem (6.8)–(6.12) in the spaces $H^0(0,T;H^l(\Omega)) \cap H^{l/2}(0,T;H^0(\Omega))$ which in fact coincide with the Sobolev–Slobodetskiĭ spaces $W_2^{l,l/2}(\Omega^T)$. Beale proved local existence for problem (6.8)–(6.12) using the theory of $H^s(0,T;X)$ spaces (where X is a Hilbert space) developed in [LMag].

As usual in order to solve (6.8)–(6.12) one has to consider an appropriate linear problem which in this case consists of equations (4.15)–(4.16) and the following boundary and initial conditions:

$$2\nu[\mathbb{S}(u)\overline{n}_0 - (\mathbb{S}(u)\overline{n}_0 \cdot \overline{n}_0)\overline{n}_0] = D \quad \text{in } S_F^T, \tag{6.13}$$

$$-q + 2\nu\mathbb{S}(u)\overline{n}_0 \cdot \overline{n}_0 - \sigma\Delta_S \int_0^t u\,dt' \cdot \overline{n}_0 = b + \sigma\int_0^t B\,dt' \quad \text{on } S_F^T, \tag{6.14}$$

$$u = 0 \quad \text{on } S_B^T, \tag{6.15}$$

$$u|_{t=0} = v_0 \quad \text{in } \Omega, \tag{6.16}$$

where $\mathbb{S}(u)$ is as before, the velocity deformation tensor and $\overline{n}_0$ is the unit outward vector normal to S_F.

In [T2] the following existence theorem for problem (4.15)–(4.16), (6.13)–(6.16) is proved.

THEOREM 6.1. *Let* $\alpha \in (1/2,1)$, $0 < T < \infty$ *and* S_F, $S_B \in W_2^{3/2+\alpha}$. *Assume that* $(F,G,v_0,(b,D),B) \in W_2^{\alpha,\alpha/2}(\Omega^T) \times W_2^{1+\alpha,(1+\alpha)/2}(\Omega^T) \times W_2^{1+\alpha}(\Omega) \times W_2^{\alpha+1/2,\alpha/2+1/4}(S_F^T) \times W_2^{\alpha-1/2,\alpha/2-1/4}(S_F^T)$, *and that* $G = \nabla\cdot R$ *with* $R \in L_2(\Omega^T)$, $R_t \in W_2^{0,\alpha/2}(\Omega^T)$. *Moreover, assume that the following compatibility conditions are satisfied*:

$$\nabla\cdot v_0 = G|_{t=0}, \quad D|_{t=0} = 2\nu[\mathbb{S}(v_0)\overline{n}_0 - (\mathbb{S}(v_0)\overline{n}_0\cdot\overline{n}_0)\overline{n}_0]|_{S_F}, \quad D\cdot\overline{n}_0 = 0, \quad v_0|_{S_B} = 0.$$

Then problem (4.15)–(4.16), (6.13)–(6.16) *has a unique solution* $u \in W_2^{2+\alpha,1+\alpha/2}(\Omega^T)$, $q \in W_2^{\alpha,\alpha/2}(\Omega^T), \nabla q \in W_2^{\alpha,\alpha/2}(\Omega^T), q \in W_2^{\alpha+1/2,\alpha/2+1/4}(S_F^T)$ *which satisfies the inequality*

$$\begin{aligned}\|u\|_{\Omega^T}^{(2+\alpha)} + \|q\|_{\Omega^T}^{(\alpha)} + \|\nabla q\|_{\Omega^T}^{(\alpha)} + \|q\|_{W_2^{\alpha+1/2,\alpha/2+1/4}(S_F^T)} \\ \le C_1(T)\{\|F\|_{\Omega^T}^{(\alpha)} + \|G\|_{W_2^{1+\alpha,(1+\alpha)/2}(S_F^T)} + \|R\|_{W_2^{0,\alpha/2+1}(\Omega^T)} \\ + T^{-\alpha/2}\|R_t\|_{L_2(\Omega^T)} + \|(b,D)\|_{W_2^{\alpha+1/2,\alpha/2+1/4}(S_F^T)} \\ + T^{-\alpha/2}\|b\|_{W_2^{1/2,0}(S_F^T)} + \sigma\|B\|_{S_F^T}^{(\alpha-1/2)} + \|v_0\|_{W_2^{1+\alpha}(\Omega)}\},\end{aligned}$$

where $C_1(T)$ *is a constant depending on* T *nondecreasingly and*

$$\begin{aligned}(\|u\|_{\Omega^T}^{(2+\alpha)})^2 &\equiv (\|u_t\|_{\Omega^T}^{(\alpha)})^2 + \sum_{|\gamma|=2}(\|D_x^\gamma u\|_{\Omega^T}^{(\alpha)})^2 + \sum_{|\gamma|=0}^{1}\|D_x^\gamma u\|_{L_2(\Omega^T)}^2, \\ (\|u\|_{\Omega^T}^{(\alpha)})^2 &\equiv \|u\|_{W_2^{\alpha,\alpha/2}(\Omega^T)}^2 + T^{-\alpha}\|u\|_{L_2(\Omega^T)}^2, \\ (\|B\|_{S_F^T}^{(\delta)})^2 &\equiv \|B\|_{W_2^{\delta,\delta/2}(S_F^T)}^2 + T^{-\delta}\|B\|_{L_2(S_F^T)}^2 \quad (0<\delta<1).\end{aligned}$$

The proof of Theorem 6.1 is closely related to the proof of Theorem 1.1 from [Sol12], however, some arguments from [TItTan] are also used. Theorem 6.1 and the method of successive approximations yield the following theorem proved by Tani in [T2].

THEOREM 6.2. *Let* $F_0 \in W_2^{5/2+\alpha}(\mathbb{R}^2)$, $b \in W_2^{3/2+\alpha}(\mathbb{R}^2)$, $\alpha \in (1/2,1)$. *Assume that* $f \in W_2^{\alpha,\alpha/2}(\mathbb{R}^3\times(0,\infty)) \cap L_\infty(0,\infty;L_2(\mathbb{R}^3))$, $f_x \in W_2^{\alpha,0}(\mathbb{R}^3\times(0,\infty))$, f_x *is Lipschitz continuous in* x *and Hölder continuous in* t *with exponent* $1/2$. *Moreover assume that* $v_0 \in W_2^{1+\alpha}(\Omega)$ *and* v_0 *satisfies the compatibility conditions*

$$\nabla\cdot v_0 = 0,\ \mathbb{S}(v_0)\overline{n}_0 - (\mathbb{S}(v_0)\overline{n}_0\cdot\overline{n}_0)\overline{n}_0|_{S_F} = 0,\ v_0|_{S_B} = 0.$$

Then there exists a unique solution (u,q) *to problem* (6.8)–(6.12) *such that* $u \in W_2^{2+\alpha,1+\alpha/2}(\Omega^{T^*})$ *and* $q, \nabla q \in W_2^{\alpha,\alpha/2}(\Omega^{T^*}), q \in W_2^{\alpha+1/2,\alpha/2+1/4}(S_F^{T^*})$ *for some* $T^*>0$.

REMARK 6.1. In the case of $\sigma=0$, a theorem analogous to Theorem 6.2 holds with the assumption $F_0 \in W_2^{5/2+\alpha}(\mathbb{R}^2)$ replaced by a weaker one: $F_0 \in W_2^{3/2+\alpha}(\mathbb{R}^2)$.

In contrast to [B1], the regularity of the solution obtained by Tani [T2] is sharp, i.e. lowest possible, which is admissible for this problem in the L_2-approach. In [B1], Beale also first studies solvability of the above linear problem with $\sigma=0$ and then by using the Banach fixed point theorem he proves that there exists $T>0$ such that problem (6.8)–(6.12) has a solution with $u \in W_2^{l,l/2}(\Omega^T)$, $q \in W_2^{l-3/2,l/2-3/4}(S_F^T)$, $\nabla q \in W_2^{l-2,l/2-1}(\Omega^T)$. He obtains this result under the assumptions that $3<l<7/2$, $v_0 \in W_2^{l-1}(\Omega)$; ξ_3, the vertical coordinate in Ω, is in $W_2^{l-3/2}(S_F)$ and appropriate compatibility conditions are satisfied. Moreover, he assumes that Ω is the image of $\Sigma = \{\zeta=(\zeta_1,\zeta_2,\zeta_3): -h<\zeta_3<0\}$ under the diffeomorphism $\overline{h}(\zeta)=\zeta+h(\zeta)$, $h\in C^5$ and $\partial_\zeta^\alpha h \to 0$ as $|\zeta|\to\infty$ for $|\alpha|\le 5$.

In [B1] existence for arbitrary $T>0$ but for sufficiently small initial data in dependence on T is also proved. This result is obtained in the same class of functions as above.

The two-dimensional case of problem (6.1)–(6.7) with $\sigma > 0$ and $f = 0$ has been examined by Allain [Al1, Al2]. In this case $\Omega \subset \mathbb{R}^2$ is a domain bounded by $S_B = \{(\xi_1, \xi_2) \in \mathbb{R}^2 : \xi_2 = h_0(\xi_1)\}$ and $S_F = \{(\xi_1, \xi_2) \in \mathbb{R}^2 : \xi_2 = h(\xi_1)\}$. The papers of Allain were written some years earlier than paper [T2] concerned with the three-dimensional case. The main result of his papers is a local existence theorem. Under the assumptions that $0 < \alpha < 1/2$, $h \in W_2^{\alpha+5/2}(\mathbb{R})$, $v_0 \in W_2^{1+\alpha}(\Omega)$, $\nabla \cdot v_0 = 0$ in Ω, $v_0 = 0$ on S_B Allain proves that there exists $T > 0$ depending on Ω, α and v_0 such that problem (6.8)–(6.12) has a unique solution such that $u \in W_2^{2+\alpha,1+\alpha/2}(\Omega^T)$, $\nabla q \in W_2^{\alpha,\alpha/2}(\Omega^T)$, $q \in W_2^{\alpha/2+1/4}(0, T; L_2(S_F))$.

Teramoto [Ter1, Ter2] studies the motion of a viscous incompressible fluid which flows down an inclined plane under the effect of gravity. The fluid is bounded from below by a fixed plane which is inclined at an angle $0 < \varphi < \pi/2$ to the horizontal plane. In order to describe this problem Teramoto chooses the following orthogonal system of coordinates: the x_1 axis is in the direction of the greatest slope down the bottom, the x_2 axis is in a direction such that the x_1x_2 plane ($x_3 = 0$) is parallel to the bottom, and the x_3 axis is upward from the x_1x_2 plane. In these coordinates the domain Ω_t occupied by the fluid at time t is defined similarly as before, but it is assumed that $b(x') = b_0$, where $b_0 > 0$ is a constant. The motion of the fluid is described by the system (6.1)–(6.2) with $f = 0$, with e_3 in (6.1) replaced by $\gamma = (\sin\varphi, 0, -\cos\varphi)$ and with conditions (6.3)–(6.7). Applying the approach from [B1], Teramoto [Ter1] obtains for $\sigma = 0$ a small time existence result analogous to Beale's result of [B1].

Paper [Ter2] is concerned with the case of $\sigma > 0$. Using Beale's idea from [B1, B2] Teramoto transforms the domain Ω_t to a fixed domain $\Omega_1 \equiv \{(x_1', x_2', x_3') \in \mathbb{R}^3 : -1 < x_3' < 0\}$ and considers problem (6.1)–(6.7) in new coordinates x_1', x_2', x_3'. Denoting by $\overline{F}_0$, $\overline{v}_0$, $\overline{v}$, $\overline{p}$ and $\overline{F}$ the functions F_0, v_0, v, p and F written in the new coordinates and then using Beale's approach, Teramoto proves the following theorem.

Theorem 6.3. *Assume $\mathcal{R} < 1/(2\sqrt{3} + 16)$, where $\mathcal{R} = \varrho_0 b_0 V_0/\nu$ (ϱ_0 is the constant density of the fluid, $V_0 = \frac{1}{2}\frac{\varrho_0}{\nu} b_0^2 g \sin\varphi$). Let $T > 0$ be arbitrary, and assume that $3 < l < 7/2$. Then there exists $\delta > 0$ such that for $\overline{F}_0$ and $\overline{v}_0$ satisfying appropriate compatibility conditions at $x_3 = 0$ and $x_3 = -1$ and the estimate*

$$\|\overline{F}_0\|_{W_2^l(R^2)} + \|\overline{v}_0\|_{W_2^{l-1/2}(\Omega_1)} \le \delta,$$

the problem considered has a solution $(\overline{v}, \overline{p}, \overline{F})$ such that $\overline{F} \in W_2^{l+1/2,l/2+1/4}(\mathbb{R}^2 \times (0,T))$, $\overline{v} \in W_2^{l,l/2}(\Omega_1^T)$, $\nabla\overline{p} \in W_2^{l-2,l/2-1}(\Omega_1^T)$, $p|_{S_{1F}} \in W_2^{l-3/2,l/2-3/4}(\mathbb{R}^2 \times (0,T))$, where $S_{1F} = \{x_3' = 0\}$.

6.1.2. *Global existence and stablity.* Global existence theorems for problem (6.1)–(6.7) are subjects of papers [B2, TTan, Syl]. All of them assume that $f = 0$. Moreover, [BNis] brings an asymptotic decay rate for a global solution guaranteed by the existence result of [B2], and [Nis] gives a review of results concerning free boundary problems for equations of fluid dynamics including global existence theorems for problem (6.1)–(6.7).

Now, we will describe the results of [B2]. We formulate the global existence theorem proved in [B2], using the notation $K^l(\Omega^T) = W_2^{l,l/2}(\Omega^T)$.

THEOREM 6.4. *Suppose that* l, k *and* T_1 *are chosen with* $3 < l < 7/2$, $k > 0$, *and* $T_1 > 0$. *Moreover, let the following compatibility conditions be satisfied*:

$$\begin{aligned} \nabla \cdot v_0 &= 0 && \text{in } \Omega \equiv \{-b(x') < x_3 < F_0(x')\}, \\ \{(v_{0i,x_j} + v_{0j,x_i})n_j\}_{\tan} &= 0 && \text{on } S_F, \\ v_0 &= 0 && \text{on } S_B, \end{aligned}$$

where "tan" *means the tangential component. Then there exists* $\delta > 0$ *such that for* F_0, v_0 *satisfying*

$$\|F_0\|_{W_2^l(R^2)} + \|v_0\|_{W_2^{l-1/2}(\Omega)} \le \delta, \tag{6.17}$$

the following existence, uniqueness and regularity statements hold:

(i) *the problem* (6.1)–(6.7) *has a solution* F, v, p, *where* F *is in* $\widetilde{K}^{l+1/2}(\mathbb{R}^2 \times \mathbb{R}_+)$, *and* v, p *are restrictions to the fluid domain* Ω_t *of functions defined on* $\mathbb{R}^3 \times \mathbb{R}_+$ *with* $v \in K^l(\mathbb{R}^3 \times \mathbb{R}_+)$, $\nabla p \in K^{l-2}(\mathbb{R}^3 \times \mathbb{R}_+)$, $p \circ F \in \widetilde{K}^{l-3/2}(\mathbb{R}^2 \times \mathbb{R}_+)$. *Here* $(p \circ F)(x', t) = p(x', F(x', t), t)$;

(ii) *for any* $T > 0$ *this solution is unique in the class of* F, v, p *with* $F \in K^{l+1/2}(\mathbb{R}^2 \times (0, T))$ *and* v, p *the restrictions to* Ω_t *of functions* $v \in K^l(\mathbb{R}^3 \times (0, T))$, $\nabla p \in K^{l-2}(\mathbb{R}^3 \times (0, T))$, $p \circ F \in K^{l-3/2}(\mathbb{R}^2 \times (0, T))$;

(iii) *the solution satisfies*: $F \in \widetilde{K}^{l+k+1/2}(\mathbb{R}^2 \times (T_1, \infty))$, $v \in K^{l+k}(\mathbb{R}^3 \times (T_1, \infty))$, $\nabla p \in K^{l+k-2}(\mathbb{R}^3 \times (T_1, \infty))$, *and* $p \circ F \in \widetilde{K}^{l+k-3/2}(\mathbb{R}^2 \times (T_1, \infty))$. *In particular, if* $k \ge 2$, *all the equations are satisfied in the classical sense for* $t > T_1$.

