Contents

1. Introduction 5
2. Preliminaries 8
3. The domain of definition of the complex Monge–Ampère operator 13
4. Maximal plurisubharmonic functions 22
5. The comparison principle 29
6. The Dirichlet problem 37
   6.1. Regular measures 37
   6.2. Singular measures 43
   6.3. The general case 46
7. Generalized boundary values 50
   7.1. Monge–Ampère boundary measures 51
   7.2. Stability of solutions and the complex Monge–Ampère type equation 56
   7.3. Subextension 63
8. The homogeneous Dirichlet problem for pluriharmonic functions 67

References 78

Abstract

The complex Monge–Ampère operator is a useful tool not only within pluripotential theory, but also in algebraic geometry, dynamical systems and Kähler geometry. In this self-contained survey we present a unified theory of Cegrell's framework for the complex Monge–Ampère operator.

Acknowledgements. The author would like to thank Per Åhag and Sławomir Kołodziej for their generous help and encouragement while writing this survey. Their valuable comments and suggestions essentially improved the final version of this survey.

2010 *Mathematics Subject Classification*: Primary 32W20; Secondary 32U15.
Key words and phrases: Complex Monge–Ampère operator, Dirichlet problem, pluripolar set, plurisubharmonic function, pluriharmonic function.

1. Introduction

Let ∂, $\bar\partial$ be the usual differential operators, i the imaginary unit, $d = \partial + \bar\partial$ and $d^c = i(\bar\partial - \partial)$. The *complex Monge–Ampère operator* is then defined by

$$(u_1, \ldots, u_n) \mapsto (dd^c u_1) \wedge \cdots \wedge (dd^c u_n) \in \mathcal{M}$$

where $u_1, \ldots, u_n \in \mathcal{PSH}(\Omega) \cap C^2(\Omega)$, and $\mathcal{M}$ is the set of Radon measures. If $u = u_1 = \cdots = u_n$, then

$$(dd^c u)^n = 4^n\, n! \det\left(\frac{\partial^2 u}{\partial z_j \partial \bar z_k}\right) dV_n,$$

where

$$dV_n = dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n = \left(\frac{i}{2}\right)^n dz_1 \wedge d\bar z_1 \wedge \cdots \wedge dz_n \wedge d\bar z_n$$

is the volume form on $\mathbb{R}^{2n}$ ($\simeq \mathbb{C}^n$), and the $n \times n$ matrix

$$\left(\frac{\partial^2 u}{\partial z_j \partial \bar z_k}\right)$$

is the complex Hessian of u. The history of the extending of the complex Monge–Ampère operator is rich and colorful so let us just mention a few results here; for further information we refer to [13, 60, 65, 70]. It follows from the work of Chern, Levine, and Nirenberg [40] that there exists a continuous extension (in the weak*-topology) of the complex Monge–Ampère operator to $\mathcal{PSH}(\Omega) \cap C(\Omega)$. Bedford and Taylor proved in their seminal article [16] that it is possible to extend the operator in question to locally bounded plurisubharmonic functions, and later they proved in [17, Proposition 3.6] that their extension is also valid for plurisubharmonic functions in the usual Sobolev space $W^{1,2}$. In the light of these positive results one might expect that it is possible to construct a continuous extension to the whole space of plurisubharmonic function. This is not the case, since Shiffman and Taylor [80] gave an example which shows that such an extension is not possible if we want to have the range of the complex Monge–Ampère operator in the space of Radon measures. Here the underlying problem is multiplication of distributions. It is worth noting that Kiselman [59] defined the complex Monge–Ampère operator with the help of multiplication of distributions in the sense of Colombeau. This approach of using more up to date distribution theory has still to be explored.

In 1998 Cegrell [30] introduced a class of negative unbounded plurisubharmonic functions in which he solved the Dirichlet problem for the complex Monge–Ampère equation. Since then the complex Monge–Ampère operator in this new setting was investigated by many mathematicians.

The goal of this survey is to present the theory of the complex Monge–Ampère operator in the framework introduced by Cegrell [30, 32, 33]. We shall use methods which are developments of the classical work for locally bounded plurisubharmonic functions. Our purpose is to collect the recent developments in the theory of the complex Monge–Ampère operator spread throughout the literature. We shall make improvements, simplify proofs, and unify the presentation to make this theory more accessible to the reader. The only demand on the reader is the knowledge of the fundamental facts concerning locally bounded plurisubharmonic functions and the Monge–Ampère operator acting on them (see e.g. [62]). To make this survey self-contained we present, in Chapter 2, the basic definitions and facts.

Let us start to give some perimeters of the setting. All definitions and notions can be found in Chapter 2 along with the necessary facts from the locally bounded case. Let $\Omega \subset \mathbb{C}^n$, $n \geq 1$, be a hyperconvex domain. In Chapter 3 we shall focus on extending the complex Monge–Ampère operator to the Cegrell class $\mathcal{E}$. The set $\mathcal{E}$ of negative plurisubharmonic functions was introduced by Cegrell in [32], and he proved that this is the largest set of non-positive plurisubharmonic functions defined on Ω for which the complex Monge–Ampère operator can be continuously extended (Theorem 3.9). Błocki proved in [22] that if $n = 2$, then

$$\mathcal{E} = \{\varphi \in \mathcal{PSH}(\Omega) \cap W^{1,2}_{\mathrm{loc}}(\Omega) : \varphi \leq 0\}.$$

Later, in [23], he obtained a complete characterization of $\mathcal{E}$ for $n \geq 1$.

Consider the following example by Kiselman [59]:

$$u(z_1, \ldots, z_n) = (-\log|z_1|)^{1/n}(|z_2|^2 + \cdots + |z_n|^2 - 1).$$

It is well-known that $u \notin \mathcal{E}$, and $u \in W^{1,2}_{\mathrm{loc}}(\Omega) \cap \mathcal{PSH}(\Omega)$ if, and only if, $n \geq 3$.

An important question is to describe the boundary values of functions in $\mathcal{E}$. To study this problem we shall need the notion of *maximal* plurisubharmonic functions introduced by Sadullaev in [79]. By using a theorem of Cegrell [34] (see Theorem 4.2 below) we are able in a convenient way to construct the smallest maximal plurisubharmonic majorant $\tilde{u}$ of a function u in $\mathcal{E}$. We set $\mathcal{N} = \{u \in \mathcal{E} : \tilde{u} = 0\}$. Functions in the Cegrell class $\mathcal{N}$ are therefore (in a way) functions with *zero boundary values*. To proceed we say that a plurisubharmonic function u is in $\mathcal{N}(H)$ $(= \mathcal{N}(\Omega, H))$, $H \in \mathcal{E}$, if there exists a function $\varphi \in \mathcal{N}$ such that

$$H \geq u \geq \varphi + H, \tag{1.1}$$

and therefore we say that any function in $\mathcal{N}(H)$ has *boundary values* H. Note that if $u \in \mathcal{N}(H)$, then $u \in \mathcal{N}(\tilde{u})$ so our interest in describing the boundary value of a function in $\mathcal{E}$ can be rephrased as: If $u \in \mathcal{E}$, then is u in $\mathcal{N}(\tilde{u})$? At the point of writing this survey, this question is not settled. In Theorem 4.10 and Proposition 6.5 we give two positive answers to the above question. In Chapter 7.1 we shall first show that the classical approach to boundary values coincides with the boundary values arising from the generalized Demailly–Hörmander boundary measures from [36] (Theorem 7.9). Then we prove that the notion of boundary values associated to inequality (1.1) is the same as the generalized Demailly–Hörmander boundary measures (Theorem 7.10). Thus, we can view the boundary values arising from (1.1) as a generalization of the classical point of view.

Now we present two main theorems from Chapter 6.

Theorem 6.22. *Assume that μ is a non-negative Radon measure. If there exists a function $w \in \mathcal{E}$ such that $\mu \leq (dd^c w)^n$, then for every maximal plurisubharmonic function $H \in \mathcal{E}$ there exists a function $u \in \mathcal{E}$ such that $w + H \leq u \leq H$ and $(dd^c u)^n = \mu$.*

Theorem 6.22 is a generalization of the Kołodziej subsolution theorem for bounded plurisubharmonic functions (Theorem 2.7). Example 6.24 shows that there exists a non-negative Radon measure μ such that there does not exist any function $u \in \mathcal{E}$ that satisfies $(dd^c u)^n = \mu$. On the other hand, it is not yet clear whether it is possible to solve the Monge–Ampère equation when the given measure has finite total mass.

Let us now give an outline of the proof of Theorem 6.22. Using the Radon–Nikodym theorem we have $\mu = \tau(dd^c w)^n$, $0 \leq \tau \leq 1$, and by the Cegrell–Lebesgue decomposition of Monge–Ampère measures [30, 32] we have

$$\mu = \tau(dd^c w)^n = \tau f (dd^c \varphi)^n + \tau\nu,$$

where $\varphi \in \mathcal{E}_0$, $0 \leq f \in L^1_{\mathrm{loc}}((dd^c\varphi)^n)$, and ν is a positive measure carried by $\{u = -\infty\}$. The measures $f(dd^c\varphi)^n$ and ν are mutually singular and therefore they will be referred to as μ's *regular* and *singular part*, respectively. Lemma 6.21 shows that we can work with the regular and singular part separately. In Chapter 6.1 we prove the following theorem.

Theorem 6.6. *Assume that μ is a non-negative Radon measure defined by $\mu = (dd^c\varphi)^n$, $\varphi \in \mathcal{N}$ with $\mu(A) = 0$ for every pluripolar set $A \subseteq \Omega$. Then for every $H \in \mathcal{E}$ such that $(dd^c H)^n \leq \mu$ there exists a uniquely determined function $u \in \mathcal{N}(H)$ such that $(dd^c u)^n = \mu$ on Ω.*

The two main tools in the proof of Theorem 6.6 are the so called *comparison principle* established in Corollary 5.10, and Kołodziej's subsolution theorem.

In the proof of Theorem 6.22 the singular part is more delicate to handle. One reason for this is that the comparison principle is not in general valid for measures that charge a pluripolar set (see e.g. [16]). Let $u \in \mathcal{E}$ and $0 \leq g \leq 1$ be a $\chi_{\{u=-\infty\}}(dd^c u)^n$-measurable function that vanishes outside $\{u = -\infty\}$. Then consider

$$u^g = \inf_{\substack{f \in T \\ f \leq g}} (\sup\{u_\tau : f \leq \tau,\ \tau \text{ is a bounded lower semicontinuous function}\})^*,$$

where u_τ is as in Definition 6.10, T is a family of certain simple functions, and w^* denotes the upper semicontinuous regularization of w. In particular, Theorem 6.15 in Chapter 6.2 implies that $(dd^c u^{\chi_E})^n = \chi_E (dd^c u)^n$, where χ_E is the characteristic function of the set E in Ω. In this way we have sufficient control over the singular part.

In Chapter 7.2 we prove a stability theorem for the complex Monge–Ampère equation in $\mathcal{N}(H)$. We also show the existence and stability of solutions of Monge–Ampère type equations. Both results generalize Cegrell and Kołodziej's work [37]. In Chapter 7.3, we present a theorem concerning subextension of plurisubharmonic functions without increasing the total Monge–Ampère mass. This type of theorem has proven to be a useful tool in, for example, approximation of plurisubharmonic functions [35], and estimating the volume of plurisubharmonic sublevel sets [5]. We shall end this survey with some applications. In Chapter 8 we shall study continuous pluriharmonic boundary values on images of product domains, i.e. we shall discuss the following question: Let $D \subset \mathbb{C}^n$ be a product domain, and let $f : \partial D \to \mathbb{R}$ be a continuous function. Under what condition does there exist a function h that is pluriharmonic on Ω_π, continuous on $\bar{\Omega}_\pi$ and $h|_{\partial\Omega_\pi} = f$? The methods presented here are not only useful within pluripotential theory, but also as a tool in algebraic geometry, dynamical systems and Kähler geometry (see e.g. [5, 51, 52, 56]).

The author's contributions are as follows:

- Chapter 2: Theorem 2.17 is from [4].
- Chapter 3: Proposition 3.16 is from [46].
- Chapter 4: Example 4.12 is from [6].
- Chapter 5: Theorems 5.7 to 5.14 are from [4].
- Chapter 6.1: Theorem 6.6 is from [4].
- Chapter 6.2: all results are from [4].
- Chapter 6.3: all results are from [4] except Example 6.24.
- Chapter 7.1: Example 7.7 was constructed by the author.
- Chapter 7.2: the results in this chapter are exclusively published in this survey.
- Chapter 7.3: all results are from [47] except Example 7.22.
- Chapter 8: all results are from [6, 8, 45].

2. Preliminaries

In this chapter we present some fundamental results concerning the complex Monge–Ampère operator: the comparison principle (Theorem 2.2), quasicontinuity of plurisubharmonic functions (Theorem 2.5), continuity of the complex Monge–Ampère operator with respect to monotone sequences (Theorem 2.1) or sequences converging in capacity (Theorem 2.6). We also introduce the Cegrell classes of negative plurisubharmonic functions $\mathcal{E}_0, \mathcal{F}, \mathcal{E}, \mathcal{E}_0(H), \mathcal{F}(H)$, and $\mathcal{E}(H)$ (Definitions 2.8, 2.9 and 2.12), and prove some basic properties.

A *domain* is an open and connected set. Recall that a bounded domain $\Omega \subseteq \mathbb{C}^n$ is called *hyperconvex* if there exists a plurisubharmonic function $\varphi : \Omega \to (-\infty, 0)$ such that the closure of the set

$$\{z \in \Omega : \varphi(z) < c\}$$

is compact in Ω for every $c \in (-\infty, 0)$, i.e., the level set $\{z \in \Omega : \varphi(z) < c\}$ is relatively compact in Ω. If there can be no misinterpretation, a sequence $[\,\cdot\,]_{j=1}^{\infty}$ will be denoted by $[\,\cdot\,]$. In this article, $\mathcal{PSH}(\Omega)$ is the family of plurisubharmonic functions defined on Ω, and $\mathcal{PSH}^-(\Omega)$ stands for the negative plurisubharmonic functions.

We start by recalling the most fundamental results concerning the complex Monge–Ampère operator acting on locally bounded plurisubharmonic functions.

The Monge–Ampère operator is continuous with respect to monotone sequences.

THEOREM 2.1 ([16]). *Let $[u_j^k]_{j=1}^{\infty}$ be a locally uniformly bounded monotone (decreasing or increasing) sequence of plurisubharmonic functions in Ω for $k = 1, \dots, n$; and let $u_j^k \to u^k \in \mathcal{PSH}(\Omega) \cap L^{\infty}_{\mathrm{loc}}(\Omega)$ as $j \to \infty$ for $k = 1, \dots, n$. Then*

$$dd^c u_j^1 \wedge \cdots \wedge dd^c u_j^n \to dd^c u^1 \wedge \cdots \wedge dd^c u^n$$

in the weak-topology.*

THEOREM 2.2 ([19], comparison principle). *Let $\Omega \subseteq \mathbb{C}^n$ be a bounded, open set and let $u, v \in \mathcal{PSH}(\Omega) \cap L^{\infty}(\Omega)$ be such that*

$$\liminf_{\substack{z \to \xi \\ z \in \Omega}} (u(z) - v(z)) \geq 0$$

for every $\xi \in \partial\Omega$. Then

$$\int_{\{u<v\}} (dd^c v)^n \leq \int_{\{u<v\}} (dd^c u)^n.$$

If in addition $u \geq v$ then

$$\int_{\Omega} (dd^c u)^n \leq \int_{\Omega} (dd^c v)^n.$$

On the other hand, if $(dd^c u)^n \leq (dd^c v)^n$, then $u \geq v$.

Let $\Omega \Subset \mathbb{C}^n$ be an open set. For every Borel set $A \subseteq \Omega$ the C_n-*capacity*, introduced by Bedford and Taylor in [19], of the set A is defined by

$$C_n(A) = C_n(A, \Omega) = \sup\left\{ \int_A (dd^c u)^n : u \in \mathcal{PSH}(\Omega), -1 \leq u \leq 0 \right\}.$$

Moreover, C_n is a subadditive Choquet capacity and it vanishes exactly on pluripolar sets. Recall that a set E in $\mathbb{C}^n$ is *pluripolar* if for any $z \in E$ there exists a neighborhood V of z and $v \in \mathcal{PSH}(V)$ such that $E \cap V \subset \{v = -\infty\}$.

DEFINITION 2.3 ([84]). A sequence $[u_j]$ of functions defined in Ω is said to *converge in capacity* to u if for any $t > 0$ and $K \Subset \Omega$,

$$\lim_{j \to \infty} C_n(K \cap \{|u - u_j| > t\}, \Omega) = 0.$$

For further information about convergence in capacity see e.g. [84].

Recall that for $E \subseteq \Omega$ the *relative extremal function* $h_{E,\Omega}$ for E in Ω is defined by

$$h_E(z) = h_{E,\Omega}(z) = \sup\{u(z) : u \in \mathcal{PSH}(\Omega),\ u < 0 \text{ and } u \leq -1 \text{ on } E\}$$

for every $z \in \Omega$.

THEOREM 2.4 ([11, 12, 19]). *Let Ω be an open set. Then for any open or compact set $E \Subset \Omega$, h_E^* is bounded plurisubharmonic function,* $\operatorname{supp}(dd^c h_E^*)^n \subset \bar{E}$ *and*

$$C_n(E, \Omega) = \int_{\Omega} (dd^c h_E^*)^n,$$

where $h_{E,\Omega}^*$ denotes the upper semicontinuous regularization of the function $h_{E,\Omega}$. Moreover, if E is an open set then $h_{E,\Omega} = h_{E,\Omega}^*$.

THEOREM 2.5 ([19], quasicontinuity of plurisubharmonic functions). *For any plurisubharmonic function u defined in Ω and any $\varepsilon > 0$ there exists an open set $U \subset \Omega$ with $C_n(U,\Omega) < \varepsilon$ such that u restricted to $\Omega \setminus U$ is continuous.*

The Monge–Ampère operator is continuous with respect to sequences of plurisubharmonic functions converging in capacity.

THEOREM 2.6 ([84]). *Let $[u_j^k]_{j=1}^{\infty}$ be a locally uniformly bounded sequence of plurisubharmonic functions in Ω for $k = 1,\dots,n$; and let $u_j^k \to u^k \in \mathcal{PSH}(\Omega) \cap L^{\infty}_{\mathrm{loc}}(\Omega)$ in capacity as $j \to \infty$ for $k = 1,\dots,n$. Then*

$$dd^c u_j^1 \wedge \dots \wedge dd^c u_j^n \to dd^c u^1 \wedge \dots \wedge dd^c u^n$$

in the weak-topology.*

The following subsolution theorem was proved by Kołodziej in [63].

THEOREM 2.7 ([63], Kołodziej's subsolution theorem). *Let Ω be a bounded hyperconvex domain in $\mathbb{C}^n$. Let $u \in \mathcal{PSH}(\Omega) \cap L^{\infty}_{\mathrm{loc}}(\Omega)$ with $\lim_{z\to w} u(z) = f(w)$ for all $w \in \partial\Omega$, where $f \in \mathcal{C}(\partial\Omega)$. If μ is a positive, finite measure such that $\mu \le (dd^c u)^n$, then there exists a unique bounded plurisubharmonic function v such that $(dd^c v)^n = \mu$ and $\lim_{z\to w} v(z) = f(w)$ for all $w \in \partial\Omega$.*

DEFINITION 2.8 ([30]). Let $\Omega \subseteq \mathbb{C}^n$ be a bounded hyperconvex domain. Define the Cegrell class $\mathcal{E}_0$ $(= \mathcal{E}_0(\Omega))$ to be the class of bounded plurisubharmonic functions φ defined on Ω such that

$$\lim_{\substack{z\to\xi \\ z\in\Omega}} \varphi(z) = 0$$

for every $\xi \in \partial\Omega$, and

$$\int_\Omega (dd^c\varphi)^n < \infty.$$

DEFINITION 2.9 ([32]). Let $\Omega \subseteq \mathbb{C}^n$ be a bounded hyperconvex domain. Define the Cegrell class $\mathcal{E}$ $(= \mathcal{E}(\Omega))$ to be the class of plurisubharmonic functions φ defined on Ω such that for each $z_0 \in \Omega$ there exists a neighborhood ω of z_0 in Ω and a decreasing sequence $[\varphi_j]$, $\varphi_j \in \mathcal{E}_0$, which converges pointwise to φ on ω and

$$\sup_j \int_\Omega (dd^c\varphi_j)^n < \infty.$$

If $[\varphi_j]$ can also be chosen such that it converges pointwise to φ on the whole Ω, then φ is said to be in the Cegrell class $\mathcal{F}$ $(= \mathcal{F}(\Omega))$.

PROPOSITION 2.10 ([32]). *Let $\mathcal{K} \in \{\mathcal{E}_0, \mathcal{F}, \mathcal{E}\}$. Then $\mathcal{K}$ is a convex cone. Moreover, if $u \in \mathcal{K}$ and $v \in \mathcal{PSH}^-(\Omega)$ then $\max(u,v) \in K$. In particular, if $u \in \mathcal{K}$ and $v \in \mathcal{PSH}^-(\Omega)$, $u \le v$ then $v \in K$.*

Proof. Observe that just from the definition of $\mathcal{F}$ and $\mathcal{E}$ it is enough to prove that $\mathcal{E}_0$ is a convex cone and for any $u \in \mathcal{E}_0$ and any negative plurisubharmonic function v we have $\max(u,v) \in \mathcal{E}_0$.

It is obvious that if $t > 0$, $u \in \mathcal{K}$ then $tu \in \mathcal{K}$. Let $u \in \mathcal{E}_0$ and v be a negative plurisubharmonic function. Then $\max(u,v) \in L^{\infty}(\Omega)$ and $\lim_{z\to\xi} \max(u,v) = 0$ for all

$\xi \in \partial\Omega$. Theorem 2.2 yields

$$\int_\Omega (dd^c \max(u,v))^n \le \int_\Omega (dd^c u)^n,$$

since $u \le \max(u,v)$.

Now take $u, v \in \mathcal{E}_0$. Since $U_t = \{u = tv\}$, for $t \in (1,2)$, is an uncountable family of disjoint sets, there exists $t \in (1,2)$ such that $\int_{\{u=tv\}} (dd^c(u+v))^n = 0$. By the comparison principle (Theorem 2.2) we obtain

$$\begin{aligned}
\int_\Omega (dd^c(u+v))^n &= \int_{\{u<tv\}} (dd^c(u+v))^n + \int_{\{u>tv\}} (dd^c(u+v))^n \\
&= \int_{\{\frac{1+t}{t}u<v+u\}} (dd^c(u+v))^n + \int_{\{u+v>(1+t)v\}} (dd^c(u+v))^n \\
&\le \Big(\frac{1+t}{t}\Big)^n \int_{\{\frac{1+t}{t}u<v+u\}} (dd^c u)^n + (1+t)^n \int_{\{u+v>(1+t)v\}} (dd^c v)^n \\
&\le 3^n \Big(\int_\Omega (dd^c u)^n + \int_\Omega (dd^c v)^n \Big).
\end{aligned}$$

This means that $u + v \in \mathcal{E}_0$. ∎

Proposition 2.11 ([32]). *Let $u \in \mathcal{E}$, and let ω be an open set such that $\omega \Subset \Omega$. Then there exists $v_\omega \in \mathcal{F}$ such that $u = v_\omega$ on ω. Moreover, if*

$$\psi = \sup\{w \in \mathcal{PSH}(\Omega) : w \le u \text{ on } \omega\},$$

then $\psi \ge u$ on Ω, $\psi = u$ on ω, and $\psi \in \mathcal{F}$.

Proof. The first part follows just from the definition of $\mathcal{E}$. To prove the second statement observe that $\psi \ge v_\omega$, so $\psi \in \mathcal{F}$ by Proposition 2.10. ∎

We are now going to define new classes of plurisubharmonic functions with a weak type of boundary values. In classical potential theory, the Riesz decomposition theorem (see [9]) says that any non-positive subharmonic function defined on a bounded domain can be written as a sum of a Green potential and a harmonic function. The smallest harmonic majorant of the Green potential is zero and the harmonic function is determined by its behavior near the boundary. Thus, one can interpret the boundary values of the given subharmonic function as the harmonic part in Riesz's decomposition theorem. A straightforward generalization of this decomposition theorem to the context of pluripotential theory is not possible. Instead we make the following definition.

Definition 2.12 ([4, 30, 32]). Let $\mathcal{K} \in \{\mathcal{E}_0, \mathcal{F}, \mathcal{E}\}$. We say that a plurisubharmonic function u defined on Ω belongs to the Cegrell class $\mathcal{K}(\Omega, H)$, $H \in \mathcal{E}$, if there exists a function $\varphi \in \mathcal{K}$ such that

$$H \ge u \ge \varphi + H.$$

Sometimes we shall simply write $\mathcal{K}(H)$ instead of $\mathcal{K}(\Omega, H)$.

In Chapter 7 we prove that under certain assumptions, the boundary values arising from $\mathcal{F}(H)$ and the classical boundary values coincide.

Definition 2.13. Let $\Omega \subseteq \mathbb{C}^n$ be a bounded domain, and $f \in \mathcal{C}(\partial\Omega)$ a real-valued function. Define

$$PB_f(z) = \sup\{w(z) : w \in \mathcal{PSH}(\Omega) : \limsup_{\substack{z\to\xi\\ z\in\Omega}} w(z) \leq f(\xi) \quad \text{for every } \xi \in \partial\Omega\}.$$

Remark 2.14.

(1) The function PB_f is nowadays usually referred to as the *Perron–Bremermann envelope*. If $\Omega \subseteq \mathbb{C}^n$ is a bounded hyperconvex domain, and $f : \partial\Omega \to \mathbb{R}$ is a continuous function, then $PB_f \in \mathcal{PSH}(\Omega)$. We shall sometimes simplify the notation by writing $u \in \mathcal{K}(f)$ if $u \in \mathcal{K}(PB_f)$ in the sense of Definition 2.12. In [44], we considered $\mathcal{K}(f)$ with upper semicontinuous boundary data f.
(2) If $u \in \mathcal{E}(H)$, then $u \in \mathcal{E}$. Furthermore, if $f \in \mathcal{C}(\partial\Omega)$, $f \leq 0$, then PB_f is a bounded negative plurisubharmonic function and therefore $\mathcal{E}(f) \subset \mathcal{E}$. Without loss of generality we only consider negative functions, since $PB_{f-c} = PB_f - c$.
(3) $\mathcal{K}(\Omega, 0) = \mathcal{K}$.

We will need the following theorem of Walsh.

Theorem 2.15 ([81], Walsh theorem). *Let $\Omega \subseteq \mathbb{C}^n$ be a bounded domain and let $f : \partial\Omega \to \mathbb{R}$ be a continuous function. If*

$$\liminf_{\substack{z\to\xi\\ z\in\Omega}} PB_f(z) = \limsup_{\substack{z\to\xi\\ z\in\Omega}} PB_f(z) = f(\xi)$$

for every $\xi \in \partial\Omega$, then $PB_f \in \mathcal{C}(\bar{\Omega})$.

Definition 2.16. A *fundamental sequence* $[\Omega_j]$ is an increasing sequence of strictly pseudoconvex subsets of Ω such that for every $j \in \mathbb{N}$ we have $\Omega_j \Subset \Omega_{j+1}$ and $\bigcup_{j=1}^{\infty} \Omega_j = \Omega$. Here $\Subset$ denotes that Ω_j is relatively compact in Ω_{j+1}.

The following theorem was proved in [32] for $H = 0$; for arbitrary $H \in \mathcal{E}$ it was proved by the author together with Per Åhag, Urban Cegrell, and Phạm Hoàng Hiệp in [4].

Theorem 2.17 ([4, 32]). *Let $H \in \mathcal{E}$ and $u \in \mathcal{PSH}(\Omega)$ be such that $u \leq H$. Then there exists a decreasing sequence $[u_j]$, $u_j \in \mathcal{E}_0(H)$, that converges pointwise to u on Ω as j tends to ∞. Moreover, if $H \in \mathcal{PSH}(\Omega) \cap \mathcal{C}(\bar{\Omega})$, then the decreasing sequence $[u_j]$ can be chosen such that $u_j \in \mathcal{E}_0(H) \cap \mathcal{C}(\bar{\Omega})$.*

Proof. First assume that $H \in \mathcal{PSH}(\Omega) \cap \mathcal{C}(\bar{\Omega})$ and let $\varphi \in \mathcal{E}_0 \cap \mathcal{C}(\bar{\Omega})$, not identically 0. Choose a fundamental sequence $[\Omega_j]$ in Ω such that for each $j \in \mathbb{N}$ we have $\varphi \geq -1/(2j^2)$ on Ω_j^c. Let $[v_j]$, $v_j \in \mathcal{PSH}(\Omega_j) \cap C^\infty(\Omega_j)$, be a decreasing sequence that converges pointwise to u as $j \to \infty$, and $v_j \leq H + 1/(2j)$ on Ω_{j+1}. Set

$$u_j' = \begin{cases} \max(v_j - 1/j, j\varphi + H) & \text{on } \Omega_j, \\ j\varphi + H & \text{on } \Omega_j^c. \end{cases}$$

Then $[u_j']$, $u_j' \in \mathcal{E}_0(H) \cap \mathcal{C}(\bar{\Omega})$, converges pointwise to u on Ω as $j \to \infty$, but $[u_j']$ is not necessarily decreasing. Let $u_j = \sup_{k\geq j} u_k'$. The construction of u_j' implies that

$$u_j' + \frac{1}{j} \geq u_{j+1}' + \frac{1}{j+1}$$

and therefore for each $j \in \mathbb{N}$ fixed it follows that

$$[\max(u'_j, u'_{j+1}, \dots, u'_{m-1}, u'_m + 1/m)]_{m=j}^{\infty}$$

decreases pointwise on Ω to u_j as $m \to \infty$. Thus, u_j is an upper semicontinuous function and we have $u_j \in \mathcal{PSH}(\Omega) \cap \mathcal{C}(\bar{\Omega})$. Moreover, $[u_j]$ is decreasing and converges pointwise to u on Ω as $j \to \infty$.

Now let $H \in \mathcal{E}$ be an arbitrary plurisubharmonic function. Since $u \leq H \leq 0$, the first part of the proof implies that there exists a decreasing sequence $[\varphi_j]$, $\varphi_j \in \mathcal{E}_0 \cap \mathcal{C}(\bar{\Omega})$, that converges pointwise to u as $j \to \infty$. If $v_j = \max(u, \varphi_j + H)$, then $[v_j]$, $v_j \in \mathcal{E}_0(H)$, is a decreasing sequence that converges pointwise to u as $j \to \infty$, and the proof is complete. ∎

Remark 2.18. If H is unbounded, then each function u_j is necessarily unbounded.

Theorem 2.19 ([32]). *$\mathcal{C}_0^{\infty}(\Omega) \subset \mathcal{E}_0 \cap \mathcal{C}(\bar{\Omega}) - \mathcal{E}_0 \cap \mathcal{C}(\bar{\Omega})$.*

Proof. Fix $\varphi \in \mathcal{E}_0 \cap \mathcal{C}(\bar{\Omega})$. Let $f \in \mathcal{C}_0^{\infty}(\Omega)$. Then there exists $k > 0$ such that

$$f + k|z|^2 \in \mathcal{PSH}(\Omega).$$

Now fix a, b such that

$$a < \inf_{\Omega} f < \sup_{\Omega}(f + k|z|^2) < b$$

and define

$$u = \max(f + k|z|^2 - b, M\varphi),$$

where $M > 0$ is chosen such that $M\varphi < a - b$ on $\operatorname{supp} f$. Then from Proposition 2.10, $u \in \mathcal{E}_0 \cap \mathcal{C}(\bar{\Omega})$ since $u \geq M\varphi$. Observe that

$$v = \max(k|z|^2 - b, M\varphi) \in \mathcal{E}_0 \cap \mathcal{C}(\bar{\Omega})$$

and $f = u - v$. This ends the proof. ∎

3. The domain of definition of the complex Monge–Ampère operator

The aim of this chapter is to extend the definition of the complex Monge–Ampère operator to the Cegrell class $\mathcal{E}$. We prove that $\mathcal{E}$ is the optimal domain of definition for the complex Monge–Ampère operator (Theorem 3.9). We also present two other results in that direction, the first one proved by Cegrell, Kołodziej and Zeriahi (Theorem 3.10) and the second by Błocki (Theorem 3.11). Several results concerning the convergence of the Monge–Ampère measures are also proved. In particular, it is shown that the complex Monge–Ampère operator is continuous with respect to monotone sequences in $\mathcal{E}$ (Corollary 3.7 and Proposition 3.8). Furthermore, we give some inequalities for the total Monge–Ampère mass (Lemma 3.12, Theorem 3.14, and Corollary 3.15). Most results in this chapter originate from [32].

In [32] Cegrell proved the following *integration by parts* theorem for negative plurisubharmonic functions.

THEOREM 3.1 ([32]). *Suppose* $u, v \in \mathcal{PSH}^-(\Omega)$, $u \not\equiv 0$, $\lim_{z\to\xi} u(z) = 0$ *for all* $\xi \in \partial\Omega$, *and* T *a positive and closed current of bidegree* $(n-1, n-1)$. *Then* $dd^c u \wedge T$ *is a well defined positive measure on* Ω. *Furthermore, if* $\int_\Omega v\, dd^c u \wedge T > -\infty$, *then* $dd^c v \wedge T$ *is also a well defined positive measure on* Ω *and*

$$\int_\Omega v\, dd^c u \wedge T \leq \int_\Omega u\, dd^c v \wedge T.$$

Moreover, if in addition $\lim_{z\to\xi} v(z) = 0$, *then* $\int_\Omega u dd^c v \wedge T > -\infty$ *and*

$$\int_\Omega v\, dd^c u \wedge T = \int_\Omega u\, dd^c v \wedge T.$$

Proof. First suppose that $u, v \in \mathcal{PSH}(\Omega) \cap \mathcal{C}(\bar{\Omega})$, $u = v = 0$ on $\partial\Omega$ and $\int_\Omega v\, dd^c u \wedge T > -\infty$. By [48], $dd^c u \wedge T$ and $dd^c v \wedge T$ are well defined positive measures on Ω. Let $\varepsilon > 0$. Then by the monotone convergence theorem

$$\int_\Omega u dd^c v \wedge T = \lim_{\varepsilon\to 0} \int_\Omega (u - \max(u, \varepsilon)) dd^c v \wedge T,$$

and

$$\int_\Omega (u - \max(u, \varepsilon)) dd^c v \wedge T = \lim_{j\to\infty} \int_\Omega (u - \max(u, \varepsilon)) * \rho_{1/j} dd^c v \wedge T,$$

where $\rho \in \mathcal{C}_0^\infty(\mathbb{C}^n)$, $\operatorname{supp}\rho = \overline{B(0,1)}$, $\int_{\mathbb{C}^n} \rho(z)\, dV_n(z) = 1$, $\rho(z) = \rho(|z|)$, $\rho_\varepsilon(z) = \frac{1}{\varepsilon^{2n}} \rho(z/\varepsilon)$.

Let $\Omega' \Subset \Omega$ be such that $\{u - \max(u, \varepsilon) \neq 0\} \subset \Omega'$. Then

$$\begin{aligned}\int_\Omega (u - \max(u, \varepsilon)) * \rho_{1/j} dd^c v \wedge T &= \int_\Omega v dd^c((u - \max(u, \varepsilon)) * \rho_{1/j}) \wedge T \\ &\geq \int_{\Omega'} v dd^c(u * \rho_{1/j}) \wedge T.\end{aligned}$$

Since v is upper semicontinuous, when $\varepsilon \to 0$ and $j \to \infty$, by the dominated convergence theorem we obtain

$$\int_\Omega v\, dd^c u \wedge T \leq \int_\Omega u\, dd^c v \wedge T$$

and similarly, using that $-\infty < \int u\, dd^c v \wedge T$, we find that

$$\int_\Omega u\, dd^c v \wedge T = \int_\Omega v dd^c u \wedge T.$$

To complete the proof of Theorem 3.1, we use Theorem 2.17 and choose $u_j, v_j \in \mathcal{E}_0 \cap \mathcal{C}(\bar{\Omega})$ such that $u_j \searrow u$, $v_j \searrow v$ as $j \to \infty$. By [48], $dd^c u \wedge T$ is a well defined positive measure on Ω. From the first part of the proof and using the dominated convergence theorem we have

$$\int_\Omega v_k dd^c u \wedge T = \lim_{j\to\infty} \int_\Omega v_k dd^c u_j \wedge T = \lim_{j\to\infty} \int_\Omega u_j dd^c v_k \wedge T = \int_\Omega u dd^c v_k \wedge T.$$

Since u is upper semicontinuous we finally get

$$\int_\Omega v dd^c u \wedge T \leq \int_\Omega u dd^c v \wedge T,$$

which ends the proof of the first part of Theorem 3.1. The second part follows from the first one. ∎

To define the complex Monge–Ampère operator for functions from the Cegrell class $\mathcal{E}$ we need the following convergence of Monge–Ampère measures of functions from the Cegrell class $\mathcal{E}_0$.

THEOREM 3.2 ([32]). *Suppose $u^k \in \mathcal{E}(\Omega)$, $1 \le k \le n$. If $u_j^k \in \mathcal{E}_0(\Omega)$ decreases to u^k as $j \to \infty$, then $dd^c u_j^1 \wedge dd^c u_j^2 \wedge \cdots \wedge dd^c u_j^n$ is convergent in the weak*-topology and the limit measure does not depend on the particular sequences $[u_j^k]_{j=1}^\infty$.*

Proof. Fix a compact set $K \subset \Omega$. From Definition 2.9 for every $z \in K$ there exist a neighborhood U of z, a decreasing sequence $[w_j^k]$, $w_j^k \in \mathcal{E}_0$, $w_j^k \to u^k$ on U such that $\sup_j \int_\Omega (dd^c w_j^k)^n < \infty$ for $k = 1, \dots, n$. Since K is a compact set we can choose finitely many neighborhoods U^s, $s = 1, \dots, N$, such that $K \subset U^1 \cup \cdots \cup U^N$. Let $[w_j^{ks}]_{j=1}^\infty$, $1 \le s \le N$, $1 \le k \le n$, be the sequence corresponding to U^s. Let $v_j^k = \sum_{s=1}^N w_j^{ks}$. Then $v_j^k \in \mathcal{E}_0$ by Proposition 2.10 and $\sup_j \int_\Omega (dd^c v_j^k)^n < \infty$. Thus, if we define $\hat{u}_j^k = \lim_{l\to\infty} \max(u_j^k, v_l^k)$, then $\sup_j \int_\Omega (dd^c \hat{u}_j^k)^n < \infty$ and $\hat{u}_j^k = u_j^k$ on the neighborhood $\bigcup_{s=1}^N U^s$ of K.

