Contents

Abstract

Let $F = (F_1, \dots, F_n) : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial mapping. By the multidegree of F we mean $\operatorname{mdeg} F = (\deg F_1, \dots, \deg F_n) \in \mathbb{N}^n$. The aim of this paper is to study the following problem (especially for $n = 3$): *for which sequence* $(d_1, \dots, d_n) \in \mathbb{N}^n$ *is there a tame automorphism* F *of* $\mathbb{C}^n$ *such that* $\operatorname{mdeg} F = (d_1, \dots, d_n)$? In other words we investigate the set $\operatorname{mdeg}(\operatorname{Tame}(\mathbb{C}^n))$, where $\operatorname{Tame}(\mathbb{C}^n)$ denotes the group of tame automorphisms of $\mathbb{C}^n$.

Since $\operatorname{mdeg}(\operatorname{Tame}(\mathbb{C}^n))$ is invariant under permutations of coordinates, we may focus on the set $\{(d_1, \dots, d_n) : d_1 \leq \dots \leq d_n\} \cap \operatorname{mdeg}(\operatorname{Tame}(\mathbb{C}^n))$.

Obviously, we have $\{(1, d_2, d_3) : 1 \leq d_2 \leq d_3\} \cap \operatorname{mdeg}(\operatorname{Tame}(\mathbb{C}^3)) = \{(1, d_2, d_3) : 1 \leq d_2 \leq d_3\}$. Not obvious, but still easy to prove is the equality $\operatorname{mdeg}(\operatorname{Tame}(\mathbb{C}^3)) \cap \{(2, d_2, d_3) : 2 \leq d_2 \leq d_3\} = \{(2, d_2, d_3) : 2 \leq d_2 \leq d_3\}$.

We give a complete description of the sets $\{(3, d_2, d_3) : 3 \leq d_2 \leq d_3\} \cap \operatorname{mdeg}(\operatorname{Tame}(\mathbb{C}^3))$ and $\{(5, d_2, d_3) : 5 \leq d_2 \leq d_3\} \cap \operatorname{mdeg}(\operatorname{Tame}(\mathbb{C}^3))$. In the examination of the last set the most difficult part is to prove that $(5, 6, 9) \notin \operatorname{mdeg}(\operatorname{Tame}(\mathbb{C}^3))$. To do this, we use the two-dimensional Jacobian Conjecture (which is true for low degrees) and the Jung–van der Kulk Theorem.

As a surprising consequence of the method used in proving that $(5, 6, 9) \notin \operatorname{mdeg}(\operatorname{Tame}(\mathbb{C}^3))$, we show that the existence of a tame automorphism F of $\mathbb{C}^3$ with $\operatorname{mdeg} F = (37, 70, 105)$ implies that the two-dimensional Jacobian Conjecture is not true.

Also, we give a complete description of the following sets: $\{(p_1, p_2, d_3) : 2 < p_1 < p_2 \leq d_3, p_1, p_2 \text{ prime numbers }\} \cap \operatorname{mdeg}(\operatorname{Tame}(\mathbb{C}^3))$, $\{(d_1, d_2, d_3) : d_1 \leq d_2 \leq d_3,\ d_1, d_2 \in 2\mathbb{N} + 1,\ \gcd(d_1, d_2) = 1\} \cap \operatorname{mdeg}(\operatorname{Tame}(\mathbb{C}^3))$. Using the description of the last set we show that $\operatorname{mdeg}(\operatorname{Aut}(\mathbb{C}^3)) \setminus \operatorname{mdeg}(\operatorname{Tame}(\mathbb{C}^3))$ is infinite.

We also obtain a (still incomplete) description of the set $\operatorname{mdeg}(\operatorname{Tame}(\mathbb{C}^3)) \cap \{(4, d_2, d_3) : 4 \leq d_2 \leq d_3\}$ and we give complete information about $\operatorname{mdeg} F^{-1}$ for $F \in \operatorname{Aut}(\mathbb{C}^2)$.

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0. Introduction

The object of principal interest in this paper is the multidegree (i.e. the sequence of the degrees of the coordinate functions) of a polynomial automorphism of the vector space $\mathbb{C}^n$. Let us mention that in the Scottish Book ([33, Problem 79]) Mazur and Orlicz posed the following question: "If $F = (F_1, \ldots, F_n) : \mathbb{C}^n \to \mathbb{C}^n$ is a one-to-one polynomial map whose inverse is also a polynomial map, is each F_i of degree one?" In other words, they asked whether every polynomial automorphism of $\mathbb{C}^n$ has multidegree $(1, \ldots, 1)$. The answer to this question is obviously "no", and in the Scotish Book itself one can find the following example: let $1 \le i \le n$ and $a = a(X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n) \in \mathbb{C}[X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n]$. Then

$$E : \mathbb{C}^n \ni (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{i-1}, x_i + a, x_{i+1}, \ldots, x_n) \in \mathbb{C}^n$$

is a polynomial automorphism with multidegree $(1, \ldots, 1, \deg a, 1, \ldots, 1)$. A map as above is called an *elementary* polynomial map. Taking finite compositions of such elementary maps and elements of the affine subgroup $\mathrm{Aff}(\mathbb{C}^n)$, i.e. the group of polynomial automorphisms $F = (F_1, \ldots, F_n) : \mathbb{C}^n \to \mathbb{C}^n$ such that $\deg F_i = 1$ for all i, we get automorphisms called *tame.*

In 1942 Jung [9] proved that each polynomial automorphism of k^2, where k is a field of characteristic zero, is tame. Later, in 1953, van der Kulk extended Jung's result to fields of arbitrary characteristic. Since then several authors have given other proofs of that result: Gutwirth [12] in 1961, Shafarevich [46] in 1966, Rentschler [42] in 1968, Makar-Limanov [32] in 1970, Nagata [36] in 1972, Abhyankar and Moh [1] in 1975, Dicks [6] in 1983, McKay and Wang [29] in 1988. The stronger statement, also called the Shafarevich–Nagata–Kombayashi theorem, saying that the group of all polynomial automorphisms of k^2 is the amalgamated product of the affine subgroup and the subgroup of de Jonquières automorphisms over their intersection, can be found in [23], [17], [36], [6], [2] and without proof in [46].

From the result of Jung and van der Kulk it also follows that if (d_1, d_2) is the multidegree of an automorphism of $\mathbb{C}^2$, then $d_1 \mid d_2$ or $d_2 \mid d_1$ (see Subsection 1.4).

Tame automorphisms are closely related to the problem of embedding of affine algebraic varieties. For example, in the proof of the famous Abhyankar–Moh–Suzuki theorem, saying that every embedding of a line in $\mathbb{C}^2$ is rectifiable (i.e. a composition of the standard embedding $\mathbb{C} \ni x \mapsto (x, 0) \in \mathbb{C}^2$ and an automorphism of $\mathbb{C}^2$), tame automorphisms play a prominent role. This result, formulated in algebraic terms as follows: if $f(T), g(T) \in k[T]$ and $k[f(T), g(T)] = k[T]$, then either $\deg f(T) \mid \deg g(T)$ or $\deg g(T) \mid \deg f(T)$, was used by Segre [45] to "prove" the Jacobian Conjecture. The problem of embeddings of affine algebraic varieties was also considered by Jelonek [13, 14, 15], Kaliman [16], Srinivas [52] and Craighero [5].

Since Jung and van der Kulk proved their theorem, many authors have tried to prove or disprove the similar result for dimension $n \geq 3$, but without any results. The most famous candidate for a so-called wild automorphism (i.e. one that is not tame) was proposed by Nagata in 1972. It took more than thirty years to prove that the Nagata automorphism

$$\sigma : \mathbb{C}^3 \ni (x, y, z) \mapsto (x + 2y(y^2 + zx) - z(y^2 + zx)^2, y - z(y^2 + zx), z) \in \mathbb{C}^3$$

is indeed wild. This remarkable result was obtained by Shestakov and Umirbaev [49]. The two main ingredients in the proof of the above result are recalled as Theorems 2.6 and 2.14 (see Subsections 2.1 and 2.3). These two theorems are also basic tools in our considerations concerning multidegrees of tame automorphisms of $\mathbb{C}^3$.

The paper is organized as follows. In Section 1 we fix notation, recall basic definitions, and discuss the multidegree of polynomial automorphisms of $\mathbb{C}^2$ (see Subsection 1.4). The discussion is based on the Jung–van der Kulk result. In Section 2 we recall the notion of a Poisson bracket of two polynomials, and two theorems due to Shestakov and Umirbaev (Theorems 2.6 and 2.14). They are the main tools used in the paper. We also prove that the degree of the Poisson bracket is an invariant of a linear change of coordinates (Lemma 2.8). This is a new result. In this section we also explain in detail that an example of a polynomial automorphism (Example 2.11) due to Shestakov and Umirbaev does not admit an elementary reduction, and recall a theorem from number theory (Theorem 2.15) that will be useful in some parts of the paper.

In Section 3 we collect some general results about multidegrees. Some of them were already published by the author: Proposition 3.1, Proposition 3.2 and Corollary 1.3 [18]. The other results in that section (except Theorem 3.14 due to Kuroda) are new. The most important results of that section are Proposition 3.2, Theorem 3.15 and Lemma 3.20.

In Section 4 we discuss tame automorphisms of $\mathbb{C}^3$ with multidegree of the form (p_1, p_2, d_3), $2 < p_1 < p_2 \leq d_3$, where p_1 and p_2 are prime numbers, and more generally, coprime odd numbers. In both cases we give a necessary and sufficient numerical condition for (p_1, p_2, d_3) to be the multidegree of a tame automorphism of $\mathbb{C}^3$. The results of that section were already published by the author [19], and by the author and J. Zygadło [22].

Section 5 presents results due to the author [20]. They concern tame automorphisms with multidegeree $(3, d_2, d_3)$, $3 \leq d_2 \leq d_3$.

The results of Sections 6 and 7 are new and concern tame automorphisms with multidegree $(4, d_2, d_3)$, $4 \leq d_2 \leq d_3$ (Section 6), and (p, d_2, d_3), $5 \leq p \leq d_2 \leq d_3$, where p is a prime (Section 7). It is of interest that in showing that there is no tame automorphism of $\mathbb{C}^3$ with multidegree $(5, 6, 9)$, we use the Jacobian Conjecture (actually the Moh theorem). On the other hand, it is very surprising that the existence of a tame automorphism of $\mathbb{C}^3$ with multidegree $(37, 70, 105)$ implies that the two-dimensional Jacobian Conjecture is false (this is proved in Section 7).

In Section 8 we present a result due to J. Zygadło [54], and in the last section we give new results on the multidegree of the inverse of a polynomial automorphism of $\mathbb{C}^2$.

1. Notation, basic definitions and two-dimensional case

1.1. Notation. We assume that $0 \in \mathbb{N}$, and we denote by $\mathbb{N}^*$, $\mathbb{Z}^*$, $\mathbb{C}^*$, respectively, $\mathbb{N}\setminus\{0\}$, $\mathbb{Z} \setminus \{0\}$, $\mathbb{C} \setminus \{0\}$. By $\mathbb{C}[X_1, \dots, X_n]$ we denote the polynomial ring in n variables over $\mathbb{C}$. In particular, $X_1, \dots, X_n$ denote variables, and $x_1, \dots, x_n$ denote coordinates in $\mathbb{C}^n$. We will work over the complex field $\mathbb{C}$, but all results remain valid over any algebraically closed field of characteristic zero.

For any $f \in \mathbb{C}[X_1, \dots, X_n]$, $\deg f$ denotes the usual total degree of f. We say that f is homogeneous if f is a sum of monomials of the same degree. We denote by $\overline{f}$ the leading form of f, i.e. the homogeneous part of f of the maximal degree. Of course, $\deg f = \deg \overline{f}$.

Moreover, $\gcd(d_1, \dots, d_n)$ and $\operatorname{lcm}(d_1, \dots, d_n)$ denote the greatest common divisor and the least common multiple of $d_1, \dots, d_n$, respectively.

1.2. Examples of polynomial automorphisms. First of all, recall that a *polynomial mapping* $F : \mathbb{C}^n \to \mathbb{C}^n$ is a mapping whose coordinate functions F_i, where $F = (F_1, \dots, F_n)$, are polynomials. By a *polynomial automorphism* of $\mathbb{C}^n$ (later, just *automorphism*) we mean a polynomial mapping $F : \mathbb{C}^n \to \mathbb{C}^n$ such that there exists a polynomial mapping $G : \mathbb{C}^n \to \mathbb{C}^n$ with $F \circ G = G \circ F = \mathrm{id}_{\mathbb{C}^n}$. We then also say that F is invertible. The group of all polynomial automorphisms of $\mathbb{C}^n$ is denoted by $\operatorname{Aut}(\mathbb{C}^n)$.

Polynomial automorphisms play a prominent role in affine algebraic geometry [33, 47]. Typical problems are the Jacobian Problem [3, 4, 9, 23, 36, 37, 38, 39, 40], existence of wild automorphisms [8, 49, 50, 51], the inverse formula [28, 29, 30, 35] or stable tameness [48].

There are some special kinds of polynomial automorphisms of $\mathbb{C}^n$:

- *Affine polynomial automorphisms*, i.e. polynomial automorphisms $F = (F_1, \dots, F_n)$ such that $\deg F_i = 1$ for $i = 1, \dots, n$. The set of all such automorphisms will be denoted $\operatorname{Aff}(\mathbb{C}^n)$; it is a subgroup of $\operatorname{Aut}(\mathbb{C}^n)$.
- *Linear automorphisms*, i.e. affine automorphisms $F : \mathbb{C}^n \to \mathbb{C}^n$ such that $F(0, \dots, 0) = (0, \dots, 0)$. This is of course the same as the general linear group, denoted $GL_n(\mathbb{C})$.
- *Elementary automorphisms*, i.e. maps of the form

$$F : \mathbb{C}^n \ni (x_1, \dots, x_n) \mapsto (x_1, \dots, x_i + f(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n), \dots, x_n) \in \mathbb{C}^n$$

for some $i \in \{1, \dots, n\}$ and $f \in \mathbb{C}[X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_n]$. One can easily see that

$$F^{-1}(x_1, \dots, x_n) = (x_1, \dots, x_i - f(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n), \dots, x_n).$$

- *Triangular automorphisms*, i.e. maps of the form

$$F : \mathbb{C}^n \ni (x_1, \dots, x_n) \mapsto (x_1, x_2 + f_1(x_1), \dots, x_n + f_{n-1}(x_1, \dots, x_{n-1})) \in \mathbb{C}^n, \quad (1.1)$$

where $f_1 \in \mathbb{C}[X_1], f_2 \in \mathbb{C}[X_1, X_2], \dots, f_{n-1} \in \mathbb{C}[X_1, \dots, X_{n-1}]$. One can check that F is invertible and

$$F^{-1}\left(\left\{\begin{matrix} x_1 \\ x_2 \\ x_3 \\ \vdots \end{matrix}\right\}\right) = \left\{\begin{matrix} x_1 \\ x_2 - f_1(x_1) \\ x_3 - f_2(x_1, x_2 - f_1(x_1)) \\ \vdots \end{matrix}\right\}.$$

We will also say that F is triangular if F is of the form (1.1) after some permutation of variables.

- *De Jonquières automorphisms*, i.e. mappings of the form

$$F:\mathbb{C}^n \ni \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} \mapsto \begin{Bmatrix} a_1x_1 + f_1(x_2,\ldots,x_n) \\ a_2x_2 + f_2(x_3,\ldots,x_n) \\ \vdots \\ a_nx_n + f_n \end{Bmatrix} \in \mathbb{C}^n, \tag{1.2}$$

  where $a_i \in \mathbb{C}^*$, $f_i \in \mathbb{C}[X_{i+1},\ldots,X_n]$ for all $1 \le i \le n-1$ and $f_n \in \mathbb{C}$. We then write $F \in J(\mathbb{C}^n)$. As for triangular mappings, one can check that if $F \in J(\mathbb{C}^n)$, then F is invertible. Also, one can verify that $J(\mathbb{C}^n)$ is a subgroup of $\mathrm{Aut}(\mathbb{C}^n)$.
- *Tame automorphisms*, i.e. compositions of a finite number of affine and triangular automorphisms. Sometimes a tame automorphism is defined as a composition of a finite number of affine and elementary automorphisms, or as a composition of a finite number of affine and de Jonquières automorphisms. One can check that all these definitions are equivalent.

To end this section, recall that for any polynomial mapping $F:\mathbb{C}^n \to \mathbb{C}^n$ we have the $\mathbb{C}$-homomorphism $F^*:\mathbb{C}[X_1,\ldots,X_n] \to \mathbb{C}[X_1,\ldots,X_n]$ defined by

$$F^*:\mathbb{C}[X_1,\ldots,X_n] \ni h \mapsto h\circ F \in \mathbb{C}[X_1,\ldots,X_n],$$

and for any $\mathbb{C}$-homomorphism $\Phi:\mathbb{C}[X_1,\ldots,X_n] \to \mathbb{C}[X_1,\ldots,X_n]$ we have the polynomial mapping $\Phi_*:\mathbb{C}^n \to \mathbb{C}^n$ defined as

$$\Phi_*:\mathbb{C}^n \ni (x_1,\ldots,x_n) \mapsto (F_1(x_1,\ldots,x_n),\ldots,F_n(x_1,\ldots,x_n)) \in \mathbb{C}^n,$$

where $F_i = \Phi(X_i)$. Moreover, recall that $(F^*)_* = F, (\Phi_*)^* = \Phi$, and F, is an automorphism if and only if F^* is a $\mathbb{C}$-automorphism of $\mathbb{C}[X_1,\ldots,X_n]$. Thus one can translate the notions of affine, linear, elementary, triangular and tame automorphisms of $\mathbb{C}^n$ into the language of $\mathbb{C}$-automorphisms of $\mathbb{C}[X_1,\ldots,X_n]$.

1.3. Degree, bidegree and multidegree. Let $F=(F_1,\ldots,F_n):\mathbb{C}^n \to \mathbb{C}^n$ be any polynomial map. By the *degree* of F, denoted $\deg F$, we mean the number

$$\deg F = \max\{\deg F_1,\ldots,\deg F_n\},$$

and by the *multidegree* of F, denoted $\mathrm{mdeg}\, F$, we mean the sequence of natural numbers

$$\mathrm{mdeg}\, F = (\deg F_1,\ldots,\deg F_n).$$

For $n=2$ the multidegree is called the *bidegree*, and denoted bideg (see e.g. [7]).

For a fixed $n \in \mathbb{N}$, we will also consider the mappings

$$\deg:\mathrm{End}(\mathbb{C}^n) \ni F \mapsto \deg F \in \mathbb{N}, \qquad \mathrm{mdeg}:\mathrm{End}(\mathbb{C}^n) \ni F \mapsto \mathrm{mdeg} F \in \mathbb{N}^n,$$

where $\mathrm{End}(\mathbb{C}^n)$ denotes the set of all polynomial mappings $\mathbb{C}^n \to \mathbb{C}^n$.

One of the main goals of this paper is to obtain a description of the sets

$$\mathrm{mdeg}(\mathrm{Aut}(\mathbb{C}^n)), \mathrm{mdeg}(\mathrm{Tame}(\mathbb{C}^n)) \subset \mathbb{N}^n.$$

If $n = 1$ the answer is

$$\operatorname{mdeg}(\operatorname{Aut}(\mathbb{C}^1)) = \operatorname{mdeg}(\operatorname{Tame}(\mathbb{C}^1)) = \{1\}.$$

The description for $n = 2$, based on a theorem of Jung and van der Kulk, will be given in the next subsection. The question for $n \geq 3$ is much more complicated, and will be investigated in the rest of the paper. The very first result in this direction says that $(3, 4, 5) \notin \operatorname{mdeg}(\operatorname{Tame}(\mathbb{C}^3))$ [18]. The next results obtained by the author [19, 20, 22] are also included.

Since for any $(F_1, \ldots, F_n) \in \operatorname{Aut}(\mathbb{C}^n)$ we have $\deg F_i \geq 1$, $i = 1, \ldots, n$, and since for any permutation σ of $\{1, \ldots, n\}$ and any sequence $(d_1, \ldots, d_n) \in \mathbb{N}^n$ we have

$$(d_1, \ldots, d_n) \in \operatorname{mdeg}(\operatorname{Tame}(\mathbb{C}^n)) \Leftrightarrow (d_{\sigma(1)}, \ldots, d_{\sigma(n)}) \in \operatorname{mdeg}(\operatorname{Tame}(\mathbb{C}^n))$$

and

$$(d_1, \ldots, d_n) \in \operatorname{mdeg}(\operatorname{Aut}(\mathbb{C}^n)) \Leftrightarrow (d_{\sigma(1)}, \ldots, d_{\sigma(n)}) \in \operatorname{mdeg}(\operatorname{Aut}(\mathbb{C}^n)),$$

in our considerations we can always assume that $1 \leq d_1 \leq \cdots \leq d_n$. In other words, we will consider the sets

$$\operatorname{mdeg}(\operatorname{Tame}(\mathbb{C}^n)) \cap \{(d_1, \ldots, d_n) : 1 \leq d_1 \leq \cdots \leq d_n\} \subset \mathbb{N}^n$$

and

$$\operatorname{mdeg}(\operatorname{Aut}(\mathbb{C}^n)) \cap \{(d_1, \ldots, d_n) : 1 \leq d_1 \leq \cdots \leq d_n\} \subset \mathbb{N}^n.$$

1.4. Jung and van der Kulk result. Before giving a description of the set $\operatorname{mdeg}(\operatorname{Tame}(\mathbb{C}^2))$, we recall the following two classical results.

PROPOSITION 1.1 ([7, Cor. 5.1.3]). $\operatorname{Tame}(\mathbb{C}^2)$ *is the amalgamated product of* $\operatorname{Aff}(\mathbb{C}^2)$ *and* $J(\mathbb{C}^2)$ *over their intersection, i.e.* $\operatorname{Tame}(\mathbb{C}^2)$ *is generated by these two groups and if* $\tau_i \in J(\mathbb{C}^2) \setminus \operatorname{Aff}(\mathbb{C}^2)$ *and* $\lambda_i \in \operatorname{Aff}(\mathbb{C}^2) \setminus J(\mathbb{C}^2)$, *then* $\tau_1 \circ \lambda_1 \circ \cdots \circ \tau_n \circ \lambda_n \circ \tau_{n+1}$ *does not belong to* $\operatorname{Aff}(\mathbb{C}^2)$.

Let us here recall the definition of an amalgamated product, following [43].

DEFINITION 1.2. Let G be a group and let A, B be two subgroups with $C = A \cap B$. We denote by Φ (resp. Ψ) a complete set of representatives of the left coset space A/C (resp. B/C) subject only to the restriction that the representative of C itself is the neutral element of G. We say that G is an *amalgamated product* of A and B over C if every element $g \in G$ can be written uniquely as $g = \varphi_0 \psi_1 \varphi_1 \psi_2 \cdots \varphi_{n-1} \psi_n \varphi_n \gamma$ for suitable $n \in \mathbb{N}$, $\varphi_0, \ldots, \varphi_n \in \Phi$, $\psi_1, \ldots, \psi_n \in \Psi$, $\gamma \in C$, where only φ_0, φ_n and γ may be the neutral element.

The second result is the following

COROLLARY 1.3 ([7, Cor. 5.1.6]). *Let* $F = (F_1, F_2) \in \operatorname{Tame}(\mathbb{C}^2)$ *with* $\operatorname{bideg} F = (d_1, d_2)$. *Let* h_i *denote the homogeneous component of* F_i *of degree* d_i. *Then:*

(a) $d_1 | d_2$ *or* $d_2 | d_1$.
(b) *If* $\deg F > 1$, *then we have:*
  (i) *if* $d_1 < d_2$, *then* $h_2 = c h_1^{d_2/d_1}$ *for some* $c \in \mathbb{C}$,
  (ii) *if* $d_2 < d_1$, *then* $h_1 = c h_2^{d_1/d_2}$ *for some* $c \in \mathbb{C}$,

(iii) *if* $d_1 = d_2$, *then there exists* $\lambda \in \mathrm{Aff}(\mathbb{C}^2)$ *such that* $\deg \widetilde{F}_1 > \deg \widetilde{F}_2$, *where* $\widetilde{F} = (\widetilde{F}_1, \widetilde{F}_2) = \lambda \circ F$.