To prove Theorem 6.4, first, the following linear problem is considered in [B2]:

$$u_t - \nu \Delta u + \nabla q = f_0 \quad \text{in } \Omega_1, \tag{6.18}$$

$$\operatorname{div} u = 0 \quad \text{in } \Omega_1, \tag{6.19}$$

$$F_t = u_3 \quad \text{on } S_{1F}, \tag{6.20}$$

$$u_{ix_3} + u_{3x_i} = f_i, \quad i = 1, 2, \quad \text{on } S_{1F}, \tag{6.21}$$

$$q - 2\nu u_{3x_3} - (gF - \sigma \Delta_{x'} F) = f_3 \quad \text{on } S_{1F}, \tag{6.22}$$

$$u = 0 \quad \text{on } S_{1B}, \tag{6.23}$$

$$u|_{t=0} = 0, \quad F|_{t=0} = 0 \quad \text{in } \Omega_1, \tag{6.24}$$

where Ω_1 is the equilibrium domain bounded by $S_{1F} = \{x_3 = 0\}$ and $S_{1B} = \{x_3 = -b(x')\}$, $\Delta_{x'} = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2$.

The following theorem holds for problem (6.18)–(6.24) (see [B2]).

THEOREM 6.5. *Suppose* f_0 *is given in* $K_{(0)}^{l-2}(\Omega_1 \times \mathbb{R}_+)$, $l \ge 2$ *and not a half-integer, and* $f_i \in K_{(0)}^{l-3/2}(S_{1F} \times \mathbb{R}_+)$, $i = 1, 2, 3$. *Then problem* (6.18)–(6.24) *has a solution* (F, u, q), *unique among finite energy solutions with* $F \in \widetilde{K}_{(0)}^{l+1/2}(S_{1F} \times \mathbb{R}_+)$, $u \in K_{(0)}^l(\Omega_1 \times \mathbb{R}_+)$, $\nabla q \in K_{(0)}^{l-2}(\Omega_1 \times \mathbb{R}_+)$, $q \in \widetilde{K}_{(0)}^{l-3/2}(S_{1F} \times \mathbb{R}_+)$. *With the indicated norms, the solution satisfies*

$$\|(F, u, q)\| \le C(\|f_0\|_{K^{l-2}} + \|(f_1, f_2, f_3)\|_{K^{l-3/2}}).$$

Sketch of proof of Theorem 6.4. To apply Theorem 6.5, new coordinates are introduced. In these new coordinates problem (6.1)–(6.7) is transformed to a problem in the equilibrium domain $\Omega_1 = \{x : x' \in \mathbb{R}^2, -b(x') < x_3 < 0\}$. Let $\widetilde{F}$ be an extension of F to $\Omega_1 \times \mathbb{R}_+$. Then for each t the transformation $\psi : \Omega_1 \to \Omega_t$ is defined by

$$\psi(x_1, x_2, x_3, t) = (x_1, x_2, \widetilde{F} + x_3(1 + \widetilde{F}/b(x'))). \tag{6.25}$$

Next, for u defined on Ω_1, the function v is introduced by the relation

$$v_i = \psi_{ix_j} u_j / J \equiv \alpha_{ij} u_j, \tag{6.26}$$

where $J = 1 + \widetilde{F}/b + \widetilde{F}_{x_3}(1 + x_3/b)$ is the Jacobian determinant of $\{\psi_{ix_j}\}$. Then by using (6.26), problem (6.1)–(6.7) can be rewritten as follows:

$$u_t - \nu \Delta u + \nabla q = G_0(F, u, \nabla q) \quad \text{in } \Omega_1 \times \mathbb{R}_+, \tag{6.27}$$

$$\operatorname{div} u = 0 \quad \text{in } \Omega_1 \times \mathbb{R}_+, \tag{6.28}$$

$$u_{ix_3} + u_{3x_i} = G_i(F, u), \quad i = 1, 2, \quad \text{on } S_{1F} \times \mathbb{R}_+, \tag{6.29}$$

$$q - 2\nu u_{3x_3} - gF + \sigma \Delta_{x'} F = G_3(F, u) \quad \text{on } S_{1F} \times \mathbb{R}_+, \tag{6.30}$$

$$F_t = u_3 \quad \text{on } S_{1F} \times \mathbb{R}_+, \tag{6.31}$$

$$u = 0 \quad \text{on } S_B \times \mathbb{R}_+, \tag{6.32}$$

$$u|_{t=0} = u_0, \quad F|_{t=0} = F_0 \quad \text{in } \Omega_1, \tag{6.33}$$

where $q = p \circ \psi$; $G_i (i = 0, \ldots, 3)$ are at least quadratic; $u_0 = v_0 \circ \psi_0$, and $\psi_0 = \psi|_{t=0}$ with $\widetilde{F}$ replaced by $\widetilde{F}_0 \in W_2^{l+1/2}(\Omega_1)$ which is an extension of F_0.

The Banach fixed point theorem, assumption (6.17) and Theorem 6.5 yield a unique solution (u, F, q) of problem (6.27)–(6.33) in the class of functions determined by Theorem 6.5.

To obtain assertion (i) of the theorem, the solution (u, F, q) is extended onto $\mathbb{R}^3 \times \mathbb{R}_+$ to functions $\widehat{u}$, $\widehat{F}$, $\widehat{q}$ such that $\widehat{u} \in K^l(\mathbb{R}^3 \times \mathbb{R}_+)$, $\nabla \widehat{q} \in K^{l-2}(\mathbb{R}^3 \times \mathbb{R}_+)$ and $\widehat{F} \in K^l(\mathbb{R}^3 \times \mathbb{R}_+)$, $\nabla \widehat{F} \in K^l(\mathbb{R}^3 \times \mathbb{R}_+)$, $\widehat{F}_t \in K^l(\mathbb{R}^3 \times \mathbb{R}_+)$. Then the restrictions to $\psi(\Omega_1)$ of functions $\widehat{v}_i = (\alpha_{ij} \widehat{u}_j) \circ \psi^{-1}$ and $\widehat{p} = \widehat{q} \circ \psi^{-1}$ and the function F satisfy assertion (i) of the theorem.

The next step of the proof is to show uniqueness of the solution. By using once again a contraction mapping argument it is proved that the solution found is unique in a class of functions of a slightly lower regularity.

The increase of regularity of the solution can be shown as follows. For given initial data $F_0 \in W_2^l(S_{1F})$, $u_0 \in W_2^{l-1/2}(\Omega_1)$, $3 < l < 7/2$, the unique solution of problem (6.27)–(6.33) with the properties: $F \in \widetilde{K}^{l+1/2}(S_{1F} \times \mathbb{R}_+)$, $u \in K^l(\Omega_1 \times \mathbb{R}_+)$, $q \in \widetilde{K}^{l-3/2}(S_{1F} \times \mathbb{R}_+)$, $\nabla q \in K^{l-2}(\Omega_1 \times \mathbb{R}_+)$ has been found. Hence, for almost all t_1, $F(t_1) \in W_2^{l+1/2}(S_{1F})$ and $u(t_1) \in W_2^l(\Omega_1)$. Thus, half a derivative is gained in comparison to the initial data. Now, one can stop at such t_1 and start again. This way one can construct a solution such that $F \in \widetilde{K}^{l+1}(S_{1F} \times (t_1, \infty))$, $u \in K^{l+1/2}(\Omega_1 \times \mathbb{R}_+)$, $q \in \widetilde{K}^{l-1}(S_{1F} \times \mathbb{R}_+)$, $\nabla q \in K^{l-3/2}(\Omega_1 \times \mathbb{R}_+)$, provided certain compatibility conditions are satisfied at t_1. Uniqueness implies that the new solution coincides with the previous one for $t > t_1$. Continuing the above procedure, the desired regularity of the solution can be reached in a specified time, provided the initial data is sufficiently small. This yields the increased regularity of the solution of problem (6.1)–(6.7). ■

Tani and Tanaka [TTan] also prove global existence for problem (6.1)–(6.7) in the case $\sigma > 0$. Applying the approach from the papers of Solonnikov [Sol6, Sol8, Sol10] and using Theorem 6.2 they prove the following theorem.

THEOREM 6.6. *Let the assumptions of Theorem* 6.2 *be satisfied with* $f = 0$. *Moreover, let* $E_0 \equiv \|v_0\|_{W_2^{1+\alpha}(\Omega)} + \sigma\|F_0\|_{W_2^{5/2+\alpha}(\mathbb{R}^2)} \le \varepsilon$ *with* ε *sufficiently small. Then the solution of problem* (6.1)–(6.7) *exists for all* $t > 0$ *and satisfies*

$$\sup_{t\ge t_1}(\|v\|_{W_2^{2+\alpha}(\Omega_t)} + \|v_t\|_{W_2^{\alpha}(\Omega_t)} + \|p\|_{W_2^{1+\alpha}(\Omega_t)} + \sigma\|F\|_{W_2^{5/2+\alpha}(\mathbb{R}^2)}) \le c_0(t_1)E_0 \quad \text{for each } t_1 > 0.$$

Sketch of proof

STEP 1. The first step is to derive the following estimate for the local solution guaranteed by Theorem 6.2:

$$\begin{aligned}(6.34)\quad E(0,T^*) &\equiv \|u\|_{W_2^{2+\alpha,1+\alpha/2}(\Omega^{T^*})} + \|\nabla q\|_{W_2^{\alpha,\alpha/2}(\Omega^{T^*})} + \|q\|_{W_2^{1/2+\alpha,1/4+\alpha/2}(S^{T^*})}\\ &\le c_1(\|v_0\|_{W_2^{1+\alpha}(\Omega)} + \sigma\|F_0\|_{W_2^{5/2+\alpha}(\mathbb{R}^2)}) \equiv c_1E_0.\end{aligned}$$

Moreover, T^* increases unboundedly as E_0 tends to zero.

STEP 2. Next, the following energy conservation law is derived:

$$(6.35)\qquad \frac{d}{dt}\Big(\int_{\Omega_t} |v|^2\,dx + 2\sigma\int_{\mathbb{R}^2}(\sqrt{1+|\nabla_{x'}F|^2}-1)\,dx' + g\int_{\mathbb{R}^2}|F|^2\,dx'\Big) + \nu E(v) = 0,$$

where $\nabla_{x'} = (\partial/\partial x_1, \partial/\partial x_2)$, $E(v) = \sum_{i,j=1}^3 \int_{\Omega_t}\big(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\big)^2\,dx$.

From (6.35) follows the equality

$$\begin{aligned}(6.36)\qquad &\int_{\Omega_t}|v|^2\,dx + 2\sigma\int_{\mathbb{R}^2}(\sqrt{1+|\nabla_{x'}F|^2}-1)\,dx' + g\int_{\mathbb{R}^2}|F|^2\,dx' + \nu\int_0^t E(v)\,dt'\\ &= \int_{\Omega}|v_0|^2\,d\xi + 2\sigma\int_{\mathbb{R}^2}(\sqrt{1+|\nabla_{x'}F_0|^2}-1)\,dx' + g\int_{\mathbb{R}^2}|F_0|^2\,dx'.\end{aligned}$$

STEP 3. The next step is to obtain estimates for a solution of the equation

$$(6.37)\qquad \sigma\nabla_{x'}\cdot\Big(\frac{\nabla_{x'}F}{\sqrt{1+|\nabla_{x'}F|^2}}\Big) - gF = \Phi(x') \quad \text{on } \mathbb{R}^2.$$

LEMMA 1. *Let* $\sigma > 0$ *and* $F(\cdot,t) \in W_2^{5/2+\alpha}(\mathbb{R}^2)$ *be a solution of* (6.37) *satisfying the condition*

$$\|F\|_{W_2^{3/2+\alpha}(R^2)} \le \delta$$

with a sufficiently small δ.

(i) *If* $\Phi \in W_2^{\alpha-1/2}(\mathbb{R}^2)$, *then*

$$(6.38)\qquad \|F\|_{W_2^{3/2+\alpha}(\mathbb{R}^2)} \le c_2\|\Phi\|_{W_2^{\alpha-1/2}(\mathbb{R}^2)} + c_3\|F\|_{L_2(\mathbb{R}^2)}.$$

(ii) *If* $\Phi \in W_2^{\alpha+1/2}(\mathbb{R}^2)$, *then*

$$\|F\|_{W_2^{5/2+\alpha}(\mathbb{R}^2)} \le c_4\|\Phi\|_{W_2^{\alpha+1/2}(\mathbb{R}^2)} + c_5\|F\|_{L_2(\mathbb{R}^2)}. \tag{6.39}$$

The constants c_3, c_5 *may depend on* $\|F\|_{W_2^{3/2+\alpha}(\mathbb{R}^2)}$.

The above lemma can be proved in the same way as Theorem 4 from [Sol6].

STEP 4. The following lemma can be proved.

LEMMA 2. *Let* $u \in W_2^{2+\alpha,1+\alpha/2}(\Omega^T)$, $\nabla q \in W_2^{\alpha,\alpha/2}(\Omega^T)$, $q|_{S^T} \in W_2^{1/2+\alpha,1/4+\alpha/2}(S^T)$ *be the solution of problem* (6.8)–(6.12) *satisfying*

$$c_6(T)\|u\|_{W_2^{2+\alpha,1+\alpha/2}(\Omega^T)} \le \delta_1$$

with sufficiently small δ_1, *where* $c_6(T)$ *is a given increasing function of* T *such that* $c_6(0) = 0$.

(i) *If* $\sigma > 0$, *then*

$$\begin{aligned} U(\lambda) &\equiv \|u\|_{W_2^{2+\alpha,1+\alpha/2}(Q(\lambda))} + \|\nabla q\|_{W_2^{\alpha,\alpha/2}(Q(\lambda))} + \|q\|_{W_2^{1/2+\alpha,1/4+\alpha/2}(G(\lambda))} \\ &\le c_7(\|u\|_{L_2(Q(0))} + \|F\|_{L_2(\mathbb{R}^2\times(t_0,T))}), \end{aligned}$$

where $\lambda \in (0,1)$, $t_0 + \lambda < T$; $Q(\lambda)$ *and* $G(\lambda)$ *are defined in Section* 4 *after formula* (4.49). *Moreover*, *for* $t_1 > t_0$,

$$\sup_{t_1<t<T}(\|u\|_{W_2^{2+\alpha}(\Omega)} + \|q\|_{W_2^{1+\alpha}(\Omega)}) \le c_8(\|u\|_{L_2(Q(0))} + \|F\|_{L_2(\mathbb{R}^2\times(t_0,T))}). \tag{6.40}$$

(ii) *If* $\sigma = 0$, *then*

$$U(\lambda) \le c_9(T)(\|F_0\|_{W_2^{1/2+\alpha}(\mathbb{R}^2)} + \|u\|_{L_2(Q(0))})$$

and

$$\sup_{t_1<t<T}(\|u\|_{W_2^{2+\alpha}(\Omega)} + \|q\|_{W_2^{1+\alpha}(\Omega)}) \le c_{10}(T)(\|F_0\|_{W_2^{1/2+\alpha}(\mathbb{R}^2)} + \|u\|_{L_2(Q(0))}).$$

STEP 5. As in papers [Sol6, Sol8, Sol10] estimates (6.34), (6.36), (6.38)–(6.40), Theorem 6.2 and the procedure of extending the solution applied infinitely many times yield the assertion of the theorem. ■

The paper of Sylvester [Syl] is also devoted to global solvability of problem (6.1)–(6.7). Applying the approach of Beale she proves a global existence theorem for the case $\sigma = 0$.

Finally, the global existence and stability theorem for the two-dimensional motion of a fluid flowing down an inclined plane is proved in [NTerW].