Therefore we can assume that $\sup_j \int_\Omega (dd^c u_j^k)^n < \infty$. Then, for $h \in \mathcal{E}_0(\Omega)$,

$$\left[\int_\Omega h dd^c u_j^1 \wedge dd^c u_j^2 \wedge \cdots \wedge dd^c u_j^n\right]$$

is a decreasing sequence by Theorem 3.1 and since

$$\int_\Omega h(dd^c u_j^k)^n \ge (\inf_\Omega h) \sup_j \int_\Omega (dd^c u_j^k)^n > -\infty,$$

the limit $\lim_{j\to\infty} \int_\Omega h dd^c u_j^1 \wedge dd^c u_j^2 \wedge \cdots \wedge dd^c u_j^n$ exists for all $h \in \mathcal{E}_0$. By Theorem 2.19, $dd^c u_j^1 \wedge dd^c u_j^2 \wedge \cdots \wedge dd^c u_j^n$ is convergent in the weak*-topology.

If $[v_j^k]$ is another sequence decreasing to u^k, we get, again by Theorem 3.1,

$$\begin{aligned}
\int_\Omega h dd^c v_j^1 \wedge dd^c v_j^2 \wedge \cdots \wedge dd^c v_j^n &= \int_\Omega v_j^1 dd^c h \wedge dd^c v_j^2 \wedge \cdots \wedge dd^c v_j^n \\
&\ge \int_\Omega u^1 dd^c h \wedge dd^c v_j^2 \wedge \cdots \wedge dd^c v_j^n = \lim_{s_1\to\infty} \int_\Omega u_{s_1}^1 dd^c h \wedge dd^c v_j^2 \wedge \cdots \wedge dd^c v_j^n \\
&= \lim_{s_1\to\infty} \int_\Omega v_j^2 dd^c h \wedge dd^c u_{s_1}^1 \wedge \cdots \wedge dd^c v_j^n \ge \cdots \\
&\ge \lim_{s_1,\dots,s_n\to\infty} \int_\Omega h dd^c u_{s_1}^1 \wedge \cdots \wedge dd^c u_{s_n}^n \ge \lim_{s\to\infty} \int_\Omega h dd^c u_s^1 \wedge dd^c u_s^2 \wedge \cdots \wedge dd^c u_s^n.
\end{aligned}$$

Therefore, $\lim_{j\to\infty} \int h dd^c v_j^1 \wedge dd^c v_j^2 \wedge \cdots \wedge dd^c v_j^n$ exists and

$$\lim_{j\to\infty} \int_\Omega h dd^c v_j^1 \wedge dd^c v_j^2 \wedge \cdots \wedge dd^c v_j^n \ge \lim_{j\to\infty} \int_\Omega h dd^c u_j^1 \wedge dd^c u_j^2 \wedge \cdots \wedge dd^c u_j^n.$$

Similarly one can obtain the reverse inequality, so we conclude that the limits are equal. ■

By using Theorem 3.2 we are now able to define the complex Monge–Ampère operator on $\mathcal{E}$.

DEFINITION 3.3 ([32]). For $u^k \in \mathcal{E}, 1 \le k \le n$, we define $dd^c u^1 \wedge dd^c u^2 \wedge \cdots \wedge dd^c u^n$ to be the limit measure obtained in Theorem 3.2.

Proposition 3.4 ([32]). *Suppose $u^k \in \mathcal{F}(\Omega)$, $1 \le k \le n$, and $h \in \mathcal{PSH}^-(\Omega)$. If $u_j^k \in \mathcal{E}_0(\Omega)$ decreases to u^k as $j \to \infty$, then*

$$\lim_{j\to\infty} \int_\Omega hdd^cu_j^1 \wedge dd^cu_j^2 \wedge \cdots \wedge dd^cu_j^n = \int_\Omega hdd^cu^1 \wedge dd^cu^2 \wedge \cdots \wedge dd^cu^n. \tag{3.1}$$

Moreover, if $\int_\Omega -hdd^cu^1 \wedge dd^cu^2 \wedge \cdots \wedge dd^cu^n < \infty$ then $hdd^cu_j^1 \wedge dd^cu_j^2 \wedge \cdots \wedge dd^cu_j^n$ converges in the weak*-topology to $hdd^cu^1 \wedge dd^cu^2 \wedge \cdots \wedge dd^cu^n$ as $j \to \infty$.

Proof. Since Ω is open, $u^k \in \mathcal{F}(\Omega)$, $1 \le k \le n$, and $dd^cu_j^1 \wedge dd^cu_j^2 \wedge \cdots \wedge dd^cu_j^n$ converges in the weak*-topology to $dd^cu^1 \wedge dd^cu^2 \wedge \cdots \wedge dd^cu^n$, we have

$$\infty > \lim_{j\to\infty} \int_\Omega dd^cu_j^1 \wedge dd^cu_j^2 \wedge \cdots \wedge dd^cu_j^n \ge \int_\Omega dd^cu^1 \wedge dd^cu^2 \wedge \cdots \wedge dd^cu^n.$$

If $h \in \mathcal{E}_0 \cap \mathcal{C}(\bar\Omega)$ then by the proof of Theorem 3.2 we obtain (3.1).

Now suppose that $h \in \mathcal{PSH}^-(\Omega)$ and $\int_\Omega hdd^cu^1 \wedge dd^cu^2 \wedge \cdots \wedge dd^cu^n > -\infty$. Let $[h_j] \subset \mathcal{E}_0 \cap \mathcal{C}(\bar\Omega)$ be a decreasing sequence converging to h (see Theorem 2.17). By the monotone convergence theorem, $[\int_\Omega (-h_j) dd^cv_1 \wedge \cdots \wedge dd^cv_n]$ is an increasing sequence tending to $\int_\Omega (-h) dd^cv_1 \wedge \cdots \wedge dd^cv_n$ for any $v_1, \ldots, v_n \in \mathcal{F}$. Moreover, for any $\varphi \in \mathcal{E}_0$ the sequence $[\int_\Omega \varphi dd^cu_j^1 \wedge dd^cu_j^2 \wedge \cdots \wedge dd^cu_j^n]$ is decreasing by Theorem 3.1 and

$$\lim_{j\to\infty} \int_\Omega \varphi dd^cu_j^1 \wedge dd^cu_j^2 \wedge \cdots \wedge dd^cu_j^n = \int_\Omega \varphi dd^cu_j \wedge dd^cu_j \wedge \cdots \wedge dd^cu_j.$$

Therefore for each $j \in \mathbb{N}$ there exist $q_j, s_j \in \mathbb{N}$ such that

$$\begin{aligned}
\int_\Omega (-h) dd^cu^1 \wedge dd^cu^2 \wedge \cdots \wedge dd^cu^n &\le \frac{1}{j} - \int_\Omega h_j dd^cu^1 \wedge dd^cu^2 \wedge \cdots \wedge dd^cu^n \\
&\le \frac{2}{j} - \int_\Omega h_j dd^cu_{q_j}^1 \wedge \cdots \wedge dd^cu_{q_j}^n \le \frac{2}{j} - \int_\Omega hdd^cu_{q_j}^1 \wedge dd^cu_{q_j}^2 \wedge \cdots \wedge dd^cu_{q_j}^n \\
&\le \frac{4}{j} - \int_\Omega h_{s_j} dd^cu_{q_j}^1 \wedge dd^cu_{q_j}^2 \wedge \cdots \wedge dd^cu_{q_j}^n \\
&\le \frac{4}{j} - \int_\Omega h_{s_j} dd^cu^1 \wedge dd^cu^2 \wedge \cdots \wedge dd^cu^n \le \frac{4}{j} - \int_\Omega hdd^cu^1 \wedge dd^cu^2 \wedge \cdots \wedge dd^cu^n.
\end{aligned}$$

This proves (3.1). Note that if $\int_\Omega hdd^cu^1 \wedge dd^cu^2 \wedge \cdots \wedge dd^cu^n = -\infty$, then

$$\lim_{j\to\infty} \int_\Omega hdd^cu_j^1 \wedge dd^cu_j^2 \wedge \cdots \wedge dd^cu_j^n = -\infty.$$

This ends the proof of the first part of Proposition 3.4.

Now assume that in addition $\int_\Omega (-h) dd^cu^1 \wedge dd^cu^2 \wedge \cdots \wedge dd^cu^n < \infty$. Then from the first part of Proposition 3.4,

$$\lim_{j\to\infty} \int_\Omega hdd^cu_j^1 \wedge dd^cu_j^2 \wedge \cdots \wedge dd^cu_j^n = \int_\Omega hdd^cu^1 \wedge dd^cu^2 \wedge \cdots \wedge dd^cu^n.$$

Since h is upper semicontinuous,

$$\lim_{j\to\infty} hdd^cu_j^1 \wedge dd^cu_j^2 \wedge \cdots \wedge dd^cu_j^n \le hdd^cu^1 \wedge dd^cu^2 \wedge \cdots \wedge dd^cu^n,$$

but since both measures have the same total mass, they must be equal. ∎

Remark 3.5. A consequence of Proposition 3.4 is that if $u \in \mathcal{F}(\Omega)$, then

$$\int_\Omega (dd^c u)^n < \infty.$$

Corollary 3.6 ([2]). *Let $u, v \in \mathcal{F}$ be such that $u \le v$. Then for all $h \in \mathcal{PSH}^-(\Omega)$,*

$$\int_\Omega (-h)(dd^c v)^n \le \int_\Omega (-h)(dd^c u)^n.$$

In particular,

$$\int_\Omega (dd^c v)^n \le \int_\Omega (dd^c u)^n.$$

Proof. There exist $u_j, v_j \in \mathcal{E}_0$, $u_j \le v_j$ such that $u_j \searrow u$, $v_j \searrow v$ as $j \to \infty$, and $\sup_j \int_\Omega (dd^c u_j)^n < \infty$, $\sup_j \int_\Omega (dd^c v_j)^n < \infty$. By Theorem 2.17 there exist $h_k \in \mathcal{E}_0$ such that $h_k \searrow h$ as $k \to \infty$. By Theorem 3.1 we have

$$\begin{aligned}\int_\Omega (-h_k)(dd^c v_j)^n &= \int_\Omega (-v_j) dd^c h_k \wedge (dd^c v_j)^{n-1} \le \int_\Omega (-u_j) dd^c h_k \wedge (dd^c v_j)^{n-1} \\ &= \int_\Omega (-h_k) dd^c u_j \wedge (dd^c v_j)^{n-1} \le \cdots \le \int_\Omega (-h_k)(dd^c u_j)^n.\end{aligned}$$

Therefore by Proposition 3.4 we obtain

$$\int_\Omega (-h_k)(dd^c v)^n \le \int_\Omega (-h_k)(dd^c u)^n,$$

thus the monotone convergence theorem gives the desired inequality. To prove the second part of Corollary 3.6 take $h = -1$. ∎

Corollary 3.7 ([33]). *Suppose $u^k, u^k_j \in \mathcal{E}(\Omega), 1 \le k \le n$, $u^k_j \ge u^k$. If $h \in \mathcal{PSH}^-(\Omega) \cap L^\infty(\Omega)$ and u^k_j tends to u^k in $L^1_{\mathrm{loc}}(\Omega)$ as $j \to \infty$ then $h dd^c u^1_j \wedge dd^c u^2_j \wedge \cdots \wedge dd^c u^n_j$ converges in the weak*-topology to $h dd^c u^1 \wedge dd^c u^2 \wedge \cdots \wedge dd^c u^n$ as $j \to \infty$.*

Proof. Since the result is local we can assume that $u^k, u^k_j \in \mathcal{F}(\Omega)$. Take $w^k_j \in \mathcal{E}_0$ such that $w^k_j \searrow u^k$ and $\sup_j \int_\Omega (dd^c w^k_j)^n < \infty$. Define $v^k_j = (\sup_{s \ge j}(w^k_j, u^k_s))^*$. It follows from Proposition 3.4 that

$$\lim_{j\to\infty} \int_\Omega h dd^c v^1_j \wedge dd^c v^2_j \wedge \cdots \wedge dd^c v^n_j = \int_\Omega h dd^c u^1 \wedge dd^c u^2 \wedge \cdots \wedge dd^c u^n,$$

and $h dd^c v^1_j \wedge dd^c v^2_j \wedge \cdots \wedge dd^c u^n_j$ converges in the weak*-topology to $h dd^c u^1 \wedge dd^c u^2 \wedge \cdots \wedge dd^c u^n$ as $j \to \infty$. By Theorem 3.1 we obtain

$$\int_\Omega h dd^c v^1_j \wedge \cdots \wedge dd^c v^n_j \ge \int_\Omega h dd^c u^1_j \wedge \cdots \wedge dd^c u^n_j \ge \int_\Omega h dd^c u^1 \wedge \cdots \wedge dd^c u^n.$$

Therefore

$$\lim_{j\to\infty} \int_\Omega h dd^c v^1_j \wedge \cdots \wedge dd^c v^n_j = \lim_{j\to\infty} \int_\Omega h dd^c u^1_j \wedge \cdots \wedge dd^c u^n_j = \int_\Omega h dd^c u^1 \wedge \cdots \wedge dd^c u^n$$

and

$$\lim_{j\to\infty} dd^c v^1_j \wedge \cdots \wedge dd^c v^n_j = \lim_{j\to\infty} dd^c u^1_j \wedge \cdots \wedge dd^c u^n_j = dd^c u^1 \wedge \cdots \wedge dd^c u^n.$$

Hence

$$\lim_{j\to\infty} h dd^c u^1_j \wedge \cdots \wedge dd^c u^n_j \le h dd^c u^1 \wedge \cdots \wedge dd^c u^n$$

and since both measures have the same total mass, they must be equal. ∎

It follows from Corollary 3.7 that the complex Monge–Ampère operator is continuous with respect to decreasing sequences. The next proposition shows that it is continuous also with respect to increasing sequences. Both results are generalizations of Theorem 2.1.

PROPOSITION 3.8 ([32]). *Suppose $u^k \in \mathcal{E}(\Omega)$, $1 \leq k \leq n$. If $u_j^k \in \mathcal{E}(\Omega)$ increases to u^k as $j \to \infty$ then $dd^c u_j^1 \wedge dd^c u_j^2 \wedge \cdots \wedge dd^c u_j^n$ converges in the weak*-topology to $dd^c u^1 \wedge dd^c u^2 \wedge \cdots \wedge dd^c u^n$ as $j \to \infty$.*

Proof. Since the result is local we can assume that $u^k, u_j^k \in \mathcal{F}(\Omega)$. The same argument as in the proof of Theorem 3.2 shows that for all $h \in \mathcal{E}_0$ the sequence $[\int_\Omega h dd^c u_j^1 \wedge dd^c u_j^2 \wedge \cdots \wedge dd^c u_j^n]$ is increasing and the limit does not depend on the particular sequences $[u_j^k]$.

We prove that for $1 \leq k \leq n$, $dd^c u_j^1 \wedge \cdots \wedge dd^c u_j^k \wedge T_{n-k}$ converges in the weak*-topology to $dd^c u^1 \wedge \cdots \wedge dd^c u^k \wedge T_{n-k}$ as $j \to \infty$, where $T_{n-k} = dd^c v_{k+1} \wedge \cdots \wedge dd^c v_n$, $v_j \in \mathcal{F}$. By Theorem 2.19 it is enough to prove that for any $h \in \mathcal{E}_0$,

$$\lim_{j\to\infty} \int_\Omega h dd^c u_j^1 \wedge \cdots \wedge dd^c u_j^k \wedge T_{n-k} = \int_\Omega h dd^c u^1 \wedge \cdots \wedge dd^c u^k \wedge T_{n-k}. \tag{3.2}$$

For $k = 1$ we obtain by the monotone convergence theorem

$$\begin{aligned}\lim_{j\to\infty} \int_\Omega h dd^c u_j^1 \wedge T_{n-1} &= \lim_{j\to\infty} \int_\Omega u_j^1 dd^c h \wedge T_{n-1} = \int_\Omega u dd^c h \wedge T_{n-1} \\ &= \int_\Omega h dd^c u \wedge T_{n-1}.\end{aligned}$$

Now suppose that (3.2) is valid for $k = p$. We show that it holds for $k = p+1$. From our assumption we have

$$\begin{aligned}&\lim_{j\to\infty} \int_\Omega h dd^c u_j^1 \wedge \cdots \wedge dd^c u_j^p \wedge dd^c u^{p+1} \wedge T_{n-p-1} \\ &\qquad = \int_\Omega h dd^c u^1 \wedge \cdots \wedge dd^c u^p \wedge dd^c u^{p+1} \wedge T_{n-p-1},\end{aligned}$$

so it is enough to prove that

$$\begin{aligned}&\lim_{j\to\infty} \int_\Omega h dd^c u_j^1 \wedge \cdots \wedge dd^c u_j^p \wedge dd^c u_j^{p+1} \wedge T_{n-p-1} \\ &\qquad = \lim_{j\to\infty} \int_\Omega h dd^c u_j^1 \wedge \cdots \wedge dd^c u_j^p \wedge dd^c u^{p+1} \wedge T_{n-p-1}.\end{aligned}$$

Since the limit (3.2) does not depend on the particular sequence the above limits are equal. ■

Cegrell proved that $\mathcal{E}$ is the natural domain of definition of the complex Monge–Ampère operator (Theorem 4.5 in [32]).

THEOREM 3.9 ([32]). *The Cegrell class $\mathcal{E}$ has the following properties:*

(1) *If $u \in \mathcal{E}$, $v \in \mathcal{PSH}^-(\Omega)$ then $\max(u, v) \in \mathcal{E}$,*
(2) *If $u \in \mathcal{E}$, $\varphi_j \in \mathcal{PSH}^-(\Omega) \cap L^\infty_{\text{loc}}(\Omega)$, $\varphi_j \searrow u$ as $j \to \infty$, then the sequence $[(dd^c \varphi_j)^n]_{j=1}^\infty$ is convergent in the weak*-topology.*

Moreover, if a class $\mathcal{K} = \mathcal{K}(\Omega) \subset \mathcal{PSH}^-(\Omega)$ has properties (1) *and* (2), *then $\mathcal{K} \subset \mathcal{E}$.*

Proof. Suppose $u \in \mathcal{E}$. Then (1) holds true by Theorem 2.10 and (2) follows from Corollary 3.7.

Conversely, suppose $u \in \mathcal{K}$ and ω is open and relatively compact in Ω. By Theorem 2.17, we can find $h_j \in \mathcal{E}_0 \cap \mathcal{C}(\bar{\Omega})$, $h_j \searrow u$ on Ω as $j \to \infty$. Define

$$\hat{h}_j = \sup\{v \in \mathcal{PSH}^-(\Omega) : \ v \le h_j \text{ on } \omega\}.$$

Then $\hat{h}_j \in \mathcal{PSH}(\Omega) \cap L^\infty(\Omega)$, $\operatorname{supp}(dd^c\hat{h}_j)^n \subset \bar{\omega}$ and $\hat{h}_j \searrow u$ on ω, $\hat{h}_j$ decreases on Ω, and $\hat{h}_j \ge u$ on Ω.

Since $u \in \mathcal{K}$, (1) yields $\lim_{j\to\infty} \hat{h}_j = \hat{h} \in \mathcal{K}$ since $\hat{h} \ge u$. Therefore, by (2), $(dd^c\hat{h}_j)^n$ is convergent in the weak*-topology and since $\operatorname{supp}(dd^c\hat{h}_j)^n \subset \bar{\omega} \Subset \Omega$ it follows that $\sup_j \int_\Omega (dd^c\hat{h}_j)^n < \infty$ and we have proved that $u \in \mathcal{E}$. ∎

Another characterization of $\mathcal{E}$ was proved in [38] in terms of the so-called *φ-capacity.* For given $\varphi \in \mathcal{PSH}^-(\Omega)$, $K \subset \Omega$ we define

$$C_\varphi(K;\Omega) = \sup\bigg\{\int_K (dd^c\psi)^n : \psi \in \mathcal{PSH}^-(\Omega) \cap L^\infty(\Omega), \varphi \le \psi \le 0\bigg\}.$$

THEOREM 3.10 ([38]). *$\varphi \in \mathcal{E}(\Omega)$ if and only if for any compact set $K \subset \Omega$ we have $C_\varphi(K;\Omega) < \infty$.*

In [22], Błocki proved that $\mathcal{E} = \{\varphi \in \mathcal{PSH}(\Omega) \cap W^{1,2}_{\mathrm{loc}}(\Omega) : \varphi \le 0\}$ when $n = 2$, and showed that this equality is not valid for $n \ge 3$. Recall that $u \in W^{k,p}_{\mathrm{loc}}(\Omega)$ if $D^\alpha u \in L^p_{\mathrm{loc}}(\Omega)$, for all $|\alpha| = k$. Later, in [23], he obtained a complete characterization of $\mathcal{E}$ for $n \ge 1$.

THEOREM 3.11 ([23]). *For $u \in \mathcal{PSH}^-(\Omega)$ the following are equivalent:*

(1) *$u \in \mathcal{E}(\Omega)$;*
(2) *there exists a measure μ in Ω such that if $U \subset \Omega$ is open and a sequence $u_j \in \mathcal{PSH}(U) \cap \mathcal{C}^\infty(U)$ is decreasing to u in U then $(dd^c u_j)^n$ tends in the weak*-topology to μ in U;*
(3) *for every open $U \subset \Omega$ and any sequence $u_j \in \mathcal{PSH}(U) \cap \mathcal{C}^\infty(U)$ decreasing to u in U the sequence $[(dd^c u_j)^n]$ is locally bounded in U in the weak*-topology;*
(4) *for every open $U \subset \Omega$ and any sequence $u_j \in \mathcal{PSH}(U) \cap \mathcal{C}^\infty(U)$ decreasing to u in U the sequences*

$$|u_j|^{n-p-2} du_j \wedge d^c u_j \wedge (dd^c u_j)^p \wedge (dd^c|z|^2)^{n-p-1}, \quad p = 0, 1, \dots, n-2, \tag{3.3}$$

are locally bounded in U in the weak-topology;*

(5) *for every $z \in \Omega$ there exist an open neighborhood U of z in Ω and a sequence $u_j \in \mathcal{PSH}(U) \cap \mathcal{C}^\infty(U)$ decreasing to u in U such that the sequences* (3.3) *are locally bounded in U in the weak*-topology.*

At the end of this chapter we are going to prove some inequalities for the total Monge–Ampère mass for functions from the Cegrell class $\mathcal{F}$. We will need the following lemma proved in [67]. Inequality (3.4) was originally proved in [21].

LEMMA 3.12 ([21, 67]). *Let $u, v \in \mathcal{PSH}(\Omega) \cap L^\infty(\Omega)$ be such that $u \le v$ on Ω and $\lim_{z\to\partial\Omega}[u(z) - v(z)] = 0$. Then*

$$\int_\Omega (v-u)^k dd^c w \wedge T \le k \int_\Omega (-w)(v-u)^{k-1} dd^c u \wedge T$$

for all $w \in \mathcal{PSH}^-(\Omega) \cap L^\infty(\Omega)$ *and all positive closed currents* T *of bidegree* $(n-1, n-1)$. *In particular, if* $w \in \mathcal{PSH}^-(\Omega) \cap L^\infty(\Omega)$ *and* $\psi \in \mathcal{F}$ *then*

$$\int_\Omega (-\psi)^n (dd^c w)^n \leq n!(\sup(-w))^{n-1} \int_\Omega (-w)(dd^c\psi)^n. \tag{3.4}$$

Proof. First, assume $u, v \in \mathcal{PSH}(\Omega) \cap L^\infty(\Omega)$, $u \leq v$ on Ω and $u = v$ on $\Omega \setminus K$, $K \Subset \Omega$. Then, using the Stokes formula we obtain

$$\begin{aligned}
\int_\Omega (v-u)^k dd^c w \wedge T &= \int_\Omega w dd^c (v-u)^k \wedge T \\
&= -k(k-1) \int_\Omega (-w) d(v-u) \wedge d^c(v-u) \wedge T + k \int_\Omega (-w)(v-u)^{k-1} dd^c(u-v) \wedge T \\
&\leq k \int_\Omega (-w)(v-u)^{k-1} dd^c(u-v) \wedge T \leq k \int_\Omega (-w)(v-u)^{k-1} dd^c u \wedge T.
\end{aligned}$$

In the general case, for each $\varepsilon > 0$ we set $v_\varepsilon = \max(u, v-\varepsilon)$. Then $v_\varepsilon \nearrow v$ on Ω, $v_\varepsilon \geq u$ on Ω and $v_\varepsilon = u$ on $\Omega \setminus K$ for some $K \Subset \Omega$. Hence

$$\int_\Omega (v_\varepsilon - u)^k dd^c w \wedge T \leq k \int_\Omega (-w)(v_\varepsilon - u)^{k-1} dd^c u \wedge T.$$

Since $0 \leq v_\varepsilon - u \nearrow v - u$ as $\varepsilon \searrow 0$, letting $\varepsilon \searrow 0$ we get

$$\int_\Omega (v-u)^k dd^c w \wedge T \leq k \int_\Omega (-w)(v-u)^{k-1} dd^c u \wedge T.$$

To prove the second part of Lemma 3.12 it is enough to take a sequence $\psi_j \in \mathcal{E}_0$ decreasing to ψ from the definition of the Cegrell class $\mathcal{F}$ and apply the first part of Lemma 3.12. We get

$$\begin{aligned}
\int_\Omega (-\psi_j)^n (dd^c w)^n &\leq n \int_\Omega (-w)(-\psi_j)^{n-1} dd^c \psi_j \wedge (dd^c w)^{n-1} \\
&\leq n(\sup_\Omega(-w)) \int_\Omega (-\psi_j)^{n-1} dd^c \psi_j \wedge (dd^c w)^{n-1} \\
&\leq \cdots \leq n!(\sup(-w))^{n-1} \int_\Omega (-w)(dd^c\psi_j)^n.
\end{aligned}$$

By the monotone convergence theorem and Proposition 3.4 the proof is finished. ■

LEMMA 3.13 ([68]). *Assume that* X *is a non-empty set,* $n \geq 1$ *an integer and that* $F : X^n \to [0, \infty)$ *is a function such that* $F(x_1, \ldots, x_n) = F(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$ *for any permutation* $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$. *If*

$$F(x_1, \ldots, x_n) \leq F(x_1, x_1, x_3, \ldots, x_n)^{1/2} F(x_2, x_2, x_3, \ldots, x_n)^{1/2},$$

then for any $p, q \in \mathbb{N}$ *such that* $2 \leq p + q \leq n$ *we have*

$$\begin{aligned}
&F(\overbrace{x_1, \ldots, x_1}^{p\text{-times}}, \overbrace{x_2, \ldots, x_2}^{q\text{-times}}, x_{p+q+1}, \ldots, x_n) \\
&\qquad \leq F(\overbrace{x_1, \ldots, x_1}^{p+q\text{-times}}, x_{p+q+1}, \ldots, x_n)^{p/(p+q)} F(\overbrace{x_2, \ldots, x_2}^{p+q\text{-times}}, x_{p+q+1}, \ldots, x_n)^{q/(p+q)}.
\end{aligned}$$

Furthermore,

$$F(x_1, \ldots, x_n) \leq F(x_1, \ldots, x_1)^{1/n} \cdots F(x_n, \ldots, x_n)^{1/n}.$$

Theorem 3.14 ([32]). *Suppose $u_1,\dots,u_n\in\mathcal{F}$ and $h\in\mathcal{PSH}^-(\Omega)\cap L^\infty(\Omega)$. Let $p,q\in\mathbb{N}$ be such that $2\le p+q\le n$ and let $T=dd^cu_{p+q+1}\wedge\dots\wedge dd^cu_n$. Then*

$$\int_\Omega(-h)(dd^cu_1)^p\wedge(dd^cu_2)^q\wedge T \le\Big(\int_\Omega(-h)(dd^cu_1)^{p+q}\wedge T\Big)^{p/(p+q)}\Big(\int_\Omega(-h)(dd^cu_2)^{p+q}\wedge T\Big)^{q/(p+q)}.$$

Furthermore,

$$\int_\Omega(-h)dd^cu_1\wedge\dots\wedge dd^cu_n\le\Big(\int_\Omega(-h)(dd^cu_1)^n\Big)^{1/n}\dots\Big(\int_\Omega(-h)(dd^cu_n)^n\Big)^{1/n}.$$

Proof. Using the definition of $\mathcal{F}$ and Proposition 3.4, we see that it is enough to consider the case when $u_1,\dots,u_n\in\mathcal{E}_0$. From Theorem 2.17 there exists a sequence $h_j\in\mathcal{E}_0$ decreasing to h. Let $T'=dd^cu_3\wedge\dots\wedge dd^cu_n$. Observe that by the Cauchy–Schwarz inequality we have

$$\begin{aligned}\int_\Omega(-h_j)dd^cu_1\wedge dd^cu_2\wedge T'&\\ &=\int_\Omega(-u_1)dd^cu_2\wedge dd^ch_j\wedge T'=\int_\Omega du_1\wedge d^cu_2\wedge dd^ch_j\wedge T'\\ &\le\Big(\int_\Omega du_1\wedge d^cu_1\wedge dd^ch_j\wedge T'\Big)^{1/2}\Big(\int_\Omega du_2\wedge d^cu_2\wedge dd^ch_j\wedge T'\Big)^{1/2}\\ &=\Big(\int_\Omega(-u_1)dd^cu_1\wedge dd^ch_j\wedge T'\Big)^{1/2}\Big(\int_\Omega(-u_2)dd^cu_2\wedge dd^ch_j\wedge T'\Big)^{1/2}\\ &=\Big(\int_\Omega(-h_j)(dd^cu_1)^2\wedge T'\Big)^{1/2}\Big(\int_\Omega(-h_j)(dd^cu_2)^2\wedge T'\Big)^{1/2}.\end{aligned}$$

By the monotone convergence theorem we get, as $j\to\infty$,

$$\int_\Omega(-h)dd^cu_1\wedge dd^cu_2\wedge T'\le\Big(\int_\Omega(-h)(dd^cu_1)^2\wedge T'\Big)^{1/2}\Big(\int_\Omega(-h)(dd^cu_2)^2\wedge T'\Big)^{1/2}.$$

Now it is enough to note that Lemma 3.13 gives the desired inequalities. ∎

Corollary 3.15 ([32]). *Suppose $u_1,\dots,u_n\in\mathcal{F}$. Then*

$$\int_\Omega dd^cu_1\wedge\dots\wedge dd^cu_n\le\Big(\int_\Omega(dd^cu_1)^n\Big)^{1/n}\dots\Big(\int_\Omega(dd^cu_n)^n\Big)^{1/n}.$$

In particular, if $u,v\in\mathcal{F}$ then

$$\Big(\int_\Omega(dd^c(u+v))^n\Big)^{1/n}\le\Big(\int_\Omega(dd^cu)^n\Big)^{1/n}+\Big(\int_\Omega(dd^cv)^n\Big)^{1/n}.$$

The next proposition was proved by the author in [46], and it shows that $\mathcal{F}$ is closed with respect to convergence of plurisubharmonic functions with uniformly bounded total Monge–Ampère mass in L^1_{loc} space.

Proposition 3.16 ([46]). *Let Ω be a bounded hyperconvex domain in $\mathbb{C}^n$. If a sequence $u_j \in \mathcal{F}$ satisfies the condition*

$$\sup_j \int_\Omega (dd^c u_j)^n < \infty,$$

and if there exists $u \in \mathcal{PSH}(\Omega)$ such that $u_j \to u$ in $L^1_{\mathrm{loc}}(\Omega)$, then $u \in \mathcal{F}$.

Proof. From Theorem 2.17 there exists $w_j \in \mathcal{E}_0 \cap \mathcal{C}(\bar{\Omega})$ such that $w_j \searrow u$ as $j \to \infty$. Note that since $u_j \to u$ in $L^1_{\mathrm{loc}}(\Omega)$, we have $u = \lim_{j\to\infty} v_j$, where

$$v_j = (\sup_{k\geq j} u_k)^*.$$

Observe that v_j is a decreasing sequence, $v_j \geq u_j$, so $v_j \in \mathcal{F}$ and from Corollary 3.6 we have

$$\int_\Omega (dd^c v_j)^n \leq \int_\Omega (dd^c u_j)^n.$$

Define $\varphi_j = \max(w_j, v_j)$. Then $\varphi_j \in \mathcal{E}_0$, $\varphi_j \searrow u$ and again from Corollary 3.6 we get

$$\sup_j \int_\Omega (dd^c \varphi_j)^n \leq \sup_j \int_\Omega (dd^c v_j)^n \leq \sup_j \int_\Omega (dd^c u_j)^n < \infty,$$

which means that $u \in \mathcal{F}$. ∎

4. Maximal plurisubharmonic functions

In this chapter we will characterize maximal plurisubharmonic functions in $\mathcal{E}$ (Corollaries 4.3 and 4.6). Next we introduce the set $\mathcal{N}$ of negative plurisubharmonic functions which have the least maximal plurisubharmonic majorant equal to zero (Definition 4.5). We shall also prove some basic properties of the Cegrell class $\mathcal{N}$ (Propositions 4.7 and 4.8).

In [79], Sadullaev introduced the concept of *maximal* plurisubharmonic functions. Following Sadullaev we say that a plurisubharmonic function u is *maximal* if for every relatively compact open set ω of Ω, and for each upper semicontinuous function v on $\bar{\omega}$ such that v is plurisubharmonic on ω and $v \leq u$ on $\partial\omega$, we have $v \leq u$ on ω. The family of maximal plurisubharmonic functions defined on Ω will be denoted by $\mathcal{MPSH}(\Omega)$. If $n = 1$, then the maximal plurisubharmonic functions are precisely the harmonic functions defined on Ω. For further information on maximal plurisubharmonic functions see e.g. [62, 79].

Lemma 4.1 ([79]). *Let u be a plurisubharmonic function on Ω. The following are then equivalent:*

1. *the function u is maximal on Ω,*
2. *if $v \in \mathcal{PSH}(\Omega)$, ω is a relatively compact open subset of Ω, and $u \geq v$ on $\partial\omega$, then $u \geq v$ on Ω,*
3. *if $v \in \mathcal{PSH}(\Omega)$ is such that $\{v > u\}$ is relatively compact in Ω, then $\{v > u\} = \emptyset$.*

Proof. (1)⇒(2): This is an immediate consequence of the definition of maximal plurisubharmonic function.

(2)⇒(3): Assume that (2) holds and let $v \in \mathcal{PSH}(\Omega)$ be such that $\{v > u\}$ is relatively compact in Ω. It is then possible to choose a relatively compact open subset Ω' of Ω such that $\{v > u\}$ is relatively compact in Ω'. Then $v \le u$ on $\partial\Omega'$ and by assumption (2) it follows that $v \le u$ on $\partial\Omega$. Thus, $\{v > u\} = \emptyset$.

(3)⇒(1): Assume that (3) holds and let ω be a relatively compact open set in Ω and v an upper semicontinuous function on $\bar{\omega}$ that is plurisubharmonic on ω and $v \le u$ on $\partial\omega$. Let φ be defined by

$$\varphi(z) = \begin{cases} \max(u(z), v(z)) & \text{if } z \in \omega, \\ u(z) & \text{if } z \in \Omega \setminus \omega. \end{cases}$$

Then $\varphi \in \mathcal{PSH}(\Omega)$ and $\{\varphi > u\}$ is relatively compact in Ω. Assumption (3) now yields $\{\varphi > u\} = \emptyset$. Thus, $\varphi \le u$ everywhere, in particular on ω, which implies that u is maximal on Ω. ■

The "if" part in Theorem 4.2 was first proved in [79] under the assumption that the sequence $[u_j]$ is decreasing (for an alternative proof see [21]).

THEOREM 4.2 ([34]). *Assume that $\Omega \subseteq \mathbb{C}^n$ is a hyperconvex domain, and $u \in \mathcal{PSH}^-(\Omega)$. Then u is maximal if, and only if, there exists a sequence $[u_j]$, $u_j \in \mathcal{E}_0 \cap \mathcal{C}(\bar{\Omega})$, $u \le u_j$, which converges pointwise to u on Ω and the sequence of measures $[(dd^c u_j)^n]$ converges in the weak*-topology to 0 as j tends to ∞.*

Proof. Assume that $u \in \mathcal{MPSH}(\Omega)$, $u \le 0$. Theorem 2.17 implies that there exists a decreasing sequence $[v_k] \subset \mathcal{E}_0 \cap \mathcal{C}(\bar{\Omega})$ which converges pointwise to u as $k \to \infty$. Consider the function

$$v_k^j(z) = \sup\{\varphi(z) : \varphi \in \mathcal{PSH}(\Omega),\ \varphi \le v_k \text{ on } \Omega_j^c\},$$

where $[\Omega_j]$ is a fundamental sequence of Ω. Hence, $v_k^j \in \mathcal{E}_0$, $\int_{\Omega_j} (dd^c v_k^j)^n = 0$ and $u \le v_k^j$. Moreover, $\lim_{k\to\infty} v_k^j \le u$ on Ω_j^c. Hence $u = \lim_{k\to\infty} v_k^j$ for all j, since u is maximal. For each j it is now possible to choose k_j such that

$$v_{k_j}^j < v_j + 1/j \quad \text{on } \Omega_j.$$

We conclude the first part of this proof by letting $u_j = v_{k_j}^j$.

