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Abstract

In this memoir, we shall consider weighted convolution algebras on discrete groups and semigroups, concentrating on the group $(\mathbb{Q}, +)$ of rational numbers, the semigroup $(\mathbb{Q}^{+\bullet}, +)$ of strictly positive rational numbers, and analogous semigroups in the real line $\mathbb{R}$. In particular, we shall discuss when these algebras are Arens regular, when they are strongly Arens irregular, and when they are neither, giving a variety of examples. We introduce the notion of 'weakly diagonally bounded' weights, weakening the known concept of 'diagonally bounded' weights, and thus obtaining more examples. We shall also construct an example of a weighted convolution algebra on $\mathbb{N}$ that is neither Arens regular nor strongly Arens irregular, and an example of a weight ω on $\mathbb{Q}^{+\bullet}$ such that $\liminf_{s\to 0+} \omega(s) = 0$.

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1. Introduction

Let A be a Banach algebra, and regard A as a closed subspace of its second dual A''. Then there are two natural products on A''; they are called the *first* and *second Arens products*, and are denoted by $\Box$ and $\Diamond$, respectively. We now briefly recall the definitions; for some notations, details, and history, see below. As usual, A' and A'' are Banach A-bimodules. For $\lambda \in A'$ and $\Phi \in A''$, define $\lambda \cdot \Phi \in A'$ and $\Phi \cdot \lambda \in A'$ by

$$\langle a, \lambda \cdot \Phi \rangle = \langle \Phi, a \cdot \lambda \rangle, \quad \langle a, \Phi \cdot \lambda \rangle = \langle \Phi, \lambda \cdot a \rangle \quad (a \in A),$$

and, for $\Phi, \Psi \in A''$, define

$$\langle \Phi \Box \Psi, \lambda \rangle = \langle \Phi, \Psi \cdot \lambda \rangle, \quad \langle \Phi \Diamond \Psi, \lambda \rangle = \langle \Psi, \lambda \cdot \Phi \rangle \quad (\lambda \in A').$$

Then $(A'', \Box)$ and $(A'', \Diamond)$ are both Banach algebras containing A as a closed subalgebra.

DEFINITION 1.1. Let A be a Banach algebra. The *left* and *right topological centres* of A'' are defined by

$$\mathfrak{Z}_t^{(\ell)}(A'') = \{\Phi \in A'' : \Phi \Box \Psi = \Phi \Diamond \Psi \ (\Psi \in A'')\},$$
$$\mathfrak{Z}_t^{(r)}(A'') = \{\Phi \in A'' : \Psi \Box \Phi = \Psi \Diamond \Phi \ (\Psi \in A'')\},$$

respectively.

Thus $A \subset \mathfrak{Z}_t^{(\ell)}(A'')$ and $A \subset \mathfrak{Z}_t^{(r)}(A'')$. The map

$$L_\Psi : \Phi \mapsto \Psi \Box \Phi, \quad A'' \to A'',$$

is weak-$*$ continuous if and only if $\Psi \in \mathfrak{Z}_t^{(\ell)}(A'')$.

See [12, 13, 53, 56, 57, 58] for extensive discussions of these centres.

DEFINITION 1.2. Let A be a Banach algebra. Then A is *Arens regular* if

$$\mathfrak{Z}_t^{(\ell)}(A'') = \mathfrak{Z}_t^{(r)}(A'') = A'';$$

A is *left strongly Arens irregular* if $\mathfrak{Z}_t^{(\ell)}(A'') = A$, *right strongly Arens irregular* if $\mathfrak{Z}_t^{(r)}(A'') = A$, and *strongly Arens irregular* if it is both left and right strongly Arens irregular. A subset V of A'' is *determining for the left topological centre* of A'' if $\Phi \in A$ whenever $\Phi \in A''$ and $\Phi \Box \Psi = \Phi \Diamond \Psi \ (\Psi \in V)$.

In the case where A is commutative, as it will be in most of this memoir, we have

$$\Phi \Box \Psi = \Psi \Diamond \Phi \quad (\Phi, \Psi \in A''),$$

so that $(A'', \Diamond)$ is just the opposite algebra to $(A'', \Box)$ and $\mathfrak{Z}_t^{(\ell)}(A'')$ and $\mathfrak{Z}_t^{(r)}(A'')$ are each just the centre $\mathfrak{Z}(A'')$ of $(A'', \Box)$; in this case, A is strongly Arens irregular whenever it is left strongly Arens irregular.

Clearly A is left strongly Arens irregular if and only if A'' is determining for the left topological centre.

A closed subalgebra and a quotient of an Arens regular Banach algebra by a closed ideal are themselves Arens regular.

Our aim in this memoir is to determine the topological centre of various weighted convolution algebras on $\mathbb{Q}$ and on $\mathbb{Q}^{+\bullet}$, and also, to some extent, on $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{R}^{+\bullet}$ (where each of these sets is regarded as a discrete subsemigroup of $(\mathbb{R},+)$), showing the possibilities that can arise. For example, let S be $\mathbb{Q}$ or $\mathbb{Q}^{+\bullet}$. By [10, Corollary 1], there exists an Arens regular weight ω on S; see Examples 9.14 and 9.17, below. We shall show in this memoir that $\ell^1(S,\omega)$ is not Arens regular for many weights ω on S, and that it is strongly Arens irregular for a large class of weights ω.

In this work, certain known proofs are repeated for the sake of completeness.

In Chapter 2, we shall summarize some history concerning Arens products on the second duals of Banach algebras, Arens regularity, and strong Arens irregularity; we shall introduce some notation to be used throughout the memoir.

In Chapter 3, we shall introduce the Banach space $\ell^1(S,\omega)$ for a non-empty set S and a function $\omega : S \to \mathbb{R}^{+\bullet}$. We shall also refer to the Stone–Čech compactification βS of S.

In Chapter 4, we shall define a weight ω on a semigroup S, and then introduce the weighted convolution algebra $(\ell^1(S,\omega),\star)$ as a Banach algebra. We shall establish some basic properties of these algebras; some of these results are repeated from [18].

Let A be a Banach algebra. In Chapter 5, we shall explain what it means for a subset V of A'' to be determining for the topological centre of A''. Let ω be a weight on a semigroup S, and let T be a subset of S. We shall recall the definition of 'ω is diagonally bounded on T', and introduce the important notion of 'ω is weakly diagonally bounded on T'. We shall find that, in the case where $A_\omega := \ell^1(S,\omega)$ for a weight ω on a semigroup S and ω is weakly diagonally bounded on a suitable subset of S, there is a finite subset of A''_ω that is determining for the topological centre of A''_ω. These results are extensions of those in [13]. We shall note that every weight on the group $(\mathbb{R},+)$ is strongly Arens irregular.

In Chapter 6, we shall give various examples of weights on subsemigroups S of the group $\mathbb{R}$. A key point is that several of them are weakly diagonally bounded, rather than diagonally bounded, on suitable subsets of S.

In Chapter 7, we shall consider algebras on subsets of the unit interval $\mathbb{I}$, rather than on semigroups; these algebras are quotients of the weighted convolution algebras considered earlier. The results are proved mainly because they will be required in Chapter 8, which is devoted to a proof of the fact that $\ell^1(\mathbb{R}^{+\bullet},\omega)$ is strongly Arens irregular for each continuous weight ω on $\mathbb{R}^{+\bullet}$.

In Chapter 9, we shall consider conditions for the Arens regularity of an algebra $\ell^1(S,\omega)$; such conditions were first given by Craw and Young in [10]. A considerable number of examples of weights are given in this chapter. Also, we shall show that there is no weight on $\mathbb{R}^{+\bullet}$ such that $\ell^1(\mathbb{R}^{+\bullet},\omega)$ is semisimple and Arens regular; we leave open the possibility that there is a weight on $\mathbb{R}^{+\bullet}$ such that $\ell^1(\mathbb{R}^{+\bullet},\omega)$ is radical and Arens regular.

In Chapter 10, we shall construct a weight ω on $\mathbb{N}$ such that the algebra $\ell^1(\mathbb{N},\omega)$ is radical, but is neither Arens regular nor strongly Arens irregular; we have not found a weight ω such that $\ell^1(\mathbb{N},\omega)$ is semsimple, but neither Arens regular nor strongly Arens irregular.

Finally, in Chapter 11, we shall construct a weight ω on the semigroup $(\mathbb{Q}^{+\bullet},+)$ with $\liminf_{s\to 0+}\omega(s)=0$, and shall investigate its properties; again the fact that our weight is weakly diagonally bounded, rather than diagonally bounded, on a suitable subset of $\mathbb{Q}^{+\bullet}$ will be important.

A list of questions that we think are open and a summary of the results known to us are given at the end of the memoir.

We are grateful to Dona Strauss for some valuable discussions.

2. Background and notation

We first give some more background, some notation that we shall use, and some history of our problem; for further details, see [11, 12, 13].

Let E be a Banach space. Then the closed unit ball of E is denoted by $E_{[1]}$; the dual space of E is denoted by E', and the second dual by E''; we regard E as a closed subspace of E'' via the canonical embedding. The action of $\lambda\in E'$ on $x\in E$ and the action of $\Lambda\in E''$ on $\lambda\in E'$ are denoted by

$$\langle x,\lambda\rangle \quad\text{and}\quad \langle \Lambda,\lambda\rangle,$$

respectively. However, because of the profusion of dual spaces below, we cannot maintain full consistency in this.

The Banach algebra of all bounded linear operators on a Banach space E, with the usual operator norm, is denoted by $\mathcal{B}(E)$.

Let A be an algebra (always over the complex field $\mathbb{C}$). Then we set

$$A^{[2]}=\{ab : a,b\in A\} \quad\text{and}\quad A^2=\operatorname{lin}A^{[2]},$$

the linear span of $A^{[2]}$. The (Jacobson) radical of A is denoted by $\operatorname{rad}A$; the algebra A is *semisimple* if $\operatorname{rad}A=\{0\}$ and *radical* if $\operatorname{rad}A=A$. For $a\in A$, we set

$$L_a(b)=ab,\qquad R_a(b)=ba\qquad (b\in A).$$

In the case where A is a Banach algebra, we see that $L_a,R_a\in\mathcal{B}(A)$ for each $a\in A$. We say that $a\in A$ is [*weakly*] *compact* if both L_a and R_a are [weakly] compact operators on A. It is a theorem of Watanabe (see [59, Proposition 1.4.13]) that A is an ideal in $(A'',\Box)$ if and only if each element of A is weakly compact.

Let A be a Banach algebra. Then A' and A'' are Banach A-bimodules with respect to the maps defined as follows: for $a\in A$ and $\lambda\in A'$, we define $a\cdot\lambda$ and $\lambda\cdot a$ in A' by

$$\langle b,a\cdot\lambda\rangle=\langle ba,\lambda\rangle \quad\text{and}\quad \langle b,\lambda\cdot a\rangle=\langle ab,\lambda\rangle$$

for $b\in A$; for $a\in A$ and $\Lambda\in A''$, we define $a\cdot\Lambda$ and $\Lambda\cdot a$ in A'' by

$$\langle a\cdot\Lambda,\lambda\rangle=\langle\Lambda,\lambda\cdot a\rangle \quad\text{and}\quad \langle\Lambda\cdot a,\lambda\rangle=\langle\Lambda,a\cdot\lambda\rangle$$

for $\lambda \in A'$. Thus we see that the two Arens products, defined above, are bilinear maps $A'' \times A'' \to A''$ that extend these natural module actions, which are defined as maps $A \times A'' \to A''$ and $A'' \times A \to A''$.

The two Arens products $\Box$ and $\Diamond$ on A'' are determined by the following formulae, where all limits are taken in the weak-$*$ topology $\sigma(A'', A')$ of A''. Let $\Phi, \Psi \in A''$, and take nets (a_α) and (b_β) in A with $\Phi = \lim_\alpha a_\alpha$ and $\Psi = \lim_\beta b_\beta$. Then

$$\Phi \Box \Psi = \lim_\alpha \lim_\beta a_\alpha b_\beta, \qquad \Phi \Diamond \Psi = \lim_\beta \lim_\alpha a_\alpha b_\beta.$$

The maps

$$R_\Psi : \Phi \mapsto \Phi \Box \Psi, \qquad A'' \to A'',$$

are weak-$*$ continuous on $(A'', \Box)$ for each $\Psi \in A''$.

A criterion for the Arens regularity of a Banach algebra A has been given by Pym [60] (see also [11, Theorem 2.6.17] and [12, Chapter 3], where more details are given). Indeed, A is Arens regular if and only if, for each pair $\{(a_m), (b_n)\}$ of bounded sequences in A and each $\lambda \in A'$, the two repeated limits

$$\lim_{m\to\infty} \lim_{n\to\infty} \langle a_m b_n, \lambda \rangle \quad \text{and} \quad \lim_{n\to\infty} \lim_{m\to\infty} \langle a_m b_n, \lambda \rangle$$

are equal whenever they both exist.

For further discussions of these products, see [11, 12, 13], for example.

The pioneering work on what are now called the Arens products on the space A'' for a Banach algebra A is that of Richard Arens in 1951 [1, 2]. For example, Arens proved that $(\ell^1, \cdot)$ is Arens regular, but that $(\ell^1, \star)$ is not Arens regular.

The first systematic study of Arens products is due to Civin and Yood in 1961 [9]; in particular, they proved that, in the case where G is a locally compact abelian group, the group algebra $(L^1(G), \star)$ is Arens regular if and only if G is finite; this was established for an arbitrary locally compact group G by Young in 1973 [65]; for shorter proofs, see [62] and [54, Proposition 5.2]. It was finally proved by Lau and Losert in 1988 that $L^1(G)$ is strongly Arens irregular for each locally compact group G [50], following an earlier proof for the compact case in [48]. For another proof in a special case, see [33].

We are indebted to Craw and Young [10] for the seminal study in 1974 of the second duals of weighted group algebras; we shall recall some of their results below.

Now let S be a semigroup, with corresponding semigroup algebra $(\ell^1(S), \star)$ (see Chapter 4). It was shown by Young in [64] that $\ell^1(S)$ is Arens regular if and only if there do not exist sequences (s_m) and (t_m) in S such that the two sets $\{s_m t_n : m < n\}$ and $\{s_m t_n : m > n\}$ are distinct; see also [5]. We shall discuss this result in Chapter 9.

The first result on the strong Arens irregularity of $\ell^1(S)$ seems to be that of Butcher [8], who proved that $\ell^1(S)$ is strongly Arens irregular whenever S is a countable, cancellative, abelian semigroup such that S^* is the union of two disjoint left ideals of βS. Lau proved in [49] that $\ell^1(S)$ is strongly Arens irregular whenever the semigroup S is cancellative.

Once we know that an algebra is strongly Arens irregular, it is natural to ask which sets are determining for the left topological centre; one seeks the 'smallest possible' sets with this property. This is the topic of Chapter 5 of the present memoir, where we shall discuss such sets for weighted convolution algebras $\ell^1(S,\omega)$, and also recall the earlier results on this topic for the algebras $\ell^1(S)$. The 'best' result is that certain subsets of βS of cardinality 2 are determining for the left topological centre of $\ell^1(S)$; see [13, Chapter 12] for an extensive account.

Now let G be locally compact group. One would like to know which subsets of $L^1(G)''$ are determining for the left topological centre. It was proved by Neufang in [56] that a set which consists of the continuous linear extensions to $L^1(G)''$ of the points of βG_d is determining for the left topological centre. This result is strengthened in [14], where it is proved that the spectrum (or character space) of $L^\infty(G)$ is determining for the left topological centre of $L^1(G)''$. However, it is so far left open whether or not there is always a finite subset of $L^1(G)''$ which is determining for the left topological centre.

It is a related open question to determine whether or not the measure algebra $M(G)$ of a locally compact group G is strongly Arens irregular; this question is discussed in [25]. It is proved by Neufang in [57, Theorem 3.5] that this is the case when G is a locally compact and non-compact group (with non-measurable cardinal). However, the case where G is a compact group, and especially the case where G is the circle group $\mathbb{T}$, appears to be open; some partial results are given in [14, Chapter 9].

Let G be locally compact group, and denote by $A(G)$ the Fourier algebra of G; for the definition, see [11, Definition 4.5.29]. Then $A(G)$ is a Banach function algebra on G; in the case where G is abelian, $A(G)$ is isomorphic to the algebra $(L^1(\Gamma), \star)$, where Γ is the dual group to G.

First, we consider when $A(G)$ is Arens regular. The first result on this question is that of Lau and Wong in [54]: for an amenable locally compact group G, $A(G)$ is Arens regular if and only if G is finite. It is proved in [52] that G is discrete whenever $A(G)$ is Arens regular; further, in the case where G is discrete and G contains an infinite amenable subgroup, $A(G)$ is not Arens regular. This latter result subsumes a result of Forrest [22, Corollary 3.8] that $A(G)$ is not Arens regular whenever G contains $\mathbb{F}_2$, the free group on two generators. It remains an open problem whether or not $A(G)$ is Arens regular in the case where G is an infinite group that does not contain any infinite amenable subgroup; such groups exist.

Second, we consider when $A(G)$ is strongly Arens irregular. Lau and Losert proved in [51] that $A(G)$ is strongly Arens irregular whenever G is amenable and discrete (and in some other special cases), and this is proved in some further cases in [52]. See also [44] for an extension of this result, and [63, Corollary 2.4] for a different proof that $A(G)$ is strongly Arens irregular when G is amenable and discrete. However, it is proved by Losert in [55] that $A(G)$ is not strongly Arens irregular whenever G is a group containing $\mathbb{F}_2$; Losert explores the nature of $\mathfrak{Z}_t^{(\ell)}(A(G)'')$ in this paper.

Viktor Losert has announced in lectures some further remarkable results in this area; for example, $A(G)$ is strongly Arens irregular when G is the compact, connected group $SU(2)$, but $A(G)$ is not strongly Arens irregular when G is the related group $SU(3)$. Further, $A(G)$ is not strongly Arens irregular whenever G is the locally compact group $SL(n,\mathbb{R})$ (where $n \geq 2$). Indeed, Losert gives a description of the topological centre of $A(G)''$ at least in the case where $n = 2$; the description involves 'radial functions'.

Again, let G be locally compact group. Then there is a generalization of the Fourier algebra $A(G)$: this is the Figà-Talamanca–Herz algebra $A_p(G)$, defined for $p > 1$, so that the Fourier algebra $A(G)$ is now $A_2(G)$. For the definition, see [11, Definition 4.5.29]. Again, $A_p(G)$ is a natural Banach function algebra on G. It is shown by Forrest in [22] that G is discrete whenever $A_p(G)$ is Arens regular; it is further shown in [22] that, for many classes of discrete groups, in fact G is finite whenever $A_p(G)$ is Arens regular. This latter result is extended in [23], where it is proved that every abelian subgroup of G is finite whenever G is a group such that $A_p(G)$ is Arens regular.

There is a further generalization of the algebras $A_p(G)$. Let G be a locally compact, non-compact group, and denote by $A_p^r(G)$ the Banach function algebra $A_p(G) \cap L^r(G)$, where $p > 1$ and $r \geq 1$. These algebras are commutative, regular (in the sense of [11, Definition 4.1.6]), natural Banach function algebras on G with respect to the norm which is the sum of the norms in $A_p(G)$ and $L^r(G)$, and they are dense ideals in $A_p(G)$. It is shown by Granirer in [35] that G is necessarily discrete whenever $A_p^r(G)$ is Arens regular.

Further results on the Arens regularity of quotients of some of the above algebras are given by Graham in [29–32].

It is an important fact that every C^*-algebra A is Arens regular and that $(A'', \Box)$ is also a C^*-algebra; this was first shown by Civin and Yood in [9]. A proof of this result is given in [11, Corollary 3.2.37], a different proof is given in [7, 38.19], and there is a discussion of the result in [12, Example 4.2]. It was shown by Young in [66] that the Banach algebra $\mathcal{K}(E)$ of all compact operators on a Banach space E is Arens regular if and only if E is reflexive; see [11, Theorem 2.6.23] for a more general result. A study of the topological centres of the algebras $\mathcal{K}(E)$ in the case where the Banach space E is not reflexive is given in [12, Chapter 6].

It is also proved in [66] that a Banach space E is reflexive whenever the Banach algebra $\mathcal{B}(E)$ is Arens regular. It was proved by Daws in 2004 [15] that the converse is almost true, in that $\mathcal{B}(E)$ is Arens regular whenever E is super-reflexive. Thus the Banach algebras $\mathcal{B}(\ell^p)$, defined for $1 \leq p \leq \infty$, are Arens regular if and only if $1 < p < \infty$. For a discussion of this result, see [12, Chapter 6]. It is interesting that, for $1 < p < \infty$, the second dual algebra $(\mathcal{B}(\ell^p)'', \Box)$ is semisimple if and only if $p = 2$ [17].

Important surveys of Arens products were given by Duncan and Hosseiniun in [19] and by Filali and Singh in [21].

The two topological centres of the space A'' were first systematically studied by Lau and Ülger in 1996 [53], where they are denoted by Z_1 and Z_2, respectively; some questions that were raised in [53] were answered in [24] and [28]. Two recent preprints that present the theory of topological centres in an abstract setting are those of Hu, Neufang, and Ruan [46, 47]. Paper [46] concludes with a useful list of the 11 open problems concerning topological centres that were posed by Lau and Ülger in [53], and the present situation regarding their solution; most of these problems have now been resolved.

A number of other examples involving second duals is given in [12, Chapter 4].

A notion related to that of 'strong Arens regularity' is that of 'extreme non-Arens regularity', as introduced by Granirer in [34]; for more on this topic, see [42, 43, 45].

We now record some notation that we shall use.

We define the sets $\mathbb{N}_n = \{1, 2, \dots, n\}$ and $\mathbb{Z}_n^+ = \{0, 1, \dots, n\}$ for $n \in \mathbb{N}$. However, $\mathbb{Z}_n$ denotes the set $\{0, 1, \dots, n-1\}$, regarded as a quotient group of $(\mathbb{Z}, +)$. We shall also write $\mathbb{P}$ for the set of prime numbers. For $p, q \in \mathbb{N}$, we write $p \mid q$ if p divides q in $\mathbb{N}$ and $p \nmid q$ if p does not divide q in $\mathbb{N}$. Let $\mathbb{Q}$ be the set of rational numbers. Here when we say that $p/q \in \mathbb{Q}$, we understand that $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ and that p and q are coprime, written $(p, q) = 1$. Throughout this memoir, we shall discuss Banach algebras on semigroups S; we shall concentrate on the case where S is the group $(\mathbb{Q}, +)$ or the subsemigroup

$$\mathbb{Q}^{+\bullet} = \{p/q \in \mathbb{Q} : p, q \in \mathbb{N}\}$$

of $(\mathbb{Q}, +)$ (so that $0 \notin \mathbb{Q}^{+\bullet}$). Of course the semigroup $\mathbb{Q}^{+\bullet}$ is countable, cancellative, and abelian.

We also set $\mathbb{R}^+ = \{x \in \mathbb{R} : x \geq 0\}$, $\mathbb{R}^{+\bullet} = \{x \in \mathbb{R} : x > 0\}$, $\mathbb{I} = [0, 1]$, and $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

We write $|V|$ for the cardinality of a set V; in particular, $|\mathbb{R}| = \mathfrak{c}$.

Let S be a non-empty set. For a function $f \in \mathbb{C}^S$, we define

$$\operatorname{supp} f = \{s \in S : f(s) \neq 0\}.$$

For each $s \in S$, δ_s is the characteristic function of the singleton $\{s\}$, so that $\operatorname{supp} \delta_s = \{s\}$. Let f be a function on $\mathbb{R}^+$ with $f \neq 0$. Then we set

$$\alpha(f) = \inf \operatorname{supp} f.$$

Let $f \in \mathbb{C}^S$. As in [12], we write

$$\operatorname*{Lim}_{s \to \infty} f(s) = \zeta,$$

where $\zeta \in \mathbb{C}$, if, for each $\varepsilon > 0$, there is a finite subset F of S such that

$$|f(s) - \zeta| < \varepsilon \qquad (s \in S \setminus F),$$

so that $f \in c_0(S)$ if and only if $\operatorname{Lim}_{s \to \infty} f(s) = 0$. In the case where $f \in \mathbb{R}^S$, we also define

$$\operatorname*{Lim\,sup}_{s \to \infty} f(s) \quad \text{and} \quad \operatorname*{Lim\,inf}_{s \to \infty} f(s)$$

in $\mathbb{R} \cup \{\infty\} \cup \{-\infty\}$ similarly.

3. Examples of Banach spaces

We now introduce the Banach spaces that we shall refer to.

DEFINITION 3.1. Let S be a non-empty set, and let $\omega : S \to \mathbb{R}^{+\bullet}$ be a function. Then

$$A_\omega := \ell^1(S, \omega) = \Big\{ f = \sum_{s \in S} f(s)\delta_s : \|f\|_\omega = \sum_{s \in S} |f(s)|\omega(s) < \infty \Big\}.$$

Clearly $(\ell^1(S, \omega), \|\cdot\|_\omega)$ is a Banach space. The dual of this Banach space is

$$A'_\omega = \ell^\infty(S, 1/\omega) = \{\lambda \in \mathbb{C}^S : \sup\{|\lambda(s)|/\omega(s) : s \in S\} < \infty\},$$

with the norm also denoted by $\|\cdot\|_\omega$, so that

$$\|\lambda\|_\omega = \sup\{|\lambda(s)|/\omega(s) : s \in S\} \qquad (\lambda \in \ell^\infty(S, 1/\omega)).$$

The duality $\langle\cdot,\cdot\rangle_\omega$ between A_ω and A'_ω is defined as follows: if $f=\sum_{s\in S} f(s)\delta_s \in A_\omega$ and $\lambda=\sum_{s\in S}\lambda(s)\delta_s\in A'_\omega$, then

$$\langle f,\lambda\rangle_\omega=\sum_{s\in S} f(s)\lambda(s).$$

The space

$$E_\omega:=c_0(S,1/\omega)=\{f\in\mathbb{C}^S : \operatorname*{Lim}_{s\to\infty} f(s)/\omega(s)=0\}$$

is a closed subspace of A'_ω, and $A_\omega=E'_\omega$. The second dual space of A_ω is $(A''_\omega,\|\cdot\|_\omega)$, and the annihilator of E_ω in A''_ω is denoted by E°_ω, so that

$$A''_\omega=A_\omega\oplus E^\circ_\omega$$

as a Banach space.