6.2. The motion of a compressible fluid. The only papers concerning such motion are [TanT] and [JinPad]. Let Ω_t denote the same unknown domain as in Section 6.1. The free boundary problem considered in [TanT] is to find Ω_t, a velocity vector field v, a density ϱ and a temperature θ satisfying the following system of equations with the boundary and initial conditions:

$$\varrho[v_t + (v\cdot\nabla)v] - \operatorname{div}\mathbb{T}(v,p) = -\varrho g e_3, \qquad x \in \Omega_t,\ t \in (0,T), \tag{6.41}$$

$$\varrho_t + \operatorname{div}(\varrho v) = 0, \qquad x \in \Omega_t,\ t \in (0,T), \tag{6.42}$$

$$\begin{aligned}
&(6.43) && \varrho c_v(\theta_t + v\cdot\nabla\theta) + \theta p_\theta \operatorname{div} v \\
&&& \quad - \nabla\cdot(\varkappa\nabla\theta) - \frac{\mu}{2}\sum_{i,j=1}^{3}(v_{ix_j}+v_{jx_i})^2 \\
&&& \quad - (\nu-\mu)(\operatorname{div} v)^2 = 0, && x\in\Omega_t,\ t\in(0,T),\\
&(6.44) && \mathbb{T}\overline{n} - \sigma H\overline{n} = -p_0\overline{n}, && x\in S_F(t),\ t\in(0,T),\\
&(6.45) && v\cdot\overline{n} = \frac{\partial F/\partial t}{\sqrt{1+|\nabla' F|^2}}, && x\in S_F(t),\ t\in(0,T),\\
&(6.46) && \varkappa\nabla\theta\cdot\overline{n} = \varkappa_a(\theta_a-\theta), && x\in S_F(t),\ t\in(0,T),\\
&(6.47) && v=0, \quad \theta=\theta_b, && x\in S_B,\ t\in(0,T),\\
&(6.48) && (\nu,\theta,\varrho)|_{t=0} = (v_0(x),\theta_0(x),\varrho_0(x)), && x\in\Omega,\\
&(6.49) && F|_{t=0} = F_0(x'), && x'\in\mathbb{R}^2,
\end{aligned}$$

where $\mathbb{T}$ is the stress tensor given by (5.7); $p=p(\varrho,\theta)$ is the pressure satisfying $p_\varrho>0$, $p_\theta>0$; $\mu=\mu(\varrho,\theta)$ and $\nu=\nu(\varrho,\theta)$ are the viscosity coefficients satisfying $\nu\ge\frac{1}{3}\mu$; $c_v=c_v(\varrho,\theta)$ is the specific heat at constant volume; $\varkappa=\varkappa(\varrho,\theta)$ is the coefficient of heat conductivity; $\varkappa_a>0$ is the constant coefficient of outer heat conductivity and θ_a, θ_b are positive functions.

Tanaka and Tani [TanT] formulate two theorems concerning problem (6.41)–(6.49), but they do not prove them. They look for a solution near the equilibrium state $(v,\theta,\varrho,F) = (0,\theta_e,\varrho_e,0)$, where θ_e is a positive constant and $\varrho_e=\varrho_e(x_3)$ is determined by $\int_{\varrho_e(0)}^{\varrho_e(x_3)}\frac{p_\varrho(\eta,\theta_e)}{\eta}\,d\eta + gx_3 = 0$, $p(\varrho_e(0),\theta_e)=p_0$.

The first result is a local solvability theorem. Tanaka and Tani claim that under the assumptions that $\alpha\in(1/2,1)$, $b\in W_2^{5/2+\alpha}(\mathbb{R}^2)$; v_0, $\theta_0-\theta_e$, $\varrho_0-\varrho_e\in W_2^{2+\alpha}(\Omega)$; $F_0\in W_2^{7/2+\alpha}(\mathbb{R}^2)$; $\theta_a-\theta_e\in W_2^{4+\alpha,2+\alpha/2}(\mathbb{R}^3\times(0,T))$, $\theta_b-\theta_e\in W_2^{5/2+\alpha,5/4+\alpha/2}(S_B^T)$, $\varrho_0>0$, $\theta_0>0$, $\theta_a>0$ and under suitable compatibility conditions, there exists $T^*>0$ such that there exists a unique solution u, $\vartheta-\theta_e\in W_2^{3+\alpha,3/2+\alpha/2}(\Omega^{T^*})$, $\eta-\varrho_e\in W_2^{2+\alpha,1+\alpha/2}(\Omega^{T^*})$ (where u, ϑ, η denote v, θ, ϱ respectively, written in Lagrangian coordinates) and

$$\begin{aligned}
&\|(u,\vartheta-\theta_e)\|_{W_2^{3+\alpha,3/2+\alpha/2}(\Omega^{T^*})} + \|\eta-\varrho_e\|_{W_2^{2+\alpha,1+\alpha/2}(\Omega^{T^*})}\\
&\qquad \le c_1(\|(v_0,\theta_0-\theta_e,\varrho_0-\varrho_e)\|_{W_2^{2+\alpha}(\Omega)} + \|F_0\|_{W_2^{7/2+\alpha}(\mathbb{R}^2)}\\
&\qquad\qquad + \|\theta_a-\theta_e\|_{W_2^{4+\alpha,2+\alpha/2}(\mathbb{R}^3\times(0,T))} + \|\theta_b-\theta_e\|_{W_2^{5/2+\alpha,5/4+\alpha/2}(S_B^T)}) \equiv c_1E_{0,T^*}.
\end{aligned}$$

Moreover, $T^*\to\infty$ as $E_{0,T^*}\to 0$. Next, they assert that for $E_0\equiv E_{0,\infty}\le\varepsilon$ with ε sufficiently small, problem (6.41)–(6.49) has a unique solution (v,θ,ϱ,F) for all $t>0$ satisfying

$$\sup_{t\ge t_1}(\|(v,\theta-\theta_e)\|_{W_2^{3+\alpha}(\Omega_t)} + \|\varrho-\varrho_e\|_{W_2^{2+\alpha}(\Omega_t)} + \|F\|_{W_2^{7/2+\alpha}(\mathbb{R}^2)}) \le c_2E_0 \quad \text{for every } t_1>0.$$

The paper [JinPad] of Jin and Padula is devoted to the periodic motion of a compressible isothermal viscous gas in a domain $\Omega_t=\{(x',x_3): x'\in\mathbf{T}^2,\ 0<x_3<F(x',t)\}$. They consider system (6.41)–(6.42) with boundary conditions (6.44)–(6.45) on the free boundary $S_F(t)=\{(x',x_3): x'\in\mathbf{T}^2,\ x_3=F(x',t)\}$, with the following condition on S_B:

(6.50) $$v = 0, \quad x \in S_B = \{(x', x_3) : x' \in \mathbf{T}^2,\ x_3 = 0\},$$

and with the initial conditions

(6.51) $$F|_{t=0} = F_0(x'), \qquad x' \in \mathbf{T}^2,$$

(6.52) $$(v, \varrho)|_{t=0} = (v_0(x), \varrho_0(x)), \quad x \in \Omega \equiv \Omega_0.$$

It is assumed that $p = a\varrho$, where $a > 0$ is a constant.

The equilibrium state is here defined by

$$v_e = 0, \quad \varrho_e = \varrho_* e^{-(g/a)x_3}, \quad F_e = \frac{a}{g} \ln\left(1 + \frac{Mg}{p_0|\mathbf{T}^2|}\right)$$

with $\int_{\mathbf{T}^2 \times (0,F_e)} \varrho_e\, dx = M$, where $M = \int_\Omega \varrho_0\, dx$ is the mass of the fluid which is conserved in view of (6.42) and $\varrho_* = \frac{p_0}{a}\big(1 + \frac{Mg}{p_0|\mathbf{T}^2|}\big)$.

Let $\varrho_\sigma = \varrho - \varrho_e$, $F_\sigma = F - F_e$ and

$$\begin{aligned}\mathcal{X}_T = \{(v, \varrho_\sigma, F_\sigma) : {}& v \in L_\infty(0,T; W_2^2(\Omega_t)) \cap L_2(0,T; W_2^3(\Omega_t)),\\ & \varrho_\sigma \in L_\infty(0,T; W_2^2(\Omega_t)) \cap L_2(0,T; W_2^2(\Omega_t)),\\ & F_\sigma \in L_\infty(0,T; W_2^3(\mathbf{T}^2)) \cap L_2(0,T; W_2^3(\mathbf{T}^2)),\\ & \mathcal{N}(t) \le 1,\ |F_\sigma| < F_e/4,\ |\varrho_\sigma| < \tfrac14 \varrho_e(\tfrac54 F_e),\ 0 < t < T\},\end{aligned}$$

where $\mathcal{N}(t) \equiv \|\varrho_\sigma\|^2_{W_2^2(\Omega_t)} + \|v\|^2_{W_2^2(\Omega_t)} + \|F_\sigma\|^2_{W_2^3(\mathbf{T}^2)}$. The main result of [JinPad] is the following global existence theorem.

THEOREM 6.7. *Let $S = \partial\Omega$ be the graph of the function $F_e + F_0(x')$, $x' \in \mathbf{T}^2$. Let $(v_0, \varrho_0, F_0) \in W_2^2(\Omega) \times W_2^2(\Omega) \times W_2^3(\mathbf{T}^2)$ and let the following compatibility conditions be satisfied*:

$$v_0 = 0 \quad \text{on } x_3 = 0,$$

$$\mathbf{T}(v_0, a\varrho_0)\overline{n}_0 = (\sigma H(0) - p_0)\overline{n}_0 \quad \text{on } x_3 = F_0(x'),$$

$$\int_{\mathbf{T}^2} \int_0^{F_0(x')} \varrho_0(x)\, dx_3\, dx' = M.$$

Moreover, assume that

(6.53) $$\|v_0\|_{W_2^2(\Omega)} + \|\varrho_0 - \varrho_e\|_{W_2^2(\Omega)} + \|F_0 - F_e\|_{W_2^3(\mathbf{T}^2)} \le \varepsilon$$

with $\varepsilon > 0$ sufficiently small. Then problem (6.41)–(6.42), (6.44)–(6.45), (6.50)–(6.52) *has a unique solution $(v, \varrho_\sigma, F_\sigma) \in \mathcal{X}_T$ for all $T < \infty$, satisfying the inequalities*

$$\begin{aligned}&\|\varrho_\sigma\|^2_{W_2^2(\Omega_t)} + \|v\|^2_{W_2^2(\Omega_t)} + \|F_\sigma\|^2_{W_2^3(\mathbf{T}^2)}\\ &\qquad \le c(\|\varrho_0 - \varrho_e\|^2_{W_2^2(\Omega)} + \|v_0\|^2_{W_2^2(\Omega)} + \|F_0 - F_e\|^2_{W_2^3(\mathbf{T}^2)})e^{-bt} \quad \text{for } t \in (0,\infty)\end{aligned}$$

and

$$\begin{aligned}&\int_0^t (\|\varrho_\sigma\|^2_{W_2^2(\Omega_{t'})} + \|v\|^2_{W_2^3(\Omega_{t'})} + \|F_\sigma\|^2_{W_2^3(\mathbf{T}^2)})\, dt'\\ &\qquad \le c(\|\varrho_0 - \varrho_e\|^2_{W_2^2(\Omega)} + \|v_0\|^2_{W_2^2(\Omega)} + \|F_0 - F_e\|^2_{W_2^3(\mathbf{T}^2)}) \quad \text{for } t \in (0,\infty),\end{aligned}$$

with some positive constants b and c independent of t.

The method of proof of Theorem 6.7 is similar to the methods applied for free boundary problems discussed in Section 5 and relies on deriving an appropriate differential inequality which allows extending the local solution for $t \in \mathbb{R}_+$. Thus, to obtain global existence for problem (6.41)–(6.42), (6.44)–(6.45), (6.50)–(6.52) Jin and Padula assume that the solution exists locally in $\mathcal{X}_T$ for some $T > 0$. Then they prove the inequality

$$\frac{d}{dt}\overline{\varphi} + \overline{\Phi} \le c_1\sqrt{\varphi}\,\Phi \quad \text{for } t \le T, \tag{6.54}$$

where

$$\varphi(t) = \|\varrho_\sigma\|^2_{W^2_2(\Omega_t)} + \|v\|^2_{W^2_2(\Omega_t)} + \|F\|^2_{W^3_2(\mathbf{T}^2)} + \|\varrho_{\sigma t}\|^2_{L_2(\Omega_t)} + \|v_t\|^2_{L_2(\Omega_t)} + \|F_t\|^2_{W^1_2(\mathbf{T}^2)},$$

$$\Phi(t) = \|\varrho_\sigma\|^2_{W^2_2(\Omega_t)} + \|v\|^2_{W^3_2(\Omega_t)} + \|v_t\|^2_{W^1_2(\Omega_t)} + \|F\|^2_{W^3_2(\mathbf{T}^2)};$$

$\overline{\varphi}$ and $\overline{\Phi}$ are certain functions equivalent to φ and Φ respectively, i.e.

$$c_2\overline{\varphi} \le \varphi \le c_3\overline{\varphi}, \tag{6.55}$$

$$c_4\overline{\Phi} \le \Phi \le c_5\overline{\Phi} \tag{6.56}$$

for some positive constants c_i $(i = 1, \dots, 4)$ which are independent of t.

Since $\overline{\Phi} \ge c_6\overline{\varphi}$, the differential inequality (6.54) together with (6.55)–(6.56) and the assumption (6.53) yield the assertion of Theorem 6.7.

A global existence and asymptotic result for the linearization of the barotropic problem without surface tension about an equilibrium solution, i.e. problem (6.41)–(6.42), (6.44)–(6.45) with $\sigma = 0$, (6.50)–(6.52) can be found in [St].

7. Final discussion

7.1. Differences in approach to drop problems and surface waves problems. In the previous sections we have described two different free boundary problems with respect to the geometry of the domain Ω_t, i.e. drop problems and surface waves problems. In a drop problem, Ω_t is a bounded domain of $\mathbb{R}^n$ $(n = 2, 3)$ with boundary S_t, all of which is free. On the other hand, a surface waves problem is considered in an unbounded domain $\Omega_t \subset \mathbb{R}^n$, the boundary of which consists of two parts: S_1, the fixed part of the boundary S_t, and S_{2t}, the free part of S_t depending on time t.

This difference in the geometry of Ω_t as well as in the nature of its boundary S_t causes some differences in the approach to the above mentioned problems. This is apparent already in the first papers concerning the free boundary problems discussed, i.e. in Beale's paper [B2], the first one devoted to the global existence result for surface waves problem (6.1)–(6.7), and in Solonnikov's paper [Sol6], the first to bring a global existence theorem for a drop problem.

Usually, the general approach to such free boundary problems is to transform a given problem to a problem in a fixed domain. There are two possible such transformations. One of them bases on introducing Lagrangian coordinates (4.10) and this way transforming the problem to the initial domain. In the other, one transforms the problem to an equilibrium domain. In [B2] Beale used the second method and as a consequence examined the surface waves problem (6.1)–(6.7) with $\sigma > 0$ in the equilibrium domain $\Omega_1 = \{x : x' \in \mathbb{R}^2, -b(x') < x_3 < 0\}$. He applied transformation (6.25) which is intimately connected with the surface waves problems, i.e. with the special geometry of the domain Ω_t and with the fact that one part of the boundary S_t is fixed. In the case of drop problems, starting from paper [Sol6], the transformation connecting Eulerian and Lagrangian coordinates is usually used. Following [Sol6], most of the existence results for drop problems both in the incompressible and compressible cases were obtained by using at least partly Lagrangian coordinates.

Solonnikov and Tani [SolT3] applied transformation (5.62) which transformed the drop problem (5.1)–(5.5) to a problem in the equilibrium domain $B_{R_e} = \{y : |y| < R_e\}$. However, it should be underlined that this transformation was used to obtain only one estimate useful in the proof of global existence, i.e. the estimate from Step 4 of the proof of Theorem 5.5. The other parts of the proof of Theorem 5.5 are based on using either Eulerian or Lagrangian coordinates.