For the converse, assume that there exists a sequence $[u_j]$, $u_j \in \mathcal{E}_0 \cap \mathcal{C}(\bar{\Omega})$, $u \le u_j$, which converges pointwise to u and the sequence of measures $[(dd^c u_j)^n]$ converges in the weak*-topology to 0 as $j \to \infty$. Let also ω be a relatively compact open set in Ω and v an upper semicontinuous function on $\bar{\omega}$ that is plurisubharmonic on ω and $v \le u$ on $\partial\omega$. By the proof of Lemma 4.1 we may assume that $v \le u$ on $\Omega \setminus \omega$. To complete the proof we must now prove that $v \le u$ on Ω. Consider the function

$$u^j(z) = \sup\{\varphi(z) : \varphi \in \mathcal{PSH}(\Omega),\ \varphi \le u_j \text{ on } \Omega'^c\},$$

where Ω' is a strictly pseudoconvex set such that $\omega \Subset \Omega' \Subset \Omega$. There exist $A, B > 0$ such that $\alpha(z) = A(|z|^2 - B) \le 0$ in Ω and $(dd^c\alpha)^n = dV_n$, where dV_n is Lebesgue measure in $\mathbb{C}^n$. Lemma 3.12 implies that

$$\int_\Omega (u^j - u_j)^n dV_n = \int_{\Omega'} (u^j - u_j)^n (dd^c\alpha)^n \le n!(\sup_{\Omega'}(-\alpha))^n \int_{\Omega'} (dd^c u_j)^n,$$

and therefore u^j converges to u in $L^1_{\mathrm{loc}}(\Omega)$ as $j \to \infty$, since u_j converges pointwise to u on Ω and by assumption $[(dd^c u_j)^n]$ converges in the weak*-topology to 0. Therefore there exists a subsequence $[u^{j_k}]$ that converges to u a.e. dV_n as $j_k \to \infty$. We now have $v \le u^{j_k}$ on Ω'^c and since u^{j_k} is maximal on Ω' we get $v \le u^{j_k}$ on Ω'. Hence, $v \le \lim_{j_k\to\infty} u^{j_k}$ almost everywhere on Ω', which completes the proof. ∎

As a direct consequence of Theorem 4.2 we get Corollaries 4.3 and 4.6 below. Corollary 4.3 was proved in [79] for $u \in \mathcal{PSH}(\Omega) \cap \mathcal{C}(\Omega)$ and as a consequence of results in [16] and [19] it also holds for $u \in \mathcal{PSH}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$. If $u \in \mathcal{E}$, then Corollary 4.3 was first proved in [22, Proposition 2.2].

Corollary 4.3 ([22]). *Let $u \in \mathcal{E}$. Then $(dd^c u)^n = 0$ if, and only if, u is maximal.*

Proof. Assume that $(dd^c u)^n = 0$. Theorem 2.17 implies that there exists a decreasing sequence $[u_j] \subset \mathcal{E}_0 \cap \mathcal{C}(\bar\Omega)$ which converges pointwise to u as $j \to \infty$, and $[(dd^c u_j)^n]$ converges in the weak*-topology to $(dd^c u)^n$, since $u \in \mathcal{E}$. The assumption that $(dd^c u)^n = 0$ and Theorem 4.2 then conclude the first part of the proof. Conversely, assume that u is maximal plurisubharmonic. Then by Theorem 4.2 there exists a sequence $[u_j]$, $u_j \in \mathcal{E}_0 \cap \mathcal{C}(\bar\Omega)$, $u \le u_j$, which converges pointwise to u on Ω and the sequence of measures $[(dd^c u_j)^n]$ converges in the weak*-topology to 0 as $j \to \infty$. Hence, $(dd^c u)^n = 0$. ∎

Definition 4.4 ([33]). Let $u \in \mathcal{PSH}(\Omega)$, $u \le 0$, and let $[\Omega_j]$ be a fundamental sequence. Set

$$u^j = \sup\{\varphi \in \mathcal{PSH}(\Omega) : \varphi \le u \text{ on } \Omega_j^c\},$$

where Ω_j^c denotes the complement of Ω_j in Ω.

Let $[\Omega_j]$ be a fundamental sequence and let $u \in \mathcal{PSH}(\Omega)$, $u \le 0$. Then $u^j \in \mathcal{PSH}(\Omega)$ and $u^j = u$ on Ω_j^c. Definition 4.4 implies that $[u^j]$ is an increasing sequence and therefore $\lim_{j\to\infty} u^j$ exists q.e. (quasi-everywhere, i.e. everywhere except on a set of C_n capacity zero) on Ω. Hence, the function $\tilde u$ defined by

$$\tilde u = (\lim_{j\to\infty} u^j)^* \tag{4.1}$$

is plurisubharmonic on Ω. Moreover, if $u \in \mathcal{E}$, then $\tilde u \in \mathcal{E}$, since $u \le \tilde u \le 0$, and by Theorem 4.2 it follows that $\tilde u$ is maximal on Ω.

In Definition 4.5 we introduce a new class of negative plurisubharmonic functions.

Definition 4.5 ([4, 33]). Set

$$\mathcal{N} = \{u \in \mathcal{E} : \tilde u = 0\}.$$

We say that a plurisubharmonic function u defined on Ω belongs to the Cegrell class $\mathcal{N}(\Omega, H) = \mathcal{N}(H)$, $H \in \mathcal{E}$, if there exists a function $\varphi \in \mathcal{N}$ such that

$$H \ge u \ge \varphi + H.$$

Corollary 4.6 ([33]). *Let $u \in \mathcal{E}$ and $\tilde u$ be as in (4.1). Then*

$$\mathcal{E} \cap \mathcal{MPSH}(\Omega) = \{u \in \mathcal{E} : \tilde u = u\}.$$

Proposition 4.7 ([33]). *$\mathcal{N}$ is a convex cone and $\mathcal{N}$ is precisely the set of functions in $\mathcal{E}$ with smallest maximal plurisubharmonic majorant identically zero. Moreover, if $u \in \mathcal{N}$*

and $v \in \mathcal{PSH}^-(\Omega)$ *then* $\max(u,v) \in \mathcal{N}$. *In particular, if* $u \in \mathcal{N}$ *and* $v \in \mathcal{PSH}^-(\Omega)$, $u \le v$ *then* $v \in \mathcal{N}$.

Proof. Let $u, v \in \mathcal{E}$ and $\alpha \in \mathbb{R}$, $\alpha \ge 0$. Then it follows from Definition 4.4 that $\widetilde{u+v} \ge \tilde{u} + \tilde{v}$ and $\widetilde{\alpha u} = \alpha\,\tilde{u}$. Moreover, if $u \ge v$, then $\tilde{u} \ge \tilde{v}$. This ends the proof. ■

Proposition 4.8 yields a complete characterization of those functions in $\mathcal{E}$ which are also in $\mathcal{N}$.

Proposition 4.8 ([57]). *Let* $u \in \mathcal{E}$. *Then the following assertions are equivalent:*

(1) $u \in \mathcal{N}$,
(2) *there exists a plurisubharmonic function* $\varphi = \sum_{j=1}^{\infty} \varphi_j$, $\varphi_j \in \mathcal{F}$, *such that* $u \ge \varphi$ *on* Ω.

Proof. Assume that $u \in \mathcal{N}$, i.e., $\tilde{u} = 0$. The sequence $[u^j]$, where u^j is defined as in Definition 4.4, increases pointwise to $\tilde{u}$ on $\Omega \setminus A$, where A is a pluripolar subset of Ω. Hence there exists a point $a \in \Omega$ and a subsequence $[u^{j_k}]$ of $[u^j]$ with $u(a) > -\infty$ and $u^{j_k}(a) \ge -1/2^{j_k}$. To simplify the notation, $[u^j]$ and $[-1/2^j]$ will be used instead of $[u^{j_k}]$ and $[-1/2^{j_k}]$. The original sequence will not be used any more. Let ω_j be a connected and open set such that $\bar{\omega}_j \subseteq \Omega$ and for each $j \ge 1$ define

$$\varphi_j = \sup\{\psi \in \mathcal{PSH}(\Omega) : \psi \le u \text{ on } \omega_j\}.$$

In particular, this construction shows that $\varphi_j \ge u$ on Ω, $\varphi_j = u$ on ω_j, and $\varphi_j \in \mathcal{F}$ by Proposition 2.11. Set

$$\omega_j = \begin{cases} \Omega_2 & \text{if } j = 1, \\ \Omega_{j+1} \setminus \bar{\Omega}_{j-1} & \text{if } j \ge 2. \end{cases}$$

This construction implies that $\Omega = \bigcup_{j=1}^{\infty} \omega_j$, $\omega_j \subseteq \Omega_{j-1}^c$ and $\varphi_j \ge u^{j-1}$ on Ω for each $j \ge 2$. Then we have

$$\sum_{j=1}^{\infty} \varphi_j(a) = \varphi_1(a) + \sum_{j=2}^{\infty} \varphi_j(a) \ge u(a) + \sum_{j=2}^{\infty} u^{j-1}(a) \ge u(a) - \sum_{j=2}^{\infty} \frac{1}{2^{j-1}} > -\infty,$$

since $u(a) > -\infty$. Thus, the function defined by $\varphi = \sum_{j=1}^{\infty} \varphi_j$ is plurisubharmonic, since $[\sum_{j=1}^{k} \varphi_j]_{k=1}^{\infty}$ is a decreasing sequence of plurisubharmonic functions which converges pointwise to a function φ which is not identically $-\infty$ as $k \to \infty$. To complete the proof of this implication we need to prove that $u \ge \varphi$ on Ω. Let $z \in \Omega$. Then there exists a j_0, not necessarily uniquely determined, such that $z \in \omega_{j_0}$ and therefore

$$u(z) = \varphi_{j_0}(z) \ge \sum_{j=1}^{\infty} \varphi_j(z) = \varphi(z).$$

For the converse assume that $u \in \mathcal{E}$ is such that (2) holds. Let $v_k = \sum_{j=1}^{k} \varphi_j$. Then $[v_k]$, $v_k \in \mathcal{F}$, is a decreasing sequence which converges pointwise to $\varphi \in \mathcal{PSH}(\Omega)$, $\varphi \le 0$, as $k \to \infty$. The assumption that $u \ge \varphi$ and the definition of the $\tilde{}$-operator yield

$$\tilde{u} \ge \tilde{\varphi} \ge \sum_{j=k}^{\infty} \varphi_j$$

for every $k \geq 1$. Let $k \to \infty$. Then it follows that $\tilde{u}(z) = 0$, since $[v_k]$ converges pointwise to φ. Thus, $u \in \mathcal{N}$, since $u \in \mathcal{E}$ by assumption. ∎

Example 4.9 shows that Proposition 4.8 is not true if we remove the assumption that $u \in \mathcal{E}$.

EXAMPLE 4.9 ([57]). Let $\mathbb{D}^2 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < 1, |z_2| < 1\}$ be the unit polydisc in $\mathbb{C}^2$ and for every $j \in \mathbb{N}$ define

$$u_j(z_1, z_2) = \max\left(j^2 \ln |z_1|, \frac{1}{j^2} \ln |z_2|\right).$$

Then $u_j \in \mathcal{PSH}(\mathbb{D}^2)$, $\lim_{z\to\xi} u_j(z) = 0$ for every $\xi \in \partial\mathbb{D}^2$, and $(dd^c u_j)^2 = (2\pi)^2\delta_{(0,0)}$, where $\delta_{(0,0)}$ denotes the Dirac measure at $(0,0) \in \mathbb{C}^2$. Hence, $u_j \in \mathcal{F}(\mathbb{D}^2)$. Let $v_k : \mathbb{D}^2 \to \mathbb{R} \cup \{-\infty\}$ be defined by $v_k = \sum_{j=1}^k u_j$. The sequence $[v_k]$ is decreasing and for every point $(z_1, z_2) \in \mathbb{D}^2$, $z_2 \neq 0$, we have

$$\lim_{k\to\infty} v_k = \sum_{j=1}^{\infty} u_j \geq 2 \ln |z_2| > -\infty,$$

which implies that $u = \lim_{k\to\infty} v_k \in \mathcal{PSH}(\mathbb{D}^2)$. Moreover, for each $k \geq 1$,

$$0 \geq \tilde{u} \geq \sum_{j=k}^{\infty} u_k,$$

since $(\sum_{j=1}^k u_k) \in \mathcal{F} \subseteq \mathcal{N}$. Hence, $\tilde{u} = 0$ q.e. on $\mathbb{D}^2$, which implies that $\tilde{u} = 0$ everywhere on $\mathbb{D}^2$. Assume now that $u \in \mathcal{E}$. Then for every open neighborhood $\omega \Subset \mathbb{D}^2$ of $(0,0)$,

$$\int_\omega (dd^c v_k)^2 = \int_\omega (dd^c(u_1 + \cdots + u_k))^2 \geq \sum_{j=1}^{k} \int_\omega (dd^c u_j)^2 = (2\pi)^2 k.$$

Thus, $\lim_{k\to\infty} \int_\omega (dd^c v_k)^2 = \infty$, which contradicts $u \in \mathcal{E}$ (see Proposition 2.11).

THEOREM 4.10 ([33]). *Suppose $u \in \mathcal{E}$ with $\int_\Omega (dd^c u)^n < \infty$. Then $u \in \mathcal{F}(\tilde{u})$, and $u \geq \psi + \tilde{u}$ for some $\psi \in \mathcal{F}$ with $\int_\Omega (dd^c\psi)^n \leq \int_\Omega (dd^c u)^n$.*

Proof. Choose $u_j \in \mathcal{E}_0 \cap \mathcal{C}(\bar{\Omega})$ decreasing to u and let $[\Omega_j]$ be a fundamental sequence in Ω. Then for each j there is $s_j > s_{j-1}$ such that for $s \geq s_j$,

$$\int_\Omega \chi_{\Omega_j} (dd^c u_s)^n \leq \int_\Omega (dd^c u)^n + 1.$$

By the Kołodziej subsolution theorem (Theorem 2.7) there exists $v_{s,j} \in \mathcal{E}_0$ such that

$$(dd^c v_{s,j})^n = \chi_{\bar{\Omega}_j} (dd^c u_s)^n.$$

Since $(dd^c u_s)^n \leq (dd^c(u_{s,j} + u_s^j))^n$, we have $u_s \geq v_{s,j} + u_s^j$ by Theorem 2.2, so if $t \geq s \geq s_j$, then

$$u_s \geq u_t \geq v_{t,j} + u_t^j.$$

In particular,

$$u_s \geq (\sup_{t\geq s} v_{t,j})^* + u^j$$

and

$$u \geq \lim_{s\to\infty} (\sup_{t\geq s} v_{t,j})^* + u^j = \psi_j + u^j.$$

Now, $[\psi_j]$ is a decreasing sequence of functions in $\mathcal{F}$, $\int_\Omega (dd^c\psi_j)^n \leq \int_\Omega (dd^c u)^n + 1$, so from Proposition 3.16, $\psi = \lim \psi_j \in \mathcal{F}$ and since u^j increases a.e. to $\tilde{u}$ as $j \to \infty$, we have $u \geq \psi + \tilde{u}$, which completes the proof. ∎

Remark 4.11. The condition $\int_\Omega (dd^c u)^n < \infty$ is not necessary for u to be in $\mathcal{F}(\tilde{u})$. There exists a function $u \in \mathcal{E}_0(\tilde{u})$ with $\int (dd^c u)^n = \infty$ (see Example 4.12).

The following example was constructed by the author and Per Åhag in [6].

Example 4.12 ([6]). Let $\mathbb{D}^2$ be the unit polydisc in $\mathbb{C}^2$. Let $f : \partial\mathbb{D}^2 \to \mathbb{R}$ be defined by $f(z_1, z_2) = |z_2|^2$. Then $f \in C^\infty(\partial\mathbb{D}^2)$ and $PB_f(z_1, z_2) = |z_2|^2$. For each $j \in \mathbb{N}$ define $\varphi_j : \mathbb{D}^2 \to \mathbb{R}$ by $\varphi_j(z) = \varphi_j(z_1, z_2) = \max(a_j \log|z_1|, b_j \log|z_2|, c_j)$, where $a_j, b_j, c_j \in \mathbb{R}$, $a_j, b_j > 0$ and $c_j < 0$. Then $\varphi_j \in \mathcal{PSH}(\mathbb{D}^2) \cap \mathcal{C}(\overline{\mathbb{D}^2})$,

$$\lim_{(z_1,z_2)\to(\xi_1,\xi_2)} \varphi_j(z_1, z_2) = 0 \quad \text{for every } (\xi_1, \xi_2) \in \partial\mathbb{D}^2,$$

and

$$\int_{\mathbb{D}^2} (dd^c\varphi_j)^2 = (2\pi)^2 a_j b_j < \infty, \tag{4.2}$$

hence $\varphi_j \in \mathcal{E}_0$. Let $v_k : \mathbb{D}^2 \to \mathbb{R}$ be defined by $v_k = \sum_{j=1}^k \varphi_j$. Then $v_k \in \mathcal{E}_0$ and $[v_k]$ is a decreasing sequence on $\mathbb{D}^2$. Corollary 3.15 and (4.2) yield

$$\int_{\mathbb{D}^2} (dd^c v_k)^2 \leq \Bigg(\sum_{j=1}^k \bigg(\int_{\mathbb{D}^2} (dd^c\varphi_j)^2\bigg)^{1/2}\Bigg)^2 \leq (2\pi)^2 \Bigg(\sum_{j=1}^k (a_j b_j)^{1/2}\Bigg)^2. \tag{4.3}$$

Assume that

$$\sum_{j=1}^\infty (a_j b_j)^{1/2} < \infty \quad \text{and} \quad \sum_{j=1}^\infty c_j > -\infty \tag{4.4}$$

and let $v(z) = \lim_{k\to\infty} v_k(z)$. The construction of v implies that

$$\lim_{(z_1,z_2)\to(\xi_1,\xi_2)} v(z_1, z_2) = 0 \quad \text{for every } (\xi_1, \xi_2) \in \partial\mathbb{D}^2.$$

The assumptions in (4.4) imply that $v \in \mathcal{PSH}(\mathbb{D}^2) \cap L^\infty(\mathbb{D}^2)$ and by inequality (4.3) it follows that $v \in \mathcal{E}_0$. Let $u : \mathbb{D}^2 \to \mathbb{R}$ be defined by $u = v + PB_f$, hence $u = (v + |z_2|^2) \in \mathcal{E}_0(f)$. Then it follows that

$$\begin{aligned} \int_{\mathbb{D}^2} (dd^c(v_k + |z_2|^2))^2 &= \int_{\mathbb{D}^2} (dd^c v_k)^2 + 4i \int_{\mathbb{D}^2} (dd^c v_k) \wedge dz_2 \wedge d\bar{z}_2 \\ &= \int_{\mathbb{D}^2} (dd^c v_k)^2 + 32 \int_{\mathbb{D}^2} \frac{\partial^2 v_k}{\partial z_1 \partial \bar{z}_1}\, dV_2((z_1, z_2)) \end{aligned} \tag{4.5}$$

$$= \int_{\mathbb{D}^2} (dd^c v_k)^2 + 32 \int_{\mathbb{D}^2} \sum_{j=1}^{k} \frac{\partial^2 \varphi_j}{\partial z_1 \partial \bar{z}_1} \, dV_2((z_1, z_2))$$

$$\geq 32 \sum_{j=1}^{k} \int_{\mathbb{D}^2} \frac{\partial^2 \varphi_j}{\partial z_1 \partial \bar{z}_1} \, dV_2((z_1, z_2)),$$

where dV_2 is the Lebesgue measure on $\mathbb{C}^2$. Let $0 < \varepsilon < 1$ and $\mathbb{D}_\varepsilon = \{z \in \mathbb{C} : |z| < \varepsilon\}$. Choose $\chi_1, \chi_2 \in C_0^\infty(\mathbb{D})$ such that $0 \leq \chi_1, \chi_2 \leq 1$ and $\chi_1 = 1 = \chi_2$ on $\mathbb{D}_{1-\varepsilon}$. For fixed $|z_2| \leq \min(1-\varepsilon, (1-\varepsilon)^{a_j/b_j})$, it follows that

$$\int_{\mathbb{D}} \chi_1(z_1) \frac{\partial^2 \varphi_j}{\partial z_1 \partial \bar{z}_1} dV_1(z_1) = 8\pi a_j. \tag{4.6}$$

Under the assumption that $a_j \geq b_j$ inequality (4.5) together with (4.6) yield

$$\begin{aligned} \int_{\mathbb{D}^2} (dd^c(v_k + |z_2|^2))^2 &\geq 32 \sum_{j=1}^{k} \int_{\mathbb{D}^2} (\chi_1(z_1)\chi_2(z_2)) \frac{\partial^2 \varphi_j}{\partial z_1 \partial \bar{z}_1} \, dV_2((z_1, z_2)) \\ &\geq c \sum_{j=1}^{k} a_j (\min(1-\varepsilon, (1-\varepsilon)^{a_j/b_j}))^2, \end{aligned} \tag{4.7}$$

where $c > 0$ is a constant. Let $\varepsilon \to 0^+$. Then (4.7) implies that

$$\int_{\mathbb{D}^2} (dd^c(v_k + |z_2|^2))^2 \geq c \sum_{j=1}^{k} a_j.$$

Thus

$$\int_{\mathbb{D}^2} (dd^c u)^2 = \lim_{k\to\infty} \int_{\mathbb{D}^2} (dd^c(v_k + |z_2|^2))^2 \geq c \sum_{j=1}^{\infty} a_j. \tag{4.8}$$

Let $a_j = 1/j$, $b_j = 1/j^3$ and $c_j = -1/j^2$. Thus the assumptions (4.4) and $a_j \geq b_j$ are satisfied, which implies that the function defined on $\mathbb{D}^2$ by

$$u(z_1, z_2) = \sum_{j=1}^{\infty} \max\left(\frac{1}{j} \log|z_1|, \frac{1}{j^3} \log|z_2|, -\frac{1}{j^2}\right) + |z_2|^2$$

belongs to $\mathcal{E}_0(f)$ and $\int_{\mathbb{D}^2} (dd^c u)^2 = \infty$, by (4.8).

THEOREM 4.13 ([70]). *The function u belongs to $\mathcal{F}$ if, and only if, $u \in \mathcal{N}$ and*

$$\int_\Omega (dd^c u)^n < \infty.$$

Proof. If $u \in \mathcal{F}$, then it is clear that $\int_\Omega (dd^c u)^n < \infty$ and by Proposition 4.8, $u \in \mathcal{N}$. For the converse, let $u \in \mathcal{N}$ and $\int_\Omega (dd^c u)^n < \infty$. Theorem 4.10 shows that there exists a function $\varphi \in \mathcal{F}$ such that

$$\tilde{u} \geq u \geq \varphi + \tilde{u}.$$

But $u \in \mathcal{N}$, hence $\tilde{u} = 0$ and therefore $u \in \mathcal{F}$. ∎

5. The comparison principle

In this chapter we generalize Theorem 2.2 to functions in $\mathcal{F}$ and $\mathcal{N}(H)$. Those results are crucial when proving uniqueness for the Dirichlet problem in Chapter 6.1. First we prove a more general version of the comparison principle for bounded plurisubharmonic functions (Lemma 5.4), then for the functions in the Cegrell class $\mathcal{F}$ (Theorem 5.5), and finally for $\mathcal{E}$ (Theorem 5.7). Then by using the convergence result for $\mathcal{N}(H)$ (Corollary 5.12), and the decomposition theorem for the positive measures (Theorem 5.6), we shall prove the comparison principle (Corollaries 5.9 and 5.10) and the identity principle (Theorem 5.14) for the functions in $\mathcal{N}(H)$.

The results of the first part of Chapter 5 were proved by Nguyễn and Phạm in [67], and the results of the second part of this chapter were a collaboration between the present author, Per Åhag, Urban Cegrell, and Phạm Hoàng Hiệp (see [4]).

We will need the following lemma.

Lemma 5.1 ([67]). *Let μ be a Borel measure on Ω and let $f : \Omega \to \mathbb{R}$ be a positive measurable function. The following are then equivalent:*

(1) *$\mu(A) = 0$ for all Borel sets $A \subset \{f \neq 0\}$,*
(2) *for every Borel set A we have $\int_A f\, d\mu = 0$.*

Proof. (1)⇒(2): Let $A \subset \Omega$ be a Borel set. Then it follows that

$$\int_A f\, d\mu = \int_{A\setminus\{f=0\}} f\, d\mu + \int_{A\cap\{f=0\}} f\, d\mu = 0.$$

(2)⇒(1): Let $X_\delta = \{f > \delta > 0\}$ for $\delta > 0$. It is sufficient to prove that $\mu(X_\delta) = 0$ for all $\delta > 0$. Hahn's decomposition theorem implies that there exist measurable subsets X_δ^+ and X_δ^- of X_δ such that $X_\delta = X_\delta^+ \cup X_\delta^-$, $X_\delta^+ \cap X_\delta^- = \emptyset$, $\mu \geq 0$ on X_δ^+ and $\mu \leq 0$ on X_δ^-. By the assumption we have

$$\delta\mu(X_\delta^+) \leq \int_{X_\delta^+} f\, d\mu = 0, \qquad \delta\mu(X_\delta^-) \geq \int_{X_\delta^-} f\, d\mu = 0,$$

and therefore $\mu(X_\delta^+) = \mu(X_\delta^-) = 0$. Thus, $\mu = 0$ on X_δ. ∎

The following lemma was proved by Demailly for locally bounded plurisubharmonic functions ([49]).

Lemma 5.2 ([67]). *Let $u, u_1, \ldots, u_{n-1} \in \mathcal{E}$, $v \in \mathcal{PSH}^-(\Omega)$ and $T = dd^c u_1 \wedge \cdots \wedge dd^c u_{n-1}$. The two non-negative measures $dd^c \max(u, v) \wedge T$ and $dd^c u \wedge T$ then coincide on the set $\{u > v\}$.*

Proof. Let $w \in \mathcal{F}$. Since the result is local there is no loss of generality to assume that $u, u_1, \ldots, u_{n-1} \in \mathcal{F}$. Theorem 2.17 implies that for each $k = 1, \ldots, n-1$ there exists a decreasing sequence $[u_k^j]_{j=1}^\infty$, $u_k^j \in \mathcal{E}_0 \cap \mathcal{C}(\bar{\Omega})$, which converges pointwise on Ω to u_k as $j \to \infty$. Moreover, there exists a decreasing sequence $[w^j]_{j=1}^\infty$, $w^j \in \mathcal{E}_0 \cap \mathcal{C}(\bar{\Omega})$, which converges pointwise to w as $j \to \infty$. Set $T^j = dd^c u_1^j \wedge \cdots \wedge dd^c u_{n-1}^j$ and $T = dd^c u_1 \wedge \cdots \wedge dd^c u_{n-1}$. Fix $a < 0$. Then the set $\{w^j > a\}$ is open, since w^j is continuous, and therefore $dd^c \max(w^j, a) \wedge T^j = dd^c w^j \wedge T^j$ on $\{w^j > a\}$. Proposition 3.4 yields

$$\lim_{j\to\infty} \max(w-a,0)\, dd^c \max(w^j, a) \wedge T^j = \max(w-a,0)\, dd^c \max(w,a) \wedge T$$

and

$$\lim_{j\to\infty} \max(w-a,0)\, dd^c w^j \wedge T^j = \max(w-a,0)\, dd^c w \wedge T$$

in the weak*-topology. Hence,

$$\max(w-a,0)[dd^c \max(w,a) \wedge T - dd^c w \wedge T] = 0$$

and therefore it follows by Lemma 5.1 that

$$dd^c \max(w,a) \wedge T = dd^c w \wedge T \quad \text{on } \{w > a\}. \tag{5.1}$$

Fix $b < 0$. By using (5.1) with $w = \max(u, v) \in \mathcal{F}$ we get

$$dd^c \max(u,v) \wedge T = dd^c \max(u,v,b) \wedge T \quad \text{on } \{\max(u,v) > b\} \tag{5.2}$$

and with $w = u$,

$$dd^c u \wedge T = dd^c \max(u,b) \wedge T \quad \text{on } \{u > b\}. \tag{5.3}$$

The function v is upper semicontinuous, which implies that $\{b > v\}$ is open and therefore

$$dd^c \max(u,v,b) \wedge T = dd^c \max(u,b) \wedge T \quad \text{on } \{b > v\}. \tag{5.4}$$

On the set $\{u > b > v\}$ we now have $dd^c \max(u,v) \wedge T = dd^c u \wedge T$, by combining (5.2)–(5.4). To complete the proof note that $\{u > v\} = \bigcup_{b\in\mathbb{Q}^-} \{u > b > v\}$, where $\mathbb{Q}^-$ is the set of non-positive rational numbers. Thus, the non-negative measures $dd^c \max(u,v) \wedge T$ and $dd^c u \wedge T$ coincide on $\{u > v\}$. ∎

The following lemma was proved by Demailly for locally bounded plurisubharmonic functions ([49]).

Lemma 5.3 ([67]). *Let $\Omega \subseteq \mathbb{C}^n$ be a bounded hyperconvex domain.*

(1) *Let $u, v \in \mathcal{E}$ be such that $(dd^c u)^n(\{u = v = -\infty\}) = 0$. Then*

$$(dd^c \max(u,v))^n \geq \chi_{\{u\geq v\}}(dd^c u)^n + \chi_{\{u<v\}}(dd^c v)^n,$$

where χ_E denotes the characteristic function of E.

(2) *Let μ be a positive measure which vanishes on all pluripolar subsets of Ω. Suppose $u, v \in \mathcal{E}$ are such that $(dd^c u)^n \geq \mu, (dd^c v)^n \geq \mu$. Then $(dd^c \max(u,v))^n \geq \mu$.*

Proof. (1): For each $\varepsilon > 0$ put $A_\varepsilon = \{u = v - \varepsilon\} \setminus \{u = v = -\infty\}$. Since $A_\varepsilon \cap A_\delta = \emptyset$ for $\varepsilon \neq \delta$, there exist $\varepsilon_j \searrow 0$ such that $(dd^c u)^n(A_{\varepsilon_j}) = 0$ for $j \geq 1$. On the other hand, since $(dd^c u)^n(\{u = v = -\infty\}) = 0$ we have $(dd^c u)^n(\{u = v - \varepsilon_j\}) = 0$ for $j \geq 1$. It follows from Lemma 5.2 that

$$\begin{aligned}
&(dd^c \max(u, v-\varepsilon_j))^n \\
&\quad \geq \chi_{\{u>v-\varepsilon_j\}}(dd^c \max(u,v-\varepsilon_j))^n + \chi_{\{u<v-\varepsilon_j\}}(dd^c \max(u,v-\varepsilon_j))^n \\
&\quad = \chi_{\{u\geq v-\varepsilon_j\}}(dd^c u)^n + \chi_{\{u<v-\varepsilon_j\}}(dd^c v)^n \geq \chi_{\{u\geq v\}}(dd^c u)^n + \chi_{\{u<v-\varepsilon_j\}}(dd^c v)^n.
\end{aligned}$$

Since $\max(u, v-\varepsilon_j) \nearrow \max(u,v)$ and $\chi_{\{u<v-\varepsilon_j\}} \nearrow \chi_{\{u<v\}}$ as $j\to\infty$, by Proposition 3.8 we get

$$(dd^c \max(u,v))^n \geq \chi_{\{u\geq v\}}(dd^c u)^n + \chi_{\{u<v\}}(dd^c v)^n.$$

(2): The proof is similar to the proof of (1). By the same argument as in the proof of (1), there exists $\varepsilon_j \searrow 0$ such that $\mu(\{u = v-\varepsilon_j\}) = 0$ for $j\geq 1$. It follows from Lemma 5.2 that

$$\begin{aligned}(dd^c \max(u, v-\varepsilon_j))^n& \\ &\geq \chi_{\{u>v-\varepsilon_j\}}(dd^c \max(u,v-\varepsilon_j))^n + \chi_{\{u<v-\varepsilon_j\}}(dd^c \max(u,v-\varepsilon_j))^n \\ &= \chi_{\{u>v-\varepsilon_j\}}(dd^c u)^n + \chi_{\{u<v-\varepsilon_j\}}(dd^c v)^n \geq \chi_{\{u\neq v-\varepsilon_j\}}d\mu = d\mu.\end{aligned}$$

Since $\max(u, v-\varepsilon_j) \nearrow \max(u,v)$ as $j\to\infty$, by Proposition 3.8 we get

$$(dd^c \max(u,v))^n \geq d\mu. \ \blacksquare$$

Lemma 5.4 ([67]). *Let $u, v \in \mathcal{PSH}(\Omega)\cap L^\infty(\Omega)$ be such that $u\leq v$ on Ω, and*

$$\lim_{z\to\partial\Omega}[u(z)-v(z)] = 0.$$

Then for $1\leq k\leq n$, for all $w_1,\dots,w_k \in \mathcal{PSH}(\Omega)$, $-1\leq w_j\leq 0$, $j=1,\dots,k$, $w_{k+1},\dots,w_n\in\mathcal{E}$, the following inequality holds:

$$\begin{aligned}\frac{1}{k!}\int_\Omega (v-u)^k dd^c w_1\wedge\cdots\wedge dd^c w_n + \int_\Omega (-w_1)(dd^c v)^k \wedge dd^c w_{k+1}\wedge\cdots\wedge dd^c w_n& \\ \leq \int_\Omega (-w_1)(dd^c u)^k\wedge dd^c w_{k+1}\wedge\cdots\wedge dd^c w_n.&\end{aligned}\tag{5.5}$$

Proof. To simplify the notation we set $T = dd^c w_{k+1}\wedge\cdots\wedge dd^c w_n$. First, assume that $u, v\in\mathcal{PSH}(\Omega)\cap L^\infty(\Omega)$, $u\leq v$ on Ω, and $u=v$ on $\Omega\setminus K$ for some $K\Subset\Omega$. Using Lemma 3.12 we get

$$\begin{aligned}\int_\Omega (v-u)^k dd^c w_1\wedge\cdots\wedge dd^c w_n &\leq k\int_\Omega (v-u)^{k-1} dd^c w_1\wedge\cdots\wedge dd^c w_{k-1}\wedge dd^c u\wedge T \\ &\leq\cdots\leq k!\int_\Omega (v-u)dd^c w_1\wedge(dd^c u)^{k-1}\wedge T \\ &\leq k!\int_\Omega (v-u)dd^c w_1\wedge\left[\sum_{i=0}^{k-1}(dd^c u)^i\wedge(dd^c v)^{k-i-1}\right]\wedge T \\ &= k!\int_\Omega w_1 dd^c(v-u)\wedge\left[\sum_{i=0}^{k-1}(dd^c u)^i\wedge(dd^c v)^{k-i-1}\right]\wedge T \\ &= k!\int_\Omega(-w_1)[(dd^c u)^k-(dd^c v)^k]\wedge T.\end{aligned}$$

In the general case, for each $\varepsilon>0$ we put $v_\varepsilon = \max(u, v-\varepsilon)$. Then $v_\varepsilon\nearrow v$ on Ω, $v_\varepsilon\geq u$ on Ω and $v_\varepsilon = u$ on $\Omega\setminus K$ for some $K\Subset\Omega$. Hence

$$\frac{1}{k!}\int_\Omega (v_\varepsilon-u)^k dd^c w_1\wedge\cdots\wedge dd^c w_n + \int_\Omega(-w_1)(dd^c v_\varepsilon)^k\wedge T \leq \int_\Omega(-w_1)(dd^c u)^k\wedge T.$$

Observe that $0 \le v_\varepsilon - u \nearrow v - u$ and $(dd^c v_\varepsilon)^k \wedge T \to (dd^c v)^k \wedge T$ in the weak*-topology as $\varepsilon \searrow 0$, furthermore $-w_1$ is lower semicontinuous and so by letting $\varepsilon \searrow 0$ we have

$$\frac{1}{k!}\int_\Omega (v-u)^k dd^c w_1 \wedge \cdots \wedge dd^c w_n + \int_\Omega (-w_1)(dd^c v)^k \wedge T \le \int_\Omega (-w_1)(dd^c u)^k \wedge T. \ \blacksquare$$

THEOREM 5.5 ([67]). *Let $\Omega \subseteq \mathbb{C}^n$ be a bounded hyperconvex domain.*

(1) *Let $u, v \in \mathcal{F}$ be such that $u \le v$ on Ω. Then for $1 \le k \le n$, for all $w_j \in \mathcal{PSH}(\Omega)$, $-1 \le w_j \le 0$, $j = 1, \dots, k$, $w_{k+1}, \dots, w_n \in \mathcal{F}$, inequality* (5.5) *holds.*
(2) *Let $u, v \in \mathcal{E}$ be such that $u \le v$ on Ω and $u = v$ on $\Omega \setminus K$ for some $K \Subset \Omega$. Then for $1 \le k \le n$, for all $w_j \in \mathcal{PSH}(\Omega)$, $-1 \le w_j \le 0$, $j = 1, \dots, k$, $w_{k+1}, \dots, w_n \in \mathcal{E}$ inequality* (5.5) *holds.*

Proof. (1): Let $\mathcal{E}_0 \ni u_j \searrow u$ and $\mathcal{E}_0 \ni v_j \searrow v$ be decreasing sequences from the definition of $\mathcal{F}$. Replacing v_j by $\max(u_j, v_j)$ we may assume that $u_j \le v_j$ for $j \ge 1$. By Lemma 5.4,

$$\frac{1}{k!}\int_\Omega (v_j-u_t)^k dd^c w_1 \wedge \cdots \wedge dd^c w_n + \int_\Omega (-w_1)(dd^c v_j)^k \wedge dd^c w_{k+1} \wedge \cdots \wedge dd^c w_n$$
$$\le \int_\Omega (-w_1)(dd^c u_t)^k \wedge dd^c w_{k+1} \wedge \cdots \wedge dd^c w_n$$

for $t \ge j \ge 1$. By Proposition 3.4 letting $t \to \infty$ in the above inequality we have

$$\frac{1}{k!}\int_\Omega (v_j-u)^k dd^c w_1 \wedge \cdots \wedge dd^c w_n + \int_\Omega (-w_1)(dd^c v_j)^k \wedge dd^c w_{k+1} \wedge \cdots \wedge dd^c w_n$$
$$\le \int_\Omega (-w_1)(dd^c u)^k \wedge dd^c w_{k+1} \wedge \cdots \wedge dd^c w_n$$

for $j \ge 1$. Next letting $j \to \infty$ again by Proposition 3.4 we get the desired conclusion.

(2): Let G, W be open sets such that $K \Subset G \Subset W \Subset \Omega$. By Proposition 2.11 we can choose a function $\hat{v} \in \mathcal{F}$ such that $\hat{v} \ge v$ and $\hat{v} = v$ on W. Set

$$\hat{u} = \begin{cases} u & \text{on } G, \\ \hat{v} & \text{on } \Omega \setminus G. \end{cases}$$

Since $u = v = \hat{v}$ on $W \setminus K$ we have $\hat{u} \in \mathcal{PSH}^-(\Omega)$. It is easy to see that $\hat{u} \in \mathcal{F}$, $\hat{u} \le \hat{v}$ and $\hat{u} = u$ on W. By (1) we have

$$\frac{1}{k!}\int_\Omega (\hat{v}-\hat{u})^k dd^c w_1 \wedge \cdots \wedge dd^c w_n + \int_\Omega (-w_1)(dd^c \hat{v})^k \wedge dd^c w_{k+1} \wedge \cdots \wedge dd^c w_n$$
$$\le \int_\Omega (-w_1)(dd^c \hat{u})^k \wedge dd^c w_{k+1} \wedge \cdots \wedge dd^c w_n.$$

Since $\tilde{u} = \hat{v}$ on $\Omega \setminus G$ we have

$$\frac{1}{k!}\int_W (\hat{v}-\hat{u})^k dd^c w_1 \wedge \cdots \wedge dd^c w_n + \int_W (-w_1)(dd^c \hat{v})^k \wedge dd^c w_{k+1} \wedge \cdots \wedge dd^c w_n$$
$$\le \int_W (-w_1)(dd^c \hat{u})^k \wedge dd^c w_{k+1} \wedge \cdots \wedge dd^c w_n.$$

Since $\hat{u} = u$, $\hat{v} = v$ on W and $u = v$ on $\Omega \setminus K$ we obtain

$$\frac{1}{k!}\int_\Omega (v-u)^k dd^c w_1 \wedge \cdots \wedge dd^c w_n + \int_\Omega (-w_1)(dd^c v)^k \wedge dd^c w_{k+1} \wedge \cdots \wedge dd^c w_n \\ \leq \int_\Omega (-w_1)(dd^c u)^k \wedge dd^c w_{k+1} \wedge \cdots \wedge dd^c w_n. \ \blacksquare$$

We will need the following decomposition theorem for positive measures.