From the above corollary we obtain

$$\mathrm{mdeg}(\mathrm{Tame}(\mathbb{C}^2)) \cap \{(d_1, d_2) : 1 \le d_1 \le d_2\} \subset \{(d_1, d_2) \in (\mathbb{N}^*)^2 : d_1 \mid d_2\}.$$

Since for $d_1 \mid d_2$ and

$$F_1 : \mathbb{C}^2 \ni (x, y) \mapsto (x + y^{d_1}, y) \in \mathbb{C}^2, \qquad F_2 : \mathbb{C}^2 \ni (u, v) \mapsto (u, v + u^{d_2/d_1}) \in \mathbb{C}^2,$$

$F_2 \circ F_1$ is a tame automorphism of $\mathbb{C}^2$ with $\mathrm{mdeg}(F_2 \circ F_1) = (d_1, d_2)$, we see that

$$\mathrm{mdeg}(\mathrm{Tame}(\mathbb{C}^2)) \cap \{(d_1, d_2) : 1 \le d_1 \le d_2\} = \{(d_1, d_2) \in (\mathbb{N}^*)^2 : d_1 \mid d_2\}.$$

To obtain a description of the set $\mathrm{mdeg}(\mathrm{Aut}(\mathbb{C}^2))$, we also need the following result due to Jung [9] and van der Kulk [23].

Theorem 1.4 (Jung–van der Kulk, see e.g. [7, Thm. 5.1.11]). *We have* $\mathrm{Aut}(\mathbb{C}^2) = \mathrm{Tame}(\mathbb{C}^2)$. *More precisely,* $\mathrm{Aut}(\mathbb{C}^2)$ *is the amalgamated product of* $\mathrm{Aff}(\mathbb{C}^2)$ *and* $J(\mathbb{C}^2)$ *over their intersection.*

Using Theorem 1.4, we of course obtain

$$\mathrm{mdeg}(\mathrm{Aut}(\mathbb{C}^2)) = \mathrm{mdeg}(\mathrm{Tame}(\mathbb{C}^2)),$$

and so

$$\mathrm{mdeg}(\mathrm{Aut}(\mathbb{C}^2)) \cap \{(d_1, d_2) : 1 \le d_1 \le d_2\} = \{(d_1, d_2) \in (\mathbb{N}^*)^2 : d_1 \mid d_2\}.$$

A crucial result, used in the proof of the Jung–van der Kulk result, is the following lemma and the notion of elementary reduction.

Lemma 1.5 (see e.g. [7, Lem. 10.2.4]). *Let* $f, g \in \mathbb{C}[X, Y]$, $f, g \neq 0$, *be homogeneous polynomials such that* $\mathrm{Jac}(f, g) = 0$. *Then there exists a homogeneous polynomial* h *such that:*

(i) $f = c_1 h^{n_1}$ *and* $g = c_2 h^{n_2}$ *for some integers* $n_1, n_2 \ge 0$ *and* $c_1, c_2 \in \mathbb{C}^*$.
(ii) h *is not of the form* $c h_0^s$ *for any* $c \in k^*$, *any* $h_0 \in k[x, y]$ *and any integer* $s > 1$.

Recall that an automorphism $F = (F_1, \dots, F_n)$ admits an *elementary reduction* if there exists an elementary automorphism $\tau : \mathbb{C}^n \to \mathbb{C}^n$ such that for $G = (G_1, \dots, G_n) = \tau \circ F$ we have

$$\mathrm{mdeg}\, G < \mathrm{mdeg}\, F,$$

i.e.

$$\deg G_i \le \deg F_i \quad \text{for all } i = 1, \dots, n, \qquad \deg G_i < \deg F_i \quad \text{for some } i.$$

We then say that G is an elementary reduction of F. One can easily notice that F admits an elementary reduction if there exists $i \in \{1, \dots, n\}$ and a polynomial $g \in \mathbb{C}[Y_1, \dots, Y_{n-1}]$ such that

$$\deg(F_i - g(F_1, \dots, F_{i-1}, F_{i+1}, \dots, F_n)) < \deg F_i.$$

We will also need the following generalization of the above lemma.

Proposition 1.6. *Let* $f, g \in \mathbb{C}[X_1, \dots, X_n]$ *be homogeneous, algebraically dependent polynomials. Then there exists a homogeneous polynomial* $h \in \mathbb{C}[X_1, \dots, X_n]$ *such that:*

(i) *$f = c_1 h^{n_1}$ and $g = c_2 h^{n_2}$ for some integers $n_1, n_2 \geq 0$ and $c_1, c_2 \in \mathbb{C}^*$.*
(ii) *h is not of the form ch_0^s for any $c \in \mathbb{C}^*$, any $h_0 \in \mathbb{C}[X_1, \dots, X_n]$ and any integer $s > 1$.*

One can obtain the above result using Lemma 2 in [53].

2. Main tools

2.1. Poisson bracket and degree of polynomials. In this section we present the first main tool which we will use in our considerations: the Poisson bracket of two polynomials and a theorem that estimates from below the degree of a polynomial of the form $h(f, g)$, where $f, g \in \mathbb{C}[X_1, \dots, X_n]$ and $h \in \mathbb{C}[X, Y]$.

We start with the definition of a *-reduced pair.

DEFINITION 2.1 ([49, Def. 1]). A pair $f, g \in \mathbb{C}[X_1, \dots, X_n]$ is called **-reduced* if

(i) f, g are algebraically independent;
(ii) $\overline{f}, \overline{g}$ are algebraically dependent;
(iii) $\overline{f} \notin \mathbb{C}[\overline{g}]$ and $\overline{g} \notin \mathbb{C}[\overline{f}]$.

Moreover, we say that f, g is a *p-reduced pair* if f, g is a *-reduced pair with $\deg f < \deg g$ and $p = \deg f / \gcd(\deg f, \deg g)$.

One may ask whether p can be equal to 1 for a p-reduced pair f, g. The answer is given by the following

PROPOSITION 2.2. *If f, g is a p-reduced pair, then $p > 1$.*

Proof. If f, g is p-reduced, then $\overline{f}$ and $\overline{g}$ are algebraically dependent. This means, by Proposition 1.6, that there is a homogeneous polynomial h such that

$$\overline{f} = \alpha h^l \quad \text{and} \quad \overline{g} = \beta h^m$$

for some $\alpha, \beta \in \mathbb{C}^*$ and $l, m \in \mathbb{N}$. Assume that $p = \deg f / \gcd(\deg f, \deg g) = 1$. Then $l \mid m$, and so $\overline{g} = \gamma \overline{f}^r$ for $r = m/l$ and $\gamma \in \mathbb{C}^*$. This contradicts condition (iii) of Definition 2.1. ∎

For any $f, g \in \mathbb{C}[X_1, \dots, X_n]$ we denote by $[f, g]$ the *Poisson bracket* of f and g, i.e. the formal sum

$$\sum_{1 \leq i < j \leq n} \left(\frac{\partial f}{\partial X_i} \frac{\partial g}{\partial X_j} - \frac{\partial f}{\partial X_j} \frac{\partial g}{\partial X_i} \right) [X_i, X_j],$$

where $[X_i, X_j]$ are formal objects satisfying the condition

$$[X_i, X_j] = -[X_j, X_i] \quad \text{for all } i, j.$$

We also define

$$\deg\, [X_i, X_j] = 2 \quad \text{for all } i \neq j,$$

$\deg 0 = -\infty$ and

$$\deg\, [f, g] = \max_{1 \leq i < j \leq n} \deg \left\{ \left(\frac{\partial f}{\partial X_i} \frac{\partial g}{\partial X_j} - \frac{\partial f}{\partial X_j} \frac{\partial g}{\partial X_i} \right) [X_i, X_j] \right\}.$$

Since $2-\infty=-\infty$, we have

$$\deg\,[f,g]=2+\max_{1\le i<j\le n}\deg\Big(\frac{\partial f}{\partial X_i}\frac{\partial g}{\partial X_j}-\frac{\partial f}{\partial X_j}\frac{\partial g}{\partial X_i}\Big),$$

and hence

$$\deg\,[f,g]\le \deg f+\deg g. \tag{2.1}$$

Another inequality involving the degree of a Poisson bracket will be a consequence of Proposition 2.3 below, in which $\frac{\partial(F_1,\dots,F_r)}{\partial(X_1,\dots,X_n)}$ means the Jacobian matrix (not necessarily quadratic) of the mapping $(F_1,\dots,F_r):\mathbb{C}^n\to\mathbb{C}^r$.

Proposition 2.3. *If $F_1,\dots,F_r\in\mathbb{C}[X_1,\dots,X_n]$, then*

$$\operatorname{rank}\frac{\partial(F_1,\dots,F_r)}{\partial(X_1,\dots,X_n)}=\operatorname{trdeg}_{\mathbb{C}}\mathbb{C}(F_1,\dots,F_r).$$

One can deduce the above result from [27, Chap. X, Prop. 10]. The version for $r=n$ can also be found in [7, Prop. 1.2.9].

By Proposition 2.3 and the definition of the degree of a Poisson bracket we obtain the following remark.

Remark 2.4. $f,g\in\mathbb{C}[X_1,\dots,X_n]$ are algebraically independent if and only if $\deg\,[f,g]\ge 2$.

We also have the following

Remark 2.5. For any $f,g\in\mathbb{C}[X_1,\dots,X_n]$ the following conditions are equivalent:

(1) $\deg\,[f,g]=\deg f+\deg g$,
(2) $\overline{f},\overline{g}$ are algebraically independent.

Proof. Let

$$f=f_0+\dots+f_d,\qquad g=g_0+\dots+g_m$$

be the homogeneous decompositions of f and g. Since

$$[f,g]=\sum_{i,j}[f_i,g_j]=[f_d,g_m]+\sum_{i<d\text{ or }j<m}[f_i,g_j]$$

and

$$\deg\,[f_i,g_j]\le\deg f_i+\deg g_j=i+j<d+m,$$

for $i<d$ or $j<m$, it follows that

$$\deg\,[f,g]=d+m\iff\deg\,[f_d,g_m]=d+m.$$

But, since f_d and g_m are homogeneous polynomials of degrees d and m, respectively, by the definition of Poisson bracket we have

$$\deg\,[f_d,g_m]=d+m\iff[f_d,g_m]\ne 0.$$

The last condition, by Proposition 2.3, is equivalent to f_d,g_m being algebraically independent. ■

Recall the following theorem due to Shestakov and Umirbaev.

THEOREM 2.6 ([49, Thm. 2]). *Let $f, g \in \mathbb{C}[X_1, \ldots, X_n]$ be a p-reduced pair, and let $G(X,Y) \in k[X,Y]$ with $\deg_Y G(X,Y) = pq + r$, $0 \le r < p$. Then*

$$\deg G(f,g) \ge q(p \deg g - \deg g - \deg f + \deg [f,g]) + r \deg g.$$

Notice that the estimate from Theorem 2.6 is true even if the condition (ii) of Definition 2.1 is not satisfied. Indeed, if $G = \sum_{i,j} a_{i,j} X^i Y^j$, then, by the algebraic independence of $\overline{f}$ and $\overline{g}$,

$$\begin{aligned}\deg G(f,g) &= \max_{i,j} \deg(a_{i,j} f^i g^j) \ge \deg_Y G(X,Y) \cdot \deg g \\ &= (qp + r) \deg g \ge q(p \deg g - \deg f - \deg g + \deg [f,g]) + r \deg g.\end{aligned}$$

The last inequality is a consequence of the fact that $\deg [f,g] \le \deg f + \deg g$.

Notice that the above calculations are also valid for $p = 1$ (when the pair f, g does not satisfy the condition (ii) of Definition 2.1, p may be equal to one).

Thus we have the following proposition.

PROPOSITION 2.7. *Let $f, g \in \mathbb{C}[X_1, \ldots, X_n]$ satisfy conditions* (i) *and* (iii) *of Definition* 2.1. *Assume that* $\deg f < \deg g$, *put*

$$p = \frac{\deg f}{\gcd(\deg f, \deg g)},$$

and let $G(X,Y) \in \mathbb{C}[X,Y]$ with $\deg_Y G(X,Y) = pq + r$, $0 \le r < p$. Then

$$\deg G(f,g) \ge q(p \deg g - \deg g - \deg f + \deg [f,g]) + r \deg g.$$

2.2. Degree of a Poisson bracket and a linear change of coordinates. This section is devoted to showing the following lemma saying that the degree of a Poisson bracket is invariant under a linear change of coordinates.

LEMMA 2.8. *If $f, g \in \mathbb{C}[X_1, \ldots, X_n]$ and $L \in GL_n(\mathbb{C})$, then*

$$\deg [L^*(f), L^*(g)] = \deg [f,g],$$

where $L^*(h) = h \circ L$ for any $h \in \mathbb{C}[X_1, \ldots, X_n]$.

We first show

PROPOSITION 2.9. *If $f, g \in \mathbb{C}[X_1, \ldots, X_n]$ and $L : \mathbb{C}^n \to \mathbb{C}^n$ is any linear map, then*

$$\deg [L^*(f), L^*(g)] \le \deg [f,g].$$

Proof. It is easy to see that for every $h \in \mathbb{C}[X_1, \ldots, X_n]$ we have (here we allow $L^*(h_d) = 0$ even if $h_d \ne 0$)

$$[L^*(h)]_d = L^*(h_d),$$

where the subscript d denotes the homogeneous part of degree d. We also have

$$[\mathrm{Jac}^{ij}(f,g)]_d = \sum_{k+l=d+2} \mathrm{Jac}^{ij}(f_k, g_l),$$

where

$$\mathrm{Jac}^{ij}(f,g) = \mathrm{Jac}^{X_i X_j}(f,g) = \det \begin{bmatrix} \partial f/\partial X_i & \partial f/\partial X_j \\ \partial g/\partial X_i & \partial g/\partial X_j \end{bmatrix}.$$

By the above equalities we have

$$\begin{aligned}[\mathrm{Jac}^{ij}(L^*(f),L^*(g))]_d &= \sum_{k+l=d+2} \mathrm{Jac}^{ij}(L^*(f)_k, L^*(g)_l)\\ &= \sum_{k+l=d+2} \mathrm{Jac}^{ij}(L^*(f_k), L^*(g_l)). \qquad (2.2)\end{aligned}$$

Since for any $h \in \mathbb{C}[X_1,\dots,X_n]$ and $r \in \{1,\dots,n\}$ we have

$$\frac{\partial L^*(h)}{\partial X_r} = \frac{\partial (h\circ L)}{\partial X_r} = \sum_{s=1}^n \frac{\partial h}{\partial X_s}(L)\cdot a_{sr},$$

where (a_{ij}) is the matrix of the mapping L, it follows that

$$\mathrm{Jac}^{ij}(L^*(f_k),L^*(g_l)) = \det\begin{bmatrix} \sum_{r=1}^n \frac{\partial f_k}{\partial X_r}(L)\cdot a_{ri} & \sum_{r=1}^n \frac{\partial f_k}{\partial X_r}(L)\cdot a_{rj}\\ \sum_{s=1}^n \frac{\partial g_l}{\partial X_s}(L)\cdot a_{si} & \sum_{s=1}^n \frac{\partial g_l}{\partial X s}(L)\cdot a_{sj}\end{bmatrix}$$

$$\begin{aligned}&= \sum_{r,s=1}^n \frac{\partial f_k}{\partial X_r}(L)\cdot a_{ri}\cdot \frac{\partial g_l}{\partial X_s}(L)\cdot a_{sj} - \sum_{r,s=1}^n \frac{\partial f_k}{\partial X_r}(L)\cdot a_{rj}\cdot \frac{\partial g_l}{\partial X_s}(L)\cdot a_{si}\\
&= \sum_{r,s=1}^n \left[\frac{\partial f_k}{\partial X_r}(L)\cdot a_{ri}\cdot \frac{\partial g_l}{\partial X_s}(L)\cdot a_{sj} - \frac{\partial f_k}{\partial X_s}(L)\cdot a_{sj}\cdot \frac{\partial g_l}{\partial X_r}(L)\cdot a_{ri}\right]\\
&= \sum_{r,s=1}^n \mathrm{Jac}^{rs}(f_k,g_l)(L)\cdot a_{ri}a_{sj}\\
&= \sum_{1\le r<s\le n} \mathrm{Jac}^{rs}(f_k,g_l)(L)\cdot a_{ri}a_{sj} + \sum_{1\le s<r\le n} \mathrm{Jac}^{rs}(f_k,g_l)(L)\cdot a_{ri}a_{sj}\\
&= \sum_{1\le r<s\le n} \mathrm{Jac}^{rs}(f_k,g_l)(L)\cdot a_{ri}a_{sj} - \sum_{1\le r<s\le n} \mathrm{Jac}^{rs}(f_k,g_l)(L)\cdot a_{si}a_{rj}\\
&= \sum_{1\le r<s\le n} \mathrm{Jac}^{rs}(f_k,g_l)(L)\det\begin{bmatrix} a_{ri} & a_{rj}\\ a_{si} & a_{sj}\end{bmatrix}. \qquad (2.3)\end{aligned}$$

Now, by (2.2) and (2.3), we have

$$\begin{aligned}[\mathrm{Jac}^{ij}(L^*(f),L^*(g))]_d &= \sum_{k+l=d+2}\ \sum_{1\le r<s\le n} \mathrm{Jac}^{rs}(f_k,g_l)(L)\det\begin{bmatrix} a_{ri} & a_{rj}\\ a_{si} & a_{sj}\end{bmatrix}\\ &= \sum_{1\le r<s\le n}\Big(\sum_{k+l=d+2} \mathrm{Jac}^{rs}(f_k,g_l)\Big)(L)\det\begin{bmatrix} a_{ri} & a_{rj}\\ a_{si} & a_{sj}\end{bmatrix}. \qquad (2.4)\end{aligned}$$

Take any $d > \deg\,[f,g]$. Then

$$\sum_{k+l=d+2} \mathrm{Jac}^{rs}(f_k,g_l) = 0 \qquad (2.5)$$

for all pairs r,s satisfying $1\le r<s\le n$. Thus, by (2.4) and (2.5), we obtain

$$[\mathrm{Jac}^{ij}(L^*(f),L^*(g))]_d = 0 \qquad (2.6)$$

for all i,j. The above equalities (for all i,j) mean that $\deg\,[L^*(f),L^*(g)] < d$. Since we can take $d = \deg[f,g]+1, \deg[f,g]+2, \ldots$ we obtain

$$\deg[L^*(f), L^*(g)] \le \deg[f,g]. \ \blacksquare \tag{2.7}$$

Proof of Lemma 2.8. By the above proposition we only need to show that $\deg[L^*(f), L^*(g)] \ge \deg[f,g]$. But $f = (L^{-1})^*(L^*(f))$ and $g = (L^{-1})^*(L^*(g))$. So applying Proposition 2.9 to the polynomials $L^*(f), L^*(g)$ and the mapping L^{-1} we obtain

$$\deg[f,g] = \deg[(L^{-1})^*(L^*(f)), (L^{-1})^*(L^*(g))] \le \deg[L^*(f), L^*(g)]. \ \blacksquare$$

2.3. Shestakov–Umirbaev reductions. In this section we present the most remarkable result of Shestakov and Umirbaev, Theorem 2.6. The notions of reductions of types I–IV are crucial in this theorem. Thus we start with the following definitions (see [49] or [50]).

Definition 2.10. Let $\Theta = (f_1, f_2, f_3)$ be an automorphism of $A = \mathbb{C}[X,Y,Z]$ such that (for some $n \in \mathbb{N}^*$) $\deg f_1 = 2n$, $\deg f_2 = ns$, where $s \ge 3$ is an odd number, $2n < \deg f_3 \le ns$ and $\overline{f}_3 \notin \mathbb{C}[\overline{f}_1, \overline{f}_2]$. Suppose that there exists $\alpha \in \mathbb{C}^*$ such that the elements $g_1 = f_1$, $g_2 = f_2 - \alpha f_3$ satisfy the following conditions:

(i) g_1, g_2 is a 2-reduced pair and $\deg g_1 = \deg f_1$, $\deg g_2 = \deg f_2$;
(ii) the automorphism (g_1, g_2, f_3) admits an elementary reduction (g_1, g_2, g_3) with $\deg[g_1, g_3] < \deg g_2 + \deg[g_1, g_2]$.

Then we will say that Θ admits a *reduction* (g_1, g_2, g_3) *of type I.* We will also say that a polynomial automorphism $F = (F_1, F_2, F_3)$ admits a reduction of type I if for some permutation σ of $\{1,2,3\}$, the automorphism $\Theta = (F_{\sigma(1)}, F_{\sigma(2)}, F_{\sigma(3)})$ admits a reduction of type I.

Before proposing next definitions we present an example due to Shestakov and Umirbaev of a tame automorphism of $\mathbb{C}^3$ which does not admit an elementary reduction but admits a reduction of type I.

Example 2.11. Let

$$\begin{aligned}
T_1(x_1,x_2,x_3) &= (x_1, x_2 + x_1^2, x_3 + 2x_1x_2 + x_1^3),\\
T_2(y_1,y_2,y_3) &= (6y_1 + 6y_2y_3 + y_3^3, 4y_2 + y_3^2, y_3),\\
T_3(z_1,z_2,z_3) &= (z_1, z_2, z_3 + z_1^2 - z_2^3),\\
L(u_1,u_2,u_3) &= (u_1 + u_3, u_2, u_3)
\end{aligned}$$

and

$$G = T_3 \circ T_2 \circ T_1, \qquad F = L \circ G.$$

It is easy to see that

$$\operatorname{mdeg}(T_2 \circ T_1) = (9,6,3),$$

and because

$$(6y_1 + 6y_2y_3 + y_3^3)^2 - (4y_2 + y_3^2)^3 = 36y_1^2 + 72y_1y_2y_3 + 12y_1y_3^3 - 12y_2^2y_3^2 - 64y_2^3$$

and (provided that $y_1 = x_1, y_2 = x_2 + x_1^2$ and $y_3 = x_3 + 2x_1x_2 + x_1^3$)

$$\begin{aligned} 12y_1y_3^3 - 12y_2^2y_3^2 &= 12x_1(x_3 + 2x_1x_2 + x_1^3)^3 - 12(x_2 + x_1^2)^2(x_3 + 2x_1x_2 + x_1^3)^2 \\ &= 12x_3x_1^7 - 12x_1^6x_2^2 + \text{lower degree monomials,} \end{aligned}$$

we have

$$\operatorname{mdeg}(T_3 \circ T_2 \circ T_1) = (9, 6, 8) \quad \text{and so} \quad \operatorname{mdeg} F = \operatorname{mdeg}(L \circ G) = (9, 6, 8).$$

From the construction of F it is clear that F is a tame automorphism. Moreover, it does not admit an elementary reduction. Indeed, if we put $F = (F_1, F_2, F_3)$ and assume that $(F_1 - g(F_2, F_3), F_2, F_3)$, for some $g \in \mathbb{C}[X, Y]$, is an elementary reduction of (F_1, F_2, F_3) then we must have

$$\deg g(F_2, F_3) = 9. \tag{2.8}$$

But by Proposition 2.7, we have

$$\deg g(F_2, F_3) \geq q(p \cdot 8 - 6 - 8 + \deg [F_2, F_3]) + 8r, \tag{2.9}$$

where $\deg_Y g(X, Y) = qp + r$, $0 \leq r < p$, $p = 6/\gcd(6, 8) = 3$. Thus by (2.8) and (2.9) and because $p \cdot 8 - 6 - 8 + \deg [F_2, F_3] = 10 + \deg [F_2, F_3] \geq 12 > 9$, we must have $q = 0$ and $r \leq 1$. Thus g must be of the form

$$g(X, Y) = g_0(X) + g_1(X)Y. \tag{2.10}$$

Since $8\mathbb{N} \cap (6 + 8\mathbb{N}) = \emptyset$, from (2.8) and (2.10) we obtain $9 = \deg g(F_2, F_3) \in 8\mathbb{N} \cup (6 + 8\mathbb{N})$, a contradiction.

Next, if we assume that $(F_1, F_2 - g(F_3, F_1), F_3)$, for some $g \in \mathbb{C}[X, Y]$, is an elementary reduction of (F_1, F_2, F_3) then we must have

$$\deg g(F_3, F_1) = 6. \tag{2.11}$$

But by Proposition 2.7,

$$\deg g(F_3, F_1) \geq q(p \cdot 9 - 9 - 8 + \deg [F_3, F_1]) + 9r, \tag{2.12}$$

where $\deg_Y g(X, Y) = qp + r$, $0 \leq r < p$, $p = 8/\gcd(8, 9) = 8$. Because $p \cdot 9 - 9 - 8 + \deg [F_3, F_1] = 55 + \deg [F_3, F_1] \geq 57 > 8$, from (2.11) and (2.12) we obtain $q = r = 0$. This means that $g(X, Y) = g(X)$ and $\deg g(F_3, F_1) = \deg g(F_3) \in 8\mathbb{N}$. However, $6 \notin 8\mathbb{N}$.