On the other hand, it turned out that surface waves problems are more "universal" to treat because they can be considered equally easily in the equilibrium domain Ω_1 and in the initial domain $\Omega_0 \equiv \Omega$. In fact, Tanaka and Tani [TTan] proved a global existence for problem (6.1)–(6.7) with $\sigma > 0$ (i.e. the problem studied by Beale) by using the approach from [Sol6].

What is the difference between these two approaches to problem (6.1)–(6.7)? The most characteristic feature of Beale's method is that thanks to transforming the problem to the domain Ω_1 it is possible to obtain immediately global existence and uniqueness for the surface waves problem. It is only necessary to assume that the initial data are sufficiently close to an equilibrium state and to use the Banach fixed point theorem. In contrast to Beale's approach, Solonnikov's method applied to the same surface waves problem (see [TTan]) relies on rewriting the problem in Lagrangian coordinates $\xi \in \Omega$ and proving first local existence and uniqueness by means of the Banach fixed point theorem. This yields the existence of a solution in the interval $[0, T]$. Then one passes to new Lagrangian coordinates $\xi \in \Omega_T$, i.e. one transforms the problem to the domain Ω_T. For such a problem the local existence of a solution is proved in the interval $[T, 2T]$. This way, the local solution is extended step by step to a global one. Obviously, this process of prolongation is possible under the assumption that the initial data are sufficiently close to an equilibrium state.

Finally, a very special surface waves problem is worth mentioning: the problem considered in the domain $\Omega_t = \{(x', x_3) : x' \in \mathbf{T}^2, 0 < x_3 < F(x', t)\}$. Thanks to such a choice of the fixed part S_B of the boundary S_t this problem can be examined in Eulerian coordinates without transforming Ω_t to a fixed domain (see [JinPad]).

7.2. Differences in approach to incompressible and compressible problems. In this subsection we want to compare the methods applied to examine solvability of incompressible problems with those used in the compressible case. We will concentrate on the case of a fixed mass of a fluid bounded by a free surface, i.e. on a drop problem.

The differences are already apparent in the proofs of local existence and they are associated with the dissimilarity of continuity equations in the two cases. The general method of proving local existence and uniqueness seems to be the same in both cases. Namely, we consider several auxiliary linear problems for which we obtain existence and uniqueness. Then in order to obtain local existence for a nonlinear problem we usually use the method of successive approximations together with solvability of an appropriate linear problem. However, in the incompressible case, in contrast to the compressible one, the continuity equation is taken into account from the very beginning, i.e. one has to consider first the Stokes problem (4.15)–(4.18) for which Theorem 4.1 holds or problem (4.15)–(4.16), (4.18), (4.23)–(4.24) in the case of $\sigma > 0$. If $\sigma = 0$ then Theorem 4.1 is directly used to get local existence and uniqueness for problem (4.11)–(4.14) (and so (4.1)–(4.5)). In the case of $\sigma > 0$, the solvability result for problem (4.15)–(4.16), (4.18), (4.23)–(4.24) is first applied to prove a local in time existence and uniqueness for the linear problem (4.26)–(4.30) (see Theorem 4.4). Then Theorem 4.4 is used to show local existence and uniqueness of a solution to problem (4.1)–(4.5).

On the other hand, in the case of a compressible fluid, the continuity equation is at first excluded from the considerations. Thus, to prove local existence, one has to consider several auxiliary linear parabolic problems. The hyperbolic continuity equation (5.10) is taken into account only when applying the procedure of successive approximations (see for example Theorem 5.1).

In particular, for the most general compressible problem, i.e. problem (5.63)–(5.69) with μ, ν, $\varkappa$ depending on ϱ and θ we consider separately two kinds of auxiliary linear parabolic problems. One of them is connected with the equation of motion (5.63) and the other one with equation (5.65). After considering some initial auxiliary linear problems we finally study the following parabolic problems (see [Z2]):

$$\eta u_t - \operatorname{div}_w \mathbb{D}_w(u) = F \quad \text{in } \Omega^T, \tag{7.1}$$

$$\mu(\eta,\gamma)\Pi_0\Pi_w\mathbb{S}_w(u)\overline{n}_w = \Pi_0 G_1 \quad \text{in } S^T, \tag{7.2}$$

$$\overline{n}_0\cdot\mathbb{S}_w(u)\overline{n}_w - \sigma\overline{n}_0\cdot\Delta_w(t)\int_0^t u\,dt' = G_2 \quad \text{on } S^T, \tag{7.3}$$

$$u|_{t=0} = v_0 \quad \text{in } \Omega, \tag{7.4}$$

where w and η are given functions, $\mathbb{D}_w(u) = 2\mu(\eta,\gamma)\mathbb{S}_w(u) + (\nu(\eta,\gamma) - \mu(\eta,\gamma))\operatorname{div}_w uI$, $\mathbb{S}_w(u) = \frac{1}{2}\{\partial_{x_i}\xi_k\partial_{\xi_k}u_j + \partial_{x_j}\xi_k\partial_{\xi_k}u_i\}_{i,j=1,2,3}$, $\operatorname{div}_w \mathbb{D}_w(u) = \{\partial_{x_j}\xi_k\partial_{\xi_k}D_{wij}(u)\}_{i=1,2,3}$; Π_0, Π_w, $\overline{n}_w$, ∇_w, Δ_w are defined in Subsection 4.1, $G_2 = G_2^{(1)} + \sigma\int_0^t G_2^{(2)}\,dt'$ and

$$\eta c_v(\eta,\gamma)\vartheta_t - \operatorname{div}_w(\varkappa(\eta,\gamma)\nabla_w\vartheta) = K \quad \text{in } \Omega^T, \tag{7.5}$$

$$\varkappa(\eta,\gamma)\overline{n}_0\cdot\nabla_w\vartheta = \overline{\vartheta} \quad \text{on } S^T, \tag{7.6}$$

$$\vartheta|_{t=0} = \theta_0 \quad \text{in } \Omega, \tag{7.7}$$

where w,γ and η are given functions.

In [Z2] the author proves the local existence and uniqueness of a solution of problem (7.1)–(7.4) such that $u \in W_2^{2+\alpha,1+\alpha/2}(\Omega^T)$, $\alpha \in (3/4, 1)$ and u satisfies

$$
(7.8)\qquad \|u\|_{\Omega^T}^{(\alpha+2,\alpha/2+1)} \le \overline{\phi}_1(T, |1/\eta|_{L_\infty(\Omega^T)}, |\eta|_{L_\infty(\Omega^T)}) \cdot (\|F\|_{\Omega^T}^{(\alpha,\alpha/2)} + \|G_1\|_{W_2^{1/2+\alpha,1/4+\alpha/2}(S^T)} + \|G_2^{(1)}\|_{W_2^{1/2+\alpha,1/4+\alpha/2}(S^T)} + \|G_2^{(2)}\|_{S^T}^{(\alpha-1/2,\alpha/2-1/4)} + \|v_0\|_{W_2^{1+\alpha}(\Omega)}),
$$

where $\overline{\phi}_1$ is a positive continuous nondecreasing function of its arguments.

The above result is obtained under the assumptions that: $\eta \in C([0,T]; W_2^{1+\alpha}(\Omega)) \cap W_2^{1+\alpha,1/2+\alpha/2}(\Omega^T)$, $1/\eta \in L_\infty(\Omega^T)$, $\gamma \in W_2^{2+\alpha,1+\alpha/2}(\Omega^T)$, $w \in W_2^{2+\alpha,1+\alpha/2}(\Omega^T)$ (where T is sufficiently small in dependence on norms of η, w, γ) and under some other assumptions.

A similar result is derived in [Z2] for problem (7.5)–(7.7). The local solution of (7.5)–(7.7) satisfies

$$
(7.9)\qquad \|\vartheta\|_{\Omega^T}^{(\alpha+2,\alpha/2+1)} \le \overline{\phi}_2(T, |1/\eta c_v(\eta,\gamma)|_{L_\infty(\Omega^T)}, |\eta c_v(\eta,\gamma)|_{L_\infty(\Omega^T)}) \cdot (\|K\|_{\Omega^T}^{(\alpha,\alpha/2)} + \|\overline{\vartheta}\|_{W_2^{1/2+\alpha,1/4+\alpha/2}(S^T)} + \|\theta_{\sigma 0}\|_{W_2^{1+\alpha}(\Omega)}),
$$

where $\overline{\phi}_2$ is a positive continuous nondecreasing function of its arguments.

Independently of problems (7.1)–(7.4) and (7.5)–(7.7), the continuity equation (5.71) is considered. From (5.71) it follows that

$$
\eta = \varrho_0 \exp\Big(-\int_0^t \nabla_u \cdot u\, dt'\Big). \tag{7.10}
$$

A local existence and uniqueness theorem for nonlinear problem (5.63)–(5.69) is proved by using the method of successive approximations together with estimates (7.8)–(7.9) and together with an appropriate estimate for η given by (7.10).

Now, we want to discuss differences and similarities occurring in the proofs of global existence theorems in the incompressible and compressible cases. Assume that $\sigma > 0$, $k = 0$ and consider the main steps of the proof of Theorem 4.8 which yields global existence and stability for the incompressible problem (4.1)–(4.5). As usual, the successive steps of the proof rely on deriving some estimates for the local solution.

First, the conservation laws of energy (4.73) and momentum (4.72) are used to estimate the norms $\|v\|_{L_2(\Omega_t)}$ and $\|R - R_0\|_{W_2^1(S^1)}$ by the initial data. Here $R_0 \equiv R_e = \big(\frac{3}{4\pi}|\Omega|\big)^{1/3} = \big(\frac{3}{4\pi}|\Omega_t|\big)^{1/3}$ and $R = R(\omega, t)$ is the function describing the free boundary (see (7.25)). The remaining three estimates are obtained for a solution of problem (4.1)–(4.5) written in Lagrangian coordinates, i.e. for a solution of (4.11)–(4.14). All of them are derived by using only estimate (4.25) which holds for a solution of the auxiliary linear problem (4.26)–(4.30). To derive the first of these estimates, i.e. estimate (4.67), one has to treat the nonlinear problem (4.11)–(4.14) as the linear problem (4.26)–(4.30) with $w = u$. Then (4.25) yields inequality (4.67) which is the estimate of the norm $\|u\|_{W_2^{2+\alpha,1+\alpha/2}(\Omega^T)}$ (where T is the time of local existence) and of appropriate norms of q by the norms of the initial data, i.e. by $\|v_0\|_{W_2^{1+\alpha}(\Omega)} + \|H(\cdot,0) + 2/R_0\|_{W_2^{1/2+\alpha}(S)}$. Under the assumption that

$$
\|v_0\|_{W_2^{1+\alpha}(\Omega)} + \|\widetilde{R} - R_0\|_{W_2^{5/2+\alpha}(S^1)} \le \varepsilon,
$$

where $\widetilde{R}$ is defined by (7.24) and ε is sufficiently small, inequality (4.67) yields

$$T^{1/2}\|u\|_{W_2^{2+\alpha,1+\alpha/2}(\Omega^T)} \le \delta, \tag{7.11}$$

where δ is sufficiently small in dependence on ε.

The next step is to obtain an estimate of the norms of the local solution u and q, determined by Theorem 4.5, by the L_2-norms of u and $R-R_0$. This is done by applying again estimate (4.25) to the linear problem (4.76)–(4.80) which has the form of (4.26)–(4.30) with w replaced by u and u replaced by $u_\lambda \equiv u\zeta_\lambda$, where $\zeta_\lambda \in C^\infty(\mathbb{R})$, $\zeta_\lambda(t) = 1$ for $t \ge t_0 + \lambda$, $\zeta_\lambda(t) = 0$ for $t \le t_0 + \lambda/2$. As a result, inequality (4.75) is derived. In the process of deriving inequality (4.75) one has to estimate some nonlinear terms with respect to $\|u\|_{W_2^{2+\alpha,1+\alpha/2)}(\Omega^T)}$. To do this inequality (7.11) is used and therefore the right-hand side of (4.75) is only the sum of the L_2-norms of u and $R-R_0$, multiplied by a constant which is a nondecreasing function of T.

The last estimate, i.e. inequality (4.86) showing the increase of regularity of the solution u and $q-q_0$ after some time, is derived by applying estimate (4.25) to a linear problem having the form of (4.26)–(4.30) with unknown functions $u^{(s)}(\xi,t) = u_\lambda(\xi,t)-u_\lambda(\xi,t-s)$, $q^{(s)}(\xi,t) = q_\lambda(\xi,t)-q_\lambda(\xi,t-s)$ and with $w = u$. Thanks to inequality (4.86) and Theorem 7.1 one can estimate the norm $\|R-R_0\|_{W_2^{5/2+\alpha}(S^1)}$ with $\alpha \in (1/2,1)$ by the L_2-norms of u and $R-R_0$, and this way control the free boundary of the fluid.

It is proved (see [Sol13]) that the time of local existence is sufficiently large if ε is sufficiently small. Therefore choosing ε sufficiently small the local existence follows in the interval $(0,1]$. The solution can be extended to the interval $[1,2]$ by using the estimates described above. Continuing this process the solution can be extended for all $t > 0$.

Together with global existence one obtains the stability of the equilibrium state. Thus, the velocity of the fluid v remains small, the pressure remains close to $2\sigma/R_0$ and the free boundary S_t remains close to the sphere of radius R_0 for all $t > 0$.

Now, consider a general compressible problem (5.63)–(5.69) with $\sigma > 0$ and $k = 0$, $f = 0$. To compare global solvability of this problem with the incompressible case we will concentrate on the proof of Theorem 5.11 which is presented in Section 5. From that proof we can see that to show global existence in this case, we need similar estimates as in the incompressible case, although the global solvability results guaranteed by Theorems 4.8 and 5.11 are obtained in spaces of functions of different regularity. However, the way of obtaining the above mentioned estimates for a compressible fluid is different due to the different nature of the continuity equations in both cases.

As in the incompressible case, to obtain an estimate of the norm $\|v\|_{L_2(\Omega_t)}$ we use the energy conservation law (5.100). However, to be able to use it we have to prove first that under some assumptions on the data, the volume of the fluid $|\Omega_t|$ does not change much for $t \le T$ (see Lemma 5.3).

In a way similar to but more complicated than in the incompressible case we obtain an estimate of $\|R-R_e\|_{W_2^1(S^1)}$ (see Lemmas 5.7–5.9), where R_e is given in Definition 5.4.

The next difference is in the necessity of obtaining not only an estimate of the L_2-norm of v but also estimates of ϱ_σ and θ_σ in $L_2(\Omega_t)$ (see Lemma 5.5).