THEOREM 5.6 ([30, 32]). *Let μ be a positive measure in a bounded hyperconvex domain $\Omega \subset \mathbb{C}^n$. Then there exist $\varphi \in \mathcal{E}_0$, $0 \leq f \in L^1_{\mathrm{loc}}((dd^c\varphi)^n)$ and a positive measure ν carried by a pluripolar set in Ω such that*

$$\mu = f(dd^c\varphi)^n + \nu.$$

In particular, if $u \in \mathcal{E}$, then there exist $\varphi \in \mathcal{E}_0$, $0 \leq f \in L^1_{\mathrm{loc}}((dd^c\varphi)^n)$ and a positive measure ν carried by $\{u = -\infty\}$ such that

$$(dd^c u)^n = f(dd^c\varphi)^n + \nu.$$

THEOREM 5.7 ([4]). *Assume that $\Omega \subseteq \mathbb{C}^n$, $n \geq 1$, is a bounded hyperconvex domain. Let $u, v \in \mathcal{E}$ be such that $\underline{\lim}_{z\to\zeta}(u(z) - v(z)) \geq 0$ for every $\zeta \in \partial\Omega$. Then for all $w_j \in \mathcal{PSH}(\Omega)$, $-1 \leq w_j \leq 0$, $j = 1, \dots, k$, $w_{k+1}, \dots, w_n \in \mathcal{E}$, the following inequality holds:*

$$\tag{5.6} \frac{1}{k!}\int_{\{u<v\}} (v-u)^k dd^c w_1 \wedge \cdots \wedge dd^c w_n \\ + \int_{\{u<v\}} (-w_1)(dd^c v)^k \wedge dd^c w_{k+1} \wedge \cdots \wedge dd^c w_n \\ \leq \int_{\{u<v\}\cup\{u=v=-\infty\}} (-w_1)(dd^c u)^k \wedge dd^c w_{k+1} \wedge \cdots \wedge dd^c w_n.$$

Proof. Let $\varepsilon > 0$ and let $T = dd^c w_{k+1} \wedge \cdots \wedge dd^c w_n$. By using Theorem 5.5(2) for u and $v_\varepsilon = \max(u, v - \varepsilon)$ we get

$$\frac{1}{k!}\int_\Omega (v_\varepsilon - u)^k dd^c w_1 \wedge \cdots \wedge dd^c w_n + \int_\Omega (-w_1)(dd^c v_\varepsilon)^k \wedge T \leq \int_\Omega (-w_1)(dd^c u)^k \wedge T.$$

From the fact that $\{u < v_\varepsilon\} = \{u < v - \varepsilon\}$ together with Lemma 5.2 it follows that

$$\chi_{\{u<v-\varepsilon\}}(dd^c v)^k \wedge T = \chi_{\{u<v-\varepsilon\}}(dd^c \max(u, v-\varepsilon))^k \wedge T,$$

where $\chi_{\{u<v-\varepsilon\}}$ is the characteristic function of the set $\{u < v - \varepsilon\}$ in Ω. Then

$$\tag{5.7} \frac{1}{k!}\int_{\{u<v-\varepsilon\}} (v-\varepsilon-u)^k dd^c w_1 \wedge \cdots \wedge dd^c w_n + \int_{\{u<v-\varepsilon\}} (-w_1)(dd^c v)^k \wedge T \\ + \int_{\{u>v-\varepsilon\}} (-w_1)(dd^c v_\varepsilon)^k \wedge T$$

$$\leq \frac{1}{k!}\int_{\{u\leq v_\varepsilon\}}(v_\varepsilon-u)^k dd^cw_1\wedge\cdots\wedge dd^cw_n$$
$$+\int_{\{u<v-\varepsilon\}}(-w_1)(dd^cv_\varepsilon)^k\wedge T+\int_{\{u>v-\varepsilon\}}(-w_1)(dd^cv_\varepsilon)^k\wedge T$$
$$\leq \frac{1}{k!}\int_{\Omega}(v_\varepsilon-u)^k dd^cw_1\wedge\cdots\wedge dd^cw_n+\int_{\Omega}(-w_1)(dd^cv_\varepsilon)^k\wedge T$$
$$\leq \int_{\Omega}(-w_1)(dd^cu)^k\wedge T=\int_{\{u\leq v-\varepsilon\}}(-w_1)(dd^cu)^k\wedge T+\int_{\{u>v-\varepsilon\}}(-w_1)(dd^cu)^k\wedge T.$$

But $\{u>v_\varepsilon\}=\{u>v-\varepsilon\}$ and by Lemma 5.2 we have

$$\chi_{\{u>v_\varepsilon\}}(dd^cv_\varepsilon)^k\wedge T=\chi_{\{u>v_\varepsilon\}}(dd^cu)^k\wedge T,$$

and therefore from (5.7) we obtain

$$\frac{1}{k!}\int_{\{u<v-\varepsilon\}}(v-\varepsilon-u)^k dd^cw_1\wedge\cdots\wedge dd^cw_n+\int_{\{u<v-\varepsilon\}}(-w_1)(dd^cv)^k\wedge T\leq \int_{\{u\leq v-\varepsilon\}}(-w_1)(dd^cu)^k\wedge T\leq\int_{\{u<v\}\cup\{u=v=-\infty\}}(-w_1)(dd^cu)^k\wedge T, \tag{5.8}$$

since for every $\varepsilon>0$,

$$\{u\leq v-\varepsilon\}\subset\{u<v\}\cup\{u=v=-\infty\}.$$

The sequence $[\chi_{\{u<v-\varepsilon\}}]$ is increasing to $\chi_{\{u<v\}}$ as $\varepsilon\to0^+$, therefore by letting $\varepsilon\to0^+$ inequality (5.8) implies that (5.6) holds and the proof is complete. ∎

Remark 5.8. Recall that $\underline{\lim}_{z\to\zeta}(u(z)-v(z))\geq0$ for every $\zeta\in\partial\Omega$ means that for any $\varepsilon>0$ there exists a set $A\Subset\Omega$ such that $u(z)-v(z)\geq-\varepsilon$ for every $z\in\Omega\setminus A$.

By using Theorems 5.6 and 5.7 we get

Corollary 5.9 ([4]). *Assume that $\Omega\subseteq\mathbb{C}^n$ is a bounded hyperconvex domain and $H\in\mathcal{E}$. If $u\in\mathcal{N}(H)$ and $v\in\mathcal{E}$ is such that $v\leq H$ on Ω, then for all $w_j\in\mathcal{PSH}(\Omega)$, $-1\leq w_j\leq0$, $j=1,\dots,n$, (5.6) holds.*

Proof. Let $u\in\mathcal{N}(H)$. Then there exists a function $\varphi\in\mathcal{N}$ such that

$$H\geq u\geq\varphi+H.$$

Let $[\Omega_j]$ be a fundamental sequence in Ω and let φ^j be as in Definition 4.4. The assumption that $v\leq H$ implies that for $\varepsilon>0$,

$$u\geq\varphi+H=\varphi^j+H\geq\varphi^j+v-\varepsilon\quad\text{ on }\Omega_j^c.$$

Theorem 5.7 implies that

$$\frac{1}{n!}\int_{\{u<v-\varepsilon+\varphi^j\}}(v-\varepsilon+\varphi^j-u)^n dd^cw_1\wedge\cdots\wedge dd^cw_n+\int_{\{u<v-\varepsilon+\varphi^j\}}(-w_1)(dd^cv)^n\leq\int_{\{u\leq v-\varepsilon\}}(-w_1)(dd^cu)^n\,.$$

We see that

$$[\chi_{\{u<v-\varepsilon+\varphi^j\}}(v-\varepsilon+\varphi^j-u)^n]_{j=1}^{\infty}\quad\text{and}\quad[\chi_{\{u<v-\varepsilon+\varphi^j\}}]_{j=1}^{\infty} \tag{5.9}$$

are two increasing sequences of functions that converge q.e. on Ω to $\chi_{\{u<v-\varepsilon\}}(v-\varepsilon-u)^n$ and $\chi_{\{u<v-\varepsilon\}}$, respectively, as $j\to\infty$. Theorem 5.6 implies that the measures $dd^cw_1\wedge\cdots\wedge dd^cw_n$ and $\chi_{\{v>-\infty\}}(dd^cv)^n$ vanish on pluripolar sets. Therefore

$$[\chi_{\{u<v-\varepsilon+\varphi^j\}}(v-\varepsilon+\varphi^j-u)^n]_{j=1}^{\infty}\quad\text{converges to}\quad \chi_{\{u<v-\varepsilon\}}(v-\varepsilon-u)^n$$

a.e. w.r.t. $dd^cw_1\wedge\cdots\wedge dd^cw_n$, and $[\chi_{\{u<v-\varepsilon+\varphi^j\}}]_{j=1}^{\infty}$ converges to $\chi_{\{u<v-\varepsilon\}}$ a.e. w.r.t. $\chi_{\{v>-\infty\}}(dd^cv)^n$. The monotone convergence theorem yields

$$\frac{1}{n!}\int_{\{u<v-\varepsilon\}}(v-\varepsilon-u)^n dd^cw_1\wedge\cdots\wedge dd^cw_n+\int_{\{u<v-\varepsilon\}}(-w_1)(dd^cv)^n \le \int_{\{u\le v-\varepsilon\}}(-w_1)(dd^cu)^n.$$

Inequality (5.6) is now obtained by letting $\varepsilon\to 0^+$. ■

Corollary 5.10 ([4], comparison principle). *Let $u,v,H\in\mathcal{E}$ be such that $(dd^cu)^n$ vanishes on all pluripolar sets in Ω and $(dd^cu)^n\le(dd^cv)^n$. Consider the following two conditions:*

(1) $\underline{\lim}_{z\to\zeta}(u(z)-v(z))\ge 0$ *for every* $\zeta\in\partial\Omega$,
(2) $u\in\mathcal{N}(H)$, $v\le H$.

If one of the above conditions is satisfied, then $u\ge v$ on Ω.

Proof. Assume that $u,v\in\mathcal{E}$ is such that $(dd^cu)^n$ vanishes on all pluripolar sets in Ω and $(dd^cu)^n\le(dd^cv)^n$.

(1): Moreover, assume that

$$\underline{\lim}_{z\to\zeta}(u(z)-v(z))\ge 0$$

for every $\zeta\in\partial\Omega$. Let $\varepsilon>0$. Theorem 5.7 implies that

$$\begin{aligned}(5.10)\qquad &\frac{\varepsilon^n}{n!}C_n(\{u+2\varepsilon<v\})\\ &\le\sup\left\{\frac{1}{n!}\int_{\{u+2\varepsilon<v\}}(v-u-2\varepsilon)^n(dd^cw)^n : w\in\mathcal{PSH}(\Omega),\ -1\le w\le 0\right\}\\ &\le\sup\left\{\frac{1}{n!}\int_{\{u+\varepsilon<v\}}(v-u-\varepsilon)^n(dd^cw)^n : w\in\mathcal{PSH}(\Omega),\ -1\le w\le 0\right\}\\ &\le\sup\left\{\int_{\{u+\varepsilon<v\}}(-w)((dd^cu)^n-(dd^cv)^n),\ -1\le w\le 0\right\}\le 0.\end{aligned}$$

Thus, $u+2\varepsilon\ge v$. Letting $\varepsilon\to 0^+$ yields $u\ge v$ on Ω.

(2): In this case assume that $u\in\mathcal{N}(H)$ and $v\le H$. Since $u\in\mathcal{N}(H)$, there exists $\varphi\in\mathcal{N}$ such that $H+\varphi\le u\le H$. Let φ^j be as in Definition 4.4 and let $\varepsilon>0$. As in (5.10), we get $u+2\varepsilon\ge v+\varphi^j$. Let $\varepsilon\to 0^+$. Hence $u\ge v$ on Ω. ■

Lemma 5.11 ([4]). *Let $u,v\in\mathcal{N}(H)$ be such that $u\le v$ and $\int_\Omega(-\varphi)dd^cu\wedge T<\infty$, $\varphi\in\mathcal{PSH}(\Omega)$, $\varphi\le 0$. Then*

$$\int_\Omega (-\varphi) dd^c u \wedge T \geq \int_\Omega (-\varphi) dd^c v \wedge T, \tag{5.11}$$

where $T = dd^c w_2 \wedge \cdots \wedge dd^c w_n$, $w_2, \ldots, w_n \in \mathcal{E}$.

Proof. Let $[\Omega_j]$ be a fundamental sequence in Ω. As $u \in \mathcal{N}(H)$ there exists $\psi \in \mathcal{N}$ such that $H \geq u \geq \psi + H$. For each $j \in \mathbb{N}$ define $v_j = \max(u, \psi^j + v)$, where ψ^j is as in Definition 4.4. This construction implies that $v_j \in \mathcal{E}$, $v_j = u$ on Ω_j^c, $u \leq v_j$, and $[v_j]$ is an increasing sequence that converges pointwise to v q.e. on Ω as $j \to \infty$. Theorem 2.17 implies that there exists a decreasing sequence $[\varphi_k]$, $\varphi_k \in \mathcal{E}_0 \cap \mathcal{C}(\bar{\Omega})$, that converges pointwise to φ as $j \to \infty$. By Stokes' theorem, for each $s \geq j$,

$$\int_{\Omega_s} (-\varphi_k) dd^c u \wedge T - \int_{\Omega_s} (-\varphi_k) dd^c v_j \wedge T = \int_{\Omega_s} (v_j - u) dd^c \varphi_k \wedge T \geq 0.$$

By letting $s \to \infty$ we get

$$\int_\Omega (-\varphi_k) dd^c u \wedge T \geq \int_\Omega (-\varphi_k) dd^c v_j \wedge T. \tag{5.12}$$

The function φ_k is bounded and therefore Proposition 3.8 implies that $(-\varphi_k) dd^c v_j \wedge T$ converges to $(-\varphi_k) dd^c v \wedge T$ in the weak*-topology as $j \to \infty$, which yields

$$\lim_{j \to \infty} \int_\Omega (-\varphi_k) dd^c v_j \wedge T \geq \int_\Omega (-\varphi_k) dd^c v \wedge T. \tag{5.13}$$

Inequalities (5.12) and (5.13) imply (5.11) for φ_k, and the monotone convergence theorem completes the proof, when we let $k \to \infty$. ∎

Corollary 5.12 ([4]). *Let* $H \in \mathcal{E}$ *and* $\varphi \in \mathcal{PSH}(\Omega)$, $\varphi \leq 0$. *If* $[u_j]$, $u_j \in \mathcal{N}(H)$, *is a decreasing sequence that converges pointwise on* Ω *to a function* $u \in \mathcal{N}(H)$ *as* j *tends to* ∞, *then*

$$\lim_{j \to \infty} \int_\Omega (-\varphi)(dd^c u_j)^n = \int_\Omega (-\varphi)(dd^c u)^n. \tag{5.14}$$

Proof. Let $\varphi \in \mathcal{PSH}(\Omega)$, $\varphi \leq 0$, and let $u_j, u \in \mathcal{N}(H)$ be such that $u \leq u_j$. If $\int_\Omega (-\varphi)(dd^c u)^n = \infty$, then (5.14) follows immediately and therefore we can assume that $\int_\Omega (-\varphi)(dd^c u)^n < \infty$. Lemma 5.11 implies that $[\int_\Omega (-\varphi)\,(dd^c u_j)^n]$ is an increasing sequence bounded above by $\int_\Omega (-\varphi)\,(dd^c u)^n$. From Proposition 3.4 it follows that $[(-\varphi)(dd^c u_j)^n]$ converges to $(-\varphi)(dd^c u)^n$ in the weak*-topology as $j \to \infty$, and the desired limit of the total masses is valid. ∎

Lemma 5.13 ([4]). *Let* $H \in \mathcal{E}$ *and let* $u, v \in \mathcal{N}(H)$ *be such that* $u \leq v$. *Then for all* $w_j \in \mathcal{PSH}(\Omega) \cap L^\infty(\Omega)$, $-1 \leq w_j \leq 0$, $j = 1, \ldots, n$, $\int_\Omega (-w_1)(dd^c u)^n < \infty$, *we have*

$$\frac{1}{n!} \int_\Omega (v - u)^n dd^c w_1 \wedge \cdots \wedge dd^c w_n + \int_\Omega (-w_1)(dd^c v)^n \leq \int_\Omega (-w_1)(dd^c u)^n. \tag{5.15}$$

Proof. First we assume that $u, v \in \mathcal{E}_0(H)$. By definition there exists $\varphi \in \mathcal{E}_0$ such that $H \geq u \geq \varphi + H$. For each $\varepsilon > 0$ small enough choose $K \Subset \Omega$ such that $\varphi \geq -\varepsilon$ on K^c. Hence,

$$u \geq \varphi + H \geq -\varepsilon + H \geq -\varepsilon + v \quad \text{on } K^c,$$

and therefore $\max(u, v-\varepsilon) = u$ on K^c. By using Lemma 5.3 we get

$$\frac{1}{n!}\int_\Omega (\max(u, v-\varepsilon) - u)^n dd^c w_1 \wedge \cdots \wedge dd^c w_n + \int_\Omega (-w_1)(dd^c \max(u, v-\varepsilon))^n \leq \int_\Omega (-w_1)(dd^c u)^n.$$

By letting $\varepsilon \to 0^+$ we obtain (5.15) in the case when $u, v \in \mathcal{E}_0(H)$. Using this case together with Proposition 2.17 and Corollary 5.12 we complete the proof. ∎

An immediate consequence of Lemma 5.13 is the following identity principle, which plays a prominent technical role.

THEOREM 5.14 ([4]). *Let $H \in \mathcal{E}$. If $u, v \in \mathcal{N}(H)$ are such that $u \leq v$, $(dd^c u)^n = (dd^c v)^n$ and $\int_\Omega (-w)(dd^c u)^n < \infty$ for some $w \in \mathcal{E}$ which is not identically 0, then $u = v$ on Ω.*

6. The Dirichlet problem

In this chapter we study the Dirichlet problem for the complex Monge–Ampère operator in a given subset of negative plurisubharmonic functions, say $\mathcal{K}$. More precisely, for a given positive measure μ on a bounded hyperconvex domain $\Omega \subset \mathbb{C}^n$, the problem is to find a plurisubharmonic function $u \in \mathcal{K}$ solving the equation

$$(dd^c u)^n = \mu. \tag{6.1}$$

Our aim is to generalize Kołodziej's subsolution theorem from locally bounded plurisubharmonic functions to functions in $\mathcal{E}$. In Chapter 6.1, we shall prove that for a large class of measures μ there exists a unique solution to (6.1) in $\mathcal{F}(f)$ (Theorem 6.1), in $\mathcal{N}$ (Theorem 6.3), and in $\mathcal{N}(H)$ (Theorem 6.6). In Chapter 6.2 we shall consider measures carried by some pluripolar set. We provide the construction of a function u such that $(dd^c u)^n = \chi_K (dd^c v)^n$, where χ_K is the characteristic function of a pluripolar compact set K and $v \in \mathcal{E}$ (Theorem 6.10). This allows us to prove the subsolution theorem for measures carried by pluripolar sets. Namely, we shall prove that if $v \in \mathcal{E}$, then for any measure μ carried by a pluripolar set such that $\mu \leq (dd^c v)^n$ there exists $u \in \mathcal{E}$ that satisfies (6.1) (Theorem 6.15). In Chapter 6.3 we shall prove the most general version of the subsolution theorem: any measure μ dominated by the Monge–Ampère measure of a function $v \in \mathcal{E}$, $\mu \leq (dd^c v)^n$, is the Monge–Ampère measure of some function from $\mathcal{E}$ (Theorem 6.22). We also give an example of a positive measure which does not belong to the range of the complex Monge–Ampère operator (Example 6.24).

Results in Chapter 6.1 were proved in [4, 32, 33]. Almost all results from Chapter 6.2 and Chapter 6.3 were proved by the author together with Per Åhag, Urban Cegrell, and Phạm Hoàng Hiệp in [4].

6.1. Regular measures. We start by solving the Dirichlet problem in $\mathcal{F}(f)$ with continuous boundary values f.

Theorem 6.1 ([32]). *Assume that μ is a positive measure on Ω and $f \in \mathcal{C}(\partial\Omega)$ is such that $\lim_{z\to w} PB_f(z) = f(w)$ for all $w \in \partial\Omega$. If $\mu(\Omega) < \infty$ and if μ vanishes on all pluripolar sets, then there exists a unique function $u \in \mathcal{F}(f)$ such that $(dd^c u)^n = \mu$.*

Proof. Without loss of generality we can assume that $f \le 0$. It follows from Theorem 5.6 that there exist $\varphi \in \mathcal{E}_0$ and $0 \le g \in L^1((dd^c\varphi)^n)$ such that $\mu = g(dd^c\varphi)^n$. Define $\mu_j = \min(g,j)(dd^c\varphi)^n$ and observe that

$$\mu_j \le (dd^c(j^{1/n}\varphi))^n \le (dd^c(j^{1/n}\varphi + PB_f))^n.$$

By the Kołodziej subsolution theorem (Theorem 2.7) there exist $v_j \in \mathcal{E}_0$ such that $(dd^c v_j)^n = \mu_j$ and $u_j \in \mathcal{PSH}(\Omega) \cap L^\infty(\Omega)$, $(dd^c u_j)^n = \mu_j$ and $\lim_{z\to w} u_j(z) = f(w)$. Since

$$(dd^c v_j)^n = \mu_j \le \mu_{j+1} = (dd^c v_{j+1})^n$$

and $\lim v_j = \lim v_{j+1} = 0$ on $\partial\Omega$, it follows from the comparison principle (Theorem 2.2) that $v_j \ge v_{j+1}$. Similarly one can prove that $[u_j]$ is a decreasing sequence and

$$PB_f \ge u_j \ge v_j + PB_f.$$

Let $u = \lim_{j\to\infty} u_j$ and $v = \lim_{j\to\infty} v_j$. There exist $A, B > 0$ such that $\psi(z) = A(|z|^2 - B) \le 0$ in Ω and $(dd^c\psi)^n = dV_n$, where dV_n is Lebesgue measure in $\mathbb{C}^n$. It follows from Lemma 3.12 that

$$\begin{aligned}\int_\Omega (-v_j)^n dV_n = \int_\Omega (-v_j)^n (dd^c\psi)^n &\le n!(\sup_\Omega(-\psi))^{n-1} \int_\Omega (-\psi)(dd^c v_j)^n \\ &\le n!(\sup_\Omega(-\psi))^n \mu(\Omega) < \infty,\end{aligned} \tag{6.2}$$

so the sequence $[v_j]$ is convergent in $L^1_{\mathrm{loc}}(\Omega)$. This implies that u, v are plurisubharmonic functions and therefore $v \in \mathcal{F}$ and $u \in \mathcal{F}(f)$. Moreover, by Corollary 3.7, $(dd^c u)^n = \mu$. The uniqueness follows from the comparison principle (Corollary 5.10). ∎

The next lemma will help us determine when the limit of a decreasing sequence of functions from $\mathcal{E}_0$ belongs to $\mathcal{E}$.

Lemma 6.2 ([33]). *Let $u \in \mathcal{PSH}^-(\Omega)$ and let $[u_j]$ be a sequence such that $u_j \in \mathcal{E}_0$, $u_j \searrow u$ as $j \to \infty$. If there exists $\psi \in \mathcal{E}_0$ such that $\psi < 0$ and*

$$\sup_j \int_\Omega (-\psi)(dd^c u_j)^n < \infty,$$

then $u \in \mathcal{E}$.

Proof. Fix $\omega \Subset \Omega$ and define

$$v_j = \sup\{w \in \mathcal{PSH}^-(\Omega) : w|\omega \le u_j|\omega\}.$$

Then $v_j \ge u_j$, $v_j \in \mathcal{E}_0$, $v_j \searrow u$ as $j \to \infty$, on ω. Since $\operatorname{supp}(dd^c v_j)^n \subset \bar{\omega}$ we have, by Lemma 5.11,

$$\begin{aligned}\sup_j \int_\Omega (dd^c v_j)^n &\le (\inf_\omega(-\psi))^{-1} \sup_j \int_\Omega (-\psi)(dd^c v_j)^n \\ &\le (\inf_\omega(-\psi))^{-1} \sup_j \int_\Omega (-\psi)(dd^c u_j)^n < \infty,\end{aligned}$$

which implies that $\lim_{j\to\infty} v_j \in \mathcal{F}$ and therefore $u \in \mathcal{E}$. ∎

In [33] Cegrell proved Theorem 6.3 below. Here we present a slightly modified proof.

THEOREM 6.3 ([33]). *Let μ be a non-negative measure defined on Ω such that μ vanishes on pluripolar subsets of Ω and there exists $\varphi \in \mathcal{PSH}(\Omega)$, $\varphi < 0$, such that $\int_\Omega (-\varphi)\, d\mu < \infty$. Then there exists a unique $u \in \mathcal{N}$ such that $(dd^c u)^n = \mu$.*

Proof. The uniqueness follows by the comparison principle in Corollary 5.10. From Theorem 2.17 there exists a sequence $[\varphi_j]$, $\varphi_j \in \mathcal{E}_0 \cap \mathcal{C}(\bar{\Omega})$, such that $\varphi_j \searrow \varphi$. Then

$$\int_\Omega (-\varphi_j)\, d\mu \le \int_\Omega (-\varphi)\, d\mu < \infty,$$

so we can assume that $\varphi \in \mathcal{E}_0 \cap \mathcal{C}(\bar{\Omega})$.

It follows from Theorem 5.6 that there exist $\psi \in \mathcal{E}_0$ and $0 \le f \in L^1_{\mathrm{loc}}((dd^c\psi)^n)$ such that $\mu = f(dd^c\psi)^n$. Set $\mu_j = \min(f, j)(dd^c\psi)^n$ and observe that $\mu_j \le (dd^c(j^{1/n}\psi))^n$. By the Kołodziej subsolution theorem (Theorem 2.7) there exists $v_j \in \mathcal{E}_0$ such that $(dd^c v_j)^n = \mu_j$. It follows from the comparison principle (Theorem 2.2) that $v_j \searrow u$. Observe that by Lemma 3.12,

$$\begin{aligned}\sup_j \int_\Omega (-v_j)^n (dd^c\psi)^n &\le n!(\sup_\Omega(-\psi))^{n-1} \sup_j \int_\Omega (-\psi)(dd^c v_j)^n \\ &\le n!(\sup_\Omega(-\psi))^{n-1} \int_\Omega (-\psi) d\mu < \infty,\end{aligned}$$

which means that $u \in \mathcal{PSH}^-(\Omega)$. Then $u \in \mathcal{E}$ by Lemma 6.2. We shall prove that $u \in \mathcal{N}$. Let $[\Omega_k]$ be a fundamental sequence. By the Kołodziej subsolution theorem there exist $\alpha_j^k, \beta_j^k \in \mathcal{E}_0$ such that

$$(dd^c\alpha_j^k)^n = \chi_{\Omega_k} \min(f, j)(dd^c\psi)^n, \qquad (dd^c\beta_j^k)^n = (1 - \chi_{\Omega_k}) \min(f, j)(dd^c\psi)^n.$$

Note that

$$(dd^c(\alpha_j^k + \beta_j^k))^n \ge (dd^c\alpha_j^k)^n + (dd^c\beta_j^k)^n = (dd^c v_j)^n,$$

so by the comparison principle,

$$v_j \ge \alpha_j^k + \beta_j^k. \tag{6.3}$$

It follows from Theorem 2.2 that $[\alpha_j^k]_{j=1}^\infty$, $[\beta_j^k]_{j=1}^\infty$ are decreasing sequences. From the first part of the proof it follows that there exist $\alpha^k, \beta^k \in \mathcal{E}$ such that $\alpha_j^k \searrow \alpha^k$, $\beta_j^k \searrow \beta^k$ as $j \to \infty$. Therefore

$$(dd^c\alpha^k)^n = \chi_{\Omega_k}(dd^c\psi)^n, \qquad (dd^c\beta^k)^n = (1 - \chi_{\Omega_k})(dd^c\psi)^n,$$

and from (6.3),

$$u \ge \alpha^k + \beta^k.$$

It follows from Proposition 3.16 that $\alpha^k \in \mathcal{F}$. Then

$$\tilde{u} \ge \tilde{\alpha^k} + \tilde{\beta^k} = \tilde{\beta^k} \ge \beta^k.$$

To prove that $u \in \mathcal{N}$ it is enough to show that $\beta^k \to 0$ as $k \to \infty$. First note that from the comparison principle, $\beta_j^k \le \beta_j^{k+1}$, so $[\beta^k]$ is increasing. By Lemma 3.12,

$$\int_\Omega (-\beta_j^k)^n (dd^c\varphi)^n \le n! \sup_\Omega (-\varphi)^{n-1} \int_\Omega (-\varphi)(dd^c\beta_j^k)^n$$
$$= n! \sup_\Omega (-\varphi)^{n-1} \int_{\Omega\setminus\Omega_k} (-\varphi)\min(f,j)(dd^c\psi)^n,$$

and then, in view of the assumption on φ, we obtain

$$\int_\Omega (-\beta^k)^n (dd^c\varphi)^n \le n! \sup_\Omega (-\varphi)^{n-1} \int_{\Omega\setminus\Omega_k} (-\varphi)d\mu \to 0$$

as $k \to \infty$, by the monotone convergence theorem. Therefore $\lim_{k\to\infty}\beta^k = 0$ on $\operatorname{supp}(dd^c\varphi)^n$, so by the maximum principle for plurisubharmonic functions we conclude that $\lim_{k\to\infty}\beta^k = 0$ on Ω. ∎

The following example shows that the condition in Theorem 6.3 is only sufficient.

EXAMPLE 6.4 ([33]). Let $\mathbb{B} = B(0,1)$ be the unit ball in $\mathbb{C}^n$. We show that there exists $u \in \mathcal{N}(\mathbb{B}) \cap L^\infty(\mathbb{B})$ such that $\lim_{z\to w} u(z) = 0$ for all $w \in \partial\mathbb{B}$ and

$$\int_{\mathbb{B}} (-\varphi)(dd^c u)^n = \infty$$

for all $\varphi \in \mathcal{E}_0$, $\varphi \ne 0$. Let us define

$$v_j(z) = \max(j^2\log|z|, -1/j^2),$$

and observe that $v_j \in \mathcal{E}_0$, $(dd^c v_j)^n = j^{2n} d\sigma_j$, where $d\sigma_j$ is the Lebesgue measure on the sphere $S_j = \{z \in \mathbb{B} : |z| = e^{-1/j^4}\}$ and

$$\int_{S_j} (dd^c v_j)^n = j^{2n}(2\pi)^n.$$

Moreover,

$$\int_{\mathbb{B}} (-v_1)(dd^c v_j)^n = j^{2n}(2\pi)^n \frac{1}{j^4}.$$

Define $u_k = \sum_{j=1}^k v_j$. Then $u_k \in \mathcal{E}_0$ and $u_k \searrow u = \sum_{j=1}^\infty v_j$. Moreover, $u \in \mathcal{N}(\mathbb{B}) \cap L^\infty(\mathbb{B})$ by Proposition 4.8 since

$$-\sum_{j=1}^\infty \frac{1}{j^2} \le u \le 0,$$

and $\operatorname{supp}(dd^c u)^n \subset B(0,1)\setminus B(0,e^{-1})$. Fix $\varphi \in \mathcal{E}_0$, $\varphi \ne 0$. Since the function v_1 is maximal on $B(0,1)\setminus \overline{B}(0,e^{-1})$, there exists a constant c such that

$$\varphi \le cv_1$$

on $B(0,1)\setminus B(0,e^{-1})$. Therefore we have

$$\int_{\mathbb{B}} (-\varphi)(dd^c u)^n = \int_{B(0,1)\setminus B(0,e^{-1})} (-\varphi)(dd^c u)^n$$
$$\ge c\int_{B(0,1)\setminus B(0,e^{-1})} (-v_1)(dd^c u)^n = c\int_{\mathbb{B}} (-v_1)(dd^c u)^n$$
$$\ge c\sum_{j=1}^\infty \int_{\mathbb{B}} (-v_1)(dd^c v_j)^n = c\sum_{j=1}^\infty j^{2n-4}(2\pi)^n = \infty.$$

Proposition 6.5 ([69]). *If $u \in \mathcal{E}$ is such that $(dd^c u)^n$ vanishes on all pluripolar sets in Ω and there exists a function $\varphi \in \mathcal{PSH}(\Omega)$, $\varphi < 0$, such that $\int_\Omega (-\varphi)(dd^c u)^n < \infty$, then $u \in \mathcal{N}(\tilde{u})$.*

Proof. As in the proof of Theorem 6.3 we can assume that $\varphi \in \mathcal{E}_0$.

Choose $u_j \in \mathcal{E}_0 \cap \mathcal{C}(\bar{\Omega})$ decreasing to u (see Theorem 2.17) and let $[\Omega_j]$ be a fundamental sequence in Ω. Then for each j there is $s_j > s_{j-1}$ such that for $s \geq s_j$,

$$\int_\Omega (-\varphi)\chi_{\Omega_j}(dd^c u_s)^n \leq \int_\Omega (-\varphi)(dd^c u)^n + 1.$$

By Theorem 2.7 there exists $v_{s,j} \in \mathcal{E}_0$ such that $(dd^c v_{s,j})^n = \chi_{\bar{\Omega}_j}(dd^c u_s)^n$. Theorem 2.2 yields $u_s \geq v_{s,j} + u_s^j$, since $(dd^c u_s)^n \leq (dd^c(u_{s,j} + u_s^j))^n$. Hence, if $t \geq s \geq s_j$, then

$$u_s \geq u_t \geq v_{t,j} + u_t^j.$$

In particular,

$$u_s \geq (\sup_{t \geq s} v_{t,j})^* + u^j$$

and

$$u \geq \lim_{s\to\infty} (\sup_{t \geq s} v_{t,j})^* + u^j = \psi_j + u^j.$$

Now, $[\psi_j]$ is a decreasing sequence of functions in $\mathcal{F}$,

$$\int_\Omega (-\varphi)(dd^c \psi_j)^n \leq \int_\Omega (-\varphi)(dd^c u)^n + 1,$$

so from Theorem 6.3, $\psi = \lim \psi_j \in \mathcal{N}$ since

$$\int_\Omega (-\varphi)(dd^c \psi)^n \leq \int_\Omega (-\varphi)(dd^c u)^n + 1,$$

Moreover, u^j increases a.e. to $\tilde{u}$ as $j \to \infty$, and we have $u \geq \psi + \tilde{u}$. ■

In Theorem 6.6 we solve the Dirichlet problem in $\mathcal{N}(H)$ with given generalized boundary values H. Here we give a slightly modified proof compared to the original one in [4].

Theorem 6.6 ([4]). *Assume that μ is a non-negative measure defined on Ω by $\mu = (dd^c\varphi)^n$, $\varphi \in \mathcal{N}$ with $\mu(A) = 0$ for every pluripolar set $A \subseteq \Omega$. Then for every $H \in \mathcal{E}$ such that $(dd^c H)^n \leq \mu$ there exists a unique $u \in \mathcal{N}(H)$ such that $(dd^c u)^n = \mu$ on Ω.*

Proof. The uniqueness follows by the comparison principle in Corollary 5.10. We proceed with the existence part. Theorem 2.17 implies that there exists a decreasing sequence $[H_k]$, $H_k \in \mathcal{E}_0 \cap \mathcal{C}(\bar{\Omega})$, that converges pointwise to H as $j \to \infty$. Let $[\Omega_j]$ be a fundamental sequence in Ω. For each $j, k \in \mathbb{N}$ let H_k^j be as in Definition 4.4, i.e.,

$$H_k^j = \sup\{\varphi \in \mathcal{PSH}(\Omega) : \varphi \leq H_k \text{ on } \Omega_j^c\},$$

Then $H_k \leq H_k^j$, $H_k = H_k^j$ on Ω_j^c, $H_k^j \in \mathcal{E}_0(\Omega)$ and H_k^j is maximal on Ω_j. Let $f_k^j = H_k|\partial\Omega_j$. We show that $H_k^j = PB_{f_k^j}$ on Ω_j. Since H_k^j is upper semicontinuous, for all $\xi \in \partial\Omega_j$,

$$\limsup_{\Omega_j \ni z \to \xi} H_k^j(z) \leq \limsup_{\Omega \ni z \to \xi} H_k^j(z) \leq H_k^j(\xi) = H_k(\xi) = f_j^k(\xi).$$

But we have also

$$\liminf_{\Omega_j \ni z \to \xi} H_k^j(z) \ge \liminf_{\Omega_j \ni z \to \xi} H_k(z) = H_k(\xi) = f_j^k(\xi),$$

since $H_k \le H_k^j$ and H_k is continuous. Thus we have proved that

$$\lim_{\Omega_j \ni z \to \xi} H_k^j(z) = f_j^k(\xi),$$

so by the Walsh theorem (Theorem 2.15) and the fact that H_k^j is maximal on Ω_j, H_k^j is the Perron–Bremermann envelope of f_k^j on Ω_j. Consider the measure $\mu_j = \chi_{\Omega_j}\mu$ defined on Ω, where χ_{Ω_j} is the characteristic function of Ω_j in Ω. For each $j \in \mathbb{N}$ the measure μ_j is a compactly supported Borel measure defined on Ω, μ_j vanishes on all pluripolar sets in Ω and $\mu_j(\Omega_j) < \mu_j(\Omega) < \infty$. Therefore by Theorem 6.1 there exists a unique $\varphi_j \in \mathcal{F}(\Omega_j)$ such that $(dd^c\varphi_j)^n = \mu_j$ on Ω_j. Moreover, from Theorem 6.1 there exist $u_{j,k} \in \mathcal{F}(\Omega_j, f_j^k)$ such that $(dd^c u_{j,k})^n = \mu_j$ on Ω_j. The comparison principle (Corollary 5.10) implies that

$$H_k^j \ge u_{j,k} \ge \varphi_j + H_k^j \quad \text{on } \Omega_j, \tag{6.4}$$

since $(dd^c u_{j,k})^n \le (dd^c(\varphi_j + H_k^j))^n$ and H_k^j is maximal on Ω_j. The comparison principle shows that $[u_{j,k}]_{k=1}^{\infty}$ is a decreasing sequence. Let $k \to \infty$ and set $u_j = \lim_{k\to\infty} u_{j,k}$. Then (6.4) gives us that $H^j \ge u_j \ge \varphi_j + H^j$ on Ω_j, i.e., $u_j \in \mathcal{F}(\Omega_j, H^j) \subseteq \mathcal{N}(\Omega_j, H^j)$. From the assumption that $\mu \ge (dd^c H)^n$ we get $(dd^c u_j)^n = \mu_j = \chi_{\Omega_j}\mu = \mu \ge (dd^c H)^n$ on Ω_j and therefore from the comparison principle, $u_j \le H$ on Ω_j. The construction of μ_j and the fact that $[\Omega_j]$ is an increasing sequence imply that $(dd^c u_j)^n = (dd^c u_{j+1})^n$ on Ω_j. Hence $[u_j]$ is decreasing and

$$H \ge u_j \ge \varphi + H \quad \text{on } \Omega_j. \tag{6.5}$$

Thus, the function $u = (\lim_{j\to\infty} u_j) \in \mathcal{N}(\Omega, H)$ is such that $(dd^c u)^n = \mu$ on Ω. ■

Remark 6.7. Let Ω be a bounded hyperconvex domain in $\mathbb{C}^n$, and $f \in \mathcal{C}(\partial\Omega)$ be a real-valued function such that

$$\lim_{\substack{z\to\xi \\ z\in\Omega}} PB_f(z) = f(\xi) \quad \text{for all } \xi \in \partial\Omega.$$

Let $\varphi \in \mathcal{N}$ be such that $\lim_{z\to\xi} \varphi(z) = 0$ for all $\xi \in \partial\Omega$, and set $\mu = (dd^c\varphi)^n$. Assume that $\mu(A) = 0$ for every pluripolar set $A \subseteq \Omega$. Then there exists a unique $u \in \mathcal{N}(f)$ such that

$$(dd^c u)^n = \mu \text{ on } \Omega \quad \text{and} \quad \lim_{\substack{z\to\xi \\ z\in\Omega}} u(z) = f(\xi) \quad \text{for all } \xi \in \partial\Omega$$

(see (6.5)).