In the special case where $\omega(s)=1$ $(s\in S)$, we write $\ell^1(S)$ for $\ell^1(S,\omega)$, $\ell^\infty(S)$ for $\ell^\infty(S,1/\omega)$, $\|\cdot\|_1$ for $\|\cdot\|_\omega$, and $\langle\cdot,\cdot\rangle$ for $\langle\cdot,\cdot\rangle_\omega$.

We remark that a bounded linear operator on such a space $\ell^1(S,\omega)$ is compact if and only if it is weakly compact [20, Corollary IV.8.13].

The following definition is given in [12, p. 96].

Definition 3.2. For $s\in S$, the *normalized point mass* at s is defined to be $\delta_s/\omega(s)$, and is denoted by $\widetilde{\delta}_s$.

Clearly $\|\widetilde{\delta}_s\|_\omega=1$ $(s\in S)$.

The Stone–Čech compactification of a non-empty set S is βS; βS is regarded as a compact, totally disconnected space and is identified with the collection of ultrafilters on S in the standard way. We shall write $\beta\mathbb{R}_d$ for the Stone–Čech compactification of $\mathbb{R}$ when $\mathbb{R}$ has the discrete topology. For $T\subset S$, we identify βT with $\overline{T}$, the closure of T in βS, and we set $T^*=\overline{T}\setminus T$, the *growth* of T. Let $\omega: S\to\mathbb{R}^{+\bullet}$ be a function. Then we denote by βT_ω the weak-$*$ closure of $\{\widetilde{\delta}_t : t\in T\}$ in A''_ω; we regard T as a subset of βT_ω; we set $T^*_\omega=\beta T_\omega\setminus T$, the *growth* of T in βS_ω. Thus βT_ω is a closed subset of the unit ball $(A''_\omega)_{[1]}$ (in the weak-$*$ topology), and so it is a compact space.

As Banach spaces, we identify $\ell^\infty(S)$ with $C(\beta S)$ and $\ell^1(S)''$ with $M(\beta S)$, the Banach space of complex-valued, regular Borel measures on βS. For details, see [13]. The sets of positive and real-valued functions in $C(\beta S)$ are denoted by $C(\beta S)^+$ and $C_{\mathbb{R}}(\beta S)$, respectively, and the support of $\lambda\in C(\beta S)$ is $\operatorname{supp}\lambda$. The sets of positive and real-valued measures in $M(\beta S)$ are $M(\beta S)^+$ and $M_{\mathbb{R}}(\beta S)$, respectively. Each measure $\mu\in M_{\mathbb{R}}(\beta S)$ has the unique Hahn decomposition $\mu=\mu_1-\mu_2$, where $\mu_1,\mu_2\in M(\beta S)^+$, the measures μ_1 and μ_2 are mutually singular, and $\|\mu\|=\|\mu_1\|+\|\mu_2\|$.

Definition 3.3. Let S be a non-empty set, and let $\omega: S\to\mathbb{R}^{+\bullet}$ be a function. We define

$$\theta_\omega : f\mapsto f/\omega,\qquad \ell^1(S)\to A_\omega=\ell^1(S,\omega).$$

Clearly the map θ_ω is a linear isometry, and $\theta_\omega(\delta_s)=\widetilde{\delta}_s$ for each $s\in S$. The dual of θ_ω is the map

$$\theta'_\omega : \lambda\mapsto\lambda/\omega,\qquad \ell^\infty(S,1/\omega)=A'_\omega\to\ell^\infty(S)=C(\beta S),$$

and the second dual is the isometry $\theta_\omega'' : M(\beta S) \to A_\omega''$. In fact, we shall usually write θ_ω for θ_ω''; we set $(A_\omega'')_{\mathbb{R}} = \theta_\omega(M_{\mathbb{R}}(\beta S))$, so that

$$A_\omega'' = (A_\omega'')_{\mathbb{R}} + \mathrm{i}(A_\omega'')_{\mathbb{R}}.$$

Clearly the map $\theta_\omega | \beta T : \beta T \to \beta T_\omega$ is a homeomorphism for each $T \subset S$.

We further note that, if $\mu, \nu \in M(\beta S)$ are mutually singular measures, then

$$\|\theta_\omega(\mu) + \theta_\omega(\nu)\|_\omega = \|\theta_\omega(\mu)\|_\omega + \|\theta_\omega(\nu)\|_\omega. \tag{3.1}$$

4. Semigroups and weights

We now introduce the notion of a weight on a semigroup and of the associated weighted convolution algebras.

Definition 4.1. A *semigroup* is a non-empty set S with an associative product, initially denoted by juxtaposition.

In this case, we shall write $L_s : t \mapsto st$ and $R_s : t \mapsto ts$ for the left and right translation operators on S for each $s \in S$. An element $s \in S$ is *left* (respectively, *right*) *cancellable* if the maps L_s and R_s, respectively, are injective; s is *cancellable* if it is both left and right cancellable; S is *cancellative* if each $s \in S$ is cancellable. The semigroup S is *weakly cancellative* if the equations $xs = t$ and $sx = t$ have only finitely many solutions for x for each $s, t \in S$. For much more on semigroups (and semigroup algebras), see [13]. For example, we shall frequently consider the semigroups $(\mathbb{N}, +)$, $(\mathbb{Q}^{+\bullet}, +)$, and $(\mathbb{R}^{+\bullet}, +)$; of course, each is abelian and cancellative.

Let U and V be non-empty subsets of a semigroup S. Then

$$UV = \{uv : u \in U,\ v \in V\};$$

we set $U^2 = UU$.

Example 4.2. Here is a further example of a semigroup. The set $S = (\mathbb{Z}^+)^{<\omega}$ is the family of all sequences

$$x = (x_1, x_2, \dots, x_k, 0, 0, \dots),$$

where $k \in \mathbb{N}$ and $x_1, x_2, \dots, x_k \in \mathbb{Z}^+$; the set S is a semigroup with respect to coordinatewise addition, and is denoted by $((\mathbb{Z}^+)^{<\omega}, +)$.

Definition 4.3. Let S be a semigroup. A *semi-character* on S is a map $\theta : S \to \overline{\mathbb{D}}$ such that

$$\theta(st) = \theta(s)\theta(t) \quad (s, t \in S)$$

and $\theta \neq 0$. The space of semi-characters on S is denoted by Φ_S.

Definition 4.4. Let S be a semigroup. A *weight* on S is a function $\omega : S \to \mathbb{R}^{+\bullet}$ which is submultiplicative on S, in the sense that

$$\omega(st) \leq \omega(s)\omega(t) \quad (s, t \in S). \tag{4.1}$$

A weight ω on a group G is *symmetric* if

$$\omega(s^{-1}) = \omega(s) \quad (s \in G).$$

In the case where S has an identity e_S, we require also that $\omega(e_S) = 1$.

Let ω_1 and ω_2 be weights on a semigroup S. Then $\omega_1\omega_2$ is also a weight on S.

Two weights ω_1 and ω_2 on a semigroup S are said to be *equivalent* if there is a continuous isomorphism from $\ell^1(S,\omega_1)$ onto $\ell^1(S,\omega_2)$. Properties involving topological centres and Arens regularity are unaffected by a change to an equivalent weight.

Definition 4.5. Let ω be a weight on $\mathbb{R}^+$ or $\mathbb{R}^{+\bullet}$. Then ω is a *continuous weight* if ω is a continuous function (for the usual topology), and ω is a *measurable weight* if ω is a measurable function (with respect to Lebesgue measure).

Proposition 4.6. *Let ω be a weight on a semigroup S. Then $\ell^1(S,\omega)$ is a Banach algebra with respect to a product $\star$ that satisfies the condition that*

$$\delta_s \star \delta_t = \delta_{st} \quad (s,t \in S). \ \blacksquare$$

The above product $\star$ is the *convolution product* on $\ell^1(S,\omega)$. At least in the case where S is a group, the weighted convolution algebras

$$(\ell^1(S,\omega), \star, \|\cdot\|_\omega)$$

are called *Beurling algebras* [12, 61]. We shall term the Banach algebras $\ell^1(S,\omega)$ the *weighted convolution algebras on S*; the Banach algebra $\ell^1(S)$ is also called the *semigroup algebra* on S. These Banach algebras have been much discussed recently; see, for example, [13] and [16].

It is standard and easily seen that, for each semi-character $\theta \in \Phi_S$, the map

$$\sum f(s)\delta_s \mapsto \sum f(s)\theta(s), \quad \ell^1(S) \to \mathbb{C},$$

is a character on the Banach algebra $\ell^1(S)$, and that each character on $\ell^1(S)$ arises in this way; the topology of pointwise convergence on Φ_S coincides with the Gel'fand topology when Φ_S is viewed as the character space of $\ell^1(S)$.

Let S be an abelian semigroup. Then $\ell^1(S)$ is semisimple if and only if Φ_S *separates the points of* S, in the sense that, for each $s,t \in S$ with $s \neq t$, there exists $\varphi \in \Phi_S$ such that $\varphi(s) \neq \varphi(t)$ [39, Theorem 3.5], and this holds if and only if S is *separating*, in the sense that $s = t$ whenever $s,t \in S$ with $st = s^2 = t^2$ [39, Theorem 5.8]. These conditions are certainly satisfied in the case where S is cancellative.

We shall now discuss weighted convolution algebras on a semigroup S.

Let ω be a weight on S. Then an algebra $\ell^1(S,\omega)$ is commutative if and only if S is abelian.

Definition 4.7. Let ω be a weight on a semigroup S. Then ω is *radical* or *semisimple* if $A_\omega = \ell^1(S,\omega)$ is a radical or semisimple Banach algebra, respectively.

Let ω be a weight on a semigroup S. For $s \in S$, we set

$$\nu_s = \inf\{\omega(s^n)^{1/n} : n \in \mathbb{N}\}.$$

Thus $\nu_s = \lim_{n\to\infty} \omega(s^n)^{1/n}$, and ν_s is the spectral radius of δ_s in the Banach algebra A_ω; s is *quasi-nilpotent* if $\nu_s = 0$, so that δ_s is quasi-nilpotent in A_ω. Suppose that ω is a radical weight on S. Then clearly $\nu_s = 0$ $(s \in S)$. It is shown in [11, Example 2.3.13(ii)] that A_ω is a radical Banach algebra whenever $\nu_s = 0$ $(s \in S)$ and $\omega(st) = \omega(ts)$ $(s,t \in S)$.

For example, let $\mathbb{S}_2$ be the free semigroup on two generators, and let $|w|$ be the length of a word w in $\mathbb{S}_2$. Set

$$\omega(w) = \exp(-|w|^2) \quad (w \in \mathbb{S}_2).$$

Then ω satisfies the above conditions, and so ω is a radical weight on $\mathbb{S}_2$.

Note that we do not necessarily suppose that a weight ω on a semigroup S is such that $\omega(s) \geq 1$ $(s \in S)$, written '$\omega \geq 1$'. However, let G be an amenable group, and let ω be a weight on G. Then there is a weight $\widetilde{\omega}$ on G which is equivalent to ω and such that $\widetilde{\omega} \geq 1$ [12, Theorem 7.44].

Let G be a group. Then it is a famous conjecture that $\ell^1(G, \omega)$ is semisimple for each weight ω on G; this is proved in [12, Theorem 7.13] whenever G is a maximally almost periodic group and ω is an arbitrary weight on G and whenever G is an arbitrary group and ω is a symmetric weight on G, but the conjecture is open in the general case.

PROPOSITION 4.8. *Let S be an abelian semigroup, and let ω be a weight on S. Then ω is a semisimple weight if and only if S is separating and $\nu_s > 0$ $(s \in S)$.*

Proof. Set $A_\omega = \ell^1(S, \omega)$. Let $s \in S$. Of course, since A_ω is commutative, $\delta_s \in \operatorname{rad} A_\omega$ if and only if $\nu_s = 0$, and

$$\nu_s = \sup\{|\varphi(\delta_s)| : \varphi \in \Phi_\omega\},$$

where Φ_ω is the character space of A_ω. Clearly $|\varphi(\delta_s)| \leq \omega(s)$ $(s \in S)$ for each $\varphi \in \Phi_\omega$.

First suppose that S is separating and that $\nu_s > 0$ $(s \in S)$. We take $f \in \operatorname{rad} A_\omega$, so that $\varphi(f) = 0$ $(\varphi \in \Phi_\omega)$. Now take $t \in S$. Since $\nu_t > 0$, there exists $\varphi_0 \in \Phi_\omega$ such that $\varphi_0(\delta_t) \neq 0$. Define

$$g(s) = f(s)\varphi_0(\delta_s) \quad (s \in S),$$

so that $g \in \ell^1(S)$. For each $\theta \in \Phi_S$, define

$$\varphi_\theta : h \mapsto \sum\{h(s)\varphi_0(\delta_s)\theta(s) : s \in S\}, \quad A_\omega \to \mathbb{C},$$

so that $\varphi_\theta \in \Phi_\omega$. We have

$$\sum\{g(s)\theta(s) : s \in S\} = \varphi_\theta(f) = 0 \quad (\theta \in \Phi_S),$$

and so $g \in \operatorname{rad} \ell^1(S)$. Since S is separating, it follows that $g = 0$, and so $f(t) = 0$. This holds for each $t \in S$, and so $f = 0$. It follows that A_ω is semisimple, and so ω is semisimple.

Conversely, suppose that ω is semisimple. Let $s \in S$. Since $\delta_s \notin \operatorname{rad} A_\omega$, necessarily $\nu_s > 0$. Now take elements $s, t \in S$ with $st = s^2 = t^2$. Then

$$\varphi(\delta_s)\varphi(\delta_t) = \varphi(\delta_s)^2 = \varphi(\delta_t)^2 \quad (\varphi \in \Phi_\omega),$$

and so $\varphi(\delta_s) = \varphi(\delta_t)$ $(\varphi \in \Phi_\omega)$. Since A_ω is semisimple and commutative, it follows that $\delta_s = \delta_t$, and so $s = t$. Thus S is separating. ∎

DEFINITION 4.9. A *difference semigroup* in $\mathbb{R}^{+\bullet}$ is a subsemigroup S of $(\mathbb{R}^{+\bullet}, +)$ such that $a - b \in S$ whenever $a, b \in S$ and $a > b$.

For example, $\mathbb{Q}^{+\bullet}$ is a difference semigroup in $\mathbb{R}^{+\bullet}$.

PROPOSITION 4.10. *Let S be a difference semigroup in $\mathbb{R}^{+\bullet}$, and let ω be a weight on S. Then exactly one of the following two possibilities occurs:*

(i) *ω is a radical weight;*
(ii) *there is a weight $\widetilde{\omega}$ on S with $\widetilde{\omega} \geq 1$ such that $\ell^1(S, \omega)$ is isometrically isomorphic to $\ell^1(S, \widetilde{\omega})$; further ω is semisimple.*

Proof. Suppose that (i) fails, so that ω is not a radical weight. Then there exists $s_0 \in S$ such that $\nu_{s_0} > 0$, and so there exists φ in the character space of $\ell^1(S, \omega)$ with $\varphi(\delta_{s_0}) \neq 0$. Define

$$\alpha(s) = \varphi(\delta_s) \quad (s \in S),$$

so that $\alpha(st) = \alpha(s)\alpha(t)$ $(s, t \in S)$ and $|\alpha(s)| \leq \omega(s)$ $(s \in S)$.

Assume towards a contradiction that $\alpha(s) = 0$ for some $s \in S$. For $t \in S$ with $t > s$, we have $t - s \in S$ because S is a difference semigroup, and so $\alpha(t) = \alpha(t - s)\alpha(s) = 0$. For $t \in S$ with $t \leq s$, we have $nt > s$ for some $n \in \mathbb{N}$, and so $\alpha(t) = \alpha(nt)^{1/n} = 0$. Thus $\alpha(t) = 0$ $(t \in S)$, a contradiction. Hence $\alpha(s) \neq 0$ $(s \in S)$.

Define

$$\widetilde{\omega}(s) = \frac{\omega(s)}{|\alpha(s)|} \quad (s \in S).$$

Then $\widetilde{\omega}$ is a weight on S with $\widetilde{\omega} \geq 1$. For $f \in \ell^1(S, \widetilde{\omega})$, set

$$\theta(f)(s) = \frac{f(s)}{|\alpha(s)|} \quad (s \in S).$$

Then $\theta : \ell^1(S, \widetilde{\omega}) \to \ell^1(S, \omega)$ is an isometric isomorphism. Since $\widetilde{\omega} \geq 1$, we see that $\nu_s \geq 1$ $(s \in S)$, and so, by Proposition 4.8, $\ell^1(S, \widetilde{\omega})$ is semisimple. Hence ω is semisimple, and (ii) holds. ∎

We conclude that each weight on $\mathbb{N}$, on $\mathbb{Q}^{+\bullet}$, and on $\mathbb{R}^{+\bullet}$ is either radical or semisimple.

DEFINITION 4.11. Let ω be a weight on a subsemigroup S of $\mathbb{R}^{+\bullet}$. Then

$$\nu_\omega = \inf\{\omega(s)^{1/s} : s \in S\}.$$

PROPOSITION 4.12. *Let ω be a weight on $\mathbb{Q}^{+\bullet}$. Then:*

(i) *$\nu_\omega = \nu_s^{1/s}$ $(s \in \mathbb{Q}^{+\bullet})$;*
(ii) *the weight ω is semisimple if and only if $\nu_\omega > 0$, and ω is radical if and only if $\nu_\omega = 0$.*

Proof. (i) (This is part of [18, Lemma 2.5].) Clearly $\nu_\omega \leq \nu_s^{1/s}$ $(s \in \mathbb{Q}^{+\bullet})$. Now take $s \in \mathbb{Q}^{+\bullet}$. For each $\varepsilon > 0$, there exists $t \in S$ with $\omega(t)^{1/t} < \nu_\omega + \varepsilon$. Choose $m, n \in \mathbb{N}$ with $ms = nt$. Then

$$\omega(ms)^{1/ms} = \omega(nt)^{1/nt} \leq \omega(t)^{1/t} < \nu_\omega + \varepsilon,$$

and so $\nu_s^{1/s} \leq \nu_\omega + \varepsilon$. This holds for each $\varepsilon > 0$, and so $\nu_s^{1/s} \leq \nu_\omega$.

(ii) This is now immediate from Proposition 4.8 and some earlier remarks. ∎

We shall see in Example 4.18 that the above result does not necessarily hold for a weight ω on $\mathbb{R}^{+\bullet}$.

PROPOSITION 4.13. *Let S be a difference semigroup in $\mathbb{R}^{+\bullet}$, and let ω be a weight on S. Then the following conditions on ω are equivalent:*

(a) $\lim_{s\to\infty}\omega(s)^{1/s}=\nu_\omega$;

(b) *there exist $a,b\in\mathbb{R}^{+\bullet}$ with $0<a<b$ such that*

$$\sup\{\omega(s):s\in S\cap(a,b)\}<\infty.$$

The conditions imply that $\inf\{\omega(s):s\in S\cap(0,c)\}>0$ *for each* $c>0$.

Proof. The result is trivial if $S\cap(0,\varepsilon)=\emptyset$ for some $\varepsilon>0$, and so we can suppose that S is dense in $\mathbb{R}^{+\bullet}$.

The equivalence of (a) and (b) is [18, Lemma 2.6]. Indeed, it is immediate that (a) implies (b). We recall the proof that (b) implies (a).

Suppose that (b) is satisfied, with

$$\sup\{\omega(s):s\in S\cap(a,b)\}=M<\infty,$$

say. Assume towards a contradiction that ω is unbounded on $S\cap[b,d]$ for some $d>b$. Then there is a sequence (s_n) in $S\cap[b,d]$ with $\omega(s_n)\to\infty$ as $n\to\infty$; by passing to a subsequence, we may suppose that $s_n\to t_0$ as $n\to\infty$, where $t_0\in[b,d]$. Since the semigroup S is dense in $\mathbb{R}^{+\bullet}$, there exists $t_1\in S$ with $t_1\in(t_0-b,t_0-a)$. Since S is a difference semigroup, we have $s_n-t_1\in S$ $(n\in\mathbb{N})$, and so, for each sufficiently large $n\in\mathbb{N}$, we have $s_n-t_1\in S\cap(a,b)$. Hence

$$\omega(s_n)\leq\omega(s_n-t_1)\omega(t_1)\leq M\omega(t_1),$$

a contradiction. Thus ω is bounded on $S\cap[b,d]$ for each $d>b$.

Now fix $c>0$. Suppose first that $c>b$, and set

$$M_c=\sup\{\omega(s):s\in S\cap(c,2c)\},$$

so that, by the previous paragraph, $M_c<\infty$. For large $s\in S$, there exist $m\in\mathbb{N}$ and $r\in S$ with $c\leq r<2c$ and $s=mc+r$. Thus

$$\omega(s)^{1/s}\leq\omega(c)^{m/s}M_c^{1/s},$$

and so $\limsup_{s\to\infty}\omega(s)^{1/s}\leq\omega(c)^{1/c}$. Now suppose that $c\leq b$. Then there exists $n\in\mathbb{N}$ with $nc>b$, and so

$$\limsup_{s\to\infty}\omega(s)^{1/s}\leq\omega(nc)^{1/nc}\leq\omega(c)^{1/c}.$$

Thus $\limsup_{s\to\infty}\omega(s)^{1/s}\leq\nu_\omega$. Since $\omega(s)^{1/s}\geq\nu_\omega$ $(s\in S)$, we have established clause (a).

Now take $c>0$. Assume towards a contradiction that there exists a sequence (s_n) in $S\cap(0,c)$ such that $\omega(s_n)\to0$. By passing to a subsequence, we may suppose that $s_n\to x\in[0,c]$. Take $t\in S$ with $x<t<x+(b-a)/4$. Then we may suppose that

$$|s_n-t|<(b-a)/2\quad(n\in\mathbb{N}).$$

Set $s=(a+b)/2$. Then $s+t-s_n\in S\cap(a,b)$ $(n\in\mathbb{N})$, and so

$$0<\omega(s+t)\leq\omega(s_n)\omega(s+t-s_n)\leq M\omega(s_n)\to0\quad\text{as}\quad n\to\infty,$$

a contradiction. Thus $\inf\{\omega(s):s\in S\cap(0,c)\}>0$. ∎

The following proposition is [18, Lemma 2.7]. A weight on $S = \mathbb{Q}^{+\bullet}$ which does not satisfy these equivalent conditions will be discussed in Chapter 11.

PROPOSITION 4.14. *Let S be a subsemigroup of $\mathbb{R}^{+\bullet}$, and let ω be a weight on S. Then the following conditions on ω are equivalent:*

(a) *there exist $a, b \in \mathbb{R}^{+\bullet}$ with $a < b$ such that*
$$\inf\{\omega(s) : s \in S \cap (a,b)\} > 0;$$
(b) *there exists $c \in \mathbb{R}^{+\bullet}$ such that*
$$\inf\{\omega(s) : s \in S \cap (0,c)\} > 0;$$
(c) $\liminf_{s\to 0+} \omega(s) \geq 1$;
(d) $\liminf_{s\to 0+} \omega(s) > 0$.

Proof. The result is trivial if $S \cap (0,\varepsilon) = \emptyset$ for some $\varepsilon > 0$, and so we can suppose that S is dense in $\mathbb{R}^{+\bullet}$.

(a)⇒(b). Since S is dense in $\mathbb{R}^{+\bullet}$, we may suppose that $a, b \in S$ and that there exists $c \in S \cap (0, b-a)$. Assume towards a contradiction that there is a sequence (s_n) in $S \cap (0,c)$ with $\omega(s_n) \to 0$ as $n \to \infty$. For each $n \in \mathbb{N}$, we have $a + s_n \in S \cap [a,b]$ and
$$\omega(a+s_n) \leq \omega(a)\omega(s_n),$$
and so $\inf\{\omega(s) : s \in S \cap (a,b)\} = 0$, a contradiction. Thus (b) holds.

(b)⇒(c). Set $\ell = \liminf_{s\to 0+} \omega(s)$, and take (s_n) in S with $s_n \to 0$ and $\omega(s_n) \to \ell$ as $n \to \infty$. Then
$$\ell \leq \liminf_{n\to\infty} \omega(2s_n) \leq \liminf_{n\to\infty} \omega(s_n)^2 = \ell^2.$$
By (b), $\ell > 0$, and so $\ell \geq 1$.

(c)⇒(d)⇒(a). These are immediate. ∎

COROLLARY 4.15. *Let S be a difference semigroup in $\mathbb{R}^{+\bullet}$, and let ω be a weight on S such that*
$$\liminf_{s\to 0+} \omega(s) < 1.$$
Then $\sup\{\omega(s) : s \in S \cap (a,b)\} = \infty$ *for each $a, b \in \mathbb{R}$ with $0 < a < b$.*

Proof. Assume towards a contradiction that $\sup\{\omega(s) : s \in S \cap (a,b)\} < \infty$ for some $a, b \in \mathbb{R}$ with $0 < a < b$. It follows from Proposition 4.13 that $\inf\{\omega(s) : s \in S \cap (0,c)\} > 0$ for each $c > 0$, and so clause (b) of Proposition 4.14 holds. Hence clause (c) of that proposition holds, a contradiction. ∎

COROLLARY 4.16. *Let ω be a weight on $\mathbb{Q}^{+\bullet}$ such that*
$$\liminf_{s\to 0+} \omega(s) < 1.$$
Then ω is a radical weight.