Next, following the argument presented by Solonnikov for the incompressible fluid we would like to derive an estimate analogous to (4.67), i.e. an estimate of the $W_2^{4,2}$-norms of u and ϑ_σ together with an estimate of $\sup_{0\le t\le T}\|\eta_\sigma\|_{W_2^3\Omega)}+\sup_{0\le t\le T}\|\eta_{\sigma t}\|_{W_2^2(\Omega)}+\|\eta_{\sigma t}\|_{L_2(0,T;W_2^3(\Omega))}+\|\eta_{\sigma tt}\|_{L_2(0,T;W_2^1(\Omega))}$ by the sum of the W_2^3-norms of the initial functions v_0, $\varrho_{\sigma 0}=\varrho_0-\varrho_e$, $\theta_{\sigma 0}=\theta_0-\theta_e$, the W_2^1-norms of $u_t(0)$, $\vartheta_{\sigma t}(0)$, the norms $\|H(\cdot,0)+2/R_e\|_{W_2^{2+1/2}(S)}$, $\|h\|_{W_2^{2,1}(\Omega^T)}$, $\|\overline{\vartheta}\|_{W_2^{3-1/2,3/2-1/4}(S^T)}$ and $\big(\int_0^T\frac{\|D^2_{\xi,t}\overline{\vartheta}\|_{L_2(S)}}{t^{1/2}}\,dt\big)^{1/2}$. Such an estimate is easily obtained by using inequalities which hold for problems (7.1)–(7.4) and (7.5)–(7.7) and which are analogous to (7.8)–(7.9), i.e. the inequalities (see [ZZaj1])

$$\begin{aligned}(7.12)\qquad \|u\|_{W_2^{4,2}(\Omega^T)}\le{}&\overline{\phi}_3(T,|1/\eta|_{L_\infty(\Omega^T)},|\eta|_{L_\infty(\Omega^T)})\\ &\cdot\bigg[\|F\|_{W_2^{2,1}(\Omega^T)}+\|G_1\|_{W_2^{3-1/2,3/2-1/4}(S^T)}+\|G_2^{(1)}\|_{W_2^{3-1/2,3/2-1/4}(S^T)}\\ &+\bigg(\int_0^T\frac{\|D^2_{\xi,t}G_1\|_{L_2(S)}}{t^{1/2}}\,dt\bigg)^{1/2}+\bigg(\int_0^T\frac{\|D^2_{\xi,t}G_2^{(1)}\|_{L_2(S)}}{t^{1/2}}\,dt\bigg)^{1/2}\\ &+\|G_2^{(2)}\|_{W_2^{2-1/2,1-1/4}(S^T)}+\|v_0\|_{W_2^3(\Omega)}+\|u_t(0)\|_{W_2^1(\Omega)}\bigg],\end{aligned}$$

and

$$\begin{aligned}(7.13)\qquad \|\vartheta_\sigma\|_{W_2^{4,2}(\Omega^T)}\le{}&\overline{\phi}_4\bigg(T,\bigg|\frac{1}{\eta c_v(\eta,\gamma)}\bigg|_{L_\infty(\Omega^T)},|\eta c_v(\eta,\gamma)|_{L_\infty(\Omega^T)}\bigg)\\ &\cdot\bigg[\|K\|_{W_2^{2,1}(\Omega^T)}+\|\overline{\vartheta}\|_{W_2^{3-1/2,3/2-1/4}(S^T)}+\bigg(\int_0^T\frac{\|D^2_{\xi,t}\overline{\vartheta}\|_{L_2(S)}}{t^{1/2}}\,dt\bigg)^{1/2}\\ &+\|\theta_{\sigma 0}\|_{W_2^3(\Omega)}+\|\vartheta_{\sigma t}(0)\|_{W_2^1(\Omega)}\bigg]\end{aligned}$$

where $\overline{\phi}_3$ and $\overline{\phi}_4$ depend also on higher norms of η and $\eta c_v(\eta,\gamma)$. Notice that problem (5.70)–(5.75) with $g=0$ and $k=0$ implies the following problems:

$$(7.14)\qquad \eta u_t-\operatorname{div}_u\mathbb{D}(u)=\nabla_u p_\sigma \qquad \text{in } \Omega^T,$$

$$(7.15)\qquad \mu(\eta,\vartheta)\Pi_0\Pi_u\mathbb{S}_u(u)=0 \qquad \text{on } S^T,$$

$$\begin{aligned}(7.16)\qquad &\overline{n}_0\cdot\mathbb{D}_u(u)\overline{n}_u-\sigma\overline{n}_0\cdot\Delta_u(t)\int_0^t u\,dt'=\overline{n}_0\cdot\overline{n}_u p_\sigma\\ &\quad+\sigma\overline{n}_0\cdot(\Delta_u(t)-\Delta_u(0))\xi+\sigma(H(\xi,0)+2/R_e) \qquad \text{on } S^T,\end{aligned}$$

$$(7.17)\qquad u|_{t=0}=v_0 \qquad \text{in } \Omega$$

and

$$\begin{aligned}(7.18)\qquad &\eta c_v(\eta,\vartheta)\vartheta_{\sigma t}-\nabla_u\cdot(\varkappa(\eta,\vartheta)\nabla_u\vartheta_\sigma)=-\vartheta p_\vartheta(\eta,\vartheta)\nabla_u\cdot u\\ &\quad+\frac{\mu}{2}\sum_{i,j=1}^3(\xi_{x_i}\cdot\partial_\xi u_j+\xi_{x_j}\cdot\partial_\xi u_i)^2-(\nu-\mu)(\nabla_u\cdot u)^2+\eta h \qquad \text{in } \Omega^T,\end{aligned}$$

$$(7.19)\qquad \varkappa(\eta,\vartheta)\overline{n}_0\cdot\nabla_u\vartheta_\sigma=\overline{\vartheta} \qquad \text{on } S^T,$$

$$(7.20)\qquad \vartheta_\sigma|_{t=0}=\theta_{\sigma 0} \qquad \text{in } \Omega.$$

Moreover,

$$\eta_{\sigma t} + \eta\nabla_u \cdot u = 0 \quad \text{in } \Omega^T, \tag{7.21}$$

$$\eta_\sigma|_{t=0} = \varrho_{\sigma 0} \quad \text{in } \Omega. \tag{7.22}$$

From (7.21)–(7.22) it follows that

$$\eta_\sigma = \varrho_{\sigma 0} - \int_0^t \eta \operatorname{div}_u u\,dt' \quad \text{for } t \le T.$$

Hence, for T sufficiently small we have

$$\begin{aligned} &\sup_{0\le t\le T} \|\eta_\sigma\|_{W_2^3(\Omega)} + \sup_{0\le t\le T} \|\eta_{\sigma t}\|_{W_2^2(\Omega)} + \|\eta_{\sigma t}\|_{L_2(0,T;W_2^3(\Omega))} + \|\eta_{\sigma tt}\|_{L_2(0,T;W_2^1(\Omega))} \\ &\quad \le \overline{\phi}_5(T, |\eta|_{L_\infty(\Omega^T)})(\|\varrho_{\sigma 0}\|_{W_2^3(\Omega)} + \|v_0\|_{W_2^3(\Omega)} + \|u\|_{W_2^{4,2}(\Omega^T)}), \end{aligned} \tag{7.23}$$

where $\overline{\phi}_5$ is an increasing function of its arguments.

Now, treating problem (7.14)–(7.17) as the linear problem (7.1)–(7.4) with $w = u$, and problem (7.18) -(7.20) as problem (7.5)–(7.7) with $w = u$, $\gamma = \vartheta$ we obtain by using estimates (7.12)–(7.13) and (7.23) the expected estimate, i.e. (5.112). The norms of η, $1/\eta$, $\eta c_v(\eta,\vartheta)$, $1/\eta c_v(\eta,\vartheta)$ occurring as the arguments of the functions $\overline{\phi}_i$ ($i = 3, 4, 5$) are estimated by constants depending on the initial data (see [ZZaj1]).

In a similar way to the incompressible case we can estimate the highest norm of $R - R_e$, i.e., $\|R - R_e\|_{W_2^{4+1/2}(S^1)}$. To do this we use Theorem 7.1, estimate (5.139) from Lemma 5.10 and Lemma 5.8.

Inequality (5.139) is obtained by applying estimates (7.12) and (7.13) to the linear parabolic problems having the forms of (7.14)–(7.17) and (7.18)–(7.20), i.e. to the problems with $w = u$ and with the unknown functions $u^{(s)}(\xi,t) = u_\lambda(\xi,t) - u_\lambda(\xi,t-s)$ and $\vartheta_\sigma^{(s)}(\xi,t) = \vartheta_{\sigma\lambda}(\xi,t) - \vartheta_{\sigma\lambda}(\xi,t-s)$, respectively. As a result of inequality (5.139) we obtain the estimates of $\sup_{t_1<t<T} \|u\|_{W_2^4(\Omega)}$ and $\sup_{t_1\le t\le T} \|\vartheta_\sigma\|_{W_2^4(\Omega)}$ (where $t_1 > 0$) by the sum of the norms: $\|u\|_{W_2^{4,2}(\Omega^T)}$, $\|\vartheta_\sigma\|_{W_2^{4,2}(\Omega^T)}$, $\|r\|_{C_B^3(\mathbb{R}^3\times\mathbb{R}_+)}$ and $\|\overline{\theta}\|_{C_B^4(\mathbb{R}^3\times\mathbb{R}_+)}$. The estimate of $\sup_{0\le t\le T} \|\eta_\sigma\|_{W_2^3(\Omega)}$, which is also necessary, is obtained from (7.23).

As in the incompressible case, estimates implied by the conservation laws and inequalities (5.112) and (5.139) do not suffice to prove global existence. We also need an estimate analogous to (4.75). Here appears the most striking difference between the incompressible and compressible cases. Namely, in contrast to the incompressible problem we are not able to derive the expected estimate by using the linear parabolic problems (7.1)–(7.4) and (7.5)–(7.7) and the hyperbolic problem

$$\eta_t + \eta\nabla_w \cdot u = 0 \quad \text{in } \Omega^T,$$

$$\eta_t|_{t=0} = \varrho_0 \quad \text{in } \Omega,$$

taking as the unknown functions $u_\lambda = u\zeta_\lambda$, $\vartheta_\lambda = \vartheta\zeta_\lambda$, $\eta_\lambda = \eta\zeta_\lambda$, and $w = u$.

The impossibility of repeating the argument of Solonnikov [Sol6] follows from the nature of the continuity equation in this case and makes the compressible problem much more complicated. The missing estimate in this case is the differential inequality (5.145). It is derived in [ZZaj7] for the nonlinear problem (5.63)–(5.69) with $f = 0$, $k = 0$

rewritten as a problem with the unknown functions v, ϱ_σ, θ_σ. Obtaining inequality (5.145) is connected with very long and arduous calculations.

Another difference is connected with the process of extending solutions to global ones. This is more complicated in the general compressible case. Since the method of extending the solution for the heat-conducting compressible problem with surface tension is presented in detail in Section 5 (see the proof of Theorem 5.11) we only underline that the main difficulty in extending the solution is associated with the function $\overline{\psi}(t)$ given by (5.156) and occurring in (5.155). Some terms of this function are expressed in Eulerian coordinates, while others in Lagrangian coordinates $\xi \in \Omega$. Knowing that $\overline{\psi}(0) \le \alpha_0$ we have to prove that $\overline{\psi}(T) \le \alpha_0$. Moreover, since passing from the interval $[0, T]$ to $[T, 2T]$ implies also passing from estimates in Ω for the functions u, ϑ_σ, η_σ to estimates in Ω_T for v, θ_σ, ϱ_σ written in Lagrangian coordinates $\xi \in \Omega_T$, the function $\overline{\psi}(t)$ is replaced in $[T, 2T]$ by a function $\overline{\psi}_T(t)$ which has the same form as $\overline{\psi}(t)$ but its appropriate terms are expressed in Lagrangian coordinates $\xi \in \Omega_T$. Therefore, we have to show that $\overline{\psi}_T(T) \le \alpha_0$, which is the main difficulty in the proof of global existence.

In contrast to the situation of a compressible fluid, in the incompressible case a function similar to $\overline{\psi}(t)$ does not appear in estimate (4.75) and this makes the process of extending the local solution much easier.

It should be noticed that similar differences between the incompressible and compressible cases to those described above exist for drop problems without surface tension (see [Sol8], [ZZaj10], [Zaj3], [ZZaj14–15]) and for surface waves problems (see [B2], [TTan], [JinPad]).

We have presented the main similarities and differences in the approach to incompressible and compressible problems which are apparent in the existing papers connecting with these problems. However, we wish to underline that the methods used to obtain global existence and stability results for the general compressible free boundary problem are universal enough to be applied likewise to incompressible problems.

7.3. Significance of surface tension in free boundary problems. In this subsection we describe the role of surface tension in controlling the free boundary of a fluid.

Consider the free boundary problems (4.1)–(4.5), (5.1)–(5.5) and (5.63)–(5.69) with $k = 0$ and $\sigma > 0$. As usual in the problems with $\sigma > 0$ we assume:

(7.24) Ω is close to a ball and S is described by the equation $|\xi| = \widetilde{R}(\omega)$, $\omega \in S^1$,

where S^1 is the unit sphere (see Theorems 4.8, 5.3, 5.4, 5.11–5.14).

Then from the relation (4.10) connecting Lagrangian and Eulerian coordinates it follows that Ω_t is also close to a ball and S_t $(t \le T)$ is described by

$$|x| = R(\omega, t), \qquad \omega \in S^1, \tag{7.25}$$

where $R(\omega, 0) = \widetilde{R}(\omega)$ and T is the time of local existence.

The boundary conditions (4.3), (5.3) or (5.66) can be written in the form

$$H + \frac{2}{R_e} = \frac{1}{\sigma} \overline{n} \cdot \mathbb{T}(v, p_\sigma) \overline{n} \quad \text{on } S_t, \tag{7.26}$$

where in the incompressible case $R_e = R_0 = \left(\frac{3}{4\pi}|\Omega|\right)^{1/3}$, while in the compressible case R_e is given in the definition of an equilibrium solution (see Definitions 5.2, 5.4).

Using (7.25) condition (7.26) takes the form

$$H[R] + 2/R_e = h(\omega), \tag{7.27}$$

where $H[R]$ is the double mean curvature of S_t expressed in spherical coordinates, i.e.

$$H[R] = \frac{1}{R\sin\varphi_2}\left(\frac{\partial}{\partial\varphi_1}\frac{R_{\varphi_1}}{\sin\varphi_2\sqrt{R^2+|\nabla R|^2}} + \frac{\partial}{\partial\varphi_2}\frac{\sin\varphi_2 R_{\varphi_2}}{\sqrt{R^2+|\nabla R|^2}}\right) - \frac{2}{\sqrt{R^2+|\nabla R|^2}}. \tag{7.28}$$

We see that the presence of surface tension on the free boundary S_t implies that the boundary condition on S_t takes the form of the elliptic equation (7.27). Therefore, to control the free boundary in this case (and as a consequence to extend the solution for all t) we use the regularity properties of this elliptic equation. More precisely, we use the following theorem.

THEOREM 7.1. *Let $R \in W_2^{3/2+l}(S^1)$, $l \in (1/2, 1)$ be a solution of equation (7.27) satisfying*

$$\sup_{S^1} |R(\omega,t) - R_e| + \sup_{S^1} |\nabla R(\omega,t)| \le \widehat{\delta} R_e$$

with sufficiently small $\widehat{\delta}$. If $h \in W_2^s(S^1)$, $s \in [0,1]$, then

$$\|R - R_e\|_{W_2^{2+s}(S^1)} \le c_1 \|h\|_{W_2^s(S^1)} + c_2 \|R - R_e\|_{L_2(S^1)}, \tag{7.29}$$

where c_1, c_2 are constants and c_2 can depend on $\|R\|_{W_2^{l+3/2}(S^1)}$. Moreover, if $R \in W_2^{2+s}(S^1)$ and $h \in W_2^{1+s}(S^1)$, $s \in (0,\infty)$, then

$$\|R - R_e\|_{W_2^{3+s}(S^1)} \le c_3 \|h\|_{W_2^{1+s}(S^1)} + c_4 \|R - R_e\|_{L_2(S^1)}, \tag{7.30}$$

where c_3, c_4 are constants and c_4 can depend on $\|R\|_{W_2^{2+s}(S^1)}$.

Inequalities (7.29)–(7.30) are proved in [Sol6] for $s \in (0,1)$.