Remark 6.8. Assume that μ is a non-negative Radon measure defined on Ω such that μ vanishes on all pluripolar sets, and $\mu(\Omega) < \infty$. Let $H \in \mathcal{E}$ be such that $(dd^c H)^n \le \mu$. Then there exists a unique $u \in \mathcal{F}(H)$ with $(dd^c u)^n = \mu$. In particular, if $H \in \mathcal{E} \cap \mathcal{MPSH}(\Omega)$, then there always exists a unique $u \in \mathcal{F}(H)$ with $(dd^c u)^n = \mu$, since $(dd^c H)^n = 0$.

6.2. Singular measures. Inequality (6.6) below was originally proved in [49].

LEMMA 6.9 ([4, 49]). *Let $u, u_k, v \in \mathcal{E}$, $k = 1, \dots, n-1$, with $u \geq v$ on Ω and set $T = dd^c u_1 \wedge \dots \wedge dd^c u_{n-1}$. Then*

$$\chi_{\{u=-\infty\}} dd^c u \wedge T \leq \chi_{\{v=-\infty\}} dd^c v \wedge T. \tag{6.6}$$

In particular, if $(dd^c v)^n$ vanishes on pluripolar sets, then $(dd^c u)^n$ vanishes on pluripolar sets. Moreover, if $u_j, v_j \in \mathcal{E}$, $u_j \geq v_j$, for $j = 1, \dots, n$, then

$$\int_A dd^c u_1 \wedge \dots \wedge dd^c u_n \leq \int_A dd^c v_1 \wedge \dots \wedge dd^c v_n$$

for every pluripolar Borel set $A \subseteq \Omega$.

Proof. Let $\varepsilon > 0$. Set $w_j = \max((1-\varepsilon)u - j, v)$. Then $w_j = (1-\varepsilon)u - j$ on the open set $\{v < -j/\varepsilon\}$ and therefore

$$dd^c w_j \wedge T = (1-\varepsilon) dd^c u \wedge T \quad \text{on } \{v < -j/\varepsilon\}.$$

Hence $dd^c w_j \wedge T \geq (1-\varepsilon)\chi_{\{u=-\infty\}} dd^c u \wedge T$. Let $j \to \infty$. Then

$$dd^c v \wedge T \geq (1-\varepsilon)\chi_{\{u=-\infty\}} dd^c u \wedge T \quad \text{on } \Omega.$$

The proof of the first part is completed by letting $\varepsilon \to 0^+$.

The second part follows from the first one. ∎

DEFINITION 6.10 ([4]). Let $u \in \mathcal{E}$ and $0 \leq \tau$ be a bounded lower semicontinuous function. Then we define

$$u_\tau = \sup\{\varphi \in \mathcal{PSH}(\Omega) : \varphi \leq \tau^{1/n} u\}.$$

Definition 6.10 yields the following elementary properties:

(1) If $u, v \in \mathcal{E}$ with $u \leq v$, then $u_\tau \leq v_\tau$.
(2) If $u \in \mathcal{E}$, then $0 \geq u_\tau \geq \|\tau\|^{1/n}_{L^\infty(\Omega)} u \in \mathcal{E}$. Hence, by Proposition 2.10 we have $u_\tau \in \mathcal{E}$.
(3) If τ_1, τ_2 are bounded lower semicontinuous functions with $\tau_1 \leq \tau_2$, then $u_{\tau_1} \geq u_{\tau_2}$.
(4) If $u \in \mathcal{E}$, then $\operatorname{supp}(dd^c u_\tau)^n \subseteq \operatorname{supp}\tau$ and if $\operatorname{supp}\tau$ is compact then $u_\tau \in \mathcal{F}$.
(5) If $[\tau_j]$, $0 \leq \tau_j$, is an increasing sequence of bounded lower semicontinuous functions that converges pointwise to a bounded lower semicontinuous function τ as $j \to \infty$, then $[u_{\tau_j}]$ is a decreasing sequence that converges pointwise to u_τ as $j \to \infty$.

LEMMA 6.11 ([4]). *Let $u \in \mathcal{E}$ and let K be a compact pluripolar subset of Ω. Then*

$$(dd^c u_K)^n = \chi_K (dd^c u)^n,$$

where u_{χ_O} is as in Definition 6.10 and

$$u_K = (\sup\{u_{\chi_O} : K \subset O \subset \Omega,\ O \text{ is open}\})^*.$$

Proof. Choose a decreasing sequence $[O_j]$, $O_j \subseteq \Omega$, such that $K = \bigcap_j O_j$. Then $[u_{\chi_{O_j}}]$ is an increasing sequence that converges to u_K outside a pluripolar set as $j \to \infty$, and $\operatorname{supp}(dd^c u_K)^n \subseteq \bigcap \bar{O}_j = K$. We have $u_{\chi_{O_j}} = u$ on O_j, hence $(dd^c u_{\chi_{O_j}})^n \geq \chi_K (dd^c u)^n$, so $(dd^c u_K)^n \geq \chi_K (dd^c u)^n$. On the other hand, $u_K \geq u$ and therefore by Lemma 6.9,

$$\int_K (dd^c u_K)^n \le \int_K (dd^c u)^n$$

and hence $(dd^c u_K)^n = \chi_K (dd^c u)^n$. ∎

LEMMA 6.12 ([4]). *Let $u_1, \dots, u_n \in \mathcal{E}$. Then*

$$\int_A dd^c u_1 \wedge \cdots \wedge dd^c u_n \le \Big(\int_A (dd^c u_1)^n \Big)^{1/n} \cdots \Big(\int_A (dd^c u_n)^n \Big)^{1/n},$$

for every pluripolar Borel set $A \subset \Omega$.

Proof. Without loss of generality we can assume that A is a compact pluripolar set and $u_1, \dots, u_n \in \mathcal{F}$. Let $[G_j]$ be a decreasing sequence of open subsets of Ω with $\bigcap_j G_j = A$. Corollary 3.15 yields

$$\int_\Omega dd^c u_{1_{G_j}} \wedge \cdots \wedge dd^c u_{n_{G_j}} \le \Big(\int_\Omega (dd^c u_{1_{G_j}})^n \Big)^{1/n} \cdots \Big(\int_\Omega (dd^c u_{n_{G_j}})^n \Big)^{1/n}$$

For $1 \le k \le n$ we have $u_{k_{G_j}} = u_k$ on G_j and $\operatorname{supp}(dd^c u_{k_{G_j}})^n \subset \bar{G}_j \subset \bar{G}_1$, hence

$$\int_{G_j} dd^c u_1 \wedge \cdots \wedge dd^c u_n \le \Big(\int_{\bar{G}_j} (dd^c u_{1_{G_j}})^n \Big)^{1/n} \cdots \Big(\int_{\bar{G}_j} (dd^c u_{n_{G_j}})^n \Big)^{1/n}.$$

Let $j \to \infty$. Lemma 6.11 then yields the conclusion. ∎

For $u \in \mathcal{E}$ we write $\mu_u = \chi_{\{u=-\infty\}} (dd^c u)^n$ and define S to be the class of simple functions $f = \sum_{j=1}^m \alpha_j \chi_{E_j}$, $\alpha_j > 0$, where E_j are pairwise disjoint and μ-measurable such that f is compactly supported and vanishes outside $\{u = -\infty\}$. We also define T to be the subclass of simple functions $f \in S$, $f = \sum_{j=1}^m \alpha_j \chi_{E_j}$, such that E_j's are compact sets.

DEFINITION 6.13 ([4]). Let $u \in \mathcal{E}$ and $0 \le g \le 1$ be a μ_u-measurable function. We define

$$u^g = \inf_{\substack{f \in T \\ f \le g}} (\sup\{u_\tau : f \le \tau,\ \tau \text{ is a bounded lower semicontinuous function}\})^*.$$

From Definition 6.13 it follows that $u \le u^g \le 0$, so $u^g \in \mathcal{E}$ by Proposition 2.10, and if $g_1 \le g_2$, then $u^{g_1} \ge u^{g_2}$. Furthermore, if $g \in T$, then

$$u^g = (\sup\{u_\tau : g \le \tau,\ \tau \text{ is a bounded lower semicontinuous function}\})^* \in \mathcal{F}.$$

LEMMA 6.14 ([4]). *Let $u \in \mathcal{E}$ and $g \in S$. Then $u^g \in \mathcal{F}$ and $(dd^c u^g)^n = g (dd^c u)^n$.*

Proof. Assume first that $g \in T$. Then $u^g \in \mathcal{F}$ as already noted. Let $g = \sum_{k=1}^m \alpha_k \chi_{A_k}$ and consider $u_k = u^{\alpha_k \chi_{A_k}}$. Then for $1 \le k \le m$ we have $u_1 + \cdots + u_m \le u^g \le u_k$ so if $B \subseteq \bigcup_{k=1}^m A_k$, then it follows from Lemma 6.9 that

$$\int_B (dd^c u_k)^n \le \int_B (dd^c u^g)^n \le \int_B (dd^c (u_1 + \cdots + u_m))^n, \qquad 1 \le k \le m.$$

Hence, if $B \subset A_k$, then it follows from Lemma 6.11 that

$$\int_B (dd^c u_k)^n = \alpha_k \int_B (dd^c u)^n,$$

since $(dd^c u_k)^n = \alpha_k \chi_{A_k} (dd^c u)^n$. From Lemma 6.12 we have

$$\alpha_k \int_B (dd^c u)^n = \int_B (dd^c(u_1 + \cdots + u_m))^n,$$

since $(dd^c u_j)^n(B) = 0$ for all $j \neq k$. Hence,

$$\alpha_k \int_B (dd^c u)^n \leq \int_B (dd^c u^g)^n \leq \alpha_k \int_B (dd^c u)^n, \qquad 1 \leq k \leq m,$$

for all Borel sets $B \subset A_k$, $k = 1, \dots, m$. Thus $(dd^c u^g)^n = g(dd^c u)^n$.

Assume now that $g \in S$, i.e., $g = \sum_{j=1}^m \alpha_j \chi_{E_j}$, $\alpha_j > 0$, E_j are pairwise disjoint and μ_u-measurable such that g is compactly supported and vanishes outside $\{u = -\infty\}$. Choose for each E_j, $1 \leq j \leq m$, an increasing sequence $[K_j^p]_{p=1}^\infty$ of compact subsets of E_j such that $\chi_p = \sum_{j=1}^m \chi_{K_j^p}$ converges to $\chi = \sum_{j=1}^m \chi_{E_j}$ a.e. w.r.t. μ_u as $p \to \infty$. Then $\chi_p \in T$ and $g\chi_p \in T$. Furthermore, if $f \in T$ with $f \leq g$, then $f\chi_p \in T$ and $f\chi_p \leq g\chi_p$. Hence $u^{f\chi_p} \geq u^{g\chi_p}$. By the first part of the proof we have $(dd^c u^{f\chi_p})^n = f\chi_p (dd^c u)^n$ and $(dd^c u^{g\chi_p})^n = g\chi_p (dd^c u)^n$. Since $\chi_p \nearrow \chi$ and $f\chi_p \nearrow f$, $[u^{f\chi_p}]$ is a decreasing sequence and $u^f \leq u^{f\chi_p}$. Therefore there exist $\varphi \in \mathcal{E}$ such that $\lim_{p\to\infty} u^{f\chi_p} = \varphi \geq u^f$. Similarly one can prove that $\lim_{p\to\infty} u^{g\chi_p} \geq u^g$. Moreover, since $(dd^c u^f)^n = (dd^c \varphi)^n = f(dd^c u)^n$, Theorem 5.14 implies that $\varphi = u^f$. Thus,

$$u^f = \lim_{p\to\infty} u^{f\chi_p} \geq \lim_{p\to\infty} u^{g\chi_p}$$

for every $f \in T$ with $f \leq g$, so Definition 6.13 yields $u^g = \lim_{p\to\infty} u^{g\chi_p} \in \mathcal{F}$ and

$$(dd^c u^g)^n = \lim_{p\to\infty} (dd^c u^{g\chi_p})^n = \lim_{p\to\infty} g\chi_p (dd^c u)^n = g(dd^c u)^n. \ \blacksquare$$

THEOREM 6.15 ([4]). *Let $u \in \mathcal{E}$ and let $0 \leq g \leq 1$ be a μ_u-measurable function that vanishes outside $\{u = -\infty\}$. Then $u^g \in \mathcal{E}$ and $(dd^c u^g)^n = g(dd^c u)^n$.*

Proof. Let $[g_j]$, $g_j \in S$, be an increasing sequence that converges pointwise to g as $j \to \infty$. If $f \in T$ with $f \leq g$, then $\min(f, g_j) \in S$ and by Lemma 6.14 we have $(dd^c u^{\min(f,g_j)})^n = \min(f, g_j)(dd^c u)^n$. Since $\min(f, g_j) \nearrow f$, $[u^{\min(f,g_j)}]$ is a decreasing sequence and $u^f \leq u^{\min(f,g_j)}$. Therefore there exist $\varphi \in \mathcal{E}$ such that $\lim_{j\to\infty} u^{\min(f,g_j)} = \varphi \geq u^f$. Similarly one can prove that $\lim_{j\to\infty} u^{g_j} \geq u^g$. Moreover, since $(dd^c u^f)^n = (dd^c \varphi)^n = f(dd^c u)^n$, Theorem 5.14 implies that $\varphi = u^f$. Thus,

$$u^f = \lim_{j\to\infty} u^{\min(f,g_j)} \geq \lim_{j\to\infty} u^{g_j}$$

for every $f \in T$ with $f \leq g$, so Definition 6.13 yields $u^g = \lim_{j\to\infty} u^{g_j} \in \mathcal{E}$ and Lemma 6.14 implies that

$$(dd^c u^g)^n = \lim_{j\to\infty} (dd^c u^{g_j})^n = \lim_{j\to\infty} g_j (dd^c u)^n = g(dd^c u)^n. \ \blacksquare$$

REMARK 6.16. Let u and g be as in Theorem 6.15. If $(dd^c u)^n$ vanishes on pluripolar sets, then it follows from Theorem 6.15 that $u^g = 0$ on Ω.

COROLLARY 6.17 ([4]). *Let $u \in \mathcal{E}$ and f, g, $0 \leq f, g \leq 1$, be two μ_u-measurable functions which vanish outside $\{u = -\infty\}$. If $f = g$ a.e. w.r.t. μ_u, then $u^f = u^g$.*

Proof. Let $u \in \mathcal{E}$ and assume for now that $f, g \in S$. Then by Lemma 6.14 we have $u^f, u^g \in \mathcal{F}$, $u^f \geq u^{\max(f,g)}$ and

$$(dd^c u^f)^n = f(dd^c u)^n = \max(f,g)(dd^c u)^n = (dd^c u^{\max(f,g)})^n.$$

Hence, by Theorem 5.14 we have $u^f = u^{\max(f,g)}$. Similarly we get $u^g = u^{\max(f,g)}$. Thus, $u^f = u^g$.

For the general case let $[\Omega_j]$ be a fundamental sequence and let f, g, $0 \leq f, g \leq 1$, be two μ_u-measurable functions that vanish outside $\{u = -\infty\}$. Our assumption that $f = g$ a.e. w.r.t. μ_u implies that $\chi_{\Omega_j} f = \chi_{\Omega_j} g$ a.e. w.r.t. μ_u and by the first part of the proof we get $u^{\chi_{\Omega_j} f} = u^{\chi_{\Omega_j} g}$. The proof is then completed by letting $j \to \infty$. ∎

Example 6.18 shows that there exists a measure $g(dd^c u)^n$ carried by a pluripolar set that is not a discrete measure.

EXAMPLE 6.18 ([4]). Let μ be a positive measure with no atoms and with support in a compact polar subset of the unit disc $\mathbb{D}$ (see e.g. [74, p. 82] and [26, Chapter IV, Theorem 1]). Let u be the subharmonic Green potential of μ. Consider $\nu = \mu \times \cdots \times \mu$ (n-times) and $v(z_1, \ldots, z_n) = \max(u(z_1), \ldots, u(z_n))$ on $\mathbb{D} \times \cdots \times \mathbb{D}$ (n-times). Then $v \in \mathcal{F}$, $(dd^c v)^n = \nu$, ν has no atoms and it is supported by a pluripolar set.

6.3. The general case

LEMMA 6.19 ([4]). *Assume that α, β_1, β_2 are non-negative measures defined on Ω which satisfy the following conditions:*

(1) *α vanishes on every pluripolar subset of Ω,*
(2) *there exists a pluripolar sets $A \subset \Omega$ such that $\beta_1(\Omega \setminus A) = \beta_2(\Omega \setminus A) = 0$.*
(3) *for every $\rho \in \mathcal{E}_0 \cap C(\bar{\Omega})$,*

$$\int_\Omega (-\rho)\, d\beta_1 \leq \int_\Omega (-\rho)(d\alpha + d\beta_2) < \infty.$$

Then

$$\int_\Omega (-\rho)\, d\beta_1 \leq \int_\Omega (-\rho)\, d\beta_2$$

for every $\rho \in \mathcal{E}_0 \cap C(\bar{\Omega})$.

Proof. Since A is pluripolar and Ω is bounded there exists a function $\varphi \in \mathcal{PSH}(\Omega)$, $\varphi \leq 0$, such that $A \subseteq \{\varphi = -\infty\}$. Take $\rho \in \mathcal{E}_0 \cap C(\bar{\Omega})$ and set $\rho_j = \max(\rho, \varphi/j)$. Then $\int_\Omega (-\rho_j)\, d\beta_1 \leq \int_\Omega (-\rho_j)(d\alpha + d\beta_2) < \infty$ and by letting $j \to \infty$ we get

$$\int_{\{\varphi=-\infty\}} (-\rho)\, d\beta_1 \leq \int_{\{\varphi=-\infty\}} (-\rho)(d\alpha + d\beta_2).$$

But α vanishes on pluripolar sets and β_1 and β_2 are carried by sets contained in $\{\varphi = -\infty\}$. This yields the conclusion. ∎

Let $u \in \mathcal{E}$. Then by Theorem 5.6 there exist $\phi_u \in \mathcal{E}_0$ and $f_u \in L^1_{\mathrm{loc}}((dd^c \phi_u)^n)$, $f_u \geq 0$, such that $(dd^c u)^n = f_u\, (dd^c \phi_u)^n + \beta_u$. The non-negative measure β_u is such that there exists a pluripolar set $A \subseteq \Omega$ with $\beta_u(\Omega \setminus A) = 0$. In Lemma 6.20 we will use $\alpha_u = f_u\, (dd^c \phi_u)^n$ and the fact that β_u is the singular part of $(dd^c u)^n$.

LEMMA 6.20 ([4]). *Let $u, v \in \mathcal{E}$. If there exists a function $\varphi \in \mathcal{E}$ such that $(dd^c\varphi)^n$ vanishes on pluripolar sets and $|u - v| \le -\varphi$, then $\beta_u = \beta_v$.*

Proof. Let $\Omega' \Subset \Omega$. Without loss of generality we can assume that $u, v, \varphi \in \mathcal{F}$, since it is sufficient to prove that $\beta_u = \beta_v$ on Ω'. The assumption that $|u-v| \le -\varphi$ yields $v+\varphi \le u$ and therefore it follows from Lemma 5.11 that

$$\int_\Omega (-\rho)(dd^c u)^n \le \int (-\rho)(dd^c(v+\varphi))^n < \infty \tag{6.7}$$

for all $\rho \in \mathcal{E}_0$. Since $(dd^c\varphi)^n$ vanishes on pluripolar sets, by Lemma 6.12 the measure $\sum_{j=1}^n \binom{n}{j}(dd^c\varphi)^j \wedge (dd^c v)^{n-j}$ also vanishes on pluripolar sets. Therefore $\beta_{v+\varphi} = \beta_v$ and

$$\alpha_{v+\varphi} = \alpha_v + \sum_{j=1}^n \binom{n}{j}(dd^c\varphi)^j \wedge (dd^c v)^{n-j}.$$

Lemma 6.19 and inequality (6.7) yield

$$\int_\Omega (-\rho)\, d\beta_u \le \int_\Omega (-\rho)\, d\beta_v$$

for every $\rho \in \mathcal{E}_0$. In a similar manner we get

$$\int_\Omega (-\rho)\, d\beta_v \le \int_\Omega (-\rho)\, d\beta_u$$

for every $\rho \in \mathcal{E}_0$. From Theorem 2.19 it now follows that $\beta_u = \beta_v$. ∎

LEMMA 6.21 ([4]). *Let $H \in \mathcal{E} \cap \mathcal{MPSH}(\Omega)$.*

(1) *If $v \in \mathcal{N}$, $(dd^c v)^n$ is carried by a pluripolar set, and $\int_\Omega (-\rho)(dd^c v)^n < \infty$ for all $\rho \in \mathcal{E}_0 \cap \mathcal{C}(\bar\Omega)$, then*

$$u = \sup\{\varphi \in \mathcal{PSH}(\Omega) : \varphi \le \min(v, H)\} \in \mathcal{N}(H)$$

satisfies $(dd^c u)^n = (dd^c v)^n$.

(2) *Assume that $\psi \in \mathcal{N}$, $(dd^c\psi)^n$ vanishes on pluripolar sets, $v \in \mathcal{N}(H)$, $(dd^c v)^n$ is carried by a pluripolar set, and $\int_\Omega (-\rho)((dd^c\psi)^n + (dd^c v)^n) < \infty$ for all $\rho \in \mathcal{E}_0 \cap \mathcal{C}(\bar\Omega)$. If u is the function defined on Ω by*

$$u = \sup\{\varphi : \varphi \in \mathcal{B}((dd^c\psi)^n, v))\},$$

where

$$\mathcal{B}((dd^c\psi)^n, v) = \{\varphi \in \mathcal{E} : (dd^c\psi)^n \le (dd^c\varphi)^n \text{ and } \varphi \le v\},$$

then $u \in \mathcal{N}(H)$ and $(dd^c u)^n = (dd^c\psi)^n + (dd^c v)^n$.

Proof. (1): Since $\min(v, H)$ is a negative and upper semicontinuous function we have $u \in \mathcal{PSH}(\Omega)$ and $H \ge u \ge v+H$. Furthermore, $u \in \mathcal{N}(H)$, since $v \in \mathcal{N}$. By Theorem 2.17 we can choose a decreasing sequence $[v_j]$, $v_j \in \mathcal{E}_0 \cap \mathcal{C}(\bar\Omega)$, that converges pointwise to v as $j \to \infty$, and use Theorem 6.6 to solve $(dd^c w_j)^n = (dd^c v_j)^n$, $w_j \in \mathcal{N}(H)$, $j \in \mathbb{N}$. Consider

$$u_j = \sup\{\varphi \in \mathcal{PSH}(\Omega) : \varphi \le \min(v_j, H)\} \in \mathcal{E}_0(H).$$

Then $u_j \geq w_j$, so by Lemma 5.11, $\int_\Omega(-\rho)(dd^cu_j)^n \leq \int_D(-\rho)(dd^cw_j)^n$. Corollary 5.12 now yields

$$\int_\Omega(-\rho)(dd^cu)^n \leq \int_\Omega(-\rho)(dd^cv)^n \quad \text{for all } \rho\in\mathcal{E}_0\cap\mathcal{C}(\bar{\Omega}),$$

and therefore $(dd^cu)^n$ is carried by some pluripolar set. It follows from Theorem 5.6 that $(dd^cu)^n$ is carried by $\{u=-\infty\}$. Since $v\geq u\geq v+H$ it follows from Lemma 6.20 that $(dd^cu)^n=(dd^cv)^n$. Thus, part (1) is proved.

(2): The function $(\psi+v)$ belongs to $\mathcal{B}((dd^c\psi)^n,v)$ and therefore we have $v+\psi\leq u\leq v$, hence $u\in\mathcal{N}(H)$. Theorem 5.6 gives $(dd^cu)^n=\alpha+\beta$, where α and β are positive measures defined on Ω such that α vanishes on all pluripolar sets and β is carried by a pluripolar set. By the classical Choquet lemma (see e.g. [62]) there exist functions $\varphi_j\in\mathcal{B}((dd^c\psi)^n,v)$ such that $u=(\sup_j\varphi_j)^*$. Since for all j we have $(dd^c\varphi_j)^n\geq(dd^c\psi)^n$, Lemma 5.3 yields $(dd^cu)^n\geq(dd^c\psi)^n$. By Lemma 6.20 we have $\beta=(dd^cv)^n$, and we have already noted that $\alpha\geq(dd^c\psi)^n$. Theorem 2.17 implies that there exists a decreasing sequence $[v_j]$, $v_j\in\mathcal{E}_0(H)$, that converges pointwise to v as $j\to\infty$. Now,

$$\int_\Omega(-\rho)((dd^c\psi)^n+(dd^cv_j)^n)<\infty \quad \text{for all } \rho\in\mathcal{E}_0\cap\mathcal{C}(\bar{\Omega}),$$

so by Theorems 6.3 and 6.6, there exists a unique $w_j\in\mathcal{N}(H)$ such that $(dd^cw_j)^n = (dd^c\psi)^n+(dd^cv_j)^n$. It follows from the comparison principle (Corollary 5.10) that $w_j\in\mathcal{B}((dd^c\psi)^n,v_j)$, so if we let

$$u_j=\sup\{\varphi:\varphi\in\mathcal{B}((dd^c\psi)^n,v_j)\},$$

then $[u_j]$ decreases pointwise to u as $j\to\infty$. Furthermore, since $\psi+v_j\leq w_j\leq u_j$ Lemma 5.11 implies that

$$\int_\Omega(-\rho)(dd^cu_j)^n\leq\int_\Omega(-\rho)(dd^cw_j)^n=\int_\Omega(-\rho)((dd^c\psi)^n+(dd^cv_j)^n).$$

Let $j\to\infty$. Then Corollary 5.12 yields

$$\int_\Omega(-\rho)(dd^cu)^n\leq\int_\Omega(-\rho)((dd^c\psi)^n+d\beta).$$

Hence $\int_\Omega(-\rho)(d\alpha+d\beta)\leq\int_\Omega(-\rho)((dd^c\psi)^n+d\beta)$. Since we know that $\alpha\geq(dd^c\psi)^n$ it follows that for all $\rho\in\mathcal{E}_0\cap\mathcal{C}(\bar{\Omega})$ we have $\int_\Omega\rho d\alpha=\int_\Omega\rho(dd^c\psi)^n$, and therefore by Theorem 2.19, $\alpha=(dd^c\psi)^n$. Thus, the proof is complete. ∎

The main result of this chapter is the following generalization of Kołodziej's subsolution theorem.

Theorem 6.22 ([4], subsolution theorem). *Assume that μ is a non-negative measure with the decomposition* (*given by Theorem* 5.6)

$$\mu=f\,(dd^c\phi)^n+\nu,$$

where $\phi\in\mathcal{E}_0$, $f\in L^1_{\text{loc}}((dd^c\phi)^n)$, $f\geq0$ and ν is a non-negative measure carried by a pluripolar subset of Ω.

(1) *If there exists a function* $w \in \mathcal{E}$ *with* $\mu \le (dd^c w)^n$, *then there exist functions* $\psi, v \in \mathcal{E}$, $v, \psi \ge w$, *such that*

$$(dd^c\psi)^n = f(dd^c\phi)^n \quad \textit{and} \quad (dd^c v)^n = \nu,$$

where ν *is carried by* $\{v = -\infty\}$.

(2) *If there exists a function* $w \in \mathcal{E}$ *with* $\mu \le (dd^c w)^n$, *then for every* $H \in \mathcal{E} \cap \mathcal{MPSH}(\Omega)$ *there exists a function* $u \in \mathcal{E}$, $w + H \le u \le H$, *with* $(dd^c u)^n = \mu$. *In particular, if* $w \in \mathcal{N}$, *then* $u \in \mathcal{N}(H)$.

Proof. (1): Using the Radon–Nikodym theorem and the decomposition of μ we obtain

$$f(dd^c\phi)^n = \tau\chi_{\{w>-\infty\}}(dd^c w)^n \quad \text{and} \quad \nu = \tau\chi_{\{w=-\infty\}}(dd^c w)^n,$$

where $0 \le \tau \le 1$ is a Borel function. For each $j \in \mathbb{N}$, let μ_j be the measure defined by $\mu_j = \min(f, j)(dd^c\phi)^n$. Hence, $\mu_j \le (dd^c(j^{1/n}\,\psi))^n$ and therefore by Theorem 2.7 there exists a unique function $\psi_j \in \mathcal{E}_0$ such that $(dd^c\psi_j)^n = \mu_j$. The comparison principle (Corollary 5.10) implies that $\psi_j \ge w$ and that $[\psi_j]$ is a decreasing sequence. The function $\psi = \lim_{j\to\infty}\psi_j$ is then in $\mathcal{E}$ and $(dd^c\psi)^n = f\,(dd^c\phi)^n$. Theorem 6.15 implies that there exists $v \in \mathcal{E}$ such that $(dd^c v)^n = \nu$ and $v \ge w$. Thus,

$$(dd^c\psi)^n = f(dd^c\phi)^n \quad \text{and} \quad (dd^c v)^n = \nu.$$

(2): Continuing with the same notations as in (1), we choose an increasing sequence of simple functions $[g_j]$, $\operatorname{supp} g_j \Subset \Omega$, that converges to $g = \chi_{\{w=-\infty\}}\tau$ as $j \to \infty$. By Theorem 6.15 we have $w^{g_j} \in \mathcal{F}$, $(dd^c w^{g_j})^n = g_j(dd^c w)^n$ and $[w^{g_j}]$ is a decreasing sequence that converges pointwise to w^g as $j \to \infty$. Moreover, $w^g \ge w$. Hence $(dd^c w^g)^n = \chi_{\{w=-\infty\}}\tau(dd^c w)^n$. Set

$$u_j = \sup\{\varphi \in \mathcal{B}((dd^c\psi_j)^n, \min(w^{g_j}, H))\},$$

where

$$\mathcal{B}((dd^c\psi_j)^n, \min(w^{g_j}, H)) = \{\varphi \in \mathcal{E} : (dd^c\psi_j)^n \le (dd^c\varphi)^n \text{ and } \varphi \le \min(w^{g_j}, H)\}.$$

This construction implies that $[u_j]$ is a decreasing sequence. The sequence $[u_j]$ converges to some plurisubharmonic function u as $j \to \infty$, and by Lemma 6.21, $u_j \in \mathcal{N}(H)$ with $(dd^c u_j)^n = (dd^c\psi_j)^n + (dd^c w^{g_j})^n$. Furthermore, we have $w + H \le u_j \le H$. We conclude the proof by letting $j \to \infty$. ∎

Remark 6.23. Let $u_1, \dots, u_n \in \mathcal{E}$. Then it follows from the subsolution theorem (Theorem 6.22) that there exists $u \in \mathcal{E}$ such that $(dd^c u)^n = dd^c u_1 \wedge \dots \wedge dd^c u_n$.

In the following example we construct a positive measure μ for which there does not exist $u \in \mathcal{E}$ such that $(dd^c u)^n = \mu$.

Example 6.24 ([33]). Let $\mathbb{B} \subset \mathbb{C}^n$. In this example we shall construct a decreasing sequence $[u_s]$, $u_s \in \mathcal{E}_0(\mathbb{B})$, that converges pointwise to $-\infty$ as $s \to \infty$, and

$$\sup_{s\ge 1} \int_{\mathbb{B}} (\log|z|^2)^2 (dd^c u_s)^n < \infty.$$

This means that if μ is an accumulation point of $[(dd^c u_s)^n]$, then there is no $u \in \mathcal{E}(\mathbb{B})$

with $(dd^c u)^n = \mu$, since by the comparison principle (Corollary 5.10) we have $u \le u_s$, and therefore $u = -\infty$ everywhere.

For $j \ge 2$, let $a_j = 1/j^{1/2}$, and $b_j = 1/(j^{1/2}\log j)$. Then $\sum_{j=2}^{\infty} a_j^2 = \infty$, $\sum_{j=2}^{\infty} b_j^2 < \infty$, and $\sum_{j=2}^{\infty} a_j b_j = \infty$. Furthermore, $a_j \sum_{k=2}^{j} a_k \le 6$ for all $j \ge 2$. Now define

$$\varphi_j = \frac{a_j}{2\pi}\max(\log|z|, \log(1-b_j)), \qquad j \ge 2.$$

Then

$$dd^c\varphi_j \wedge dd^c\varphi_k = \begin{cases} a_j^2 d\sigma_{1-b_j}, & j = k,\\ a_j a_k d\sigma_{\max(1-b_j, 1-b_k)}, & j \ne k,\end{cases}$$

where $d\sigma_r$ is the normalized Lebesgue measure on the sphere with radius r. We then have

$$\begin{aligned}\int_{\mathbb{B}} (\log|z|^2)^2 \Big(dd^c \textstyle\sum_{j=2}^{s}\varphi_j\Big)^n &= \sum_{j,k=2}^{s} \int_{\mathbb{B}} (\log|z|^2)^2 dd^c\varphi_j \wedge dd^c\varphi_k\\ &\le 2\sum_{j=2}^{s}\sum_{k=2}^{j}\int_{\mathbb{B}} (\log|z|^2)^2 dd^c\varphi_j \wedge dd^c\varphi_k\\ &\le 2\sum_{j=2}^{s} \log(1-b_j)^2 a_j \sum_{k=1}^{j} a_k\\ &\le 12\sum_{j=2}^{s}\log(1-b_j)^2 < \infty.\end{aligned}$$

To conclude this example set $u_s = \sum_{j=2}^{s}\varphi_j$.

7. Generalized boundary values

In this chapter we shall study the boundary behavior of plurisubharmonic functions. It follows directly from the definition of $\mathcal{E}_0$ (Definition 2.8) that every function $u \in \mathcal{E}_0$ has zero boundary values in the sense that

$$\lim_{\substack{z\to\xi\\ z\in\Omega}} \varphi(z) = 0$$

for every $\xi \in \partial\Omega$. This is no longer true for functions in $\mathcal{N}$. Instead, for all $u \in \mathcal{N}$ we have

$$\limsup_{\substack{z\to\xi\\ z\in\Omega}} \varphi(z) = 0 \tag{7.1}$$

for every $\xi \in \partial\Omega$ (Theorem 7.1). On the other hand, there exists $u \in \mathcal{F}$ such that

$$\liminf_{\substack{z\to\xi\\ z\in\Omega}} \varphi(z) = -\infty$$

for every $\xi \in \partial\Omega$ (Example 7.3). For functions in $\mathcal{E}$ we cannot even expect that (7.1) holds. Therefore for a better understanding of boundary behavior of plurisubharmonic functions, following Cegrell ([36]), we introduce another concept of boundary values for plurisubharmonic functions. We prove that so called *generalized boundary values* are a natural extension of the classical notion of boundary values. This topic is discussed in Chapter 7.1.

In Chapter 7.2 we prove a stability theorem for the complex Monge–Ampère operator in $\mathcal{N}(H)$ (Theorem 7.12). Using the stability theorem we will show the existence and stability of solutions of the complex Monge–Ampère type equation (Theorems 7.16 and 7.18). Chapter 7.3 is devoted to a subextension theorem for functions in $\mathcal{F}(H)$ (Theorem 7.19).

7.1. Monge–Ampère boundary measures. In this chapter we define, using some convergence results for the Monge–Ampère measures of functions from $\mathcal{F}$ (Theorem 7.4), the boundary measure μ_u associated with $u \in \mathcal{F}$ (Definition 7.8). Furthermore, we shall show that such a construction is not possible for every function from $\mathcal{N}$ (Example 7.7). For every bounded plurisubharmonic function φ we also define the boundary values φ^u of φ with respect to the measure μ_u. We prove that under certain assumptions the boundary values arising from the Monge–Ampère boundary measures, the boundary values for functions in the Cegrell class $\mathcal{F}(H)$, and the classical boundary values coincide (Theorems 7.9 and 7.10). Most results in this chapter originate from [36].