Proof. Assume towards a contradiction that ω is not a radical weight. By Proposition 4.12(ii), $\nu_\omega > 0$, and so there exists $\delta \in (0,1)$ such that $\omega(s) > \delta^s$ $(s \in \mathbb{Q}^{+\bullet})$. But now $\inf\{\omega(s) : s \in \mathbb{Q}^{+\bullet} \cap (0,1)\} \geq \delta$, and so, by Proposition 4.14, $\liminf_{s\to 0+} \omega(s) \geq 1$, a contradiction. ∎

We prove the following standard result; more general results are given in [40, §7.4].

Proposition 4.17. *Let ω be a measurable weight on $\mathbb{R}^{+\bullet}$. Then:*

(i) $\sup\{\omega(s) : s \in (a,b)\} < \infty$ *for each* $a, b \in \mathbb{R}^{+\bullet}$ *with* $0 < a < b$;
(ii) $\inf\{\omega(s) : s \in (0,c)\} > 0$ *for each* $c > 0$;
(iii) $\liminf_{s\to 0+} \omega(s) \geq 1$.

Proof. (i) Take η with $\exp \eta = \omega$, so that $\eta : \mathbb{R}^{+\bullet} \to \mathbb{R}$ is a subadditive function. For $\alpha > 0$, set

$$E_\alpha = \{t \in (0,\alpha) : \eta(t) \geq \eta(\alpha)/2\}.$$

Then each set E_α is a measurable subset of $\mathbb{R}^+$, and $(0,\alpha) = E_\alpha \cup (\alpha - E_\alpha)$ because $\eta(\alpha) \leq \eta(s) + \eta(t)$ whenever $s, t \in (0,\alpha)$ with $s + t = \alpha$. Hence $m(E_\alpha) \geq \alpha/2$, where m denotes Lebesgue measure.

Assume towards a contradiction that η is not bounded above on the interval (a,b) for some $a, b \in \mathbb{R}$ with $0 < a < b$. For each $n \in \mathbb{N}$, set

$$F_n = \{t \in (a,b) : \eta(t) \geq n\},$$

so that F_n is a measurable subset of $\mathbb{R}^+$, and then choose $t_n \in (a,b)$ with $\eta(t_n) > 2n$. Thus, for each $n \in \mathbb{N}$, we have $E_{t_n} \subset F_n$, and so

$$m(F_n) \geq m(E_{t_n}) \geq t_n/2 > a/2.$$

Set $F = \bigcap\{F_n : n \in \mathbb{N}\}$, so that F is a measurable subset of $\mathbb{R}^+$ with $m(F) \geq a/2 > 0$. For each $t \in F$, we have $\eta(t) \geq n$ for each $n \in \mathbb{N}$, a contradiction.

Thus $\sup\{\omega(s) : s \in (a,b)\} < \infty$ for each $a, b \in \mathbb{R}$ with $0 < a < b$.

(ii) and (iii) follow from Propositions 4.13 and 4.14. ∎

Example 4.18. Let $\mathcal{H}$ be a (Hamel) basis for the linear space $\mathbb{R}$ over the field $\mathbb{Q}$; we may suppose that $1 \in \mathcal{H}$. Then, for each $x \in \mathbb{R}^\bullet$, there exist a unique $\alpha_0 \in \mathbb{Q}$ and unique numbers $\alpha_1, \dots, \alpha_n \in \mathbb{Q}^\bullet$ and elements $h_1, \dots, h_n \in \mathcal{H}$ such that

$$x = \alpha_0 1 + \alpha_1 h_1 + \cdots + \alpha_n h_n.$$

Now define $\omega(x) = \exp(\alpha_0)$. Then clearly ω is a weight on $\mathbb{R}$ with $\liminf_{s\to 0+} \omega(s) = 0$. In this case $\liminf_{s\to 0+} \omega(s) = 1$ on $\mathbb{Q}^{+\bullet}$.

However, $\nu_1 = \mathrm{e}$, and so ω is not a radical weight; indeed, it is semisimple. This shows that Corollary 4.16 does not necessarily hold for weights on $\mathbb{R}^{+\bullet}$. Further, $\nu_{h_1} = 1$, and so clause (i) of Proposition 4.12 fails in this case.

More generally, we can define $\eta(h)$ arbitrarily for $h \in \mathcal{H}$, and extend η to be an additive function on $\mathbb{R}^{+\bullet}$. Set $\omega = \exp \eta$, so that

$$\omega(x+y) = \omega(x)\omega(y) \quad (x, y \in \mathbb{R}^{+\bullet}),$$

and hence ω is a weight on $\mathbb{R}^{+\bullet}$. Clearly $\nu_h = \exp \eta(h)$ $(h \in \mathcal{H})$, and so ω is semisimple. However,

$$\nu_\omega \leq \inf\{\exp(\eta(h)/h) : h \in \mathcal{H}\},$$

and so we can arrange that $\nu_\omega = 0$. This shows that clause (ii) of Proposition 4.12 fails in this case.

For a discussion of compact elements in weighted semigroup algebras, see [18]; we shall give an elementary proof of the following proposition, which is part of [18, Corollary 3.6]. More general results were given by N. Grønbæk in his thesis [36].

Let S be an abelian semigroup, and let ω be a weight on S. Then it is clear that the collection of compact elements of A_ω forms a closed ideal in A_ω. For further remarks on this closed ideal, see [18].

PROPOSITION 4.19. *Let ω be a weight on a cancellative, abelian semigroup S.*

(i) *Let $f \in \ell^1(S,\omega)$. Then f is a compact element of $\ell^1(S,\omega)$ if and only if*

$$\operatorname*{Lim}_{t\to\infty} \sum_{s\in S} |f(s)| \frac{\omega(s+t)}{\omega(t)} = 0.$$

(ii) *The only compact element of $\ell^1(S,\omega)$ is 0 if and only if δ_s is not a compact element of $\ell^1(S,\omega)$ for each $s \in S$.*

Proof. Set $A_\omega = \ell^1(S,\omega)$.

(i) Suppose that L_f is a compact operator on A_ω. For each sequence (t_n) of distinct points of S, the sequence $(\widetilde{\delta}_{t_n})$ is contained in the unit ball of A_ω, and so the sequence $((\delta_{t_n} \star f)/\omega(t_n) : n \in \mathbb{N})$ has a convergent subsequence, corresponding to (n_j), say. Since S is cancellative, the only possible limit of such a subsequence is 0. Hence

$$\lim_{j\to\infty} \sum_{s\in S} |f(s)| \frac{\omega(s+t_{n_j})}{\omega(t_{n_j})} = \lim_{j\to\infty} \sum_{s\in S} \frac{\|\delta_{t_{n_j}} \star f\|_\omega}{\omega(t_{n_j})} = 0.$$

It follows that $\operatorname{Lim}_{t\to\infty} \sum_{s\in S} |f(s)|\omega(s+t)/\omega(t) = 0$.

For the converse, first take $s \in S$ such that $\operatorname{Lim}_{t\to\infty} \omega(s+t)/\omega(t) = 0$.

To show that δ_s is compact, it suffices to suppose that S is countable. Choose finite subsets S_j of S such that $S_j \subset S_{j+1}$ $(j \in \mathbb{N})$ and $\bigcup\{S_j : j \in \mathbb{N}\} = S$, and define

$$Q_j : g \mapsto g \mid S_j, \quad A_\omega \to A_\omega \quad (j \in \mathbb{N}).$$

Then each Q_j is a finite-rank operator on A_ω, and hence is compact. Thus each operator $L_{\delta_s} \circ Q_j$ is compact.

Now fix $\varepsilon > 0$, and choose $j_0 \in \mathbb{N}$ such that $\omega(s+t)/\omega(t) < \varepsilon$ whenever $t \in S \setminus S_{j_0}$. For each $j \geq j_0$ and $g \in (A_\omega)_{[1]}$, we have

$$\begin{aligned}\sum_{u\in S} |(\delta_s \star Q_j g)(u) - (\delta_s \star g)(u)|\omega(u) &= \sum_{t\in S} |(Q_j g - g)(t)|\omega(s+t) \\ &\leq \sum_{t\in S\setminus S_j} |g(t)|\omega(t)\, \frac{\omega(s+t)}{\omega(t)} \\ &\leq \varepsilon,\end{aligned}$$

and so $\|\delta_s \star Q_j - \delta_s\| \leq \varepsilon$ $(j \geq j_0)$. Thus $L_{\delta_s} = \lim_{j\to\infty} L_{\delta_s} \circ Q_j$ is a compact operator on A_ω, and δ_s is compact in A_ω.

Now suppose that $f \in A_\omega$ is such that

$$\operatorname*{Lim}_{t\to\infty} \sum_{s\in S} |f(s)|\omega(s+t)/\omega(t) = 0.$$

Then $\mathrm{Lim}_{t\to\infty}\,\omega(s+t)/\omega(t)=0$ for each $s\in\operatorname{supp} f$, and so δ_s is compact for each such s, whence f is compact.

(ii) This is now immediate. ∎

Let ω be a weight on a semigroup S. Then the module actions on $A'_\omega=\ell^\infty(S,1/\omega)$ are determined as follows: for each $s\in S$ and $\lambda\in A_\omega$, we have

$$(s\cdot\lambda)(t)=\lambda(ts),\qquad(\lambda\cdot s)(t)=\lambda(st)\qquad(t\in S).\tag{4.2}$$

DEFINITION 4.20. A weight ω on a semigroup S is *Arens regular* or *strongly Arens irregular on S* if the algebra $\ell^1(S,\omega)$ has the corresponding property.

The seminal discussion of the Arens regularity of weighted semigroup algebras was given by Craw and Young in [10]; we shall give their condition for a weight on a cancellative semigroup to be Arens regular in Theorem 9.4.

EXAMPLE 4.21. Let ω be a weight on a semigroup S, and again set $A_\omega=\ell^1(S,\omega)$. Suppose further that T is a semigroup and that $\theta:S\to T$ is a semigroup epimorphism. Define

$$\widetilde{\omega}(x)=\inf\{\omega(s):\theta(s)=x\}\qquad(x\in T),$$

so that $\widetilde{\omega}(x)\geq 0$ $(x\in T)$. In the case where $\widetilde{\omega}(x)>0$ $(x\in T)$, we see that $\widetilde{\omega}$ is a weight on T; it is called the *induced weight*. Set $A_{\widetilde{\omega}}=\ell^1(T,\widetilde{\omega})$. Clearly θ induces a continuous algebra epimorphism $\theta:A_\omega\to A_{\widetilde{\omega}}$ such that $\theta(\delta_s)=\delta_{\theta(s)}$ $(s\in S)$, and $A_{\widetilde{\omega}}$ is isometrically isomorphic to $A_\omega/\ker\theta$. Further, $\widetilde{\omega}$ is Arens regular whenever ω is Arens regular.

Following [10], we shall also use the following standard notation: for a weight ω on a semigroup S, we set

$$\Omega(s,t)=\frac{\omega(st)}{\omega(s)\omega(t)}\qquad(s,t\in S).\tag{4.3}$$

Thus $0<\Omega(s,t)\leq 1$ $(s,t\in S)$. Let G be a group. Then we see that

$$\Omega(s,t)\geq\frac{1}{\omega(t)\omega(t^{-1})}\qquad(s,t\in G).\tag{4.4}$$

Let ω be a weight on a semigroup S. For each $s\in S$, the function $t\mapsto\Omega(s,t)$, $S\to\mathbb{I}$, has a continuous extension to a function $\beta S\to\mathbb{I}$; the value of this function at $v\in\beta S$ is denoted by $\Omega(s,v)$. Next, the function $s\mapsto\Omega(s,v)$, $S\to\mathbb{I}$, has a continuous extension to a function $\beta S\to\mathbb{I}$; the value of this function at $u\in\beta S$ is denoted by $\Omega_\Box(u,v)$. Let $u,v\in\beta S$, say $u=\lim_\alpha s_\alpha$ and $v=\lim_\beta t_\beta$. Then we express $\Omega_\Box(u,v)$ by the repeated limit

$$\Omega_\Box(u,v)=\lim_\alpha\lim_\beta\Omega(s_\alpha,t_\beta);\tag{4.5}$$

of course the repeated limit is independent of the nets (s_α) and (t_β). Similarly, we define

$$\Omega_\Diamond(u,v)=\lim_\beta\lim_\alpha\Omega(s_\alpha,t_\beta).$$

The following well-known result is proved in detail in [12, Proposition 3.1].

THEOREM 4.22. *Let ω be a weight on a semigroup S, and let $u, v \in \beta S$. Then there are sequences (s_m) and (t_n) in S such that*

$$\Omega_{\Box}(u,v) = \lim_{m\to\infty}\lim_{n\to\infty}\Omega(s_m,t_n) \quad \textit{and} \quad \Omega_{\Diamond}(u,v) = \lim_{n\to\infty}\lim_{m\to\infty}\Omega(s_m,t_n). \ \blacksquare$$

Let S be a semigroup, let ω be a weight on S, and let $A_\omega = \ell^1(S,\omega)$ and E_ω be as above. The first and second Arens products on A_ω'' are now denoted by $\Box_\omega$ and $\Diamond_\omega$, respectively; however, we write just $\Box$ and $\Diamond$ in the special case where $\omega = 1$. Suppose that S is a weakly cancellative semigroup. Then, essentially as in [13, Theorem 4.6], we see that E_ω is a submodule of A_ω', so that A_ω is a dual Banach algebra, and that

$$(A_\omega'', \Box_\omega) = A_\omega \ltimes E_\omega^\circ$$

as a semidirect product; in this case, the Banach algebra A_ω is left strongly Arens irregular if and only if $\mathfrak{Z}_t^{(\ell)}(A_\omega'') \cap E_\omega^\circ = \{0\}$.

We remark that A_ω may be a dual Banach algebra even when S is not weakly cancellative: see Example 9.13, below.

DEFINITION 4.23. Let V be a semigroup which is also a compact topological space. Then V is a *compact right topological semigroup* if the map R_v is continuous on V for each $v \in V$.

For the theory of compact right topological semigroups, see [6, 13, 41], for example.

Let S be a semigroup. Then $(\beta S, \Box)$ is a compact right topological semigroup that is a subsemigroup of $(M(\beta S), \Box)$. This structure is discussed extensively in [13] and in the monographs [6, 41]. For $u \in \beta S$, we shall often identify u with the element δ_u, regarded as a measure in $M(\beta S)$. For example, we may regard $\theta_\omega(u)$ as an element of $\beta S_\omega \subset A_\omega''$ when $\omega : S \to \mathbb{R}^{+\bullet}$ is a function on S.

Let ω be a weight on S, and take elements $u, v \in \beta S$, say $u = \lim_\alpha s_\alpha$ and $v = \lim_\beta t_\beta$, where (s_α) and (t_β) are nets in S, and let $\lambda \in C(\beta S)$. Then

$$\langle \theta_\omega(u) \mathbin{\Box_\omega} \theta_\omega(v), \lambda\omega\rangle_\omega = \lim_\alpha \lim_\beta \langle \widetilde{\delta}_{s_\alpha} \star \widetilde{\delta}_{t_\beta}, \lambda\omega\rangle_\omega = \lim_\alpha \lim_\beta \Omega(s_\alpha, t_\beta)\langle u \mathbin{\Box} v, \lambda\rangle,$$

and so

$$\langle \theta_\omega(u) \mathbin{\Box_\omega} \theta_\omega(v), \lambda\omega\rangle_\omega = \Omega_{\Box}(u,v)\langle u \mathbin{\Box} v, \lambda\rangle \tag{4.6}$$

and

$$\|\theta_\omega(u) \mathbin{\Box_\omega} \theta_\omega(v)\|_\omega = \Omega_{\Box}(u,v). \tag{4.7}$$

It follows from (4.6) that

$$\langle \theta_\omega(\mu) \mathbin{\Box_\omega} \theta_\omega(v), \lambda\omega\rangle_\omega = 0 \tag{4.8}$$

whenever $\mu \in M(\beta S)$, $v \in \beta S$, and $\lambda \in C(\beta S)$ are such that $(\operatorname{supp}\lambda) \cap (\beta S \mathbin{\Box} v) = \emptyset$.

Similarly, $\langle \theta_\omega(u) \mathbin{\Diamond_\omega} \theta_\omega(v), \lambda\omega\rangle_\omega = \Omega_{\Diamond}(u,v)\langle u \mathbin{\Diamond} v, \lambda\rangle$ for $u, v \in \beta S$ and $\lambda \in C(\beta S)$, and so we have

$$|\langle \theta_\omega(\mu) \mathbin{\Diamond_\omega} \theta_\omega(v), \lambda\omega\rangle_\omega| \le \|\mu\| \sup_{x \in \beta S} |\langle x \mathbin{\Diamond} v, \lambda\rangle| \le \|\mu\|\,\|\lambda\| \tag{4.9}$$

for each $\mu \in M(\beta S)$, $v \in \beta S$, and $\lambda \in C(\beta S)$.

The question when $\ell^1(\mathbb{Z}, \omega)$ has the above-mentioned properties of Arens regularity and strong Arens irregularity is discussed in detail in [12, Chapter 9]. For example, whilst

$\ell^1(\mathbb{Z})$ is strongly Arens irregular, the algebras $\ell^1(\mathbb{Z},\omega_\alpha)$ are Arens regular whenever $\alpha > 0$ [10], [12, Example 9.1]; here

$$\omega_\alpha(n) = (1+|n|)^\alpha \quad (n \in \mathbb{Z}). \tag{4.10}$$

In this latter case,

$$\Omega_\Box(u,v) = \Omega_\Diamond(u,v) = 0 \quad (u,v \in \mathbb{Z}^*).$$

Further, there are several examples (e.g., [12, Examples 9.7, 9.8, 9.15, 9.16]) of weights ω on $\mathbb{Z}$ that are neither Arens regular nor strongly Arens irregular; however, no such weight on $\mathbb{N}$ was given in [12]. Necessary and sufficient conditions for ω to be Arens regular are specified in [12, Theorem 8.11], but such conditions for ω to be strongly Arens irregular are not known; some specific open questions are raised in [12]. For example, it is known, as we shall see below, that $\ell^1(\mathbb{Z},\omega)$ is strongly Arens irregular whenever

$$\liminf_{n\to\infty} \omega(n)\omega(-n) < \infty, \tag{4.11}$$

but it is open whether or not $\ell^1(\mathbb{Z},\omega)$ is necessarily strongly Arens irregular whenever ω satisfies the weaker condition that

$$\liminf_{n\to\infty} \omega(n) < \infty \quad \text{and} \quad \liminf_{n\to\infty} \omega(-n) < \infty.$$

5. Sets determining for the topological centre

The following definition is taken from [13, Definition 12.3]. Recall that we regard a Banach algebra A as being a subset of A''.

Definition 5.1. Let A be a Banach algebra. A subset V of A'' is *determining for the left topological centre* of A'' if $\Phi \in A$ whenever $\Phi \in A''$ and $\Phi \Box \Psi = \Phi \Diamond \Psi$ $(\Psi \in V)$.

Clearly A is left strongly Arens irregular if and only if A'' is determining for the left topological centre. In the case where A is commutative, we say that V is *determining for the topological centre*, and so V is determining for the topological centre if $\Phi \in A$ whenever Φ commutes in $(A'',\Box)$ with each element of V.

Let S be a semigroup, and set $A = \ell^1(S)$. In [13, Chapter 12], it is shown that certain 'small' subsets of $\beta S \subset M(\beta S)$ are determining for the left topological centre of $A'' = M(\beta S)$. For example, it is proved in [13, Theorem 12.15] that, for each infinite semigroup S such that S is 'weakly cancellative and nearly right cancellative' (which includes the case where S is cancellative, and, in particular, where S is a group), there is a two-element subset $V = \{a,b\}$ of βS such that V is determining for the left topological centre. (It follows that A is strongly Arens irregular whenever S is cancellative; however, [13, Example 7.33] shows that this is not necessarily the case when S is just weakly cancellative.) Properties of such points a and b are investigated; for example, it is shown in [13, Theorem 12.8] that, in the case where S is infinite, countable, and cancellative and P and Q are infinite subsets of S, we may choose $a \in P^*$ and $b \in Q^*$. In this section, we shall show that a modification of the arguments in [13] gives a similar result for certain Beurling algebras $\ell^1(G,\omega)$ and more general weighted convolution algebras on semigroups.

The following definition was given in [12, Definition 7.41].

DEFINITION 5.2. Let G be an infinite group, let $\omega : G \to \mathbb{R}^{+\bullet}$ be a function on G, and let T be a subset of G. The function ω is *diagonally bounded* on T if

$$d_T := \sup\{\omega(t)\omega(t^{-1}) : t \in T\} < \infty. \tag{5.1}$$

In the case where ω is a weight on G, it follows from (4.4) that condition (5.1) is satisfied if and only if

$$1/d_T \leq \Omega(s,t) \leq 1 \quad (s \in G,\, t \in T). \tag{5.2}$$

This condition has proved to be significant in several contexts. For example, a Banach algebra $\ell^1(G,\omega)$ is amenable if and only if G is an amenable group and ω is diagonally bounded on G [38]; for a new proof of this latter result, see [26].

Clearly a weight ω on $\mathbb{Z}$ is diagonally bounded on an infinite subset of $\mathbb{Z}$ if and only if

$$\liminf_{n\to\infty} \omega(n)\omega(-n) < \infty,$$

as in (4.11).

We now extend the above definition.

DEFINITION 5.3. Let S be a semigroup, let ω be a weight on S, and let T be a subset of S. Then:

(i) ω is *diagonally bounded* on T, with *bound* $d_T > 0$, if

$$1/d_T \leq \Omega(s,t) \leq 1 \quad (s \in S,\, t \in T); \tag{5.3}$$

(ii) ω is *weakly diagonally bounded* on T, with *bound* $c_T > 0$, if

$$1/c_T \leq \Omega_\Box(u,v) \leq 1 \quad (u \in S^*,\, v \in T^*).$$

Clearly a diagonally bounded weight on a semigroup is weakly diagonally bounded, and a weight on a group G is diagonally bounded on a subset T of G in the sense of Definition 5.3(i) if and only if it is diagonally bounded on T in the old sense. Trivially the weight $\omega = 1$ is diagonally bounded on each semigroup S.

Let S be a semigroup. A weight ω is weakly diagonally bounded on a subset T of S, with bound $c_T > 0$, if and only if

$$\operatorname*{Lim\,inf}_{s\in S,\, s\to\infty}\ \operatorname*{Lim\,inf}_{t\in T,\, t\to\infty}\ \Omega(s,t) \geq 1/c_T;$$

explicitly, this holds if and only if, for each $\varepsilon > 0$, there is a cofinite subset S_0 of S such that, for each $s \in S_0$, there is a cofinite subset T_s of T with

$$\frac{1}{c_T}(1-\varepsilon) \leq \Omega(s,t) \leq 1 \quad (t \in T_s). \tag{5.4}$$

We shall give a variety of examples of weakly diagonally bounded weights below.

Let ω be a weight on an infinite, countable, cancellative semigroup S such as $\mathbb{Q}$ or $\mathbb{Q}^{+\bullet}$, and suppose that ω is weakly diagonally bounded on an infinite subset of S. We do not know whether or not there is always a weight equivalent $\widetilde{\omega}$ to ω on S such that $\widetilde{\omega}$

is weakly diagonally bounded on an infinite subset T of S, with bound $c_T < 2$, or even $c_T = 1$.

Let S be an infinite semigroup, and let ω be a weight on S that is weakly diagonally bounded on an infinite subset T of S, with bound $c_T > 0$. Suppose that $\lambda \in C(\beta S)$, $u \in S^*$, and $v \in T^*$ are such that $\langle u \,\Box\, v, \lambda\rangle \geq 0$. Then it follows from (4.6) that

$$\langle \theta_\omega(u) \,\Box_\omega\, \theta_\omega(v), \lambda\omega\rangle_\omega \geq \frac{1}{c_T}\,\langle u \,\Box\, v, \lambda\rangle,$$

and so, since

$$\Big\{\sum_{i=1}^{n} \alpha_i \delta_{u_i} : \alpha_1, \dots, \alpha_n \in \mathbb{R}^+, u_1, \dots, u_n \in S^*,\, n \in \mathbb{N}\Big\}$$

is weak-$*$ dense in $M(S^*)^+$ and the map R_Ψ is continuous on $(A''_\omega, \Box)$ for $\Psi = \theta_\omega(v)$, we see that

$$\langle \theta_\omega(\mu) \,\Box_\omega\, \theta_\omega(v), \lambda\omega\rangle_\omega \geq \frac{1}{c_T}\,\langle \mu \,\Box\, v, \lambda\rangle \tag{5.5}$$

for each $\mu \in M(S^*)^+$, $v \in T^*$, and $\lambda \in C(\beta S)^+$.