Proof of Theorem 7.1. Cover S^1 by a finite number of domains S' having sufficiently small diameters. Take a function $\zeta = \zeta(\varphi)$ such that $\zeta = 1$ on S', $\zeta = 0$ on $S^1 \setminus S''$, $\overline{S'} \subset S''$ and $0 \le \zeta \le 1$. Next, set $R_* = R - R_e$, $\widetilde{R}_* = \zeta R_*$ and choose $\varphi^0 \in S^1$. Then, by applying the formula

$$\frac{1}{R_e} - \frac{1}{\sqrt{R^2+|\nabla R|^2}} = \frac{(R-R_e)(R+R_e) + |\nabla R|^2}{R_e\sqrt{R^2+|\nabla R|^2}(R_e + \sqrt{R^2+|\nabla R|^2})} \equiv A,$$

equation (7.27) takes the form of the following elliptic equation:

$$\sum_{\gamma,\delta=1}^{2} A_{\gamma\delta}(\varphi^0)\frac{\partial^2 \widetilde{R}_*}{\partial\varphi_\gamma\partial\varphi_\delta} + \sum_{\gamma=1}^{2} A_\gamma(\varphi^0)\frac{\partial \widetilde{R}_*}{\partial\varphi_\gamma} = F,$$

where

$$
\begin{aligned}
F \equiv & \sum_{\gamma,\delta=1}^{2} (A_{\gamma\delta}(\varphi^0) - A_{\gamma\delta}(\varphi)) \frac{\partial^2 \widetilde{R}_*}{\partial\varphi_\gamma \partial\varphi_\delta} + 2 \sum_{\gamma\delta=1}^{2} A_{\gamma\delta}(\varphi) \frac{\partial\zeta}{\partial\varphi_\gamma} \frac{\partial R_*}{\partial\varphi_\delta} \\
& + R_* \sum_{\gamma,\delta=1}^{2} A_{\gamma\delta}(\varphi) \frac{\partial^2\zeta}{\partial\varphi_\gamma \partial\varphi_\delta} + \sum_{\gamma=1}^{2} (A_\gamma(\varphi_0) - A_\gamma(\varphi)) \frac{\partial \widetilde{R}_*}{\partial\varphi_\gamma} \\
& + R_* \sum_{\gamma=1}^{2} A_\gamma(\varphi) \frac{\partial\zeta}{\partial\varphi_\gamma} - 2A\zeta + h\zeta.
\end{aligned}
$$

Spherical coordinates has been chosen so that $\sin\varphi_1 \ge c_0 > 0$. Therefore $A_{\gamma\delta}, A_\gamma, A \in W_2^{1/2+l}(S^1)$, and these coefficients do not depend on $R_{\varphi\varphi}$, $\varphi = (\varphi_1, \varphi_2)$.

Let us extend these coefficients to $\mathbb{R}^2$ $(-\infty < \varphi_1, \varphi_2 < \infty)$ in the same class of functions and so that

$$
\sup_{\varphi\in\mathbb{R}^2} |A_{\gamma\delta}(\varphi) - A_{\gamma\delta}(\varphi^0)| \le c \sup_{\varphi\in S''} |A_{\gamma\delta}(\varphi) - A_{\gamma\delta}(\varphi^0)|.
$$

The regularity theory of linear elliptic equations yields the estimate

$$
\|\widetilde{R}_*\|_{W_2^{2+s}(S'')} \le c(\|F\|_{W_2^s(S'')} + \|\widetilde{R}_*\|_{L_2(S'')}) \quad \text{for } s \in [0,\infty).
$$

In order to obtain (7.29) and (7.30) it suffices to estimate the terms of F.

Let first $s = 0$. First, consider the term $F_1 = \sum_{\gamma,\delta=1}^{2} (A_{\gamma\delta}(\varphi^0) - A_{\gamma\delta}(\varphi)) \frac{\partial^2 \widetilde{R}_*}{\partial\varphi_\gamma \partial\varphi_\delta}$. We have

$$
\begin{aligned}
\|F_1\|_{L_2(S'')} &\le \sum_{\gamma,\delta=1}^{2} \sup_{\varphi\in S''} |A_{\gamma\delta}(\varphi) - A_{\gamma\delta}(\varphi_0)| \bigg\| \frac{\partial^2 \widetilde{R}_*}{\partial\varphi_\gamma \partial\varphi_\delta} \bigg\|_{L_2(S'')} \\
&\le c\lambda^\beta \sum_{\gamma,\delta=1}^{2} \|A_{\gamma\delta}\|_{W_2^{l+1/2}(S^1)} \|\widetilde{R}_*\|_{W_2^2(S'')},
\end{aligned}
$$

where $0 < \beta \le l - 1/2$ and $\lambda = \mathrm{diam} S''$ is sufficiently small.

Now consider $F_2 = 2\sum_{\gamma,\delta=1}^{2} A_{\gamma\delta}(\varphi) \frac{\partial\zeta}{\partial\varphi_\gamma} \frac{\partial R_*}{\partial\varphi_\delta}$. We obtain

$$
\begin{aligned}
\|F_2\|_{L_2(S'')} &\le c \sum_{\gamma,\delta=1}^{2} \|A_{\gamma\delta}(\varphi)\|_{W_2^{1/2+l}(S^1)} \bigg\| \frac{\partial R_*}{\partial\varphi_\gamma} \bigg\|_{L_4(S'')} \\
&\le c \sum_{\gamma,\delta=1}^{2} \|A_{\gamma\delta}(\varphi)\|_{W_2^{1/2+l}(S^1)} (\varepsilon \|R_*\|_{W_2^2(S'')} + c(\varepsilon) \|R_*\|_{L_2(S'')}),
\end{aligned}
$$

where we have used the interpolation inequality from Lemma 2.1.

The other terms of F are estimated in the same way. Therefore, assuming that λ is sufficiently small we get

$$
\|R_*\|_{W_2^2(S')} \le c_1 \|h\|_{L_2(S'')} + c_2 (\|R_*\|_{L_2(S'')} + \varepsilon \|R_*\|_{W_2^2(S'')}). \tag{7.31}
$$

Summing estimates (7.31) over all S' and assuming that ε is sufficiently small we obtain (7.29).

Now, let $s=1$. Then we have to estimate $\|F\|_{W_2^1(S'')}$. We have

$$\|F_1\|_{W_2^1(S'')} \le c\Big[\|F_1\|_{L_2(S'')} + \sum_{\gamma,\delta=1}^{2} \sup_{\varphi\in S''} |A_{\gamma\delta}(\varphi) - A_{\gamma\delta}(\varphi^0)| \Big(\sum_{\alpha=1}^{2} \Big\|\frac{\partial^3 \widetilde{R}_*}{\partial\varphi_\gamma\partial\varphi_\delta\partial\varphi_\alpha}\Big\|_{L_2(S'')}\Big)$$
$$+ \sum_{\alpha,\gamma,\delta=1}^{2} \Big\|\frac{\partial A_{\gamma\delta}}{\partial\varphi_\alpha}\Big\|_{L_{p_1}(S'')} \Big\|\frac{\partial^2 \widetilde{R}_*}{\partial\varphi_\gamma\partial\varphi_\delta}\Big\|_{L_{p_2}(S'')}\Big]$$
$$\le c(\lambda^\beta+\varepsilon)\sum_{\gamma,\delta=1}^{2} \|A_{\gamma\delta}\|_{W_2^{1/2+l}(S'')} \|\widetilde{R}_*\|_{W_2^3(S'')} + c(\varepsilon)\|\widetilde{R}_*\|_{L_2(S'')},$$

where the last inequality holds for some p_1 and p_2.

The other terms of F are estimated similarly. Therefore inequality (7.29) holds for $s=1$.

Estimate (7.29) for $s\in(0,1)$ is proved in [Sol6]. For example, the norm $\|F_1\|_{W_2^s(\mathbb{R}^2)}$ is estimated in [Sol6] as follows:

$$\|F_1\|_{W_2^s(\mathbb{R}^2)} = \|F_1\|_{L_2(\mathbb{R}^2)} + \Big(\int_{\mathbb{R}^2} \|\Delta(\psi)F_1\|^2_{L_2(\mathbb{R}^2)} \frac{d\psi}{|\psi|^{2+2s}}\Big)^{1/2}$$
$$\le c\lambda^\beta \sum_{\gamma,\delta=1}^{2} \|A_{\gamma\delta}\|_{W_2^{l+1/2}(\mathbb{R}^2)} \Big[\|\widetilde{R}_*\|_{W_2^2(\mathbb{R}^2)} + \Big(\int_{\mathbb{R}^2} \Big\|\Delta(\psi)\frac{\partial^2\widetilde{R}_*}{\partial\varphi_\gamma\partial\varphi_\delta}\Big\|^2_{L_2(\mathbb{R}^2)} \frac{d\psi}{|\psi|^{2+2s}}\Big)^{1/2}\Big]$$
$$+ c\sum_{\gamma,\delta=1}^{2} \Big\|\frac{\partial^2\widetilde{R}_*}{\partial\varphi_\gamma\partial\varphi_\delta}\Big\|_{L_{2p_1}(\mathbb{R}^2)} \Big(\int_{\mathbb{R}^2} \|\Delta(\psi)A_{\gamma\delta}\|^{2/p_2}_{L_2(\mathbb{R}^2)} \|\Delta(\psi)A_{\gamma\delta}\|^{2/p_1}_{W_2^1(\mathbb{R}^2)} \frac{d\psi}{|\psi|^{2+2s}}\Big)^{1/2},$$

where $\Delta(\psi)f = f(\varphi+\psi) - f(\varphi)$, $p_2 > 1/s$, $1/p_1 = 1 - 1/p_2$. Using the fact that

$$\Big\|\frac{\partial^2\widetilde{R}_*}{\partial\varphi_1\partial\varphi_2}\Big\|_{L_{2p_1}(\mathbb{R}^2)} \le c\Big\|\frac{\partial^2\widetilde{R}_*}{\partial\varphi_1\partial\varphi_2}\Big\|_{W_2^{s'}(\mathbb{R}^2)}, \qquad s' = s - 1/p_2,$$

and

$$\Big(\int_{\mathbb{R}^2} \|\Delta(\psi)A_{\gamma\delta}\|^{2/p_2}_{L_2(\mathbb{R}^2)} \|\Delta(\psi)A_{\gamma\delta}\|^{2/p_1}_{W_2^1(\mathbb{R}^2)} \frac{d\psi}{|\psi|^{2+2s}}\Big)^{1/2}$$
$$\le \Big(\int_{\mathbb{R}^2} \|\Delta(\psi)A_{\gamma\delta}\|^2_{L_2(\mathbb{R}^2)} \frac{d\psi}{|\psi|^{2+2r}}\Big)^{1/p_2} \Big(\int_{\mathbb{R}^2} \|\Delta(\psi)A_{\gamma\delta}\|^2_{W_2^1(\mathbb{R}^2)} \frac{d\psi}{|\psi|^{1+2l}}\Big)^{1/p_1}$$
$$\le c\|A_{\gamma\delta}\|_{W_2^{l+1/2}(\mathbb{R}^2)}, \qquad r\in(0,1),\ r/p_2 + (l-1/2)/p_1 = s,$$

we get

$$\|F_1\|_{W_2^s(\mathbb{R}^2)} \le c(\lambda^\beta+\varepsilon)\sum_{\gamma,\delta=1}^{2} \|A_{\gamma\delta}\|_{W_2^{l+1/2}(\mathbb{R}^2)} \|\widetilde{R}_*\|_{W_2^{2+s}(\mathbb{R}^2)} + c(\varepsilon)\sum_{\gamma,\delta=1}^{2} \|\widetilde{R}_*\|_{L_2(\mathbb{R}^2)}$$

for sufficiently small λ and ε.

The other terms of F can be estimated similarly. This way inequality (7.29) is proved. Inequality (7.30) can be derived by using similar calculations. ∎

Theorem 7.1 is essential to all proofs of global existence and stability for problems in which Ω_t is a domain occupied by a fixed mass of fluid bounded by a free boundary with surface tension. If we assume that $S \in W_2^{3+s}$ for some $s > 0$ and if we can prove that for all $t > 0$ the right-hand side of (7.30) is finite, we obtain by Theorem 7.1 the same regularity of the boundary for all $t > 0$, i.e. $S_t \in W_2^{3+s}$. Moreover, for data sufficiently close to an equilibrium state we can usually prove that the free boundary S_t remains close to a ball of radius R_e for all t.

Thus, Theorem 7.1 enables us to control the free boundary of the fluid. Obviously, we have to find appropriate estimates of the terms occurring on the right-hand sides of (7.29) and (7.30). The possibility of obtaining an estimate of $\|R - R_e\|_{L_2(S^1)}$ is always a consequence of the conservation laws of energy, momentum and mass (in the compressible case) or volume (in the incompressible case). Since the forms of the energy conservation laws differ for different fluids, estimates derived for $\|R - R_e\|_{L_2(S^1)}$ are also different. However, in all these cases it is assumed that the barycentre of the initial domain coincides with the origin of coordinates. This assumption together with the assumption that the momentum of the initial domain is equal to zero and together with the momentum conservation law implies that also the barycentre of Ω_t for $t \le T$ (T is the time of local existence) coincides with the origin.

By (7.26) we see that in order to estimate the norms $\|h\|_{W_2^s(S^1)}$ or $\|h\|_{W_2^{1+s}(S^1)}$ we need estimates of v and p_σ. Such estimates are derived in different ways in dependence on what motion we consider. More details about these estimates are given in Sections 4 and 5.

For surface waves problems, a theorem analogous to Theorem 7.1 can be proved for the function $F(x', t)$ describing the free boundary in this case (see Theorem 6.7 of Section 6). The function F satisfies then elliptic equation (6.37).

7.4. How to control the free boundary in drop problems without surface tension? In this subsection we consider problems (4.1)–(4.5), (5.1)–(5.5) and (5.63)–(5.69) with $k = 0$, $f = 0$ and $\sigma = 0$. In this case the Laplace–Beltrami operator $\Delta_{S_t}(t)$ does not appear in boundary conditions (4.3), (5.3) and (5.66). Therefore, we cannot use the regularity properties of elliptic equations as in the case of $\sigma > 0$. Since we cannot apply Theorem 7.1, the way of controlling the free boundary in such problems is quite different. In the case of $\sigma = 0$, the following differential inequality can be proved:

$$\frac{d\overline{\varphi}}{dt} + c_1 \Phi \le 0 \quad \text{for } t \le T, \tag{7.32}$$

where T is the time of local existence; $c_1 > 0$ is a constant.

For a compressible heat-conducting fluid, $\overline{\varphi} = \overline{\varphi}(t)$ is a function equivalent to

$$\varphi(t) = \|v(t)\|^2_{X(\Omega_t)} + \|\theta_\sigma(t)\|^2_{X(\Omega_t)} + \|\varrho_\sigma(t)\|^2_{X(\Omega_t)},$$

where $X(\Omega_t)$ is a certain function space, usually of Sobolev type; $\theta_\sigma = \theta - \theta_e$, $\varrho_\sigma = \varrho - \varrho_e$; (v, θ, ϱ) is the local solution of problem (5.63)–(5.69); θ_e and ϱ_e are the constants defined in Definition 5.3. Moreover,

$$\Phi(t) = \|v(t)\|^2_{Y(\Omega_t)} + \|\theta_\sigma(t)\|^2_{Y(\Omega_t)} + \|\varrho_\sigma(t)\|^2_{Z(\Omega_t)},$$

where $Y(\Omega_t)$ and $Z(\Omega_t)$ are spaces such that

$$\Phi \geq c_2\varphi. \tag{7.33}$$

For a barotropic compressible fluid $\overline{\varphi}$ is equivalent to

$$\varphi(t) = \|v(t)\|^2_{X(\Omega_t)} + \|\varrho_\sigma(t)\|^2_{X(\Omega_t)},$$

where (v,ϱ) is the local solution of problem (5.1)–(5.5), ϱ_e is the constant defined in Definition 5.1. In this case

$$\Phi(t) = \|v(t)\|^2_{Y(\Omega_t)} + \|\varrho_\sigma(t)\|^2_{Z(\Omega_t)}.$$

Finally, for an incompressible fluid $\overline{\varphi}$ is equivalent to $\varphi(t) = \|v(t)\|^2_{X(\Omega_t)}$ and $\Phi(t) = \|v(t)\|^2_{Y(\Omega_t)}$, where v is the local solution of (4.1)–(4.5).

Thus, we have

$$c_3\varphi(t) \leq \overline{\varphi}(t) \leq c_4\varphi(t) \quad \text{for } t \leq T, \tag{7.34}$$

where in the general heat-conducting case the constants c_3, $c_4 > 0$ depend on ϱ_1, ϱ_2, θ_1, θ_2, μ, ν, $\varkappa$, c_v, p, $T\int_0^T \|v\|^2_{W_2^3(\Omega_t)}\,dt$, and ϱ_1, ϱ_2, θ_1, θ_2 are positive constants such that

$$\varrho_1 < \varrho(x,t) < \varrho_2, \quad \theta_1 < \theta(x,t) < \theta_2 \quad \text{for } x \in \overline{\Omega}_t,\ t \in [0,T]. \tag{7.35}$$

The constant c_1 in (7.32) depends on the same quantities as c_3 and c_4. It depends also on $\|S_t\|_{W_2^{5/2}}$ and the constants from imbedding theorems and Korn inequalities which depend on Ω_t, $t \leq T$.