We start with the classical boundary values of negative plurisubharmonic functions. We prove the following theorem.

THEOREM 7.1 ([3]). *If $u \in \mathcal{N}$ then $\limsup_{z\to\xi} u(z) = 0$ for all $\xi \in \partial\Omega$.*

Proof. Let $u \in \mathcal{N}$. Suppose that there exists $\xi \in \partial\Omega$ such that $\limsup_{z\to\xi} u(z) < 0$. Then there exists $f \in \mathcal{C}(\partial\Omega)$ such that $u^* \le f \le 0$ and $f(\xi) < 0$. Since Ω is a bounded hyperconvex domain, there exists a harmonic function h in Ω, continuous on $\bar{\Omega}$ such that $h = f$ on $\partial\Omega$ (see [9]). The smallest maximal plurisubharmonic majorant of $u \in \mathcal{N}$ is $\tilde{u} = 0$, by the definition of the Cegrell class $\mathcal{N}$. Since $0 \ge h \ge \tilde{u}$, we have $h = 0$, but $h(\xi) < 0$ and we obtain a contradiction. ∎

PROPOSITION 7.2 ([32]). *For every pluripolar set E in Ω there exists $u \in \mathcal{F}$ such that $E \subset \{u = -\infty\}$.*

Proof. Since E is pluripolar, there exists a sequence $[U_j]$ of open sets $U_j \subset \Omega$ such that $E \subset U_j$ and $C_n(U_j) \le 1/2^j$. Let $[\Omega_j]$ be the fundamental sequence. Define by $h_j = h_{U_j\cap\Omega_j}$ the relative extremal function of $U_j \cap \Omega_j$. Then $h_j \in \mathcal{E}_0$, $-1 \le h_j \le 0$ and $h_j = -1$ on $U_j \cap \Omega_j$. Let $u_k = \sum_{j=1}^k h_j$. Fix $v \in \mathcal{E}_0$, $-1 \le v < 0$ in Ω. Then from Theorem 3.14, Lemma 3.12 and Theorem 2.4 we have

$$\begin{aligned}\Big(\int_\Omega (-u_k)^n (dd^c v)^n\Big)^{1/n} &\le (n!)^{1/n}\Big(\int_\Omega (-v)(dd^c u_k)^n\Big)^{1/n}\\ &\le (n!)^{1/n}\sum_{j=1}^k \Big(\int_\Omega (-v)(dd^c h_j)^n\Big)^{1/n} \le (n!)^{1/n}\sum_{j=1}^k \Big(\int_\Omega (dd^c h_j)^n\Big)^{1/n}\\ &\le (n!)^{1/n}\sum_{j=1}^k C_n(U_j\cap\Omega_j)^{1/n} \le (n!)^{1/n}\sum_{j=1}^k C_n(U_j)^{1/n} \le (n!)^{1/n}\sum_{j=1}^\infty 2^{-j/n} < \infty.\end{aligned}$$

Then $u = \lim_{k\to\infty} u_k$ is a negative plurisubharmonic function and $u \in \mathcal{F}$ since by Theorem 3.14,

$$\Big(\int_\Omega (dd^c u)^n\Big)^{1/n} \le \sum_{j=1}^\infty \Big(\int_\Omega (dd^c h_j)^n\Big)^{1/n} = \sum_{j=1}^\infty 2^{-j/n} < \infty.$$

To finish the proof note that if $z \in E$, then there exists j_0 such that $z \in \Omega_{j_0}$ and then $z \in U_j \cap \Omega_j$ for all $j \geq j_0$. Therefore $h_j(z) = -1$ for $j \geq j_0$, so $u(z) = \sum_{j=1}^{\infty} h_j(z) = -\infty$. ∎

Example 7.3 ([2]). We construct a function $u \in \mathcal{F}$ such that $\liminf_{z \to \xi} u(z) = -\infty$ for all $\xi \in \partial\Omega$. Let $E = \{z_j, j \in \mathbb{N}\} \subset \Omega$ be such that $\partial\Omega \subset \bar{E}$. Since E is a pluripolar set, from Proposition 7.2 there exists $u \in \mathcal{F}$ such that $E \subset \{u = -\infty\}$. Therefore $\liminf_{z \to \xi} u(z) = -\infty$ for all $\xi \in \partial\Omega$.

Let $[\Omega_j]$ be a fundamental sequence for Ω. Let $u \in \mathcal{F}(\Omega)$ and let $[u^j]$ be the sequence from Definition 4.4. Then $u \leq u^j \leq u^{j+1} \leq 0$ and $u^j \in \mathcal{F}(\Omega)$. Moreover, by Stokes' theorem $\int_\Omega (dd^c u^j)^n = \int_\Omega (dd^c u)^n$ and also $\operatorname{supp}(dd^c u^j)^n \subset \Omega \setminus \Omega_j$.

The following theorem allows us to define a boundary measure associated with a function in $\mathcal{F}$.

Theorem 7.4 ([36]). *Let $u \in \mathcal{F}(\Omega)$. Then there exists a measure μ_u on $\partial\Omega$ such that $(dd^c u^j)^n$ is convergent in the weak*-topology to μ_u. For all $\varphi \in \mathcal{PSH}(\Omega) \cap L^\infty(\Omega)$ the limit*

$$\lim_{j\to\infty} \int_\Omega \varphi (dd^c u^j)^n$$

exists. Moreover, if $\psi \in \mathcal{PSH}(\Omega) \cap L^\infty(\Omega)$ then the sequence $[\psi(dd^c u^j)^n]$ is convergent in the weak-topology.*

Proof. Let U be a strictly pseudoconvex set containing $\bar{\Omega}$. Let $\varphi \in \mathcal{PSH}(\Omega) \cap L^\infty(\Omega)$. Since $\int_\Omega (dd^c u^j)^n = \int_\Omega (dd^c u)^n$, without loss of generality we can assume that $\varphi \leq 0$. By Theorem 2.17 there exists a decreasing sequence $[\varphi_k] \subset \mathcal{E}_0 \cap \mathcal{C}(\overline{\Omega})$ converging to φ. Then by Theorem 3.1 we have

$$-\infty < \int_\Omega \varphi_k (dd^c u)^n \leq \int_\Omega \varphi_k (dd^c u^j)^n \leq \int_\Omega \varphi_k (dd^c u^{j+1})^n$$

and using the monotone convergence theorem we find that for $\varphi \in \mathcal{PSH}(\Omega) \cap L^\infty(\Omega)$,

$$-\infty < \int_\Omega \varphi (dd^c u)^n \leq \int_\Omega \varphi (dd^c u^j)^n \leq \int_\Omega \varphi (dd^c u^{j+1})^n \leq \sup_\Omega \varphi \int_\Omega (dd^c u)^n. \tag{7.2}$$

Thus $\int_\Omega \varphi (dd^c u^j)^n$ is a bounded increasing sequence, so $\lim_{j\to\infty} \int_\Omega \varphi (dd^c u^j)^n$ exists. In particular, this limit exists for all $\varphi \in \mathcal{C}_0^\infty(U) \subset \mathcal{E}_0(U) \cap \mathcal{C}(\bar{U}) - \mathcal{E}_0(U) \cap \mathcal{C}(\bar{U})$ by Theorem 2.19, which means that it holds for any $\varphi \in \mathcal{C}_0$. This implies that the limit $\lim_{j\to\infty} (dd^c u^j)^n$ defines a positive functional on $\mathcal{C}_0$, so by the Riesz theorem there exists a positive measure μ_u on U such that $(dd^c u^j)^n$ is convergent in the weak*-topology to μ_u. It follows from the construction that $\operatorname{supp} \mu_u \subset \partial\Omega$.

Fix $\psi \in \mathcal{PSH}(\Omega) \cap L^\infty(\Omega)$. To prove the second part of Theorem 7.4 it is enough, by Theorem 2.19, to prove that for any $\varphi \in \mathcal{PSH}^-(\Omega) \cap L^\infty(\Omega)$ the limit

$$\lim_{j\to\infty} \int_\Omega \psi\varphi (dd^c u^j)^n$$

exists. For given φ, ψ there exist $a, b > 0$ such that $\varphi + a \geq 0$, $\psi + b \geq 0$, and then $(\varphi + a)^2, (\psi + b)^2, (\varphi + \psi + a + b)^2 \in \mathcal{PSH}(\Omega) \cap L^\infty(\Omega)$. Let w denote any of those functions. Then by the first part of the proof, $\lim_{j\to\infty} \int_\Omega w (dd^c u^j)^n$ exists. Then the limit exists for $w = (\varphi + a)(\psi + b)$ and finally for $w = \psi\varphi$. This ends the proof. ∎

EXAMPLE 7.5 ([36]). Let $\mathbb{B}$ be the unit ball in $\mathbb{C}^n$, and let $u(z) = \log|z|$. In this example, we show that $\mu_u = d\sigma$. Let $\Omega_j = B(0, e^{-1/j})$ be a fundamental sequence. Then $u^j(z) = \max(\log|z|, -1/j)$. Therefore, $(dd^c u^j)^n = d\sigma_j$, where $d\sigma_j$ is the Lebesgue measure on $\partial B(0, e^{-1/j})$. Hence, $\mu_u = d\sigma$, since $(dd^c u^j)^n$ tends to $d\sigma$ in the weak*-topology.

EXAMPLE 7.6 ([36]). Let $\mathbb{D}^2$ be the unit polydisc in $\mathbb{C}^2$, and let u be the function defined on $\mathbb{D}$ by

$$u(z, w) = \max(\log|z|, \log|w|).$$

We shall prove that $\mu_u = d\sigma \otimes d\sigma$, where $d\sigma$ is the Lebesgue measure on $\partial\mathbb{D}$. Let $\Omega_j = \mathbb{D}_{1-1/j} \times \mathbb{D}_{1-1/j}$ be the fundamental sequence. Then

$$u^j(z, w) = \max(\log|z|, \log|w|, \log(1 - 1/j)).$$

Hence, $(dd^c u^j)^n = d\sigma_j \otimes d\sigma_j$, where $d\sigma_j$ is the Lebesgue measure on $\partial\mathbb{D}_{1-1/j}$, which means that $\mu_u = d\sigma \otimes d\sigma$ since $(dd^c u^j)^n$ tends to $d\sigma \otimes d\sigma$ in the weak*-topology.

Theorem 7.4 is not valid for functions in $\mathcal{N}$ (Example 7.7), and therefore it is not possible to define the boundary measure μ_u for $u \in \mathcal{N}$.

EXAMPLE 7.7. Let $\mathbb{B}$ be the unit ball in $\mathbb{C}^n$. Define $u_j(z) = \max(\log|z|, -1/2^j)$. Then $u_j \in \mathcal{E}_0$, $(dd^c u_j)^n = (2\pi)^n d\sigma_j$, where $d\sigma_j$ is the Lebesgue measure on the sphere $\{|z| = e^{-1/2^j}\}$. Let $u = \sum_{j=1}^{\infty} u_j$. Then $u \in \mathcal{N}$ by Proposition 4.8 since $u(0) = -1$. Moreover,

$$\int_{\mathbb{B}} (dd^c u)^n \geq \sum_{j=1}^{\infty} \int_{\mathbb{B}} (dd^c u_j)^n = \sum_{j=1}^{\infty} (2\pi)^n = \infty.$$

Let $\Omega_j = B(0, e^{-1/2^j})$ be the fundamental sequence. Note that

$$u^j = \sum_{k=j}^{\infty} u_k,$$

and therefore $\int_{\mathbb{B}} (dd^c u^j)^n = \infty$.

Based on Theorem 7.4 we can state the following definition.

DEFINITION 7.8 ([36]). For $u \in \mathcal{F}(\Omega)$ and $\varphi \in \mathcal{PSH}(\Omega) \cap L^\infty(\Omega)$, let φ^u be the function in $L^\infty(\partial\Omega, \mu_u)$ such that

$$\lim_{j\to\infty} \varphi (dd^c u^j)^n = \varphi^u d\mu_u.$$

We may consider φ^u as the boundary values of φ with respect to μ_u. Note that at least formally φ^u depends on both φ and u. However, the following theorems describe some situations when this definition agrees with other notions of boundary values. In Theorem 7.9, we present a slightly modified proof compared to the original one.

THEOREM 7.9 ([36]). *Let $u \in \mathcal{F}(\Omega)$ be such that $(dd^c u)^n$ vanishes on pluripolar sets in Ω, and $\varphi \in \mathcal{PSH}^-(W) \cap L^\infty(W)$, where W is a bounded hyperconvex domain containing $\bar{\Omega}$. Then $\varphi^u = \varphi|_{\partial\Omega}$ almost everywhere with respect to μ_u.*

Proof. Let $\varphi \in \mathcal{PSH}^-(W) \cap L^\infty(W)$. By Theorem 2.17 there exists $\varphi_k \in \mathcal{E}_0 \cap \mathcal{C}(W)$ such that $\varphi_k \searrow \varphi$. Then by Theorem 7.4,

$$\lim_{j\to\infty} \int_\Omega \varphi (dd^c u^j)^n \leq \lim_{j\to\infty} \int_\Omega \varphi_k (dd^c u^j)^n = \int_{\partial\Omega} \varphi_k d\mu_u.$$

By letting $k \to \infty$ we obtain

$$\lim_{j\to\infty} \int_\Omega \varphi (dd^c u^j)^n \leq \int_{\partial\Omega} \varphi d\mu_u. \tag{7.3}$$

Fix $f \in \mathcal{C}(\bar{\Omega})$, $f \geq 0$. It follows as above that

$$\int_{\partial\Omega} f\varphi^u \, d\mu_u = \lim_{j\to\infty} \int_\Omega f\varphi (dd^c u^j)^n \leq \int_{\partial\Omega} f\varphi \, d\mu_u.$$

Thus $\varphi^u \leq \varphi$ a.e w.r.t. μ_u, so it remains to prove that $\int_{\partial\Omega} \varphi^u \, d\mu_u = \int_{\partial\Omega} \varphi \, d\mu_u$. Choose a hyperconvex set U such that $\Omega \Subset U \Subset W$. For a given $\varepsilon > 0$ by the quasicontinuity of plurisubharmonic functions (Theorem 2.5) there exist an open set $U_\varepsilon \subset U$, $C_n(U_\varepsilon, U) < \varepsilon$ and $\varphi_\varepsilon \in \mathcal{C}_0(U)$ such that $\inf_U \varphi \leq \varphi_\varepsilon \leq 0$ and $U \setminus U_\varepsilon \subset \{z \in U : \varphi(z) = \varphi_\varepsilon(z)\}$. It follows that

$$\begin{aligned}
\int_{\partial\Omega} \varphi^u d\mu_u &= \lim_{j\to\infty} \int_\Omega \varphi (dd^c u^j)^n = \lim_{j\to\infty} \int_{\Omega\cap U_\varepsilon} \varphi (dd^c u^j)^n + \lim_{j\to\infty} \int_{\Omega\setminus U_\varepsilon} \varphi_\varepsilon (dd^c u^j)^n \\
&\geq \lim_{j\to\infty} \int_{\Omega\cap U_\varepsilon} \varphi (dd^c u^j)^n + \int_{\partial\Omega} \varphi_\varepsilon d\mu_u \\
&= \lim_{j\to\infty} \int_{\Omega\cap U_\varepsilon} \varphi (dd^c u^j)^n + \int_{\partial\Omega\cap U_\varepsilon} \varphi_\varepsilon d\mu_u + \int_{\partial\Omega\setminus U_\varepsilon} \varphi d\mu_u \\
&\geq \lim_{j\to\infty} \int_{\Omega\cap U_\varepsilon} \varphi (dd^c u^j)^n + \int_{\partial\Omega\cap U_\varepsilon} \varphi_\varepsilon d\mu_u + \int_{\partial\Omega} \varphi d\mu_u.
\end{aligned} \tag{7.4}$$

Let $h_\varepsilon = \sup\{\psi \in \mathcal{PSH}^-(W) : \psi|U_\varepsilon \leq -1\}$ be the relative extremal function for U_ε. Then by (7.2), (7.3) and (7.4) we obtain

$$\begin{aligned}
0 &\geq \int_{\partial\Omega} \varphi^u d\mu_u - \int_{\partial\Omega} \varphi d\mu_u \geq \lim_{j\to\infty} \int_{\Omega\cap U_\varepsilon} \varphi (dd^c u^j)^n + \int_{\partial\Omega\cap U_\varepsilon} \varphi_\varepsilon d\mu_u \\
&\geq \inf_U \varphi \left(\lim_{j\to\infty} \int_{\Omega\cap U_\varepsilon} (dd^c u^j)^n + \int_{\partial\Omega\cap U_\varepsilon} d\mu_u \right) \\
&= -\inf_U \varphi \left(\lim_{j\to\infty} \int_{\Omega\cap U_\varepsilon} h_\varepsilon (dd^c u^j)^n + \int_{\partial\Omega\cap U_\varepsilon} h_\varepsilon d\mu_u \right) \\
&\geq -\inf_U \varphi \left(\lim_{j\to\infty} \int_\Omega h_\varepsilon (dd^c u^j)^n + \int_{\partial\Omega} h_\varepsilon d\mu_u \right) \geq -2 \inf_U \varphi \int_\Omega h_\varepsilon (dd^c u)^n.
\end{aligned}$$

To finish the proof it is now enough to show that

$$\lim_{\varepsilon\to 0} \int_\Omega h_\varepsilon (dd^c u)^n = 0.$$

Since $(dd^c u)^n$ does not put mass on pluripolar sets, by Theorem 5.6 there exist $\psi \in \mathcal{E}_0(W)$, $f \geq 0$, $f \in L^1((dd^c\psi)^n)$ such that $\mu = f(dd^c\psi)^n$. Define $\mu_k = \min(f, k)(dd^c\psi)^n$ for $k \in \mathbb{N}$. Then by the Kołodziej subsolution theorem (Theorem 2.7) there exists $\psi_k \in \mathcal{E}_0(W)$ such that $(dd^c\psi_k)^n = \mu_k$. Therefore by Corollary 3.15 and Theorem 2.4,

$$\begin{aligned}\int_\Omega (-h_\varepsilon)d\mu_k &= \int_W (-h_\varepsilon)(dd^c\psi_k)^n = \int_W (-\psi_k)dd^c h_\varepsilon \wedge (dd^c\psi_k)^n \\ &\le \sup_W(-\psi_k)\Big(\int_W (dd^c h_\varepsilon)^n\Big)^{1/n}\Big(\int_W (dd^c\psi_k)^n\Big)^{(n-1)/n} \\ &\le \mu_k(W)^{(n-1)/n}\sup_W(-\psi_k)C_n(U_\varepsilon, W)^{1/n} \\ &< C\mu_k(W)^{(n-1)/n}\sup_W(-\psi_k)C_n(U_\varepsilon, U)^{1/n} \\ &= C(k)\varepsilon^{1/n},\end{aligned}$$

where the constants $C, C(k)$ do not depend on ε. So we have proved that

$$\lim_{\varepsilon\to 0}\int_W h_\varepsilon d\mu_k = 0.$$

Hence by the monotone convergence theorem

$$\begin{aligned}0 \ge \lim_{\varepsilon\to 0}\int_W h_\varepsilon(dd^c u)^n &= \lim_{\varepsilon\to 0}\int_W h_\varepsilon d\mu_k + \lim_{\varepsilon\to 0}\int_W h_\varepsilon(f-\min(k,f))(dd^c\psi)^n \\ &\ge \int_W (\min(k,f)-f)(dd^c\psi)^n \to 0, \qquad k\to\infty,\end{aligned}$$

since $-1 \le h_\varepsilon \le 0$. ∎

THEOREM 7.10 ([36]). *Let H be a bounded maximal plurisubharmonic function and let $u \in \mathcal{F}(\Omega)$ be such that $(dd^c u)^n$ vanishes on pluripolar sets in Ω. Then for every $\varphi \in \mathcal{F}(\Omega, H)$ such that $\int_\Omega \varphi(dd^c u)^n > -\infty$ we have $\varphi^u = H^u$.*

Proof. Note that by Theorem 7.4 the limit $\lim_{j\to\infty} H(dd^c u^j)^n$ exists. We have to prove that

$$\lim_{j\to\infty}\varphi(dd^c u^j)^n = \lim_{j\to\infty} H(dd^c u^j)^n.$$

By the same argument as in the proof of Theorem 7.4 it is enough to prove that for all $w \in \mathcal{PSH}^-(\Omega)\cap L^\infty(\Omega)$ we have

$$\lim_{j\to\infty}\int_\Omega w\varphi(dd^c u^j)^n = \lim_{j\to\infty}\int_\Omega wH(dd^c u^j)^n.$$

Since $\varphi \in \mathcal{F}(\Omega, H)$, there exists $\psi \in \mathcal{F}(\Omega)$ such that $\psi + H \le \varphi \le H$. We may assume that $\psi \ge \varphi$ (taking $\psi_0 = \max(\psi, \varphi)$ if necessary) and that $-1 \le w, H \le 0$. We have

$$\int_\Omega w\varphi(dd^c u^j)^n = \int_\Omega w(\varphi - H)(dd^c u^j)^n + \int_\Omega wH(dd^c u^j)^n.$$

Note that by Theorem 3.1,

$$\begin{aligned}0 \le \int_\Omega w(\varphi - H)(dd^c u^j)^n &= \int_\Omega (-w)(H-\varphi)(dd^c u^j)^n \le \int_\Omega (-w)(-\psi)(dd^c u^j)^n \\ \le \int_\Omega (-\psi)(dd^c u^j)^n &= \int_\Omega (-u^j)dd^c\psi\wedge(dd^c u^j)^{n-1} \le \int_\Omega (-u)dd^c\psi\wedge(dd^c u^j)^{n-1}\end{aligned}$$

$$= \int_\Omega (-u^j) dd^c\psi \wedge dd^c u \wedge (dd^c u^j)^{n-2} \le \cdots \le \int_\Omega (-u^j) dd^c\psi \wedge (dd^c u)^{n-1} = I_j$$
$$\le \int_\Omega (-u) dd^c\psi \wedge (dd^c u)^{n-1} = \int_\Omega (-\psi)(dd^c u)^n \le \int_\Omega (-\varphi)(dd^c u)^n < \infty.$$

Since u^j increases to zero outside a pluripolar set and the measure $dd^c\psi \wedge (dd^c u)^{n-1}$ does not put mass on pluripolar sets, we have $I_j \searrow 0$ as $j \to \infty$. This ends the proof. ■

REMARK 7.11. If $\varphi \in L^\infty(\Omega)$, then $\int_\Omega \varphi(dd^c u)^n > -\infty$ for every $u \in \mathcal{F}$. Furthermore, $\psi \ge \varphi$. This implies that ψ is bounded, and therefore the measure $dd^c\psi \wedge (dd^c u)^{n-1}$ does not put mass on any pluripolar set. Thus, for any bounded function $\varphi \in \mathcal{F}(\Omega, H)$ we have

$$\varphi^u \mu_u = \lim_{j\to\infty} \varphi(dd^c u^j)^n = \lim_{j\to\infty} H(dd^c u^j)^n = H^u \mu_u$$

for every $u \in \mathcal{F}$.

7.2. Stability of solutions and the complex Monge–Ampère type equation. In [37], Cegrell and Kołodziej proved a stability theorem for the complex Monge–Ampère equation in $\mathcal{F}(PB_g)$, where g is a continuous function on $\partial\Omega$ such that $\lim_{z\to\xi} PB_g(z) = g(\xi)$ for all $\xi \in \partial\Omega$. Consider the Monge–Ampère equation

$$(dd^c u)^n = \mu. \tag{7.5}$$

By *stability of solutions* to (7.5) in a class $\mathcal{K}$ we mean that after a small perturbation μ' of the measure μ one still can find a solution u' to the equation $(dd^c u')^n = \mu'$ that also belongs to $\mathcal{K}$. Furthermore u' should be in some sense close to u. We prove the stability theorem in $\mathcal{N}(H)$ (Theorem 7.12).

In [28], Cegrell proved that convergence in the sense of distributions of plurisubharmonic functions does not in general imply convergence of their Monge–Ampère measures. In other words, the complex Monge–Ampère operator is not continuous on the set of plurisubharmonic functions equipped with its natural topology, i.e. weak topology. Recall that a sequence of plurisubharmonic functions that converges in the sense of distributions is also convergent in L^p_{loc} for any $p \in [1, \infty)$, i.e. it is weakly convergent (see e.g. [58]). Later Lelong proved that every locally bounded plurisubharmonic function defined on a bounded pseudoconvex domain can be approximated in the weak topology by continuous maximal plurisubharmonic functions [66]. It was proved by Xing [84], Cegrell [31], Phạm [71], Åhag and the author [7], among others, that it is better to consider other types of convergence for plurisubharmonic function to ensure continuity of the complex Monge–Ampère operator. In particular, Cegrell proved that if a sequence of plurisubharmonic functions is bounded from below by a function from the Cegrell class $\mathcal{E}$ and if it is convergent in capacity then the corresponding Monge–Ampère measures are weak* convergent. We shall give an example, using the stability theorem, that on a certain set of plurisubharmonic functions weak convergence is equivalent to convergence in capacity (Theorem 7.15).

Bedford and Taylor proved in [18] the existence of a solution to the following Monge–Ampère type equation:

$$(dd^c u)^n = F(u(z), z)\mu. \tag{7.6}$$

They assumed that μ is the Lebesgue measure, and $F^{1/n} \geq 0$ is bounded, continuous, convex, and increasing in the first variable. Later in [29], Cegrell showed that the convexity and monotonicity conditions are superfluous. The case when F is smooth was proved in [24]. In [64], Kołodziej proved existence and uniqueness of solution to (7.6) when F is a bounded, nonnegative function that is nondecreasing and continuous in the first variable. Furthermore, μ was assumed to be a Monge–Ampère measure generated by some bounded plurisubharmonic function. The underlying domain of (7.6) has so far been assumed to be strictly pseudoconvex. A generalization to hyperconvex domains was made by Cegrell and Kołodziej in [37]. There assumptions were that μ is finite, vanishing on pluripolar sets, and $F \geq 0$ is continuous in the first variable, upper bounded by a function from $L^1(d\mu)$. Here we shall prove existence and stability of the solution of the Monge–Ampère type equation (7.6) for some unbounded measures μ (Theorems 7.16 and 7.18).

The results in this chapter are exclusively published in this survey.

The aim of this chapter is the following stability result:

Theorem 7.12 (Stability theorem). *Let Ω be a bounded hyperconvex domain, and let μ be a measure vanishing on pluripolar sets such that there exists $v \in \mathcal{N}$ with $\mu = (dd^c v)^n$, and let $H \in \mathcal{E} \cap \mathcal{MPSH}(\Omega)$. Let $0 \leq f, f_j \leq 1$ be measurable functions such that $f_j\mu \to f\mu$ in the weak*-topology as $j \to \infty$. Then for $u_j, u \in \mathcal{N}(H)$ which solve*

$$(dd^c u_j)^n = f_j\mu, \qquad (dd^c u)^n = f\mu$$

we have $u_j \to u$ in capacity.

Theorem 7.12 is a generalization of Cegrell and Kołodziej's stability theorem ([37]). From quasi-continuity of plurisubharmonic functions and the monotone convergence theorem we have the following well-known lemma ([84]).

Lemma 7.13. *If $[u_j]$ is a monotone sequence of plurisubharmonic functions converging to a plurisubharmonic function u, then $u_j \to u$ in capacity as $j \to \infty$.*

Proof. Fix $K \Subset \Omega$, $\varepsilon > 0$, $\delta > 0$. By Theorem 2.5 there exists an open set $G \subset \Omega$ such that $C_n(G) < \delta/2$, and functions u_j, u are continuous on G^c. Thus, $|u - u_j| \searrow 0$ locally uniformly on G^c. Therefore, there exists j_0 such that for all $j \geq j_0$, on $K \setminus G$ we have

$$|u - u_j| \leq \frac{\delta\varepsilon}{2C_n(K)}.$$

For all $w \in \mathcal{PSH}(\Omega)$, $-1 \leq w \leq 0$,

$$\begin{aligned}\int_{\{|u_j-u|>\varepsilon\}\cap K} (dd^c w)^n &\leq \int_G (dd^c w)^n + \frac{1}{\varepsilon}\int_{K\setminus G} |u - u_j|(dd^c w)^n \\ &\leq C_n(G) + \frac{1}{\varepsilon}\frac{\delta\varepsilon}{2C_n(K)}\int_{K\setminus G}(dd^c w)^n \leq \delta.\end{aligned}$$

This means that, after taking the supremum over all $w \in \mathcal{PSH}(\Omega)$, $-1 \leq w \leq 0$,

$$C_n(\{|u_j - u| > \varepsilon\} \cap K) \leq \delta,$$

and the proof is complete. ∎

Proposition 7.14. *Let Ω be a bounded hyperconvex domain, and let $u \in \mathcal{E}$ be such that $\mu = (dd^c u)^n$ vanishes on pluripolar sets. Let $[\Omega_k]$ be the fundamental sequence. The following two assertions are then equivalent:*

(1) *there exists $\varphi \in \mathcal{N}$ such that $(dd^c\varphi)^n = \mu$,*
(2) *there exists a sequence $[u_k]$ such that $u_k \in \mathcal{E}$, $(dd^c u_k)^n = (1 - \chi_{\Omega_k})\mu$, and $u_k \nearrow 0$ a.e. in Ω.*

Proof. (1)⇒(2): Assume that $u \in \mathcal{N}$. Since $(1-\chi_{\Omega_k})\mu \le \mu = (dd^c u)^n$, by the subsolution theorem (Theorem 6.22) there exists $u_k \in \mathcal{N}$ such that $(dd^c u_k)^n = (1 - \chi_{\Omega_k})\mu$. The comparison principle (Corollary 5.10) implies that the sequence $[u_k]$ is increasing. It remains to show that $u_k \nearrow 0$ a.e. To prove this take another fundamental sequence $[\omega_k]$ such that

$$\omega_k \Subset \Omega_k \Subset \omega_{k+1} \Subset \Omega_{k+1},$$

and define

$$v_k = \sup\{w \in \mathcal{E} : w \le u \text{ on } \omega_k^c\}.$$

Since $v_k = u$ on ω_k^c, we have $(dd^c v_k)^n \ge (dd^c u_k)^n$, and from the comparison principle we obtain $u_k \ge v_k$. Since $u \in \mathcal{N}$, by definition $v_k \nearrow 0$ a.e. and therefore $u_k \nearrow 0$ a.e. as $k \to \infty$.

(2)⇒(1): Assume that there exists a sequence $[u_k]$ such that $u_k \in \mathcal{E}$, $(dd^c u_k)^n = (1-\chi_{\Omega_k})\mu$ and $u_k \nearrow 0$ a.e. in Ω. It follows from Theorem 5.6 that there exist $\psi \in \mathcal{E}_0$ and $0 \le f \in L^1_{\text{loc}}((dd^c\psi)^n)$ such that $\mu = f(dd^c\psi)^n$. By the Kołodziej subsolution theorem (Theorem 2.7) there exist $u_k^j \in \mathcal{E}_0$ such that

$$(dd^c u_k^j)^n = \min(f, j)(1 - \chi_{\Omega_k})(dd^c\psi)^n.$$

It follows from the comparison principle that $[u_k^j]$ is a decreasing sequence for fixed k, and $u_k^j \ge u_k$. Then there exists $v_k \in \mathcal{E}$ such that $(dd^c v_k)^n = (1-\chi_{\Omega_k})\mu$ and $u_k^j \searrow v_k \ge u_k$ as $j \to \infty$. Furthermore, again by Corollary 5.10, $[v_k]$ is an increasing sequence and $v_k \nearrow 0$ a.e. as $k \to \infty$, since $v_k \ge u_k$.

By the Kołodziej subsolution theorem there exists $\varphi_k^j \in \mathcal{E}_0$ such that

$$(dd^c u_k^j)^n = \min(f, j)\chi_{\Omega_k}(dd^c\psi)^n.$$

It follows from the comparison principle that $[\varphi_k^j]$ is a decreasing sequence for fixed j, and $\varphi_k^j \searrow \varphi^j$ as $k \to \infty$, where $\varphi^j \in \mathcal{F}$ and $(dd^c\varphi^j)^n = \min(f, j)(dd^c\psi)^n$. Similarly $[\varphi_k^j]$ is a decreasing sequence, for fixed k, and $\varphi_k^j \searrow \varphi_k$ as $j \to \infty$, where $\varphi_k \in \mathcal{F}$ and $(dd^c\varphi_k)^n = \chi_{\Omega_k}(dd^c\psi)^n$.

The comparison principle implies that $[\varphi^j]$ is a decreasing sequence and $\varphi^j \ge u$, and therefore there exists $\varphi \in \mathcal{E}$ such that $\varphi^j \searrow \varphi \ge u$ and $(dd^c\varphi)^n = \mu$. Now we prove that $\varphi \in \mathcal{N}$. Note that

$$(dd^c(\varphi_k^j + u_k^j))^n \ge (dd^c\varphi_k^j)^n + (dd^c u_k^j)^n = (dd^c\varphi^j)^n,$$

so by the comparison principle we have

$$\varphi^j \ge \varphi_k^j + u_k^j.$$

Letting $j \to \infty$ we obtain

$$\varphi \geq \varphi_k + v_k,$$

and since $\tilde{\varphi_k} = 0$ we have

$$\tilde{\varphi} \geq \tilde{\varphi}_k + \tilde{v}_k \geq v_k \nearrow 0, \qquad k \to \infty,$$

which means that $\tilde{\varphi} = 0$ a.e, so $\tilde{\varphi} \equiv 0$ and $\varphi \in \mathcal{N}$. ∎

Proof of Theorem 7.12. First note that it follows from the subsolution theorem (Theorem 6.22) that there exist $u_j, u \in \mathcal{N}(H)$ such that $(dd^c u_j)^n = f_j \mu$ and $(dd^c u)^n = f\mu$. Moreover, the comparison principle (Corollary 5.10) implies that u_j, u are uniquely determined.

Let $[\Omega_k]$ be a fundamental sequence. By Theorem 6.6 there exist $u_j^k, u^k \in \mathcal{F}(H)$ such that

$$(dd^c u_j^k)^n = \chi_{\Omega_k} f_j \mu, \qquad (dd^c u^k)^n = \chi_{\Omega_k} f\mu.$$

It follows from the comparison principle that $[u_j^k]$, $[u^k]$ are decreasing sequences and $u_j^k \searrow u_j$, $u^k \searrow u$, as $k \to \infty$. By the subsolution theorem there exists $v_k \in \mathcal{N}$ such that $(dd^c v_k)^n = (1 - \chi_{\Omega_k})\mu$. By our assumptions $\mu = (dd^c v)^n$ with $v \in \mathcal{N}$, so by Proposition 7.14, $v_k \nearrow 0$. Observe that

$$(dd^c u_j)^n = f_j \mu \leq (1 - \chi_{\Omega_k})\mu + f_j \chi_{\Omega_k} \mu = (dd^c v_k)^n + (dd^c u_j^k)^n \leq (dd^c(v_k + u_j^k))^n$$

and then by the comparison principle $u_j \geq v_k + u_j^k$. Therefore

$$u - u_j \leq (u^k - u_j^k) + (u_j^k - u_j) \leq u^k - u_j^k - v_k \tag{7.7}$$

and

$$u_j - u \leq u_j^k - u = (u_j^k - u^k) + (u^k - u). \tag{7.8}$$

Now fix $K \Subset \Omega$ and $\delta, \varepsilon > 0$. Since $[v_k]$, $[u^k]$ are monotone sequences, by Lemma 7.13 $v_k \to 0$ and $u^k \to u$ in capacity as $k \to \infty$, and so there exists k_0 such that for all $k \geq k_0$,

$$C_n(\{v_k < -\varepsilon/4\} \cap K) < \delta/4, \qquad C_n(\{|u^k - u| > \varepsilon/4\} \cap K) < \delta/4. \tag{7.9}$$

Now for fixed k we prove that $u_j^k \to u^k$ in capacity as $j \to \infty$. Note that by Corollary 5.9 we have for all $w \in \mathcal{PSH}(\Omega)$, $-1 \leq w \leq 0$,

$$\begin{aligned} \int_{\{u^k > u_j^k\}} (u^k - u_j^k)^n (dd^c w)^n &\leq n! \int_{\{u^k > u_j^k\}} (-w)((dd^c u_j^k)^n - (dd^c u^k)^n) \\ &\leq n! \int_{\Omega_k \cap \{u^k > u_j^k\}} (-w)|f_j - f| d\mu, \end{aligned} \tag{7.10}$$

and similarly

$$\int_{\{u_j^k > u^k\}} (u_j^k - u^k)^n (dd^c w)^n \leq n! \int_{\Omega_k \cap \{u_j^k > u^k\}} (-w)|f - f_j| d\mu. \tag{7.11}$$

Now fix $K \Subset \Omega$, $\varepsilon > 0$ and $w \in \mathcal{PSH}(\Omega)$, $-1 \leq w \leq 0$. Then by (7.10) and (7.11) we obtain

$$\begin{aligned}\int_{\{|u_j^k-u^k|>\varepsilon\}\cap K}(dd^cw)^n &\le \frac{1}{\varepsilon^n}\int_{\{|u_j^k-u^k|>\varepsilon\}\cap K}|u_j^k-u^k|^n(dd^cw)^n\\ &\le \frac{1}{\varepsilon^n}\int_{\{u_j^k>u^k\}}(u_j^k-u^k)^n(dd^cw)^n+\frac{1}{\varepsilon^n}\int_{\{u_j^k>u^k\}}(u_j^k-u^k)^n(dd^cw)^n\\ &\le \frac{n!}{\varepsilon^n}\int_{\Omega_k}(-w)|f_j-f|d\mu \le \frac{n!}{\varepsilon^n}\int_{\Omega_k}|f_j-f|d\mu.\end{aligned}$$

Since $f_j\mu \to f\mu$, taking supremum over all $w \in \mathcal{PSH}(\Omega)$, $-1 \le w \le 0$, we get $u_j^k \to u^k$ in capacity as $j \to \infty$. Fix $k \ge k_0$. Then there exists j_0 such that for all $j \ge j_0$,

$$C_n(\{|u_j^k-u^k|>\varepsilon/4\}\cap K)<\delta/4. \tag{7.12}$$

Inequalities (7.7), (7.8), (7.9), and (7.12) yield, for $j \ge j_0$,

$$\begin{aligned}C_n(\{|u_j-u|>\varepsilon\}\cap K) &\le 2C_n(\{|u_j^k-u^k|>\varepsilon/4\}\cap K)\\ &\quad + C_n(\{|u^k-u|>\varepsilon/4\}\cap K)+C_n(\{v_k<-\varepsilon/4\}\cap K)<\delta.\end{aligned}$$

This ends the proof. ∎

Corollary 7.15. *Let Ω be a bounded hyperconvex domain, let μ be a measure vanishing on pluripolar sets such that there exists $v \in \mathcal{N}$ with $\mu = (dd^cv)^n$, and let $H \in \mathcal{E} \cap \mathcal{MPSH}(\Omega)$. Let $\mathcal{A}(\mu, H)$ denote the set of all solutions of the Dirichlet problem*

$$u \in \mathcal{N}(H), \qquad (dd^cu)^n = g\mu,$$

where g varies over all μ-measurable function with $0 \le g \le 1$. Then weak convergence and convergence in capacity are equivalent in $\mathcal{A}(\mu, H)$.