Now take $\mu \in M_{\mathbb{R}}(S^*)$, and suppose that μ has the Hahn decomposition $\mu = \mu_1 - \mu_2$, say. Take $\varepsilon > 0$. Then there are $\nu_1, \nu_2 \in M(S^*)^+$, with supports K_1 and K_2, respectively, in βS such that $K_1 \cap K_2 = \emptyset$ and $\|\mu_1 - \nu_1\| + \|\mu_2 - \nu_2\| < \varepsilon$. Take open neighbourhoods U_1 and U_2 of K_1 and K_2, respectively, in βS such that $U_1 \cap U_2 = \emptyset$, and then take functions $\lambda_1, \lambda_2 \in C(\beta S)^+_{[1]}$ with $\operatorname{supp} \lambda_j \subset U_j$ and $\langle \nu_j, \lambda_j\rangle \geq (1-\varepsilon)\|\nu_j\|$ for $j = 1, 2$. Set $\lambda = \lambda_1 - \lambda_2$, so that $\lambda \in C_{\mathbb{R}}(\beta S)_{[1]}$ and

$$\langle \nu_1 - \nu_2, \lambda\rangle = \langle \nu_1, \lambda_1\rangle + \langle \nu_2, \lambda_2\rangle.$$

We have

$$\langle \mu, \lambda\rangle > \langle \nu_1, \lambda_1\rangle + \langle \nu_2, \lambda_2\rangle - \varepsilon > (1-\varepsilon)\|\mu\| - 4\varepsilon. \tag{5.6}$$

Let $\mu \in M_{\mathbb{R}}(\beta S)$ have the Hahn decomposition $\mu = \mu_1 - \mu_2$, and take $a \in T^*$ such that a is right cancellable in $(\beta S, \Box)$. Then $\mu \,\Box\, a \in M_{\mathbb{R}}(\beta S)$, and the Hahn decomposition of $\mu \,\Box\, a$ is $\mu_1 \,\Box\, a - \mu_2 \,\Box\, a$. By [13, Proposition 4.4(iii)], we have $\|\mu \,\Box\, a\| = \|\mu\|$. Take $\varepsilon > 0$. Then there exists an element $\lambda = \lambda_1 - \lambda_2 \in C_{\mathbb{R}}(\beta S)_{[1]}$ with the above properties. We have

$$\begin{aligned}
\langle \theta_\omega(\mu) \,\Box_\omega\, \theta_\omega(a), \lambda\omega\rangle_\omega &= \sum_{j=1}^{2} \langle \theta_\omega(\mu_j) \,\Box_\omega\, \theta_\omega(a), \lambda_j\omega\rangle_\omega \quad \text{by (4.8)}\\
&\geq \frac{1}{c_T}\sum_{j=1}^{2} \langle \mu_j \,\Box\, a, \lambda_j\rangle \quad \text{by (5.5)}\\
&> \frac{1-\varepsilon}{c_T}\,\|\mu \,\Box\, a\| - \frac{4\varepsilon}{c_T} \quad \text{by (5.6)}\\
&= \frac{1-\varepsilon}{c_T}\,\|\mu\| - \frac{4\varepsilon}{c_T}.
\end{aligned}$$

Let K be a clopen subset of βS with $S \,\Box\, a \subset K$. It follows from (4.8) that, by replacing λ by $\lambda | K$, we may suppose that $\operatorname{supp} \lambda \subset K$.

Thus we have obtained the following result.

PROPOSITION 5.4. *Let ω be a weight on an infinite semigroup S such that ω is weakly diagonally bounded on an infinite subset T of S, with bound c_T. Let $\mu \in M_{\mathbb{R}}(\beta S)$ with $\|\mu\| = 1$, and let $a \in T^*$ be right cancellable in βS, with $S \,\Box\, a \subset K$, where K is a clopen subset of βS. Set $\Phi = \theta_\omega(\mu) \in A''_\omega$ and $v = \theta_\omega(a)$. Then, for each $\varepsilon > 0$, there exists $\lambda \in C_{\mathbb{R}}(\beta S)_{[1]}$ with* $\operatorname{supp} \lambda \subset K$ *such that*

$$\frac{1}{c_T}(1-\varepsilon) \leq \langle \Phi \,\Box_\omega\, v, \lambda\omega\rangle_\omega \leq \|\Phi \,\Box_\omega\, v\|_\omega \leq 1. \ \blacksquare \tag{5.7}$$

We shall now generally restrict ourselves to consideration of countable, cancellative semigroups, but we shall make a few remarks about the uncountable semigroups $\mathbb{R}$ and $\mathbb{R}^{+\bullet}$; some more general definitions and results for arbitrary locally compact groups and more general semigroups are given in [13] and [58].

It is noted in [12] that a straightforward modification of the proof [12, Theorem 11.9] shows the following (the result is also given in [58] with a different proof). Let G be an infinite, countable group, and let ω be weight on G such that ω is diagonally bounded on an infinite subset of G. Then ω is strongly Arens irregular. We shall modify a proof given in [13] that shows that, under the same hypotheses, there is a finite subset of A''_ω that is determining for the left topological centre.

In fact, we shall prove our result under the weaker hypothesis that ω is weakly diagonally bounded, rather than just diagonally bounded, on an infinite subset of G; examples to be given below will show that this change considerably extends the collection of weights for which the result is known. In particular, we shall show in Example 6.4 that there is a weight ω on the group $(\mathbb{Q}, +)$ such that ω is weakly diagonally bounded on an infinite subset of $\mathbb{Q}$, but that ω is not diagonally bounded on any infinite subset of $\mathbb{Q}$.

The preliminary proposition is proved by modifying part of the proof of [13, Theorem 12.7]; we give the details. (The result in [13] is exactly the case $n = 2$ of the following proof.)

PROPOSITION 5.5. *Let S be an infinite, countable, cancellative semigroup, let T be an infinite subset of S, and let $n \in \mathbb{N}$. Then there exist $a_1, \dots, a_n \in T^*$ and pairwise disjoint, infinite subsets $U_1, \dots, U_n$ of S such that $a_1, \dots, a_n$ are right cancellative in βS and*

$$(\beta S \,\Box\, a_i) \cap (\beta S \,\Box\, a_j) = \emptyset \qquad (i, j \in \mathbb{N}_n \text{ with } i \neq j),$$

and $S \,\Box\, a_i \subset \overline{U}_i$ for $i \in \mathbb{N}_n$. Further, for each $x \in S^*$, the set $(x \diamond S) \cap \overline{U}_r$ is non-empty for at most one value of $r \in \mathbb{N}_n$.

Proof. Indeed, for this we may suppose that S has an identity e_S.

We enumerate S as a sequence $(s_k : k \in \mathbb{Z}^+)$, where $s_0 = e_S$. For $s = s_j$ and $t = s_k$ in S, we set $s \preccurlyeq t$ if $j \leq k$ and $s \prec t$ if $j < k$; for $t \in S$, we set $[t] = \{s \in S : s \preccurlyeq t\}$; for a subset F of S, we set

$$[F] = \bigcup\{[t] : t \in F\},$$

so that $[F]$ is finite whenever F is finite.

We shall construct a certain sequence (t_k) in S by induction. First, set $t_0 = s_0 = e_S$. Once $t_0, \dots, t_k$ have been defined, set

$$T_k = \{s_0, \dots, s_k, t_0, \dots, t_k\}.$$

The sequence (t_k) will satisfy the following two conditions for each $k \in \mathbb{Z}^+$:

(i) $sT_k \cap [T_k] = \emptyset$ whenever $s \in S$ with $t_{k+1} \preccurlyeq s$;
(ii) $rs \prec tt_{k+1}$ whenever $r, s, t \in T_k$.

Take $k \in \mathbb{Z}^+$, and assume that $t_0, \dots, t_k$ have been specified in S. Since S is cancellative, the set

$$\bigcup_{j \le k} \{s \in S : t_j s \in T_k^2\}$$

is finite. We choose t_{k+1} to be any element of S that is not in the above set. It is clear that clauses (i) and (ii) are satisfied, and so the inductive construction continues.

Note that $s_k \preccurlyeq t_k$ $(k \in \mathbb{Z}^+)$, and so $\bigcup\{[t_k] : k \in \mathbb{Z}^+\} = S$.

Define $\varphi : S \to \mathbb{Z}^+$ by setting

$$\varphi(s) = \min\{k \in \mathbb{Z}^+ : s \in [t_k]\} \quad (s \in S).$$

Suppose that $\varphi(s) = j \in \mathbb{N}$. It follows from (ii) that we have $t_{k-1} \prec st_k \prec t_{k+1}$ whenever $k > j$, and so

$$\varphi(st_k) \in \{k, k+1\} \quad (k > j). \tag{5.8}$$

Now suppose that $j \ge 2$ and $k \le j - 2$. Then $u \prec ss_k$ for each $u \in T_{j-2}$ by (i), above, because $sT_{j-2} \cap [T_{j-2}] = \emptyset$; further, $ss_k \prec t_{k+1}$ by (ii), because $s \in T_j$. Thus $\varphi(ss_k) \in \{j-1, j, j+1\}$, and so we can conclude that

$$\varphi(ss_k) \in \{j-1, j, j+1\} \quad (k + 2 \le j). \tag{5.9}$$

For each $s \in S$, set $\gamma(s) \equiv \varphi(s)$ taken modulo $4n$. Then $\gamma : S \to \mathbb{Z}_{4n}$ is a (continuous) map into a finite set, and so it has a continuous extension, also denoted by γ, to a map $\gamma : \beta S \to \mathbb{Z}_{4n}$. It follows from (5.9) that

$$\gamma(x \mathbin{\Box} s) \in \{\gamma(x) - 1, \gamma(x), \gamma(x) + 1\} \quad (x \in S^*,\ s \in S), \tag{5.10}$$

where addition and subtraction in $\mathbb{Z}_{4n}$ are taken modulo $4n$.

For $r = 1, \dots, n$, define

$$A_r = \{t_k : \gamma(t_k) = 4(r-1) + 1\}, \qquad U_r = A_i \cup \{t_k : \gamma(t_k) = 4(r-1) + 2\},$$

so that each of $A_1, \dots, A_n$ is an infinite subset of S, $A_r \subset U_r$ for each $r \in \mathbb{N}_n$, and the sets $\{U_1, \dots, U_n\}$ are pairwise disjoint. For each $r = 1, \dots, n$, choose $a_r \in A_r^*$.

We *claim* that each a_r is right cancellable in $(\beta S, \Box)$. Indeed, let u_1 and u_2 be distinct points of βS, and take $N_1, N_2 \subset S$ such that $N_1 \cap N_2 = \emptyset$, $u_1 \in \overline{N}_1$, and $u_2 \in \overline{N}_2$. For $j = 1, 2$, set

$$Y_j = \{s_i t_k : s_i \in N_j,\ t_k \in A_r,\ i < k\},$$

so that $Y_j \in u_j \mathbin{\Box} a_r$. Take $i_1, i_2, k_1, k_2 \in \mathbb{N}$ with $i_1 < k_1$, $i_2 < k_2$, and $i_1 \ne i_2$. Then $s_{i_1} t_{k_1} \ne s_{i_2} t_{k_2}$; this holds for $k_1 < k_2$ by (ii), and for $k_1 = k_2$ because S is cancellative. It follows that $Y_1 \cap Y_2 = \emptyset$, and so $u_1 \mathbin{\Box} a \ne u_2 \mathbin{\Box} a$, as required for the claim.

It follows from (5.8) that $S \mathbin{\Box} a_i \subset \overline{U}_i$ for $i \in \mathbb{N}_n$.

Take $x \in S^*$. Then $\gamma(x) = i$ for some $i \in \mathbb{Z}_{4n}$. Take $s \in S$, and suppose that $x \diamond s \in \overline{U}_r$, where $r \in \mathbb{N}_n$. Then it follows from (5.10) that

$$i \in \{4(r-1), 4(r-1) + 1, 4(r-1) + 2, 4(r-1) + 3\}.$$

Thus there is at most one value of $r \in \mathbb{N}_n$ such that $(x \diamond S) \cap \overline{U}_r \ne \emptyset$. ∎

THEOREM 5.6. *Let S be an infinite, countable, cancellative semigroup, and let ω be a weight on S such that ω is weakly diagonally bounded on an infinite subset T of S, with bound c_T. Take $n \in \mathbb{N}$ with $n > c_T$. Then there is a subset V of T_ω^* with $|V| = n$ such that V consists of right cancellable elements of $(\beta S, \Box)$ and V is determining for the left topological centre of $\ell^1(S,\omega)''$.*

Proof. Let $a_1, \dots, a_n \in T^*$ and $U_1, \dots, U_n \subset S$ be as specified in Proposition 5.5.

For $r \in \mathbb{N}_n$, we set $v_r = \theta_\omega(a_r) \in A_\omega''$, and we define $V = \{v_1, \dots, v_n\}$, so that $|V| = n$.

Let $\Phi \in (A_\omega'')_\mathbb{R}$ with $\|\Phi\|_\omega = 1$, take $\varepsilon > 0$, and let $r \in \mathbb{N}_n$. It follows from Proposition 5.4 that there exists $\lambda_r \in C_\mathbb{R}(\beta S)_{[1]}$ with $\operatorname{supp} \lambda_r \subset \overline{U}_r$ such that

$$\frac{1}{c_T}(1-\varepsilon) \le \langle \Phi \,\Box_\omega\, v_r,\, \lambda_r\omega\rangle_\omega. \tag{5.11}$$

Let $f \in \operatorname{lin}\{\delta_u : u \in S^*\}$ with $\|f\|_1 \le 1$. Then we can write

$$f = f_1 + \cdots + f_n,$$

where, for each $r \in \mathbb{N}_n$, we have $f_r \in \operatorname{lin}\{\delta_u : u \in S^*\}$ and $u \diamond S \subset \overline{U}_r$ whenever $u \in \operatorname{supp} f_r$; we have $\sum_{r=1}^n \|f_r\|_1 \le 1$, and so there exists $j \in \mathbb{N}_n$ such that $\|f_j\|_1 \le 1/n$. It follows from (4.9) that

$$|\langle \theta_\omega(f) \diamond_\omega \theta_\omega(s), \lambda_j\omega\rangle_\omega| \le 1/n \quad (s \in S). \tag{5.12}$$

Now take $\mu \in M(S^*)_{[1]}$, say $\mu = \lim_\alpha g_\alpha$, where

$$g_\alpha \in \operatorname{lin}\{\delta_u : u \in S^*\}$$

with $\|g_\alpha\|_1 \le 1$ for each α. For each α, there exists $j_\alpha \in \mathbb{N}_n$ such that (5.12) holds (with g_α for f and j_α for j), and so, by passing to a subnet, we may suppose that there exists $j \in \mathbb{N}_n$ such that (5.12) holds (with g_α for f) for each α. Since the map $\Phi \mapsto \Phi \diamond_\omega \theta_\omega(s)$ is weak-$*$ continuous on A_ω'' for each $s \in S$, it follows that there exists $j \in \mathbb{N}_n$ such that

$$|\langle \theta_\omega(\mu) \diamond_\omega \theta_\omega(s), \lambda_j\omega\rangle_\omega| \le 1/n \quad (s \in S).$$

The map $\Phi \mapsto \theta_\omega(\mu) \diamond_\omega \Phi$ is weak-$*$ continuous on A_ω'' for each $\mu \in M(S^*)$, and so we see that, for each $\Phi \in (E_\omega^\circ)_\mathbb{R}$ with $\|\Phi\|_\omega \le 1$, there exists $j \in \mathbb{N}_n$ such that

$$|\langle \Phi \diamond_\omega v_j, \lambda_j\omega\rangle_\omega| \le 1/n. \tag{5.13}$$

Take $\Phi \in E_\omega^\circ$ with $\Phi \,\Box_\omega\, v = \Phi \diamond_\omega v$ $(v \in V)$, say $\Phi = \Phi_1 + \mathrm{i}\Phi_2$, where $\Phi_1, \Phi_2 \in (E_\omega^\circ)_\mathbb{R}$. Assume towards a contradiction that $\Phi_1 \ne 0$, say $\|\Phi_1\|_\omega = 1$. It follows from (5.11) and (5.13) that there exists $j \in \mathbb{N}_n$ such that

$$(1-\varepsilon)/c_T < \langle \Phi_1 \,\Box_\omega\, v_j, \lambda_j\omega\rangle_\omega = |\langle \Phi \diamond_\omega v_j, \lambda_j\omega\rangle_\omega| \le 1/n,$$

which is indeed a contradiction for suitably small $\varepsilon > 0$ because $n > c_T$. Thus $\Phi_1 = 0$. Similarly, $\Phi_2 = 0$, and so $\Phi = 0$.

We have proved that V is determining for the left topological centre of $\ell^1(S,\omega)''$. ∎

We have proved in the above theorem that there is an appropriate subset V of T_ω^* with $|V| = n > c_T$. We do not know if we can always find sets with smaller cardinality with the required properties.

Corollary 5.7. *Let S be an infinite, countable, abelian, cancellative semigroup, and let ω be a weight on S such that ω is weakly diagonally bounded on an infinite subset of S. Then ω is strongly Arens irregular on S. In particular, $\ell^1(S)$ is semisimple and strongly Arens irregular.*

Proof. By the theorem, $\ell^1(S,\omega)$ is left strongly Arens irregular. Since $\ell^1(S,\omega)$ is commutative, ω is strongly Arens irregular on S. ∎

We state as a further corollary a result that is contained within a trivial modification of the above proof.

Corollary 5.8. *Let S be an infinite, countable, cancellative semigroup, and let ω be a weight on S such that ω is weakly diagonally bounded on an infinite subset W of S, with bound c_W. Suppose that $\mu \in M_{\mathbb{R}}(S^*)_{[1]}$ and $\varepsilon > 0$. Then there exist elements $a \in W^*$ and $\lambda \in \ell^\infty_{\mathbb{R}}(S,\omega)$ with $\|\lambda\|_\omega = 1$ such that*

$$\langle \theta_\omega(\mu) \,\Box_\omega\, \theta_\omega(a), \lambda\rangle_\omega > \frac{1}{c_W}\,(1-\varepsilon) \quad \textit{and} \quad |\langle \theta_\omega(\mu) \diamond_\omega \theta_\omega(a), \lambda\rangle_\omega| < \varepsilon. \ \blacksquare$$

The following is a special case of the above corollary.

Corollary 5.9. *Let W be an infinite subset of $\mathbb{Q}^{+\bullet}$. Take $\varepsilon > 0$ and $\mu \in M_{\mathbb{R}}((\mathbb{Q}^{+\bullet})^*)_{[1]}$. Then there exist $a \in (\mathbb{Q}^{+\bullet})^*$ and $\lambda \in \ell^\infty_{\mathbb{R}}(\mathbb{Q}^{+\bullet})$ with $\|\lambda\| = 1$ such that*

$$\langle \mu \,\Box\, a, \lambda\rangle > 1-\varepsilon \quad \textit{and} \quad |\langle a \,\Box\, \mu, \lambda\rangle| < \varepsilon. \ \blacksquare$$

Finally, we make some remarks concerning uncountable semigroups. It can be checked that a modification of the proof of [13, Theorem 12.15] by the introduction of a weight, in the same manner as in the modification of [13, Theorem 12.7] to give Theorem 5.6, will give the following two results.

Theorem 5.10. *Let S be an infinite, cancellative semigroup, and let ω be a weight on S such that ω is weakly diagonally bounded on a subset W of S for which $|W| = |S|$. Then ω is strongly Arens irregular.* ∎

Before giving the next theorem, we clarify some terminology. We have taken the real line $\mathbb{R}$ to have the discrete topology, and have formed the Stone–Čech compactification $\beta\mathbb{R}_d$. Take an element $a \in \beta\mathbb{R}_d$ such that $a \in \mathbb{I}^*$. Then a is an ultrafilter on $\mathbb{I}$, and so we have the notion of whether or not this filter is convergent. Since $\mathbb{I}$ is compact (in the usual topology on $\mathbb{I}$), a must converge in this topology to a point of $\mathbb{I}$, say $a \to s_0$. This means that each open interval in $\mathbb{R}$ that contains s_0 is a member of a.

Theorem 5.11. *Let $\varepsilon > 0$ and $\mu \in M_{\mathbb{R}}((\mathbb{R}^{+\bullet})^*)_{[1]}$. Then there exist elements $a \in \mathbb{I}^*$ and $\lambda \in \ell^\infty_{\mathbb{R}}(\mathbb{R}^{+\bullet})$ with $\|\lambda\| = 1$ such that $\|\mu \,\Box\, a\| = \|\mu\|$ and $a \to 0$ and such that*

$$\langle \mu \,\Box\, a, \lambda\rangle > 1-\varepsilon \quad \textit{and} \quad |\langle a \,\Box\, \mu, \lambda\rangle| < \varepsilon. \tag{5.14}$$

In particular, $\mu \,\Box\, a \neq a \,\Box\, \mu$.

Proof. The modification of the proof of [13, Theorem 12.7] consists in dividing the set T of that proof into n parts, rather than two parts (as in Theorem 5.6), where $1/n < \varepsilon$. The proof gives an element $a \in \mathbb{I}^*$ such that a is right cancellable in $(\beta\mathbb{R}_d, \Box)$ and an element

$\lambda \in \ell^\infty_{\mathbb{R}}(\mathbb{R})$ with $\|\lambda\| = 1$ such that λ and a satisfy (5.14). Since a is right cancellable, it follows from [13, Proposition 4.4(iii)] that $\|\mu \,\Box\, a\| = \|\mu\|$.

Since a is an ultrafilter on $\mathbb{I}$, there is a point $s_0 \in \mathbb{I}$ such that $a \to s_0$. By replacing a by $a - s_0$ (which does not change any of the above properties), we may suppose that $a \to 0$. Since a is an ultrafilter on $\mathbb{R}$ and $(-1,1) \in a$, either $[0,1) \in a$ or $(-1,0] \in a$; in the latter case, we replace a by $-a$ (which again does not change any of the above properties). Thus we may suppose that $[0,1) \in a$, and hence that $a \in \mathbb{I}^*$.

Since $\mu \in M_{\mathbb{R}}(\mathbb{R}^{+\bullet})$, we may suppose that $\lambda \in \ell^\infty_{\mathbb{R}}(\mathbb{R}^{+\bullet})$. ∎

In fact, the following weaker result than Theorem 5.10 is sufficient to obtain a result about the group $\mathbb{R}$. Theorem 5.12 was already proved in [12, Corollary 11.10]; the result in Corollary 5.13 generalizes [10, Corollary 1], which states that ω is not Arens regular for each weight ω on $\mathbb{R}$.

Theorem 5.12. *Let G be a group, and let ω be a weight on G such that ω is diagonally bounded on a subset W of G for which $|W| = |G|$. Then ω is strongly Arens irregular.* ∎

Corollary 5.13. *Let ω be a weight on the group $(\mathbb{R}, +)$. Then ω is strongly Arens irregular.*

Proof. For $n \in \mathbb{N}$, set $R_n = \{t \in \mathbb{R} : \omega(t)\omega(-t) \le n\}$. Assume that $|R_n| < \mathfrak{c}$ for each $n \in \mathbb{N}$. Then

$$|\mathbb{R}| = \left|\bigcup\{R_n : n \in \mathbb{N}\}\right| < \mathfrak{c},$$

a contradiction. Thus ω is diagonally bounded on a subset W of $\mathbb{R}$ for which $|W| = \mathfrak{c}$, and so ω is strongly Arens irregular. ∎

The above result does not apply to weights on the semigroup $(\mathbb{R}^{+\bullet}, +)$; in Example 9.10, we shall give a weight on $\mathbb{R}^{+\bullet}$ which is neither Arens regular nor strongly Arens irregular. However, we shall prove in Theorem 8.1 that a continuous weight on $\mathbb{R}^+$ is strongly Arens irregular, and in Corollary 9.19 that a semisimple weight on $\mathbb{R}^{+\bullet}$ cannot be Arens regular.

6. Examples of weights

We now give some examples of weights on the semigroups that we are considering. For a large collection of weights on $\mathbb{Z}$, see [12].

Example 6.1. Let S be a subsemigroup of $\mathbb{R}^+$ or $\mathbb{R}^{+\bullet}$ such that S contains a sequence (t_n) with $t_n \searrow 0$, and set $T = \{t_n : n \in \mathbb{N}\}$, an infinite subset of S. Suppose that ω is a continuous weight on $\mathbb{R}^+$, so that $\omega | S$ is a weight on S.

We *claim* that ω (as a weight on S) is weakly diagonally bounded on T, with bound $c_T = 1$. Indeed, fix $\varepsilon > 0$. For each $s \in S$, there exists $n_s \in \mathbb{N}$ such that

$$\omega(s + t_n) > (1 - \varepsilon)\omega(s) \quad \text{and} \quad \omega(t_n) < 1 + \varepsilon$$

for each $n \geq n_s$, where we recall that $\omega(1) = 1$. Let $T_s = \{t_n : n \geq n_s\}$, so that T_s is a cofinite subset of T. For each $t \in T_s$, we have

$$\Omega(s,t) = \frac{\omega(s+t)}{\omega(s)\omega(t)} \geq \frac{1-\varepsilon}{1+\varepsilon},$$

giving the claim.

In the case where S is countable, it follows from Theorem 5.6 that there is a two-element subset of T^*_ω that is determining for the topological centre of $\ell^1(S,\omega)''$, and so ω is strongly Arens irregular.

For example, set $\omega(s) = \exp(-s^2)$ $(s \in \mathbb{Q}^{+\bullet})$. Clearly ω is a continuous weight on $\mathbb{R}^+$, and ω is radical; by the above remarks, ω is strongly Arens irregular as a weight on $\mathbb{Q}^{+\bullet}$.

However, we cannot apply this argument and Theorem 5.10 to the weight function $\omega : s \mapsto \exp(-s^2)$ on $\mathbb{R}^{+\bullet}$ because this function is not weakly diagonally bounded on any uncountable subset of $\mathbb{R}^{+\bullet}$. Indeed, note that

$$\Omega(s,t) = \exp(-2st) \quad (s,t \in \mathbb{R}^{+\bullet}).$$

Assume towards a contradiction that T is an uncountable subset of $\mathbb{R}^{+\bullet}$ such that ω is weakly diagonally bounded on T, with bound $c > 0$. Then there exists $t_0 > 0$ such that $T \cap (t_0, \infty)$ is infinite. There is a cofinite subset S_0 of $\mathbb{R}^{+\bullet}$ such that, for each $s \in S$, there is a cofinite subset T_s of T with $\exp(-2st) \geq 1/2c$ $(t \in T_s)$. Choose $s_0 \in S$ with $\exp(-2s_0t_0) < 1/2c$, and then choose $t_1 \in T_{s_0} \cap (t_0, \infty)$ with $\exp(-2s_0t_1) \geq 1/2c$. We have

$$\frac{1}{2c} \leq \exp(-2s_0t_1) \leq \exp(-2s_0t_0) < \frac{1}{2c},$$

a contradiction.