Finally, if we assume that $(F_1, F_2, F_3 - g(F_2, F_1))$, for some $g \in \mathbb{C}[X, Y]$, is an elementary reduction of (F_1, F_2, F_3) then

$$\deg g(F_2, F_1) = 8. \tag{2.13}$$

As before, by Proposition 2.7,

$$\deg g(F_2, F_1) \geq q(p \cdot 9 - 9 - 6 + \deg [F_2, F_1]) + 9r, \tag{2.14}$$

where $\deg_Y g(X, Y) = qp + r$, $0 \leq r < p$, $p = 6/\gcd(6, 9) = 2$. In this case $p \cdot 9 - 9 - 6 = 3$ is not large enough for our purpose but $\deg [F_2, F_1]$ is. Indeed,

$$\begin{aligned} \frac{\partial F_1}{\partial x_i} &= \frac{\partial u_1}{\partial x_i} + \frac{\partial u_3}{\partial x_i} = \frac{\partial z_1}{\partial x_i} + \frac{\partial z_3}{\partial x_i} + 2z_1 \frac{\partial z_1}{\partial x_i} - 3z_2^2 \frac{\partial z_2}{\partial x_i}, \\ \frac{\partial F_2}{\partial x_i} &= \frac{\partial u_2}{\partial x_i} = \frac{\partial z_2}{\partial x_i}. \end{aligned}$$

Thus, for $1 \le i < j \le 3$,

$$\begin{aligned}\frac{\partial F_1}{\partial x_i}\frac{\partial F_2}{\partial x_j} - \frac{\partial F_1}{\partial x_j}\frac{\partial F_2}{\partial x_i} &= \left(\frac{\partial z_1}{\partial x_i} + \frac{\partial z_3}{\partial x_i} + 2z_1\frac{\partial z_1}{\partial x_i} - 3z_2^2\frac{\partial z_2}{\partial x_i}\right)\frac{\partial z_2}{\partial x_j}\\ &\quad - \left(\frac{\partial z_1}{\partial x_j} + \frac{\partial z_3}{\partial x_j} + 2z_1\frac{\partial z_1}{\partial x_j} - 3z_2^2\frac{\partial z_2}{\partial x_j}\right)\frac{\partial z_2}{\partial x_i}\\ &= \left(\frac{\partial z_1}{\partial x_i}\frac{\partial z_2}{\partial x_j} - \frac{\partial z_1}{\partial x_j}\frac{\partial z_2}{\partial x_i}\right) + \left(\frac{\partial z_3}{\partial x_i}\frac{\partial z_2}{\partial x_j} - \frac{\partial z_3}{\partial x_j}\frac{\partial z_2}{\partial x_i}\right)\\ &\quad + 2z_1\left(\frac{\partial z_1}{\partial x_i}\frac{\partial z_2}{\partial x_j} - \frac{\partial z_1}{\partial x_j}\frac{\partial z_2}{\partial x_i}\right). \qquad (2.15)\end{aligned}$$

Since z_1, z_2, z_3 are algebraically independent, by Corollary 2.3 for at least one pair i, j, $1 \le i < j \le 3$, we have

$$\frac{\partial z_1}{\partial x_i}\frac{\partial z_2}{\partial x_j} - \frac{\partial z_1}{\partial x_j}\frac{\partial z_2}{\partial x_i} \ne 0.$$

And since $\deg z_1 = 9$, for that pair i, j we have

$$\deg 2z_1\left(\frac{\partial z_1}{\partial x_i}\frac{\partial z_2}{\partial x_j} - \frac{\partial z_1}{\partial x_j}\frac{\partial z_2}{\partial x_i}\right) \ge 9. \qquad (2.16)$$

Of course we also have

$$\deg 2z_1\left(\frac{\partial z_1}{\partial x_i}\frac{\partial z_2}{\partial x_j} - \frac{\partial z_1}{\partial x_j}\frac{\partial z_2}{\partial x_i}\right) > \deg\left(\frac{\partial z_1}{\partial x_i}\frac{\partial z_2}{\partial x_j} - \frac{\partial z_1}{\partial x_j}\frac{\partial z_2}{\partial x_i}\right). \qquad (2.17)$$

Since moreover

$$\frac{\partial z_2}{\partial x_i} = 4\frac{\partial y_2}{\partial x_i} + 2y_3\frac{\partial y_3}{\partial x_i}, \qquad \frac{\partial z_3}{\partial x_i} = \frac{\partial y_3}{\partial x_i}$$

and

$$\deg y_2 = \deg(x_2 + x_1^2) = 2, \qquad \deg y_3 = \deg(x_3 + 2x_1x_2 + x_1^3) = 3,$$

it follows that

$$\begin{aligned}\frac{\partial z_2}{\partial x_i}\frac{\partial z_3}{\partial x_j} - \frac{\partial z_2}{\partial x_j}\frac{\partial z_3}{\partial x_i} &= \left(4\frac{\partial y_2}{\partial x_i} + 2y_3\frac{\partial y_3}{\partial x_i}\right)\frac{\partial y_3}{\partial x_j} - \left(4\frac{\partial y_2}{\partial x_j} + 2y_3\frac{\partial y_3}{\partial x_j}\right)\frac{\partial y_3}{\partial x_i}\\ &= 4\left(\frac{\partial y_2}{\partial x_i}\frac{\partial y_3}{\partial x_j} - \frac{\partial y_2}{\partial x_j}\frac{\partial y_3}{\partial x_i}\right),\end{aligned}$$

and so

$$\deg\left(\frac{\partial z_2}{\partial x_i}\frac{\partial z_3}{\partial x_j} - \frac{\partial z_2}{\partial x_j}\frac{\partial z_3}{\partial x_i}\right) = \deg\left(\frac{\partial y_2}{\partial x_i}\frac{\partial y_3}{\partial x_j} - \frac{\partial y_2}{\partial x_j}\frac{\partial y_3}{\partial x_i}\right) \le 3. \qquad (2.18)$$

Finally, by (2.15)–(2.18),

$$\deg\,[F_1, F_2] \ge 11. \qquad (2.19)$$

Now, using (2.19) and (2.14) we find that

$$\deg g(F_2, F_1) \ge q \cdot 14 + 9r. \qquad (2.20)$$

Thus, by (2.20) and (2.13), we have $q = r = 0$. This means that $g(X, Y) = g(X)$ and $\deg g(F_2, F_1) = \deg g(F_2) \in 6\mathbb{N}$, contrary to $8 \notin 6\mathbb{N}$.

For more information about polynomial automorphisms which admit reductions of type I see [25].

Definition 2.12. Let $\Theta = (f_1, f_2, f_3)$ be an automorphism of $A = \mathbb{C}[X, Y, Z]$ such that (for some $n \in \mathbb{N}^*$) $\deg f_1 = 2n$, $\deg f_2 = 3n$, $\frac{3}{2}n < \deg f_3 \leq 2n$ and $\overline{f}_1$, $\overline{f}_3$ are linearly independent. Suppose that there exist $\alpha, \beta \in \mathbb{C}$ with $(\alpha, \beta) \neq (0,0)$ such that the elements $g_1 = f_1 - \alpha f_3$, $g_2 = f_2 - \beta f_3$ satisfy the following conditions:

(i) g_1, g_2 is a 2-reduced pair and $\deg g_1 = \deg f_1$, $\deg g_2 = \deg f_2$;
(ii) the automorphism (g_1, g_2, f_3) admits an elementary reduction (g_1, g_2, g_3) with $\deg [g_1, g_3] < \deg g_2 + \deg [g_1, g_2]$.

Then we will say that Θ admits a *reduction* (g_1, g_2, g_3) *of type II.* We will also say that a polynomial automorphism $F = (F_1, F_2, F_3)$ admits a reduction of type II if for some permutation σ of $\{1, 2, 3\}$, the automorphism $\Theta = (F_{\sigma(1)}, F_{\sigma(2)}, F_{\sigma(3)})$ admits a reduction of type II.

Definition 2.13. Let $\Theta = (f_1, f_2, f_3)$ be an automorphism of $A = \mathbb{C}[X, Y, Z]$ such that (for some $n \in \mathbb{N}^*$) $\deg f_1 = 2n$, and either

$$\deg f_2 = 3n, \quad n < \deg f_3 \leq 3n/2,$$

or

$$5n/2 < \deg f_2 \leq 3n, \quad \deg f_3 = 3n/2.$$

Suppose that there exist $\alpha, \beta, \gamma \in \mathbb{C}$ such that the elements $g_1 = f_1 - \beta f_3$, $g_2 = f_2 - \gamma f_3 - \alpha f_3^2$ satisfy the following conditions:

(i) g_1, g_2 is a 2-reduced pair and $\deg g_1 = 2n$, $\deg g_2 = 3n$;
(ii) there exists g_3 of the form $g_3 = \sigma f_3 + g$, where $\sigma \in \mathbb{C}^*$, $g \in \mathbb{C}[g_1, g_2]$, such that $\deg g_3 \leq \frac{3}{2}n$, $\deg [g_1, g_3] < 3n + \deg [g_1, g_2]$.

If $(\alpha, \beta, \gamma) \neq (0, 0, 0)$ and $\deg g_3 < n + \deg [g_1, g_2]$, then we will say that Θ admits a *reduction* (g_1, g_2, g_3) *of type III.* On the other hand, if there exists $\mu \in \mathbb{C}^*$ such that $\deg(g_2 - \mu g_3^2) \leq 2n$, then we will say that Θ admits a *reduction* $(g_1, g_2 - \mu g_3^2, g_3)$ *of type* IV.

We will also say that a polynomial automorphism $F = (F_1, F_2, F_3)$ admits a reduction of type III (resp. IV) if for some permutation σ of $\{1, 2, 3\}$, the automorphism $\Theta = (F_{\sigma(1)}, F_{\sigma(2)}, F_{\sigma(3)})$ admits a reduction of type III (resp. IV).

Now, we can present the above mentioned theorem.

Theorem 2.14 ([49, Thm. 3]). *Let $F = (F_1, F_2, F_3)$ be a tame automorphism of $\mathbb{C}^3$. If $\deg F_1 + \deg F_2 + \deg F_3 > 3$ (in other words, if F is not an affine automorphism), then F admits either an elementary reduction or a reduction of one of types I–IV.*

2.4. Some number theory. We will use the following result from number theory, connected with the so-called coin problem or Frobenius problem.

Theorem 2.15 (see e.g. [10]). *If d_1, d_2 are positive integers such that* $\gcd(d_1, d_2) = 1$, *then for every integer $k \geq (d_1 - 1)(d_2 - 1)$ there are $k_1, k_2 \in \mathbb{N}$ such that*

$$k = k_1 d_1 + k_2 d_2.$$

Moreover $(d_1 - 1)(d_2 - 1) - 1 \notin d_1\mathbb{N} + d_2\mathbb{N}$.

The proof of the above theorem can be found in the number theory literature, but for the convenience of the reader we give it here. In the proof we will write $M(d_1, d_2)$ for the minimal $s \in \mathbb{N}$ such that $\{s, s+1, \dots\} \subset d_1\mathbb{N} + d_2\mathbb{N}$. Let us mention that the so-called *Frobenius number* (the maximal $s \in \mathbb{N}$ such that $s \notin d_1\mathbb{N}+d_2\mathbb{N}$) is equal to $M(d_1, d_2)-1$.

Proof. Without loss of generality we can assume that $1 < d_1 \leq d_2$. Indeed, if $d_1 = 1$, then $d_1\mathbb{N}+d_2\mathbb{N} = \mathbb{N}$ and $(d_1 - 1)(d_2 - 1) = 0$. Thus for any $r = 1, \dots, d_1 - 1$ there are integers $k_{1,r}, k_{2,r} \in \mathbb{Z}$ such that

$$k_{1,r}d_1 + k_{2,r}d_2 = r.$$

Since $d_1, d_2, r > 0$ and $r < d_1 \leq d_2$, we have $k_{1,r}k_{2,r} < 0$. Moreover, since $(k_{1,r} - d_2)d_1 + (k_{2,r}+d_1)d_2 = k_{1,r}d_1+k_{2,r}d_2 = r$, we can assume that $k_{2,r} > 0$. Notice that we can assume even more, namely that $k_{2,r} > 0$ and $k_{1,r} \geq 1 - d_2$. Indeed, let $k_{1,r}, k_{2,r} \in \mathbb{Z}$ be such that $k_{1,r}d_1 + k_{2,r}d_2 = r$, $k_{2,r} > 0$ and there are no $k'_{1,r}, k'_{2,r} \in \mathbb{Z}$ such that $k'_{1,r}d_1 + k'_{2,r}d_2 = r$, $k'_{2,r} > 0$ and $k'_{2,r} < k_{2,r}$. Then, since $(k_{1,r}+d_2)d_1+(k_{2,r}-d_1)d_2 = k_{1,r}d_1+k_{2,r}d_2 = r$, we have $k_{2,r} - d_1 \leq 0$ (since $r < d_1 \leq d_2$ we actually have $k_{2,r} - d_1 < 0$). Thus $k_{1,r} + d_2 > 0$, and so $k_{1,r} \geq 1 - d_2$.

It is easy to see that to show that any natural number $k \geq (d_1 - 1)(d_2 - 1)$ is in $d_1\mathbb{N} + d_2\mathbb{N}$, we only need to show that

$$(d_1 - 1)(d_2 - 1), (d_1 - 1)(d_2 - 1) + 1, \dots, (d_1 - 1)(d_2 - 1) + d_1 - 1 \in d_1\mathbb{N} + d_2\mathbb{N}.$$

First,

$$\begin{aligned}(d_1 - 1)(d_2 - 1) &= (d_2 - 1)d_1 - d_2 + 1 = (d_2 - 1)d_1 - d_2 + k_{1,1}d_1 + k_{2,1}d_2 \\ &= (d_2 - 1 + k_{1,1})d_1 + (k_{2,1} - 1)d_2 \in d_1\mathbb{N} + d_2\mathbb{N},\end{aligned}$$

because $k_{1,1} \geq 1 - d_2$ and $k_{2,1} > 0$. Similarly, we show that $(d_1 - 1)(d_2 - 1) + 1 = (d_2 - 1)d_1 - d_2 + 2, \dots, (d_1 - 1)(d_2 - 1) + d_1 - 2 = (d_2 - 1)d_1 - d_2 + (d_1 - 1) \in d_1\mathbb{N} + d_2\mathbb{N}$. To see that $(d_1 - 1)(d_2 - 1) + d_1 - 1 \in d_1\mathbb{N} + d_2\mathbb{N}$ we write

$$(d_1 - 1)(d_2 - 1) + d_1 - 1 = d_1d_2 - d_1 - d_2 + 1 + d_1 - 1 = (d_1 - 1)d_2.$$

Thus we have shown that $M(d_1, d_2) \leq (d_1 - 1)(d_2 - 1)$.

To prove that $M(d_1, d_2) = (d_1-1)(d_2-1)$ it is enough to show that $(d_1-1)(d_2-1)-1 \notin d_1\mathbb{N} + d_2\mathbb{N}$. Since $(d_2 - 1)d_1 - d_2 = (d_1 - 1)(d_2 - 1) - 1$ and $\operatorname{lcm}(d_1, d_2) = d_1d_2$, it follows that

$$\{(k_1, k_2) \in \mathbb{Z}^2 : k_1d_1 + k_2d_2 = (d_1 - 1)(d_2 - 1) - 1\} = \{(d_2 - 1 - ld_2, ld_1 - 1) : l \in \mathbb{Z}\}.$$

But $\{(d_2 - 1 - ld_2, ld_1 - 1) : l \in \mathbb{Z}\} \cap \mathbb{N}^2 = \emptyset$. This ends the proof. ∎

3. Some useful results

3.1. Some simple remarks. In this section we make some simple but useful remarks about existence of automorphisms and tame automorphisms with given multidegree.

PROPOSITION 3.1 ([18, Prop. 2.1]). *If for $1 \leq d_1 \leq \dots \leq d_n$ there is a sequence of integers $1 \leq i_1 < \dots < i_m \leq n$ such that there exists an automorphism G of $\mathbb{C}^m$ with* mdeg $G = (d_{i_1}, \ldots, d_{i_m})$, *then there exists an automorphism* F *of* $\mathbb{C}^n$ *with* mdeg $F = (d_1, \ldots, d_n)$. *Moreover, if* G *is tame, then* F *can also be found tame.*

Proof. Without loss of generality we can assume that $m < n$. Let $1 \le j_1 < \cdots < j_{n-m} \le n$ be such that $\{i_1, \ldots, i_m\} \cup \{j_1, \ldots, j_{n-m}\} = \{1, \ldots, n\}$. Then, of course, $\{i_1, \ldots, i_m\} \cap \{j_1, \ldots, j_{n-m}\} = \emptyset$. Consider the mapping $h = (h_1, \ldots, h_n) : \mathbb{C}^n \to \mathbb{C}^n$ given by

$$h_k(x_1, \ldots, x_n) = \begin{cases} x_k & \text{for } k \in \{i_1, \ldots, i_m\}, \\ x_k + (x_{i_1})^{d_k} & \text{for } k \in \{j_1, \ldots, j_{n-m}\}. \end{cases}$$

Of course h is an automorphism of $\mathbb{C}^n$ and $\deg h_k = d_k$ for $k \in \{j_1, \ldots, j_{n-m}\}$.

Consider also the mapping $g = (g_1, \ldots, g_n) : \mathbb{C}^n \to \mathbb{C}^n$ given by

$$g_k(u_1, \ldots, u_n) = \begin{cases} G_l(u_{i_1}, \ldots, u_{i_m}) & \text{for } k = i_l, \\ u_k & \text{for } k \in \{j_1, \ldots, j_{n-m}\}. \end{cases}$$

Then g is an automorphism of $\mathbb{C}^n$ and $\deg g_k = d_k$ for $k \in \{i_1, \ldots, i_m\}$.

Now $F = g \circ h$ is an automorphism of $\mathbb{C}^n$ (tame when G is tame) with mdeg $F = (d_1, \ldots, d_n)$. ■

PROPOSITION 3.2 ([18, Prop. 2.2]). *If for a sequence of integers* $1 \le d_1 \le \cdots \le d_n$ *there is* $i \in \{1, \ldots, n\}$ *such that*

$$d_i = \sum_{j=1}^{i-1} k_j d_j \quad \textit{with } k_j \in \mathbb{N},$$

then there exists a tame automorphism F *of* $\mathbb{C}^n$ *with* mdeg$F = (d_1, \ldots, d_n)$.

Proof. Define $h = (h_1, \ldots, h_n) : \mathbb{C}^n \to \mathbb{C}^n$ and $g = (g_1, \ldots, g_n) : \mathbb{C}^n \to \mathbb{C}^n$ by

$$h_k(x_1, \ldots, x_n) = \begin{cases} x_k & \text{for } k = i, \\ x_k + x_i^{d_k} & \text{for } k \neq i, \end{cases}$$

and

$$g_k(u_1, \ldots, u_n) = \begin{cases} u_k + u_1^{k_1} \cdots u_{i-1}^{k_{i-1}} & \text{for } k = i, \\ u_k & \text{for } k \neq i. \end{cases}$$

It is easy to see that $F = g \circ h$ is a tame automorphism with mdeg$F = (d_1, \ldots, d_n)$. ■

The above proposition implies the following result.

COROLLARY 3.3 ([18, Cor. 2.3]). *If* $1 \le d_1 \le \cdots \le d_n$ *is a sequence of integers with* $d_1 \le n-1$, *then there exists a tame automorphism* F *of* $\mathbb{C}^n$ *with* mdeg $F = (d_1, \ldots, d_n)$.

Proof. Let $r_i \in \{0, 1, \ldots, d_1 - 1\}$, for $i = 2, \ldots, n$, be such that $d_i \equiv r_i \pmod{d_1}$. If there is an $i \in \{2, \ldots, n\}$ such that $r_i = 0$, then $d_i = kd_1$ for some $k \in \mathbb{N}^*$ and by Proposition 3.2, there exists a tame automorphism F of $\mathbb{C}^n$ with the desired properties.

Thus assume that $r_i \neq 0$ for all $i = 2, \ldots, n$. Since $d_1 - 1 < n - 1$, there are $i, j \in \{2, \ldots, n\}$, $i \neq j$, such that $r_i = r_j$. Without loss of generality we can assume that $i < j$. Then $d_j = d_i + kd_1$ for some $k \in \mathbb{N}$, and by Proposition 3.2 there exists a tame automorphism F of $\mathbb{C}^n$ with the desired properties. ■

The above corollary can be improved as follows.

THEOREM 3.4. *If $1 \le d_1 \le \cdots \le d_n$ is a sequence of integers with*

$$\frac{d_1}{\gcd(d_1, \dots, d_n)} \le n - 1,$$

then there exists a tame automorphism F of $\mathbb{C}^n$ with $\mathrm{mdeg} F = (d_1, \dots, d_n)$.

Proof. Let $d = \gcd(d_1, \dots, d_n)$. Then the numbers $r_2, \dots, r_n$ defined as in the proof of Corollary 3.3 satisfy $r_i \in \{0, d, 2d, \dots, d_1 - d\}$ for $i = 2, \dots, n$. Since the number of elements of the set $\{0, d, 2d, \dots, d_1 - d\}$ is equal to

$$\frac{d_1}{\gcd(d_1, \dots, d_n)} \le n - 1,$$

we can use the same arguments as in the proof of Corollary 3.3. ■

Combining Theorem 3.4 and Proposition 3.1 we obtain the following result.

COROLLARY 3.5. *If for $1 \le d_1 \le \cdots \le d_n$ there is a sequence of integers $1 \le i_1 < \cdots < i_m \le n$ such that*

$$\frac{d_{i_1}}{\gcd(d_{i_1}, \dots, d_{i_m})} \le m - 1,$$

then there exists a tame automorphism F of $\mathbb{C}^n$ with $\mathrm{mdeg} F = (d_1, \dots, d_n)$.

3.2. Reducibility of type I and II. Now we will show that in our considerations we do not need to pay attention to reducibility of type I and II.

LEMMA 3.6. *Let $(d_1, d_2, d_3) \neq (1, 1, 1)$, $d_1 \le d_2 \le d_3$, be a sequence of positive integers. If there is an automorphism (resp. a tame automorphism) $F : \mathbb{C}^3 \to \mathbb{C}^3$ such that F admits a reduction of type I or II and* $\mathrm{mdeg}\, F = (d_1, d_2, d_3)$, *then there is also an automorphism (resp. a tame automorphism) $\widetilde{F} : \mathbb{C}^3 \to \mathbb{C}^3$ such that $\widetilde{F}$ admits an elementary reduction and* $\mathrm{mdeg}\, \widetilde{F} = (d_1, d_2, d_3)$. *Moreover, if $F(0,0,0) = (0,0,0)$, then $\widetilde{F}$ can also be found such that $\widetilde{F}(0,0,0) = (0,0,0)$.*

Proof. Assume that $F = (F_1, F_2, F_3)$ admits a reduction of type I. By Definition 2.10 there is a permutation σ of $\{1, 2, 3\}$ and $\alpha \in \mathbb{C}^*$ such that the elements $g_1 = F_{\sigma(1)}$, $g_2 = F_{\sigma(2)} - \alpha F_{\sigma(3)}$ satisfy the following conditions:

(i) g_1, g_2 is a 2-reduced pair and $\deg g_1 = \deg F_{\sigma(1)}$, $\deg g_2 = \deg F_{\sigma(2)}$;
(ii) the automorphism $(g_1, g_2, F_{\sigma(3)})$ admits an elementary reduction of the form (g_1, g_2, g_3).

For simplicity of notation (and without loss of generality) we assume that $\sigma = \mathrm{id}_{\{1,2,3\}}$. Thus we can take $\widetilde{F} = (g_1, g_2, F_3)$.

If F admits a reduction of type II we can use a similar construction to obtain an automorphism $\widetilde{F}$.