Proof. We follow the proof given in [37]. Suppose that $u_j \to u$ in $L^1_{\text{loc}}(\Omega)$ and let $(dd^cu_j)^n = f_j\mu$. Choose a subsequence f_{j_k} converging to some f in the weak*-topology. Then by Theorem 7.12, $(dd^cu)^n = f\mu$ and $u_{j_k} \to u$ in capacity. This argument works for any subsequence of the original sequence, and therefore $u_j \to u$ in capacity. ∎

The stability theorem (Theorem 7.12) coupled with the Schauder–Tikhonov fixed point theorem (see e.g. [78]) allow us to show a very general existence theorem for the complex equation of Monge–Ampère type (7.6).

Theorem 7.16. *Let Ω be a bounded hyperconvex domain, μ a non-negative measure vanishing on pluripolar sets, and let $H \in \mathcal{E} \cap \mathcal{MPSH}(\Omega)$. Assume also that $F(x, z) \ge 0$ is a $dx \times d\mu$-measurable function on $(-\infty, 0] \times \Omega$ that is continuous in the x variable. Consider the following two conditions:*

(1) *there exist $\varphi \in \mathcal{PSH}^-(\Omega)$ and $g \in L^1((-\varphi)\mu)$ such that*

$$0 \le F(x, z) \le g(z),$$

(2) *there exist $w \in \mathcal{N}$ such that $\mu = (dd^cw)^n$, and a bounded function g such that*

$$0 \le F(x, z) \le g(z).$$

If one of the above conditions is satisfied, then there exists a function $u \in \mathcal{N}(H)$ that satisfies

$$(dd^c u)^n = F(u(z), z)\, \mu.$$

Furthermore, if F is a nondecreasing function in the first variable, then the solution u is uniquely determined.

Proof. We follow the proof given in [37].

(1): By Theorem 6.3 there exists a unique $\psi_0 \in \mathcal{N}$ such that $(dd^c\psi_0)^n = g\,\mu$, and therefore Theorem 6.6 implies that there exists a unique $\psi_1 \in \mathcal{N}(H)$ such that $(dd^c\psi_1)^n = g\mu$. Set

$$K = \{u \in \mathcal{N}(H) : u \geq \psi_1\}.$$

The set K is convex, and compact in the L^1_{loc}-topology. Let us define a map $\mathcal{T} : K \to K$ so that

$$\text{if } (dd^c v)^n = F(u(z), z)\, \mu, \text{ then } \mathcal{T}(u) = v.$$

Note that if $u \in K$, then $F(u(z), z)\, \mu \leq (dd^c\psi_0)^n$. Theorem 6.6 implies that there exists a unique $v \in \mathcal{N}(H)$ such that $(dd^c v)^n = F(u(z), z)\, \mu$, and by Corollary 5.10 we have $v \geq \psi_1$. Thus, $v \in K$, i.e. $\mathcal{T}$ is well defined.

We proceed to prove that $\mathcal{T}$ is continuous, and then the Schauder–Tikhonov fixed point theorem concludes the existence part of the proof. Assume that $u_j \in K$ with $u_j \to u$. By [37] there exists a subsequence (still denoted by $[u_j]$) converging to u in $L^1_{\mathrm{loc}}(d\mu)$. Theorem 7.12 applied to the measure $g\mu$ implies that the sequence $v_j = \mathcal{T}(u_j)$ converges in capacity to some $v \in K$. Since $v_j, v \in K$ we can use [31] to find that $(dd^c v_j)^n$ tends to $(dd^c v)^n$ in the weak*-topology. Hence,

$$(dd^c v)^n = \lim_{j\to\infty} (dd^c v_j)^n = \lim_{j\to\infty} F(u_j(z), z)\, \mu = F(u(z), z)\, \mu = (dd^c\mathcal{T}(u))^n,$$

which implies that $v = \mathcal{T}(u)$ by Corollary 5.10. Thus, $\lim_{j\to\infty}\mathcal{T}(u_j) = \mathcal{T}(u)$, i.e. $\mathcal{T}$ is continuous.

We now proceed with the uniqueness part. Assume that F is nondecreasing in the first variable, and there exist $u, v \in \mathcal{N}(H)$ such that

$$(dd^c u)^n = F(u(z), z)\, \mu \quad \text{and} \quad (dd^c v)^n = F(v(z), z)\, \mu.$$

On the set $\{z \in \Omega : u(z) < v(z)\}$ we have

$$(dd^c u)^n = F(u(z), z)\, \mu \leq F(v(z), z)\, \mu = (dd^c v)^n.$$

Corollary 5.9 yields

$$\int_{\{u<v\}} (dd^c v)^n \leq \int_{\{u<v\}} (dd^c u)^n,$$

hence $(dd^c u)^n = (dd^c v)^n$ on $\{z \in \Omega : u(z) < v(z)\}$. In a similar manner, we get $(dd^c u)^n = (dd^c v)^n$ on $\{z \in \Omega : u(z) > v(z)\}$. Furthermore, on $\{u = v\}$ we have

$$(dd^c u)^n = F(u(z), z)\, \mu = F(v(z), z)\, \mu = (dd^c v)^n.$$

Hence, $(dd^c u)^n = (dd^c v)^n$ on Ω. Thus $u = v$, by Corollary 5.10.

(2): Let g be a bounded function such that $0 \le F(x,z) \le g(z)$. By Theorem 6.6 there exist unique $\psi_0 \in \mathcal{N}$ and $\psi_1 \in \mathcal{N}(H)$ with

$$(dd^c\psi_0)^n = (dd^c\psi_1)^n = g\,d\mu.$$

The rest of the proof is the same as for (1). ∎

Remark 7.17. Condition (1) in the above theorem is satisfied by a smaller class of measures, but with a wider class of functions F. On the other hand, condition (2) gives a more general class of measures, but then we must assume a more restrictive conditions on F. Note that Theorem 7.16 is a generalization of the corresponding result in [37], since for a finite measure μ it is enough to take $\varphi = -1$ in condition (1).

We end this chapter by proving a stability theorem for the Monge–Ampère type equation.

Theorem 7.18. *Let Ω be a bounded hyperconvex domain, let μ be a non-negative measure vanishing on pluripolar sets and let $H \in \mathcal{E} \cap \mathcal{MPSH}(\Omega)$. Suppose also that $F(x,z) \ge 0$ is a $dx \times d\mu$-measurable function on $(-\infty, 0] \times \Omega$ which is continuous and non-decreasing in x. Assume that one of the conditions below is satisfied:*

(1) *there exist $\varphi \in \mathcal{PSH}^-(\Omega)$ and $g \in L^1((-\varphi)\mu)$ such that*

$$0 \le F(x,z) \le g(z),$$

(2) *there exist $w \in \mathcal{N}$ such that $\mu = (dd^c w)^n$ and a bounded function g such that*

$$0 \le F(x,z) \le g(z).$$

Let $0 \le f, f_j \le 1$ be measurable functions such that $f_j\mu \to f\mu$ in the weak-topology as $j \to \infty$. Then for $u_j, u \in \mathcal{N}(H)$ which solve*

$$(dd^c u_j)^n = F(u_j(z), z) f_j(z)\mu, \qquad (dd^c u)^n = F(u(z), z) f(z)\mu$$

we have $u_j \to u$ in capacity.

Proof. Theorem 7.16 yields unique functions $u_j \in \mathcal{N}(H)$ such that

$$(dd^c u_j)^n = F(u_j(z), z) f_j(z)\mu.$$

Theorem 6.6 together with Corollary 5.10 implies that there exists a unique function $\psi \in \mathcal{N}(H)$ such that $(dd^c\psi)^n = g\,\mu$, and $u_j \ge \psi$, since $(dd^c u_j)^n = F(u_j(z), z) f_j(z) d\mu \le g\,\mu = (dd^c\psi)^n$. Therefore there exists a subsequence, still denoted by $[u_j]$, that converges to u in the weak topology. Furthermore, $H \ge u \ge \psi$, since $H \ge u_j \ge \psi$. Thus, $u \in \mathcal{N}(H)$. Corollary 7.15 yields $u_j \to u$ in capacity, and then [31] implies that $[(dd^c u_j)^n]$ tends to $(dd^c u)^n$ in the weak*-topology. Passing to a subsequence, still denoted by $[f_j]$, we may assume that $f_j \to f$ pointwise a.e. w.r.t. $[\mu]$. The dominated convergence theorem gives us

$$(dd^c u)^n = \lim_{j\to\infty} (dd^c u_j)^n = \lim_{j\to\infty} F(u_j(z), z) f_j(z)\mu = F(u(z), z) f(z)\mu.$$

Hence, u is a solution to

$$(dd^c u)^n = F(u(z), z) f(z)\mu.$$

Since this argument works for any subsequence taken from the original sequence, we conclude that $u_j \to u$ in capacity. ∎

7.3. Subextension. The problem of finding the domains of existence for plurisubharmonic functions was considered by Bedford and Burns ([14]), and by Cegrell ([27]), i.e. they studied domains for which there exists a plurisubharmonic function that cannot be extended to any larger domain. They proved that any smooth bounded domain in $\mathbb{C}^n$ satisfying a certain non-degeneracy condition on the Levi form is a domain of existence. Plurisubharmonic functions are defined as functions satisfying certain inequalities, and therefore it is more natural to consider for them the subextension problem: for $\Omega_1 \subset \Omega_2 \Subset \mathbb{C}^n$ and for a given $u \in \mathcal{PSH}(\Omega_1)$ find $v \in \mathcal{PSH}(\Omega_2)$ such that $v \leq u$ on Ω_1. Bedford and Taylor improved an example by Fornæss and Sibony [55] by constructing a smooth negative plurisubharmonic function on an arbitrary bounded domain in $\mathbb{C}^n$ with C^2-boundary that does not subextend [20]. Cegrell and Zeriahi [39] proved that plurisubharmonic functions with bounded Monge–Ampère mass on a bounded hyperconvex domain admit a plurisubharmonic subextension to any larger bounded hyperconvex domain with control of the Monge–Ampère mass. Cegrell, Kołodziej and Zeriahi [38] proved several results showing that plurisubharmonic functions with bounded total Monge–Ampère mass admit global plurisubharmonic subextension with logarithmic growth at infinity. We prove that subextension is possible in $\mathcal{F}(H)$ (Theorem 7.19 and Proposition 8.17). Wiklund constructed an example that shows that subextension is not possible in $\mathcal{N}$ (Example 7.22). Subextension theorems proved their usefulness for example in estimation of the volume of sublevel sets of plurisubharmonic functions and in estimation of the integrability index for plurisubharmonic functions ([5, 7, 39], see e.g. [41, 75] for further applications).

This chapter is mainly based on the joint work of the author and Lisa Hed in [47]. Here we shall prove the following theorem.

Theorem 7.19 ([47]). *Let Ω_1 and Ω_2 be two bounded hyperconvex domains such that $\Omega_1 \subset \Omega_2 \subset \mathbb{C}^n$, $n \geq 1$ and let $F \in \mathcal{E}(\Omega_1)$, $G \in \mathcal{E}(\Omega_2) \cap \mathcal{MPSH}(\Omega_2)$ be such that*

$$F \geq G \quad \text{on } \Omega_1. \tag{7.13}$$

If $u \in \mathcal{F}(\Omega_1, F)$, then there exists $v \in \mathcal{F}(\Omega_2, G)$ such that $v \leq u$ on Ω_1 and

$$\int_{\Omega_2} (dd^c v)^n \leq \int_{\Omega_1} (dd^c u)^n.$$

Without the control of the total Monge–Ampère mass, the subextension in $\mathcal{F}(H)$, $H \in \mathcal{E}$, would follow as in the second part of the proof of Theorem 7.19 by using Theorem 2.2 in [39]. At this point it is not known if the assumption that $G \in \mathcal{MPSH}(\Omega_2)$ is necessary, but we observe that it is necessary that $\int_{\Omega_2} (dd^c G)^n \leq \int_{\Omega_1} (dd^c F)^n$.

To prove Theorem 7.19 we need the following proposition.

Proposition 7.20 ([47]). *Let $H \in \mathcal{E}$. If $u \in \mathcal{F}(H)$ is such that*

$$\int_{\Omega} (dd^c u)^n < \infty, \tag{7.14}$$

then there exists a decreasing sequence $[u_j]$, $u_j \in \mathcal{E}_0(H)$, that converges pointwise to u as

j tends to ∞, and

$$\sup_j \int_\Omega (dd^c u_j)^n < \infty. \tag{7.15}$$

Furthermore, if $[u_j]$, $u_j \in \mathcal{F}(H)$, is a decreasing sequence that converges pointwise to a function u as j tends to ∞, such that (7.15) *is satisfied, then $u \in \mathcal{F}(H)$ and* (7.14) *holds.*

Proof. Assume that $u \in \mathcal{F}(H)$ is such that (7.14) holds. It follows from Theorem 2.17 that there exists a decreasing sequence $[u_j]$, $u_j \in \mathcal{E}_0(H)$, that converges pointwise to u on Ω as $j \to \infty$. By Corollary 5.12 and assumption (7.14) we have

$$\sup_j \int_\Omega (dd^c u_j)^n < \infty.$$

Now assume first that $[u_j]$, $u_j \in \mathcal{E}_0(H)$, is a decreasing sequence such that (7.15) holds and $[u_j]$ converges pointwise to a function u as $j \to \infty$. From (7.15) and Lemma 5.11 we have $\int_\Omega (dd^c H)^n < \infty$, since $u_j, H \in \mathcal{F}(H)$ and $u_j \leq H$. Theorem 4.10 implies that $H \in \mathcal{F}(\tilde{H})$, where $\tilde{H}$ is defined as in Definition 4.4. Hence, we can, without loss of generality, assume that $(dd^c H)^n = 0$. The measure $(dd^c u_j)^n$ has finite total mass and vanishes on pluripolar sets by Lemma 6.20. Therefore Theorem 6.6 implies that there exists a unique $\varphi_j \in \mathcal{F}$ such that $(dd^c \varphi_j)^n = (dd^c u_j)^n$. Furthermore,

$$(dd^c(\varphi_j + H))^n \geq (dd^c u_j)^n.$$

Thus, $u_j \geq \varphi_j + H$, by the comparison principle (Corollary 5.10). Let φ_j' be the function defined by $\varphi_j' = (\sup_{k \geq j} \varphi_k)^*$. This construction implies that $[\varphi_j']$, $\varphi_j' \in \mathcal{F}$, is a decreasing sequence and

$$\sup_j \int_\Omega (dd^c \varphi_j')^n \leq \sup_j \int_\Omega (dd^c \varphi_j)^n < \infty,$$

by (7.15) and the fact that $(dd^c \varphi_j)^n = (dd^c u_j)^n$. Thus, by Proposition 3.16, $\varphi = (\lim_{j\to\infty} \varphi_j') \in \mathcal{F}$. For every $k \in \mathbb{N}$ we have $u_j \geq u_{(j+k)} \geq \varphi_{(j+k)} + H$. Hence, for every $j \in \mathbb{N}$ we have $u_j \geq \varphi + H$. By letting $j \to \infty$ we get $u \in \mathcal{F}(H)$. Now (7.15) and Corollary 5.12 imply that

$$\int_\Omega (dd^c u)^n = \lim_{j\to\infty} \int_\Omega (dd^c u_j)^n < \infty.$$

If $u_j \in \mathcal{F}(H)$ only, we can take $\psi \in \mathcal{E}_0(\Omega)$, $\psi \neq 0$ and define

$$u_j' = \max\{u_j, j\psi + H\}.$$

Since $j\psi + H \in \mathcal{E}_0(H)$ for every fixed j, we know that $u_j' \in \mathcal{E}_0(H)$. By the construction, $u_j' \searrow u$ as $j \to \infty$ and then Lemma 5.11 and (7.15) imply that $\int_\Omega (dd^c u_j')^n \leq \int_\Omega (dd^c u_j)^n$. It follows from (7.15) that

$$\sup_j \int_\Omega (dd^c u_j')^n < \infty$$

and the result follows. ∎

Proof of Theorem 7.19. Let $u \in \mathcal{F}(\Omega_1, F)$. First assume that

$$\int_{\Omega_1} (dd^c u)^n < \infty. \tag{7.16}$$

This assumption and Lemma 5.11 imply that $\int_{\Omega_1} (dd^c F)^n < \infty$, since $u, F \in \mathcal{F}(\Omega_1, F)$ and $u \le F$. Theorem 4.10 implies that $F \in \mathcal{F}(\Omega_1, \tilde{F})$, where $\tilde{F}$ is defined as in Definition 4.4. Hence, we can, without loss of generality, assume that $(dd^c F)^n = 0$. Proposition 7.20 implies that there exists a decreasing sequence $[u_j]$, $u_j \in \mathcal{E}_0(\Omega_1, F)$, which converges pointwise to u on Ω_1 as $j \to \infty$, and

$$\sup_j \int_{\Omega_1} (dd^c u_j)^n < \infty. \tag{7.17}$$

Consider the measure $\mu_j = \chi_{\Omega_1}(dd^c u_j)^n$ defined on Ω_2, where χ_{Ω_1} is the characteristic function defined in Ω_2 for the set Ω_1. The measure μ_j is a Borel measure in Ω_2 and it vanishes on pluripolar sets by Lemma 6.20. Moreover, from (7.17) it follows that $\mu_j(\Omega_2) < \infty$. Theorem 6.6 together with Theorem 4.10 implies that there exists a unique $\psi_j \in \mathcal{F}(\Omega_2, G)$ such that $(dd^c \psi_j)^n = \mu_j$ on Ω_2. Theorem 5.6 implies that there exist $w_j \in \mathcal{E}_0(\Omega_2, 0)$ and $\varphi_j \in L^1(\Omega_2, (dd^c w_j)^n)$, $\varphi_j \ge 0$, such that $\mu_j = \varphi_j (dd^c w_j)^n$ on Ω_2. For $k \in \mathbb{N}$ let the measure μ_{jk} be defined on Ω_2 by

$$\mu_{jk} = \min(\varphi_j, k)(dd^c w_j)^n.$$

It follows from Theorems 6.6 and 4.10 that there exist decreasing sequences $[\psi_{jk}]_{k=1}^\infty$, $\psi_{jk} \in \mathcal{F}(\Omega_2, G)$, $[\varphi_{jk}]_{k=1}^\infty$, $\varphi_{jk} \in \mathcal{F}(\Omega_1, F)$, such that

$$(dd^c \psi_{jk})^n = \mu_{jk} \quad \text{on } \Omega_2 \quad \text{and} \quad (dd^c \varphi_{jk})^n = \mu_{jk} \quad \text{on } \Omega_1.$$

Furthermore, $[\psi_{jk}]_{k=1}^\infty$ converges pointwise to ψ_j on Ω_2 and $[\varphi_{jk}]_{k=1}^\infty$ converges pointwise to u_j on Ω_1 as $k \to \infty$. The comparison principle (Corollary 5.10) and (7.13) yield

$$\psi_{jk} \le \varphi_{jk} \quad \text{on } \Omega_1.$$

Thus, $\psi_j \le u_j$ on Ω_1. For each $j \in \mathbb{N}$ let $v_j = (\sup_{l \ge j} \psi_l)^*$. By the construction we have $v_j \in \mathcal{F}(\Omega_2, G)$,

$$v_j \le u_j \quad \text{on } \Omega_1, \tag{7.18}$$

and $v_j \ge \psi_j$ on Ω_2 and therefore it follows that

$$\int_{\Omega_2} (dd^c v_j)^n \le \int_{\Omega_2} (dd^c \psi_j)^n = \int_{\Omega_1} (dd^c u_j)^n,$$

hence

$$\sup_j \int_{\Omega_2} (dd^c v_j)^n \le \sup_j \int_{\Omega_1} (dd^c u_j)^n < \infty. \tag{7.19}$$

Thus, $\lim_{j\to\infty} v_j \in \mathcal{F}(\Omega_2, G)$, by Proposition 7.20. Let $v = \lim_{j\to\infty} v_j$. Then it follows from (7.18) that $v \le u$ on Ω_1 and by (7.19) and Corollary 5.12 we have

$$\int_{\Omega_2} (dd^c v)^n \le \int_{\Omega_1} (dd^c u)^n,$$

which completes the proof in this case.

Now assume that $u \in \mathcal{F}(\Omega_1, F)$ is such that

$$\int_{\Omega_1} (dd^c u)^n = \infty. \tag{7.20}$$

Then it suffices to construct a function v in $\mathcal{F}(\Omega_2, G)$ such that $v \leq u$ on Ω_1. By definition there exists $u' \in \mathcal{F}(\Omega_1, 0)$ such that

$$F \geq u \geq u' + F.$$

From the first part of the proof there exists $v' \in \mathcal{F}(\Omega_2, 0)$ such that $v' \leq u'$ on Ω_1. Now let $v = v' + G$. Then $v \in \mathcal{F}(\Omega_2, G)$ and it follows by assumption (7.13) that

$$u \geq u' + F \geq v' + G = v$$

on Ω_1. Thus the proof is complete. ∎

The next example shows that condition (7.13) in Theorem 7.19 is relevant.

Example 7.21 ([7]). Let $\Omega_1 = \mathbb{D} \times \mathbb{D}$ and $\Omega_2 = \mathbb{D}_2 \times \mathbb{D}_2$ be polydiscs in $\mathbb{C}^2$. Then define $f(z, w) = |z|^2$ on $\partial\Omega_1$, and $g(z, w) = |w|^2$ on $\partial\Omega_2$. If $u(z, w) = |z|^2$, then $(dd^c u)^2 = 0$ on Ω_1. Now suppose that there exists $v \in \mathcal{F}(\Omega_2, |w|^2)$ such that $v \leq u$ with

$$\int_{\Omega_2} (dd^c v)^2 \leq \int_{\Omega_1} (dd^c u)^2 = 0.$$

This means that v must be a maximal plurisubharmonic function. In other words, v is the solution to the Dirichlet problem

$$\limsup_{(z,w)\to\partial\Omega_2} v(z, w) = |w|^2, \qquad (dd^c v)^2 = 0.$$

By uniqueness we have $v(z, w) = |w|^2$. Hence, $v(0, 1/2) = 1/4 > 0 = u(0, 1/2)$, which is impossible.

The following example shows that there exists a function in $\mathcal{N} \setminus \mathcal{F}$ that cannot be subextended.

Example 7.22 ([57, 83]). Let Ω be a bounded hyperconvex domain in $\mathbb{C}^n$ and let $z \in \Omega$. Recall that the pluricomplex Green function with the pole at z is defined as follows:

$$g_\Omega(z, w) = \sup\big\{u(w) : u \in \mathcal{PSH}(\Omega),\ u \leq 0,\ \big|u(\xi) - \log|\xi - z|\big| \leq C \text{ near } z\big\}.$$

It is well known that $g_\Omega(z, \cdot) \in \mathcal{PSH}(\Omega) \cap \mathcal{C}(\Omega \setminus \{z\})$, $g_\Omega(z, w) = 0$ for $w \in \partial\Omega$ and $(dd^c g_\Omega(z, \cdot))^n = (2\pi)^n \delta_z$, where δ_z is the Dirac measure at z (see [62]). Carlehed, Cegrell and Wikstöm proved in [25] that for every $z_0 \in \partial\Omega$ there exists a pluripolar set $E \subset \Omega$ such that

$$\limsup_{z\to z_0} g_\Omega(z, w) = 0$$

for every $w \in \Omega \setminus E$. Take $p \notin E$. Then choose $\{w_j\} \subset \Omega$ such that $g(p, w_j) > -1/j^3$ and $w_j \to w \in \partial\Omega$. Let u be the function defined on Ω by

$$u(z) = \sum_{j=1}^{\infty} j\, g(z, w_j).$$

Note that $u \in \mathcal{PSH}^-(\Omega)$ since $u(p) > -\infty$. Proposition 4.8 yields $u \in \mathcal{N}(\Omega)$. Recall also the definition of the Lelong number:

$$\nu(u, z_0) = \lim_{r\to 0} \frac{1}{(2\pi)^n} \int_{B(z_0,r)} dd^c u \wedge (dd^c \log|z - z_0|)^{n-1}.$$

It is well known that $\nu(u, z_0)$ is finite for any plurisubharmonic function.

Now suppose that v is a subextension of u to a domain $\Omega' \supset \Omega$ and $w \in \Omega'$. Since $v \le u$, we have $\nu(v, w_j) \ge \nu(u, w_j) \ge j$ and therefore $\nu(v, w) = \infty$, a contradiction which means that u cannot be subextended to a larger domain.

8. The homogeneous Dirichlet problem for pluriharmonic functions

Pluriharmonic functions, as locally real parts of holomorphic functions, play an important role in complex analysis. It is well known that for a continuous function $f : \partial\Omega \to \mathbb{R}$ there does not always exist a pluriharmonic function u which is continuous on $\bar{\Omega}$ such that $u|_{\partial\Omega} = f$. This Dirichlet problem has been extensively studied for the case of smoothly bounded domains, like the unit ball, strictly pseudoconvex domains or the unit polydisc. Bedford [10] and Bedford and Federbush [15] proved that on the smooth boundary of a strictly pseudoconvex domain any smooth function satisfying a certain tangential equation can be extended inside the domain to a pluriharmonic function (see also [77]). In [76] Rudin proved that a continuous function on the boundary of a polydisc with some of its Fourier coefficients vanishing can be extended inside a polydisc to a pluriharmonic function. We shall give a complete characterization of this Dirichlet problem in a domain which is a proper image of bounded hyperconvex product domains (Theorem 8.1). Nearly all results in this chapter were proved by the author in [45].

The aim of this chapter is the following theorem:

Theorem 8.1 ([45]). *Let D_j be a bounded hyperconvex domain in $\mathbb{C}^{n_j}$, $n_j \ge 1$. Set $D = D_1 \times \cdots \times D_s$, $j = 1, \ldots, s$, $s \ge 3$. Moreover, let U be an open neighborhood of $\bar{D}$, let $\pi : U \to \mathbb{C}^n$, $n = n_1 + \cdots + n_s$, be a proper holomorphic map and let $\Omega_\pi = \pi(D)$. If $f : \partial\Omega_\pi \to \mathbb{R}$, $n \ge 3$, is a continuous function, then the following assertions are equivalent:*

(1) *there exists a function h that is pluriharmonic on Ω_π, continuous on $\bar{\Omega}_\pi$ and $h|_{\partial\Omega_\pi} = f$,*
(2) *f is pluriharmonic on $\partial\Omega_\pi$ in the sense of Definition* 8.8,
(3) *the Perron–Bremermann envelope PB_f is pluriharmonic on Ω_π and*

$$PB_{-f} = -PB_f,$$

(4) *f is a compliant function* (*see Definition* 8.2),
(5)

$$\lim_{\Omega_\pi \ni z \to \xi} (PB_f + PB_{-f})(z) = 0 \quad \text{for every } \xi \in \partial\Omega_\pi, \tag{8.1}$$

(6) *for every* $z_0 \in \partial\Omega_\pi$ *and every Jensen measure* $\mu \in \mathcal{J}_{z_0}$ *with barycenter* z_0 *we have*

$$f(z_0) = \int_{\partial\Omega_\pi} f\, d\mu.$$

Definition 8.2 ([2, 30]). Let $f : \partial\Omega \to \mathbb{R}$ be a continuous function such that

$$\lim_{\substack{z\to\xi \\ z\in\Omega}} PB_f(z) = f(\xi)$$

for every $\xi \in \partial\Omega$. If

$$PB_f + PB_{-f} \in \mathcal{E}_0$$

then the function f is said to be *compliant.*

The notion of compliant functions was first introduced in [2].

Remark 8.3. It was proved in [6] that the compliant functions form a linear space. Moreover, if f, g are compliaint functions then

$$\begin{aligned} &PB_{f+g} + PB_{-f-g} \geq PB_f + PB_{-f} + PB_g + PB_{-g}, \\ &PB_{tf} = |t| PB_f, \end{aligned} \tag{8.2}$$

where $t \in \mathbb{R}$.

Proposition 8.4 ([6]). *Let* $\mathcal{K} \in \{\mathcal{E}_0, \mathcal{F}\}$ *and let* f *be a compliant function. If* $u \in \mathcal{K}(f)$, *then*

$$\int_\Omega (dd^c u)^n < \infty,$$

and $u + PB_{-f} \in \mathcal{K}$.

Proof. Since f is a compliant function, $PB_f + PB_{-f} \in \mathcal{E}_0$. If $u \in \mathcal{K}(f)$ then there exists $v \in \mathcal{K}$ such that $PB_f \geq u \geq v + PB_f$, so $u + PB_{-f} \geq v + PB_f + PB_{-f}$, so $u + PB_{-f} \in \mathcal{K}$. Moreover,

$$\int_\Omega (dd^c u)^n \leq \int_\Omega (dd^c (u + PB_{-f}))^n < \infty.$$

This ends the proof. ∎

Proposition 8.5 ([6]). *If* $D \subseteq \mathbb{C}^n$ *is a bounded, strictly pseudoconvex domain with* C^2-*boundary and* $f \in C^2(\partial D)$, *then* f *is a compliant function.*

Proof. The domain D is in particular B-regular, the function f is continuous and therefore $PB_f + PB_{-f} = 0$ on the boundary ∂D. There exists an open neighborhood U of D and a strictly plurisubharmonic C^2-function $\rho : U \to \mathbb{R}$ such that $\rho = 0$ on ∂D, since D is a strictly pseudoconvex domain with C^2-boundary. By Theorem I in [82] there exists a C^2-function $\hat{f} : \mathbb{C}^n \to \mathbb{R}$ such that $\hat{f} = f$ on ∂D. Choose $A > 0$ such that $u = \hat{f} + A\rho \in \mathcal{PSH}(D)$ and $B > 0$ such that $v = -\hat{f} + B\rho \in \mathcal{PSH}(D)$. Hence, $u, v \in \mathcal{PSH}(U) \cap C^2(U)$, $u = -v = f$ on ∂D. Thus

$$\int_D (dd^c(u+v))^n = \int_D \sum_{k=0}^n \binom{n}{k} (dd^c u)^{n-k} \wedge (dd^c v)^k < \infty. \tag{8.3}$$

The construction of PB_f and PB_{-f} implies that $u+v \le PB_f + PB_{-f}$, hence

$$\int_D (dd^c(PB_f + PB_{-f}))^n \le \int_D (dd^c(u+v))^n < \infty,$$

by (8.3) and Theorem 2.2. This ends the proof. ■

Example 8.6 shows that for any $\alpha < 1$ there exists $f \in \mathcal{C}^{1,\alpha}(\partial\mathbb{B})$ such that f is not a compliant function. Recall that $f \in \mathcal{C}^{k,\alpha}$, $0 < \alpha \le 1$, if $f \in \mathcal{C}^k$ and for any $|\beta| = k$ function $D^\beta f$ is α-Hölder continuous.

Example 8.6 ([6]). Let $\mathbb{B} \subseteq \mathbb{C}^n$ be the unit ball and $z = (z', z_n) \in \mathbb{B}$. For fixed $0 < p < 1$, let $f_p : \partial\mathbb{B} \to \mathbb{R}$ be defined by $f_p(z', z_n) = |z_n|^{2p}$. Then

$$PB_{f_p}(z', z_n) = |z_n|^{2p} \quad \text{and} \quad PB_{-f_p}(z', z_n) = -(1-|z'|^2)^p.$$

Set $A = \{(z', z_n) \in \mathbb{B} : z_n = 0\} \cup \{(z', z_n) \in \mathbb{B} : |z'| = 1\}$. The set A is pluripolar, hence

$$\begin{aligned}\int_{\mathbb{B}} (dd^c(PB_{f_p} + PB_{-f_p}))^n &= \int_{\mathbb{B}\setminus A} (dd^c(PB_{f_p} + PB_{-f_p}))^n \\ &= C\int_0^1 r^{2np-2n+1}(1-r^2)^{n-1}dr,\end{aligned}$$

where $C > 0$ is a constant only depending on n and p. Thus f_p is not compliant if, and only if, $p \le (n-1)/n$. For $n > 2$, the function $f_{(n-1)/n}$ belongs to $C^{1,1-2/n}(\partial\mathbb{B})$.

Let D_j be a bounded hyperconvex domain in $\mathbb{C}^{n_j}$, $n_j \ge 1$ and set

$$D = D_1 \times \cdots \times D_s \subseteq \mathbb{C}^n,$$

where $n = n_1 + \cdots + n_s$. For an open neighborhood U of $\bar{D}$ and a proper holomorphic map $\pi : U \to \mathbb{C}^n$ we use the notation $\Omega_\pi = \pi(D)$.

Let $I_k = (j_1, \ldots, j_k)$ be an increasing multi-index of length k: $1 \le j_1 < \cdots < j_k \le s$, where $1 \le k \le s$. Denote

$$\Lambda^{I_k} = D_1 \times \cdots \times \overbrace{\partial D_{j_1}}^{j_1} \times \cdots \times \overbrace{\partial D_{j_k}}^{j_k} \times \cdots \times D_s \quad \text{and} \quad \Lambda_\pi^{I_k} = \pi(\Lambda^{I_k}).$$

Hence,

$$\partial D = \bigcup_{I_k} \Lambda^{I_k} \quad \text{and} \quad \partial\Omega_\pi = \pi(\partial D) = \bigcup_{I_k} \Lambda_\pi^{I_k}.$$

Finally, let the distinguished boundary of D be denoted by ∂D^+, i.e.

$$\partial D^+ = \partial D_1 \times \cdots \times \partial D_s.$$

Proposition 8.7 ([45]). *The domain Ω_π is hyperconvex.*

Proof. For every j, $1 \le j \le s$, the set D_j is hyperconvex in $\mathbb{C}^{n_j}$ and φ_j is an exhaustion function for D_j. Let u be the function defined on D by

$$u(\zeta_1, \ldots, \zeta_s) = \max\{\varphi_1(\zeta_1), \ldots, \varphi_s(\zeta_s)\},$$

hence u is a plurisubharmonic exhaustion function for the set D in $\mathbb{C}^n$. Let us now define

$$\varphi(w) = \max\{u(z) : z \in \pi^{-1}(w)\}.$$

From [61] it follows that φ is a plurisubharmonic exhaustion function for Ω_π. Thus, Ω_π is hyperconvex. ■

DEFINITION 8.8 ([8, 45]). A continuous function $u : \partial\Omega_\pi \to \mathbb{R}$ is *pluriharmonic* if u is pluriharmonic on every $\Lambda_\pi^{I_k}$ for every increasing multi-index I_k of length $k \in \{1, \dots, s-1\}$ i.e., for every $1 \le j_1 < \dots < j_k \le s$, $k \in \{1, \dots, s-1\}$ and for every $w_{j_1} \in \partial D_{j_1}, \dots, w_{j_k} \in \partial D_{j_k}$, the function defined by

$$u_{w_{j_1},\dots,w_{j_k}} : (z_1, \dots, z_{s-k}) \mapsto u \circ \pi(z_1, \dots, w_{j_1}, \dots, w_{j_k}, \dots, z_{s-k}) \tag{8.4}$$

is pluriharmonic on the set

$$D_{I_k} = D_1 \times \dots \times \widehat{\partial D_{j_1}} \times \dots \times \widehat{\partial D_{j_k}} \times \dots \times D_s.$$

In a similar manner an upper semicontinuous function $u : \partial\Omega_\pi \to \mathbb{R} \cup \{-\infty\}$ is *plurisubharmonic* if u is plurisubharmonic on every $\Lambda_\pi^{I_k}$ for every increasing multi-index I_k of length $k \in \{1, \dots, s-1\}$. The identically $-\infty$ function is by fiat not considered to be plurisubharmonic.

REMARK 8.9. Note that, if we take $\pi = \mathrm{id}_D$ in Definition 8.8, then a continuous function u is pluriharmonic on ∂D if for every increasing multi-index I_k the restriction of u to Λ^{I_k} is pluriharmonic.

DEFINITION 8.10. Let $\Omega \subseteq \mathbb{C}^n$ be a bounded domain and let μ be a non-negative, regular Borel measure on $\bar{\Omega}$. The measure μ is a *Jensen measure with barycenter at* $z \in \bar{\Omega}$ for continuous plurisubharmonic functions if

$$u(z) \le \int_{\bar{\Omega}} u \, d\mu$$

for every continuous function $u \in \mathcal{PSH}(\Omega)$. The set of all Jensen measures with barycenter at z for continuous plurisubharmonic functions will be denoted by $\mathcal{J}_z$.

It is clear that $\{\delta_z\} \subset \mathcal{J}_z$, where δ_z denotes the Dirac measure at z. If Ω is hyperconvex domain, then $\operatorname{supp}\mu \subset \partial\Omega$ for all $z \in \partial\Omega$ and all $\mu \in \mathcal{J}_z$ (see [25]).

LEMMA 8.11 ([45]). *Let D be a bounded hyperconvex domain in $\mathbb{C}^n$, and let U be an open neighborhood of $\bar{D}$. Let $\pi : U \to \mathbb{C}^n$ be a proper holomorphic map and let $\Omega_\pi = \pi(D)$. If $f : \partial\Omega_\pi \to \mathbb{R}$ is a continuous function such that $PB_{f\circ\pi} \in \mathcal{PSH}(D) \cap \mathcal{C}(\bar{D})$, $PB_{f\circ\pi} = f \circ \pi$ on ∂D, then*

$$PB_{f\circ\pi} = PB_f \circ \pi.$$

Furthermore, $PB_{f\circ\pi}$ is pluriharmonic in D if, and only if, PB_f is pluriharmonic on Ω_π.