This leaves open the question whether or not the weight

$$\omega : s \mapsto \exp(-s^2), \quad \mathbb{R}^{+\bullet} \to \mathbb{R}^{+\bullet},$$

is strongly Arens irregular; this will be resolved in Example 8.2, below.

Example 6.2. An interesting example of a weight ω on $\mathbb{Z}$ is given in [12, Example 9.17]; it is due to J. F. Feinstein. Each $n \in \mathbb{Z} \setminus \{0\}$ can be written in the form

$$n = \sum_{j=1}^{r} \varepsilon_j 2^{a_j}, \tag{6.1}$$

where $\varepsilon_j \in \{-1, 1\}$ and $a_j \in \mathbb{Z}^+$ for each $j \in \mathbb{N}$, and where

$$a_1 \geq \cdots \geq a_r.$$

We define $\eta(n)$ to be the minimum value of $r \in \mathbb{N}$ that can arise in (6.1), and take $\eta(0) = 0$; we set $\omega = \exp \eta$, so that ω is a semisimple weight on $\mathbb{Z}$. Set $T = \{2^k : k \in \mathbb{N}\}$, an infinite subset of $\mathbb{Z}$. Then $\eta(n) = 1$ $(n \in T)$, and so ω is diagonally bounded on T with $d_T = \mathrm{e}^2$. However, ω is unbounded on $\mathbb{Z}$.

We *claim* that ω is weakly diagonally bounded on T with bound $c_T = 1$. Indeed, take n as in (6.1), with $r = \eta(n)$. For each $k \in \mathbb{N}$ with $2^k > 2^{a_1+1}$, we have $\eta(n + 2^k) = r + 1$, and so $\Omega(n, 2^k) = 1$, giving the claim.

We conclude from Corollary 5.7 that $\ell^1(\mathbb{Z},\omega)$ is strongly Arens irregular, a fact that was already proved in [12, Example 9.17], and now that there is a two-element subset of $\mathbb{Z}^*_\omega$ that is determining for the topological centre of $\ell^1(\mathbb{Z},\omega)''$.

We remark that it was left open in [12] whether or not, with this particular weight ω, the Banach algebra $(\ell^1(\mathbb{Z},\omega)'',\Box)$ is semisimple; this is still an open question.

Example 6.3. We next consider a special weight on the group $(\mathbb{Q},+)$ (or the semigroup $(\mathbb{Q}^{+\bullet},+)$). Let $f:\mathbb{N}\to\mathbb{R}$ be an increasing function with

$$f(1)=1 \quad \text{and} \quad f(mn)\le f(m)f(n) \quad (m,n\in\mathbb{N}).$$

For example, we can take

$$f(n)=n \quad \text{or} \quad f(n)=1+\log n \quad \text{for } n\in\mathbb{N}. \tag{6.2}$$

Then define

$$\omega(p/q)=f(q) \quad (p/q\in\mathbb{Q}^\bullet), \tag{6.3}$$

where we recall that we are supposing that $(p,q)=1$ in the expression p/q. Also set $\omega(0)=1$.

Clearly each such ω is a symmetric weight on $\mathbb{Q}$ with $\omega\ge 1$; also $\omega|\mathbb{Q}^{+\bullet}$ is a semisimple weight.

For each $n\in\mathbb{N}$ and $p/q\in\mathbb{Q}^\bullet$, we have $\omega(n+p/q)=\omega((nq+p)/q)$ and $(nq+p,q)=1$, and so $\Omega(n,p/q)=1$. Thus ω is diagonally bounded on $\mathbb{N}$, with $d_{\mathbb{N}}=1$.

We conclude that each such Banach algebra of the form $\ell^1(\mathbb{Q},\omega)$ or $\ell^1(\mathbb{Q}^{+\bullet},\omega)$ is strongly Arens irregular; further, there is a two-element subset of $\mathbb{Q}^*_\omega$ or $(\mathbb{Q}^{+\bullet})^*_\omega$, respectively, that is determining for the topological centre of $\ell^1(\mathbb{Q},\omega)''$ or of $\ell^1(\mathbb{Q}^{+\bullet},\omega)''$.

Now set $T=\{1/r : r\in\mathbb{P}\}$. In the case where $f(n)=n$ $(n\in\mathbb{N})$, the weight ω is not diagonally bounded on T. Indeed, take $r\in\mathbb{P}$ with $r>2$, and set $s=t=1/r$. Then

$$\omega(s+t)=r \quad \text{and} \quad \omega(s)=\omega(t)=r,$$

and so $\Omega(s,t)=1/r$. Thus $\inf\{\Omega(s,t):s,t\in T\}=0$, and so (5.3) fails for any $d_T>0$.

However, we *claim* that ω is weakly diagonally bounded on T, with bound $c_T=1$. Indeed, given $s=p/q\in\mathbb{Q}$, let $T_s=\{1/r : r\in\mathbb{P},\ r>q\}$, a cofinite subset of T. For $t=1/r\in T_s$, we have

$$s+t=\frac{pr+q}{qr} \quad \text{with} \quad (pr+q,qr)=1,$$

and so $\omega(s+t)=qr=\omega(s)\omega(t)$. Thus $\Omega(s,t)=1$ $(s\in S,\ t\in T_s)$, giving the claim. The fact that $\inf T=0$ will be important later.

Example 6.4. We consider another special weight on the group $(\mathbb{Q},+)$ (or the semigroup $(\mathbb{Q}^{+\bullet},+)$). Let

$$\omega(p/q)=1+|p|+q \quad (p/q\in\mathbb{Q}^\bullet), \tag{6.4}$$

and set $\omega(0)=1$. Again ω is a symmetric weight on $\mathbb{Q}$ with $\omega\ge 1$; also $\omega|\mathbb{Q}^{+\bullet}$ is a semisimple weight. The weight ω is not bounded, and hence not diagonally bounded, on any infinite subset of $\mathbb{Q}$.

Again set $T = \{1/r : r \in \mathbb{P}\}$. We *claim* that ω is weakly diagonally bounded on T, with bound $c_T = 1$. Indeed, given $\varepsilon > 0$, we see that

$$\frac{|p| + q}{1 + |p| + q} > 1 - \varepsilon$$

for all save finitely many $p/q \in \mathbb{Q}$, say for $p/q \in S_0$, where S_0 is a certain cofinite subset of $\mathbb{Q}$. Given $s = p/q \in S_0$, take $t = 1/r$, where $r \in \mathbb{P}$ with $r > q$. Then

$$\Omega(s, 1/r) = \frac{1 + |pr + q| + qr}{(1 + |p| + q)(2 + r)} \to \frac{|p| + q}{1 + |p| + q} \quad \text{as } r \to \infty,$$

and so $\Omega(s, 1/r) > 1 - \varepsilon$ ($s \in S_0$, $t \in T_s$) for a certain cofinite subset T_s of T, giving the claim.

We again conclude that there is a two-element subset of $\mathbb{Q}^*_\omega$ or of $(\mathbb{Q}^{+\bullet})^*_\omega$ that is determining for the appropriate topological centre, so that $\ell^1(\mathbb{Q}, \omega)$ and $\ell^1(\mathbb{Q}^{+\bullet}, \omega)$ are strongly Arens irregular. However,

$$\omega(n) = 2 + |n| \quad (n \in \mathbb{Z} \setminus \{0\}),$$

and so the closed subalgebra $\ell^1(\mathbb{Z}, \omega)$ of $\ell^1(\mathbb{Q}, \omega)$ is Arens regular.

EXAMPLE 6.5. It is clear that a radical weight ω on $\mathbb{N}$ cannot be diagonally bounded on any infinite subset of $\mathbb{N}$. For let ω be a radical weight on $\mathbb{N}$, and assume towards a contradiction that $\Omega(1, n) \geq \delta > 0$ for infinitely many values of $n \in \mathbb{N}$. Then

$$\nu_\omega = \lim_{n\to\infty} \omega(n)^{1/n} \geq \liminf_{n\to\infty} \frac{\omega(n+1)}{\omega(n)} \geq \omega(1)\delta > 0,$$

a contradiction of the fact that ω is radical.

However, we shall now exhibit a radical weight ω on $\mathbb{N}$ such that ω is weakly diagonally bounded on an infinite subset of $\mathbb{N}$.

We first inductively define a sequence $(m_k : k \in \mathbb{N})$ in $\mathbb{N}$ by setting $m_1 = 1$ and $m_{k+1} = 2m_k + 1$ $(k \in \mathbb{N})$, and then we define a function $\eta : \mathbb{N} \to \mathbb{N}$ by setting $\eta(1) = 1$ and by using the inductive formula

$$\eta(m_k + 1) = \max\{2\eta(m_k), (2m_k + 1)k\},$$
$$\eta(m_k + 1 + r) = \eta(m_k + 1) + \eta(r) \quad (r \in \mathbb{N}_{m_k}),$$

for $k \in \mathbb{N}$. Clearly η is an increasing function on $\mathbb{N}$.

We *claim* that $\eta(m+n) \geq \eta(m)+\eta(n)$ $(m, n \in \mathbb{N})$. This is trivially true for $m = n = 1$. Assume that it is true whenever $m + n \leq m_k$, where $k \in \mathbb{N}$, and take $m, n \in \mathbb{N}$ with $m + n \leq m_{k+1}$; we may suppose that $m + n \geq m_k + 1$. First, suppose that $m, n \leq m_k$. Then

$$\eta(m + n) \geq \eta(m_k + 1) \geq 2\eta(m_k) \geq \eta(m) + \eta(n).$$

Second, suppose that $m \geq m_k + 1$, say $m = m_k + 1 + r$, where $r \in \mathbb{Z}^+_{m_k}$. Then $n \leq m_k$ and

$$\eta(m + n) = \eta(m_k + 1) + \eta(r + n) \geq \eta(m_k + 1) + \eta(r) + \eta(n) = \eta(m) + \eta(n).$$

This establishes the claim.

Set $\omega = \exp(-\eta)$. Then, by the claim, ω is a weight on $\mathbb{N}$.

Let $k \in \mathbb{N}$. For each $n \in \mathbb{N}$ with $m_k + 1 \le n \le m_{k+1}$, we have

$$\frac{\eta(n)}{n} \ge \frac{\eta(m_k+1)}{m_{k+1}} = \frac{\eta(m_k+1)}{2m_k+1} \ge k,$$

and so $\omega(n)^{1/n} \le \exp(-k)$. Thus $\lim_{n\to\infty} \omega(n)^{1/n} = 0$, and ω is radical.

Let $T = \{m_k + 1 : k \in \mathbb{N}\}$, so that T is an infinite subset of $\mathbb{N}$. For each $m \in \mathbb{N}$, set $T_m = \{n \in T : n > m\}$, a cofinite subset of T. For $m \in \mathbb{N}$ and $n = m_k + 1 \in T_m$, we have

$$\Omega(m,n) = \exp(\eta(m) + \eta(m_k+1) - \eta(m_k+1+m)) = 1,$$

and so ω is weakly diagonally bounded on T, with $c_T = 1$.

We conclude from Theorem 5.6 that there is a two-element subset of $\mathbb{N}^*_\omega$ that is determining for the topological centre of $\ell^1(\mathbb{N},\omega)''$, and then from Corollary 5.7 that ω is strongly Arens irregular.

7. Algebras on subsets of $\mathbb{I}$

Throughout this section, we take S to be either $(\mathbb{Q}^{+\bullet},+)$ or $(\mathbb{R}^{+\bullet},+)$ and $T = S \cap (0,1]$, and let ω be a weight on the semigroup S. We denote by A_ω and B_ω the Banach spaces $\ell^1(S,\omega)$ and $\ell^1(T,\omega)$, respectively, and we regard B_ω as a closed, complemented subspace of A_ω, so that B''_ω is a closed, complemented subspace of A''_ω; as usual, $(A_\omega,\star)$ is a weighted convolution algebra on S. We write A and B for A_ω and B_ω, respectively, in the special case where $\omega = 1$.

Set

$$I_\omega = \{f \in A_\omega : \alpha(f) \ge 1,\ f(1) = 0\} \cup \{0\},$$

so that I_ω is a closed ideal in A_ω. We define the product $\star_B$ on B_ω by identifying B_ω with the quotient Banach algebra A_ω/I_ω. Thus, for $s,t \in T$, we have $\delta_s \star_B \delta_t = \delta_{s+t}$ if $s+t \le 1$ and $\delta_s \star_B \delta_t = 0$ if $s+t > 1$. Each $f \in B_\omega$ with $\alpha(f) > 0$ is nilpotent, and so B_ω is a radical Banach algebra.

We remark that, although the Banach algebra B has a rather innocent appearance, there are apparently challenging open questions about it. For example, it is not known whether or not $B = B^2$. Further, it is not known whether or not B contains a non-zero semigroup over $\mathbb{Q}^{+\bullet}$ (in the terminology of [11]).

It is easy to see that the only compact element of B is 0, and so certainly B is not an ideal in $(B'',\Box)$.

The first Arens products in A''_ω and B''_ω are denoted by $\Box_\omega$ and $\Box^B_\omega$, respectively. Set

$$F_\omega = c_0(T, 1/\omega),$$

so that B_ω is a dual Banach algebra, with predual F_ω, and hence $B''_\omega = B_\omega \ltimes F^\circ_\omega$, as before, where F°_ω is the annihilator of F_ω in B''_ω. To show that B_ω is strongly Arens irregular, it suffices to show that, for each $\mu \in F^\circ_\omega$ with $\mu \ne 0$, there exists $\nu \in F^\circ_\omega$ with $\mu \,\Box^B_\omega\, \nu \ne \nu \,\Box^B_\omega\, \mu$. We shall do more than this in the case where $S = \mathbb{Q}^{+\bullet}$ under a mild condition on ω.

Let $\mu \in F_\omega^\circ$, say $\mu = \lim_\alpha f_\alpha$, where (f_α) is a net in B_ω. We may suppose that $\operatorname{supp} f_\alpha$ is finite for each α. Further, we have

$$\lim_\alpha f_\alpha(t) = 0 \qquad (t \in T),$$

and so we may suppose that $\operatorname{supp} f_\alpha \subset (0,1)$.

THEOREM 7.1. *Let ω be a weight on the semigroup $\mathbb{Q}^{+\bullet}$. Suppose that ω is weakly diagonally bounded, with bound $c_W < 2$, on a subset W of $\mathbb{Q}^{+\bullet}$ for which* $\inf W = 0$. *Set $B_\omega = \ell^1(T,\omega)$, and let $\mu \in F_\omega^\circ$ with $\mu \neq 0$. Then there exists $v \in W_\omega^*$ such that*

$$\mu \,\Box_\omega^B\, v \neq v \,\Box_\omega^B\, \mu.$$

In particular, B_ω is strongly Arens irregular.

Proof. We may suppose that $\mu \in (F_\omega^\circ)_{\mathbb{R}}$ and that $\|\mu\|_\omega = 1$, and we may regard μ as an element of A_ω'' in the above notation. Since $\inf W = 0$, we may also suppose that W is a sequence that decreases to 0. Choose $\widetilde{\mu} \in M_{\mathbb{R}}((\mathbb{Q}^{+\bullet})^*)_{[1]}$ with $\theta_\omega(\widetilde{\mu}) = \mu$; we have $\widetilde{\mu} \neq 0$.

Fix $\varepsilon > 0$ with

$$\left(\frac{2}{c_W} + 1\right)\varepsilon < \frac{2}{c_W} - 1. \tag{7.1}$$

By Corollary 5.8, there exist $a \in W^*$ and $\lambda \in (A_\omega')_{\mathbb{R}}$ with $\|\lambda\|_\omega = 1$ such that

$$\frac{1}{c_W}(1-\varepsilon) < r \leq 1 \quad \text{and} \quad |s| < \varepsilon,$$

where we set $r = \langle \theta_\omega(\widetilde{\mu}) \,\Box_\omega\, \theta_\omega(a), \lambda\rangle_\omega$ and $s = \langle \theta_\omega(a) \,\Box_\omega\, \theta_\omega(\widetilde{\mu}), \lambda\rangle_\omega$. Set $v = \theta_\omega(a) \in W_\omega^*$, so that

$$r = \langle \mu \,\Box_\omega\, v, \lambda\rangle_\omega \quad \text{and} \quad s = \langle v \,\Box_\omega\, \mu, \lambda\rangle_\omega.$$

Finally, set $\lambda_1 = \lambda|(0,1]$ and $\lambda_2 = \lambda|(1,\infty)$, so that $\lambda_1 \in B_\omega'$.

We first *claim* that $r = \langle \mu \,\Box_\omega^B\, v, \lambda_1\rangle_\omega$. Indeed,

$$r = \lim_m \lim_n \langle f_m \star \widetilde{\delta}_{t_n}, \lambda\rangle_\omega$$

for certain sequences (f_m) in B_ω and (t_n) in W. We may suppose that $\operatorname{supp} f_m \subset (0,1)$ for each $m \in \mathbb{N}$, and so, for each fixed $m \in \mathbb{N}$, we have $\operatorname{supp}(f_m \star \widetilde{\delta}_{t_n}) \subset (0,1)$ eventually because $t_n \to 0$. It follows that

$$\begin{aligned} r &= \lim_{m\to\infty} \lim_{n\to\infty} \langle f_m \star \widetilde{\delta}_{t_n}, \lambda\rangle_\omega \\ &= \lim_{m\to\infty} \lim_{n\to\infty} \langle f_m \star_B \widetilde{\delta}_{t_n}, \lambda\rangle = \langle \mu \,\Box_\omega^B\, v, \lambda_1\rangle_\omega, \end{aligned}$$

as required.

Now set $\alpha = \langle v \,\Box_\omega\, \mu, \lambda_1\rangle = \langle v \,\Box_\omega^B\, \mu, \lambda_1\rangle$ and $\beta = \langle v \,\Box_\omega\, \mu, \lambda_2\rangle$, so that $\alpha + \beta = s$. For each $\zeta \in \mathbb{T}$, we have $\|\lambda_1 + \zeta\lambda_2\|_\omega = 1$ in A_ω'. But

$$|\alpha + \zeta\beta| \leq \|\lambda_1 + \zeta\lambda_2\|$$

because $\|v \,\Box_\omega\, \mu\|_\omega \leq 1$, and so we have $|\alpha + \zeta\beta| \leq 1$ $(\zeta \in \mathbb{T})$. This shows that $|\alpha| + |\beta| \leq 1$.

Assume towards a contradiction that $\alpha = r$. Then

$$1 \geq |\alpha| + |\beta| = r + |s - r| \geq 2r - |s| > \frac{2}{c_W}(1-\varepsilon) - \varepsilon,$$

a contradiction of (7.1). Thus $\alpha \neq r$. This implies that

$$\langle \mu \,\Box_\omega^B\, v, \lambda_1 \rangle_\omega \neq \langle v \,\Box_\omega^B\, \mu, \lambda_1 \rangle_\omega,$$

and so $\mu \,\Box_\omega^B\, v \neq v \,\Box_\omega^B\, \mu$, as required. ∎

We do not know whether or not the constraint '$c_W < 2$' is necessary in the above theorem.

We claim that an amalgamation of the proof of Theorem 7.1 with results in Example 6.1 proves the following result. We omit the details, which are notationally complicated.

THEOREM 7.2. *Let ω_1 be a continuous weight on $\mathbb{R}^+$, and let ω_2 be a weight on $\mathbb{Q}^{+\bullet}$ such that ω_2 is weakly diagonally bounded, with bound $c_W < 2$, on a subset W of $\mathbb{Q}^{+\bullet}$ for which* $\inf W = 0$. *Set $\omega = \omega_1\omega_2$. Then A_ω is strongly Arens irregular.* ∎

We remark that it will be shown in Example 9.16, below, that it is not sufficient to suppose that ω_2 is weakly diagonally bounded on $\mathbb{N}$ for the conclusion to hold.

EXAMPLE 7.3. For $s = p/q \in \mathbb{Q}^{+\bullet}$, set

$$\omega(p/q) = q\exp(-s^2) \quad \text{or} \quad \omega(p/q) = (1+p+q)\exp(-s^2).$$

Then ω is a radical, strongly Arens irregular weight on the semigroup $(\mathbb{Q}^{+\bullet}, +)$.

We now give a similar result to Theorem 7.1 in the case where $S = \mathbb{R}^{+\bullet}$, rather than $S = \mathbb{Q}^{+\bullet}$; we cannot apply Corollary 5.8 directly because $\mathbb{R}^{+\bullet}$ is not countable. We give the result only in the case where the weight is 1. As before, set $B = (\ell^1(\mathbb{I}), \star_B)$ and $F = c_0(\mathbb{I})$; $\Box_B$ denotes the first Arens product in B''.

THEOREM 7.4. *Let $B = (\ell^1(\mathbb{I}), \star_B)$, and take $\mu \in F^\circ$ with $\mu \neq 0$. Then there exists $a \in (0,1)^*$ with $a \to 0$ and $\mu \,\Box_B\, a \neq a \,\Box_B\, \mu$. In particular, B is strongly Arens irregular.*

Proof. We may suppose that $\mu \in F_{\mathbb{R}}^\circ$ and that $\|\mu\| = 1$; we regard μ as an element of $M_{\mathbb{R}}(\mathbb{R}^*) \setminus \ell^1(\mathbb{R})$.

By Theorem 5.11, there exists $a \in \mathbb{I}^*$ such that $a \to 0$ and

$$\|\mu \,\Box\, a\| = \|\mu\| = 1, \tag{7.2}$$

and $\mu \,\Box\, a \neq a \,\Box\, \mu$, where $\Box$ denotes the first Arens product in $M(\beta\mathbb{R}_d)$.

We first *claim* that

$$\mu \,\Box\, a = \mu \,\Box_B\, a. \tag{7.3}$$

Indeed, fix $\varepsilon > 0$, and take $f \in \ell^1([0,1))$. Then there exists $\eta > 0$ such that

$$\|f \star \delta_r - (f \star \delta_r) | \mathbb{I}\| < \varepsilon \qquad (0 < r < \eta).$$

Since $a \to 0$, it follows that $\|f \,\Box\, a - f \,\Box_B\, a\| \leq \varepsilon$. Since $\ell^1([0,1))$ is weak-$*$ dense in $M(\mathbb{I}^*)$, it follows that $\|\mu \,\Box\, a - \mu \,\Box_B\, a\| \leq \varepsilon$. But this holds true for each $\varepsilon > 0$, and so the claim is proved.

Assume towards a contradiction that $\mu \,\Box_B\, a = a \,\Box_B\, \mu$. Then it follows from (7.2) and (7.3) that

$$\|a \,\Box_B\, \mu\| = 1. \tag{7.4}$$

Our second *claim* is that, given $\varepsilon > 0$, there exists $t \in (0,1)$ such that $|\mu((t,1)^*)| < \varepsilon$. Indeed, assume that this is not the case. Then there exists $\varepsilon > 0$ such that $|\mu((t,1)^*)| \geq \varepsilon$ for each $t \in (0,1)$. Thus we have $\|(\delta_r \star \mu)|\mathbb{I}\| < 1 - \varepsilon$. Since the map R_μ is weak-$*$ continuous on $(B'', \Box_B)$, it follows that $\|a \Box_B \mu\| \leq 1 - \varepsilon$, a contradiction of (7.4). We have established the second claim.

For each $t \in (0,1)$, set $\mu_t = \mu|(0,t)^*$. By the second claim, $\lim_{t\to 1-} \|\mu_t - \mu\| = 0$, and so

$$\lim_{t\to 1-} \|a \Box_B \mu_t - a \Box_B \mu\| = 0 \quad \text{and} \quad \lim_{t\to 1-} \|a \Box \mu_t - a \Box \mu\| = 0.$$

But $a \Box_B \mu_t = a \Box \mu_t$ for each $t \in (0,1)$, again because $a \in \mathbb{I}^*$ and $a \to 0$, and so $a \Box_B \mu = a \Box \mu$.

Since $\mu \Box a \neq a \Box \mu$, we conclude that $a \Box_B \mu \neq \mu \Box_B a$, and so B is strongly Arens irregular. ∎

8. Continuous weights on $\mathbb{R}^+$

Let ω be a continuous weight on $\mathbb{R}^+$ or $\mathbb{R}^{+\bullet}$. In this section we shall prove that $\ell^1(\mathbb{R}^{+\bullet}, \omega)$ is strongly Arens irregular.

In the following proof, we shall use a function $\kappa : \mathbb{R}^{+\bullet} \to \mathbb{I}$, defined by

$$\kappa(t) = \limsup_{s\to\infty} \Omega(s,t) \qquad (t \in \mathbb{R}^{+\bullet}).$$

Let (r_n) be a sequence in $\mathbb{R}^{+\bullet}$ such that $r_n \searrow 0$. We note that, by passing to a subsequence, we may suppose that there are two cases: either

(i) there exists $n_0 \in \mathbb{N}$ such that $\kappa(r_n) > 4/5$ $(n \geq n_0)$, or
(ii) $\kappa(r_n) \leq 4/5$ $(n \in \mathbb{N})$.

For example, cases (i) and (ii) arise when

$$\omega(s) = 1 + s \quad \text{and} \quad \omega(s) = \exp(-s^2) \qquad \text{for } s \in \mathbb{R}^{+\bullet},$$

respectively.

We again set $A = (\ell^1(\mathbb{R}^+), \star)$ and $B = (\ell^1(\mathbb{I}), \star)$ in the notation of Chapter 7.