Since $\widetilde{F} = G \circ F$, where

$$G : \mathbb{C}^3 \ni \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} \mapsto \begin{Bmatrix} x \\ y - \alpha z \\ z \end{Bmatrix} \in \mathbb{C}^3 \quad \text{(for type I)}$$

or

$$G : \mathbb{C}^3 \ni \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} \mapsto \begin{Bmatrix} x - \alpha z \\ y - \beta z \\ z \end{Bmatrix} \in \mathbb{C}^3 \quad \text{(for type II)}$$

$\widetilde{F}$ is tame if and only if F is tame. It is also clear that $\widetilde{F}(0,0,0) = (0,0,0)$ when $F(0,0,0) = (0,0,0)$. ■

The above lemma also implies the following

PROPOSITION 3.7. *Let $(d_1, d_2, d_3) \neq (1,1,1)$, $d_1 \leq d_2 \leq d_3$, be a sequence of positive integers. If there is a tame automorphism $F : \mathbb{C}^3 \to \mathbb{C}^3$ with* $\operatorname{mdeg} F = (d_1, d_2, d_3)$, *then there is also a tame automorphism $\widetilde{F} : \mathbb{C}^3 \to \mathbb{C}^3$ such that* $\operatorname{mdeg} \widetilde{F} = (d_1, d_2, d_3)$ *and $\widetilde{F}$ admits either an elementary reduction or a reduction of type III or IV. Moreover we can require that $\widetilde{F}(0,0,0) = (0,0,0)$.*

Proof. Let $F = (F_1, F_2, F_3) : \mathbb{C}^3 \to \mathbb{C}^3$ be any tame automorphism with $\operatorname{mdeg} F = (d_1, d_2, d_3)$ and let $T : \mathbb{C}^3 \to \mathbb{C}^3$ be the translation given by

$$T : \mathbb{C}^3 \ni (x, y, z) \mapsto (x - F_1(0), y - F_2(0), z - F_3(0)) \in \mathbb{C}^3.$$

Then obviously $T \circ F$ is a tame automorphism of $\mathbb{C}^3$ such that $\operatorname{mdeg}(T \circ F) = \operatorname{mdeg} F = (d_1, d_2, d_3)$ and $(T \circ F)(0,0,0) = (0,0,0)$. If $T \circ F$ admits either an elementary reduction or a reduction of type III or IV, then we take $\widetilde{F} = T \circ F$. And if $T \circ F$ admits a reduction of type I or II, then we can use Lemma 3.6. ■

In particular Proposition 3.7 says that reductions of type I and II are irrelevant for our considerations. To be precise we formulate the following

THEOREM 3.8. *Let $(d_1, d_2, d_3) \neq (1,1,1)$, $d_1 \leq d_2 \leq d_3$, be a sequence of positive integers. To prove that there is no tame automorphism of $\mathbb{C}^3$ with multidegree (d_1, d_2, d_3) it is enough to show that a (hypothetical) automorphism F of $\mathbb{C}^3$ with* $\operatorname{mdeg} F = (d_1, d_2, d_3)$ *admits neither an elementary reduction nor a reduction of type III or IV. Moreover, we can restrict our attention to automorphisms F with $F(0,0,0) = (0,0,0)$.*

To end this section, let us look again at Example 2.11. If F is the automorphism from that example, then $\operatorname{mdeg} F = (9, 6, 8)$ or $(6, 8, 9)$ after permutation of coordinates. This automorphism does not admit an elementary reduction and admits a reduction of type I. One can easily see that (in the notation of Example 2.11)

$$T_2 \circ T_1 = T_3^{-1} \circ L^{-1} \circ F$$

is a reduction of type I of F. Moreover for $\widetilde{F} = L^{-1} \circ F$ we have

$$\operatorname{mdeg} \widetilde{F} = \operatorname{mdeg} F$$

and $T_3^{-1} \circ \widetilde{F}$ is an elementary reduction of $\widetilde{F}$.

3.3. Reducibility of type III. First of all notice that if $1 \leq d_1 \leq d_2 \leq d_3$ are such that $\operatorname{mdeg} F = (d_1, d_2, d_3)$ for some automorphism F that admits a reduction of type III, then by Definition 2.13 there is $n \in \mathbb{N}^*$ such that

$$d_{\sigma(1)} = 2n$$

and either

$$d_{\sigma(2)} = 3n, \quad n < d_{\sigma(3)} \leq 3n/2,$$

or

$$5n/2 < d_{\sigma(2)} \leq 3n, \quad d_{\sigma(3)} = 3n/2$$

for some permutation σ, of $\{1, 2, 3\}$. Since $\frac{3}{2}n < 2n < \min\{\frac{5}{2}n, 3n\}$, we must actually have

$$d_2 = 2n$$

and either

$$d_3 = 3n, \quad n < d_1 \leq 3n/2,$$

or

$$5n/2 < d_3 \leq 3n, \quad d_1 = 3n/2.$$

Thus we have the following remark.

REMARK 3.9. If an automorphism F of $\mathbb{C}^3$ with $\operatorname{mdeg} F = (d_1, d_2, d_3)$, $1 \leq d_1 \leq d_2 \leq d_3$, admits a reduction of type III, then

(1) $2 \mid d_2$,
(2) $3 \mid d_1$ or $d_3/d_2 = 3/2$.

Because of the remark above it is natural to consider the situation of the following lemma.

LEMMA 3.10. *Let $(d_1, d_2, d_3) \neq (1, 1, 1)$, $d_1 \leq d_2 \leq d_3$, be a sequence of positive integers such that $d_3/d_2 = 3/2$. If there is an automorphism (resp. a tame automorphism) $F : \mathbb{C}^3 \to \mathbb{C}^3$ such that F admits a reduction of type III and* $\operatorname{mdeg} F = (d_1, d_2, d_3)$, *then there is also an automorphism (resp. a tame automorphism) $\widetilde{F} : \mathbb{C}^3 \to \mathbb{C}^3$ such that $\widetilde{F}$ admits an elementary reduction and* $\operatorname{mdeg} \widetilde{F} = (d_1, d_2, d_3)$. *Moreover, if $F(0, 0, 0) = (0, 0, 0)$, then $\widetilde{F}$ can also be found such that $\widetilde{F}(0, 0, 0) = (0, 0, 0)$.*

In the proof of this lemma we will use the following result.

LEMMA 3.11 ([50, Cor. 4]). *If an automorphism (g_1, g_2, g_3) is a reduction of type III of an automorphism (f_1, f_2, f_3), then*

$$\deg g_1 + \deg g_2 + \deg g_3 < \deg f_1 + \deg f_2 + \deg f_3.$$

Proof of Lemma 3.10. Assume that $F = (F_1, F_2, F_3)$ admits a reduction of type III. By the above considerations, the conditions of Definition 2.13 must be satisfied for the automorphism $\theta = (f_1, f_2, f_3) = (F_2, F_3, F_1)$. Also by Definition 2.13 there are $n \in \mathbb{N}^*$ and $\alpha, \beta, \gamma \in \mathbb{C}$, $(\alpha, \beta, \gamma) \neq (0, 0, 0)$, such that the elements $g_1 = f_1 - \beta f_3$, $g_2 = f_2 - \gamma f_3 - \alpha f_3^2$ satisfy the following conditions:

(i) g_1, g_2 is a 2-reduced pair and $\deg g_1 = 2n, \deg g_2 = 3n$;
(ii) there exists g_3 of the form $g_3 = \sigma f_3 + g$, where $\sigma \in \mathbb{C}^*$, $g \in \mathbb{C}[g_1, g_2]$, such that $\deg g_3 \leq \frac{3}{2}n, \deg\,[g_1, g_3] < 3n + \deg\,[g_1, g_2]$;
(iii) $\deg g_3 < n + \deg\,[g_1, g_2]$.

Let us notice that apart from $g_3 = \sigma f_3 + g$, we can also take $\widetilde{g}_3 = f_3 + \frac{1}{\sigma} g = f_3 + \widetilde{g}$, with $\widetilde{g} = \frac{1}{\sigma} g \in \mathbb{C}[g_1, g_2]$.

Since in our situation, i.e. $d_3/d_2 = 3/2$, we have $d_2 = 2n$, $d_3 = 3n$ and hence $\deg F_2 = \deg f_1 = 2n = \deg g_1$ and $\deg F_3 = \deg f_2 = 3n = \deg g_2$, the lemma above yields $\deg g_3 < \deg f_3 = \deg F_1 = d_1$. This means that the automorphism (g_1, g_2, f_3), and hence $\widetilde{F} = (F_1, g_1, g_2)$, admits an elementary reduction. Of course $\operatorname{mdeg}(F_1, g_1, g_2) = \operatorname{mdeg}(F_1, F_2, F_3)$.

Since $\widetilde{F} = T_2 \circ T_1 \circ F$, where the mappings

$$T_1 : \mathbb{C}^3 \ni \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} \mapsto \begin{Bmatrix} x \\ y - \beta x \\ z - \gamma x - \alpha x^2 \end{Bmatrix} \in \mathbb{C}^3$$

and

$$T_2 : \mathbb{C}^3 \ni \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} \mapsto \begin{Bmatrix} x + \widetilde{g}(y, z) \\ y \\ z \end{Bmatrix} \in \mathbb{C}^3$$

are triangular automorphisms, $\widetilde{F}$ is tame if and only if F is tame.

Since $\deg F_1 > 0$, also $\deg \widetilde{g} > 0$, and hence $\overline{\widetilde{g}} = \overline{\widetilde{g} - a}$ for all $a \in \mathbb{C}$. Thus we can assume that $\widetilde{g}(0,0) = 0$. Then $\widetilde{F}(0,0,0) = (0,0,0)$ when $F(0,0,0) = (0,0,0)$. ■

By Lemma 3.10 we also have the following result.

Proposition 3.12. *Let $(d_1, d_2, d_3) \neq (1,1,1)$, $d_1 \le d_2 \le d_3$, be a sequence of positive integers such that $d_3/d_2 = 3/2$. If there is a tame automorphism $F : \mathbb{C}^3 \to \mathbb{C}^3$ such that* $\operatorname{mdeg} F = (d_1, d_2, d_3)$, *then there is also a tame automorphism $\widetilde{F} : \mathbb{C}^3 \to \mathbb{C}^3$ such that $\widetilde{F}$ admits either a reduction of type IV or an elementary reduction and* $\operatorname{mdeg} \widetilde{F} = (d_1, d_2, d_3)$. *Moreover we can require that $\widetilde{F}(0,0,0) = (0,0,0)$.*

Proof. As in the proof of Proposition 3.7, we consider the automorphism $T \circ F$. Then we have three cases: (I) $T \circ F$ admits a reduction of type IV or an elementary reduction; (II) $T \circ F$ admits reduction of type III; (III) $T \circ F$ admits a reduction of type I or II. In the first case we put $\widetilde{F} = T \circ F$, in the second case we use Lemma 3.10 and in the third case we use Lemma 3.6. ■

The above proposition means that whenever $d_3/d_2 = 3/2$, reductions of type I, II and III are irrelevant for our considerations. More precisely, we have the following

Theorem 3.13. *Let $(d_1, d_2, d_3) \neq (1,1,1)$, $d_1 \le d_2 \le d_3$, be a sequence of positive integers such that $d_3/d_2 = 3/2$ or $3 \nmid d_1$. To prove that there is no tame automorphism of $\mathbb{C}^3$ with multidegree (d_1, d_2, d_3) it is enough to show that a (hypothetical) automorphism F of $\mathbb{C}^3$ with* $\operatorname{mdeg} F = (d_1, d_2, d_3)$ *admits neither a reduction of type IV nor an elementary reduction. Moreover, we can restrict our attention to automorphisms $F : \mathbb{C}^3 \to \mathbb{C}^3$ such that $F(0,0,0) = (0,0,0)$.*

Proof. Take any $\widetilde{F} \in \operatorname{Tame}(\mathbb{C}^3)$ with $\operatorname{mdeg} \widetilde{F} = (d_1, d_2, d_3)$. By Theorem 3.8 we can assume that $\widetilde{F}$ admits either an elementary reduction or a reduction of type III or IV.

If $\widetilde{F}$ admits a reduction of type III, then by Remark 3.9 and by the assumptions we have $d_3/d_2 = 3/2$. Thus we can use Proposition 3.12. ■

3.4. Reducibility of type IV and Kuroda's result. In the previous sections we have proved that from our point of view reductions of type I and II are irrelevant. The same is true for reductions of type III under an additional assumption (see Theorem 3.13).

The following result due to Kuroda says that reduction of type IV is also irrelevant for our aim.

THEOREM 3.14 ([26, Thm. 7.1]). *No tame automorphism of* $\mathbb{C}^3$ *admits a reduction of type IV.*

Thus we have the following

THEOREM 3.15. *Let* $(d_1, d_2, d_3) \neq (1,1,1)$, $d_1 \leq d_2 \leq d_3$, *be a sequence of positive integers. To prove that there is no tame automorphism* F *of* $\mathbb{C}^3$ *with* $\operatorname{mdeg} F = (d_1, d_2, d_3)$ *it is enough to show that a (hypothetical) automorphism* F *of* $\mathbb{C}^3$ *with* $\operatorname{mdeg} F = (d_1, d_2, d_3)$ *admits neither a reduction of type III nor an elementary reduction. Moreover, if we additionally assume that* $d_3/d_2 = 3/2$ *or* $3 \nmid d_1$, *then it is enough to show that no (hypothetical) automorphism of* $\mathbb{C}^3$ *with multidegree* (d_1, d_2, d_3) *admits an elementary reduction. In both cases we can restrict our attention to automorphisms* $F : \mathbb{C}^3 \to \mathbb{C}^3$ *such that* $F(0,0,0) = (0,0,0)$.

Proof. The proof is similar to the proof of Theorem 3.13. ∎

3.5. Reducibility and linear change of coordinates. Now we make some remarks that will be useful in considerations of some special cases. The main result of this section says that we can restrict our attention to automorphisms whose linear part is the identity map.

LEMMA 3.16. *If an automorphism* (F_1, F_2, F_3) *admits an elementary reduction, then so does* $(F_1, F_2, F_3) \circ L$ *for every* $L \in GL_3(\mathbb{C})$.

Proof. Without loss of generality we can assume that (F_1, F_2, F_3) admits an elementary reduction of the form $(F_1 - G(F_2, F_3), F_2, F_3)$. It is easy to see that $(F_1 \circ L - G(F_2 \circ L, F_3 \circ L), F_2 \circ L, F_3 \circ L) = (F_1 - G(F_2, F_3), F_2, F_3) \circ L$ is an elementary reduction of $(F_1, F_2, F_3) \circ L = (F_1 \circ L, F_2 \circ L, F_3 \circ L)$. ∎

We also have the following obvious lemma.

LEMMA 3.17. *For every mapping* $F : \mathbb{C}^n \to \mathbb{C}^n$ *and every* $L \in GL_n(\mathbb{C})$ *we have*

$$\operatorname{mdeg}(F \circ L) = \operatorname{mdeg} F.$$

Combining the above two lemmas we obtain the following result.

THEOREM 3.18. *For every sequence of positive integers* $(d_1, \ldots, d_n) \neq (1, \ldots, 1)$, *if there is a tame automorphism* $F : \mathbb{C}^n \to \mathbb{C}^n$ *such that* F *admits an elementary reduction,* $F(0, \ldots, 0) = (0, \ldots, 0)$ *and* $\operatorname{mdeg} F = (d_1, \ldots, d_n)$, *then there is also a tame automorphism* $\widetilde{F} : \mathbb{C}^n \to \mathbb{C}^n$ *such that* $\widetilde{F}$ *admits an elementary reduction,* $\operatorname{mdeg} \widetilde{F} = (d_1, \ldots, d_n)$, $\widetilde{F}(0, \ldots, 0) = (0, \ldots, 0)$ *and the linear part of* $\widetilde{F}$, *is equal to* $\operatorname{id}_{\mathbb{C}^n}$.

Proof. Let L be the linear part of F. Since $F \in \operatorname{Aut}(\mathbb{C}^n)$, we have $L \in GL_n(\mathbb{C})$. The linear part of $F \circ L^{-1}$ is equal to $\operatorname{id}_{\mathbb{C}^n}$. We also have $(F \circ L^{-1})(0, \ldots, 0) = F(0, \ldots, 0) = (0, \ldots, 0)$. ∎

3.6. Relationship between the degree of the Poisson bracket and the number of variables. The main result of this section is Lemma 3.20 below. We start with the following

LEMMA 3.19. *Let $f, g \in \mathbb{C}[X_1, \dots, X_n]$ be such that*

$$f = X_1 + f_2 + \cdots + f_l, \quad g = X_2 + g_2 + \cdots + g_m,$$

where f_i, g_i are homogeneous forms of degree i. If $\deg [f, g] = 2$ and f does not involve X_i, where $i > 2$, then g does not involve X_i either.

Proof. The assumption $\deg [f, g] = 2$ implies that for all $1 \leq k < l \leq n$ we have

$$\deg \operatorname{Jac}^{X_k X_l}(f, g) \leq 0.$$

In particular,

$$\deg \operatorname{Jac}^{X_1 X_i}(f, g) \leq 0,$$

but

$$\operatorname{Jac}^{X_1 X_i}(f, g) = \frac{\partial f}{\partial X_1}\frac{\partial g}{\partial X_i} - \frac{\partial f}{\partial X_i}\frac{\partial g}{\partial X_1} = \frac{\partial f}{\partial X_1}\frac{\partial g}{\partial X_i}.$$

Thus $\deg \frac{\partial g}{\partial X_i} \leq 0$. In other words if g involves X_i then X_i occurs in the linear part of g. But this contradicts the assumptions. ∎

Now we are in a position to prove the following lemma that is one of the main ingredients in proving, for instance, that $(5, 6, 9) \notin \operatorname{mdeg}(\operatorname{Tame}(\mathbb{C}^3))$.

LEMMA 3.20. *Let $f, g \in \mathbb{C}[X_1, \dots, X_n]$ be such that*

$$f = X_1 + f_2 + \cdots + f_l, \quad g = X_2 + g_2 + \cdots + g_m,$$

where f_i, g_i are homogeneous forms of degree i. If $\deg [f, g] = 2$, then $f, g \in \mathbb{C}[X_1, X_2]$.

Proof. Without loss of generality we can assume that $l \leq m$. Let $i > 2$ be arbitrary. Let us notice that

$$[\operatorname{Jac}^{X_1 X_i}(f, g)]_1 = \operatorname{Jac}^{X_1 X_i}(X_1, g_2) + \operatorname{Jac}^{X_1 X_i}(f_2, X_2) = \frac{\partial g_2}{\partial X_i}$$

and

$$[\operatorname{Jac}^{X_2 X_i}(f, g)]_1 = \operatorname{Jac}^{X_2 X_i}(X_1, g_2) + \operatorname{Jac}^{X_2 X_i}(f_2, X_2) = -\frac{\partial f_2}{\partial X_i},$$

where $[\operatorname{Jac}^{X_k X_l}(f, g)]_d$ is the homogeneous part of degree d of $\operatorname{Jac}^{X_k X_l}(f, g)$. But the assumption $\deg [f, g] = 2$ means in particular that $[\operatorname{Jac}^{X_1 X_i}(f, g)]_1 = 0$ and $[\operatorname{Jac}^{X_2 X_i}(f, g)]_1 = 0$. Thus we obtain

$$\frac{\partial g_2}{\partial X_i} = 0, \quad \frac{\partial f_2}{\partial X_i} = 0,$$

and so f_2, g_2 do not involve X_i. It follows that

$$\begin{aligned}[\operatorname{Jac}^{X_1 X_i}(f, g)]_2 &= \operatorname{Jac}^{X_1 X_i}(X_1, g_3) + \operatorname{Jac}^{X_1 X_i}(f_2, g_2) + \operatorname{Jac}^{X_1 X_i}(f_3, X_2) \\ &= \operatorname{Jac}^{X_1 X_i}(X_1, g_3) = \frac{\partial g_3}{\partial X_i}\end{aligned}$$

and

$$[\mathrm{Jac}^{X_2X_i}(f,g)]_2 = \mathrm{Jac}^{X_2X_i}(X_1,g_3) + \mathrm{Jac}^{X_2x_i}(f_2,g_2) + \mathrm{Jac}^{X_2X_i}(f_3,X_2)$$
$$= \mathrm{Jac}^{X_2X_i}(f_3,X_2) = -\frac{\partial f_3}{\partial X_i}.$$

Since $\deg[f,g]=2$ implies $[\mathrm{Jac}^{x_1x_i}(f,g)]_2=0$ and $[\mathrm{Jac}^{x_2x_i}(f,g)]_2=0$, we see that

$$\frac{\partial g_3}{\partial X_i}=0, \quad \frac{\partial f_3}{\partial X_i}=0,$$

and so f_3, g_3 do not involve X_i.

Proceeding inductively, when we know that $f_2,\dots,f_{l-1},g_2,\dots,g_{l-1}$ do not involve X_i, we obtain

$$[\mathrm{Jac}^{X_1X_i}(f,g)]_{n-1} = \mathrm{Jac}^{X_1X_i}(X_1,g_n) + \dots + \mathrm{Jac}^{X_1X_i}(f_n,X_2)$$
$$= \mathrm{Jac}^{X_1X_i}(X_1,g_n) = \frac{\partial g_n}{\partial X_i}$$

and

$$[\mathrm{Jac}^{X_2X_i}(f,g)]_{n-1} = \mathrm{Jac}^{X_2X_i}(X_1,g_n) + \dots + \mathrm{Jac}^{X_2X_i}(f_n,X_2)$$
$$= \mathrm{Jac}^{X_2X_i}(f_n,X_2) = -\frac{\partial f_n}{\partial X_i}.$$

By the assumption $\deg[f,g]=2$, as before we find that f_n and g_n do not involve X_i. Therefore f does not involve X_i. To deduce that g does not involve X_i either, we can use Lemma 3.19. ∎

By similar arguments one can prove the following

THEOREM 3.21. *Let $f,g\in\mathbb{C}[X_1,\dots,X_n]$ be such that*

$$f = X_1 + f_2 + \dots + f_l, \quad g = X_2 + g_2 + \dots + g_m,$$

where f_i, g_i are homogeneous forms of degree i. If $\deg[f,g]=d\le\min\{l,m\}$, $d\ge 2$, and f_i, g_i, for $i=1,\dots,d-1$, do not involve X_r, where $r>2$, then f and g do not involve X_r.

The results of Lemma 3.20 and Theorem 3.21 can be generalized as follows.

THEOREM 3.22. *Let $f,g\in\mathbb{C}[X_1,\dots,X_n]$ be such that*

$$f = f_1 + f_2 + \dots + f_l, \quad g = g_1 + g_2 + \dots g_m,$$

where f_i, g_i are homogeneous forms of degree i. If f_1, g_1 are linearly independent, $\deg[f,g]$ $= d\le\min\{l,m\}$, *$d\ge 2$, and f_i, g_i, for $i=1,\dots,d-1$, do not involve X_r, where $r>2$, then f and g do not involve X_r.*

Proof. Let $l_3,\dots,l_{n-1}\in\mathbb{C}[X_1,\dots,X_{r-1},X_{r+1},\dots,X_n]$ be linear forms such that $f_1,g_1,$ $l_3,\dots,l_{n-1}$ are linearly independent. Then $f_1,g_1,l_3,\dots,l_{n-1},X_r$ are also linearly independent. Let $L=(f_1,g_1,l_3,\dots,l_{n-1},X_r):\mathbb{C}^n\to\mathbb{C}^n$. Of course $L,L^{-1}\in GL_n(\mathbb{C})$, and by Lemma 2.8, $\deg[f\circ L^{-1},g\circ L^{-1}]=\deg[f,g]=d$. One can also check that $(f\circ L^{-1})_1=X_1,(g\circ L^{-1})_1=X_2$ and that $(f\circ L^{-1})_i,(g\circ L^{-1})_i$, for $i=1,\dots,d-1$, do not involve X_r. Thus by Theorem 3.21, $f\circ L^{-1}, g\circ L^{-1}$ do not involve X_r either. And one can easily check that the same is true for $f=(f\circ L^{-1})\circ L$ and $g=(g\circ L^{-1})\circ L$. ∎

4. The case (p_1, p_2, d_3) and its generalization

4.1. The case (p_1, p_2, d_3). Here we investigate the set

$$\{(p_1, p_2, d_3) : 3 \le p_1 < p_2 \le d_3,\ p_1, p_2 \text{ prime numbers } \} \cap \operatorname{mdeg}(\operatorname{Tame}(\mathbb{C}^3)).$$

The complete description of this set is given in the following theorem.

THEOREM 4.1 ([19, Thm. 1.1]). *Let $d_3 \ge p_2 > p_1 \ge 3$ be integers. If p_1 and p_2 are primes, then $(p_1, p_2, d_3) \in \operatorname{mdeg}(\operatorname{Tame}(\mathbb{C}^3))$ if and only if $d_3 \in p_1\mathbb{N} + p_2\mathbb{N}$.*

Proof. If $d_3 \in p_1\mathbb{N}+p_2\mathbb{N}$, then by Proposition 3.2, there exists a tame automorphism $F \in \operatorname{Tame}(\mathbb{C}^3)$ such that $\operatorname{mdeg} F = (p_1, p_2, d_3)$. Conversely, let $d_3 \notin p_1\mathbb{N}+p_2\mathbb{N}$ and assume, to the contrary, that there are tame automorphisms F of $\mathbb{C}^3$ such that $\operatorname{mdeg} F = (p_1, p_2, d_3)$. By Theorem 3.15, we only need to show that such automorphisms do not admit an elementary reduction or a reduction of type III. Since $p_2 > 3$ is a prime, $2 \nmid p_2$. Hence by Remark 3.9, no automorphism F of $\mathbb{C}^3$ with $\operatorname{mdeg} F = (p_1, p_2, d_3)$ admits a reduction of type III.