Proof. We start by defining $g = f \circ \pi : \partial D \to \mathbb{R}$. By our assumption the Perron–Bremermann envelope PB_g is plurisubharmonic on D and continuous on $\bar{D}$. Define

$$\varphi(w) = \max\{PB_g(z) : z \in \pi^{-1}(w)\}.$$

From [61] it follows that $\varphi \in \mathcal{PSH}(\Omega_\pi) \cap \mathcal{C}(\Omega_\pi)$. We prove that $\varphi|_{\partial\Omega_\pi} = f$. Let $\Omega_\pi \ni w_j \to w \in \partial\Omega_\pi$. Then there exist finitely many $z_j^1, \dots, z_j^{k_j} \in \pi^{-1}(w_j)$. Take $z_j^{l_j}$ such that $\varphi(w_j) = PB_g(z_j^{l_j})$. Since π is a proper map, we have $z_j^{l_j} \to z_0 \in \partial D$ and then $\varphi(w_j) = PB_g(z_j^{l_j}) \to PB_g(z_0) = g(z_0) = f(\pi(z_0)) = f(w_0)$.

Hence $\varphi \leq PB_f \in \mathcal{PSH}(\Omega_\pi) \cap \mathcal{C}(\bar{\Omega}_\pi)$, by the Walsh theorem (Theorem 2.15). Therefore $PB_f \circ \pi \in \mathcal{PSH}(D)$ and $(PB_f \circ \pi)|_{\partial D} = g$. Thus, for $z \in \pi^{-1}(w)$ we get $(PB_f \circ \pi)(z) \leq PB_g(z)$ and then $PB_f(w) \leq \varphi(w)$, which implies that $\varphi = PB_f$. Therefore

$$PB_{f \circ \pi} = PB_f \circ \pi,$$

since both functions are maximal with the same boundary values g.

Now we prove the second part of Lemma 8.11. From the first part it is clear that if PB_f is pluriharmonic on Ω_π, then $PB_{f\circ\pi}$ is pluriharmonic on D.

Now assume that PB_g is pluriharmonic on D. Note that $PB_g = -PB_{-g}$, since PB_g is pluriharmonic on D and continuous on $\bar{D}$ and by the first part of the proof,

$$\begin{aligned} PB_f(w) &= \max\{PB_g(z) : z \in \pi^{-1}(w)\} \\ &= \max\{-PB_{-g}(z) : z \in \pi^{-1}(w)\} \\ &= -\min\{PB_{-g}(z) : z \in \pi^{-1}(w)\}. \end{aligned}$$

In a similar manner we get $PB_{-f}(w) = \max\{PB_{-g}(z) : z \in \pi^{-1}(w)\}$. Combining these two representations we obtain

$$\begin{aligned} 0 \geq PB_f + PB_{-f} = {} & \max\{PB_{-g}(z) : z \in \pi^{-1}(w)\} \\ & - \min\{PB_{-g}(z) : z \in \pi^{-1}(w)\} \geq 0, \end{aligned}$$

and so $PB_f = -PB_{-f}$, which means that PB_f is pluriharmonic. ∎

Proof of Theorem 8.1. The following implications are straightforward: (3)⇒(1), (3)⇒(4), and (4)⇒(5).

(1)⇒(2): Let I_k be an increasing multi-index of length $k \in \{1, \ldots, s-1\}$, and let $w_{j_1} \in \partial D_{j_1}, \ldots, w_{j_k} \in \partial D_{j_k}$. Let $f_{w_{j_1},\ldots,w_{j_k}} : D_{I_k} \to \mathbb{R} \cup \{-\infty\}$ be defined as in (8.4). We need to prove that this function is pluriharmonic under the assumption that there exists $u \in \mathcal{PSH}(\Omega_\pi) \cap \mathcal{C}(\bar{\Omega}_\pi)$ such that $u|_{\partial\Omega_\pi} = f$. Take a sequence $[(w_{j_1}^m, \ldots, w_{j_k}^m)]_{m=1}^\infty$ in $D_{j_1} \times \cdots \times D_{j_k}$ which converges to $(w_{j_1}, \ldots, w_{j_k})$ as $m \to \infty$. Moreover, let $[u_m]$ be the sequence of real-valued functions on D_{I_k} defined by

$$u_m(z_1, \ldots, z_{s-k}) = u \circ \pi(z_1, \ldots w_{j_1}^m, \ldots, w_{j_k}^m, \ldots, z_{s-k}).$$

This construction implies that u_m is pluriharmonic on D_{I_k}, and continuous up to the boundary. The sequence $[u_m]$ converges uniformly to $f_{w_{j_1},\ldots,w_{j_k}}$ on D_{I_k} as $m \to \infty$, and hence f is pluriharmonic in the sense of Definition 8.8.

(5)⇒(6): First we prove that assumption (8.1) implies that

$$\lim_{\substack{z\to\zeta \\ z\in\Omega_\pi}} PB_f(z) = f(\zeta) \quad \text{and} \quad \lim_{\substack{z\to\xi \\ z\in\Omega_\pi}} PB_{-f}(z) = -f(\xi)$$

for every $\zeta, \xi \in \partial\Omega_\pi$. Assume that this is not the case, for example there exists a $\xi \in \partial\Omega_\pi$ such that $\limsup_{z\to\xi} PB_f(z) < f(\xi)$. This assumption yields

$$\begin{aligned} 0 = \lim_{\substack{z\to\xi \\ z\in\Omega_\pi}} (PB_f + PB_{-f})(z) &= \limsup_{\substack{z\to\xi \\ z\in\Omega_\pi}} (PB_f + PB_{-f})(z) \\ &\leq \limsup_{\substack{z\to\xi \\ z\in\Omega_\pi}} PB_f(z) + \limsup_{\substack{z\to\xi \\ z\in\Omega_\pi}} PB_{-f}(z) < f(\xi) - f(\xi) = 0, \end{aligned}$$

a contradiction, hence $\limsup PB_f = f$ and $\limsup PB_{-f} = -f$ on $\partial\Omega$. Assume now that there exists $\zeta \in \partial\Omega_\pi$ such that $\liminf_{z\to\zeta} PB_f(z) < f(\zeta)$. Then there exists a sequence $[z_j]$ in Ω_π which converges to ζ such that $\lim_{j\to\infty} PB_f(z_j) < f(\zeta)$, hence

$$\begin{aligned} 0 &= \lim_{j\to\infty} (PB_f + PB_{-f})(z_j) = \liminf_{j\to\infty} (PB_f + PB_{-f})(z_j) \\ &= \lim_{j\to\infty} PB_f(z_j) + \liminf_{j\to\infty} PB_{-f}(z_j) < f(\zeta) - f(\zeta) = 0, \end{aligned}$$

a contradiction once again. Now it follows by Walsh's theorem (Theorem 2.15) that $PB_f, PB_{-f} \in \mathcal{C}(\bar{\Omega}_\pi)$. Fix $z_0 \in \partial\Omega_\pi$ and take $\mu \in \mathcal{J}_{z_0}$. Then

$$f(z_0) = PB_f(z_0) \le \int_{\bar{\Omega}_\pi} PB_f\, d\mu.$$

Thus

$$f(z_0) \le \inf\Big\{\int_{\bar{\Omega}_\pi} PB_f\, d\mu : \mu \in \mathcal{J}_{z_0}\Big\}.$$

If $\mu = \delta_{z_0}$, then we obtain

$$f(z_0) = \inf\Big\{\int_{\bar{\Omega}_\pi} PB_f\, d\mu : \mu \in \mathcal{J}_{z_0}\Big\}.$$

In a similar manner the corresponding formula for $-f$ can be obtained, and therefore

$$\sup\Big\{\int_{\bar{\Omega}_\pi} -PB_{-f}\, d\mu : \mu \in \mathcal{J}_{z_0}\Big\} = -\inf\Big\{\int_{\bar{\Omega}_\pi} PB_{-f}\, d\mu : \mu \in \mathcal{J}_{z_0}\Big\} = f(z_0).$$

The maximum principle for plurisubharmonic functions and assumption (8.1) yield

$$\inf\Big\{\int_{\bar{\Omega}_\pi} PB_f\, d\mu : \mu \in \mathcal{J}_{z_0}\Big\} = f(z_0) \ge \sup\Big\{\int_{\bar{\Omega}_\pi} PB_f\, d\mu : \mu \in \mathcal{J}_{z_0}\Big\}.$$

Thus, for every $z_0 \in \partial\Omega$ and every $\mu \in \mathcal{J}_{z_0}$ we have

$$f(z_0) = \int_{\bar{\Omega}_\pi} PB_f\, d\mu = \int_{\partial\Omega_\pi} f\, d\mu$$

since Ω_π is hyperconvex and $PB_f = f$ on the boundary $\partial\Omega_\pi$.

(6)$\Rightarrow$(2): Let I_k be an increasing multi-index $1 \le j_1 < \cdots < j_k \le s$ of length $k \in \{1, \ldots, s-1\}$ and $z_0 \in \Lambda^{I_k}$. Take any complex line l through z_0 and $r > 0$ such that $z_0 + \mathbb{D}_r \subset l \cap \Lambda^{I_k}$, where $\mathbb{D}_r = \{z \in \mathbb{C} : |z| < r\}$. Since the Lebesgue measure dV_1 on the unit disc $\mathbb{D}$ is a Jensen measure at z_0, the measure $\mu_\pi(A) = V_1(\pi^{-1}(A))$, where $A \subset z_0 + r\mathbb{D}$, is a Jensen measure at $\pi(z_0)$. We have by assumption

$$f(\pi(z_0)) = \int_{\pi(z_0+\mathbb{D}_r)} f d\mu_\pi = \int_{z_0+\mathbb{D}_r} f \circ \pi\, dV_1,$$

which implies that f is harmonic on $\pi(z_0 + \mathbb{D}_r)$ and therefore f is pluriharmonic on $\partial\Omega_\pi$.

(2)$\Rightarrow$(3): Let $g = f \circ \pi : \partial D \to \mathbb{R}$. This definition implies that g is a pluriharmonic on ∂D and we will prove that PB_g pluriharmonic on D, continuous on $\bar{D}$ and $PB_g = g$ on ∂D. Therefore Lemma 8.11 will finish the proof.

Assume that $g : \partial D \to \mathbb{R}$ is a continuous function and let u be defined by

$$u(z_1, \ldots, z_s) = \int_{\partial D^+} g(t_1, \ldots, t_s)\, d\omega_{z_1}(t_1) \cdots d\omega_{z_s}(t_s),$$

where ω_{z_j} is the harmonic measure relative D_j and z_j. The function u is s-harmonic on D, continuous on $\bar{D}$ and $u|_{\partial D} = g$. We will now show that u is pluriharmonic on D. Let $z_0 = (z_1, \dots, z_s) \in D$, $X = (X_1, \dots, X_s) \in \mathbb{C}^n$ ($X_j \in \mathbb{C}^{n_j}$) be such that $X_s = 0$ and choose $r > 0$ such that $\{z_0 + \zeta X : \zeta \in \mathbb{C}, |\zeta| < r\} = z_0 + X\mathbb{D}_r \subseteq D$. For every $w_1 \in D_1, \dots, w_{s-1} \in D_{s-1}, w_s \in \partial D_s$, $w' = (w_1, \dots, w_{s-1})$, $X' = (X_1, \dots, X_{s-1})$, $\zeta \in B_r$, $t' = (t_1, \dots, t_{s-1})$, where $t_j \in D_j$, $1 \le j \le s-1$ denote $d\omega'_{w'+\zeta X'}(t') = d\omega_{w_1+\zeta X_1}(t_1) \cdots d\omega_{w_{s-1}+\zeta X_{s-1}}(t_{s-1})$. The assumption that g is pluriharmonic in the sense of Definition 8.8 implies in particular that g is pluriharmonic on $D_1 \times \cdots \times D_{s-1} \times \{w_s\}$, hence

$$\frac{1}{\pi r^2} \int_{\mathbb{D}_r} g(w' + \zeta X', w_s)\, d\lambda(\zeta) = g(w', w_s) = \int_{\partial D^+} g\, d\omega_{w_1} \cdots d\omega_{w_s},$$

and

$$g(w' + \zeta X', w_s) = \int_{\partial D_1 \times \cdots \times \partial D_{s-1}} g(t', w_s)\, d\omega'_{w'+\zeta X'}$$

Therefore,

$$\begin{aligned}
\frac{1}{\pi r^2} \int_{\mathbb{D}_r} u(z_0 + \zeta X)\, d\lambda(\zeta) &= \frac{1}{\pi r^2} \int_{\mathbb{D}_r} \int_{\partial D^+} g(t', t_s)\, d\omega'_{z'+\zeta X'}(t') d\omega_{z_s}(t_s)\, d\lambda(\zeta) \\
&= \int_{\partial D_s} \frac{1}{\pi r^2} \int_{\mathbb{D}_r} \int_{\partial D_1 \times \cdots \times \partial D_{s-1}} g(t', t_s)\, d\omega'_{z'+\zeta X'}(t') d\lambda(\zeta) d\omega_{z_s}(t_s) \\
&= \int_{\partial D_s} \frac{1}{\pi r^2} \int_{\mathbb{D}_r} g(z' + \zeta X', t_s) d\lambda(\zeta) d\omega_{z_s}(t_s) = \int_{\partial D_s} g(z', t_s) d\omega_{z_s}(t_s) \\
&= \int_{\partial D_1 \times \cdots \times \partial D_s} g(t_1, \dots, t_s)\, d\omega_{z_1}(t_1) \cdots d\omega_{z_s}(t_s) = u(z_0),
\end{aligned}$$

which proves that u is pluriharmonic on $D_1 \times \cdots \times D_{s-1} \times \{z_s\}$ for all $z_s \in D_s$. By repeating the same argument for $X \in \mathbb{C}^n$ such that $X_k = 0$, $1 \le k \le s-1$, we reach the conclusion that for each k fixed, $1 \le k \le s$, the function u is pluriharmonic on the set

$$D_1 \times \cdots \times \{z_k\} \times \cdots \times D_s \subseteq D$$

for all $z_k \in D_k$. This means that u is pluriharmonic on D since the Levi form of u is

$$\begin{aligned}
&\mathcal{L}u(z_1, \dots, z_s)(X_1, \dots, X_s) \\
&\quad = \frac{1}{s-2} \sum_{j=1}^{s} \mathcal{L}(u \circ \rho_j)(z_1, \dots, z_{j-1}, z_{j+1}, \dots, z_s)(X_1, \dots, X_{j-1}, X_{j+1}, \dots, X_s) \\
&\qquad - \frac{1}{s-2} \sum_{k=1}^{n_1+\cdots+n_s} \frac{\partial^2 u}{\partial z_k \partial \bar{z}_k}(z_1, \dots, z_s)|X_k|^2 = 0,
\end{aligned}$$

where $(z_1, \dots, z_s) \in D$, $X = (X_1, \dots, X_s) \in \mathbb{C}^n$, $X_j \in \mathbb{C}^{n_j}$, $\rho_j(z_1, \dots, \widehat{z_j}, \dots, z_s) = (z_1, \dots, z_j, \dots, z_s)$, $X_j \in \mathbb{C}^{n_j}$ and $s \ge 3$.

Since u is pluriharmonic on D, continuous on $\bar{D}$ and $u|_{\partial D} = f$, we have $(dd^c u)^n = 0$ and therefore $u = PB_f$. By the same arguments we see that the function v defined by

$$v(z_1, \dots, z_s) = \int_{\partial D^+} (-g(t_1, \dots, t_s))\, d\omega_{z_1}(t_1) \cdots d\omega_{z_s}(t_s)$$

is pluriharmonic on D, continuous on $\bar{D}$ and $v|_{\partial D} = -f$. Thus $v = PB_{-f}$, which implies that $PB_{-f} = -PB_f$ on D, by the construction of u and v. ■

Remark 8.12. Similarly to the proof of Theorem 8.1 the author proved in [45] that a continuous function $f : \partial\Omega_\pi \to \mathbb{R}$ can be extended inside the domain $\Omega_\pi := \pi(D)$ to a plurisubharmonic function continuous up to the boundary if and only if f is plurisubharmonic in the sense of Definition 8.8. Here $\pi : U \to \mathbb{C}^n$ is a proper holomorphic map, U is an open neighborhood of the closure of a bounded hyperconvex domain $D = D_1 \times \cdots \times D_n$ in $\mathbb{C}^n$, $D_j \Subset \mathbb{C}$, $n \geq 2$.

Example 8.13 ([8]). Let $\mathbb{D}$ be the open unit disc in $\mathbb{C}$. If we take $D_j = \mathbb{D}$ and $\pi = \mathrm{id}$, then $\Omega_\pi = \mathbb{D}^n$.

Let $\pi_n = (\pi_{n,1}, \ldots, \pi_{n,n}) : \mathbb{C}^n \to \mathbb{C}^n$, $n \geq 1$, be defined as follows:

$$\pi_{n,k}(z_1, \ldots, z_n) = \sum_{1 \leq j_1 < \cdots < j_k \leq n} z_{j_1} \cdots z_{j_k}$$

for $1 \leq k \leq n$. Then π_n is a proper holomorphic mapping and so too is $\pi_n|_{\mathbb{D}^n} : \mathbb{D}^n \to \pi_n(\mathbb{D}^n)$. Then $\mathbb{G}_n = \pi_n(\mathbb{D}^n)$, where the domain $\mathbb{G}_n$ is the so called *symmetrized polydisc* (see e.g. [1, 42, 43, 53, 54]).

Example 8.14 below shows that in the case when $n = 2$ the implication from (2) to (3) in Theorem 8.1 is, in general, not true.

Example 8.14 ([73]). Let $\mathbb{D}^2 = \{(z, w) \in \mathbb{C}^2 : |z| < 1,\ |w| < 1\}$ be the unit polydisc in $\mathbb{C}^2$ and let $f : \partial\mathbb{D}^2 \to \mathbb{R}$ be defined by

$$f(\zeta, \xi) = \mathrm{Re}(\zeta\bar{\xi}).$$

The function f is pluriharmonic on $\partial\mathbb{D}^2$ in the sense of Definition 8.8; we will prove that condition (3) in Theorem 8.1 is not true for f.

Let $\phi : \mathbb{D} \to \mathbb{D}^2$ be an analytic disc (i.e. an injective holomorphic map) with $\phi(0) = (a_1, a_2)$. If $v \in \mathcal{PSH}(\mathbb{D}^2) \cap \mathcal{C}(\overline{\mathbb{D}^2})$ satisfies $v = f$ on $\partial\mathbb{D}^2$, then

$$v(a_1, a_2) \leq \frac{1}{2\pi} \int_0^{2\pi} f(\phi(e^{i\theta}))\, d\theta.$$

Furthermore, using a theorem by Poletsky ([72]) we have

$$\begin{aligned} PB_f(z) &= \sup\{v(z) : u \in \mathcal{PSH}(\mathbb{D}^2), \limsup_{\xi \to w} v(\xi) \leq f(w), \forall w \in \partial\mathbb{D}^2\} \\ &= \inf\left\{ \frac{1}{2\pi} \int_0^{2\pi} f(\phi(e^{i\theta}))d\theta : \phi \text{ an analytic disc, } \phi(0) = z \right\}. \end{aligned}$$

We restrict our considerations to analytic discs of the form $\phi(\zeta) = (B_1(\zeta), B_2(\zeta))$, where

$$B_j(\zeta) = \frac{c_j\zeta + a_j}{1 + \bar{a}_j c_j \zeta}, \qquad j = 1, 2,$$

and $a_1, a_2 \in \mathbb{D}$, $|c_1| = |c_2| = 1$. Note that $\phi(0) = (a_1, a_2)$. By the classical residue theorem,

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} f(\phi(e^{i\theta}))\,d\theta &= \frac{1}{2\pi}\operatorname{Re}\Big(\int_0^{2\pi} B_1(e^{i\theta})\overline{B_2(e^{i\theta})}\,d\theta\Big)\\ &= \frac{1}{2\pi}\operatorname{Re}\int_0^{2\pi}\frac{c_1e^{i\theta}+a_1}{1+\bar a_1c_1e^{i\theta}}\,\frac{1+\bar a_2c_2e^{i\theta}}{c_2e^{i\theta}+a_2}\,d\theta\\ &= \operatorname{Re}\bigg(\frac{1}{2\pi i}\int_{|\zeta|=1}\frac{(c_1\zeta+a_1)(1+\bar a_2c_2\zeta)}{(1+\bar a_1c_1\zeta)(c_2\zeta+a_2)\zeta}\,d\zeta\bigg)\\ &= \operatorname{Re}\bigg(\frac{a_1}{a_2}+\frac{(c_1(\frac{-a_2}{c_2})+a_1)(1+\bar a_2c_2(\frac{-a_2}{c_2}))}{(1+\bar a_1c_1(\frac{-a_2}{c_2}))c_2(\frac{-a_2}{c_2})}\bigg)\\ &= \operatorname{Re}\bigg(\frac{c_1(1-|a_1|^2-|a_2|^2)+a_1\bar a_2c_2}{c_2-\bar a_1a_2c_1}\bigg)=\operatorname{Re}\Big(\frac{g\alpha+k}{\alpha-h}\Big),\end{aligned}$$

where $\alpha=c_2/c_1$, $k=1-|a_1|^2-|a_2|^2$, $g=a_1\bar a_2$ and $h=\bar a_1a_2$. We want to calculate

$$\inf_{|c_1|=|c_2|=1}\frac{1}{2\pi}\int_0^{2\pi} f(\phi(e^{i\theta}))\,d\theta=\inf_{|\alpha|=1}\operatorname{Re}\Big(\frac{g\alpha+k}{\alpha-h}\Big).$$

Note that the function

$$\Psi(\alpha)=\frac{g\alpha+k}{\alpha-h}$$

maps $\partial\mathbb{D}$ to $\partial\mathbb{D}(w_0,r)$, where

$$\begin{aligned}w_0&=\frac{g+k\bar h}{1-|h|^2}=\frac{a_1\bar a_2(2-|a_1|^2-|a_2|^2)}{1-|a_1a_2|^2},\\ r&=\frac{|k+gh|}{1-|h|^2}=\frac{(1-|a_1|^2)(1-|a_2|^2)}{1-|a_1a_2|^2}.\end{aligned}$$

Hence

$$\begin{aligned}\inf_{|\alpha|=1}\operatorname{Re}\Big(\frac{g\alpha+k}{\alpha-h}\Big)&=\operatorname{Re}(w_0)-r\\ &=\frac{\operatorname{Re}(a_1\bar a_2)(2-|a_1|^2-|a_2|^2)}{1-|a_1a_2|^2}-\frac{(1-|a_1|^2)(1-|a_2|^2)}{1-|a_1a_2|^2}=u(a_1,a_2).\end{aligned}$$

We will show that $u\in\mathcal{PSH}(\mathbb{D}^2)\cap\mathcal{C}(\overline{\mathbb{D}^2})$, $u(z,w)=\operatorname{Re}(z,w)$ on $\partial\mathbb{D}^2$ and $(dd^cu)^2=0$, which will imply that $u=PB_f$. It is clear that $u\in\mathcal{C}^\infty(\mathbb{D}^2)\cap\mathcal{C}(\overline{\mathbb{D}^2})$, and

$$\lim_{(z,w)\to(\zeta,\xi)}u(z,w)=\operatorname{Re}(\zeta\bar\xi)\quad\text{ for every }(\zeta,\xi)\in\partial\mathbb{D}^2.$$

We have

$$\begin{aligned}u_{z\bar z}(z,w)&=\frac{\partial^2u}{\partial z\partial\bar z}(z,w)=\frac{(1-|w|^2)^2|1-z\bar w|^2}{(1-|z|^2|w|^2)^3},\\ u_{z\bar w}(z,w)&=\frac{\partial^2u}{\partial z\partial\bar w}(z,w)=\frac{(1-|z|^2)(1-|w|^2)(1-\bar zw)^2}{(1-|z|^2|w|^2)^3},\\ u_{\bar zw}(z,w)&=\frac{\partial^2u}{\partial\bar z\partial w}(z,w)=\frac{(1-|z|^2)(1-|w|^2)(1-z\bar w)^2}{(1-|z|^2|w|^2)^3},\\ u_{w\bar w}(z,w)&=\frac{\partial^2u}{\partial w\partial\bar w}(z,w)=\frac{(1-|z|^2)^2|1-\bar zw|^2}{(1-|z|^2|w|^2)^3}.\end{aligned}$$

Since $u_{z\bar{z}} \geq 0$, $u_{w\bar{w}} \geq 0$ and

$$\det \begin{pmatrix} u_{z\bar{z}}(z,w) & u_{z\bar{w}}(z,w) \\ u_{\bar{z}w}(z,w) & u_{w\bar{w}}(z,w) \end{pmatrix} = 0,$$

it follows that u is a maximal plurisubharmonic function $\mathbb{D}^2$ and $u = PB_f$. We will next obtain an explicit formula for PB_{-f}. Let $F(z,w) = (-z,w)$. Then $-\operatorname{Re}(z\bar{w}) = \operatorname{Re}(z\bar{w}) \circ F$ and $PB_{-f} = PB_f \circ F$ by Lemma 8.11. Thus,

$$PB_{-f}(z,w) = \frac{-\operatorname{Re}(z\bar{w})(2-|z|^2-|w|^2)}{1-|zw|^2} - \frac{(1-|z|^2)(1-|w|^2)}{1-|zw|^2}$$

and we get $PB_f + PB_{-f} \neq 0$, hence PB_f is not pluriharmonic on $\mathbb{D}^2$ and therefore condition (3) in Theorem 8.1 is not true for f.

Example 8.15 below shows that there exists a compliant function f for which PB_f is not pluriharmonic. Note that when $n \geq 3$ this is not possible (Theorem 8.1).

Example 8.15 ([8]). Let $\mathbb{D}^2 = \{(z,w) \in \mathbb{C}^2 : |z| < 1, |w| < 1\}$ be the unit polydisc in $\mathbb{C}^2$ and let $f(z,w) = \operatorname{Re}(z\bar{w})$ be as in Example 8.14. Since

$$(PB_f + PB_{-f})(z,w) = -2\frac{(1-|z|^2)(1-|w|^2)}{1-|zw|^2},$$

it is clear that

$$\lim_{\substack{z\to\xi \\ z\in\Omega}} (PB_f + PB_{-f})(z) = 0$$

for every $\xi \in \partial\Omega$. By straightforward calculations we get

$$(dd^c(PB_f + PB_{-f}))^2 = 128\frac{(1-|z|^2)^2(1-|w|^2)^2}{(1-|z|^2|w|^2)^4}\, dV_2$$

and

$$\int_{\mathbb{D}^2} (dd^c(PB_f + PB_{-f}))^2 = 128 \int_{\mathbb{D}^2} \frac{(1-|z|^2)^2(1-|w|^2)^2}{(1-|z|^2|w|^2)^4}\, dV_2(z,w) = \frac{64\pi^2}{3},$$

where dV_2 is the Lebesgue measure on $\mathbb{C}^2$. Thus, f is a compliant function on $\partial\mathbb{D}^2$.

In [77] Rudin proved the following theorem which characterizes those continuous functions on the boundary of the unit bidisc in $\mathbb{C}^2$ which can be extended inside the bidisc to pluriharmonic functions continuous up to the boundary.

Theorem 8.16. *Let $f \in \mathcal{C}(\partial\mathbb{D} \times \partial\mathbb{D})$, and let $d\sigma$ be the normalized Lebesgue measure on $\partial\mathbb{D}$. Then the Poisson integral of f defined by*

$$P[f](z_1,z_2) = \int_{\partial\mathbb{D}\times\partial\mathbb{D}} \frac{(1-|z_1|^2)(1-|z_2|^2)}{|w_1-z_1|^2|w_2-z_2|^2} f(w_1,w_2)\, d\sigma(w_1)\, d\sigma(w_2)$$

is a 2-harmonic function (i.e. harmonic in each variable separately) on $\mathbb{D}^2$ which is continuous on $\overline{\mathbb{D}^2}$. Furthermore, $P[f]$ is pluriharmonic function in $\mathbb{D}^2$ if, and only if,

$$\int_{\partial\mathbb{D}\times\partial\mathbb{D}} w_1^{k_1}\bar{w}_2^{k_2} f(w_1,w_2)\, d\sigma(w_1)\, d\sigma(w_2) = 0$$

for every $k_1, k_2 \in \mathbb{N}$. Moreover, if u is a 2-harmonic function on $\mathbb{D}^2$, and continuous on $\overline{\mathbb{D}^2}$, then $u = P[u]$.

Using Rudin's result we obtain a similar result for $\partial\Omega_\pi$:

Proposition 8.17. *Let U be an open neighborhood of the closure of $\mathbb{D}^2$ in $\mathbb{C}^2$, and let $\pi : U \to \mathbb{C}^2$ be a proper holomorphic map. Let $\Omega_\pi := \pi(\mathbb{D}^2)$, and let $f : \partial\Omega_\pi \to \mathbb{R}$ be a continuous function. The following are then equivalent:*

(1) *there exists a function u which is pluriharmonic on Ω_π, continuous on $\bar{\Omega}_\pi$ and $u|_{\partial\Omega_\pi} = f$,*
(2) *f is harmonic in the sense of Definition 8.8 and satisfies*

$$\int_{\partial\mathbb{D}\times\partial\mathbb{D}} w_1^{k_1}\bar{w}_2^{k_2} f(\pi(w_1,w_2))\,d\sigma(w_1)\,d\sigma(w_2) = 0 \tag{8.5}$$

for every $k_1, k_2 \in \mathbb{N}$.

Proof. (1)⇒(2): Similarly to the proof of the implication (1)⇒(2) in Theorem 8.1 one can show that f is harmonic in the sense of Definition 8.8. By assumption it follows also that PB_f is a pluriharmonic function on Ω_π. Therefore by Lemma 8.11, $PB_{f\circ\pi} = PB_f \circ \pi$ and $PB_f \circ \pi$ is pluriharmonic on $\mathbb{D}^2$, continuous on $\overline{\mathbb{D}^2}$, and $PB_f \circ \pi = f \circ \pi$ on $\partial\mathbb{D}^2$. Then $PB_{f\circ\pi} = P[f\circ\pi]$ is the Poisson integral of $f\circ\pi$, so by Theorem 8.16 we conclude that (8.5) holds.

(2)⇒(1): Assume that f satisfies (2), so (8.5) holds for every $k_1, k_2 \in \mathbb{N}$. By Theorem 8.16 again, $P[f\circ\pi]$ is pluriharmonic on $\mathbb{D}^2$. The assumption that f is harmonic in the sense of Definition 8.8 implies that $P[f\circ\pi] = PB_{f\circ\pi}$ is pluriharmonic on $\mathbb{D}^2$, continuous on $\partial\mathbb{D}^2$, and $PB_{f\circ\pi} = f\circ\pi$ on $\partial\mathbb{D}^2$. Therefore by Lemma 8.11, PB_f is pluriharmonic on Ω_π and the proof is complete. ∎

We end this chapter by proving a sufficient condition for a continuous function defined on $\partial\mathbb{D}^2$ to be compliant.

Proposition 8.18 ([8]). *If $f : \partial\mathbb{D}^2 \to \mathbb{R}$ is a pluriharmonic function in the sense of Definition 8.8 which satisfies*

$$\sum_{k_1,k_2=0}^{\infty} \sqrt{k_1k_2}\,|a_{k_1,k_2}| < \infty,$$

then f is compliant on $\mathbb{D}^2$. Here

$$a_{k_1,k_2} = \int_{\partial\mathbb{D}\times\partial\mathbb{D}} w_1^{k_1}\bar{w}_2^{k_2} f(w_1,w_2)\,d\sigma(w_1)\,d\sigma(w_2),$$

where $d\sigma$ is normalized Lebesgue measure on $\partial\mathbb{D}$.

Proof. For all integers $k, l \ge 1$ let $f_{k,l} = \mathrm{Re}(\zeta^k\bar{\xi}^l)$ and $g_{k,l} = \mathrm{Im}(\zeta^k\bar{\xi}^l)$. Then it follows by Example 8.15 that

$$\begin{aligned}\int_{\mathbb{D}^2} (dd^c(PB_{f_{k,l}} + PB_{-f_{k,l}}))^2 &= \int_{\mathbb{D}^2} (dd^c(PB_{g_{k,l}} + PB_{-g_{k,l}}))^2 \\ &= kl\int_{\mathbb{D}^2} (dd^c(PB_{f_{1,1}} + PB_{-f_{1,1}}))^2 = \frac{64\pi^2 kl}{3},\end{aligned}$$

and therefore $f_{k,l}$ and $g_{k,l}$ are compliant. Let

$$u(z_1,z_2)=\int_{\partial\mathbb{D}\times\partial\mathbb{D}}\frac{(1-|z_1|^2)(1-|z_2|^2)}{|w_1-z_1|^2|w_2-z_2|^2}f(w_1,w_2)d\sigma(w_1)d\sigma(w_2),$$

Then u is 2-harmonic on $\mathbb{D}$, continuous on $\bar{\mathbb{D}}$ and $u|_{\partial\mathbb{D}}=f$. Note that u is, in general, not pluriharmonic. Then there exists a holomorphic function U defined on $\mathbb{D}^2$ such that

$$u(z_1,z_2)=\operatorname{Re}(U)+\sum_{k_1,k_2=0}^{\infty}a_{k_1,k_2}\bar{z}_1^{k_1}z_2^{k_2}+\sum_{k_1,k_2=0}^{\infty}\bar{a}_{k_1,k_2}z_1^{k_1}\bar{z}_2^{k_2}$$

(see e.g. [77]), hence

$$f(z_1,z_2)=\operatorname{Re}(U)+\sum_{k_1,k_2=0}^{\infty}b_{k_1,k_2}\operatorname{Re}(z_1^{k_1}\bar{z}_2^{k_2})+\sum_{k_1,k_2=0}^{\infty}c_{k_1,k_2}\operatorname{Im}(z_1^{k_1}\bar{z}_2^{k_2}),$$

where $b_{k_1,k_2}=2\operatorname{Re}(a_{k_1,k_2})$ and $c_{k_1,k_2}=-2\operatorname{Im}(a_{k_1,k_2})$. By (8.2) we have

$$PB_f+PB_{-f}\geq\sum_{k_1,k_2=0}^{\infty}(|b_{k_1,k_2}|v(f_{k_1,k_2})+|c_{k_1,k_2}|v(g_{k_1,k_2})),$$

where $v(f_{k_1,k_2})=PB_{f_{k_1,k_2}}+PB_{-f_{k_1,k_2}}$ and $v(g_{k_1,k_2})=PB_{g_{k_1,k_2}}+PB_{-g_{k_1,k_2}}$. Now it follows from Corollaries 3.6 and 3.15 that

$$\begin{aligned}
&\Big(\int_{\mathbb{D}^2}(dd^c(PB_f+PB_{-f}))^2\Big)^{1/2}\\
&\quad\leq\bigg(\int_{\mathbb{D}^2}\Big(dd^c\sum_{k_1,k_2=0}^{\infty}(|b_{k_1,k_2}|v(f_{k_1,k_2})+|c_{k_1,k_2}|v(g_{k_1,k_2}))\Big)^2\bigg)^{1/2}\\
&\quad\leq\sum_{k_1,k_2=0}^{\infty}|b_{k_1,k_2}|\Big(\int_{\mathbb{D}^2}(dd^cv(f_{k_1,k_2}))^2\Big)^{1/2}+\sum_{k_1,k_2=0}^{\infty}|c_{k_1,k_2}|\Big(\int_{\mathbb{D}^2}(dd^cv(g_{k_1,k_2}))^2\Big)^{1/2}\\
&\quad\leq\frac{8\pi}{\sqrt{3}}\sum_{k_1,k_2=0}^{\infty}\sqrt{k_1k_2}(|b_{k_1,k_2}|+|c_{k_1,k_2}|)\leq\frac{16\sqrt{2}\pi}{\sqrt{3}}\sum_{k_1,k_2=0}^{\infty}\sqrt{k_1k_2}\,|a_{k_1,k_2}|<\infty.
\end{aligned}$$

Thus, f is compliant. ■

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List of symbols

$\mathbb{N}$	the set of natural numbers
$\mathbb{Q}$	the set of rational numbers
$\mathbb{R}$	the field of real numbers
$\mathbb{C}$	the field of complex numbers
$\mathbb{D}_r$	$\{z \in \mathbb{C} : \|z\| < r\}$, $r > 0$
$\mathbb{D} = \mathbb{D}_1$	the unit disc in $\mathbb{C}$
$\mathbb{B}$	the unit ball in $\mathbb{C}^n$
$B(a, r)$	the ball with center at a and radius r
A^c	the complement of the set A
$A \Subset B$	the set A is relatively compact in the set B
χ_A	the characteristic function of the set A
$u * v$	the convolution of the functions u and v
dV_n	the Lebesgue measure defined on $\mathbb{R}^{2n}$ ($\simeq \mathbb{C}^n$)
$d\sigma$	the Lebesgue measure on $\partial\mathbb{B}$
a.e.	almost everywhere
δ_z	the Dirac measure at z
$\operatorname{supp}\mu$	the support of the measure μ
v^*	the upper semicontinuous regularization of v
$\tilde{u}$	the smallest maximal plurisubharmonic majorant of a plurisubharmonic function u
$\mathcal{L}u$	the Levi form of u
$\mathcal{PSH}(\Omega)$	the family of plurisubharmonic functions on Ω
$\mathcal{PSH}^-(\Omega)$	the family of non-positive plurisubharmonic functions on Ω
$\mathcal{MPSH}(\Omega)$	the family of maximal plurisubharmonic functions on Ω
$\mathcal{E}_0$	see Definition 2.8, page 10
$\mathcal{F}$	see Definition 2.9, page 10
$\mathcal{N}$	see Definition 4.5, page 24
$\mathcal{E}$	see Definition 2.9, page 10
$\mathcal{K}(f)$	the definition is stated on page 12
$\mathcal{K}(H)$	see Definition 2.12, page 11
PB_f	the Perron–Bremermann envelope (Definition 2.13, page 12)
C_n	the Bedford–Taylor $\mathbb{C}^n$-capacity
q.e.	quasi-everywhere, i.e. everywhere except on a set of C_n-capacity zero
h_E	the relative extremal function of the set E
μ_u	the Monge–Ampère boundary measure associated with a function $u \in \mathcal{F}$
φ^u	generalized boundary values of a bounded plurisubharmonic function φ with respect the boundary measure μ_u with $u \in \mathcal{F}$
$W^{k,p}_{\mathrm{loc}}(\Omega)$	the set of functions u defined in Ω such that $D^\alpha u \in L^p_{\mathrm{loc}}(\Omega)$ for all $\|\alpha\| = k$
$\mathcal{C}^{k,\alpha}(\Omega)$	the set of functions u such that $u \in \mathcal{C}^k(\Omega)$ and for any $\|\beta\| = k$, $D^\beta u$ is α-Hölder continuous, $0 < \alpha \leq 1$
$\mathcal{J}_z$	the set of all Jensen measures with barycenter at z