Theorem 8.1. *Let ω be a continuous weight on $\mathbb{R}^+$, and set $A_\omega = \ell^1(\mathbb{R}^{+\bullet}, \omega)$. For each $\mu \in E_\omega^\circ$ with $\mu \neq 0$, there exists an element $v \in (\mathbb{R}^{+\bullet})_\omega^*$ with $\mu \Box_\omega v \neq v \Box_\omega \mu$. In particular, A_ω is strongly Arens irregular.*

Proof. Assume towards a contradiction that there exists $\mu \in (E_\omega^\circ)_\mathbb{R}$ with $\|\mu\|_\omega = 1$ such that $\mu \Box_\omega v = v \Box_\omega \mu$ for each $v \in (\mathbb{R}^{+\bullet})_\omega^*$.

We make a first reduction. For each $t > 0$, set

$$I_t = \{f \in A_\omega : \alpha(f) \geq t,\ f(t) = 0\} \cup \{0\},$$

so that I_t is a closed ideal in A_ω. The quotient map is

$$\pi_t : A_\omega \to A_\omega/I_t =: A_t,$$

and this map extends to a map $\pi_t'' : A_\omega'' \to A_t''$. Since ω is continuous on the closed interval $[0,t]$ of $\mathbb{R}$, there exist constants $m, M > 0$ such that

$$m \le \omega(s) \le M \qquad (0 \le s \le t).$$

This shows that A_t is isomorphic to $\ell^1(\mathbb{Q} \cap (0,t])$ by a linear homeomorphism, and hence A_t is isomorphic to B by a linear homeomorphism.

Assume towards a contradiction that there exists $t > 0$ with

$$\nu := \pi_t''(\mu) \ne 0.$$

Then $\nu \in A_t'' \setminus A_t$, and so, by Theorem 7.4, there exists $a_t \in (0,t)^*$ with $\nu \,\Box\, a_t \ne a_t \,\Box\, \nu$ in A_t''. Choose $v \in (0,t)^*_\omega$ such that $\pi''(v) = a_t$. Then necessarily $\mu \,\Box_\omega\, v \ne v \,\Box_\omega\, \mu$ in A_ω'', a contradiction. Thus we may suppose that we have $\pi_t''(\mu) = 0$ for each $t > 0$.

Take $\widetilde{\mu} \in (A'')_{\mathbb{R}}$ such that $\theta_\omega(\widetilde{\mu}) = \mu$; we have $\|\widetilde{\mu}\| = 1$.

By Theorem 5.11 (with $\varepsilon = 1/10$), there exist elements $a \in \mathbb{I}^*$ with $a \to 0$ and $\widetilde{\lambda} \in \ell^\infty_{\mathbb{R}}(\mathbb{R}^{+\bullet})$ with $\|\widetilde{\lambda}\| = 1$ such that

$$\langle \widetilde{\mu} \,\Box\, a, \widetilde{\lambda}\rangle > 9/10, \quad |\langle a \,\Box\, \widetilde{\mu}, \widetilde{\lambda}\rangle| < 1/10.$$

Set $\lambda = \widetilde{\lambda}\omega$ and $v = \theta_\omega(a)$, so that $\lambda \in A_\omega'$ with $\|\lambda\|_\omega = 1$ and $v \in \mathbb{I}^*_\omega$. We have

$$\langle \widetilde{\mu} \,\Box\, a, \widetilde{\lambda}\rangle - \langle \mu \,\Box_\omega\, v, \lambda\rangle = \lim_{m\to\infty} \lim_{n\to\infty} \sum_s \widetilde{\lambda}(s + r_n)\widetilde{f}_m(s)(1 - \Omega(s, r_n)) \tag{8.1}$$

and

$$\langle a \,\Box\, \widetilde{\mu}, \widetilde{\lambda}\rangle - \langle v \,\Box_\omega\, \mu, \lambda\rangle = \lim_{n\to\infty} \lim_{m\to\infty} \sum_s \widetilde{\lambda}(s + r_n)\widetilde{f}_m(s)(1 - \Omega(s, r_n)) \tag{8.2}$$

for some sequence (f_m) of real-valued functions in $A_{[1]}$ of finite support and some sequence (r_n) in $(0,1)$ with $r_n \searrow 0$; the details of the construction of these sequences are given in [12, Proposition 3.1], and inspection of that proof shows that we may suppose that $r_n \searrow 0$ and that $r_1 < t_0$ in case (i), above. By our first reduction, $\lim_{m\to\infty} \alpha(f_m) = \infty$.

By the hypothesis on ω, we have $\lim_{t\to 0+} \Omega(s,t) = 1$ $(s \in \mathbb{R}^{+\bullet})$, and so, for each $\varepsilon > 0$ and $m \in \mathbb{N}$, there exists $n_0 \in \mathbb{N}$ such that

$$|1 - \Omega(s, r_n)| < \varepsilon \qquad (n \ge n_0,\ s \in \operatorname{supp} f_m).$$

It follows that

$$|\langle \widetilde{\mu} \,\Box\, a, \widetilde{\lambda}\rangle - \langle \mu \,\Box_\omega\, v, \lambda\rangle| \le \limsup_{m\to\infty} \varepsilon \sum_{s\in\mathbb{R}^{+\bullet}} |\widetilde{f}_m(s)| = \varepsilon.$$

The above holds for each $\varepsilon > 0$, and so $\langle \widetilde{\mu} \,\Box\, a, \widetilde{\lambda}\rangle = \langle \mu \,\Box_\omega\, v, \lambda\rangle$. Thus $\langle \mu \,\Box_\omega\, v, \lambda\rangle \ge 9/10$.

We now compare $\langle a \,\Box\, \widetilde{\mu}, \widetilde{\lambda}\rangle$ and $\langle v \,\Box_\omega\, \mu, \lambda\rangle$.

First, suppose that case (i) occurs. Then it follows from (8.2) that

$$\begin{aligned} |\langle v \,\Box_\omega\, \mu, \lambda\rangle| &\le |\langle a \,\Box\, \widetilde{\mu}, \widetilde{\lambda}\rangle| + \limsup_{n\to\infty} \limsup_{m\to\infty} \sum_{s\in\mathbb{R}^{+\bullet}} |f_m(s)|(1 - \Omega(s, r_n)) \\ &\le \frac{1}{10} + \limsup_{n\to\infty} \limsup_{m\to\infty} \sum_{s\in\mathbb{R}^{+\bullet}} |f_m(s)|(1 - \Omega(s, r_n)). \end{aligned}$$

For $n \in \mathbb{N}$, choose $m_n \in \mathbb{N}$ so that $\Omega(s, r_m) > 4/5$ whenever $m \geq m_n$ and $s \in \operatorname{supp} f_m$; this uses the fact that $\lim_{m\to\infty} \alpha(f_m) = \infty$. Thus

$$|\langle v \,\Box_\omega\, \mu, \lambda\rangle| \leq \frac{1}{10} + \frac{1}{5} = \frac{3}{10} < \frac{9}{10} \leq \langle \mu \,\Box_\omega\, v, \lambda\rangle,$$

and so $\mu \,\Box_\omega\, v \neq v \,\Box_\omega\, \mu$, a contradiction.

Second, suppose that case (ii) occurs. Then we see directly that

$$|\langle v \,\Box_\omega\, \mu, \lambda\rangle| \leq \limsup_{n\to\infty} \limsup_{s\to\infty} \Omega(s, t_n) = \limsup_{n\to\infty} \kappa(t_n) \leq \frac{4}{5} < \frac{9}{10} \leq \langle \mu \,\Box_\omega\, v, \lambda\rangle,$$

and so again $\mu \,\Box_\omega\, v \neq v \,\Box_\omega\, \mu$, a contradiction.

This concludes the proof. ■

Example 8.2. The following question was left open in Example 6.1: Is the weight

$$\omega : s \mapsto \exp(-s^2), \qquad \mathbb{R}^{+\bullet} \to \mathbb{R}^{+\bullet},$$

strongly Arens irregular? We now see from the above theorem that this is the case. The weight ω is a radical weight.

9. Conditions for Arens regularity

A condition for the Arens regularity of weighted convolution algebras on cancellative semigroups was given as [5, Corollary (3.8)(i)]; results for more general semigroups are given elsewhere in [5]. The first definition is from [12, Definition 3.2(ii)]; more general versions are given in [5].

Definition 9.1. Let X and Y be non-empty sets, and let $f : X \times Y \to \mathbb{C}$ be a function. Then f *0-clusters on* $X \times Y$ if

$$\lim_{m\to\infty} \lim_{n\to\infty} f(x_m, y_n) = \lim_{n\to\infty} \lim_{m\to\infty} f(x_m, y_n) = 0$$

whenever (x_m) and (y_n) are sequences of distinct elements of X and Y, respectively, and both repeated limits exist.

Suppose that $f : X \times Y \to \mathbb{R}^+$ is bounded and that (x_m) and (y_n) are sequences in X and Y, respectively. Then, by taking successive subsequences of (x_m) and (y_n), we can suppose that the two repeated limits of $(f(x_m, y_n) : m, n \in \mathbb{N})$ exist. Thus f fails to 0-cluster on $X \times Y$ if there exist sequences (x_m) and (y_n) of distinct elements in X and Y, respectively, such that

$$\liminf_{m\to\infty} \liminf_{n\to\infty} f(x_m, y_n) > 0.$$

In particular, f fails to 0-cluster on $X \times Y$ if there exist $\delta > 0$ and sequences as above such that

$$f(x_m, y_n) \geq \delta \quad \text{whenever} \quad m, n \in \mathbb{N} \text{ with } m < n.$$

The following theorem is proved by exactly the argument of [12, Theorem 8.8]; the result originates in the work of Craw and Young in [10, Theorem 1].

Theorem 9.2. *Let ω be a weight on a semigroup S, and let X and Y be infinite subsets of S. Suppose that Ω 0-clusters on $X \times Y$, and let $\mu \in M(X_\omega^*)$ and $\nu \in M(Y_\omega^*)$. Then $\mu \,\Box_\omega\, \nu = 0$.* ∎

Corollary 9.3. *Let ω be a weight on an abelian semigroup S, and let X be an infinite subset of S. Suppose that Ω 0-clusters on $X \times S$. Then ω is not strongly Arens irregular.*

Proof. Since S is abelian, Ω also 0-clusters on $S \times X$.

Let $\mu \in M(X_\omega^*)$. Then $\mu \,\Box_\omega\, \nu = \nu \,\Box_\omega\, \mu$ ($\nu \in A_\omega$) and

$$\mu \,\Box_\omega\, \nu = \nu \,\Box_\omega\, \mu = 0 \quad (\nu \in E_\omega^\circ).$$

There exists $\mu \in M(X_\omega^*) \setminus \{0\}$, and then $\mu \in \mathfrak{Z}(A_\omega'') \setminus A_\omega$. Thus A_ω is not strongly Arens irregular. ∎

The following theorem of Craw and Young is [10, Theorem 1]; see [12, Chapter 8] for related results.

Theorem 9.4. *Let ω be a weight on a cancellative semigroup S. Then ω is Arens regular if and only if Ω 0-clusters on $S \times S$.* ∎

Corollary 9.5. *Let ω_1 be an Arens regular weight on a cancellative semigroup S, and let ω_2 be any weight on S. Then $\omega_1\omega_2$ is an Arens regular weight on S.*

Proof. Since ω_1 is an Arens regular weight, the corresponding function Ω_1 0-clusters on $S \times S$. But then $\Omega_1\Omega_2$, which corresponds to $\omega_1\omega_2$, 0-clusters on $S \times S$, and so $\omega_1\omega_2$ is Arens regular. ∎

The corollary below contrasts with [27, Theorem 2.2], which states the following. Let ω be a continuous weight on $\mathbb{R}^+$ such that every element of the Banach algebra $(L^1(\mathbb{R}^+, \omega), \star)$ is compact. Then $L^1(\mathbb{R}^+, \omega)$ is strongly Arens irregular.

Corollary 9.6. *Let S be a subsemigroup of $(\mathbb{R}, +)$, and let ω be a weight on S such that each element of $\ell^1(S, \omega)$ is compact. Then ω is Arens regular.*

Proof. By Proposition 4.19(i),

$$\operatorname*{Lim}_{t \to \infty} \omega(s+t)/\omega(t) = 0 \quad (s \in S).$$

Let (s_m) and (t_n) be sequences of distinct elements of $\mathbb{Q}^{+\bullet}$. Then

$$\lim_{n \to \infty} \Omega(s_m, t_n) = 0 \quad (m \in \mathbb{N}),$$

and so Ω 0-clusters on $S \times S$; by the theorem, ω is Arens regular. ∎

Corollary 9.7. *Let S be a dense, difference subsemigroup of $\mathbb{R}^{+\bullet}$, and let ω be a weight on S such that ω is bounded on $S \cap (a, b)$ for some $a, b \in \mathbb{R}$ with $0 < a < b$. Then ω is not Arens regular.*

Proof. By hypothesis, there exists $M > 0$ with $\omega(s) \le M$ ($s \in S \cap (a, b)$). By Proposition 4.13, there exist $\delta > 0$ such that $\omega(s+t) > \delta$ ($s, t \in S \cap (0, b)$). Let (s_m) and (t_n) be sequences of distinct elements of $S \cap (a, b)$. Then

$$\Omega(s_m, t_n) \ge \delta/M^2 > 0 \quad (m, n \in \mathbb{N}),$$

and so Ω does not 0-cluster on the set $S \times S$. By Theorem 9.4, ω is not Arens regular. ∎

Example 9.15, below, will show that such a weight ω is not necessarily strongly Arens irregular.

EXAMPLE 9.8. Let $\omega(n) = \exp(-n^2)$ $(n \in \mathbb{N})$. Then ω is a radical, Arens regular weight on $\mathbb{N}$. Indeed, we see that

$$\Omega(m,n) = \exp(-2mn) \quad (m,n \in \mathbb{N}),$$

and so Ω 0-clusters on $\mathbb{N} \times \mathbb{N}$.

EXAMPLE 9.9. Let $\omega_\alpha(n) = (1+|n|)^\alpha$ $(n \in \mathbb{Z})$, where $\alpha > 0$. Then ω_α is a weight on $\mathbb{Z}$. Clearly ω_α is Arens regular, as remarked in [10]. Similarly, $\omega_\alpha|\mathbb{N}$ is a semisimple, Arens regular weight on $\mathbb{N}$. Of course, $\ell^1(\mathbb{Z})$ and $\ell^1(\mathbb{N})$ are strongly Arens irregular.

EXAMPLE 9.10. Let $\omega(s) = \exp(1/s)$ $(s \in \mathbb{Q}^{+\bullet})$. Then ω is a semisimple weight on $\mathbb{Q}^{+\bullet}$, and

$$\Omega(s,t) \leq \exp(-1/s) \quad (s \in \mathbb{Q}^{+\bullet}).$$

It follows that ω is not weakly diagonally bounded on any infinite subset of $\mathbb{Q}^{+\bullet}$.

Since ω is bounded on the set $[1,\infty)$, ω is not Arens regular by Corollary 9.7.

Set $X = \{1/m : m \in \mathbb{N}\}$, an infinite subset of $\mathbb{Q}^{+\bullet}$. For $t \in \mathbb{Q}^{+\bullet}$ and $m \in \mathbb{N}$, we have $\Omega(1/m,t) \leq \exp(-m)$. Thus Ω 0-clusters on $X \times \mathbb{Q}^{+\bullet}$, and so, by Corollary 9.3, ω is not strongly Arens irregular.

Now let $\omega(s) = \exp(1/s)$ $(s \in \mathbb{R}^{+\bullet})$, so that ω is a semisimple weight on $\mathbb{R}^{+\bullet}$. By the same argument, ω is neither Arens regular nor strongly Arens irregular on $\mathbb{R}^{+\bullet}$.

By replacing ω by the weight given by

$$\omega(s) = \exp\left(-s^2 + \frac{1}{s}\right),$$

we obtain a radical weight on $\mathbb{Q}^{+\bullet}$ or $\mathbb{R}^{+\bullet}$ that is neither Arens regular nor strongly Arens irregular.

We give another example of the same phenomenon; it is a preliminary to the more important Example 9.12.

EXAMPLE 9.11. For $m \in \mathbb{Z}^+$ and $p/q \in \mathbb{Q}^{+\bullet} \cap (0,1)$, set

$$\omega(m) = 1+m, \qquad \omega(m+p/q) = 1+m+q.$$

First, we *claim* that $\omega(s+t) \leq \omega(s)\omega(t)$ for $s,t \in \mathbb{Q}^+$. This is immediate in the case where $s,t \in \mathbb{Z}^+$. Now suppose that $s = m+p/q$ and $t = n+u/v$, where $m,n \in \mathbb{Z}^+$ and p/q, $u/v \in \mathbb{Q}^{+\bullet} \cap (0,1)$. Then

$$\begin{aligned}\omega(s+t) &\leq 2+m+n+qv \leq (2+m)(1+n)+q(1+v)\\ &\leq (1+m+q)(1+n+v) = \omega(s)\omega(t).\end{aligned}$$

Similarly our inequality holds when $s = m \in \mathbb{Z}^+$ and $t = n+u/v$, with $n \in \mathbb{Z}^+$ and $u/v \in \mathbb{Q}^{+\bullet} \cap (0,1)$. Thus the claim holds.

Since $\omega(s) \geq 1$ $(s \in \mathbb{Q}^+)$, the weight ω is semisimple on $\mathbb{Q}^+$.

Second, we *claim* that Ω does not 0-cluster on $\mathbb{Q}^+ \times \mathbb{Q}^+$, and hence, by Theorem 9.4, ω is not Arens regular. Indeed, let (q_m) be a strictly increasing sequence of prime numbers, and set $x_m = y_m = 1/q_m$ ($m \in \mathbb{N}$). Then $\Omega(x_m, y_n) = 1$ whenever $m, n \in \mathbb{N}$ with $m \neq n$, and so the claim holds.

Third, we *claim* that Ω does 0-cluster on the set $\mathbb{N} \times \mathbb{Q}^+$, and hence, by Corollary 9.3, ω is not strongly Arens irregular. Indeed, for each $n \in \mathbb{Z}^+$ and $p/q \in \mathbb{Q}^{+\bullet} \cap (0,1)$, set

$$\varphi(n + p/q) = n + q$$

(with $\varphi(n) = n$); we note that $\varphi(y_n) \to \infty$ as $n \to \infty$ for every sequence (y_n) of distinct elements of $\mathbb{Q}^+$. Let (x_m) be a sequence of distinct elements of $\mathbb{N}$, so that $x_m \to \infty$ as $m \to \infty$. Then

$$\Omega(x_m, y_n) = \frac{1 + x_m + \varphi(y_n)}{(1 + x_m)(1 + \varphi(y_n))},$$

and so both repeated limits of $(\Omega(x_m, y_n) : m, n \in \mathbb{N})$ are 0.

We conclude that ω is a semisimple weight on $\mathbb{Q}^+$ or $\mathbb{Q}^{+\bullet}$ that is neither Arens regular nor strongly Arens irregular.

Example 9.12. Let ω be the weight on $\mathbb{Q}^+$ specified in Example 9.11, and extend ω to $\mathbb{Q}$ by setting $\omega(-s) = \omega(s)$ ($s \in \mathbb{Q}^{+\bullet}$). Thus

$$\omega(-n + p/q) = n + q \qquad (n \in \mathbb{N},\ p/q \in \mathbb{Q}^{+\bullet} \cap (0,1)).$$

To show that ω is submultiplicative on $\mathbb{Q}$, we must verify that

$$\omega(s - t) \leq \omega(s)\omega(t) \qquad (s, t \in \mathbb{Q}^{+\bullet}). \tag{9.1}$$

Certainly $\omega|\mathbb{Z}$ is submultiplicative. Take $s = m + p/q$ and $t = n + u/v$, as above. Then

$$\begin{aligned}\omega(s - t) &\leq 1 + |m - n| + qv \leq 1 + m + n + qv \\ &\leq (1 + m + q)(1 + n + v) = \omega(s)\omega(t).\end{aligned}$$

Similarly (9.1) holds in the cases where $s = m$ and $t = n + u/v$ and where $s = m + p/q$ and $t = n$. Thus we have verified (9.1).

As before, Ω does not 0-cluster on $\mathbb{Q}^+ \times \mathbb{Q}^+$, and hence, by Theorem 9.4, ω is not Arens regular.

We already know that Ω 0-clusters on $\mathbb{N} \times \mathbb{Q}^+$. Again as before, take (x_m) and (y_n) to be sequences of distinct elements of $\mathbb{N}$ and $\mathbb{Q}^+$, respectively. Then

$$\Omega(x_m, -y_n) \leq \frac{1 + x_m + \varphi(y_n)}{(1 + x_m)(1 + \varphi(y_n))},$$

in the earlier notation, and so both repeated limits of $(\Omega(x_m, y_n) : m, n \in \mathbb{N})$ are 0. This shows that Ω 0-clusters on $\mathbb{N} \times \mathbb{Q}$, and hence that ω is not strongly Arens irregular.

We conclude that ω is a symmetric weight on $\mathbb{Q}$ that is neither Arens regular nor strongly Arens irregular.

Example 9.13. For $m, n \in \mathbb{N}$, set $m \wedge n = \min\{m, n\}$. Then

$$\mathbb{N}_\wedge = (\mathbb{N}, \wedge)$$

is an abelian semigroup, as in [13, Example 3.36]. The semigroup $\mathbb{N}_\wedge$ is not weakly cancellative.

Let $\omega : \mathbb{N} \to [1, \infty)$ be any function. Then ω is a weight on $\mathbb{N}_\wedge$. Set $E_\omega = c_0(\mathbb{N}, 1/\omega)$, as before. For $m \in \mathbb{N}$ and $\lambda = (\lambda_n : n \in \mathbb{N}) \in E_\omega$, we have

$$\limsup_{n\to\infty} \frac{(\widetilde{\delta}_m \cdot \lambda)(n)}{\omega(n)} = \limsup_{n\to\infty} \frac{\lambda_m}{\omega(m)\omega(n)} \le \limsup_{n\to\infty} \frac{\|\lambda\|_\omega}{\omega(n)}. \tag{9.2}$$

Now suppose that $\omega(n) \to \infty$ as $n \to \infty$. Then

$$\widetilde{\delta}_m \cdot \lambda \in E_\omega \quad (m \in \mathbb{N}).$$

Thus E_ω is an A_ω-submodule of A'_ω, and so A_ω is a dual Banach algebra.

Further, let (s_m) and (t_n) be two sequences of distinct elements of $\mathbb{N}$, so that

$$\lim_{m\to\infty} s_m = \lim_{n\to\infty} t_n = \infty.$$

For each fixed $m \in \mathbb{N}$, we have

$$\limsup_{n\to\infty} \Omega(s_m, t_n) = \limsup_{n\to\infty} \frac{1}{\omega(t_n)} = 0,$$

and so Ω 0-clusters on $\mathbb{N} \times \mathbb{N}$. By Theorem 9.4, A_ω is Arens regular.

On the other hand, suppose that $\liminf_{n\to\infty} \omega(n) < \infty$. Then there exist $C > 0$ and an infinite subset T of $\mathbb{N}$ such that $\omega(n) < C$ $(n \in T)$. For each $m \in \mathbb{N}$, set $T_m = \{n \in T : n \ge m\}$, a cofinite subset of T. Then

$$\Omega(m, n) = 1/\omega(n) \ge 1/C \quad (n \in T_m),$$

and so ω is weakly diagonally bounded on T. By Corollary 5.7, ω is strongly Arens irregular on $\mathbb{N}$.

The present example generalizes [13, Example 7.3.2].

The following example, giving a radical, Arens regular weight on $\mathbb{Q}^{+\bullet}$, slightly extends [10, Corollary 1]; a more explicit example of such a weight will be given in Example 9.17, below.

EXAMPLE 9.14. Let S be the semigroup $((\mathbb{Z}^+)^{<\omega}, +)$, as above. Set

$$\omega(x) = 1 + x_1 + 2x_2 + \cdots + kx_k \quad (x = (x_1, x_2, \dots, x_k, 0, 0, \dots) \in S).$$

Then it is immediately checked that ω is a weight on S and that $\omega(x^m) \to \infty$ as $m \to \infty$ for each sequence (x^m) of distinct elements of S.

We have

$$\Omega(x, y) = \frac{\omega(x) + \omega(y) - 1}{\omega(x)\omega(y)} \quad (x, y \in S),$$

and so Ω 0-clusters on $S \times S$. Thus ω is Arens regular on S.

The map

$$\theta : (x_1, x_2, \dots, x_k, 0, 0, \dots) \mapsto \sum_{j=1}^{k} \frac{x_j}{j}, \qquad S \to \mathbb{Q}^{+\bullet},$$

is a semigroup epimorphism. Clearly $\omega(x) \ge 1$ $(x \in S)$, and so we obtain an induced weight $\widetilde{\omega}$ which is Arens regular on $\mathbb{Q}^{+\bullet}$, as in Example 4.21. Since $\widetilde{\omega} \ge 1$, the weight $\widetilde{\omega}$ is semisimple.

Similarly, the map

$$\theta : (x_1, x_2, \dots, x_k, 0, 0, \dots) \mapsto \sum_{j=1}^{k} \frac{x_{2j} - x_{2j-1}}{j}, \qquad S \to \mathbb{Q},$$

is a semigroup epimorphism, and so we obtain an Arens regular weight on the group $\mathbb{Q}$.

By multiplying the above weight $\widetilde{\omega}$ on $\mathbb{Q}^{+\bullet}$ by the weight $s \mapsto \exp(-s^2)$, we obtain a radical, Arens regular weight on $\mathbb{Q}^{+\bullet}$.

Example 9.15. We give an example of a bounded, radical weight ω on $\mathbb{Q}^{+\bullet}$ such that ω is neither Arens regular nor strongly Arens irregular.

Indeed, set $\omega(s) = 1/(m+1)!$ whenever $m < s \leq m+1$ for some $m \in \mathbb{Z}^+$. To see that ω is a weight on $\mathbb{Q}^{+\bullet}$, let $s, t \in \mathbb{Q}^{+\bullet}$, and take $m, n \in \mathbb{Z}^+$ such that $m < s \leq m+1$ and $n < t \leq n+1$. Then $m + n < s + t \leq m + n + 2$, and so

$$\omega(s+t) \leq \frac{1}{(m+n+1)!} \leq \frac{1}{(m+1)!} \frac{1}{(n+1)!} = \omega(s)\omega(t),$$

as required.