Assume, to the contrary, that there is an automorphism $F = (F_1, F_2, F_3)$ of $\mathbb{C}^3$ with $\operatorname{mdeg} F = (p_1, p_2, d_3)$ that admits an elementary reduction. Notice that, by Theorem 2.15,

$$d_3 < (p_1 - 1)(p_2 - 1). \tag{4.1}$$

Assume that

$$(F_1, F_2, F_3 - g(F_1, F_2)),$$

where $g \in \mathbb{C}[X, Y]$, is an elementary reduction of (F_1, F_2, F_3). Then we have $\deg g(F_1, F_2) = \deg F_3 = d_3$. But, by Proposition 2.7,

$$\deg g(F_1, F_2) \ge q(p_1 p_2 - p_1 - p_2 + \deg [F_1, F_2]) + r p_2,$$

where $\deg_Y g(X, Y) = q p_1 + r$ with $0 \le r < p_1$. Since F_1, F_2 are algebraically independent, $\deg [F_1, F_2] \ge 2$ and so

$$p_1 p_2 - p_1 - p_2 + \deg [F_1, F_2] \ge p_1 p_2 - p_1 - p_2 + 2 > (p_1 - 1)(p_2 - 1).$$

This and (4.1) imply that $q = 0$, and that

$$g(X, Y) = \sum_{i=0}^{p_1 - 1} g_i(X) Y^i.$$

Since $\operatorname{lcm}(p_1, p_2) = p_1 p_2$, the sets

$$p_1\mathbb{N}, p_2 + p_1\mathbb{N}, \ldots, (p_1 - 1)p_2 + p_1\mathbb{N}$$

are pairwise disjoint. This yields

$$\deg\Big(\sum_{i=0}^{p_1 - 1} g_i(F_1) F_2^i\Big) = \max_{i=0,\ldots,p_1-1} (\deg F_1 \deg g_i + i \deg F_2),$$

and so

$$d_3 = \deg g(F_1, F_2) \in \bigcup_{r=0}^{p_1 - 1} (r p_2 + p_1\mathbb{N}) \subset p_1\mathbb{N} + p_2\mathbb{N},$$

a contradiction.

Now, assume that

$$(F_1, F_2 - g(F_1, F_3), F_3)$$

is an elementary reduction of $F = (F_1, F_2, F_3)$. Since $d_3 \notin p_1\mathbb{N} + p_2\mathbb{N}$, we have $p_1 \nmid d_3$ and $\gcd(p_1, d_3) = 1$. This means, by Proposition 2.7, that

$$\deg g(F_1, F_3) \geq q(p_1 d_3 - d_3 - p_1 + \deg [F_1, F_3]) + r d_3,$$

where $\deg_Y g(X, Y) = qp_1 + r$ with $0 \leq r < p_1$. Since $p_1 d_3 - d_3 - p_1 + \deg [F_1, F_3] \geq p_1 d_3 - 2d_3 \geq d_3 > p_2$ and since we want to have $\deg g(F_1, F_3) = p_2$, we conclude that $q = r = 0$. This means that $g(X, Y) = g(X)$, and so $p_2 = \deg g(F_1) \in p_1\mathbb{N}$, a contradiction.

Finally, if we assume that $(F_1 - g(F_2, F_3), F_2, F_3)$ is an elementary reduction of (F_1, F_2, F_3), then we obtain a contradiction in the same way as in the previous case. ∎

Corollary 4.2. *We have*

$$\{(p_1, p_2, d_3) : 3 \leq p_1 < p_2 \leq d_3,\ p_1, p_2 \text{ primes } \} \cap \operatorname{mdeg}(\operatorname{Tame}(\mathbb{C}^3))$$
$$= \{(p_1, p_2, d_3) : 3 \leq p_1 < p_2 \leq d_3,\ p_1, p_2 \text{ primes},\ d_3 \in p_1\mathbb{N} + p_2\mathbb{N}\}.$$

4.2. Some consequences

Theorem 4.3 ([19, Thm. 3.1]). *Let $p_2 > 3$ be a prime and $d_3 \geq p_2$ be an integer. Then $(3, p_2, d_3) \in \operatorname{mdeg}(\operatorname{Tame}(\mathbb{C}^3))$ if and only if $d_3 \notin \{2p_2 - 3k : k = 1, \ldots, [p_2/3]\}$.*

Proof. Since $p_2 > 3$ is a prime, $p_2 \equiv r \pmod 3$ for some $r \in \{1, 2\}$. It is easy to see that if $d_3 \geq p_2$ and $d_3 \equiv 0 \pmod 3$ or $d_3 \equiv r \pmod 3$, then $d_3 \in 3\mathbb{N} + p_2\mathbb{N}$. Thus, by Theorem 2.15,

$$2(p_2 - 1) - 1 \neq 0, r \pmod 3.$$

Take any d_3 such that $p_2 \leq d_3 \leq 2p_2 - 3$ and $d_3 \neq 0, r \pmod 3$. Since $d_3 \leq 2p_2 - 3$ and $d_3 \equiv 2p_2 - 3 \pmod 3$, we see that $d_3 \notin 3\mathbb{N} + p_2\mathbb{N}$, because otherwise we would have $2p_2 - 3 \in 3\mathbb{N} + p_2\mathbb{N}$, contrary to Theorem 2.15. Thus

$$\{d_3 \in \mathbb{N} \mid d_3 \geq p_2, d_3 \notin 3\mathbb{N} + p_2\mathbb{N}\} = \{d_3 \in \mathbb{N} \mid p_2 \leq d_3 \leq 2p_2 - 3, d_3 \equiv 2p_2 - 3 \pmod 3\}$$
$$= \{2p_2 - 3k \mid k = 1, \ldots, [p_2/3]\}. \ ∎$$

Theorem 4.4 ([19, Thm. 3.2]).

(a) *If $d_3 \geq 7$, then $(5, 7, d_3) \in \operatorname{mdeg}(\operatorname{Tame}(\mathbb{C}^3))$ if and only if*

$$d_3 \neq 8, 9, 11, 13, 16, 18, 23.$$

(b) *If $d_3 \geq 11$, then $(5, 11, d_3) \in \operatorname{mdeg}(\operatorname{Tame}(\mathbb{C}^3))$ if and only if*

$$d_3 \neq 12, 13, 14, 17, 18, 19, 23, 24, 28, 29, 34, 39.$$

(c) *If $d_3 \geq 13$, then $(5, 13, d_3) \in \operatorname{mdeg}(\operatorname{Tame}(\mathbb{C}^3))$ if and only if*

$$d_3 \neq 14, 16, 17, 19, 21, 22, 24, 27, 29, 32, 34, 37, 42, 47.$$

(d) *If $d_3 \geq 11$, then $(7, 11, d_3) \in \operatorname{mdeg}(\operatorname{Tame}(\mathbb{C}^3))$ if and only if*

$$d_3 \neq 12, 13, 15, 16, 17, 19, 20, 23, 24, 26, 27, 30, 31, 34, 37, 38, 41, 45, 48, 52, 59.$$

Proof. This is a consequence of Theorems 2.15 and 4.1. For example to prove (a), by Theorems 2.15 and 4.1 we only have to check which numbers among $7, 8, \ldots, 23 = (5-1)(7-1)-1$ are elements of the set $5\mathbb{N}+7\mathbb{N}$. ■

4.3. Generalization. Here we generalize Theorem 4.1.

THEOREM 4.5 ([22, Thm. 2.1]). *Let $d_3 \geq d_2 > d_1 \geq 3$ be integers. If d_1 and d_2 are odd and* $\gcd(d_1, d_2) = 1$, *then* $(d_1, d_2, d_3) \in \operatorname{mdeg}(\operatorname{Tame}(\mathbb{C}^3))$ *if and only if* $d_3 \in d_1\mathbb{N} + d_2\mathbb{N}$.

Proof. The proof is a modification of the proof of Theorem 4.1. As before, if $d_3 \in d_1\mathbb{N} + d_2\mathbb{N}$, then by Proposition 3.2, there is a tame automorphism F of $\mathbb{C}^3$ such that $\operatorname{mdeg} F = (d_1, d_2, d_3)$.

Moreover, as in the proof of Theorem 4.1, we only need to show that no automorphism F of $\mathbb{C}^3$ with $\operatorname{mdeg} F = (d_1, d_2, d_3)$ admits an elementary reduction when $d_3 \notin d_1\mathbb{N}+d_2\mathbb{N}$. As before, suppose otherwise.

If we assume that $(F_1, F_2, F_3 - g(F_1, F_2))$, where $g \in \mathbb{C}[X, Y]$, is an elementary reduction of (F_1, F_2, F_3), then we can proceed exactly in the same way as in the proof of Theorem 4.1.

Assume that $(F_1, F_2 - g(F_1, F_3), F_3)$ is an elementary reduction of (F_1, F_2, F_3). Since $d_3 \notin d_1\mathbb{N} + d_2\mathbb{N}$, we have $d_1 \nmid d_3$, so

$$p = \frac{d_1}{\gcd(d_1, d_3)} > 1.$$

Since d_1, is odd, we also have $p \neq 2$. Thus by Proposition 2.7,

$$\deg g(F_1, F_3) \geq q(pd_3 - d_3 - d_1 + \deg [F_1, F_3]) + rd_3,$$

where $\deg_Y g(X, Y) = qp + r$ with $0 \leq r < p$. Since $p \geq 3$, we see that $pd_3 - d_3 - d_1 + \deg [F_1, F_3] \geq 2d_3 - d_1 + 2 > d_3$. Since we want to have $\deg g(F_1, F_3) = d_2$, it follows that $q = r = 0$, and hence $g(X, Y) = g(X)$. This means that $d_2 = \deg g(F_1) \in d_1\mathbb{N}$, contradicting $\gcd(d_1, d_2) = 1$ and $1 < d_1$.

Finally, if we assume that $(F_1 - g(F_2, F_3), F_2, F_3)$ is an elementary reduction of (F_1, F_2, F_3), then we obtain a contradiction as in the previous case. ■

COROLLARY 4.6. *We have*

$$\begin{aligned}\{(d_1, d_2, d_3) : d_1 \leq d_2 \leq d_3,\ d_1, d_2 \text{ odd and } \gcd(d_1, d_2) = 1\} \cap \operatorname{mdeg}(\operatorname{Tame}(\mathbb{C}^3)) \\ = \{(d_1, d_2, d_3) : d_1 \leq d_2 \leq d_3,\ d_1, d_2 \text{ odd and } \gcd(d_1, d_2) = 1,\ d_3 \in d_1\mathbb{N} + d_2\mathbb{N}\}.\end{aligned}$$

4.4. The set $\operatorname{mdeg}(\operatorname{Aut}(\mathbb{C}^3)) \setminus \operatorname{mdeg}(\operatorname{Tame}(\mathbb{C}^3))$**.** In this subsection we say a few words about relations between $\operatorname{mdeg}(\operatorname{Tame}(\mathbb{C}^3))$ and $\operatorname{mdeg}(\operatorname{Aut}(\mathbb{C}^3))$. Obviously,

$$\operatorname{mdeg}(\operatorname{Tame}(\mathbb{C}^3)) \subset \operatorname{mdeg}(\operatorname{Aut}(\mathbb{C}^3))$$

and, more generally,

$$\operatorname{mdeg}(\operatorname{Tame}(\mathbb{C}^n)) \subset \operatorname{mdeg}(\operatorname{Aut}(\mathbb{C}^n)).$$

The question is whether the inclusion is strict. In dimension two the answer is negative due to Jung [9] and van der Kulk [23]. Namely we have

$$\operatorname{mdeg}(\operatorname{Tame}(\mathbb{C}^2)) = \operatorname{mdeg}(\operatorname{Aut}(\mathbb{C}^2)) = \{(d_1, d_2) : d_1 \mid d_2 \text{ or } d_2 \mid d_1\}.$$

Let us notice that the result of Shestakov and Umirbaev [50] about wildness of Nagata's example does not imply a positive answer in dimension three. The problem is that Nagata's example is of multidegree $(5,3,1) \in \operatorname{mdeg}(\operatorname{Tame}(\mathbb{C}^3))$. In spite of that, the answer is positive. Actually we will show that $\operatorname{mdeg}(\operatorname{Aut}(\mathbb{C}^3)) \setminus \operatorname{mdeg}(\operatorname{Tame}(\mathbb{C}^3))$ has infinitely many elements.

Let

$$N : \mathbb{C}^3 \ni (x,y,z) \mapsto (x + 2y(y^2+zx) - z(y^2+zx)^2, y - z(y^2+zx), z) \in \mathbb{C}^3$$

be Nagata's example and let

$$T : \mathbb{C}^3 \ni (x,y,z) \mapsto (z,y,x) \in \mathbb{C}^3.$$

We start with the following lemma.

LEMMA 4.7 ([22, Lem. 3.1]). *For all* $n \in \mathbb{N}$ *we have* $\operatorname{mdeg}((T\circ N)^n) = (4n-3, 4n-1, 4n+1)$.

Proof. We have $T \circ N(x,y,z) = (z, y - z(y^2+zx), x + 2y(y^2+zx) - z(y^2+zx)^2)$, so the assertion is true for $n = 1$. Let $(f_n, g_n, h_n) = (T\circ N)^n$ for $f_n, g_n, h_n \in \mathbb{C}[X,Y,Z]$. One can see that $g_1^2 + h_1 f_1 = Y^2 + ZX$, and by induction $g_n^2 + h_n f_n = Y^2 + ZX$ for any $n \in \mathbb{N}^*$. Thus

$$\begin{aligned}(f_{n+1}, g_{n+1}, h_{n+1}) &= (T\circ N) \circ (f_n, g_n, h_n)\\ &= (h_n, g_n - h_n(g_n^2 + h_n f_n), f_n + 2g_n(g_n^2 + h_n f_n) - h_n(g_n^2+h_nf_n)^2)\\ &= (h_n, g_n - h_n(Y^2+ZX), f_n + 2g_n(Y^2+ZX) - h_n(Y^2+ZX)^2).\end{aligned}$$

So if we assume that $\operatorname{mdeg}(f_n, g_n, h_n) = (4n-3, 4n-1, 4n+1)$, we obtain

$$\begin{aligned}\operatorname{mdeg}(f_{n+1}, g_{n+1}, h_{n+1}) &= (4n+1, (4n+1)+2, (4n+1)+2\cdot 2)\\ &= (4(n+1)-3, 4(n+1)-1, 4(n+1)+1). \ \blacksquare\end{aligned}$$

By the above lemma and Theorem 4.5 we obtain the following

THEOREM 4.8 ([22, Thm. 3.2]). *For every* $n \in \mathbb{N}$ *the automorphism* $(T\circ N)^n$ *is wild.*

Proof. For $n = 1$ this is the result of Shestakov and Umirbaev [49, 50]. So assume that $n \geq 2$. The numbers $4n-3, 4n-1$ are odd and $\gcd(4n-3, 4n-1) = \gcd(4n-3, 2) = 1$. Since $4n-3 > 2$, we see that $4n+1 \notin (4n-3)\mathbb{N} + (4n-1)\mathbb{N}$. Hence, by Theorem 4.5, $(4n-3, 4n-1, 4n+1) \notin \operatorname{mdeg}(\operatorname{Tame}(\mathbb{C}^3))$ for $n > 1$. This proves that $(T\circ N)^n$ is not a tame automorphism. $\blacksquare$

Let us notice that we have also proved that

$$\{(4n-3, 4n-1, 4n+1) : n \in \mathbb{N}, n \geq 2\} \subset \operatorname{mdeg}(\operatorname{Aut}(\mathbb{C}^3)) \setminus \operatorname{mdeg}(\operatorname{Tame}(\mathbb{C}^3)).$$

This gives the following result.

THEOREM 4.9 ([22, Thm. 1.1]). *The set* $\operatorname{mdeg}(\operatorname{Aut}(\mathbb{C}^3)) \setminus \operatorname{mdeg}(\operatorname{Tame}(\mathbb{C}^3))$ *is infinite.*

5. The case $(3, d_2, d_3)$

In this section we give a complete description of the set

$$\{(3, d_2, d_3) : 3 \le d_2 \le d_3\} \cap \operatorname{mdeg}(\operatorname{Tame}(\mathbb{C}^3)).$$

This description is given by the following

THEOREM 5.1 ([20, Thm. 1.1]). *If* $3 \le d_2 \le d_3$, *then* $(3, d_2, d_3) \in \operatorname{mdeg}(\operatorname{Tame}(\mathbb{C}^3))$ *if and only if* $3 \mid d_2$ *or* $d_3 \in 3\mathbb{N} + d_2\mathbb{N}$.

Proof. By Corollary 3.2, if $3 \mid d_2$ or $d_3 \in 3\mathbb{N} + d_2\mathbb{N}$, there exists a tame automorphism $F : \mathbb{C}^3 \to \mathbb{C}^3$ such that $\operatorname{mdeg} F = (3, d_2, d_3)$. Conversely, assume that $3 \nmid d_2$ and $d_3 \notin 3\mathbb{N} + d_2\mathbb{N}$.

Since $3 \nmid d_2$, we have $\gcd(3, d_2) = 1$. Hence Theorem 2.15 implies that for all $k \ge (3-1)(d_2-1) = 2d_2 - 2$ we have $k \in 3\mathbb{N} + d_2\mathbb{N}$. Thus, since $d_3 \notin 3\mathbb{N} + d_2\mathbb{N}$, we have

$$d_3 < 2d_2 - 2. \tag{5.1}$$

By Theorem 3.15 it is enough to show that automorphisms F of $\mathbb{C}^3$ with $\operatorname{mdeg} F = (3, d_2, d_3)$ do not admit an elementary reduction or a reduction of type III. Notice also that, since $d_1 = 3$ and d_2 can be even, we cannot use Remark 3.9 to infer that automorphisms F of $\mathbb{C}^3$ with $\operatorname{mdeg} F = (3, d_2, d_3)$ do not admit a reduction of type III.

Assume that an automorphism $F = (F_1, F_2, F_3) : \mathbb{C}^3 \to \mathbb{C}^3$ with $\operatorname{mdeg} F = (3, d_2, d_3)$ admits a reduction of type III. Then by Definition 2.13 there is a permutation σ of $\{1, 2, 3\}$ and $n \in \mathbb{N}^*$ such that $\deg F_{\sigma(1)} = 2n$, and either

$$\deg F_{\sigma(2)} = 3n, \quad n < \deg F_{\sigma(3)} \le 3n/2, \tag{5.2}$$

or

$$5n/2 < \deg F_{\sigma(2)} \le 3n, \quad \deg F_{\sigma(3)} = 3n/2. \tag{5.3}$$

Since $\frac{3}{2}n < 2n < \min\{\frac{5}{2}n, 3n\}$, we have $d_2 = 2n$ and either

$$d_3 = 3n, \quad n < 3 \le 3n/2,$$

or

$$5n/2 < d_3 \le 3n, \quad 3 = 3n/2.$$

Thus $n = 2$ and so $5 < d_3 \le 6$, that is, $d_3 = 6$. This contradicts $d_3 \notin 3\mathbb{N} + d_2\mathbb{N}$.

Now, assume that $(F_1, F_2, F_3 - g(F_1, F_2))$, where $g \in \mathbb{C}[X, Y]$, is an elementary reduction of (F_1, F_2, F_3). Then $\deg g(F_1, F_2) = \deg F_3 = d_3$. Since $\gcd(3, d_2) = 1$, by Proposition 2.7 we have

$$\deg g(F_1, F_2) \ge q(3d_2 - d_2 - 3 + \deg [F_1, F_2]) + rd_2,$$

where $\deg_Y g(X, Y) = 3q + r$ with $0 \le r < 3$. Since F_1, F_2 are algebraically independent, $\deg [F_1, F_2] \ge 2$ and so $3d_2 - d_2 - 3 + \deg [F_1, F_2] \ge 2d_2 - 1$. Then (5.1) implies $q = 0$. Also by (5.1) we must have $r < 2$. Thus $g(X, Y) = g_0(X) + g_1(X)Y$. Since $3\mathbb{N} \cap (d_2 + 3\mathbb{N}) = \emptyset$, we deduce that $\deg g(F_1, F_2) \in 3\mathbb{N} \cup (d_2 + 3\mathbb{N}) \subset 3\mathbb{N} + d_2\mathbb{N}$, contrary to assumption.

Now, assume that $(F_1, F_2 - g(F_1, F_3), F_3)$ is an elementary reduction of (F_1, F_2, F_3). Then $\deg g(F_1, F_3) = d_2$. Since $d_3 \notin 3\mathbb{N} + d_2\mathbb{N}$, it follows that $\gcd(3, d_3) = 1$. Then by

Proposition 2.7 we have

$$\deg g(F_1, F_3) \geq q(3d_3 - d_3 - 3 + \deg[F_1, F_3]) + rd_3,$$

where $\deg_Y g(X,Y) = 3q+r$ with $0 \leq r < 3$. Since $3d_3-d_3-3+\deg[F_1,F_3] \geq 2d_3-1 > d_2$, we infer that $q = 0$. Since also $d_3 > d_2$ (because $d_3 \geq d_2$ and $d_3 \notin 3\mathbb{N} + d_2\mathbb{N}$), we see that $r = 0$. Thus $g(X,Y) = g(X)$, and $\deg g(F_1, F_3) = \deg g(F_1) \in 3\mathbb{N}$, a contradiction.

Finally, assume that $(F_1 - g(F_2, F_3), F_2, F_3)$ is an elementary reduction of (F_1, F_2, F_3). Then $\deg g(F_2, F_3) = 3$. Let

$$p = \frac{d_2}{\gcd(d_2, d_3)}.$$

Since $d_3 \notin 3\mathbb{N} + d_2\mathbb{N}$, we obtain $d_2 \nmid d_3$, and hence $p > 1$. By Proposition 2.7,

$$\deg g(F_2, F_3) \geq q(pd_3 - d_2 - d_3 + \deg[F_1, F_3]) + rd_3,$$

where $\deg_Y g(X,Y) = qp+r$ with $0 \leq r < p$. Since $d_3 > 3$, it follows that $r = 0$. Consider the case $p \geq 3$. Then $pd_3 - d_2 - d_3 + \deg[F_1, F_3] \geq d_3 + \deg[F_1, F_3] > 3$. Thus we must have $q = 0$. Hence $g(X,Y) = g(X)$, and $3 = \deg g(F_2, F_3) = \deg g(F_2) \in d_2\mathbb{N}$. This contradicts $d_2 \neq 3$ (we have assumed that $3 \nmid d_2$).

Consider now the case $p = 2$. Since $p = 2$, we have, for some $n \in \mathbb{N}$, $d_2 = 2n$ and $d_3 = ns$, where $s \geq 3$ is odd. Since also $d_2 > 3$, it follows that $n \geq 2$. This means that $d_3 - d_2 \geq 2$, and $2d_3 - d_3 - d_2 + \deg[F_1, F_3] = d_3 - d_2 + \deg[F_1, F_3] \geq 4 > 3$. Thus, also in this case we have $q = 0$. As before this leads to a contradiction. ∎

COROLLARY 5.2. *We have*

$$\{(3, d_2, d_3) : 3 \leq d_2 \leq d_3\} \cap \operatorname{mdeg}(\operatorname{Tame}(\mathbb{C}^3)) = \{(3, d_2, d_3) : 3 \leq d_2 \leq d_3,\ 3 \mid d_2 \text{ or } d_3 \in 3\mathbb{N} + d_2\mathbb{N}\}.$$

6. The case $(4, d_2, d_3)$

In this section we give a partial description of the set

$$\{(4, d_2, d_3) : 4 \leq d_2 \leq d_3\} \cap \operatorname{mdeg}(\operatorname{Tame}(\mathbb{C}^3)).$$

This description will be given separately for four cases: (I) d_2, d_3 both even, (II) d_2, d_3 both odd, (III) d_2 even and d_3 odd, (IV) d_2 odd and d_3 even.

6.1. The case $(4, \text{even}, \text{even})$**.** This is the easiest case, summarised as follows.

THEOREM 6.1. *For all even numbers* $d_3 \geq d_2 \geq 4$, $(4, d_2, d_3) \in \operatorname{mdeg}(\operatorname{Tame}(\mathbb{C}^3))$.

Proof. Since all numbers $4, d_2, d_3$ are even, we have $\gcd(4, d_2, d_3) \in \{2, 4\}$. Thus $4/\gcd(4, d_2, d_3) \leq 2$ and we can use Theorem 3.4. ∎

6.2. The case $(4, \text{odd}, \text{odd})$**.** In this subsection we give a complete description of the set

$$\{(4, d_2, d_3) : 4 \leq d_2 \leq d_3,\ d_2, d_3 \in 2\mathbb{N} + 1\} \cap \operatorname{mdeg}(\operatorname{Tame}(\mathbb{C}^3)).$$

We will show the following

Theorem 6.2. *Let $d_3 \geq d_2 \geq 4$ be odd numbers. Then $(4, d_2, d_3) \in \operatorname{mdeg}(\operatorname{Tame}(\mathbb{C}^3))$ if and only if $d_3 \in 4\mathbb{N} + d_2\mathbb{N}$.*

Proof. By Proposition 3.2 it is enough to show the "only if" part. Thus, assume that $d_3 \notin 4\mathbb{N} + d_2\mathbb{N}$. Since d_2 is odd, we have $\gcd(4, d_2) = 1$, and so, by Theorem 2.15,

$$d_3 < (4-1)(d_2-1) = 3d_2 - 3. \tag{6.1}$$

By Remark 3.9 and Theorem 3.15, it is enough to show that no automorphism $F = (F_1, F_2, F_3)$ of $\mathbb{C}^3$ with $\operatorname{mdeg} F = (4, d_2, d_3)$ admits an elementary reduction.