Inequalities (7.32)–(7.34) imply

$$\frac{d\overline{\varphi}}{dt} + c_5\overline{\varphi} \leq 0 \quad \text{for } t \leq T. \tag{7.36}$$

Hence

$$\overline{\varphi}(t) \leq \overline{\varphi}(0)e^{-c_5 t} \quad \text{for } t \leq T \tag{7.37}$$

and

$$\varphi(t) \leq \frac{c_4}{c_3}\varphi(0)e^{-c_5 t} \quad \text{for } t \leq T. \tag{7.38}$$

Moreover

$$\overline{\varphi}(t) + c_1\int_0^t \Phi(t')\,dt' \leq \overline{\varphi}(0) \quad \text{for } t \leq T. \tag{7.39}$$

In the incompressible case it suffices to take $X(\Omega_t) = L_2(\Omega_t)$ and $Y(\Omega_t) = W_2^1(\Omega_t)$, which follows from the general strategy applied to incompressible free boundary problems (see Subsection 7.2). Then $\overline{\varphi}(t) = \varphi(t) = \|v(t)\|^2_{L_2(\Omega_t)}$ and inequalities (7.32) and (7.36) follow from the energy conservation law and the Korn inequality (see [Sol8] or Theorem 4.2 in Section 4). However, the solvability of problem (4.1)–(4.5) is proved in spaces of functions $v(t)$ more regular than $L_2(\Omega_t)$ (see [Sol8]).

In contrast to the incompressible case, inequality (7.36) for compressible fluids is obtained in the same function spaces in which the solvability of problems (5.1)–(5.5) or (5.63)–(5.69) is proved (see [ZZaj10, ZZaj15], see also Section 5). In the compressible case, the spaces $X(\Omega_t)$, $Y(\Omega_t)$ and $Z(\Omega_t)$ with the lowest possible regularity of functions v, θ_σ, ϱ_σ are defined as follows:

$$X(\Omega_t) = \Big\{w : \sum_{i=0}^{2} \|\partial_t^i w(t)\|_{W_2^{2-i}(\Omega_t)} < \infty\Big\}, \tag{7.40}$$

$$Y(\Omega_t) = \Big\{w : \sum_{i=0}^{2} \|\partial_t^i w(t)\|_{W_2^{3-i}(\Omega_t)} < \infty\Big\} \tag{7.41}$$

and

$$Z(\Omega_t) = \{w : \|w(t)\|^2_{W_2^2(\Omega_t)} + \|w_t(t)\|^2_{W_2^2(\Omega_t)} + \|w_{tt}(t)\|^2_{W_2^1(\Omega_t)} < \infty\}.$$

Obviously, the method applied to prove global existence for free boundary compressible problems can also be used to problem (4.1)–(4.5). Thus, we can derive inequality (7.32) in the incompressible case with $X(\Omega_t)$ and $Y(\Omega_t)$ defined by (7.40) and (7.41), respectively.

Now, we will show that inequalities (7.37) and (7.39) allow us to control the free boundary S_t if the data $\varphi(0)$ is sufficiently small. We control the free boundary and extend the local solution from $[0,T]$ to $\mathbb{R}_+$ step by step. First, assuming that the initial conditions ϱ_0, θ_0 and the equilibrium solution (ϱ_e, θ_e) (see Definition 5.3) satisfy inequalities (7.35) and moreover assuming that $\varphi(0) \le \varepsilon$ with ε sufficiently small, we prove by using estimate (5.87) (which holds for the local solution) that estimates (7.35) are satisfied for $x \in \overline{\Omega}_t$, $t \in [0,T]$.

Furthermore, the same inequality (5.87) yields, for $t \le T$,

$$|x-\xi| \le \Big|\int_0^t u(\xi,t')\,dt'\Big| \le c_6 \Big\|\int_0^t u\,dt'\Big\|_{W_2^3(\Omega)} \le c_6 T^{1/2}\|u\|_{\mathcal{A}_{T,\Omega}} \le c_7(T)T^{1/2}\varepsilon^{1/2}, \tag{7.42}$$

where c_7 is an increasing continuous function of T.

Therefore, assuming that $S \in W_2^{5/2}$ we see that $S_t \in W_2^{5/2}$ for $t \le T$ and by (7.42), the volume and shape of Ω_t $(t \le T)$ do not change much if ε is sufficiently small. Hence we can derive inequalities (7.32) and (7.36) with the constants c_1 and c_5 which in fact do not depend on $T\int_0^T \|v\|^2_{W_2^3(\Omega_t)}\,dt$ and Ω_t for $t \le T$, but depend on the other quantities mentioned above.

Moreover, estimate (7.42) and inequalities (7.35) for $x \in \overline{\Omega}_t$, $t \in [0,T]$, imply that if we assume (7.34) for $\varphi(0)$ and $\overline{\varphi}(0)$ with c_3, c_4 depending on ϱ_1, ϱ_2, θ_1, θ_2, μ, ν, c_v, p, $\varkappa$, we obtain this estimate for $\varphi(t)$ and $\overline{\varphi}(t)$ and all $t \le T$ with the same constants c_3, c_4.

Hence, by (7.37) we get (7.38). As a consequence, we have

$$\overline{\varphi}(t) \le c_4\varepsilon \qquad \text{for } t \le T,$$
$$\varphi(t) \le \frac{c_4}{c_3}\varepsilon \qquad \text{for } t \le T.$$

Therefore, for sufficiently small ε the solution can be extended to $[T, 2T]$. Moreover, the local solution satisfies in $[T,2T]$ inequality (5.87) with $\varphi(0)$ replaced by $\varphi(T)$, with $(u_T,\ \vartheta_{T\sigma},\ \eta_{T\sigma})$ denoting $(v,\ \theta_\sigma,\ \varrho_\sigma)$ written in Lagrangian coordinates $\xi_T \in \Omega_T$ (i.e. $\xi_T = \xi + \int_0^T u(\xi,t')\,dt'$) and with the norms of the spaces $\mathcal{A}_{T,\Omega}$ and $\mathcal{B}_{T,\Omega}$ replaced by those of $\mathcal{A}_{T,\Omega_T}$ and $\mathcal{B}_{T,\Omega_T}$. The spaces $\mathcal{A}_{T,\Omega_{iT}}$ and $\mathcal{B}_{T,\Omega_{iT}}$, $i \in \mathbb{N}\cup\{0\}$, are defined as follows:

$$\mathcal{A}_{T,\Omega_{iT}} \equiv \mathcal{B}_{T,\Omega_{iT}} \cap L_2(iT,(i+1)T;W_2^3(\Omega_{iT})),$$

$$\begin{aligned}\mathcal{B}_{T,\Omega_{iT}} \equiv \{w \in C([iT,(i+1)T];W_2^2(\Omega_{iT})) :&\\ w_t \in C([iT,(i+1)T];W_2^1(\Omega_{iT})) \cap L_2(iT,(i+1)T;W_2^2(\Omega_{iT})),&\\ w_{tt} \in C([iT,(i+1)T];L_2(\Omega_{iT})) \cap L_2(iT,(i+1)T;W_2^1(\Omega_{iT}))\}&\end{aligned}$$

for $i \in \mathbb{N}$ and $\mathcal{A}_{T,\Omega_{0T}} \equiv \mathcal{A}_{T,\Omega}$, $\mathcal{B}_{T,\Omega_{0T}} \equiv \mathcal{B}_{T,\Omega}$.

Namely, inequality (5.87) has the following form in $[T,2T]$:

$$\|u_T\|^2_{\mathcal{A}_{T,\Omega_T}} + \|\vartheta_{T\sigma}\|^2_{\mathcal{A}_{T,\Omega_T}} + \|\eta_{T\sigma}\|^2_{\mathcal{B}_{T,\Omega_T}} \le C_1(T)\varphi(T). \tag{7.43}$$

Thanks to (7.43) we prove that estimate (7.34) holds for $x \in \overline{\Omega}_t$, $t \in [0,2T]$. Moreover, we prove that the volume and shape of Ω_t change in $[0,2T]$ no more than they do in $[0,T]$. In fact, by (7.39), (7.43), (5.87) and (7.38) we have, for sufficiently small ε and $t \le 2T$,

$$\begin{aligned}|x-\xi| &= \Big|\int_0^t u(\xi,t')\,dt'\Big| \le \Big\|\int_0^t u(\xi,t')\,dt'\Big\|_{L_\infty(\Omega)} \le c_6\Big\|\int_0^t u(\xi,t')\,dt'\Big\|_{W_2^3(\Omega)} \\ &\le c_6\Big(\int_0^T \|u(\xi,t')\|_{W_2^3(\Omega)}\,dt' + \int_T^{2T} \|u(\xi,t')\|_{W_2^3(\Omega)}\,dt'\Big) \\ &\le c_6T^{1/2}\Big[c_8\Big(\int_0^T \Phi(t')\,dt'\Big) + c_9\|u_T\|_{A_{T,\Omega_T}}\Big] \\ &\le c_6T^{1/2}\varepsilon^{1/2}\Big(c_8c_4 + c_{10}(T)\frac{c_4}{c_3}\Big) \le c_6\overline{\varepsilon}.\end{aligned} \tag{7.44}$$

It follows from (7.44) that for ε sufficiently small the volume and shape of Ω_t do not change much in $[0,2T]$. Thus, we are able to derive the differential inequality for $T \le t \le 2T$ with the same constant c_1 as before. However, it should be underlined that in order to obtain (7.32) we partly use Lagrangian coordinates and therefore some terms of $\overline{\varphi}$ are expressed in Lagrangian coordinates $\xi \in \Omega$, while the others in Eulerian coordinates. For this reason, after passing to $[T,2T]$, we derive the differential inequality (7.32) for a function $\overline{\varphi}_T$ which has the same form as $\overline{\varphi}$ but has appropriate terms expressed in Lagrangian coordinates $\xi_T \in \Omega_T$.

From the form of $\overline{\varphi}_T$ and from (7.44) it follows that $\overline{\varphi}_T$ satisfies estimate (7.34) for $T \le t \le 2T$.

Therefore the differential inequality for $\overline{\varphi}_T$, i.e.

$$\frac{d\overline{\varphi}_T}{dt} + c_1\Phi \le 0$$

implies (7.36), (7.37) and (7.39) for $T \le t \le 2T$ with $\overline{\varphi}$ replaced by $\overline{\varphi}_T$.

Moreover, if ε is sufficiently small then

$$\overline{\varphi}_T(T) \le (1 + c_{11}T^{1/2}\|u\|_{\mathcal{A}_{T,\Omega}})\overline{\varphi}(T).$$

Hence, by (7.37) and (5.87)

$$\overline{\varphi}_T(T) \le \left[1 + c_{11}\left(\frac{c_4}{c_3}TC_1(T)\varepsilon\right)^{1/2}\right]c_4\varepsilon e^{-c_5T} \equiv c_{12}c_4\varepsilon e^{-c_5T} < c_4\varepsilon, \tag{7.45}$$

if we assume that ε is so small that $c_{12}e^{-c_5T} < 1$. By (7.45) and the inequalities (7.34) and (7.37) written for $\overline{\varphi}_T$, we obtain

$$\overline{\varphi}_T(t) \le c_4\varepsilon \quad \text{for } T \le t \le 2T, \tag{7.46}$$

$$\varphi(t) \le \frac{c_4}{c_3}\varepsilon \quad \text{for } T \le t \le 2T. \tag{7.47}$$

Estimates (7.46)–(7.47) allow us to extend the solution to the interval $[2T, 3T]$.

Continuing the above process, assume that there exists a solution in $[0, lT]$, $l \ge 3$, satisfying:

$$\|u_{jT}\|^2_{\mathcal{A}_{T,\Omega_{jT}}} + \|\vartheta_{jT\sigma}\|^2_{\mathcal{A}_{T,\Omega_{jT}}} + \|\eta_{jT\sigma}\|^2_{\mathcal{B}_{T,\Omega_{jT}}} \le C_1(T)\varphi(jT), \quad j = 0, \dots, l-1,$$

$$\overline{\varphi}_{jT}(t) \le c_4\varepsilon \qquad \text{for } jT \le t \le (j+1)T,\ j = 0, \dots, l-2,$$

$$\varphi(t) \le \frac{c_4}{c_3}\varepsilon \qquad \text{for } t \le (l-1)T,$$

$$\overline{\varphi}_{jT}(t) + c_1\int_{jT}^{t}\Phi(t')\,dt' \le \overline{\varphi}_{jT}(jT) \quad \text{for } jT \le t \le (j+1)T, j = 0, \dots, l-2,$$

where u_{jT}, $\vartheta_{jT\sigma}$, $\eta_{jT\sigma}$ denote v, θ_σ, ϱ_σ written in Lagrangian coordinates $\xi_{jT} \in \Omega_{jT}$; $\overline{\varphi}_{0T} \equiv \overline{\varphi}$; $\overline{\varphi}_{jT}$ has the same form as $\overline{\varphi}$ and appropriate terms of $\overline{\varphi}_{jT}$ are written in Lagrangian coordinates $\xi_{jT} \in \Omega_{jT}$.

Assume also that the volume and shape of Ω_t change in $[0, (l-1)T]$ no more than they do in $[0, T]$ and that

$$\Big\|\int_0^t u(\xi, t')\,dt'\Big\|_{W_2^3(\Omega)} \le \overline{\varepsilon} \quad \text{for } t \le (l-1)T$$

with sufficiently small $\overline{\varepsilon}$. From the above assumptions it follows that estimate (7.34) holds with $\overline{\varphi}$ replaced by $\overline{\varphi}_{jT}$ for $jT \le t \le (j+1)T$, $j = 0, \dots, l-2$. Moreover, the form of $\overline{\varphi}_{jT}$ $(j = 1, \dots, l-2)$ and the smallness of ε imply

$$\overline{\varphi}_{jT}(jT) \le (1 + c_{11}T^{1/2}\|u\|_{\mathcal{A}_{T,\Omega_{(j-1)T}}})\overline{\varphi}_{(j-1)T}(jT), \quad j = 1, \dots, l-2.$$

Hence, assuming that $\overline{\varepsilon}$ is sufficiently small we obtain, for $0 \le t \le lT$,

$$|x - \xi| = \Big|\int_0^t u(\xi, t')\,dt'\Big| \le \Big\|\int_0^t u(\xi, t')\,dt'\Big\|_{L_\infty(\Omega)}$$

$$\le c_6\Big\|\int_0^t u(\xi, t')\,dt'\Big\|_{W_2^3(\Omega)} \le c_6\Big(\sum_{j=0}^{l-2}\Big\|\int_{jT}^{(j+1)T} u(\xi, t')\,dt'\Big\|_{W_2^3(\Omega)} + \Big\|\int_{(l-1)T}^{lT} u(\xi, t')\,dt'\Big\|_{W_2^3(\Omega)}\Big)$$

$$\le c_6T^{1/2}\Big[c_8\sum_{j=0}^{l-2}\Big(\int_{jT}^{(j+1)T}\Phi(t')\,dt'\Big)^{1/2} + c_9\|u_{(l-1)T}\|_{A_{T,\Omega_{(l-1)T}}}\Big]$$

$$\leq c_6T^{1/2}\bigg[c_8\sum_{j=0}^{l-2}(\overline{\varphi}_{jT}(jT))^{1/2}+c_{10}(T)\frac{c_4}{c_3}\varepsilon^{1/2}\bigg]$$

$$\leq c_6T^{1/2}\bigg\{c_8[\overline{\varphi}(0)(1+c_{12}e^{-c_5T}+c_{12}^2e^{-2c_5T}+\ldots)]^{1/2}+c_{10}(T)\frac{c_4}{c_3}\varepsilon^{1/2}\bigg\}$$

$$\leq c_6T^{1/2}\varepsilon^{1/2}\bigg[\frac{c_8c_4^{1/2}}{(1-c_{12}e^{-c_5T})^{1/2}}+c_{10}(T)\frac{c_4}{c_3}\bigg]\leq c_6\overline{\varepsilon},$$

if ε is sufficiently small in dependence on $\overline{\varepsilon}$.