For $m, n \in \mathbb{N}$, set $s_m = 1/m$ and $t_n = 1/n$. Then $\Omega(s_m, t_n) = 1$ for $m, n \geq 2$, and so Ω does not 0-cluster on $\mathbb{Q}^{+\bullet} \times \mathbb{Q}^{+\bullet}$; by Theorem 9.4, ω is not Arens regular. Since $\Omega(m, t) \leq 1/m$ ($m \in \mathbb{N}$, $t \in \mathbb{Q}^{+\bullet}$), it is immediate that Ω 0-clusters on $\mathbb{N} \times \mathbb{Q}^{+\bullet}$, and so, by Corollary 9.3, ω is not strongly Arens irregular.

Example 9.16. Let $\omega_1(s) = \exp(-s^2)$ ($s \in \mathbb{R}^+$). Then ω_1 is a continuous weight function on $\mathbb{R}^+$. Let

$$\omega_2(p/q) = 1 + \log q \qquad (p/q \in \mathbb{Q}^{+\bullet}).$$

Then, as in Example 6.3, ω_2 is a weight on $\mathbb{Q}^{+\bullet}$ with $\omega_2 \geq 1$, and ω_2 is diagonally bounded on $\mathbb{N}$, with $d_{\mathbb{N}} = 1$; by Corollary 5.7, ω_2 is strongly Arens irregular. (Indeed, by Theorem 5.6, there is a two-element subset V of $\mathbb{N}^*_{\omega_2}$ that is determining for the topological centre of A_{ω_2}.)

However, we *claim* that $\omega := \omega_1\omega_2$ is not strongly Arens irregular on $\mathbb{Q}^{+\bullet}$; cf. a remark in Chapter 7. To see this, we apply the condition of Corollary 9.3, taking

$$X = \{1 + 1/n : n \in \mathbb{N}\},$$

an infinite subset of $\mathbb{Q}^{+\bullet}$.

Let (x_m) and (y_n) be two sequences of distinct elements of X and $\mathbb{Q}^{+\bullet}$, respectively, say $x_m = p_m/q_m$ ($m \in \mathbb{N}$) and $y_n = r_n/s_n$ ($n \in \mathbb{N}$). We have $x_m \geq 1$ ($m \in \mathbb{N}$), and so

$$\Omega(x_m, y_n) \leq \frac{1 + \log q_m + \log s_n}{(1 + \log q_m)(1 + \log s_n)} \exp(-2y_n) \leq \exp(-2y_n)$$

for each $m, n \in \mathbb{N}$. Necessarily $q_m \to \infty$ as $m \to \infty$. By passing to a subsequence, we may suppose that (y_n) either converges to a point of $\mathbb{R}^+$ or diverges to ∞.

First, suppose that $\lim_{n\to\infty} y_n = \infty$. Since $\Omega(x_m, y_n) \leq \exp(-2y_n)$, certainly

$$\lim_{m\to\infty} \lim_{n\to\infty} \Omega(x_m, y_n) = \lim_{n\to\infty} \lim_{m\to\infty} \Omega(x_m, y_n) = 0.$$

Second, suppose that $\lim_{n\to\infty} y_n < \infty$. By passing to a further subsequence, we may suppose that $s_n \to \infty$ as $n \to \infty$, and so

$$\limsup_{m\to\infty} \limsup_{n\to\infty} \Omega(x_m, y_n) \le \limsup_{m\to\infty} \frac{1}{1+\log q_m} = 0.$$

Thus $\lim_{m\to\infty} \lim_{n\to\infty} \Omega(x_m, y_n) = 0$. Similarly,

$$\lim_{n\to\infty} \lim_{m\to\infty} \Omega(x_m, y_n) = 0.$$

It follows that Ω 0-clusters on $X \times \mathbb{Q}^{+\bullet}$, and so, by Corollary 9.3, ω is not strongly Arens irregular, as claimed.

We also *claim* that the weight ω is not Arens regular on $\mathbb{Q}^{+\bullet}$. To see this, we take $x_m = 1/p_m$ $(m \in \mathbb{N})$, where (p_m) is a strictly increasing sequence of prime numbers, and let $y_n = n$ $(n \in \mathbb{N})$. Then

$$\Omega(x_m, y_n) = \exp(-2n/p_m) \qquad (m, n \in \mathbb{N}),$$

and so

$$\lim_{m\to\infty} \Omega(x_m, y_n) = 1 \qquad (n \in \mathbb{N}) \quad \text{and} \quad \lim_{n\to\infty} \Omega(x_m, y_n) = 0 \qquad (m \in \mathbb{N}).$$

Thus both repeated limits of the double sequence $(\Omega(x_m, y_n) : m, n \in \mathbb{N})$ exist and they are unequal, so that Ω does not 0-cluster on $\mathbb{Q}^{+\bullet} \times \mathbb{Q}^{+\bullet}$. By Theorem 9.4, ω is not Arens regular on $\mathbb{Q}^{+\bullet}$.

In summary, the weight

$$\omega : p/q \mapsto (1+\log q)\exp(-p^2/q^2), \qquad \mathbb{Q}^{+\bullet} \to \mathbb{R}^{+\bullet},$$

is a radical weight on $\mathbb{Q}^{+\bullet}$ that is neither Arens regular nor strongly Arens irregular.

We cannot see an example in which ω_1 is a continuous weight function on $\mathbb{R}^+$, ω_2 is a strongly Arens irregular weight on $\mathbb{Q}^{+\bullet}$, and $\omega_1\omega_2$ is Arens regular on $\mathbb{Q}^{+\bullet}$.

Example 9.17. We now show that there are a continuous weight function ω_1 on $\mathbb{R}^{+\bullet}$ and a strongly Arens irregular weight ω_2 on $\mathbb{Q}^{+\bullet}$ such that $\omega := \omega_1\omega_2$ is Arens regular on $\mathbb{Q}^{+\bullet}$.

First, set

$$\omega_1(s) = \exp(-s^2 + 1/s) \qquad (s \in \mathbb{R}^{+\bullet}),$$

as in Example 9.9, so that ω_1 is a continuous weight function on $\mathbb{R}^{+\bullet}$, and take

$$\omega_2 : p/q \mapsto 1 + \log q$$

to be as in Example 9.16, so that ω_2 is a strongly Arens irregular weight on $\mathbb{Q}^{+\bullet}$. Set $\omega := \omega_1\omega_2$; we must show that ω 0-clusters on $\mathbb{Q}^{+\bullet} \times \mathbb{Q}^{+\bullet}$.

Again, we take (x_m) and (y_n) to be two sequences of distinct elements of $\mathbb{Q}^{+\bullet}$, say $x_m = p_m/q_m$ $(m \in \mathbb{N})$ and $y_n = r_n/s_n$ $(n \in \mathbb{N})$, as above. For each $m, n \in \mathbb{N}$, the number $\Omega(x_m, y_n)$ is not greater than

$$\frac{1+\log q_m + \log s_n}{(1+\log q_m)(1+\log s_n)} \exp\left(-2x_m y_n + \frac{1}{x_m+y_n} - \frac{1}{x_m} - \frac{1}{y_n}\right)$$

By passing to subsequences, we may suppose that both of (x_m) and (y_n) either converge to a point of $\mathbb{R}^+$ or diverge to ∞.

The argument of Example 9.16 shows that the function Ω 0-clusters on $\mathbb{Q}^{+\bullet} \times \mathbb{Q}^{+\bullet}$ in the case where $\lim_{m\to\infty} x_m < \infty$, and so we may suppose that $\lim_{m\to\infty} x_m = \infty$. But now

$$\limsup_{m\to\infty} \Omega(x_m, y_n) \leq \limsup_{m\to\infty} \exp(-2x_m y_n) = 0$$

for each $n \in \mathbb{N}$, and so $\lim_{n\to\infty} \lim_{m\to\infty} \Omega(x_m, y_n) = 0$. In the case where $\lim_{n\to\infty} y_n > 0$, it follows similarly that

$$\lim_{m\to\infty} \lim_{n\to\infty} \Omega(x_m, y_n) = 0.$$

In the case where $\lim_{n\to\infty} y_n = 0$, we have

$$\limsup_{n\to\infty} \Omega(x_m, y_n) \leq \limsup_{n\to\infty} \exp\left(\frac{1}{x_m + y_n} - \frac{1}{x_m} - \frac{1}{y_n}\right) = 0,$$

and so $\lim_{m\to\infty} \lim_{n\to\infty} \Omega(x_m, y_n) = 0$.

Thus Ω 0-clusters on $\mathbb{Q}^{+\bullet} \times \mathbb{Q}^{+\bullet}$, and so ω is Arens regular.

We conclude this section by considering whether or not there can be an Arens regular weight on the semigroup $(\mathbb{R}^{+\bullet}, +)$; this question was left open in the seminal paper of Craw and Young [10]. Our partial answer is given by essentially the proof of Theorem 5.12.

THEOREM 9.18. *Let ω be a weight on $\mathbb{R}^{+\bullet}$ such that $\liminf_{s\to 0+} \omega(s) > 0$. Then ω is not Arens regular.*

Proof. By Proposition 4.14, (d)$\Rightarrow$(b), there exist $c \in \mathbb{R}^{+\bullet}$ and $\delta > 0$ such that

$$\inf\{\omega(s) : s \in (0, c)\} \geq \delta.$$

For $n \in \mathbb{N}$, set $R_n = \{t \in (0, c) : \omega(t) \leq n\}$. Since $|(0, c)| = \mathfrak{c}$, there exists $n_0 \in \mathbb{N}$ such that $|R_{n_0}| = \mathfrak{c}$; in particular, R_{n_0} is infinite, and so there exists a sequence (r_n) of distinct points in R_{n_0}. Clearly,

$$\Omega(r_m, r_n) \geq \delta/n_0^2 \quad (m, n \in \mathbb{N}).$$

This shows that Ω fails to 0-cluster on $\mathbb{R}^{+\bullet} \times \mathbb{R}^{+\bullet}$, and so, by Theorem 9.4, ω is not Arens regular. ∎

COROLLARY 9.19. *Let ω be a weight on $\mathbb{R}^{+\bullet}$ that is either measurable or semisimple. Then ω is not Arens regular.*

Proof. Suppose that ω is measurable. Then it follows from the theorem and Proposition 4.17 that ω is not Arens regular.

Suppose that ω is semisimple. Then it follows from Proposition 4.10 that there is a weight $\widetilde{\omega}$ on $\mathbb{R}^{+\bullet}$ with $\widetilde{\omega} \geq 1$ such that $\ell^1(\mathbb{R}^{+\bullet}, \omega)$ is isometrically isomorphic to $\ell^1(\mathbb{R}^{+\bullet}, \widetilde{\omega})$. By the theorem, $\widetilde{\omega}$ is not Arens regular, and hence ω is not Arens regular. ∎

We believe that it is true that no weight on $\mathbb{R}^{+\bullet}$ is Arens regular, but we cannot see this yet.

10. A weight on $\mathbb{N}$

In this section we shall exhibit a radical weight on $\mathbb{N}$ that is neither Arens regular nor strongly Arens irregular. We observe that no such example is given in [12]. A somewhat related example is given in [4, Chapter 2] (see also [3]), but that example can be shown to be Arens regular.

We proceed through some preliminary results and notation.

A *base point* in $\mathbb{N}$ is a number of the form 2^k, where $k \in \mathbb{Z}^+$, together with numbers of the form $2^k + 2^j$, where $k \in \mathbb{N}$ and $j \in \mathbb{Z}^+_{k-1}$, so our initial base points are

$$1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 16, \dots.$$

The *special base points* are

$$m_k = 2^k + 2^{k-1} \quad (k \in \mathbb{N}),$$

and we set $X = \{m_k : k \in \mathbb{N}\}$. We also fix an increasing sequence $(\alpha_k : k \in \mathbb{Z}^+)$ in $\mathbb{R}^+$, with $\alpha_0 = 0$. In fact, $(\alpha_k : k \in \mathbb{Z}^+)$ is rapidly increasing, in that we require that $\alpha_1 = 4$ and that

$$\alpha_{k+1} \geq (2^{2k+3} + 1)\alpha_k \quad (k \in \mathbb{Z}^+). \tag{10.1}$$

We note that $\alpha_k \geq 4^k$ $(k \in \mathbb{N})$. For convenience, we also set

$$\beta_k = 2^{k+2}\alpha_k \geq \alpha_k + \alpha_{k-1} \quad (k \in \mathbb{N}).$$

We define a function η at our base points by

$$\eta(2^k) = \alpha_k \quad (k \in \mathbb{Z}^+), \qquad \eta(2^k + 2^j) = \alpha_k + \alpha_j \quad (k \in \mathbb{N},\ j \in \mathbb{Z}^+_{k-1}).$$

We then define (the graph of) η to be linear between $(b_1, \eta(b_1))$ and $(b_2, \eta(b_2))$ whenever b_1 and b_2 are adjacent base points, save that we define $\eta(t)$ for $t \in (m_k, 2^{k+1})$ by linear interpolation between $(m_k, \eta(m_k))$ and $(2^{k+1}, \beta_k)$. We also set $\eta(t) = 0$ $(t \in \mathbb{I})$. Thus we have defined $\eta : \mathbb{R}^+ \to \mathbb{R}^+$. Clearly η is an increasing function on $\mathbb{R}^+$. Note that

$$\alpha_{k+1} - \beta_k \geq (2^{2k+3} - 2^{k+2})\alpha_k \geq 2^{3k+2} \quad (k \in \mathbb{N}), \tag{10.2}$$

and so η has a 'big jump' from $2^{k+1} - 1$ to 2^{k+1} for large $k \in \mathbb{N}$.

LEMMA 10.1. *Let $k \in \mathbb{N}$, and let $s, t \in \mathbb{N}$ with $s + t < 2^k$. Then*

$$\eta(2^k + s + t) - \eta(2^k + s) - \eta(t) \geq \max\{\eta(s/4), \eta(t/4)\} \geq 0.$$

Proof. We set $C = \eta(2^k + s + t) - \eta(2^k + s) - \eta(t)$.

First, suppose that $2^{j_1} + 2^{j_2} \leq s < 2^{j_1} + 2^{j_2+1}$ and that

$$2^{k_1} + 2^{k_2} \leq t < 2^{k_1} + 2^{k_2+1},$$

where $j_2 < j_1 < k$ and $k_2 < k_1 < k$. We also suppose that $j_1 \geq k_1$, that $s + t - 2^{j_1} < 2^{j_1}$, and that $j_1 < k - 1$. Then

$$2^k + s + t \in (2^k + 2^{j_1}, 2^k + 2^{j_1+1}),$$

where $2^k + 2^{j_1}$ and $2^k + 2^{j_1+1}$ are adjacent base points (and $2^k + 2^{j_1+1} \notin \{2^n : n \in \mathbb{N}\}$), and so

$$\eta(2^k + s + t) = \alpha_k + \alpha_{j_1} + \frac{s + t - 2^{j_1}}{2^{j_1}}(\alpha_{j_1+1} - \alpha_{j_1}).$$

Further, $2^k + s \in (2^k + 2^{j_1}, 2^k + 2^{j_1+1})$, and so

$$\eta(2^k + s) = \alpha_k + \alpha_{j_1} + \frac{s - 2^{j_1}}{2^{j_1}}\,(\alpha_{j_1+1} - \alpha_{j_1}).$$

Finally, $\eta(t) < \beta_{k_1} \le 2^{j_1+2}\alpha_{j_1}$. Thus

$$C \ge \frac{t}{2^{j_1}}\,(\alpha_{j_1+1} - \alpha_{j_1}) - 2^{j_1+2}\alpha_{j_1} \ge \frac{1}{2^{j_1}} \cdot 2^{2j_1+3}\alpha_{j_1} - 2^{j_1+2}\alpha_{j_1} \ge 2^{j_1}\alpha_{j_1}$$

by (10.1), and so $C \ge \eta(s/2)$ in this case, noting that $s < 2^{j_1+1}$. It follows that we have $\eta(s/2) < \eta(2^{j_1}) = \alpha_{j_1}$.

We continue to suppose that we are considering the case where

$$2^{j_1} + 2^{j_2} \le s < 2^{j_1} + 2^{j_2+1} \quad \text{and} \quad 2^{k_1} + 2^{k_2} \le t < 2^{k_1} + 2^{k_2+1},$$

where $j_2 < j_1 < k$ and $k_2 < k_1 < k$, that $j_1 \ge k_1$, and that $s + t - 2^{j_1} < 2^{j_1}$, but we now take $j_1 = k - 1$. Then essentially the same estimates give

$$\begin{aligned} C &\ge \frac{t}{2^{k-1}}\,(\beta_k - \alpha_{k-1}) - 2^{k+1}\alpha_{k-1} \ge \frac{1}{2^{k-1}}\,(2^{k+2}\alpha_k - \alpha_{k-1}) - 2^{k+1}\alpha_{k-1} \\ &\ge (2^{2k+6} - 2^{k+1})\alpha_{k-1} \ge 2^k\alpha_{k-1} > \eta(s/2). \end{aligned}$$

We continue to suppose that $2^{j_1}+2^{j_2} \le s < 2^{j_1}+2^{j_2+1}$ and $2^{k_1}+2^{k_2} \le t < 2^{k_1}+2^{k_2+1}$, where $j_2 < j_1 < k$ and $k_2 < k_1 < k$, and that $j_1 \ge k_1$, but we now allow $s + t - 2^{j_1} \ge 2^{j_1}$ (necessarily $2^{j_2} + 2^{k_1} + 2^{k_2} \le 3 \cdot 2^{j_1}$). Then $\eta(2^k + s + t) \ge \alpha_k + \alpha_{j_1+1}$, and the estimates for $\eta(2^k + s)$ and $\eta(t)$ are unchanged, and so we see that

$$\begin{aligned} C &\ge \alpha_{j_1+1} - \alpha_{j_1} - \frac{1}{2}\,(\alpha_{j_1+1} - \alpha_{j_1}) - 2^{j_1+1}\alpha_{j_1} \ge \frac{1}{2}\,\alpha_{j_1+1} - (2^{j_1+1} + 1)\alpha_{j_1} \\ &\ge (2^{2j_1+2} - 2^{j_1+1})\alpha_{j_1} \ge 2^{j_1}\alpha_{j_1} \end{aligned}$$

by (10.1), and so again $C \ge \eta(s/2)$ in this case.

Finally, it may be that s has the form 2^{j_1} or $2^{j_1} + 1$, where $j_1 < k$, rather than satisfy the inequalities $2^{j_1} + 2^{j_2} \le s < 2^{j_1} + 2^{j_2+1}$, and still $j_1 \ge k_1$. In this case, we carry through all the above estimates, replacing the term 2^{j_2} by 0 to obtain the same conclusions. Similarly, in the case where t has the form 2^{k_1} or $2^{k_1} + 1$, where $k_1 < k$, we replace the term 2^{k_2} by 0 in the above.

In all the above cases, we have $C \ge \eta(s/2) \ge \eta(s/4)$. But $t \le 2s$ (because $j_1 \ge k_1$), and so $\eta(t/4) \le \eta(s/2) \le C$. Thus the lemma holds in this case.

The formulae are not symmetric in s and t, but nevertheless essentially the same estimates apply in the case where $j_1 \le k_1$ to imply the same conclusion. ■

THEOREM 10.2. *There is a radical weight on $\mathbb{N}$ that is neither Arens regular nor strongly Arens irregular.*

Proof. Let η be the function $\eta : \mathbb{N} \to \mathbb{R}^+$ defined above.

We first verify that

$$\eta(s + t) \ge \eta(s) + \eta(t) \quad (s, t \in \mathbb{N}). \tag{10.3}$$

The result is vacuously true when $s + t = 1$. Now assume inductively that (10.3) holds whenever $s, t \in \mathbb{N}$ with $s+t \le 2^k - 1$, where $k \in \mathbb{N}$, and suppose that $2^k \le s+t \le 2^{k+1} - 1$.

We may suppose that $s \geq t$. First, consider the case where $s \leq 2^k - 1$. Then

$$\eta(s+t) \geq \eta(2^k) = \alpha_k \geq 2^{k+2}\alpha_{k-1} + 1 > 2\beta_{k-1} \geq 2\eta(2^k - 1) \geq \eta(s) + \eta(t).$$

Second, suppose that $s \geq 2^k$, say $s = 2^k + r$, where $r \in \mathbb{Z}^+_{2^k-1}$. Then $r + t < 2^k$, and so Lemma 10.1 shows that

$$\eta(s+t) - \eta(s) - \eta(t) = \eta(2^k + r + t) - \eta(2^k + r) - \eta(t) \geq 0,$$

as required. Thus (10.3) is proved by induction on k.

We set $\omega = \exp(-\eta)$, so that ω is a weight on $\mathbb{N}$.

We have $\eta(2^k)/2^k = \alpha_k/2^k \geq 2^k$, and so $\lim_{n\to\infty} \omega(n)^{1/n} = 0$. Thus ω is a radical weight.

We *claim* that ω is not Arens regular. Indeed, take $x_m = y_m = 2^m$ for $m \in \mathbb{N}$. Then (x_m) and (y_n) are sequences of distinct elements of $\mathbb{N}$ with $\Omega(x_m, y_n) = 1$ whenever $m, n \in \mathbb{N}$ with $m \neq n$, and so Ω fails to 0-cluster on $\mathbb{N} \times \mathbb{N}$. By Theorem 9.4, ω is not Arens regular.

We next *claim* that Ω 0-clusters on $X \times \mathbb{N}$, and so, by Corollary 9.3, ω is not strongly Arens irregular.

Take $y, k \in \mathbb{N}$, and set

$$C_{y,k} = \eta(m_k + y) - \eta(m_k) - \eta(y) = \eta(2^k + 2^{k-1} + y) - \eta(2^k + 2^{k-1}) - \eta(y).$$

First, fix $y \in \mathbb{N}$, and consider $\liminf_{k\to\infty} C_{y,k}$. We may suppose when calculating $C_{y,k}$ that $y < 2^{k-1}$ and $\eta(y) < \alpha_{k-1}$, and so

$$C_{y,k} \geq \frac{y}{2^{k-1}}\,(\beta_k - \alpha_k - \alpha_{k-1}) - \alpha_{k-1} \geq \frac{\beta_k - \alpha_k}{2^k} - 2\alpha_{k-1} \geq 7\alpha_k - 3\alpha_{k-1} \geq 4\alpha_{k-1}.$$

Thus $\lim_{k\to\infty} C_{y,k} = \infty$.

Second, fix $k \in \mathbb{N}$, and consider $\liminf_{y\to\infty} C_{y,k}$. We may suppose that $y = 2^j + s$, where $j > k+2$ and $s \in \mathbb{N}$ with $s < 2^j$. Then Lemma 10.1 applies with j for k and with $t = m_k$ in the case where $s + m_k < 2^j$ to show that

$$C_{y,k} = \eta(2^j + s + m_k) - \eta(y) - \eta(m_k) \geq \eta(m_k/4) \geq \eta(2^{k-2}) = \alpha_{k-2}.$$

In the case where $s + m_k \geq 2^j$, we have

$$C_{y,k} \geq \alpha_{j+1} - \beta_j - \alpha_k - \alpha_{k-1} \geq 4\alpha_j - \alpha_k - \alpha_{k-1}$$

by (10.2). It follows that, in each case, $C_{y,k} \geq \alpha_{k-2}$. Thus

$$\lim_{k\to\infty} \liminf_{y\to\infty} C_{y,k} = \infty,$$

and so indeed Ω 0-clusters on $X \times \mathbb{N}$. ■

We should like to find a modification of the above weight ω to give a semisimple weight on $\mathbb{N}$ that is neither Arens regular nor strongly Arens irregular. Unfortunately we cannot see how to manufacture such a weight.

11. A strange weight on $\mathbb{Q}^{+\bullet}$

In this section, our main goal is to construct a weight ω on $\mathbb{Q}^{+\bullet}$ with $\liminf_{s\to 0+}\omega(s) = 0$. In Proposition 4.14, we gave equivalent conditions for this property. We prove directly in clause (i) that our weight ω is radical, but it follows from Corollary 4.16 that any weight on $\mathbb{Q}^{+\bullet}$ satisfying clause (iii) must be radical.

An attempt to exhibit such a weight is given in [11, p. 159]. Unfortunately, the attempt fails, and the construction indicated is not correct; we apologise for this.

A Banach algebra $(A, \|\cdot\|)$ is *uniformly radical* (see [11, Definition 2.3.11]) if

$$\lim_{n\to\infty} \sup\{\|a^n\|^{1/n} : a \in A_{[1]}\} = 0;$$

a weight ω on a semigroup S is *uniformly radical* if the Banach algebra $\ell^1(S,\omega)$ is uniformly radical.

Theorem 11.1. *There exists a weight ω on $\mathbb{Q}^{+\bullet}$ with the following properties:*

(i) *ω is radical, but not uniformly radical;*
(ii) $\limsup_{s\to 0+}\omega(s) = \infty$*;*
(iii) $\liminf_{s\to 0+}\omega(s) = 0$*;*
(iv) $\inf\{\omega(s) : s \in \mathbb{Q}^{+\bullet} \cap (a,b)\} = 0$ *for each a, b with $0 < a < b$;*
(v) $\sup\{\omega(s) : s \in \mathbb{Q}^{+\bullet} \cap (a,b)\} = \infty$ *for each a, b with $0 < a < b$;*
(vi) *the only compact element in $\ell^1(\mathbb{Q}^{+\bullet},\omega)$ is 0;*
(vii) *ω is strongly Arens irregular.*

The proof of this theorem will be given in several steps, in which we maintain the same notation. We shall obtain various other properties of ω en route to the main theorem.