Assume, to the contrary, that $(F_1, F_2, F_3 - g(F_1, F_2))$, where $g \in \mathbb{C}[X, Y]$, is an elementary reduction of such an F. Then

$$\deg g(F_1, F_2) = d_3. \tag{6.2}$$

By Proposition 2.7,

$$\deg g(F_1, F_2) \geq q(pd_2 - d_2 - 4 + \deg [F_1, F_2]) + rd_2, \tag{6.3}$$

where $\deg_Y g(X, Y) = pq + r, 0 \leq r < p$ and $p = 4/\gcd(4, d_2) = 4$. Since $pd_2 - d_2 - 4 + \deg [F_1, F_2] = 3d_2 - 4 + \deg [F_1, F_2] \geq 3d_2 - 2$, by (6.1)–(6.3) we have $q = 0$ and $r \leq 2$. This means that $g(X, Y)$ is of the form

$$g(X, Y) = g_0(X) + g_1(X)Y + g_2(X)Y^2.$$

Since the sets $4\mathbb{N}, d_2 + 4\mathbb{N}$ and $2d_2 + 4\mathbb{N}$ are pairwise disjoint (because $\operatorname{lcm}(4, d_2) = 4d_2$), it follows that

$$d_3 = \deg g(F_1, F_2) \in 4\mathbb{N} \cup (d_2 + 4\mathbb{N}) \cup (2d_2 + 4\mathbb{N}).$$

This contradicts $d_3 \notin 4\mathbb{N} + d_2\mathbb{N}$.

Now, assume that $(F_1, F_2 - g(F_1, F_3), F_3)$ is an elementary reduction of F. Then

$$\deg g(F_1, F_3) = d_2. \tag{6.4}$$

But, by Proposition 2.7 we have

$$\deg g(F_1, F_3) \geq q(pd_3 - d_3 - 4 + \deg [F_1, F_3]) + rd_3, \tag{6.5}$$

where $\deg_Y g(X, Y) = pq + r, 0 \leq r < p$ and $p = 4/\gcd(4, d_2) = 4$. Since $d_3 > d_2 > 4$, we see that $pd_3 - d_3 - 4 + \deg [F_1, F_3] > 2d_3 > d_2$. Hence by (6.4) and (6.5), $q = r = 0$. This means that $g(X, Y) = g(X)$ and so $d_2 = \deg g(F_1, F_3) = \deg g(F_1) \in 4\mathbb{N}$. This contradicts the assumption that d_2 is odd.

Finally, assume that $(F_1 - g(F_2, F_3), F_2, F_3)$ is an elementary reduction of F. Then

$$\deg g(F_2, F_3) = 4. \tag{6.6}$$

By Proposition 2.7,

$$\deg g(F_1, F_3) \geq q(pd_3 - d_3 - d_2 + \deg [F_2, F_3]) + rd_3, \tag{6.7}$$

where $\deg_Y g(X, Y) = pq + r, 0 \leq r < p$ and $p = d_2/\gcd(d_2, d_3)$. Since $d_3 > 4$, by (6.6) and (6.7) we have $r = 0$. Since also $2 \nmid d_2$ and $d_2 \nmid d_3$ (because $d_3 \notin 4\mathbb{N} + d_2\mathbb{N}$), we conclude that $p = d_2/\gcd(d_2, d_3) \geq 3$ and $pd_3 - d_3 - d_2 + \deg [F_2, F_3] > d_3 > 4$. Thus $q = 0$. Then we obtain a contradiction as in the previous case. ■

COROLLARY 6.3. *We have*

$$\{(4,d_2,d_3) : 4 \le d_2 \le d_3,\ d_2,d_3 \in 2\mathbb{N}+1\} \cap \mathrm{mdeg}(\mathrm{Tame}(\mathbb{C}^3)) = \{(4,d_2,d_3) : 4 \le d_2 \le d_3,\ d_2,d_3 \in 2\mathbb{N}+1,\ d_3 \in 4\mathbb{N}+d_2\mathbb{N}\}.$$

6.3. The case $(4, \mathrm{even}, \mathrm{odd})$**.** We start with two examples (or rather two series of examples).

EXAMPLE 6.4. Since

$$(X+Z^4)^3 = Z^{12} + 3XZ^8 + 3X^2Z^4 + X^3, \qquad (Y+Z^6)^2 = Z^{12} + 2YZ^6 + Y^2,$$

we see that

$$\deg\,[(Y+Z^6)^2 - (X+Z^4)^3] = 9.$$

Thus, for any $k \in \mathbb{N}$,

$$\deg\,[(Y+Z^6)^2 - (X+Z^4)^3](X+Z^4)^k = 9+4k.$$

This means that

$$\mathrm{mdeg}(F_2 \circ F_1) = (4,6,9+4k),$$

where

$$F_1(x,y,z) = (x+z^4, y+z^6, z), \qquad F_2(u,v,w) = (u,v,w+(v^2-u^3)u^k).$$

EXAMPLE 6.5. Since

$$(X+Z^4)^3 = Z^{12} + 3XZ^8 + 3X^2Z^4 + X^3,$$
$$(Y+\tfrac{3}{2}XZ^2+Z^6)^2 = Z^{12} + 3XZ^8 + 2YZ^6 + \tfrac{9}{4}X^2Z^4 + 3YXZ^2 + Y^2,$$

it follows that

$$\deg\,[(Y+\tfrac{3}{2}XZ^2+Z^6)^2 - (X+Z^4)^3] = 7,$$

and

$$\deg\,[(Y+\tfrac{3}{2}XZ^2+Z^6)^2 - (X+Z^4)^3](X+Z^4)^k = 7+4k.$$

Thus we have

$$\mathrm{mdeg}(F_2 \circ F_1) = (4,6,7+4k),$$

where

$$F_1(x,y,z) = (x+z^4, y+\tfrac{3}{2}xz^2+z^6, z), \qquad F_2(u,v,w) = (u,v,w+(v^2-u^3)u^k).$$

Combining the above examples and Theorem 6.1 we obtain the following

PROPOSITION 6.6. *For any integer $d_3 \ge 6$ we have $(4,6,d_3) \in \mathrm{mdeg}(\mathrm{Tame}(\mathbb{C}^3))$.*

In the same manner one can prove

PROPOSITION 6.7. *For any integer $d_3 \ge 10$ we have $(4,10,d_3) \in \mathrm{mdeg}(\mathrm{Tame}(\mathbb{C}^3))$.*

Using Corollary 3.3 we obtain

PROPOSITION 6.8. *For $k = 1,2,\dots$ and any integer $d_3 \ge 4k$ we have $(4,4k,d_3) \in \mathrm{mdeg}(\mathrm{Tame}(\mathbb{C}^3))$.*

The next proposition gives partial information about multidegrees of the form $(4,4k+2,d_3)$, where $k = 3,4,\dots$ and $d_3 \ge 4k+2$.

Proposition 6.9. *For any integers $k \geq 3$ and $d_3 \geq 5k+1$ we have $(4, 4k+2, d_3) \in \operatorname{mdeg}(\operatorname{Tame}(\mathbb{C}^3))$.*

Proof. Let us notice that

$$(X+Z^4)^{2k+1} = \sum_{l=0}^{2k+1} \binom{2k+1}{l} X^l Z^{8k+4-4l}$$

and

$$\Big(Y + Z^r + \sum_{l=0}^{k} a_l X^l Z^{4k+2-4l}\Big)^2 = Y^2 + 2YZ^r + Z^{2r} + 2Y \sum_{l=0}^{k} a_l X^l Z^{4k+2-4l}$$
$$+ 2Z^r \sum_{l=0}^{k} a_l X^l Z^{4k+2-4l}$$
$$+ \sum_{s=0}^{2k} \Big(\sum_{l+m=s,\, l,m \in \{0,\dots,k\}} a_l a_m \Big) X^s Z^{8k+4-4s}.$$

We will consider the cases $r = k-1, k, k+1$ and $k+2$. Thus we have:

$$\deg 2YZ^r \leq k+3 < 5k+1,$$
$$\deg Z^{2r} \leq 2k+4 < 5k+1,$$
$$\deg 2Y \sum_{l=0}^{k} a_l X^l Z^{4k+2-4l} \leq 4k+3 < 5k+1,$$
$$\deg 2Z^r \sum_{l=2}^{k} a_l X^l Z^{4k+2-4l} \leq 5k-2 < 5k+1.$$

This means that the only summands of the polynomial

$$(X+Z^4)^{2k+1} - \Big(Y + Z^r + \sum_{l=0}^{k} a_l X^l Z^{4k+2-4l}\Big)^2 \tag{6.8}$$

of degree greater than or equal to $5k+1$ are:

$$(1-a_0^2)Z^{8k+4},$$
$$\left[\binom{2k+1}{1} - 2a_0a_1\right] XZ^{8k},$$
$$\left[\binom{2k+1}{2} - (2a_0a_2 + a_1^2)\right] X^2 Z^{8k-4},$$
$$\vdots$$
$$\left[\binom{2k+1}{k} - (a_0a_k + a_1a_{k-1} + \cdots + a_{k-1}a_1 + a_ka_0)\right] X^k Z^{4k+4},$$
$$2a_0 z^{4k+2+r}$$

and (only in the case $r = k+2$)

$$2a_1 XZ^{4k-2+r}.$$

Since we can recursively solve the following system of equations (notice that we can take $a_0 = 1$):

$$
\begin{aligned}
1 - a_0^2 &= 0,\\
\binom{2k+1}{1} - 2a_0a_1 &= 0,\\
\binom{2k+1}{2} - (2a_0a_2 + a_1^2) &= 0,\\
&\vdots\\
\binom{2k+1}{k} - (a_0a_k + a_1a_{k-1} + \cdots + a_{k-1}a_1 + a_ka_0) &= 0,
\end{aligned}
$$

it follows that we can choose $a_0, a_1, \ldots, a_k$ so that the degree of the polynomial (6.8) is equal to

$$\deg(2a_0Z^{4k+2+r}) = 4k + 2 + r.$$

Taking $r = k-1, k, k+1$ and $k+2$ we obtain polynomials of degree equal to $5k+1$, $5k+2, 5k+3$ and $5k+4$, respectively.

Now, it is easy to see that taking

$$
\begin{aligned}
F(x, y, z) &= \Big(x + z^4, y + z^r + \sum_{l=0}^{k} a_l x^l z^{4k+2-4l}, z\Big),\\
G(u, v, w) &= (u, v, w + (u^{4k+1} - v^2)u^q),
\end{aligned}
$$

where $q = 0, 1, \ldots$, we obtain

$$\operatorname{mdeg}(G \circ F) = (4, 4k + 2, 4k + 2 + r + 4q).$$

Since for any $d_3 \geq 5k + 1$ we can find $r \in \{k - 1, k, k + 1, k + 2\}$ and $q \in \mathbb{N}$ such that $4k + 2 + r + 4q = d_3$, the result follows. ■

6.4. The case $(4, \text{odd}, \text{even})$**.** In this subsection we give an almost complete description of the set

$$\{(4, d_2, d_3) : 4 \leq d_2 \leq d_3,\ d_2 \in 2\mathbb{N} + 1,\ d_3 \in 2\mathbb{N}\} \cap \operatorname{mdeg}(\operatorname{Tame}(\mathbb{C}^3)).$$

Namely we have the following result.

THEOREM 6.10. *If $d_2 \geq 5$ is odd and $d_3 \geq d_2$ is even such that $d_3 - d_2 \neq 1$, then $(4, d_2, d_3) \in \operatorname{mdeg}(\operatorname{Tame}(\mathbb{C}^3))$ if and only if $d_3 \in 4\mathbb{N} + d_2\mathbb{N}$.*

Proof. If $d_3 \in 4\mathbb{N} + d_2\mathbb{N}$, then by Proposition 3.2, $(4, d_2, d_3) \in \operatorname{mdeg}(\operatorname{Tame}(\mathbb{C}^3))$. Conversely, assume that $d_3 \notin 4\mathbb{N} + d_2\mathbb{N}$. Since d_2 is odd, by Remark 3.9 and Theorem 3.15 it is enough to show that no automorphism $F = (F_1, F_2, F_3) : \mathbb{C}^3 \to \mathbb{C}^3$ with $\operatorname{mdeg} F = (4, d_2, d_3)$ admits an elementary reduction.

Assume that $(F_1, F_2, F_3 - g(F_1, F_2))$, where $g \in \mathbb{C}[X, Y]$, is such a reduction. Thus

$$\deg g(F_1, F_2) = d_3,$$

and by Proposition 2.7,

$$\deg g(F_1, F_2) \geq q(pd_2 - d_2 - 4 + \deg [F_1, F_2]) + rd_2,$$

where $\deg_Y g(X, Y) = pq + r, 0 \leq r < p$ and $p = 4/\gcd(4, d_2) = 4$. Since $d_3 \notin 4\mathbb{N} + d_2\mathbb{N}$ and $\gcd(4, d_2) = 1$, we have (as in the proof of Theorem 6.2)

$$d_3 < 3d_2 - 3. \tag{6.9}$$

Thus we can repeat the arguments from the corresponding case in the proof of Theorem 6.2 to obtain a contradiction.

Now, assume that $(F_1, F_2 - g(F_1, F_3), F_3)$ is an elementary reduction of F. Then

$$\deg g(F_1, F_3) = d_2, \tag{6.10}$$

and by Proposition 2.7,

$$\deg g(F_1, F_3) \geq q(pd_3 - d_3 - 4 + \deg [F_1, F_2]) + rd_3, \tag{6.11}$$

where $\deg_Y g(X, Y) = pq + r, 0 \leq r < p$ and $p = 4/\gcd(4, d_3) = 2$ (because d_3 is even and $d_3 \notin 4\mathbb{N}+d_2\mathbb{N}$). Thus $pd_3 - d_3 - 4 + \deg [F_1, F_2] \geq d_3 - 2$. But by the assumptions $d_3 - d_2 \geq 0$ is an odd number different from 1. So $d_2 \leq d_3 - 3$, and then $pd_2 - d_2 - 4 + \deg [F_1, F_2] > d_2$. Consequently, by (6.10) and (6.11), $q = 0$. Since also $r = 0$ (because $d_3 > d_2$), we see that $g(X, Y) = g(X)$, and so

$$d_2 = \deg g(F_1, F_3) = \deg g(F_1) \in 4\mathbb{N}.$$

This contradicts the assumption that d_2 is odd.

In the last case we can repeat the arguments from the corresponding case in the proof of Theorem 6.2. ■

COROLLARY 6.11. *If $d_2 \geq 5$ is odd and $d_2 \equiv 3 \pmod 4$, and $d_3 \geq d_2$ is even, then $(4, d_2, d_3) \in \operatorname{mdeg}(\operatorname{Tame}(\mathbb{C}^3))$ if and only if $d_3 \in 4\mathbb{N} + d_2\mathbb{N}$.*

Proof. Notice that if $d_3 - d_2 = 1$, then $4 \mid d_3$. Thus $d_3 \in 4\mathbb{N} + d_2\mathbb{N}$ and by Proposition 3.2, $(4, d_2, d_3) \in \operatorname{mdeg}(\operatorname{Tame}(\mathbb{C}^3))$. In the case $d_3 - d_2 > 1$, we can use Theorem 6.10. ■

By the above corollary, to complete the description of the set

$$\{(4, d_2, d_3) : 4 \leq d_2 \leq d_3,\ d_2 \in 2\mathbb{N} + 1,\ d_3 \in 2\mathbb{N}\} \cap \operatorname{mdeg}(\operatorname{Tame}(\mathbb{C}^3))$$

it is enough to consider the triples of the form (1)

$$(4, 4k + 1, 4k + 2) \quad \text{for } k = 1, 2, \ldots$$

Moreover, using the arguments from the proof of Theorem 6.10, one can show

PROPOSITION 6.12. *Let $k \in \mathbb{N}^*$. If there exists a tame automorphism $\widetilde{F}$ of $\mathbb{C}^3$ with $\operatorname{mdeg} \widetilde{F} = (4, 4k + 1, 4k + 2)$, then there is also a tame automorphism $F = (F_1, F_2, F_3)$ of $\mathbb{C}^3$ with $\operatorname{mdeg} F = (4, 4k + 1, 4k + 2)$ that admits an elementary reduction $(F_1, F_2 - g(F_1, F_3), F_3)$ for some $g \in \mathbb{C}[X, Y]$. Moreover, for such F we have $\deg [F_1, F_3] \leq 3$.*

(1) Recently, the author proved that $(4, 5, 6) \notin \operatorname{mdeg}(\operatorname{Tame}(\mathbb{C}^3))$ [21]. The method developed in [21] seems to be useful in other cases. For example, the author believes that, for $k = 1, 2, \ldots$, we have $(4, 4k + 1, 4k + 2) \notin \operatorname{mdeg}(\operatorname{Tame}(\mathbb{C}^3))$. He also believes that this can be proved by the above mentioned method.

Using arguments from the proof of Theorem 7.3 one can also show that $\deg\,[F_1, F_3] = 3$ when $k < 25$.

7. The cases (p, d_2, d_3) and $(5, d_2, d_3)$

7.1. The general case. Now we generalize, in a sense, the results of the section 'The case $(3, d_2, d_3)$'. This generalization is not complete. The first, general result is

THEOREM 7.1. *Let $2 \le p \le d_2 \le d_3$ be integers, and let p be a prime. If*

(1) $d_3/d_2 \neq 3/2$, *or*
(2) $d_3/d_2 = 3/2$ *and* $d_2/2 > p - 2$,

then $(p, d_2, d_3) \in \operatorname{mdeg}(\operatorname{Tame}(\mathbb{C}^3))$ *if and only if* $p \mid d_2$ *or* $d_3 \in p\mathbb{N} + d_2\mathbb{N}$.

Proof. By Corollary 3.2, if $p \mid d_2$ or $d_3 \in p\mathbb{N}+d_2\mathbb{N}$, then there exists a tame automorphism $F : \mathbb{C}^3 \to \mathbb{C}^3$ such that $\operatorname{mdeg} F = (p, d_2, d_3)$. Conversely, assume that $p \nmid d_2$ and $d_3 \notin p\mathbb{N} + d_2\mathbb{N}$ and (1) or (2) holds.

In particular $p < d_2 < d_3$. By Theorems 5.1 and 3.3, we can assume that $p > 3$. Indeed, for $p = 2$, by Corollary 3.3 we have $(2, d_2, d_3) \in \operatorname{mdeg}(\operatorname{Tame}(\mathbb{C}^3))$ for all integers $2 \le d_2 \le d_3$. Also the condition $2 \mid d_2$ or $d_3 \in 2\mathbb{N} + d_2\mathbb{N}$ is satisfied for all integers $2 \le d_2 \le d_3$. For $p = 3$ we simply use Theorem 5.1. So assume that $p > 3$. By Theorem 3.15 it is enough to show that no automorphism $F = (F_1, F_2, F_3) : \mathbb{C}^3 \to \mathbb{C}^3$ with $\operatorname{mdeg} F = (p, d_2, d_3)$ admits an elementary reduction (notice that $3 \nmid p$).

Assume, to the contrary, that there is such a reduction. Since $p \nmid d_2$, we have $\gcd(p, d_2) = 1$. So by Theorem 2.15 we have $k \in p\mathbb{N} + d_2\mathbb{N}$ for all $k \ge (p-1)(d_2-1) = pd_2 - d_2 - p + 1$. Thus

$$d_3 < pd_2 - d_2 - p + 1, \tag{7.1}$$

since $d_3 \notin p\mathbb{N} + d_2\mathbb{N}$.

Assume that $(F_1, F_2, F_3 - g(F_1, F_2))$ is an elementary reduction of (F_1, F_2, F_3). Hence $\deg g(F_1, F_2) = \deg F_3 = d_3$. Since $\gcd(p, d_2) = 1$, we see that $p/\gcd(p, d_2) = p$, and so by Proposition 2.7,

$$\deg g(F_1, F_2) \ge q(pd_2 - d_2 - p + \deg\,[F_1, F_2]) + rd_2,$$

where $\deg_Y g(X, Y) = pq + r$ with $0 \le r < p$. Since F_1, F_2 are algebraically independent, $\deg\,[F_1, F_2] \ge 2$ and $pd_2 - d_2 - p + \deg\,[F_1, F_2] \ge pd_2 - d_2 - p + 2$. Then by (7.1) it follows that $q = 0$. Thus

$$g(X, Y) = \sum_{i=0}^{p-1} g_i(X) Y^i.$$

Since $\operatorname{lcm}(p, d_2) = pd_2$, the sets

$$p\mathbb{N},\ d_2 + p\mathbb{N},\ \ldots,\ (p-1)d_2 + p\mathbb{N}$$

are pairwise disjoint. So

$$\deg\Big(\sum_{i=0}^{p-1} g_i(F_1) F_2^i\Big) = \max_{i=0,\ldots,p-1} (\deg F_1 \deg g_i + i \deg F_2)$$

and

$$d_3 = \deg g(F_1, F_2) = \deg\Big(\sum_{i=0}^{p-1} g_i(F_1)F_2^i\Big) \in \bigcup_{r=0}^{p-1}(rd_2 + p\mathbb{N}) \subset p\mathbb{N} + d_2\mathbb{N},$$

a contradiction.

Now assume that $(F_1, F_2 - g(F_1, F_3), F_3)$ is an elementary reduction of (F_1, F_2, F_3). Since $d_3 \notin p\mathbb{N} + d_2\mathbb{N}$, we have $p \nmid d_3$ and $\gcd(p, d_3) = 1$. Hence by Proposition 2.7,

$$\deg g(F_1, F_3) \geq q(pd_3 - d_3 - p + \deg\,[F_1, F_3]) + rd_3,$$

where $\deg_Y g(X, Y) = qp + r$ with $0 \leq r < p$. Since $pd_3 - d_3 - p + \deg\,[F_1, F_3] \geq pd_3 - 2d_3 \geq 3d_3 > d_2$ and since we want to have $\deg g(F_1, F_3) = p_2$, we conclude that $q = r = 0$. This means that $g(X, Y) = g(X)$, and so

$$d_2 = \deg g(F_1, F_2) = \deg g(F_1) \in p\mathbb{N} \subset p\mathbb{N} + d_2\mathbb{N},$$

a contradiction.

Finally, assume that $(F_1 - g(F_2, F_3), F_2, F_3)$ is an elementary reduction of (F_1, F_2, F_3). Thus we have $\deg g(F_2, F_3) = p$. Let

$$\widetilde{p} = \frac{d_2}{\gcd(d_2, d_3)}.$$

Since $d_3 \notin p\mathbb{N} + d_2\mathbb{N}$, we see that $d_2 \nmid d_3$, and so $\widetilde{p} > 1$. By Proposition 2.7,

$$\deg g(F_2, F_3) \geq q(\widetilde{p}d_3 - d_2 - d_3 + \deg\,[F_1, F_3]) + rd_3,$$

where $\deg_Y g(X, Y) = q\widetilde{p} + r$ with $0 \leq r < \widetilde{p}$. Since $d_3 > p$ (because $d_3 > d_2 > p$), we see that $r = 0$. Consider the case $\widetilde{p} \geq 3$. Then $\widetilde{p}d_3 - d_2 - d_3 + \deg\,[F_1, F_3] \geq d_3 + \deg\,[F_1, F_3] > p$. Thus we must have $q = 0$. Hence $g(X, Y) = g(X)$ and

$$p = \deg g(F_2, F_3) = \deg g(F_2) \in d_2\mathbb{N}.$$

This contradicts $d_2 \neq p$ (we have assumed that $p \nmid d_2$).

Now, consider the case $\widetilde{p} = 2$. Then, for some $n \in \mathbb{N}^*$, $d_2 = 2n$ and $d_3 = ns$, where $s \geq 3$ is odd. Consider first the case $s > 3$. Then

$$\begin{aligned} 2d_3 - d_3 - d_2 + \deg\,[F_1, F_3] &= d_3 - d_2 + \deg\,[F_1, F_3] \\ &= (s-2)n + \deg\,[F_1, F_3] > d_2 > p. \end{aligned}$$

Thus we have $q = 0$. As before this leads to a contradiction.

Now, consider the case $s = 3$. This is the case when we use the second statement of the assumption (2). Since $d_2 = 2n$ and $d_3 = 3n$, we see that $d_3/d_2 = 3/2$. Hence (1) is not satisfied. Thus, the assumption (2) is satisfied and so $n = d_2/2 > p - 2$. Hence

$$\begin{aligned} 2d_3 - d_3 - d_2 + \deg\,[F_1, F_3] &= d_3 - d_2 + \deg\,[F_1, F_3] \\ &\geq n + 2 > p. \end{aligned}$$

So, also in this case we have $q = 0$. As before this leads to a contradiction. ∎

For small prime numbers p the above theorem gives, for example, the following results.