Thus, the volume and shape of Ω_t change in $[0,lT]$ no more than they do in $[0,(l-1)T]$. These changes are as small as we want if we assume that $\overline{\varepsilon}$ is sufficiently small.

This way we can control the free boundary of Ω_t in the case of $\sigma=0$. At the same time, this way we can extend the solution to a global one.

The above method of controlling the free boundary and proving global existence in this case is presented in [ZZaj15].

Remark 7.1. In order to prove differential inequality (7.32) we proceed as in the case of $\sigma>0$, that is, we use systems (5.142) and (5.143) with $\sigma=0$ in the boundary condition. We use these systems step by step in the process of extending the solution to $\mathbb{R}_+$. Therefore, if we want to prove the differential inequality in the interval $[kT,(k+1)T]$, $k\geq 1$, we have to choose the covering of Ω_{kT} and the family of fuctions $\{\zeta_{kT,i}\}_{i\in\mathcal{M}\cup\mathcal{N}}$ such that $\zeta_{kT,i}$ has the support in an appropriate domain of the covering.

In contrast to the case of $\sigma>0$ (see the proof of Theorem 5.11) we choose a covering $\bigcup_{i\in\mathcal{M}\cup\mathcal{N}}\widetilde{\Omega}_{kT,i}$ of Ω_{kT} such that

$$\widetilde{\Omega}_{kT,i}=\Big\{\xi_{kT}\in\Omega_{kT}:\xi_{kT}=\xi+\int_0^{kT}u\,dt',\,\xi\in\widetilde{\Omega}_i\Big\},\qquad i\in\mathcal{M},$$

and

$$\widetilde{\Omega}_{kT,i}\cap\Omega_T=\Big\{\xi_{kT}\in\Omega_{kT}:\xi_{kT}=\xi+\int_0^{kT}u\,dt',\,\xi\in\widetilde{\Omega}_i\cap\Omega\Big\},\qquad i\in\mathcal{N},$$

where $\bigcup_{i\in\mathcal{M}\cup\mathcal{N}}\widetilde{\Omega}_i$ is a covering of Ω.

We prove the differential inequality for $t\in[kT,(k+1)T]$ in order to extend the solution to the interval $[(k+1)T,(k+2)T]$. Since we know that $\|\int_0^{kT}u\,dt'\|_{W_2^3(\Omega)}\leq\overline{\varepsilon}$ for $k\geq 1$, we can associate with the covering $\bigcup_{i\in\mathcal{M}\cup\mathcal{N}}\widetilde{\Omega}_{kT,i}$ the same family of functions $\{\zeta_i\}$ as with $\bigcup_{i\in\mathcal{M}\cup\mathcal{N}}\widetilde{\Omega}_i$.

In fact, assuming that $\overline{\varepsilon}$ is sufficiently small we have $\operatorname{supp}\zeta_i\subset\widetilde{\Omega}_{kT,i}$ and $0<n_0\leq\sum_{i\in\mathcal{M}\cup\mathcal{N}}\zeta_i(\xi_{kT})\leq N_0$ for $\zeta_{kT}\in\Omega_{kT}$, where n_0 and N_0 are sufficiently close to 1.

Alternatively, we can take the family $\{\zeta_{kT,i}\}_{i\in\mathcal{M}\cup\mathcal{N}}$ such that $\zeta_{kT,i}(\xi_{kT})=\zeta_i(\xi_{kT}-\int_0^{kT}u\,dt')$. Then obviously $\operatorname{supp}\zeta_{kT,i}\subset\widetilde{\Omega}_{kT,i}$ and $\sum_{i\in\mathcal{M}\cup\mathcal{N}}\zeta_i(\xi_{kT})=1$ for $\xi_{kT}\in\widetilde{\Omega}_{kT,i}$.

7.5. Difficulties connected with the self-gravitational force in drop problems. Now consider problems (4.1)–(4.5) and (5.1)–(5.5) with $k>0$. The only global existence result with the self-gravitational force taken into account has been proved by

Solonnikov [Sol10]; his paper concerns problem (4.1)–(4.5) with $k > 0$ and $\sigma > 0$. The assumption that $\sigma > 0$ is essential because it enables control of the free boundary as we now describe.

Assume that $p_0 = 0$, $f = 0$ and substitute $p' = p - \frac{2\sigma}{R_0} - k(U - U_0|_{|x|=R_0})$, where $R_0 = (\frac{3}{4\pi}|\Omega|)^{1/3}$, $U_0(x) = \int_{|y|<R_0} \frac{dy}{|x-y|}$, $U_0|_{|x|=R_0} = \frac{4}{3}\pi R_0^3$. Then problem (4.1)–(4.5) takes the form

$$v_t + (v\cdot\nabla)v - \nu\Delta v + \nabla p' = 0, \qquad x \in \Omega_t,\ t \in (0,T), \tag{7.48}$$

$$\operatorname{div} v = 0, \qquad x \in \Omega_t,\ t \in (0,T), \tag{7.49}$$

$$\mathbb{T}(v,p')\overline{n} - \sigma H\overline{n} = \left(kU + \frac{2\sigma}{R_0} - \frac{4}{3}\pi k R_0^2\right)\overline{n}, \qquad x \in S_t,\ t \in (0,T), \tag{7.50}$$

$$v\cdot\overline{n} = -\phi_t/|\nabla\phi|, \qquad x \in S_t,\ t \in (0,T), \tag{7.51}$$

$$v(x,0) = v_0(x), \qquad x \in \Omega. \tag{7.52}$$

Rewrite boundary condition (7.50) as

$$H + \frac{2}{R_0} = \frac{1}{\sigma}\overline{n}\cdot\mathbb{T}(v,p')\overline{n} - \frac{k}{\sigma}\left(U - \frac{4}{3}\pi R_0^2\right), \qquad x \in S_t. \tag{7.53}$$

Hence, assuming that condition (7.24) is satisfied (and hence also (7.25)), condition (7.53) takes the form

$$H[R] + \frac{2}{R_0} = h(\omega) + \frac{1}{\sigma}(U(R\omega) - U_0(R_0\omega)) \equiv \overline{h}(\omega), \qquad \omega \in S^1, \tag{7.54}$$

where $H[R]$ is the double mean curvature of S_t given in spherical coordinates by (7.28), and $h(\omega) = \frac{1}{\sigma}\overline{n}\cdot\mathbb{T}(v,p')\overline{n}|_{x=R\omega}$.

Equation (7.54) can be transformed to an integral-differential equation with the unknown function $R - R_0$.

From the assumptions of [Sol10] it follows that $R(\cdot,t) \in W_2^{3/2+\alpha}(S^1)$, $\alpha \in (1/2,1)$ and

$$\sup_{S^1}|R(\omega,t) - R_0| + \sup_{S^1}|\nabla R(\omega,t)| \le \widehat{\delta}R_0 \tag{7.55}$$

with sufficiently small $\widehat{\delta}$, where $t \le T$, T is the time of local existence. Hence, in view of Theorem 7.1 the solution R of equation (7.54) satisfies the estimate

$$\begin{aligned} \|R - R_0\|_{W_2^{5/2+\alpha}(S^1)} &\le c_1\|\overline{h}\|_{W_2^{1/2+\alpha}(S^1)} + c_2\|R - R_0\|_{L_2(S^1)} \\ &\le c_1\|h\|_{W_2^{1/2+\alpha}(S^1)} + c_1\|U(R\omega) - U_0(R\omega)\|_{W_2^{1/2+\alpha}(S^1)} \\ &\quad + c_2\|R - R_0\|_{L_2(S^1)} \quad \text{for } t \le T. \end{aligned} \tag{7.56}$$

The third term on the right-hand side of (7.56) can be estimated by using the following lemma.

LEMMA 7.1 (see Lemma 2.4 of [Sol10]). *Let $R \in W_2^{3/2+\alpha}(S^1)$ and let* (7.56) *with $\widehat{\delta} \le 1/10$ be satisfied. Then*

$$\|U(R\omega) - U_0(R_0\omega)\|_{W_2^{1/2+\alpha}(S^1)} \le c_3(\sup_{S^1}|R - R_0| + \sup_{S^1}|\nabla R|) \le c_4\|R - R_0\|_{W_2^{3/2+\alpha}(S^1)}.$$

Lemma 7.1 applied to (7.56), together with the interpolation inequality

$$\|R-R_0\|_{W_2^{3/2+\alpha}(S^1)} \le \varepsilon_1\|R-R_0\|_{W_2^{5/2+\alpha}(S^1)} + c(\varepsilon_1)\|R-R_0\|_{L_2(S^1)},$$

yields the estimate, for sufficiently small ε_1,

$$\|R-R_0\|_{W_2^{5/2+\alpha}(S^1)} \le c_5\|h\|_{W_2^{1/2+\alpha}(S^1)} + c_6\|R-R_0\|_{L_2(S^1)}. \tag{7.57}$$

Estimate (7.57) has exactly the form of inequality (7.30). Thanks to this estimate we can control the free boundary in time if we have estimates for the terms of the right-hand side of (7.57). Solonnikov proved in [Sol10] that under the assumption $\|v_0\|_{W_2^{1+\alpha}(\Omega)} + \|\widetilde{R}(\omega)-R_0\|_{W_2^{5/2+\alpha}(S^1)} \le \varepsilon$, where ε is sufficiently small, the right-hand side of (7.57) is also small for $t_1 \le t \le T$, $t_1 > 0$ (see inequalities (4.90) and (4.91) of this paper) and $S_t \in W_2^{5/2+\alpha}$ for $0 \le t \le T$.

It seems that in the case of compressible fluid, the above method of controlling the free boundary can also be applied, at least for barotropic fluids.

In contrast to the situation described above there are no global existence results for the case $k > 0$ and $\sigma = 0$, both for incompressible and compressible fluids.

Assume now that $k > 0$, $\sigma = 0$ and substitute in problem (4.1)–(4.5): $p' = p - k(U - U_0|_{|x|=R_0})$. Then problem (4.1)–(4.5) takes the form of problem (7.48)–(7.49), (7.51)–(7.52) with boundary condition (7.51) replaced by

$$\mathbb{T}(v,p')\overline{n} = k(U - U_0|_{|x|=R_0})\overline{n}, \quad x \in S_t,\ t \in (0,T). \tag{7.58}$$

The lack of an elliptic operator in equation (7.58) makes it useless for deriving an estimate similar to (7.57). Therefore, it seems that the only way of controlling the free boundary in this case is to do this by deriving differential inequality (7.32). However, so far attempts of obtaining such an inequality have failed.

7.6. Final remarks. Most of the results on free boundary problems for Navier–Stokes equations reviewed above are existence and stability theorems. However, in the previous sections we also mentioned some asymptotic results. Much has been done in this field, in the one-dimensional and spherically symmetric cases. The most characteristic feature of the asymptotic results in the one-dimensional case is that all of them require the assumption that the external pressure P (see boundary conditions (3.4) or (3.42)) is positive. This assumption is crucial to the proofs of the asymptotic convergence of solutions of the one-dimensional problems to stationary solutions.

In the two- and three-dimensional cases there are asymptotic results for incompressible and compressible barotropic fluids. For drop problems such results have been obtained by Solonnikov in [Sol9] in the case of an incompressible fluid with $\sigma > 0$, $k = 0$, and [Sol10] for the incompressible case with $\sigma > 0$, $k > 0$, and by Solonnikov and Tani [SolT3] for the compressible barotropic case with $\sigma > 0$, $k = 0$. All of these results are similar, i.e. it is proved that as $t \to \infty$ a solution of a free boundary problem tends to a quasi-stationary solution corresponding to a rotation of the fluid as a rigid body about an axis which is parallel to the angular momentum vector. These asymptotic results are obtained under the assumption that the initial angular momentum vector which is equal to $m = \int_\Omega (v_0 \times \xi)\, d\xi$ in the incompressible case and to $m = \int_\Omega \varrho_0(v_0 \times \xi)\, d\xi$

in the compressible case, is sufficiently small. By the angular momentum conservation law, the angular momentum of the fluid remains small and equal to m for all $t > 0$. Thus, $v(x,t) \to v_\infty(x)$, $R(\omega,t) \to R_\infty(\omega)$ as $t \to \infty$ and $p(x,t) \to p_\infty(x)$ as $t \to \infty$ (in the incompressible case) or $\varrho(x,t) \to \varrho_\infty(x)$ as $t \to \infty$ (in the compressible barotropic case). The above convergences are uniform convergences with respect to x or ω. Moreover, it is shown that the first derivatives of v with respect to x tend uniformly to the first derivatives of v_∞, and the first and second derivatives of R tend uniformly to the first and second derivatives of R_∞. In all the cases the rotational velocity v_∞ of the fluid is small in dependence on the smallness of $|m|$ (see the end of Section 4), and in the compressible barotropic case also in dependence on the smallness of $|M - (4/3)\pi R_e^3 \varrho_e|$ (see Theorem 5.5). The pressure p_∞ is the sum of two terms: one of them is small in dependence on m and the other is a constant which depends on the specific problem. Similarly, ϱ_∞ in the compressible case is a sum of two such terms. The function R_∞ in each case has to be found from an equation implied by one of the boundary conditions: (4.3) or (5.3). In the simplest two-dimensional case with $k = 0$ it follows that $R_\infty = R_0 = \big(\frac{3}{4\pi}|\Omega|\big)^{1/3}$, i.e. the domain Ω_∞ is the ball $|x| < R_0$.

Now, we want to mention the asymptotic results for the surface waves problems with $\sigma > 0$. They were obtained by Beale and Nishida [BNis] for the three-dimensional incompressible motion, by Nishida and Teramoto [NTer] for the two-dimensional motion of an incompressible fluid flowing down an inclined plane under the influence of gravity, and by Jin and Padula [JinPad] for a compressible flow. All of these results show the asymptotic convergence of solutions in appropriate norms to equilibrium states together with decay rates. For example, for the problem considered in [Jin Pad] this decay rate is exponential (see Theorem 6.8).

We would like to end this paper with a general remark concerning all free boundary drop problems with $\sigma > 0$. To derive some estimates useful for the proofs of global existence as well as to examine the asymptotic behaviour of solutions of free boundary drop problems it is usually assumed that the total initial momentum of the fluid vanishes and that the barycentre of the initial domain coincides with the origin, i.e.

$$\int_\Omega v_0\, d\xi = 0, \quad \int_\Omega \xi\, d\xi = 0 \tag{7.59}$$

in the incompressible case, and

$$\int_\Omega \varrho_0 v_0\, d\xi = 0, \quad \int_\Omega \varrho_0 \xi\, d\xi = 0 \tag{7.60}$$

in the compressible case.

From the laws of conservation of momentum and barycentre and from (7.59) or (7.60) it follows then that the total momentum of the fluid vanishes and the barycentre of the fluid coincides with the origin for all $t > 0$.

However, it turns out that assumptions (7.59) or (7.60) are not restrictive because by applying the transformation

$$x' = x - Vt, \quad v' = v - V,$$

where $V = \frac{1}{|\Omega|}\int_\Omega v_0\ d\xi$ in the incompressible case (see [Sol6]) and $V = (\int_\Omega \varrho_0\, d\xi)^{-1} \times \int_\Omega \varrho_0 v_0\, d\xi$ in the compressible case (see [SolT3]), we obtain $\int_\Omega v_0'\, d\xi = 0$ and $\int_\Omega \varrho_0 v_0'\, d\xi = 0$, respectively. Hence, the coordinates of the barycentre are conserved, i.e. $\int_{\Omega_t} x'\, dx' = \int_\Omega \xi\, d\xi$ or $\int_{\Omega_t} \varrho x'\, dx' = \int_\Omega \varrho_0 \xi\, d\xi$ and we can always place the barycentre of the initial domain at the origin.

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