First, we define our weight ω. To attain our main aim of (iii), it is sufficient to exhibit a mapping $\eta : \mathbb{Q}^{+\bullet} \to \mathbb{R}$ such that

$$\eta(s+t) \le \eta(s) + \eta(t) \quad (s, t \in \mathbb{Q}^{+\bullet})$$

and

$$\liminf_{s\to 0+}\eta(s) = -\infty,$$

for we then set $\omega = \exp\eta$. In this case, clause (iv) of Theorem 11.1 will follow from Proposition 4.14 and clause (v) will follow from Corollary 4.15.

Choose a strictly increasing sequence (q_j) of prime numbers with $q_1 \ge 3$ such that

$$q_j/q_{j+1} \searrow 0 \quad \text{as} \quad j \to \infty \quad \text{and} \quad q_{j+1} > (j+1)q_j \quad (j \in \mathbb{N}).$$

Of course, $q_n > n$ $(n \in \mathbb{N})$.

As before, set $S = (\mathbb{Z}^+)^{<\omega}$; we again suppose that, in the representation of

$$\alpha = (\alpha_1, \dots, \alpha_m) = \sum_{j=1}^{m} \alpha_j\delta_j \in S,$$

we have $\alpha_m \ge 1$ (unless $\alpha = 0$).

Given two elements $\alpha = (\alpha_1, \dots, \alpha_m) \in S$ and $\beta = (\beta_1, \dots, \beta_n) \in S$, we set

$$N(\alpha, \beta) = \sum_{i=1}^{m} \alpha_i \mathrm{e}^i - \sum_{j=1}^{n} j\beta_j$$

(thus defining m and n when α and β are non-zero), and

$$\theta(\alpha, \beta) = \sum_{i=1}^{m} \frac{\alpha_i}{i} + \sum_{j=1}^{n} \beta_j \frac{q_j}{q_{j+1}}.$$

We note that, for each $x = a/b \in \mathbb{Q}^{+\bullet}$, there exist $\alpha, \beta \in S$ such that $\theta(\alpha, \beta) = x$. Indeed, we can take $\alpha = a\delta_b$ and $\beta = 0$.

As a preliminary remark, we note that the function $\varphi : t \mapsto \mathrm{e}^t - t^2$ on $\mathbb{R}$ has derivative

$$\varphi'(t) = \mathrm{e}^t - 2t \geq 1 - t + \frac{t^2}{2} = \frac{1}{2}\left((t-1)^2 + 1\right) \geq \frac{1}{2},$$

and so φ is increasing on $\mathbb{R}$. In particular, $\mathrm{e}^t - t^2 \geq 1$ for $t \geq 0$. Also

$$\varphi(s) \geq \varphi(2t) \geq \mathrm{e}^t \quad (s \geq 2t,\ t \geq 1) \tag{11.1}$$

because $\varphi(2) > 0$ and every derivative of the function $t \mapsto \mathrm{e}^{2t} - 4t^2 - \mathrm{e}^t$ at $t = 1$ is positive.

Lemma 11.2. *Let $x \in \mathbb{Q}^{+\bullet}$ and take $\alpha, \beta \in S$ with $\theta(\alpha, \beta) = x = a/b$, where $a, b \in \mathbb{N}$ and $(a, b) = 1$, and set $r = [x]$.*

(i) *Suppose that $1 \leq n \leq x$. Then*

$$N(\alpha, \beta) \geq -x^2 q_{r+1}/q_r.$$

(ii) *Suppose that $q_n > x$. Then*

$$\beta_n \leq \sum_{j=1}^{n} \beta_j < q_{n+1}.$$

(iii) *Suppose that $n \geq 1$ and $b \geq q_{n+1}$. Then*

$$N(\alpha, \beta) \geq -\sum_{j=1}^{n} j\beta_j \geq -a \sum_{j=1}^{a} j.$$

(iv) *Suppose that $n \geq x$ and $q_{n+1} \nmid b$. Then*

$$N(\alpha, \beta) \geq \exp(q_{n+1}) - q_{n+1}^2 \geq 1.$$

(v) *Suppose that $k \in \mathbb{N}$, that q_{k+1} is the largest prime factor of b, and that $x \leq n < k$. Then $N(\alpha, \beta) \geq 1$.*

Proof. (i) We have

$$x = \theta(\alpha, \beta) \geq \sum_{j=1}^{n} \beta_j \frac{q_j}{q_{j+1}} \geq \Big(\sum_{j=1}^{n} \beta_j\Big) \frac{q_n}{q_{n+1}} \geq \Big(\sum_{j=1}^{n} \beta_j\Big) \frac{q_r}{q_{r+1}}$$

because $r \geq n \geq 1$, and so

$$N(\alpha, \beta) \geq -\sum_{j=1}^{n} j\beta_j \geq -r \sum_{j=1}^{n} \beta_j \geq -\frac{xrq_{r+1}}{q_r} \geq -\frac{x^2 q_{r+1}}{q_r}.$$

(ii) Assume that $\sum_{j=1}^{n}\beta_j \geq q_{n+1}$. Then
$$x \geq \sum_{j=1}^{n}\beta_j\,\frac{q_j}{q_{j+1}} \geq \Big(\sum_{j=1}^{n}\beta_j\Big)\frac{q_n}{q_{n+1}} \geq q_n,$$
a contradiction. Hence the result holds.

(iii) For each $j \in \mathbb{N}_n$, we have
$$\frac{a}{b} \geq \beta_j\,\frac{q_j}{q_{j+1}} \geq \beta_j\,\frac{q_n}{q_{n+1}}.$$
But $b \geq q_{n+1}$, and so $a \geq \beta_j q_n \geq \beta_j$. In particular, $a \geq \beta_n q_n \geq q_n > n$. Hence
$$N(\alpha,\beta) \geq -\sum_{j=1}^{n} j\beta_j \geq -a\sum_{j=1}^{n} j \geq -a\sum_{j=1}^{a} j.$$

(iv) First assume that $\alpha = 0$ or that $\alpha \neq 0$ and $m < q_{n+1}$. Then we multiply both sides of the equation $\theta(\alpha,\beta) = a/b$ by $b\cdot q_{n+1}!$ and rearrange to see that
$$a\cdot q_{n+1}! - b\sum_{i=1}^{m}\alpha_i\,\frac{q_{n+1}!}{i} - b\sum_{j=1}^{n-1}\beta_j q_j\,\frac{q_{n+1}!}{q_{j+1}} = \beta_n b q_n(q_{n+1}-1)!.$$
The left-hand side of this equation is a multiple of q_{n+1}. However, q_{n+1} is certainly not a factor of $q_n(q_{n+1}-1)!$, q_{n+1} is not a factor of b by hypothesis, and $q_{n+1} > \beta_n$ by (ii). Thus the right-hand side of this equation is not a multiple of q_{n+1}, a contradiction. Hence $m \geq q_{n+1}$ (and $\alpha_m \geq 1$).

As above, we have
$$\sum_{j=1}^{n}\beta_j \leq \frac{xq_{n+1}}{q_n}.$$
Since $x \leq n$ and $n < q_n < q_{n+1}$, we have
$$\sum_{j=1}^{n} j\beta_j \leq n\sum_{j=1}^{n}\beta_j \leq \frac{nxq_{n+1}}{q_n} \leq q_{n+1}^2.$$
Therefore
$$N(\alpha,\beta) \geq \alpha_m \mathrm{e}^m - q_{n+1}^2 \geq \exp(q_{n+1}) - q_{n+1}^2 \geq 1$$
by the preliminary remark.

(v) First, assume that $\alpha = 0$ or that $\alpha \neq 0$ and $m \leq q_{k+1} - 1$. Then
$$(q_{k+1}-1)!\cdot\theta(\alpha,\beta) = \sum_{i=1}^{m}\frac{\alpha_i}{i}\,(q_{k+1}-1)! + \sum_{j=1}^{n}\beta_j\,\frac{q_j}{q_{j+1}}\,(q_{k+1}-1)!$$
is an integer, but $(q_{k+1}-1)!\cdot a/b$ is not an integer because q_{k+1} is not a factor of $a\cdot(q_{k+1}-1)!$, a contradiction of the fact that $\theta(\alpha,\beta) = a/b$. Hence $m \geq q_{k+1}$. By (ii), $\sum_{j=1}^{n}\beta_j < q_{n+1}$, and so
$$\sum_{j=1}^{n} j\beta_j < nq_{n+1} < q_k^2.$$
Thus $N(\alpha,\beta) \geq \exp(q_{k+1}) - q_k^2 \geq \exp(q_{k+1}) - q_{k+1}^2 \geq 1$, as required. ∎

COROLLARY 11.3. *For each $x \in \mathbb{Q}^{+\bullet}$, there exists $f(x) \in \mathbb{R}$ such that $N(\alpha, \beta) \geq f(x)$ whenever $\theta(\alpha, \beta) = x$.*

Proof. If $\beta = 0$, then $N(\alpha, \beta) \geq 0$, and so we may suppose that $\beta \neq 0$, and hence that $n \geq 1$. If $x \geq n$, then $N(\alpha, \beta) \geq -x^2 q_{r+1}/q_r$ by (i). If $x \leq n$, then $N(\alpha, \beta) \geq -a \sum_{j=1}^{a} j$ by (iii) and (iv). ∎

DEFINITION 11.4. For each $x \in \mathbb{Q}^{+\bullet}$, set

$$\eta(x) = \inf\{N(\alpha, \beta) : \theta(\alpha, \beta) = x\}.$$

The above corollary shows that $\eta(x) \in \mathbb{R}$ for each $x \in \mathbb{Q}^{+\bullet}$. It is immediate that η is subadditive on $\mathbb{Q}^{+\bullet}$, and so $\omega = \exp \eta$ is a weight on $\mathbb{Q}^{+\bullet}$.

LEMMA 11.5. *The weight ω is radical.*

Proof. Let $k \in \mathbb{N}$. Take $\alpha = 0$ and $\beta = q_{k+1}\delta_k \in S$. Then $\theta(\alpha, \beta) = q_k$, and so we have $\eta(q_k) \leq N(\alpha, \beta) = -kq_{k+1}$. Thus

$$\omega(q_k)^{1/q_k} \leq \exp(-kq_{k+1}/q_k) < \exp(-k),$$

and so $\nu_\omega = 0$. By Proposition 4.12(ii), ω is radical. ∎

LEMMA 11.6. *Let $k \in \mathbb{N}$.*

(i) *$\eta(1/k) = \mathrm{e}^k$, and so $\limsup_{s \to 0+} \omega(s) = \infty$.*
(ii) *$\eta(q_k/q_{k+1}) = -k$, and so $\liminf_{s \to 0+} \omega(s) = 0$.*

Proof. (i) Certainly $\eta(1/k) \leq \mathrm{e}^k$; we must show that $\eta(1/k) \geq \mathrm{e}^k$.

Take $\alpha, \beta \in S$ with $\theta(\alpha, \beta) = 1/k$. If $\beta = 0$, then $\alpha \neq 0$ and $m \geq k$, so that

$$N(\alpha, \beta) \geq \mathrm{e}^m \geq \mathrm{e}^k.$$

Thus we may suppose that $n \geq 1 \geq 1/k$. We have

$$\frac{1}{k} \geq \beta_n \frac{q_n}{q_{n+1}} \geq \frac{2}{q_{n+1}},$$

and so $2k \leq q_{n+1}$. By Lemma 11.2(iv) (with $a = 1$ and $b = k$) and (11.1), we have

$$N(\alpha, \beta) \geq \exp(q_{n+1}) - q_{n+1}^2 \geq \mathrm{e}^k$$

by a preliminary remark, as required.

(ii) Certainly $\eta(q_k/q_{k+1}) \leq -k$, which already implies that

$$\liminf_{s \to 0+} \omega(s) = 0.$$

We shall show that $\eta(q_k/q_{k+1}) \geq -k$.

Take $\alpha, \beta \in S$ with $\theta(\alpha, \beta) = q_k/q_{k+1}$. If $\beta = 0$, then clearly $N(\alpha, \beta) \geq 0 > -k$, and so we may suppose that $n \geq 1$. Since (q_j/q_{j+1}) is a strictly decreasing sequence, necessarily $n \geq k$ and $\beta_1 = \cdots = \beta_{k-1} = 0$ (in the case where $k \geq 2$). If $n = k$, then $\alpha = 0$ and $\beta_k = 1$, and so $N(\alpha, \beta) = -k$. If $n > k$, then $n > q_k/q_{k+1}$ and $q_{n+1} \nmid q_{k+1}$, and so, by Lemma 11.2(iv), $N(\alpha, \beta) \geq 1$. Thus $\eta(q_k/q_{k+1}) \geq -k$, as required. ∎

We denote by Q the set of elements in $\mathbb{Q}^{+\bullet}$ of the form q_k/q_{k+1} for some $k \in \mathbb{N}$, where (q_k) is our specified sequence.

Lemma 11.7.

(i) *Let $s, t \in Q$. Then $\omega(s+t) = \omega(s)\omega(t)$.*
(ii) *Let $s_1, \dots, s_n \in Q$ with $s_1 + \dots + s_n < 1$. Then*

$$\omega(s_1 + \dots + s_n) = \omega(s_1) \cdots \omega(s_n).$$

Proof. (i) We may suppose that $s = q_k/q_{k+1}$ and $t = q_\ell/q_{\ell+1}$ for some $k, \ell \in \mathbb{N}$ with $k \le \ell$, so that $\eta(s) + \eta(t) = -k - \ell$. We must show that $\eta(s+t) \ge -k - \ell$.

Choose $\alpha, \beta \in S$ with

$$\theta(\alpha, \beta) = s + t = \frac{q_k}{q_{k+1}} + \frac{q_\ell}{q_{\ell+1}} = \frac{a}{b},$$

where $a = q_k q_{\ell+1} + q_\ell q_{k+1}$ and $b = q_{k+1} q_{\ell+1}$ with $(a, b) = 1$ because $q_{k+1} \ge 3$.

First, suppose that $\beta = 0$. Then $N(\alpha, \beta) \ge 0 \ge -k - \ell$.

Second, suppose that $\beta \ne 0$. Then $n \ge 1 > s + t$ because we have $s, t \in (0, 1/2)$. If $q_{n+1} \nmid q_{k+1} q_{\ell+1}$, then $N(\alpha, \beta) \ge 0$ by Lemma 11.2(iv), and so certainly $N(\alpha, \beta) \ge -k - \ell$. If $q_{n+1} \mid q_{k+1} q_{\ell+1}$, then either $n = k$ or $n = \ell$. Suppose that $n = k$. Then

$$\sum_{i=1}^{m} \frac{\alpha_i}{i} + \sum_{j=1}^{k-1} \beta_j \frac{q_j}{q_{j+1}} + (\beta_k - 1) \frac{q_k}{q_{k+1}} = \frac{q_\ell}{q_{\ell+1}},$$

and so $N(\alpha, \beta) + k \ge -\ell$, whence $N(\alpha, \beta) \ge -k - \ell$. A similar argument applies in the case where $n = \ell$.

(ii) This proceeds by induction on $n \in \mathbb{N}$. The case where $n = 2$ is proved in (i), and the inductive step is essentially the same. ∎

The following result will be strengthened later.

Corollary 11.8. *The weight ω is not Arens regular.*

Proof. Take $x_m = y_m = q_m/q_{m+1}$ $(m \in \mathbb{N})$. Then

$$\Omega(x_m, y_n) = 1 \qquad (m, n \in \mathbb{N})$$

by Lemma 11.7(i), and so Ω does not 0-cluster on $\mathbb{Q}^{+\bullet} \times \mathbb{Q}^{+\bullet}$. ∎

Corollary 11.9. *The weight ω is not uniformly radical.*

Proof. For $n \in \mathbb{N}$, choose $k \in \mathbb{N}$ with $nq_k/q_{k+1} < 1$; such a k exists because $q_j/q_{j+1} \searrow 0$ as $j \to \infty$. Set $a = \delta_s/\omega(s)$, where $s = q_k/q_{k+1}$, so that $a \in (A_\omega)_{[1]}$. Then clearly $\|a^n\|_\omega = \omega(ns)/\omega(s)^n$, and so, by Lemma 11.7(ii), $\|a^n\|_\omega = 1$. It follows that A_ω is not uniformly radical. ∎

Lemma 11.10. *The only compact element in $\ell^1(\mathbb{Q}^{+\bullet}, \omega)$ is 0.*

Proof. By Proposition 4.19(ii), we must show that no element δ_s for $s \in \mathbb{Q}^{+\bullet}$ is compact. Since δ_{s_2} is compact whenever δ_{s_1} is compact and $s_2 \ge s_1$ in $\mathbb{Q}^{+\bullet}$, it suffices to show that δ_s is not compact whenever $s = q_\ell$ for some $\ell \in \mathbb{N}$.

Fix such an element s. By Proposition 4.19(i), we must show that there exists $M > 0$ such that

$$\{t \in \mathbb{Q}^{+\bullet} : \eta(t) - \eta(s+t) \le M\}$$

is infinite. In fact, we shall take $M = (s+1)^2 q_{s+1}/q_s$, and show that

$$\eta(t) - \eta(s+t) = \eta(q_k/q_{k+1}) - \eta(q_\ell + q_k/q_{k+1}) \le M \qquad (t \in Q)$$

for each $k \in \mathbb{N}$. By Lemma 11.6(ii), $\eta(q_k/q_{k+1}) = -k$ $(k \in \mathbb{N})$, and so we must show that $\eta(q_\ell + q_k/q_{k+1}) \ge -M - k$ $(k \in \mathbb{N})$.

Let $k \in \mathbb{N}$, and suppose that $\theta(\alpha, \beta) = q_\ell + q_k/q_{k+1}$, where, again,

$$\theta(\alpha,\beta) = \sum_{i=1}^{m} \frac{\alpha_i}{i} + \sum_{j=1}^{n} \beta_j \, \frac{q_j}{q_{j+1}}.$$

If $\beta = 0$, then $N(\alpha,\beta) \ge 0$. If $1 \le n \le s+t$, then

$$N(\alpha,\beta) \ge -(s+t)^2 q_{s+1}/q_s$$

by Lemma 11.2(i), and so $N(\alpha,\beta) \ge -M$. If $n \ge s+t$ and $q_{n+1} \nmid q_{k+1}$, then $N(\alpha,\beta) \ge 1$ by Lemma 11.2(iv). Finally, suppose that $n \ge s+t$ and $q_{n+1} \mid q_{k+1}$. Then necessarily $n = k$ and $s = \theta(\alpha, \beta - \delta_k)$. By Lemma 11.2(iv), $1 \le N(\alpha, \beta-\delta_k)$, and so $N(\alpha,\beta) \ge 1-k$. Thus, in each case, $N(\alpha,\beta) \ge -M-k$, and so $\eta(q_\ell + q_k/q_{k+1}) \ge -M-k$, as required. ∎

LEMMA 11.11. *The weight ω is weakly diagonally bounded, with bound $c_Q = 1$, on the infinite subset Q of $\mathbb{Q}^{+\bullet}$.*

Proof. It is sufficient to show that, for each $s \in \mathbb{Q}^{+\bullet}$, there exists $k_s \in \mathbb{N}$ such that

$$\eta(s + q_k/q_{k+1}) + k = \eta(s) \qquad (k \ge k_s), \tag{11.2}$$

where we recall that $\eta(q_k/q_{k+1}) = -k$ $(k \in \mathbb{N})$ by Lemma 11.6(ii).

In fact, set $s = a/b$, and choose $k_s \in \mathbb{N}$ such that

$$k_s \ge (s+1)^2 q_{r+1}/q_r + \eta(s) + b,$$

where $r = [s+1] \in \mathbb{N}$. We fix $k \ge k_s$, and set $x = s + q_k/q_{k+1}$; the denominator of x is bq_{k+1}. We note that $0 \ge \eta(s) - k$, and also that $[x] \le [s+1] = r$, so that $q_{[x]+1}/q_{[x]} \le q_{r+1}/q_r$.

Take $\alpha, \beta \in S$ with $\theta(\alpha,\beta) = x$: we shall show that

$$N(\alpha,\beta) \ge \eta(s) - k.$$

If $\beta = 0$, then $N(\alpha,\beta) \ge 0$.

If $\beta \ne 0$ and $1 \le n \le x$, then $N(\alpha,\beta) \ge -x^2 q_{[x]+1}/q_{[x]}$ by Lemma 11.2(i), and so

$$N(\alpha,\beta) \ge -(s+1)^2 q_{r+1}/q_r \ge \eta(s) - k.$$

If $n \ge x$ and $q_{n+1} \nmid bq_{k+1}$, then $N(\alpha,\beta) \ge 1$ by Lemma 11.2(iv).

If $n \ge x$ and $q_{n+1} \mid bq_{k+1}$, then either $q_{n+1} \mid b$ and $n < k$ or $n = k$, noting that we have $q_{k+1} > k+1 > b$. In the case where $n < k$, certainly q_{k+1} is the largest prime factor of bq_{k+1}, and so $N(\alpha,\beta) \ge 1$ by Lemma 11.2(v). In the case where $n = k$, we have $\beta\theta(\alpha, \beta - \delta_k) = x - q_k/q_{k+1} = s$, and so

$$\eta(s) \le N(\alpha, \beta - \delta_k) = N(\alpha,\beta) + k.$$

Thus $N(\alpha,\beta) \ge \eta(s) - k$.

We have verified the required inequality in each case. ∎

Corollary 11.12. *The weight ω is strongly Arens irregular.*

Proof. This now follows from the above lemma by using Corollary 5.7. ■

Proof of Theorem 11.1. This is contained within the union of the above results. ■

12. Open questions

We believe that the following questions are open.

1. Let ω be a weight on an infinite, countable, cancellative semigroup S such as $\mathbb{Q}$ or $\mathbb{Q}^{+\bullet}$. Suppose that ω is weakly diagonally bounded on an infinite subset of S. Is there an equivalent weight $\widetilde{\omega}$ to ω on S such that $\widetilde{\omega}$ is weakly diagonally bounded on an infinite subset T of S, with bound $c_T < 2$, or even $c_T = 1$?
2. Let G be a group, and let ω be a weight on G. Is the Banach algebra $\ell^1(G, \omega)$ always semisimple?
3. Let ω be a weight on $\mathbb{Z}$ such that
$$\liminf_{n\to\infty} \omega(n) < \infty \quad \text{and} \quad \liminf_{n\to\infty} \omega(-n) < \infty.$$
Is $\ell^1(\mathbb{Z}, \omega)$ necessarily strongly Arens irregular?
4. Let ω be a weight on $S = \mathbb{Q}$ or $S = \mathbb{Q}^{+\bullet}$. Suppose that there is a finite subset V of S^*_ω such that V is determining for the topological centre of $\ell^1(S, \omega)$. Is there a subset W of S^*_ω such that $|W| = 2$ and W is determining for the topological centre of $\ell^1(S, \omega)$?
5. Is the constraint '$c_W < 2$' in Theorem 7.1 necessary?
6. Is there an example such that ω_1 is a continuous weight function on $\mathbb{R}^+$, ω_2 is a strongly Arens irregular weight on $\mathbb{Q}^{+\bullet}$, and $\omega_1\omega_2$ is Arens regular on $\mathbb{Q}^{+\bullet}$?
7. Is there a semisimple weight on $\mathbb{N}$ that is neither Arens regular nor strongly Arens irregular?
8. Is there an Arens regular weight on $\mathbb{R}^{+\bullet}$? Such a weight must be radical and non-measurable.

13. Summary

We conclude with a summary of the examples that we have found. Usually, references are to examples in the present memoir, but there are some references to the memoir [12]. The word 'neither' means 'neither Arens regular nor strongly Arens irregular'. All semigroups are subsemigroups of $(\mathbb{R}, +)$. Recall that each weight on each of the semigroups $\mathbb{N}$, $\mathbb{Q}^{+\bullet}$, and $\mathbb{R}^{+\bullet}$ is either radical or semisimple, and that each weight on each of the groups $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ is semisimple.

S is	ω is	ω is	Examples
$\mathbb{N}$	radical	Arens regular	Example 9.8
		strongly Arens irregular	Example 6.5
		neither	Theorem 10.2
$\mathbb{N}$	semisimple	Arens regular	Example 9.9
		strongly Arens irregular	$\ell^1(\mathbb{N})$, Example 9.9,
		neither	**Not known**
$\mathbb{Z}$	semisimple	Arens regular	Example 9.9 [12, Examples 9.1, 9.13] [12, Example 9.14]
		strongly Arens irregular	Examples 6.2, 9.9 [12, Example 9.6]
		neither	[12, Examples 9.7, 9.8] [12, Examples 9.15, 9.16]
$\mathbb{Q}^{+\bullet}$	radical	Arens regular	Examples 9.14, 9.17
		strongly Arens irregular	Examples 6.1, 7.3 Theorem 11.1
		neither	Examples 9.10, 9.15, 9.16
$\mathbb{Q}^{+\bullet}$	semisimple	Arens regular	Example 9.14
		strongly Arens irregular	$\ell^1(\mathbb{Q}^{+\bullet})$, Examples 6.3, 6.4
		neither	Examples 9.10, 9.11
$\mathbb{Q}$	semisimple	Arens regular	Example 9.14
		strongly Arens irregular	$\ell^1(\mathbb{Q})$, Examples 6.3, 6.4
		neither	Example 9.12
$\mathbb{R}^{+\bullet}$	radical	Arens regular	**Not known**
		strongly Arens irregular	Example 8.2
		neither	Example 9.10
$\mathbb{R}^{+\bullet}$	semisimple	Arens regular	Not possible, by Corollary 9.19
		strongly Arens irregular	$\ell^1(\mathbb{R}^{+\bullet})$
		neither	Example 9.10
$\mathbb{R}$	semisimple	strongly Arens irregular	Always, by Corollary 5.13

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