COROLLARY 7.2.

(a) *If* $(5, d_2, d_3) \neq (5, 6, 9)$ *and* $5 \leq d_2 \leq d_3$*, then* $(5, d_2, d_3) \in \operatorname{mdeg}(\operatorname{Tame}(\mathbb{C}^3))$ *if and only if* $5 \mid d_2$ *or* $d_3 \in 5\mathbb{N} + d_2\mathbb{N}$.
(b) *If* $(7, d_2, d_3) \notin \{(7, 8, 12), (7, 10, 15)\}$ *and* $7 \leq d_2 \leq d_3$*, then we have* $(7, d_2, d_3) \in \operatorname{mdeg}(\operatorname{Tame}(\mathbb{C}^3))$ *if and only if* $7 \mid d_2$ *or* $d_3 \in 7\mathbb{N} + d_2\mathbb{N}$.
(c) *If* $(11, d_2, d_3) \notin \{(11, 12, 18), (11, 14, 21), (11, 16, 24), (11, 18, 27)\}$ *and* $11 \leq d_2 \leq d_3$*, then* $(11, d_2, d_3) \in \operatorname{mdeg}(\operatorname{Tame}(\mathbb{C}^3))$ *if and only if* $11 \mid d_2$ *or* $d_3 \in 11\mathbb{N} + d_2\mathbb{N}$.
(d) *If* $(13, d_2, d_3) \notin \{(13, 14, 21), (13, 16, 24), (13, 18, 27), (13, 20, 30), (13, 22, 33)\}$ *and* $13 \leq d_2 \leq d_3$*, then* $(13, d_2, d_3) \in \operatorname{mdeg}(\operatorname{Tame}(\mathbb{C}^3))$ *if and only if* $13 \mid d_2$ *or* $d_3 \in 13\mathbb{N} + d_2\mathbb{N}$.

Proof. One can easily check that, for example, for $p = 11$ the only triples of the form $(11, d_2, d_3)$ with $11 \leq d_2 \leq d_3$ that satisfy neither condition (1) nor (2) of the above theorem are $(11, 12, 18), (11, 14, 21), (11, 16, 24)$ and $(11, 18, 27)$. ∎

The point (a) of the above corollary yields an almost complete description of the set

$$\{(5, d_2, d_3) : 5 \leq d_2 \leq d_3\} \cap \operatorname{mdeg}(\operatorname{Tame}(\mathbb{C}^3)). \tag{7.2}$$

The only thing that we do not know yet is whether $(5, 6, 9)$ is an element of this set. One can, of course, notice that $9 \notin 5\mathbb{N} + 6\mathbb{N}$. In the next section we show that $(5, 6, 9) \notin \operatorname{mdeg}(\operatorname{Tame}(\mathbb{C}^3))$, completing the description of the set (7.2).

7.2. Tame automorphism of $\mathbb{C}^3$ with multidegree equal $(5, 6, 9)$ and the Jacobian Conjecture. Our main purpose in this section is to prove the following result.

THEOREM 7.3. *There is no tame automorphism of* $\mathbb{C}^3$ *with multidegree* $(5, 6, 9)$.

Before we give the proof of the above theorem we recall some positive results about the Jacobian Conjecture in dimension two. In the proof of the theorem we use one of such results but for completeness we recall a little more.

The first one is the following result due to Magnus [31].

THEOREM 7.4 (Magnus, see also [7, Thm. 10.2.24]). *Let* $F = (P, Q)$ *be a Keller map* (*i.e. such that* $\operatorname{Jac} F = 1$). *If* $\gcd(\deg P, \deg Q) = 1$ *then* F *is invertible and* $\deg P = 1$ *or* $\deg Q = 1$.

The next, also due to Magnus, is the following corollary of the above theorem.

COROLLARY 7.5 (Magnus, see e.g. [7]). *If* $F = (P, Q)$ *is a Keller map and* $\deg P$ *or* $\deg Q$ *is a prime number, then* F *is invertible.*

Later Applegate, Onishi and Nagata improved the result of Magnus.

THEOREM 7.6 (Applegate, Onishi, Nagata, see e.g. [3, 4] or [7]). *Let* $F = (P, Q)$ *be a Keller map and* $d = \gcd(\deg P, \deg Q)$. *If* $d \leq 8$ *or* d *is a prime, then* F *is invertible.*

The last result we recall here is due to Moh [34].

THEOREM 7.7 (see also [7]). *Let* $F : \mathbb{C}^2 \to \mathbb{C}^2$ *be a Keller map with* $\deg F \leq 101$. *Then* F *is invertible.*

Now we can give the proof of Theorem 7.3.

Proof of Theorem 7.3. By Theorem 3.15, it is enough to show that no (hypothetical) automorphism F of $\mathbb{C}^3$ with $\operatorname{mdeg} F = (5, 6, 9)$ admits an elementary reduction. Moreover, it is enough to show this for automorphisms $F = (F_1, F_2, F_3) : \mathbb{C}^3 \to \mathbb{C}^3$ such that $F(0, 0, 0) = (0, 0, 0)$.

Assume that $(F_1, F_2, F_3 - g(F_1, F_2))$, where $g \in \mathbb{C}[X, Y]$, is an elementary reduction of F. Then

$$\deg g(F_1, F_2) = \deg F_3 = 9. \tag{7.3}$$

By Proposition 2.7,

$$\deg g(F_1, F_2) \geq q(5 \cdot 6 - 6 - 5 + \deg [F_1, F_2]) + 6r, \tag{7.4}$$

where $\deg_Y g(X, Y) = 5q + r$, with $0 \leq r < 5$. Since $5 \cdot 6 - 6 - 5 + \deg [F_1, F_2] \geq 19 + \deg [F_1, F_2] > 9$, by (7.3) and (7.4) we have $q = 0$. Also by (7.3) and (7.4) we have $r < 2$. Thus $g(X, Y) = g_0(X) + Y g_0(X)$, and since $5\mathbb{N} \cap (6 + 5\mathbb{N}) = \emptyset$, it follows that

$$9 = \deg g(F_1, F_2) \in 5\mathbb{N} \cup (6 + 5\mathbb{N}),$$

a contradiction.

Now, assume that $(F_1, F_2 - g(F_1, F_3), F_3)$ is an elementary reduction of (F_1, F_2, F_3). Then

$$\deg g(F_1, F_3) = \deg F_2 = 6. \tag{7.5}$$

By Proposition 2.7,

$$\deg g(F_1, F_3) \geq q(5 \cdot 9 - 9 - 5 + \deg [F_1, F_3]) + 9r, \tag{7.6}$$

where $\deg_Y g(X, Y) = 5q + r$, with $0 \leq r < 5$. Since $5 \cdot 9 - 9 - 5 + \deg [F_1, F_3] \geq 31 + \deg [F_1, F_3] > 6$, we have $q = r = 0$. This means that $g(X, Y) = g(X)$, and so

$$\deg g(F_1, F_2) = \deg g(F_1) \in 5\mathbb{N},$$

a contradiction.

Finally, assume that $(F_1 - g(F_2, F_3), F_2, F_3)$ is an elementary reduction of (F_1, F_2, F_3). By Theorem 3.15, we can also assume that $F(0, 0, 0) = (0, 0, 0)$. We have

$$\deg g(F_2, F_3) = \deg F_1 = 5 \tag{7.7}$$

and by Proposition 2.7,

$$\deg g(F_2, F_3) \geq q(p \cdot 9 - 9 - 6 + \deg [F_2, F_3]) + 9r, \tag{7.8}$$

where $\deg_Y g(X, Y) = qp + r$, with $0 \leq r < p$ and $p = 6/\gcd(6, 9) = 2$. By (7.7) and (7.8), $r = 0$.

Consider the case $\deg [F_2, F_3] > 2$. Then $p \cdot 9 - 9 - 6 + \deg [F_2, F_3] = 3 + \deg [F_2, F_3] > 5$, and then by (7.7) and (7.8) we see that $q = 0$. Thus in this case, we have $g(X, Y) = g(X)$, and so $\deg g(F_2, F_3) = \deg g(F_2) \in 6\mathbb{N}$. This contradicts (7.7).

Now, consider the case $\deg [F_2, F_3] = 2$ (since F_2, F_3 are algebraically independent, we have $\deg [F_2, F_3] \geq 2$). Let L be the linear part of the automorphism F. Since $F(0, 0, 0) =$

$(0,0,0)$, the linear part of $F \circ L^{-1}$ is the identity map $\mathrm{id}_{\mathbb{C}^3}$. Thus

$$F_2 \circ L^{-1} = X_2 + \text{higher degree summands}, \qquad (7.9)$$
$$F_3 \circ L^{-1} = X_3 + \text{higher degree summands}.$$

Since, by Lemma 2.8,

$$\deg\,[F_2 \circ L^{-1}, F_3 \circ L^{-1}] = \deg\,[F_2, F_3] = 2,$$

it follows, by Lemma 3.20, that

$$F_2 \circ L^{-1}, F_3 \circ L^{-1} \in \mathbb{C}[X_2, X_3].$$

But $\deg\,[F_2 \circ L^{-1}, F_3 \circ L^{-1}] = 2$ means that

$$\mathrm{Jac}(F_2 \circ L^{-1}, F_3 \circ L^{-1}) \in \mathbb{C}^*$$

(of course we consider here $F_2 \circ L^{-1}, F_3 \circ L^{-1}$ as functions of two variables X_2, X_3). By Lemma 3.17 we have $\deg(F_2 \circ L^{-1}) = 6, \deg(F_3 \circ L^{-1}) = 9$. Then, by Theorem 7.7, the map $(F_2 \circ L^{-1}, F_3 \circ L^{-1}) : \mathbb{C}^2 \to \mathbb{C}^2$ is an automorphism. But $6 \nmid 9$ contradicts the Jung–van der Kulk theorem (see Theorem 1.4 and Corollary 1.3). ∎

By Theorem 7.3 and Corollary 7.2(a) we obtain the following result.

Corollary 7.8. *We have*

$$\{(5, d_2, d_3) : 5 \le d_2 \le d_3\} \cap \mathrm{mdeg}(\mathrm{Tame}(\mathbb{C}^3))$$
$$= \{(5, d_2, d_3) : 5 \le d_2 \le d_3,\ 5 \,|\, d_2 \text{ or } d_3 \in 5\mathbb{N} + d_2\mathbb{N}\}.$$

7.3. The case $(p, 2(p-2), 3(p-2))$. In the same manner as we proved Theorem 7.3 one can show the following

Theorem 7.9. *Let $p \ge 5$ be a prime such that $p \le 35$. Then $(p, 2(p-2), 3(p-3)) \notin \mathrm{mdeg}(\mathrm{Tame}(\mathbb{C}^3))$.*

Proof. Since $3(p-2) \le 101$, one can use Theorem 7.7 and repeat the arguments from the proof of Theorem 7.3. ∎

By the above theorem and Corollary 7.2 we obtain

Corollary 7.10. *We have*

$$[\{(7, d_2, d_3) : 7 \le d_2 \le d_3\} \cap \mathrm{mdeg}(\mathrm{Tame}(\mathbb{C}^3))] \setminus \{(7, 8, 12)\}$$
$$= \{(7, d_2, d_3) : 7 \le d_2 \le d_3,\ 7 \,|\, d_2 \text{ or } d_3 \in 7\mathbb{N} + d_2\mathbb{N}\}.$$

The above corollary means that in order to obtain a complete description of the set $\{(7, d_2, d_3) : 7 \le d_2 \le d_3\} \cap \mathrm{mdeg}(\mathrm{Tame}(\mathbb{C}^3))$ we “only” need to know whether $(7, 8, 12) \in \mathrm{mdeg}(\mathrm{Tame}(\mathbb{C}^3))$.

To end this subsection notice the following result.

Theorem 7.11. *The Jacobian Conjecture for dimension two implies that for every prime $p \ge 5$ we have $(p, 2(p-2), 3(p-2)) \notin \mathrm{mdeg}(\mathrm{Tame}(\mathbb{C}^3))$.*

Proof. If we assume that the Jacobian Conjecture for dimension two holds true, then one can repeat the arguments from the proof of Theorem 7.3. ∎

Corollary 7.12. *If there is a tame automorphism F of $\mathbb{C}^3$ with* $\operatorname{mdeg} F = (p, 2(p-2), 3(p-2))$, *where $p > 35$ is a prime, then the Jacobian Conjecture for dimension two is false.*

Proof. This is a consequence of Theorems 7.9 and 7.11. ∎

In particular we have

Theorem 7.13. *If there is a tame automorphism F of $\mathbb{C}^3$ with* $\operatorname{mdeg} F = (37, 70, 105)$, *then the two-dimensional Jacobian Conjecture is false.*

8. Finiteness results

Let us consider the set

$$T_{a,b}^{(n)} = \{(d_1, \dots, d_n) \in (\mathbb{N}^*)^n : d_1 \le \dots \le d_n,\ d_1 = a,\ d_2 = b\} \setminus \operatorname{mdeg}(\operatorname{Tame}(\mathbb{C}^n)).$$

Of course, by Jung–van der Kulk's result, $T_{a,b}^{(2)} = \{(a,b)\}$ if $a \nmid b$, and $T_{a,b}^{(2)} = \emptyset$ if $a \mid b$. Thus $\# T_{a,b}^{(2)} \le 1 < +\infty$ for all $1 \le a \le b$. We will show that also for $n \ge 3$ the set $T_{a,b}^{(n)}$ is finite. For $n = 3$ this result is due to Zygadło [54].

Theorem 8.1. *For all integers $1 \le a \le b$ the set $T_{a,b}^{(3)}$ is finite. Moreover,*

$$T_{a,b}^{(3)} \subset \{(a, b, d_3) : d_3 < \operatorname{lcm}(a,b) - r\},$$

where $r = \min\{b-1, (a-1)(\lfloor b/a \rfloor + 1)\}$.

The original proof of the above theorem due to Zygadło can be found in [54], but we give here another, simpler proof. It is based on the proof of Proposition 6.9, but there are also similarities to the proof in [54].

Proof. First of all notice that without loss of generality we can assume that $1 < a < b$. Indeed, if $a = 1$, or $a = b$, then by Proposition 3.2 we have $T_{a,b}^{(3)} = \emptyset$. Thus up to the end of the proof we assume that $1 < a < b$.

Let $d = \gcd(a,b)$. Then $a = d\widetilde{a}$, $b = d\widetilde{b}$, where $\widetilde{a}, \widetilde{b} \in \mathbb{N}^*$ are coprime. We have $\operatorname{lcm}(a,b) = ab/\gcd(a,b) = a\widetilde{b} = b\widetilde{a}$. Let us notice that

$$(X + Z^a)^{\widetilde{b}} = \sum_{l=0}^{\widetilde{b}} \binom{\widetilde{b}}{l} X^l Z^{a\widetilde{b} - la} \tag{8.1}$$

and

$$\Big(Y + Z^p + \sum_{l=0}^{\lfloor b/a \rfloor} a_l X^l Z^{b-la}\Big)^{\widetilde{a}} = \sum_{s=1}^{\widetilde{a}} \binom{\widetilde{a}}{s} (Y + Z^p)^s \Big(\sum_{l=0}^{\lfloor b/a \rfloor} a_l X^l Z^{b-la}\Big)^{\widetilde{a}-s} + \Big(\sum_{l=0}^{\lfloor b/a \rfloor} a_l X^l Z^{b-la}\Big)^{\widetilde{a}}. \tag{8.2}$$

If we take $p < b$, then

$$\deg\Big[\sum_{s=1}^{\widetilde{a}}\binom{\widetilde{a}}{s}(Y+Z^p)^s\Big(\sum_{l=0}^{\lfloor b/a\rfloor}a_lX^lZ^{b-la}\Big)^{\widetilde{a}-s}\Big]\le p+b(\widetilde{a}-1),$$

and since $Z^{p+b(\widetilde{a}-1)}$ can be obtained in the above polynomial in only one way, we actually have (provided that $a_0\neq 0$)

$$\deg\Big[\sum_{s=1}^{\widetilde{a}}\binom{\widetilde{a}}{s}(Y+Z^p)^s\Big(\sum_{l=0}^{\lfloor b/a\rfloor}a_lX^lZ^{b-la}\Big)^{\widetilde{a}-s}\Big]=p+b(\widetilde{a}-1). \tag{8.3}$$

In the following, we will take $p\in\{1,\dots,b-1\}$ such that $p+b(\widetilde{a}-1)\in\{\operatorname{lcm}(a,b)-r,\dots,\operatorname{lcm}(a,b)-r+(a-1)\}$. This is possible, because $b(\widetilde{a}-1)+1\le\operatorname{lcm}(a,b)-r$ and $\operatorname{lcm}(a,b)-r+(a-1)<\operatorname{lcm}(a,b)=b\widetilde{a}$.

Now, using (8.1)–(8.3) we find that the summands of degree greater than $p+b(\widetilde{a}-1)$, in the polynomial

$$(X+Z^a)^{\widetilde{b}}-\Big(Y+Z^p+\sum_{l=0}^{\lfloor b/a\rfloor}a_lX^lZ^{b-la}\Big)^{\widetilde{a}}$$

are

$$(1-a_0^{\widetilde{a}})Z^{a\widetilde{b}},$$
$$\Big[\binom{\widetilde{b}}{1}-\binom{\widetilde{a}}{1}a_0^{\widetilde{a}-1}a_1\Big]XZ^{a(\widetilde{b}-1)},$$
$$\Big[\binom{\widetilde{b}}{2}-\binom{\widetilde{a}}{2}a_0^{\widetilde{a}-2}a_1^2-\binom{\widetilde{a}}{1}a_0^{\widetilde{a}-1}a_2\Big]X^2Z^{a(\widetilde{b}-2)},$$

and for $k=3,\dots,\lfloor b/a\rfloor$,

$$\Big[\binom{\widetilde{b}}{k}-\Big(\sum_{l_1+\dots+l_{\widetilde{a}}=k,\,l_i<k}a_{l_1}\cdots a_{l_{\widetilde{a}}}\Big)-\binom{\widetilde{a}}{1}a_0^{\widetilde{a}-1}a_k\Big]X^kZ^{a(\widetilde{b}-k)}.$$

Thus we can recursively choose coefficients $a_0,\dots,a_{\lfloor b/a\rfloor}$ so that all expressions in the brackets above are equal to zero. Since also in the polynomial

$$(X+Z^a)^{\widetilde{b}}-\Big(\sum_{l=0}^{\lfloor b/a\rfloor}a_lX^lZ^{b-la}\Big)^{\widetilde{a}}$$

there are no summands belonging to $\mathbb{C}[Z]\setminus\mathbb{C}$ (provided that $a_0=1$), we have

$$\deg\Big[(X+Z^a)^{\widetilde{b}}-\Big(Y+Z^p+\sum_{l=0}^{\lfloor b/a\rfloor}a_lX^lZ^{b-la}\Big)^{\widetilde{a}}\Big]=p+b(\widetilde{a}-1).$$

Now, let $d_3\ge\operatorname{lcm}(a,b)-r$ be arbitrary. Then there are $p\in\{1,\dots,b-1\}$ and $q\in\mathbb{N}$ such that $p+b(\widetilde{a}-1)\in\{\operatorname{lcm}(a,b)-r,\dots,\operatorname{lcm}(a,b)-r+(a-1)\}$ and $d_3=p+b(\widetilde{a}-1)+qa$. By the above considerations we obtain

$$\operatorname{mdeg}(G\circ F)=(a,b,d_3),$$

where

$$F(x,y,z)=\Big(x+z^a,\, y+z^p+\sum_{l=0}^{\lfloor b/a\rfloor} a_l x^l z^{b-la},\, z\Big),$$

$$G(u,v,w)=(u,v,w+(u^{\widetilde{b}}-v^{\widetilde{a}})u^q). \ \blacksquare$$

Corollary 8.2. *For $n\in\mathbb{N}$, $n\geq 3$, and all integers $1\leq a\leq b$ the set $T_{a,b}^{(n)}$ is finite. Moreover,*

$$T_{a,b}^{(3)}\subset\{(a,b,d_3,\ldots,d_n)\in(\mathbb{N}^*)^n : d_3,\ldots,d_n<\operatorname{lcm}(a,b)-r\},$$

where r is defined as in Theorem 8.1.

Proof. If for some $i\in\{3,\ldots,n\}$ we have $d_i\geq \operatorname{lcm}(a,b)-r$ (actually we can think that $i=n$, because $d_3\leq\cdots\leq d_n$) then by Theorem 8.1, there exists a tame automorphism $F:\mathbb{C}^3\to\mathbb{C}^3$ such that $\operatorname{mdeg} F=(a,b,d_i)$. Now we use Proposition 3.2. ■

9. Multidegree of the inverse of a polynomial automorphism of $\mathbb{C}^2$

In [44] Rusek and Winiarski proved that $\deg F^{-1}\leq(\deg F)^{n-1}$ for all automorphisms F of $\mathbb{C}^n$ and hence $\deg F^{-1}=\deg F$ for $n=2$. Here we give complete information about $\operatorname{mdeg} F^{-1}$ for $F\in\operatorname{Aut}(\mathbb{C}^2)$.

9.1. Multidegree and length of automorphisms of $\mathbb{C}^2$. Here we establish the relations between the multidegree of a given automorphism of $\mathbb{C}^2$ and its length (Theorem 9.5). We start with the following technical (cf. [11, Lem. 2])

Lemma 9.1. *If $(P,Q)\in\operatorname{Aut}(\mathbb{C}^2)$ is such that $\deg P<\deg Q$, then there is a polynomial $f\in\mathbb{C}[T]$ with $\deg f>1$ such that:*

(1) $\deg(Q-f(P))<\deg P$ *if* $\deg P>1$,
(2) $\deg(Q-f(P))=1$ *if* $\deg P=1$.

Proof. Since $\deg Q>\deg P\geq 1$, we have $\deg Q+\deg P>2$ and $\operatorname{Jac}(\overline{P},\overline{Q})=0$ (because $\operatorname{Jac}(P,Q)\in\mathbb{C}^*$). By Lemma 1.5,

$$\overline{P}=\alpha h^{n_1},\qquad \overline{Q}=\beta h^{n_2}$$

for some $\alpha,\beta\in\mathbb{C}^*$, $n_1,n_2\in\mathbb{N}^*$ and some homogeneous polynomial $h\in\mathbb{C}[X,Y]$. Since $\deg\overline{P}\mid\deg\overline{Q}$, we have $n_1\mid n_2$ and so $\overline{Q}=c_1\overline{P}^{k_1}$ for some $c_1\in\mathbb{C}^*$ and $k_1=n_2/n_1$. Now $\deg(Q-c_1P^{k_1})<\deg Q$, and if $\deg(Q-c_1P^{k_1})<\deg P$ or $\deg(Q-c_1P^{k_1})=\deg P=1$, then we are done. And, if $\deg(Q-c_1P^{k_1})>\deg P$ or $\deg(Q-c_1P^{k_1})=\deg P>1$, then we can repeat the above arguments for $\overline{Q-c_1P^{k_1}}$ and $\overline{P}$ to obtain $c_2\in\mathbb{C}^*$ and $k_2<k_1$ such that $\overline{Q-c_1P^{k_1}}=c_2\overline{P}^{k_2}$. Then

$$\deg(Q-c_1P^{k_1}-c_2P^{k_2})<\deg(Q-c_1P^{k_1})$$

and we can proceed inductively. ■

Now we can prove the following (cf. [11, Thm. 1])

Proposition 9.2. *If $F \in \mathrm{Aut}(\mathbb{C}^2)$, then there is a number $l \in \mathbb{N}$ (possibly zero), affine automorphisms L_1, L_2 of $\mathbb{C}^2$ and triangular automorphisms $T_1, \ldots, T_l$ of the forms*

$$T_i : \mathbb{C}^2 \ni (x, y) \mapsto (x, y + f_i(x)) \in \mathbb{C}^2 \quad \text{for } i = 1, 3, \ldots, \tag{9.1}$$

$$T_i : \mathbb{C}^2 \ni (x, y) \mapsto (x + f_i(y), y) \in \mathbb{C}^2 \quad \text{for } i = 2, 4, \ldots, \tag{9.2}$$

with $\deg f_i > 1$, *such that*

$$F = L_2 \circ T_l \circ \cdots \circ T_1 \circ L_1.$$

Moreover, the number l is unique, and one can require that T_i, $i = 1, \ldots, l$, are of the form (9.1) *for even i and of the form* (9.2) *for odd i.*

Proof. Let $F = (F_1, F_2)$. If $\deg F_1 = \deg F_2 = 1$, then F is an affine mapping and we have $F = L_2 \circ L_1$ for $L_2 = \mathrm{id}_{\mathbb{C}^2}$ and $L_1 = F$.

If $\deg F_1 = \deg F_2 > 1$, then $\mathrm{Jac}(\overline{F_1}, \overline{F_2}) = 0$ (because $\mathrm{Jac}(F_1, F_2) \in \mathbb{C}^*$). Thus, by Lemma 1.5,

$$\overline{F_1} = \alpha h^n, \quad \overline{F_2} = \beta h^n$$

for some $\alpha, \beta \in \mathbb{C}^*, n \in \mathbb{N}^*$ and some homogeneous polynomial $h \in \mathbb{C}[X, Y]$. Let $L_2(x, y) = (x + (\alpha/\beta)y, y)$ and

$$(G_1, G_2) = L_2^{-1} \circ F.$$

Then $\deg G_2 = \deg F_2$ (actually $G_2 = F_2$) and $\deg G_1 < \deg G_2$. Hence we can assume that $\deg F_1 \neq \deg F_2$, and without loss of generality that $\deg F_1 < \deg F_2$ (if $\deg F_1 > \deg F_2$, then for $(G_1, G_2) = L_2^{-1} \circ F$, where $L_2(x, y) = (y, x)$, we have $\deg G_1 < \deg G_2$).

By Lemma 9.1, we obtain a polynomial $f \in \mathbb{C}[T]$, $\deg f > 1$, such that for $T_1(x, y) = (x, y + f(x))$ and $(G_1, G_2) = T_1^{-1} \circ F$ we have $\deg G_2 < \deg G_1$ or $\deg G_2 = \deg G_1 = 1$. In the second case (G_1, G_2) is an affine map and for $L_1 = (G_1, G_2)$ we have $F = T_1 \circ L_1$, so we are done. And in the first case we can use Lemma 9.1 once again and proceed inductively.

Thus we can assume that $F = \widetilde{L}_2 \circ \widetilde{T}_1 \circ \cdots \circ \widetilde{T}_l \circ \widetilde{L}_1$, where $\widetilde{L}_1, \widetilde{L}_2 \in \mathrm{Aff}(\mathbb{C}^2)$ and $\widetilde{T}_i$ are of the forms (9.1), (9.2). Let us set

$$T_i = \begin{cases} \widetilde{T}_{l+1-i} & \text{for odd } l, \\ L \circ \widetilde{T}_{l+1-i} \circ L & \text{for even } l, \end{cases}$$

$$L_1 = \begin{cases} \widetilde{L}_1 & \text{for odd } l, \\ L \circ \widetilde{L}_1 & \text{for even } l, \end{cases} \qquad L_2 = \begin{cases} \widetilde{L}_2 & \text{for odd } l, \\ \widetilde{L}_2 \circ L & \text{for even } l, \end{cases}$$

where $L(x, y) = (y, x)$. Then one can check that $F = L_2 \circ T_l \circ \cdots \circ T_1 \circ L_1$.

To see that l is unique it is enough to notice that $L \circ T_j \circ L \in J(\mathbb{C}^2) \setminus \mathrm{Aff}(\mathbb{C}^2)$, $j = 1, 3, \ldots$, and $T_j \in J(\mathbb{C}^2) \setminus \mathrm{Aff}(\mathbb{C}^2)$, $j = 2, 4, \ldots$, and so

$$F = \widehat{L}_2 \circ \cdots \circ L \circ (L \circ T_3 \circ L) \circ L \circ T_2 \circ L \circ (L \circ T_1 \circ L) \circ (L \circ L_1)$$

is the amalgamated representation of F for suitable sets Φ and Ψ (see Definition 1.2, Proposition 1.1 and [7, Cor. 5.1.3]), where

$$\widehat{L}_2 = \begin{cases} \widetilde{L}_2 & \text{for even } l, \\ \widetilde{L}_2 \circ L & \text{for odd } l. \end{cases}$$

To see that the last statement holds true, one can write

$$F = (L_2 \circ L) \circ (L \circ T_l \circ L) \circ \cdots \circ (L \circ T_1 \circ L) \circ (L \circ L_1). \ \blacksquare$$

DEFINITION 9.3 (see e.g. [11, p. 612]). Let $F \in \mathrm{Aut}(\mathbb{C}^2)$ be a polynomial automorphism. The number l from Proposition 9.2 is called the *length* of F and denoted $\operatorname{length} F$.

In what follows we will use the following numerical object.

DEFINITION 9.4. Let $k \in \mathbb{N}^*$ and let $k = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ be its prime decomposition. Then we denote by $l(k)$ the number $\alpha_1 + \cdots + \alpha_r$.

Obviously, $l(k_1k_2) = l(k_1) + l(k_2)$ for all $k_1, k_2 \in \mathbb{N}^*$, and $l(k) \geq 1$ for $k > 1$.

THEOREM 9.5. *Let* $F \in \mathrm{Aut}(\mathbb{C}^2)$. *Then:*

(1) *if* $\operatorname{length} F = 1$, *then* $\operatorname{mdeg} F \in \{(1,d), (d,1), (d,d)\}$, *where* $1 < d$,
(2) *if* $\operatorname{length} F = 2$, *then either* $\operatorname{mdeg} F \in \{(d_1,d_2), (d_2,d_1)\}$ *with* $1 < d_1 < d_2$, $d_1 \mid d_2$, *or* $\operatorname{mdeg} F = (d,d)$ *with* $l(d) \geq 2$ *(in particular* $d > 1$ *is a composite number),*
(3) *if* $\operatorname{length} F \geq 3$, *then either* $\operatorname{mdeg} F \in \{(d_1,d_2), (d_2,d_1)\}$ *with* $1 < d_1 < d_2$, $d_1 \mid d_2$, $l(d_1) \geq \operatorname{length} F - 1$, *or* $\operatorname{mdeg} F = (d,d)$ *with* $l(d) \geq \operatorname{length} F$.

Proof. (1) Since $\operatorname{length} F = 1$, we have $F = L_2 \circ T \circ L_1$, where $L_1, L_2 \in \mathrm{Aff}(\mathbb{C}^2)$ and T is of the form $T : \mathbb{C}^2 \ni (x,y) \mapsto (x, y + f(x)) \in \mathbb{C}^2$ with $\deg f > 1$. Thus $\operatorname{mdeg}(T \circ L_1) = (1,d)$, where $d = \deg f$, and then one can easily check that $\operatorname{mdeg}(L_2 \circ T \circ L_1) \in \{(1,d), (d,1), (d,d)\}$.

(2) Since $\operatorname{length} F = 2$, we have $F = L_2 \circ T_2 \circ T_1 \circ L_1$, where $L_1, L_2 \in \mathrm{Aff}(\mathbb{C}^2)$ and T_1, T_2 are of the form

$$T_1 : \mathbb{C}^2 \ni (x,y) \mapsto (x, y + f_1(x)) \in \mathbb{C}^2, \qquad T_2 : \mathbb{C}^2 \ni (x,y) \mapsto (x + f_2(y), y) \in \mathbb{C}^2,$$

with $\deg f_1, \deg f_2 > 1$. Thus $\operatorname{mdeg}(T_1 \circ L_1) = (1, \deg f_1)$, and then $\operatorname{mdeg}(T_2 \circ T_1 \circ L_1) = (d_2, d_1)$, where $d_1 = \deg f_1$, $d_2 = \deg f_2 \cdot \deg f_1$. Since $\deg f_1, \deg f_2 > 1$, it follows that $l(d_2) = l(\deg f_1) + l(\deg f_2) \geq 2$. Now, one can easily see that $\operatorname{mdeg}(L_2 \circ T_2 \circ T_1 \circ L_1) \in \{(d_1,d_2), (d_2,d_1), (d_2,d_2)\}$.

(3) Since $l = \operatorname{length} F \geq 3$, we have $F = L_2 \circ T_l \circ \cdots \circ T_1 \circ L_1$, where $L_1, L_2 \in \mathrm{Aff}(\mathbb{C}^2)$ and $T_1, \ldots, T_l$ are of the form

$$T_i : \mathbb{C}^2 \ni (x,y) \mapsto (x + f_i(y), y) \in \mathbb{C}^2$$

for even i, and

$$T_i : \mathbb{C}^2 \ni (x,y) \mapsto (x, y + f_i(x)) \in \mathbb{C}^2$$

for odd i, with $\deg f_i > 1$ for $i = 1, \ldots, l$. Now, one can easily check that (see also [7, Lem. 5.1.2])

$$\operatorname{mdeg}(T_l \circ \cdots \circ T_1 \circ L_1) = \begin{cases} (\prod_{j=1}^{l} \deg f_j, \prod_{j=1}^{l-1} \deg f_j) & \text{for even } l, \\ (\prod_{j=1}^{l-1} \deg f_j, \prod_{j=1}^{l} \deg f_j) & \text{for odd } l. \end{cases}$$

Let

$$d_2 = \prod_{j=1}^{l} \deg f_j \quad \text{and} \quad d_1 = \prod_{j=1}^{l-1} \deg f_j.$$

Then $\operatorname{mdeg}(T_l \circ \cdots \circ T_1 \circ L_1) = (d_1, d_2)$ for odd l, and $\operatorname{mdeg}(T_l \circ \cdots \circ T_1 \circ L_1) = (d_2, d_1)$ for even l.

Since $\deg f_i > 1$ for $i = 1, \ldots, l$, we have

$$l(d_1) \geq l(\deg f_1) + \cdots + l(\deg f_{l-1}) \geq l - 1,$$

and

$$l(d_2) \geq l(\deg f_1) + \cdots + l(\deg f_l) \geq l.$$

Of course, as in the previous case, we have

$$\operatorname{mdeg}(L_2 \circ T_l \circ \cdots \circ T_1 \circ L_1) \in \{(d_1, d_2), (d_2, d_1), (d_2, d_2)\}. \ ■$$

THEOREM 9.6. *Let* $F \in \operatorname{Aut}(\mathbb{C}^2)$ *be a polynomial automorphism with* $\operatorname{mdeg} F = (d_1, d_2)$, $d_1 \leq d_2$. *Then* $\operatorname{length} F \leq \min\{l(d_2), l(d_1) + 1\}$.

Proof. This is a consequence of Theorem 9.5. ■

9.2. The case of length 1. Here we consider the situation when $\operatorname{length} F = 1$. Because of Theorem 9.5, this simple situation is described by the following result.

THEOREM 9.7. *Let* $F \in \operatorname{Aut}(\mathbb{C}^2)$, *where* $\operatorname{length} F = 1$ *and* $\operatorname{mdeg} F \in \{(1, d), (d, d)\}$ *with* $1 < d$. *Then*

$$\operatorname{mdeg} F^{-1} \in \{(1, d), (d, 1), (d, d)\}.$$

Proof. Since $\operatorname{length} F = 1$, we have $F = L_2 \circ T \circ L_1$, where T is a triangular automorphism of the form $T : \mathbb{C}^2 \ni (x, y) \mapsto (x, y + f(x)) \in \mathbb{C}^2$ with $\deg f > 1$, and $L_1, L_2 \in \operatorname{Aff}(\mathbb{C}^2)$. Notice that $\deg f = \deg T = \deg F = d$. Thus $\operatorname{mdeg}(T^{-1} \circ L_2^{-1}) = (1, d)$. Now, it is easy to see that

$$\operatorname{mdeg} F^{-1} = \operatorname{mdeg}(L_1^{-1} \circ T^{-1} \circ L_2^{-1}) \in \{(1, d), (d, 1), (d, d)\}. \ ■$$

The following two examples show that all possibilities described in the above theorem are realized.

EXAMPLE 9.8. Let $d \in \mathbb{N} \setminus \{0, 1\}$. Put

$$F_a = T, \quad F_b = T \circ L_b, \quad F_c = T \circ L_c,$$

where $T(x, y) = (x, y + x^d)$, $L_b(x, y) = (y, x)$ and $L_c(x, y) = (x + y, y)$. One can check that

$$\operatorname{mdeg} F_a = \operatorname{mdeg} F_b = \operatorname{mdeg} F_c = (1, d),$$
$$\operatorname{mdeg} F_a^{-1} = (1, d), \quad \operatorname{mdeg} F_b^{-1} = (d, 1), \quad \operatorname{mdeg} F_c^{-1} = (d, d).$$

EXAMPLE 9.9. Let $d \in \mathbb{N} \setminus \{0, 1\}$ and put

$$F_a = L_c \circ T, \quad F_b = L_c \circ T \circ L_b, \quad F_c = L_c \circ T \circ L_c,$$

where T, L_b and L_c are as in the previous example. One can check that

$$\operatorname{mdeg} F_a = \operatorname{mdeg} F_b = \operatorname{mdeg} F_c = (d, d),$$
$$\operatorname{mdeg} F_a^{-1} = (1, d), \quad \operatorname{mdeg} F_b^{-1} = (d, 1), \quad \operatorname{mdeg} F_c^{-1} = (d, d).$$

9.3. The case (d_1, d_2). Here we investigate the situation when $\operatorname{mdeg} F = (d_1, d_2)$, $d_1 \neq d_2$ and $\operatorname{length} F > 1$. Of course, without loss of generality, we can assume that $d_1 < d_2$. Because of Theorem 9.5, the situation is described by the following two theorems.

THEOREM 9.10. *Let* $F \in \operatorname{Aut}(\mathbb{C}^2)$, *where* $\operatorname{length} F = 2$ *and* $\operatorname{mdeg} F = (d_1, d_2)$ *with* $1 < d_1 < d_2$, $d_1 \mid d_2$. *Then*

$$\operatorname{mdeg} F^{-1} \in \{(d_2, d_2/d_1), (d_2/d_1, d_2), (d_2, d_2)\}.$$

Proof. Since $\operatorname{length} F = 2$, we have $F = L_2 \circ T_2 \circ T_1 \circ L_1$, where T_1, T_2 are triangular (and non-affine) automorphisms and $L_1, L_2 \in \operatorname{Aff}(\mathbb{C}^2)$. We can assume that T_1 and T_2 are of the form

$$T_1 : \mathbb{C}^2 \ni (x, y) \mapsto (x + f_1(y), y) \in \mathbb{C}^2, \quad T_2 : \mathbb{C}^2 \ni (x, y) \mapsto (x, y + f_2(x)) \in \mathbb{C}^2.$$

Then $\operatorname{mdeg}(T_1 \circ L_1) = (\deg f_1, 1)$ and $\operatorname{mdeg}(T_2 \circ T_1 \circ L_1) = (\deg f_1, \deg f_2 \cdot \deg f_1)$. Thus, we have $\deg f_1 = d_1$ and $\deg f_2 = d_2/d_1$. Now one can easily check that

$$\operatorname{mdeg}(T_2^{-1} \circ L_2^{-1}) = (1, \deg f_2) = (1, d_2/d_1),$$
$$\operatorname{mdeg}(T_1^{-1} \circ T_2^{-1} \circ L_2^{-1}) = (\deg f_2 \cdot \deg f_1, \deg f_2) = (d_2, d_2/d_1).$$

Since $F^{-1} = L_1^{-1} \circ T_1^{-1} \circ T_2^{-1} \circ L_2^{-1}$, the result follows. ■

The following example shows that all possibilities described in the above theorem are realized.

EXAMPLE 9.11. Let $d_1, d_2 \in \mathbb{N}$ be such that $1 < d_1 < d_2$, $d_1 \mid d_2$. Put

$$T_1 : \mathbb{C}^2 \ni (x, y) \mapsto (x + y^{d_1}, y) \in \mathbb{C}^2, \quad T_2 : \mathbb{C}^2 \ni (x, y) \mapsto (x, y + x^{\delta}) \in \mathbb{C}^2,$$

where $\delta = d_2/d_1$, and

$$F_a = T_2 \circ T_1, \quad F_b = T_2 \circ T_1 \circ L_b, \quad F_c = T_2 \circ T_1 \circ L_c,$$

where $L_b(x, y) = (y, x)$ and $L_c(x, y) = (x, y + x)$. One can check that

$$\operatorname{mdeg} F_a = \operatorname{mdeg} F_b = \operatorname{mdeg} F_c = (d_1, d_2),$$
$$\operatorname{mdeg} F_a^{-1} = (d_2, d_2/d_1), \quad \operatorname{mdeg} F_b^{-1} = (d_2/d_1, d_2), \quad \operatorname{mdeg} F_c^{-1} = (d_2, d_2).$$

THEOREM 9.12. *Let* $F \in \operatorname{Aut}(\mathbb{C}^2)$, *where* $\operatorname{length} F \geq 3$ *and* $\operatorname{mdeg} F = (d_1, d_2)$ *with* $1 < d_1 < d_2$, $d_1 \mid d_2$. *Then*

$$\operatorname{mdeg} F^{-1} \in \{(d_2, d_2/a), (d_2/a, d_2), (d_2, d_2) : a \in \mathcal{A}_F\},$$

where $\mathcal{A}_F = \{a : 1 < a < d_1,\ a \mid d_1,\ l(d_1/a) \geq \operatorname{length} F - 2\}$.

Proof. Let $l = \operatorname{length} F$. Then F can be written in the form

$$F = L_2 \circ T_l \circ \cdots \circ T_1 \circ L_1,$$

where $T_1, \ldots, T_l$ are triangular (and non-affine) automorphisms and $L_1, L_2 \in \operatorname{Aff}(\mathbb{C}^2)$. We can assume that T_i are of the form

$$T_i : \mathbb{C}^2 \ni (x, y) \mapsto (x + f_i(y), y) \in \mathbb{C}^2 \quad \text{for odd } i,$$
$$T_i : \mathbb{C}^2 \ni (x, y) \mapsto (x, y + f_i(x)) \in \mathbb{C}^2 \quad \text{for even } i.$$

Now, one can check that

$$\operatorname{mdeg}(T_l \circ \cdots \circ T_1 \circ L_1) = \begin{cases} (\prod_{j=1}^{l} \deg f_j, \prod_{j=1}^{l-1} \deg f_j) & \text{for odd } l, \\ (\prod_{j=1}^{l-1} \deg f_j, \prod_{j=1}^{l} \deg f_j) & \text{for even } l. \end{cases}$$

In both cases we have

$$\prod_{j=1}^{l} \deg f_j = d_2 \quad \text{and} \quad \prod_{j=1}^{l-1} \deg f_j = d_1.$$

Let $a = \deg f_1$. Since T_i are not affine, $\deg f_i > 1$. Since also $l \geq 3$ (in other words, $l - 1 > 1$), a is a proper divisor of d_1 and $l(d_1/a) = l(\deg f_2 \cdots \deg f_{l-1}) \geq l - 2$.

Now, one can check that

$$\operatorname{mdeg}(T_1^{-1} \circ \cdots \circ T_l^{-1} \circ L_2^{-1}) = \Big(\prod_{j=1}^{l} \deg f_j, \prod_{j=2}^{l} \deg f_j\Big) = (d_2, d_2/a).$$

Since $F^{-1} = L_1^{-1} \circ T_1^{-1} \circ \cdots \circ T_l^{-1} \circ L_2^{-1}$, the result follows. ■

Also in this case all possibilities are realized, as the following example shows.

EXAMPLE 9.13. Let $d_1, d_2 \in \mathbb{N}$ be such that $1 < d_1 < d_2$, $d_1 \mid d_2$, and let $l \leq l(d_1) + 1$ be an even number. Assume also that a is a proper divisor of d_1 such that $l(d_1/a) \geq l - 2$. Take positive integers $a_2, \ldots, a_{l-1}$ such that

$$d_1 = a \cdot a_2 \cdots a_{l-1}.$$

Such integers exist, because $l(d_1/a) \geq l - 2$. Now put:

$$\begin{aligned} T_1 &: \mathbb{C}^2 \ni (x, y) \mapsto (x + y^a, y) \in \mathbb{C}^2, \\ T_2 &: \mathbb{C}^2 \ni (x, y) \mapsto (x, y + x^{a_2}) \in \mathbb{C}^2, \\ T_3 &: \mathbb{C}^2 \ni (x, y) \mapsto (x + y^{a_3}, y) \in \mathbb{C}^2, \\ &\vdots \\ T_{l-1} &: \mathbb{C}^2 \ni (x, y) \mapsto (x + y^{a_{l-1}}, y) \in \mathbb{C}^2, \\ T_l &: \mathbb{C}^2 \ni (x, y) \mapsto (x, y + x^{\delta}) \in \mathbb{C}^2, \end{aligned}$$

where $\delta = d_2/d_1$. Also set

$$F_a = T_l \circ \cdots \circ T_1, \qquad F_b = T_l \circ \cdots \circ T_1 \circ L_b, \qquad F_c = T_l \circ \cdots \circ T_1 \circ L_c,$$

where L_b and L_c are defined as in the previous example. One can check that

$$\operatorname{mdeg} F_a = \operatorname{mdeg} F_b = \operatorname{mdeg} F_c = (d_1, d_2), \qquad \operatorname{length} F = l.$$

It is also easy to see that

$$\operatorname{mdeg} F_a^{-1} = (d_2, d_2/a), \quad \operatorname{mdeg} F_b^{-1} = (d_2/a, d_2), \quad \operatorname{mdeg} F_c^{-1} = (d_2, d_2).$$

In a similar way one can obtain an example when l is odd.

The following example shows an application of Theorem 9.12.

EXAMPLE 9.14. Let $F \in \operatorname{Aut}(\mathbb{C}^2)$ be such that $\operatorname{mdeg} F = (60, 120)$. Since $l(60) = l(2^2 \cdot 3 \cdot 5) = 4$, we have $\operatorname{length} F \leq 5$.

If $\operatorname{length} F = 3$, then

$$\mathcal{A}_F = \{2, 3, 5, 4, 6, 10, 15, 12, 20, 30\},$$

and so, by Theorem 9.12,

$$\begin{aligned}\operatorname{mdeg} F^{-1} \in \{&(120, 60), (120, 40), (120, 24), (120, 30), (120, 20),\\ &(120, 12), (120, 8), (120, 10), (120, 6), (120, 4), (60, 120),\\ &(40, 120), (24, 120), (30, 120), (20, 120), (12, 120).\\ &(8, 120), (10, 120), (6, 120), (4, 120), (120, 120)\}.\end{aligned}$$

If $\operatorname{length} F = 4$, then

$$\mathcal{A}_F = \{2, 3, 5, 4, 6, 10, 15\},$$

and so, by Theorem 9.12,

$$\begin{aligned}\operatorname{mdeg} F^{-1} \in \{&(120, 60), (120, 40), (120, 24), (120, 30), (120, 20),\\ &(120, 12), (120, 8), (60, 120), (40, 120), (24, 120),\\ &(30, 120), (20, 120), (12, 120), (8, 120), (120, 120)\}.\end{aligned}$$

If $\operatorname{length} F = 5$, then

$$\mathcal{A}_F = \{2, 3, 5\},$$

and so, by Theorem 9.12,

$$\operatorname{mdeg} F^{-1} \in \{(120, 60), (120, 40), (120, 24), (60, 120), (40, 120), (24, 120), (120, 120)\}.$$

Moreover, by the previous example, all the listed possibilities are realized.

9.4. The case (d, d)**.** Using similar arguments to those in the proof of Theorem 9.12 one can prove the following

THEOREM 9.15. *Let* $F \in \operatorname{Aut}(\mathbb{C}^2)$, *where* $\operatorname{length} F \geq 2$ *and* $\operatorname{mdeg} F = (d, d)$ *with* $1 < d$. *Then*

$$\operatorname{mdeg} F^{-1} \in \{(d, d/a), (d/a, d), (d, d) : a \in \mathcal{A}_F\},$$

where $\mathcal{A}_F = \{a : 1 < a < d,\ a \mid d,\ l(d/a) \geq \operatorname{length} F - 1\}$.

Also in this case all the possibilities are realized, as the following example shows (this example is a modification of the example given after Theorem 9.12).

EXAMPLE 9.16. Let $d \in \mathbb{N}$ and $l \geq 2$ be an even number such that $l \leq l(d)$. Assume also that a is a proper divisor of d such that $l(d/a) \geq l - 1$. Take positive integers $a_2, \ldots, a_l$ such that

$$d = a \cdot a_2 \cdots a_l.$$

Such integers exist, because $l(d/a) \geq l - 1$. Let $T_1, \ldots, T_{l-1}$ be defined as in Example 9.13 and put

$$T_l : \mathbb{C}^2 \ni (x, y) \mapsto (x, y + x^{a_l}) \in \mathbb{C}^2.$$

Also set

$$F_a = L \circ T_l \circ \cdots \circ T_1, \quad F_b = L \circ T_l \circ \ldots \circ T_1 \circ L_b, \quad F_c = L \circ T_l \circ \cdots \circ T_1 \circ L_c,$$

where $L_b(x,y) = (y,x)$, $L_c(x,y) = (x, y+x)$ and $L(x,y) = (x+y, y)$. Then one can check that

$$\operatorname{mdeg} F_a = \operatorname{mdeg} F_b = \operatorname{mdeg} F_c = (d,d), \quad \operatorname{length} F = l,$$
$$\operatorname{mdeg} F_a^{-1} = (d, d/a), \quad \operatorname{mdeg} F_b^{-1} = (d/a, d), \quad \operatorname{mdeg} F_c^{-1} = (d,d).$$

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